2.6 Adjoint BVP and IVP - Unit 2 Lect 3

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Let $a, b \in \mathbb{R}$ with $a < b$. Consider the inner product space $\mathcal{C}^2([a,b]) \subset L^2([a,b])$ of complex-valued functions defined on $[a,b]$. This space is endowed with its natural inner product

$$\langle u, v \rangle := \int_a^b \bar{u}(x) v(x)\, dx, \qquad \forall u, v \in \mathcal{C}^2([a,b]). \tag{2.8}$$

On that space, we act with the class of second order linear differential operators L, defined by

$$L := p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x), \quad p_k \in \mathcal{C}^{2-k}([a,b]), \quad p_k \text{ complex-valued, } k \in \{0,1,2\}, \tag{2.9}$$

and focus on differential equations

$$Lu(x) = f(x), \ a < x < b.$$

Here $f(x)$ is a given source term. In this context, the domain $\mathcal{D}(L) \subset \mathcal{C}^2([a,b])$ encodes boundary or initial conditions on solutions.

D.L IPDL

- **BVP on $[a, b]$**: $\mathcal{D}(L) := \{u : u \in \mathcal{C}^2([a,b]) \subset L^2([a,b]),\ B_1(u) = B_2(u) = 0\}$ where the most general boundary conditions we will use are given by

$$\boxed{\begin{aligned} B_1(u) &= \alpha_1 u(a) + \beta_1 u'(a) + \eta_1 u(b) + \kappa_1 u'(b) = 0, \\ B_2(u) &= \alpha_2 u(a) + \beta_2 u'(a) + \eta_2 u(b) + \kappa_2 u'(b) = 0, \end{aligned}} \quad ? \tag{2.10}$$

  for $(\alpha_1, \beta_1, \eta_1, \kappa_1)$ and $(\alpha_2, \beta_2, \eta_2, \kappa_2)$ two linearly independent constant four-vectors.

  One has, for $j \in \{1, 2\}$,

$$B_j u_1 = B_j u_2 = 0 \Rightarrow B_j(c_1 u_1 + c_2 u_2) = c_1 B_j(u_1) + c_2 B_j(u_2) = 0 \text{ for all } c_1, c_2 \in \mathbb{R} \text{ or } \mathbb{C},$$

  so the linear boundary conditions respect the vector space structure of $\mathcal{C}^2([a,b])$.
  Common boundary conditions:

$B_1(u) = 0$	$B_2(u) = 0$	Name
$u(a) = 0$,	$u(b) = 0$	(Dirichlet)
$u'(a) = 0$,	$u'(b) = 0$	(Neumann)
$u(a) = u(b)$,	$u'(a) = u'(b)$	(periodic)
$\alpha_1 u(a) + \beta_1 u'(a) = 0$,	$\eta_2 u(b) + \kappa_2 u'(b) = 0$	(mixed)

  Note: $\mathcal{D}(L)$ is dense in $L^2([a,b])$ since $\mathcal{C}^\infty([a,b]) \subset \mathcal{C}^2([a,b])$ and $\mathcal{C}^\infty([a,b])$ is dense in $L^2([a,b])$.

- **IVP on $[a, b]$**:

$$\mathcal{D}(L) := \{u : u \in \mathcal{C}^2([a,b]) \subset L^2([a,b]),\ u(a) = 0, u'(a) := \left.\frac{du}{dx}\right|_{x=a} = 0\}. \tag{2.11}$$

- **Green's formula and formal adjoint L^* of the operator L:**

Proposition 2.26 (Green's formula) *Consider the differential operator L defined in (2.9). For every $u, v \in \mathcal{C}^2([a,b])$, one has*

$$\langle Lu, v\rangle = \langle u, L^* v\rangle + \underbrace{\Big[\overline{p_0}\,(v\overline{u'} - v'\overline{u}) + (\overline{p_1} - \overline{p_0'})v\overline{u}\Big]_a^b}_{\text{boundary terms}}, \tag{2.12}$$

where

$$L^* := \overline{p_0}\frac{d^2}{dx^2} + (2\overline{p_0}' - \overline{p_1})\frac{d}{dx} + \overline{p_0}'' - \overline{p_1}' + \overline{p_2} \tag{2.13}$$

*is the **formal adjoint** of the operator L.*

Proof: Use integration by parts.

$$\begin{aligned}
\langle Lu, v\rangle &= \int_a^b \overline{Lu}(x)\,v(x)\,dx = \int_a^b \Big(\overline{p_0}(x)\overline{u''}(x) + \overline{p_1}(x)\overline{u'}(x) + \overline{p_2}(x)\overline{u}(x)\Big)\,v(x)\,dx,\\
&\overset{(1)}{=} \Big[\overline{p_0}\,v\,\overline{u'} + \overline{p_1}\,v\,\overline{u}\Big]_a^b - \int_a^b \Big((\overline{p_0}(x)\,v(x))'\,\overline{u'}(x) + (\overline{p_1}(x)\,v(x))'\,\overline{u}(x) - \overline{p_2}(x)\,\overline{u}(x)\,v(x)\Big)\,dx\\
&\overset{(2)}{=} \Big[\overline{p_0}\,v\,\overline{u'} + \overline{p_1}\,v\,\overline{u} \,-\, (\overline{p_0}\,v)'\,\overline{u}\Big]_a^b + \int_a^b \overline{u}(x)\,\{\,(\overline{p_0}(x)v(x))'' - (\overline{p_1}(x)v(x))' + \overline{p_2}(x)v(x)\,\}\,dx.
\end{aligned}$$

where equality (1) results from integrating by parts once the first two terms of the integral, and equality (2) from integrating the first term by parts a second time. But

$$\Big[\overline{p_0}\,v\,\overline{u'} + \overline{p_1}\,v\,\overline{u} \,-\, (\overline{p_0}\,v)'\,\overline{u}\Big]_a^b = \Big[\overline{p_0}\,(v\overline{u'} - v'\overline{u}) + (\overline{p_1} - \overline{p_0'})\,v\overline{u}\Big]_a^b$$

and

$$\begin{aligned}
(\overline{p_0}(x)v(x))'' - (\overline{p_1}(x)v(x))' + \overline{p_2}(x)v(x) &= \overline{p_0}(x)''v(x) + (2\overline{p_0}' - \overline{p_1})v'(x) + (\overline{p_0}'' - \overline{p_1}' + \overline{p_2})v(x)\\
&:= L^*v(x),
\end{aligned}$$

so we have

$$\begin{aligned}
\langle Lu, v\rangle &= \Big[\overline{p_0}\,(v\overline{u'} - v'\overline{u}) + (\overline{p_1} - \overline{p_0'})\,v\overline{u}\Big]_a^b + \int_a^b \overline{u}(x)\,L^*v(x)\,dx\\
&= \Big[\overline{p_0}\,(v\overline{u'} - v'\overline{u}) + (\overline{p_1} - \overline{p_0'})\,v\overline{u}\Big]_a^b + \langle u, L^*v\,\rangle.
\end{aligned}$$

■

Remark 2.27 The linear operator L^* is called the 'formal adjoint' of the operator L defined in (2.9) because its domain $\mathcal{D}(L^*)$ has not been specified. □

- **Obtaining $\mathcal{D}(L^*)$ when $(L, \mathcal{D}(L))$ yields a BVP or IVP**

We are interested in domains $\mathcal{D}(L^*)$ which force the boundary terms in Green's formula to vanish, resulting in the equality

$$\langle Lu, v\rangle = \langle u, L^*v\,\rangle,$$

which is consistent with the definition of adjoint operator given in Definition 2.25.

Definition 2.28 *Consider a BVP which involves the linear operator* $(L, \mathcal{D}(L))$*, where the domain* $\mathcal{D}(L)$ *is given by (2.10). The* ***adjoint linear operator*** $(\boldsymbol{L^*}, \boldsymbol{\mathcal{D}(L^*)})$ *is given by the formal adjoint* L^* *defined in (2.13) and its domain* $\mathcal{D}(L^*)$*, which consists of all functions* v *whose boundary conditions ensure the vanishing of the boundary terms in (2.12), i.e.*

$$\left[\overline{p_0}\,(v\overline{u'} - v'\overline{u}) + (\overline{p_1} - \overline{p_0'})v\overline{u}\right]_a^b = 0, \quad \forall u \in \mathcal{C}^2([a,b]) \text{ such that } B_1(u) = B_2(u) = 0. \quad (2.14)$$

It is customary to call the set of boundary conditions satisfied by the functions $v \in \mathcal{D}(L^*)$ *the* ***adjoint boundary conditions*** *and to denote them symbolically as* $B_1^*(v) = B_2^*(v) = 0$*.*

Example 2.29 Consider the BVP problem for $L = \frac{d^2}{dx^2}$ and Dirichlet boundary conditions, i.e. $B_1(u) := u(a) = 0, B_2(u) := u(b) = 0$. The domain of L is given by

$$\mathcal{D}(L) = \{u : u \in \mathcal{C}^2([a,b]) \subset L^2([a,b]) : u(a) = u(b) = 0\}.$$

Since L is obtained from setting $p_0 = 1, p_1 = p_2 = 0$ in (2.9), it is immediate from the definition of L^* in (2.13)) that $L = L^*$. But what is $\mathcal{D}(L^*)$?

To determine $\mathcal{D}(L^*)$, we use (2.14). Since $p_0 = 1 = \overline{p}_0$ and $p_1 = p_2 = 0$ in this example, we have

$$\left[\overline{p_0}\,(v\overline{u'} - v'\overline{u}) + (\overline{p_1} - \overline{p_0'})v\overline{u}\right]_a^b = v(b)\overline{u'(b)} - v'(b)\overline{u(b)} - \{v(a)\overline{u'(a)} - v'(a)\overline{u(a)}\,\},$$

but $u(a) = 0 \Rightarrow \overline{u(a)} = 0$ and $u(b) = 0 \Rightarrow \overline{u(b)} = 0$. Therefore, the boundary terms in (2.12) become

$$v(b)\overline{u'(b)} - v(a)\overline{u'(a)} = 0.$$

For these to vanish, one must have $v(a) = v(b) = 0$, as there is no reason why $u'(a)$ and $u'(b)$ should vanish for all functions u. Hence the domain of the operator L^* is

$$\mathcal{D}(L^*) = \{v : v \in \mathcal{C}^2([a,b]) \subset L^2([a,b]) : v(a) = 0 \text{ and } v(b) = 0\}. \qquad \square$$

Definition 2.30 *Consider an IVP which involves the linear operator* $(L, \mathcal{D}(L))$*, where* $\mathcal{D}(L)$ *is given by*

$$\mathcal{D}(L) = \{u : u \in \mathcal{C}^2([a,b]) \subset L^2([a,b]),\ u(a) = 0, u'(a) = 0\}.$$

The adjoint linear operator $(L^*, \mathcal{D}(L^*))$ *is given by the formal adjoint* L^* *defined in (2.13) and its domain* $\mathcal{D}(L^*)$*, which consists of all functions* v *satisfying a minimal set of conditions that ensure the vanishing of the boundary terms in (2.12), i.e.*

$$\left[\overline{p_0}\,(v\overline{u'} - v'\overline{u}) + (\overline{p_1} - \overline{p_0'})v\overline{u}\right]_a^b = 0, \qquad \forall u \in \mathcal{C}^2([a,b]) \text{ such that } u(a) = u'(a) = 0. \quad (2.15)$$

Example 2.31 Let L be as in the previous example, but impose initial conditions instead, $u(a) = 0, u'(a) = 0$, i.e. $\mathcal{D}(L) = \{u : u \in \mathcal{C}^2([a,b]) : u(a) = u'(a) = 0\}$. Green's formula yields

$$\langle u'', v\rangle - \langle u, v''\rangle = [\,v\overline{u}' - v'\overline{u}\,]_a^b = v(b)\overline{u'}(b) - v'(b)\overline{u}(b),$$

since $u(a) = u'(a) = 0 \Rightarrow \overline{u}(a) = \overline{u'}(a) = 0$. Since we cannot expect that, in general, the functions in $\mathcal{D}(L)$ obey $u(b) = u'(b) = 0$ or equivalently, $\overline{u}(b) = \overline{u'}(b) = 0$, we need to restrict $\mathcal{D}(L^*)$ to functions that satisfy $v(b) = v'(b) = 0$, i.e. $\mathcal{D}(L^*) = \{v : v \in \mathcal{C}^2([a,b]) : v(b) = v'(b) = 0\}$. $\square$

End of Unit 2 Lect 3

2.7 Self-adjoint BVP - Unit 2 Lect 4

Definition 2.32 *The BVP $Lu(x) = f(x),\ B_1(u) = B_2(u) = 0$ is **self-adjoint** if $L = L^*$ AND if the adjoint boundary conditions coincide with the boundary conditions of the original BVP. This is equivalent to say that $\mathcal{D}(L) = \mathcal{D}(L^*)$, and so that the linear operator $(L, \mathcal{D}(L))$ is self-adjoint.*

In the Physics literature, self-adjoint linear operators such as those considered here are called 'Hermitian'. The eigenvalues and eigenfunctions of self-adjoint linear operators $(L, \mathcal{D}(L))$ have very nice properties, which are the infinite-dimensional analogue of those of Hermitian matrices.

Example 2.33 Consider the BVP of Example 2.29. The adjoint boundary conditions $v(a) = v(b) = 0$ coincide with the boundary conditons $u(a) = u(b) = 0$, and $\mathcal{D}(L) = \mathcal{D}(L^*)$. Since we also have $L = L^*$, the BVP for $L = \frac{d^2}{dx^2}$ with Dirichlet boundary conditions is self-adjoint. This amounts to say that the unbounded differential operator $(L, \mathcal{D}(L))$ is self-adjoint. □

Example 2.34 Consider the IVP of Example 2.31. There, $\mathcal{D}(L^*) \neq \mathcal{D}(L)$, as the adjoint of the initial conditions (at $x = a$ are final conditions (at $x = b$), so the IVP for $L = \frac{d^2}{dx^2}$ is not self-adjoint, which amounts to say that the operator $(L, \mathcal{D}(L))$ is not self-adjoint, despite the fact that $L = L^* = \frac{d^2}{dx^2}$.

2.8 Sturm-Liouville operators

Let us now examine the conditions under which a second order linear differential operator L of type (2.9) is equal to L^*, and is therefore formally self-adjoint. We have

$$Lu(x) = L^*u(x) \Leftrightarrow p_0 = \overline{p_0},\ p_1 = 2\overline{p_0}' - \overline{p_1},\ p_2 = \overline{p_0}'' - \overline{p_1}' + \overline{p_2},$$

that is, p_0 must be a real-valued function, $Re(p_1) = p_0'$ and $Im(p_2) = \frac{1}{2}Im(p_1')$. Hence, a self-adjoint operator L of type (2.9) is determined by three functions: $p_0, Im(p_1)$ and $Re(p_2)$.
Note: If the coefficients of the self-adjoint operator L are real-valued functions, then L is determined by two real-valued functions $\boldsymbol{p_0}$ and $\boldsymbol{p_2}$.

Definition 2.35 *Let L be a second order linear differential operator of type (2.9)* **with real-valued coefficients**. *If $p_1 = p_0'$, then*

$$L^* = p_0 \frac{d^2}{dx} + p_0' \frac{d}{dx} + p_2 = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2 = L. \tag{2.16}$$

*In this case the operator L is said to be **formally self-adjoint** with respect to the inner product (2.8). We use the following notation for a formally self-adjoint second order linear differential operator,*

$$\mathfrak{L} := L = L^* = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2. \tag{2.17}$$

Note that if $\mathfrak{L}$ is formally self-adjoint, so is $(-\mathfrak{L})$.

Definition 2.36 *Formally self-adjoint operators of the type $\mathfrak{L}$ in (2.17) are called **Sturm-Liouville operators**.*

Example 2.37 The differential operator $\mathfrak{L} = (x^2 - 1)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$ is of the form (2.9) with real-valued functions $p_0(x) = x^2 - 1$, $p_1(x) = 2x$ and $p_2(x) = 0$. Since $p_0'(x) = 2x = p_1(x)$, $\mathfrak{L}$ is formally self-adjoint w.r.t. the inner product (2.8). This operator is directly related to the Legendre differential equation, which may be written as $\mathfrak{L}u = \lambda u$ for λ constant. You have already studied aspects of the Legendre ODE in Calculus I (Many Variable Calculus, Epiphany 2023 Topic 2). $\square$

How can we relate a linear second order differential operator to a Sturm-Liouville operator?

Any operator L of the form (2.9) with real-valued coefficients, if not formally self-adjoint, is related to a formally self-adjoint operator obtained by multiplying L by an appropriate real-valued function $\rho(x)$. Indeed, with

$$\rho\, L = \rho\, p_0 \frac{d^2}{dx} + \rho\, p_1 \frac{d}{dx} + \rho\, p_2,$$

we have, according to the definition (2.13) of the formal adjoint of a given operator,

$$(\rho\, L)^* = \rho\, p_0 \frac{d^2}{dx} + (2(\rho\, p_0)' - \rho\, p_1) \frac{d}{dx} + (\rho\, p_0)'' - (\rho\, p_1)' + \rho\, p_2.$$

If

$$(\rho\, p_0)' = \rho\, p_1, \quad \text{i.e.} \quad \frac{\rho'}{\rho} = \frac{p_1 - p_0'}{p_0}, \tag{2.18}$$

one easily checks that $(\rho\, L)^* = \rho\, L$. The equation (2.18) may be integrated to provide the function ρ required for transforming the differential operator L into a self-adjoint operator $\mathfrak{L}$. Indeed, provided $p_0 \neq 0$, we have

$$\rho = p_0^{-1} \exp\left(\int \frac{p_1}{p_0}\, dx \right). \tag{2.19}$$

To summarise, given an operator L of the general form (2.9), the operator

$$\mathfrak{L} := \rho\, L = \frac{d}{dx}\left(\rho p_0 \frac{d}{dx} \right) + \rho\, p_2 \quad \text{with } \rho = p_0^{-1} \exp\left(\int \frac{p_1}{p_0}\, dx \right)$$

is formally self-adjoint for the inner product (2.8). Note that if $p_1 = p_0'$, the factor $\rho = 1$ and L is formally self-adjoint for the same inner product as expected.

Example 2.38 Consider the linear differential operator $L = \dfrac{d^2}{dx^2} - 2x\dfrac{d}{dx} - 2\alpha$ for α any constant. This operator is not formally self-adjoint since $p_0(x) = 1$ and $p_1(x) = -2x \neq p_0'(x)$. However, according to (2.19), by multiplying L on the left by the function

$$\rho(x) = \exp\left(\int (-2x)\, dx \right) = e^{-x^2},$$

we obtain a formally self-adjoint operator

$$\mathfrak{L} = \rho\, L = e^{-x^2} \frac{d^2}{dx^2} - 2x e^{-x^2} \frac{d}{dx} - 2\alpha e^{-x^2}.$$

Indeed, for $\mathfrak{L}$, $p_0(x) = e^{-x^2}$ and $p_1(x) = -2xe^{-x^2} = p_0'(x)$. $\square$

The Sturm-Liouville theory, which we study next, is set up for self-adjoint, second-order differential operators, and we may use the above technique to render our differential operators self-adjoint.

End of Unit 2 Lect 4