MATH 8450 – SPRING 2024 – HOMEWORK 1

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Problem 1: The Legendre-Fenchel Transform for true functions

Let F be a smooth (infinitely differentiable) function from $\mathbb{R}^n$ to $\mathbb{R}$. We will say that F is strictly convex if $\forall x \in \mathbb{R}^n$, there exists $\varepsilon(x) > 0$ such that the matrix

$$\mathrm{Hess}(F)(x) - \varepsilon(x) I$$

is positive semidefinite. Here $\mathrm{Hess}(F)(x)$ denotes the Hessian of F at x, i.e., the matrix with entries

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(x)$$

with $1 \le i, j \le n$. We will say that F is uniformly strictly convex (USC) if one can pick $\varepsilon(x)$ so that it is independent of x.

1) Consider the map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \phi(x) = \begin{pmatrix} \frac{\partial F}{\partial x_1}(x) \\ \vdots \\ \frac{\partial F}{\partial x_n}(x) \end{pmatrix} .$$

Show that if F is USC, then ϕ is a diffeomorphism, i.e., it is smooth, bijective, and its inverse is smooth. Hints: use the inverse function theorem (as in the "Baby Rudin") to invert locally and deal with all the smoothness issues. Then concentrate on the real issue: proving the map is globally bijective. For this, consider the maximization problem

$$\sup_{x \in \mathbb{R}^n} \ [xp - F(x)]$$

where $xp = \sum_{i=1}^n x_i p_i$, where the p_i's are fixed real numbers.

2) Does injectivity hold, and does surjectivity hold if we only assume that F is strictly convex instead of USC. Hint: experiment with $n = 1$.

3) For F a USC function, we denote by ψ the inverse of the previous map and define the function $F^* : \mathbb{R}^n \to \mathbb{R}$ given by

$$F^*(p) = xp - F(x)$$

where $x = \psi(p)$. This is called the Legendre-Fenchel transform of F. Show that F^* is strictly convex. We will say F is uniformly strictly convex above and below (USCAB) if it is USC and the map $x \mapsto \mathrm{Hess}(F)(x)$ is bounded from $\mathbb{R}^n$ into the space of real $n \times n$ matrices. Show that if F is USCAB, then so is F^*. Show that, in this case, $F^{**} = F$, namely, the Legendre-Fenchel transform is an involution.

4) What happens if we are dealing with a smooth multiparameter family of functions F_λ, of rather $F(\lambda, \cdot)$? Compute the partial derivatives $\frac{\partial F^*}{\partial \lambda}$.

Problem 2: The Legendre transform for FPSs masquerading as maps

Let $F \in R[[X_1, \ldots, X_n]]$ be a formal power series with coefficients in a commutative ring with unit R. Suppose that 2 is invertible in R. Suppose F has no linear part, i.e.,

$$F = \sum_{\alpha \in \mathbb{N}^n} F_\alpha X^\alpha$$

where $F_\alpha = 0$ if $|\alpha| = 1$. Consider the Hessian at the origin

$$H = \left(\left. \frac{\partial^2 F}{\partial X_i \partial X_j} \right|_{X:=0} \right) .$$

Suppose that H is invertible, i.e., $H \in \mathrm{GL}_n(R)$, i.e., there exists a matrix B with entries in R such that $RB = I$ (with the unit of R on the diagonal), i.e., $\det(H)$ is invertible in R. Consider the formal power series $f_1, \ldots, f_n \in R[[X_1, \ldots, X_n]]$ given by

$$f_i = \frac{\partial F}{\partial X_i} .$$

1) Use the inverse function theorem for FPSs from the lecture, in order to show that there exist a unique tuple of FPSs $g_1, \ldots, g_n$ in some new formal variables $P_1, \ldots, P_n$, such that, under composition of FPSs, we have

$$f_i(g_1(P), \ldots, g_n(P)) = P_i$$

for all i, as equalities in $R[[P_1, \ldots, P_n]]$.

2) Consider $F^* \in R[[P_1, \ldots, P_n]]$ defined by

$$F^* = \left(\sum_{i=1}^{n} g_i(P)\ P_i \right) - F(g_1(P), \ldots, g_n(P))$$

where the g's are the ones from the previous question. Compute the first partial derivatives of F^*. Show that we can repeat the Legendre transform operation and produce F^{**}, and the latter is the original FPS F.

3) What about families? Hint: take for R a ring of FPSs in some other hidden variables $\lambda_1.\lambda_2, \ldots$