1: The First Problem

See the spread sheet.

2: The second problem

From proposition 4.1, and Theorem 2.3 we have

$$\begin{aligned}
\sum_{j=1}^{n} j^4 &= \sum_{j=1}^{n} \left(24\binom{j}{4} + 36\binom{j}{3} + 14\binom{j}{2} + \binom{j}{1}\right) \\
&= 24\sum_{j=0}^{n}\binom{j}{4} + 36\sum_{j=0}^{n}\binom{j}{3} + 14\sum_{j=0}^{n}\binom{j}{2} + \sum_{j=0}^{n}\binom{j}{1} \\
&= 24\binom{n+1}{5} + 36\binom{n+1}{4} + 14\binom{n+1}{3} + \binom{n+1}{2} \\
&= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)
\end{aligned}$$

3: The third problem

We claim that a permutation w has 1 and 2 in the same cycle *if and only if* (why?) the corresponding encoding of permutation $(b_1, b_2, \cdots, b_n)$ have b_2 being $b_2 = 1$. In this case, $(b_1, b_2, \cdots, b_n)$ maps bijectively to $(b_1, b_3, \cdots, b_n) \in \{0\} \times \{0, 1, 2\} \times \cdots \times \{0, 1, 2, \cdots, n-1\}$. Thus,

$$a(w) = \beta(b_1) + \beta(b_3) + \cdots + \beta(b_n)$$

whose corresponding generating function is concluded as followed

$$\sum_{k=1}^{n} a(n,k)x^k = \sum_{w \in S_n} x^{a(w)} = x(x+2)\cdots(x+n-1)$$

4: The fourth problem

Let

$$A(x) := \sum_{k=1}^{n} a(n,k)x^k = x(x+2)\cdots(x+n-1)$$

$$\begin{aligned}
\sum_{k=0}^{n-1} t(n,k+1)x^{k+1} &= \sum_{k=1}^{n}(c(n,k) - a(n,k))x^k \\
&= \sum_{k=1}^{n} c(n,k)x^k - \sum_{k=1}^{n} a(n,k)x^k \\
&= (x+1)A(x) - A(x) \\
&= A(x)(x+1-1) = xA(x) \\
&= \sum_{k=1}^{n} a(n,k)x^{k+1}
\end{aligned}$$

For any $1 \leq k \leq n-1$, comparing the coefficient of $k+1$ both sides, we get the result.

5: The fifth problem

Select m elements $a_1, a_2, \cdots, a_{n-m}$ (which has $\binom{n}{n-m} = \binom{n}{m}$ ways) from

$$[n+1] = \{1, \cdots, n+1\}$$

$[n+1] - \{a_1, a_2, \cdots, a_{n-m}\}$ has $S(m,k)$ Construct a block $\{n+1, a_1, \cdots, a_{n-m}\}$

$$\{\{1,2,3\}, \{5,6\}, \{7\}, \cdots \{n+1, a_1, \cdots, a_{n-m}\}\}$$

Together,

6: The sixth problem

Note the number of unordered pairs of an n-element set is $\binom{n}{2}$. And #inversion =#(unordered pairs) - #($\{\{i,j\} : i < j \wedge \omega(i) < \omega(j)\}$)

$J(n,k)$ denote the number of permutations of $[n]$ with k non-inversions It is clear that $I(n,k) = J(n,k)$.

We unpack the definition of $I(n,k)$.

$$I(n,k) := \#\{\omega : i < j \wedge \omega(i) > \omega(j) \wedge i,j \in \omega \wedge \omega \in S_n\} = \#I$$

$$J(n,k) := \#\{\omega : i < j \wedge \omega(i) < \omega(j) \wedge i,j \in \omega \wedge \omega \in S_n\} = \#J$$

We construct a bijection $\phi : I \to J$ such that

$$\phi((w_1, w_2, \cdots, w_n)) = (w_n, w_{n-1}, \cdots, w_1) = (w'_1, w'_2, \cdots, w'_n)$$

in this case we have that $w'_i > w'_j$ if and only if $w_i < w_j$ and $w'_i < w'_j$ if and only if $w_i > w_j$ for all $i, j \in [n] \wedge i < j$ Therefore,

$$I(n,k) = J(n,k)$$

Thus

$$I(n,k) = J\left(n, \binom{n}{2} - k\right) = I\left(n, \binom{n}{2} - k\right)$$

7: The seventh problem

Find an explicit formula for $I(n,3)$ where $n \geq 3$.

We have recursion formula

$$\begin{aligned} I(n,3) &= I(n-1,0) + I(n-1,1) + I(n-1,2) + I(n-1,3) \\ &= 1 + (n-2) + I(n-1,2) + I(n-1,3) \\ &= n - 1 + I(n-1,2) + I(n-1,3) \end{aligned}$$

while

$$\begin{aligned} I(n,2) &= I(n-1,2) + I(n-1,1) + I(n-1,0) \\ &= I(n-1,2) + n - 2 + 1 \\ &= I(n-1,2) + n - 1 \end{aligned}$$

and $I(2,2) = 0$

Thus

$$I(n,2) = \frac{(n-2)(n+1)}{2}$$

Hence

$$\begin{aligned} I(n,3) &= I(n-1,3) + \frac{n(n-3)}{2} + n - 1 \\ &= I(n-1,3) + \frac{(n+1)(n-2)}{2} \end{aligned}$$

With $I(3,3) = 1$, we have

$$I(n,3) = \frac{n(n^2-7)}{6}$$

8: The Last Question

We decorate the weighted generating function

$$h(x,y) := \sum_{\omega\in S} x^{c(\omega)} y^{n-c(\omega)} = \sum_{k=1}^{n} x^k y^{n-k}$$

Replace it with the original one in the proof of Theorem 3.9 we have

$$C(x,y) := \sum_{k=1}^{n} c(n,k) x^k y^{n-k} = \prod_{k=0}^{n-1} (x+ky)$$

$$\text{we have } C(1,x) = \sum_{k=1}^{n} C(n,k) x^{n-k} = \prod_{k=0}^{n-1} (1+kx) = \prod_{k=1}^{n-1} (1+kx)$$

3:

$$A_n = 2A_{n-1} + 6A_{n-2}$$

with $A_0 = 2, A_1 = 2$

5:

Proof: Let $c_1 \cdots c_n$ be a string, and # be the number of string statisfying the condition.

$$\begin{aligned} a_n = \#c_1c_2\cdots c_{n-1}c_n &= \underbrace{\#c_1\cdots c_{n-1}2}_{a_{n-1}} \\ &+ \underbrace{\#c_1\cdots c_{n-2}20}_{a_{n-2}} \\ &+ \underbrace{\#c_1\cdots c_{n-2}21}_{a_{n-2}} \\ &= a_{n-1} + 2a_{n-2} \end{aligned}$$

Male and Female:

Let F_n and M_n be the population of males and females respectively. We have the recurrrence relation.

$$(F_n, M_n) = (F_{n-1}+, F_{n-1})$$

Final Question:

Suppose we have a sequence of length $n \geq 2$. Then we consider that this sequence ends with

$$a_1 \cdots a_{n-1} A$$

where A is not Red. Then we have h_{n-1} in this case

$$a_1 \cdots a_{n-1} R$$

where R is red, then a_{n-1} must not be Red. So in this case a_{n-1} has 2 possibility.
Therefore

$$\begin{aligned}
h_n &= \underbrace{\# a_1 \cdots a_{n-1}\text{Blue}}_{h_{n-1}} + \underbrace{\# a_1 \cdots a_{n-1}\text{White}}_{h_{n-1}} \\
&+ \underbrace{\# a_1 \cdots a_{n-2}\text{Blue Red}}_{h_{n-2}} + \underbrace{\# a_1 \cdots a_{n-2}\text{White Red}}_{h_{n-2}} \\
&= 2h_{n-1} + 2h_{n-2}
\end{aligned}$$