Notation and infrastructure for Notes on Combinatorics

The following is standard (as of 2022):
$\mathbb{N}$ = the non-negative integers
$\mathbb{N}_+$ = the positive integers
$\mathbb{Z}$ = all the integers (pos., neg., and zero)
$\mathbb{R}$ = the real numbers
$\mathbb{C}$ = the complex numbers (a **field**)
$\mathbb{H}$ = the quaternions (a **division algebra**)
$\mathbb{O}$ = the octonions (nearly a division algebra but **non-associative!**)
For $n \in \mathbb{N}$, $[n] := \{j \in \mathbb{N}_+ : 1 \leq j \leq n\}$, which is empty if $n = 0$.
If $x \in \mathbb{R}$, then $\lfloor x \rfloor := \max\{j \in \mathbb{Z} : j \leq x\}$ and $\lceil x \rceil := \min\{j \in \mathbb{Z} : j \geq x\}$.
For S, T sets, $S \cup T := \{x : x \in S \text{ or } x \in T\}$, $S \cap T := \{x : x \in S \text{ and } x \in T\}$,
$S \setminus T := \{x : x \in S \text{ and } x \notin T\}$, and $S \coprod T := S \cup T$ and $S \cap T = \emptyset$.
If $S, T \subseteq U$ (U fixed), then $S^c := U \setminus S$ and $S \setminus T = S \cap T^c$.
If $S_1, \ldots, S_k$ are sets, then $S_1 \times S_2 \times \cdots \times S_k := \prod_{i=1}^k S_i := \{(x_1, \ldots, x_k) : x_i \in S_i\}$.
For $f : S \to T$, $f(S) := \{f(s) : s \in S\}$, $f^{-1}(t) := \{s \in S : f(s) = t\}$, $t \in T$.
If $A, B \subseteq \mathbb{C}$ (or any vector space), $A + B := \{a + b : a \in A,\ b \in B\}$.
If S is a finite set, then $\#S$ or $|S|$ denotes the number of elements in S, its *cardinality*.
Rule of addition: $\#(S \coprod T) = \#S + \#T$.

Rule of multiplication: $\#(S \times T) = \#S \cdot \#T$.

Principle of induction: Let $S(n)$ be a logical statement (True or False, T or F) for each $n \in \mathbb{N}$. If $S(0)$ is T and $\forall n \in \mathbb{N},\ S(n) \implies S(n+1)$, then $\forall n \in \mathbb{N}$, $S(n)$ is T.

Pigeonhole Principle (PHP): If $m < n$, $\#S = m$, $\#T = n$, and $f : T \to S$, then $\exists t \in T$ s.t. $\#f^{-1}(t) > 1$. Also, if $g : S \to T$, then $\exists t \in T$ s.t. $f^{-1}(t) = \emptyset$.

Principle of Inclusion/Exclusion (PIE): Let A, B, C be finite sets. Then

$$\#(A \cup B \cup C) = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Principle of Bijection: Two sets have the same number of elements iff there is a bijection from one to the other.

Notes on Combinatorics

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1 Functional composition

Let $k : X \to Y$ be a function. We call k **injective** if $k(x) = k(x') \implies x = x'$. Equivalently, k is an injection if $x \neq x' \implies k(x) \neq k(x')$. We call k **surjective** if for every $y \in Y$ there is an $x \in X$ such that $k(x) = y$. A **bijection** is a function which is both injective and surjective. For example the **identity function** $Id := Id_X : X \to X$, which is defined by $Id(x) = x$ for all x in X, is a bijection.

Let $f : A \to B$ and $g : B \to C$; the pair (g, f) is called **composable** and the function $g \circ f : A \to C$ is the **functional composition** of f *followed by* g if $\forall a \in A$

$$(g \circ f)(a) := (gf)(a) := g(f(a)).$$

So "$\circ$" is a multiplication of composable functions. The parentheses can be dropped because *this binary operation is associative.* (**Question**: Can you find a connection between "functional composition" and "compositions of a positive integer"?)

A function $h : X \to X$ is an **involution** if $h^2 := h \circ h = Id_X$; e.g., conjugation.

The **inverse** of $k : X \to Y$ is a function $m : Y \to X$ such that $m \circ k = Id_X$ and $k \circ m = Id_Y$. *If k is a bijection, then k has an inverse.* Indeed, let y be in Y. Then since k is surjective, there exists $x \in X$ with $k(x) = y$. Further, since k is injective, there is only one such element. Hence, defining $m(y) = x$, we have $k \circ m = Id_Y$. And $m(k(x)) = x$ so $m \circ k = Id_X$. (The claim also follows from (3.3), (3.4) below.)

Establish the following claims for $f : A \to B$ and $g : B \to C$. The key parts are (3.6) and (3.8) which use (3.3) and (3.4) but the other parts are also interesting.

(3.1) If f and g are both injective, so is $g \circ f$.
(3.2) If f and g are both surjective, so is $g \circ f$.
(3.3) If $g \circ f$ is injective, then f is injective.
(3.4) If $g \circ f$ is surjective, then g is surjective.
(3.5) If $g \circ f$ is inj and f is surj, then g is inj.
(3.6) If $h : X \to X$ is an involution, then h is a bij.

(3.7) Find A, B, $f : A \to B$, $g : B \to A$ s.t. $gf : A \to A$ is inj and g *not* injective.
(3.8) If $k : X \to Y$ has an inverse, then k is bijective.

Here are the proofs.

(3.1) *If f and g are both injective, so is $g \circ f$.*
Pf. Let $a \neq a'$. Then $fa \neq fa'$ (as f is injective) so $gfa \neq gfa'$.

(3.2) *If f and g are both surjective, so is $g \circ f$.*
Pf. For $c \in C$, choose $b \in B$ such that $gb = c$ as g is surjective. Now choose $a \in A$ such that $fa = b$. Then $gfa = c$ so gf is onto.

(3.3) *If $g \circ f$ is injective, then f is injective.*
Pf. Suppose $a \neq a'$. Then, as gf is injective, $gfa \neq gfa'$; hence, $fa \neq fa'$.

(3.4) *If $g \circ f$ is surjective, then g is surjective.*
Pf. Let $c \in C$. As gf is surjective, $\exists\ a \in A$ s.t. $gfa = c$. Hence, g is surjective.

(3.5) *If $g \circ f$ is inj and f is surj, then g is inj.*
Pf. By (3.3) f is injective so if also surj, then f is bijective. Let m be the inverse of f. Then $g = g \circ Id = gfm = (gf) \circ m$ injective by (3.1).

(3.6) *If $h : X \to X$ is an involution, then h is a bij.*
Pf. As Id is a bijection so both inj and surj, we are done by (3.3) and (3.4).

(3.7) *Find A, B, $f : A \to B$, $g : B \to A$ s.t. $gf : A \to A$ is inj and g not injective.*
Pf. 1. Put $A := \{1\}$, $B := \{1, 2\}$, $f(1) = 1$. We don't need to specify g as there is a unique function to a set with one element. Observe $gf = Id_A$. But $g(1) = g(2)$.
Pf. 2. Put $A = B = \mathbb{Z}$, the set of integers, $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(i) = 2i$, and $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(j) := \lfloor j/2 \rfloor$. Then $gf = Id_{\mathbb{Z}}$ and $g(2k+1) = g(2k)$ for all k.

(3.8) *If $k : X \to Y$ has an inverse, then k is bijective.*
Pf. Let m satisfy $mk = Id_X$ and $km = Id_Y$. By (3.3), (3.4), k is inj and surj, resp.

Call f **right-cancellable** if $g_1 f = g_2 f \implies g_1 = g_2$. For sets and functions, right-cancellable is equivalent to surj. (But for topological spaces and continuous maps, right cancellable is equivalent to being onto a subset with closure the whole space.) For sets and functions, **left-cancellable** is equivalent to being injective (see below).

Theorem 1. *f is surjective $\iff$ f is right-cancellable.*

Proof. To show $\implies$, suppose $g_1 \neq g_2$. Then $\exists b$ s.t. $g_1(b) \neq g_2(b)$. As f is surj, $\exists a$ s.t. $fa = b$. Hence, $g_1 fa \neq g_2 fa$ so, by contraposition, f is right-cancellable.
For the reverse implication, suppose f is *not* surjective. So B has at least two elements and one of them, say b_0, is not in $f(A) := \{fa : a \in A\}$. Let $C := \{1, 2\}$ and define $g_1, g_2 : B \to C$ by $g_1(b) = 1\ \forall b$ and $g_2(b) = 1\ \forall b \neq b_0$ and $g_2(b_0) = 2$. Then $g_1 f = g_2 f$ but $g_1 \neq g_2$ so f not right-cancellable. ∎

Ex-FunComp-1 Prove that f is injective iff it is left-cancellable.
Ex-FunComp-2 If $m, m' : Y \to X$ are both inverse to $k : X \to Y$, then $m = m'$.

2 Fibonacci

Consider the set $B_n := \{0, 1\}^n$ which consists of all **binary sequences** of length $n \geq 0$. The **cardinality** $\# B_n$ (number of elements) of B_n is 2^n. How many of these sequences contain no two consecutive 0s and no two consecutive 1s? If $n \geq 1$, there is a first coordinate and the successive coordinates must alternate between the two values. So only two sequences satisfy the constraint, 01010101... and 10101010... .

Things are more interesting if we take only the first constraint. Let s_n denote the *number of binary sequences of length n with no two zeros consecutive*; e.g., $s_1 = \#\{(0), (1)\} = 2$ and $s_2 = \#\{(01), (10), (11)\} = 3$. Also

$$s_n = s_{n-1} + s_{n-2} \text{ for } n \geq 3. \tag{1}$$

Indeed, if the first bit is 0, then the second bit is 1, so the string is determined by its last $n - 2$ bits, and there are s_{n-2} such length-$(n-2)$ binary strings with no-2-consecutive-0s. If the first bit is 1, there are $n-1$ bits left so s_{n-1} possible bit-strings. As the possibilities are mutually exclusive, we add s_{n-2} and s_{n-1} to get s_n.

This interesting recursion was discovered long before computers by Fibonacci. Let $F_0 := 0$, $F_1 := 1$, and for $n \geq 2$, put $F_n := F_{n-1} + F_{n-2}$. The resulting sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

is known as the **Fibonacci sequence**. It has many surprising properties. For $n \geq 1$,

$$F_1 + \cdots + F_n = F_{n+2} - 1. \tag{2}$$

This is true for $n = 1$. The proof proceeds by induction. If true for $n = k$, then

$$F_1 + \cdots + F_k + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1.$$

Another nice property is that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for $n \geq 1$. The basis case $n = 1$ is true as $0 - 1 = (-1)^1$ and it suffices to show that for $n \geq 2$,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)(F_nF_{n-2} - F_{n-1}^2) = F_{n-1}^2 - F_nF_{n-2}.$$

As $F_{n+1} = F_{n-1} + F_n$ and $F_{n-2} = F_n - F_{n-1}$, we do get the equality.

Ex. Fib-1. Find a formula for $F_nF_{n-1} - F_{n+1}F_{n-2}$ and prove that it holds.
Ex. Fib-2. Show that $s_n = F_{n+2}$.

3 Binomial coefficients

Let $\binom{n}{k}$ denote the number of ways to choose k elements from a set of n. As $S \subseteq [n]$ is determined by its complement, $\binom{n}{k} = \binom{n}{n-k}$, and we soon show $\binom{n}{k} = n!/(n-k)!k!$.

Theorem 2 (Binomial Theorem)**.** $(x+y)^n = \sum_{i=0}^n \binom{n}{k} x^k y^{n-k}$.

Proof. We write out the n-th power of the binomial $x + y$ and use distributivity:

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y) = \sum \prod_{i=1}^n a_i,$$

where the sum is over all ways to choose $a_1, \ldots, a_n$ from $\{x, y\}$. There are $\binom{n}{k}$ ways to choose x for k of the a_i; the remaining $n - k$ choices must be y. ■

Corollary 1. (a) $2^n = \sum_{k=0}^n \binom{n}{k}$; (b) $0 = \sum_{k=0}^n (-1)^k \binom{n}{k}$; $3^n = \sum_{k=0}^n 2^k \binom{n}{k}$.

Proof. Take $(x, y) = (1, 1), (-1, 1), (2, 1)$, respectively, in the Binomial Theorem. ■

Binomial coefficients also have a recursion.

Theorem 3. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Proof. Given the formula for $\binom{n}{k}$, this is easy algebra. Instead, we count. Let $\star \in [n]$ be some fixed element. A k-element subset of $[n]$ that *contains* $\star$ is completely determined by the $k-1$-elements chosen from $[n] - \star := [n] \setminus \{\star\}$, and there are $\binom{n-1}{k-1}$ ways this can happen. But a k-element subset that *doesn't contain* $\star$ can be formed in $\binom{n-1}{k}$ ways. As the two alternatives are mutually exclusive, we add the results to get the recursion. ■

Ex. Binom-1 Show that for $n \geq 1$, $\sum_{k=1}^n k\binom{n}{k} = n2^{n-1}$.
Ex. Binom-2 Prove that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \cdots + \binom{k}{k} = \sum_{j=k}^n \binom{j}{k}$ for $0 \leq k \leq n$.

4 Permutations, samples, and combinations

A **k-combination** from $[n]$ is a **subset** *of* k *distinct elements* from $[n]$; the set of all such subsets is denoted $C(n,k)$ or $\binom{[n]}{k}$, and $\#\binom{[n]}{k} = \binom{n}{k}$, the binomial coefficient.

A **k-sample** from $[n]$ is a **sequence** *of* k *not-necessarily-distinct elements* in $[n]$. For instance, a 6-sample from 9 is 422812. By the rule of multiplication, there are n^k choices of k-samples from $[n]$. The k samples are just elements in $Fun([k],[n])$.

A **k-permutation** from $[n]$ is a **sequence** *of* k *distinct elements* in $[n]$. A k-permutation from $[n]$ is an injective function from $[k]$ to $[n]$ and, again by the rule of multiplications, there are

$$n(n-1)\cdots(n-k+1) = n!/(n-k)!$$

such functions. Denote the set of k-permutations from $[n]$ by $P(n,k)$. Forgetting order, each k-permutation from $[n]$ determines a unique k-element subset of $[n]$ while each k-subset is produced by any of the $k!$ possible orderings of its elements, so

$$\binom{n}{k} = \#P(n,k)/k! = n!/(n-k)!k! \tag{3}$$

Another variant is a **circular permutation** which "circularizes" a linear ordering by considering the first element to follow the last element. To make this clear, we need to formalize the notions of relation and equivalence relation.

A **relation** R on a set X is any set of ordered pairs from X; that is, $R \subseteq X \times X$. R is **reflexive** if $(x,x) \in R$, $\forall x \in R$; that is, if R contains $\Delta_X := \{(x,x) : x \in X\}$ (the "diagonal" of $X \times X$); R is **symmetric** if $(x,y) \in R \implies (y,x) \in R$, $\forall x,y \in X$; that is, $op(R) = R$, where $op : X \times X \to X \times X$ maps (x,y) to (y,x); and R is **transitive** if $(x,y),(y,z) \in R \implies (x,z) \in R$, $\forall x,y,z \in X$. A relation on X which is reflexive, symmetric, and transitive is called an **equivalence relation**.

The key property of an equivalence relation is that it provides a partition. Let R be an equivalence relation on X. For x in X, let $\widehat{x}_R := \widehat{x} := \{y \in X : (x,y) \in R\} \subseteq X$ be the **equivalence class** of x w.r.t. R. Then

$$\widehat{x} \cap \widehat{x'} \neq \emptyset \iff \widehat{x} = \widehat{x'}.$$

Indeed, if z is in the intersection, x and x' are both related to z so x and x' are equivalent. Hence. the two subsets are identical. Let $X/R := \{\widehat{x}_R : x \in X\}$. If all equivalence classes have the same cardinality k, then $\#(X/R) = \#(X)/k$. We call F a **relation from** S **to** T if $F \subseteq S \times T$; it is a **function** if $(s,t),(s,t') \in F \implies t = t'$.

Call two permutations in $P(n,k)$ **circularly equivalent** if one is a rotation of the other. This *is* an equivalence relation R_c on $P(n,k)$ and $P_c(n,k) := P(n,k)/R_c$.

A permutation of length k has exactly k distinct rotations; e.g., the equivalence class of $(1,3,5)$ is $\{(1,3,5),(3,5,1),(5,1,3)\}$. (The permutation $(1,5,3)$ and its rotations are not included.) We call the elements of $P_c(n,k)$ the **circular permutations** of length k from $[n]$ and we have

$$\#P_c(n,k) = \#P(n,k)/k = n!/(n-k)!k. \tag{4}$$

Ex-PermComb-1 If a club of 12 people chooses Pres., Vice-Pres., and Treasurer, and then chooses 2 representatives to a senate, in how many ways can this occur?
Ex-PermComb-2 How many samples of 2 people can be drawn from the club?
Ex-PermComb-3 Check $a \equiv b$ iff $a-b$ is divisible by 3 is an equivalence relation on $\mathbb{Z}$, the integers. What are the equivalence classes? Note that $\#(\mathbb{Z}/\equiv)$ is finite!
Ex-PermComb-4 In how many ways can the sequence 1123566 be rearranged?

5 Compositions of an integer

If $n \in \mathbb{N}$, then a **strict composition** of $[n]$ of length k is an ordered sum

$$n = a_1 + a_2 + \cdots + a_k, \text{ with } a_i \in \mathbb{N},\ 1 \le i \le k. \tag{5}$$

The a_i are called the **parts** or **summands**; e.g., $3 = 3 = 1+2 = 2+1 = 1+1+1$ so 3 has four strict compositions. By the PHP, $k \le n$.

Theorem 4. *The set $sc(n)$ of* all *strict compositions of n has 2^{n-1} elements.*

Proof. Write n "1"s in a row and in the $n-1$ gaps, place either a comma or a "+". The set of choices has 2^{n-1} elements (by the rule of multiplication) and the choices are in one-to-one correspondence with the strict partitions. ■

Ex-StrComp-1 Write the sequence of "1"s, commas, and "+"s for 8 = 4+1+1+2.

Drilling down, let $sc(n,k)$ be the set of strict compositions of $[n]$ *into exactly k parts.*

Theorem 5. *For $1 \le k \le n$, $\#sc(n,k) = \binom{n-1}{k-1}$*

Proof. Represent the summands by stars (so if r is one of the k summands, then it is represented by a string of r stars) and represent the $(k-1)$ +'s by vertical bars, which must appear in one of the $n-1$ gaps between the stars. ■

Of course, $\#sc(n) = \sum_{k=1}^{n} \#sc(n,k)$ which gives us a different derivation of Cor. 1a.

Let $wc(n,k)$ denote the set of **weak compositions** of $[n]$ with length k, *where we allow $a_i = 0$ for some of the parts*. For instance, $\#wc(2,3) = 6$; explicitly,

$$2 = 2+0+0 = 0+2+0 = 0+0+2 = 1+1+0 = 1+0+1 = 0+1+1.$$

Theorem 6. *For $k \geq 1, n \geq 1$, $\#wc(n,k) = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.*

Proof. The set of weak compositions is in one-one correspondence with the set of strings of $k-1$ bars and n stars, where bars may appear arbitrarily, so enumerated by the binomial coefficient. ∎

For instance $2+0+0$ is represented by $\star\star|||$ while $|\star|\star$ represents $0+1+1$. Note that no strict composition of n has more than n parts but weak compositions could have arbitrarily many zero-summands.

Here is a nice result which was asked on a quiz.

Theorem 7. *For $n \geq 1$, let $sc_{1,2}(n)$ be the set of compositions of n where each part is 1 or 2. Then $\#sc_{1,2}(n) = F_{n+1}$.*

Proof. Take $(a_1, \ldots, a_k) \in sc_{1,2}(n)$. If $a_1 = 1$, then $(a_2, \ldots, a_k) \in sc_{1,2}(n-1)$; if $a_1 = 2$, then $(a_2, \ldots, a_k) \in sc_{1,2}(n-2)$. Hence, the function $\#sc_{1,2}$ satisfies the recursion $\#sc_{1,2}(n) = \#sc_{1,2}(n-1) + \#sc_{1,2}(n-2)$. But $\#sc_{1,2}(1) = 1 = F_2$ and $\#sc_{1,2}(2) = 2 = F_3$, so $\#sc_{1,2}$ is Fibonacci shifted one ahead: $\#sc_{1,2}(n) = F_{n+1}$. ∎

If $sc_{1,2}(n,k) := sc_{1,2}(n) \cap sc(n,k)$, then $sc_{1,2}(n) = \coprod_{k=\lceil n/2 \rceil}^{n} sc_{1,2}(n,k)$, where "$\coprod$" means disjoint union iterated from the lower to the upper index as for sum.

Ex-Comp-1 Why is $\lceil n/2 \rceil$ the least possible value for k? Give an example!

Instead, consider the set $sc_{1,2,t}(n)$ which is the family of $1,2$-compositions of n that have *exactly t parts equal to 2*. The possible values for t are 0 (all parts equal 1) up to $t = \lfloor n/2 \rfloor$. (Do you see why?) Observe that any element $x \in sc_{1,2,t}(n)$ has length $n - 2t + t = n - t$ (why?) and is completely determined by saying where the t 2's go.
Ex-Comp-2 Let F_n be Fibonacci-n. Write F_n as a sum of binomial coefficients.
Ex-Comp-3 Let $sc_{1,2,3}(n)$ be the set of compositions of n where all parts are 1, 2 or 3. Find a recursion for $sc_{1,2,3}(n)$. Calculate $sc_{1,2,3}(n)$ for $n = 1, \ldots, 4$. Identify the integer sequence $sc_{1,2,3}(n)$.
Ex-Comp-4 Prove the binomial identity for $n, k \geq 1$,

$$\binom{n+k-1}{k-1} = \sum_{i=0}^{k-1} \binom{k}{i} \binom{n-1}{k-i-1}.$$

6 Partitions of an integer and the 12-fold way

Partitions differ from compositions in that order of the summands doesn't matter; a sequence $(a_1, \cdots, a_k)$ s.t. $n = a_1 + \cdots + a_k$ and $a_1 \geq \cdots \geq a_k$ is a **k-partition** of n. As a standard (and very useful) convention, we assume that the sums are listed in non-increasing order, so $a_k = \min_{1 \leq i \leq k} a_i$.

Let $p_k(n)$ denote the number of k partitions.

Theorem 8. *The number of ways to place n unlabeled balls into k unlabeled boxes with ≥ 1 ball per box (bpb) is $p_k(n)$.*

Proof. Start with a placement. Although the boxes are unlabeled, we may count the number of balls in each and then put the boxes into an order such that the number of balls in each is non-increasing. This gives a partition of n into k parts and the correspondence is 1-to-1 and onto. ■

This theorem is 1/12 of the 12-fold way. There are two possibilities for balls (labeled L or unlabeleded U) and two possibilities for boxes (also L or U). Further, we may consider cases ≤ 1 bpb, ≥ 1 bpb, and $\geq 0 \equiv$ *arb* number of balls per box. In the last case, there is no constraint. As $2 \times 2 \times 3$ we get 12. Other cases can often be derived from the 12 standard cases.

We write for the sets of mappings in the 12-fold way

$$(A_n \longrightarrow B_k)_\kappa, \text{ where } A, B \in \{L, U\},$$

n, k are numbers of balls and boxes, and κ is the constraint so placement is *arbitrary*.

From Theorem 8,

$$\#(U_n \longrightarrow U_k)_{\geq 1} = p_k(n).$$

When the number of balls per box is arbitrary, we can reduce to the ≥ 1 case by keeping track of how many of the unlabeled boxes are non-empty.

Theorem 9. $\#(U_n \longrightarrow U_k)_{arb} = \sum_{i=1}^{k} p_i(n)$.

For the ≥ 1 bpb case of $U \longrightarrow L$, strict compositions describe the placements.

Theorem 10. $\#(U_n \longrightarrow L_k)_{\geq 1} = \#sc(n,k) = \binom{n-1}{k-1}$.

For unconstrained bpb case of $U \longrightarrow L$, weak compositions describe the placements.

Theorem 11. $\#(U_n \longrightarrow L_k)_{arb} = \#wc(n,k) = \binom{n+k-1}{k-1}$

Ex-12-fold-1 Work out all four cases when ≤ 1 bpb.
Ex-12-fold-2 Find $\#(L_n \longrightarrow L_k)_{arb}$.
For the remaining three cases, we need another gadget.

7 Stirling numbers of the second kind

A **partition of a set** S is a family of *non-empty* subsets which are *pairwise disjoint* and *have union* S. For example, $[5] = \{1,4\} \cup \{2,3,5\}$. The members of the family are called the **parts** of the partition. A partition of the set $[n]$ produces a partition of the integer n but more than one partition of set $[n]$ can give rise to the same partition of integer n. For instance, both the set-partition $[5] = \{2,3\} \cup \{1,4,5\}$ and the one given above yield the integer partition $5 = 3 + 2$.

Let $\Pi([n], k)$ be the family of all partitions of $[n]$ into k parts and denote by $s^{II}(n,k) := \#\Pi([n], k)$. This is called the **Stirling number of the second kind**.

Theorem 12. $(L_n \longrightarrow U_k)_{\geq 1} = s^{II}(n,k)$.

Proof. Think of the labeled balls as the members of $[n]$. Then placement of balls in boxes partitions $[n]$ into k non-empty parts as we require ≥ 1 bpb. ∎

By considering the number of non-empty boxes, one has the following.

Theorem 13. $(L_n \longrightarrow U_k)_{arb} = \sum_{i=1}^{k} s^{II}(n,i)$.

Finally, for the L to L case, the order of parts in the partition is significant and there are $k!$ orderings of the k parts as the parts, being disjoint non-empty subsets, must all be distinct. Hence, we get

Theorem 14. $\#(L_n \longrightarrow L_k)_{\geq 1} = k!\, s^{II}(n,k)$.

Ex-stirling2-1: Find $g(n,k) := \#(L_n \longrightarrow U_k)_{\geq 2}$ for all $n, k \in \mathbb{N}$.
Ex-stirling2-2: Find $h(n,k) := \#(L_n \longrightarrow U_k)_{=1}$ for all $n, k \in \mathbb{N}$.
Ex-stirling2-3: In what way do the $L \longrightarrow L$ cases count functions? Explain.
Ex-stirling2-4: Prove that $s^{II}(n,k) = ks^{II}(n-1,k) + s^{II}(n-1,k-1)$.
Ex-stirling2-5: Show that $s^{II}(n,2) = 2^{n-1} - 1$.

The Stirling numbers are named for James Stirling who also found the formula

$$\lim_{n\to\infty} n!/\sqrt{2\pi n}(n/e)^n = 1; \text{ that is, } n! \sim \sqrt{2\pi n}(n/e)^n, \quad e = 2.71828\ldots$$

(see for example, https://mathshistory.st-andrews.ac.uk/Biographies/Stirling).
In the next section, we define Stirling numbers of the *first* kind and illustrate PIE both w.r.t. decompositions of permutations into cyclic permutations.