UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
Computer Science and Applied Mathematics

APPM2007
Lagrangian Mechanics

Class Test

Date: - **Student Number:** ____________________

Duration: 120 minutes **ID. Number:** ____________________

Instructions

This is a mock examination to give students an idea of the sorts of questions to expect in the final examination and to practice answering questions outside of the examination setting. Note

- This examination may *not* be submitted for credit.
- Model solutions shall *not* be supplied for this mock examination.

For maximum benefit, students should work through this mock examination on their own, under simulated examination conditions. Note the following general instructions,

- Read all the questions carefully.
- Answer all questions.
- Show all working.
- This test comprises 3 questions.
- There are 60 points available, and 60 points is 100%.
- A formula sheet is attached at the end of this test paper.

Question 1 — General Theory (10 Marks)

Answer the following question on the general theory of Lagrangian Mechanics.

[**1.1**] Explain how Lagrangian Mechanics uses the idea of state machines. **(2 Marks)**

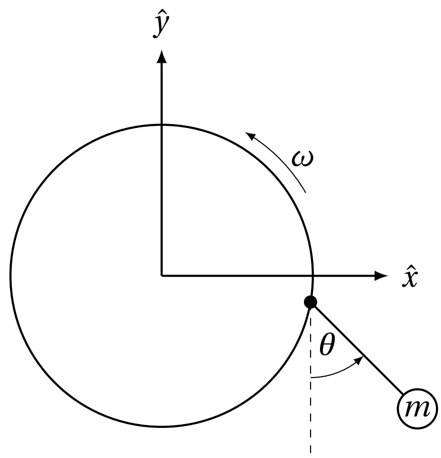
[**1.2**] List any two advantages of using Lagrangian approach to classical mechanics over the standard Newtonian approach. **(2 Marks)**

[**1.3**] Explain the role of the Lagrangian and the Action in the formulation of Lagrangian Mechanics. **(4 Marks)**

[**1.4**] What is a generalized coordinate? **(2 Marks)**

Question 2 — Pendulum on a Wheel (29 Marks)

Consider the following mechanical setup in two dimensions. A bead of mass m_{bead} is attached to a massless rotating wheel of radius l_1. The wheel rotates in a vertical plane with an angular velocity ω measured with respect to the horizontal. The end of a massless rod of length l_2 is attached to the bead and hangs vertically downward from the bead. The rod is free to swing around the bead. As the rod moves, the angular displacement of the rod from the vertical line passing through the center of the bead is measured by the angle theta. At the opposite end of the rod is a ball of mass m_{ball}. See the diagram below, and answer the following questions.



[**2.1**] Determine the position of the bead at time t **(2 Marks)**

[**2.2**] Determine the position of the ball with respect to the bead at a time t in terms of the angular displacement θ. **(2 Marks)**

[**2.3**] Determine the position of the ball with respect to the origin. **(2 Marks)**

[**2.4**] Show that the kinetic energy of the bead is

$$T_{\text{bead}} = \frac{1}{2} m_{\text{bead}} l_1^2 \omega^2.$$

(2 Marks)

[**2.5**] Show that the kinetic energy of the ball is

$$T_{\text{ball}} = \frac{1}{2} m_{\text{ball}} \left(l_1^2 \omega^2 + l_2^2 \dot{\theta}^2 - 2 l_1 l_2 \omega \sin(t\omega - \theta) \dot{\theta} \right).$$

(5 Marks)

[**2.6**] Determine the potential energy of the bead. **(2 Marks)**

[**2.7**] Determine the potential energy of the ball. **(3 Marks)**

[**2.8**] Show that a compatable Lagrangian for this system is

$$\mathscr{L} = \frac{1}{2} m_{\text{bead}} \left(l_1^2 \omega^2 \right) + \frac{1}{2} m_{\text{ball}} \left(l_1^2 \omega^2 + l_2^2 \dot{\theta}^2 - 2 l_1 l_2 \omega \sin(t\omega - \theta) \dot{\theta} \right)$$
$$- g \left((m_{\text{bead}} + m_{\text{ball}}) l_1 \sin(t\omega) - m_{\text{ball}} l_2 \cos(\theta) \right).$$

(2 Marks)

[**2.9**] Compute the Euler-Lagrange Equation for this system. **(5 Marks)**

[**2.10**] How does the equation of motion change when the mass of the bead is doubled and the mass of the ball is halved? Does this align with your expectation of the behaviour of a pendulum? Explain your answers. **(4 Marks)**

Question 3 — Particle-on-a-Spring-on-a-Table (21 Marks)

A particle of mass m is moving on a smooth horizontal table. The (inextensible) string attached to the particle passes through a hole in the centre of the table and is fastened to a light (extensible) spring, with spring constant, as shown in Figure 1. The spring itself is fixed to the ground and is at its rest length when the particle is at the centre of the table. Let the centre of the table be the origin of our reference coordinate system. Choose as generalized coordinates the polar coordinates (r, θ) of the particle. Suppose that the origin of the table coodinate system coincides with the position of the hole.

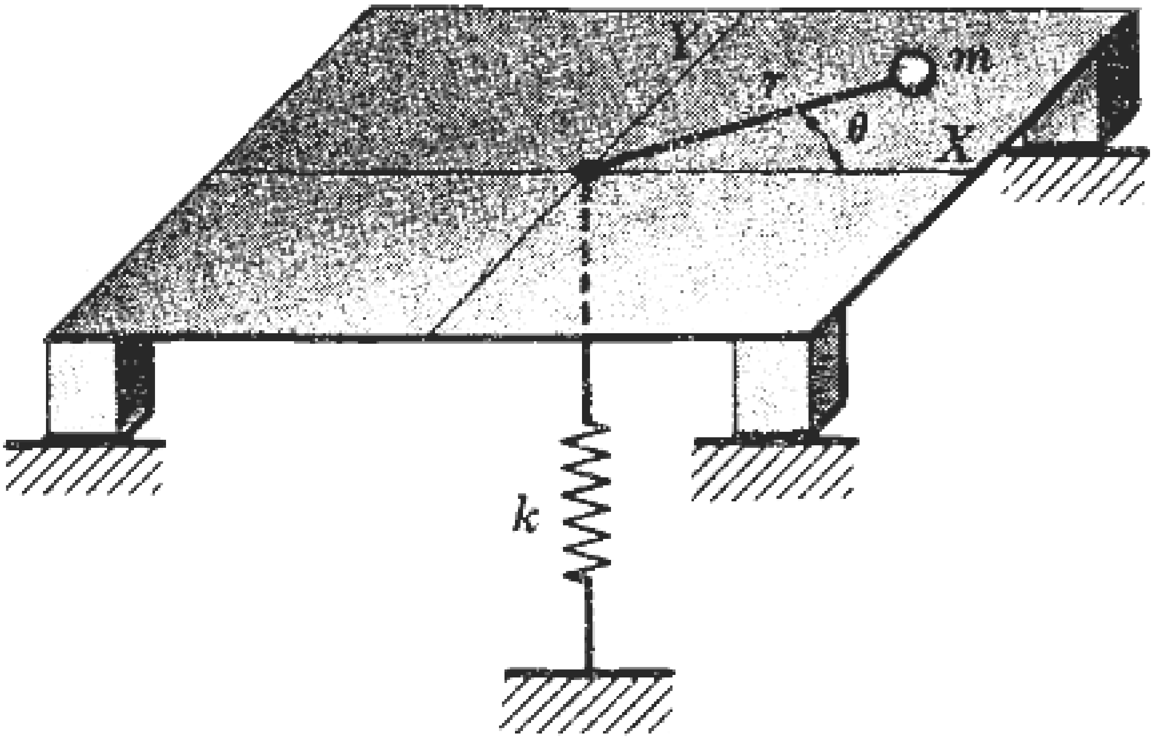


Figure 1: This diagram is adapted from D.A. Wells "Lagrangian Dynamics".

[**3.1**] Write transformation equations giving the x, y and z coordinates of the particle in terms of the generalized coordinates. **(2 Marks)**

[**3.2**] Write down the Kinetic Energy of the particle in terms of the generalized coordinates. **(2 Marks)**

[**3.3**] What is the *gravitational* potential energy of the particle? **(1 Marks)**

[**3.4**] What is the *spring* potential energy of the particle? **(2 Marks)**

[**3.5**] Determine a Lagrangian for the system. **(1 Marks)**

[**3.6**] Identify a cyclic coordinate. **(1 Marks)**

[**3.7**] Show that

$$\dot{\theta} = \frac{j}{mr^2}$$

(1 Marks)

[**3.8**] Show that the equation of motion for this particle in terms of j is

$$m\ddot{r} = \frac{j^2}{mr^3} - kr$$

where $j = \dot{\theta}(0)$ is a constant. **(2 Marks)**

[**3.9**] Show that the particle will move on a circular orbit around the hole when

$$r(0) = \left(\frac{\dot{\theta}^2}{mk}\right)^{1/4} \quad \text{and} \quad \dot{r}(0) = 0.$$

(2 Marks)

[**3.10**] Re-express the Lagrangian for this system in terms of x and y coordinates and show that corresponding Euler-Lagrange equations in terms of x and y are

$$m\ddot{x} + kx = 0 \quad \text{and} \quad m\ddot{y} + ky = 0.$$

(5 Marks)

[**3.11**] Show that the particle moves on an ellipse around the hole in the table **(2 Marks)**

Formula sheet

$$1 = \cos^2(\theta) + \sin^2(\theta)$$
$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

$$\cosh(A \pm B) = \cosh(A)\cosh(B) \pm \sinh(A)\sinh(B)$$
$$\sinh(A \pm B) = \sinh(A)\cosh(B) \pm \cosh(A)\sinh(B)$$
$$\tanh(A \pm B) = \frac{\tanh(A) \pm \tanh(B)}{1 \pm \tanh(A)\tanh(B)}$$

$$\vec{a}\cdot\vec{a} = \|\vec{a}\|^2, \quad \vec{a}\cdot\vec{b} = \|\vec{a}\|\left\|\vec{b}\right\|\cos(\theta) = \sum_{i=1}^{N} a^i b^i$$
$$\vec{v} = \dot{\vec{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{x} \qquad \vec{a} = \dot{\vec{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{v} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{x}$$

$$T = \frac{1}{2} m g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \qquad x^{\mu}, x^{\nu} = \left\{t, r, \theta, \phi\right\} \qquad \dot{x}^{\mu}, \dot{x}^{\nu} = \left\{\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}\right\}$$

$$\int \mathrm{d}x\ \sin^2(x) = \frac{1}{2}x - \frac{1}{4}\sin(2x), \quad \int \mathrm{d}x\ \cos^2(x) = \frac{1}{2}x + \frac{1}{4}\sin(2x), \quad \int \mathrm{d}x\ \cos(x)\sin(x) = -\frac{1}{2}\cos^2(x)$$

$$\hat{I} = \int_V \mathrm{d}m\,\vec{r}_\perp\cdot\vec{r}_\perp = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{xx} = \int_V \mathrm{d}m\left(y^2 + z^2\right), \qquad I_{xy} = I_{yx} = -\int_V \mathrm{d}m\,xy,$$
$$I_{yy} = \int_V \mathrm{d}m\left(x^2 + z^2\right), \qquad I_{xz} = I_{zx} = -\int_V \mathrm{d}m\,xz,$$
$$I_{zz} = \int_V \mathrm{d}m\left(x^2 + y^2\right), \qquad I_{yz} = I_{zy} = -\int_V \mathrm{d}m\,yz.$$

$$T_{\mathrm{lin}} = \frac{1}{2} m\vec{v}\cdot\vec{v} \qquad T_{\mathrm{rot}} = \frac{1}{2}\vec{\omega}^{\top}\hat{I}\vec{\omega} \qquad U = mgh_{\mathrm{cm}}$$