THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2022

School of Mathematics and Statistics

MAST10006 Calculus 2

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 30 pages (including this page) with 14 questions and 118 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten or printed).
- No calculators are permitted. No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- Marks may be awarded for correct use of appropriate mathematical techniques; accuracy and validity of any calculations or algebraic manipulations; clear justification or explanation of techniques and rules used; clear communication of mathematical ideas through diagrams; and use of correct mathematical notation and terminology.
- Label all important features, axes, axis intercepts and asymptotes in all graphs. State if you use any standard limits, limit laws, continuity, limit theorems or series tests.
- Formulas from the Calculus 2 formula sheet may be used without further justification. Other formulas should be justified or proved before use.
- If you are writing answers on the exam or masked exam and you need more space, use blank paper. Note this in the answer box, so the marker knows.
- If you are only writing on blank A4 paper, the first page must contain only your student number, subject code and subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

Question 1 (13 marks)

(a) Evaluate the limit $\lim_{\theta \to 0} \cos\left(\frac{\cosh(\theta) - 1}{\theta}\right)$

(b) Evaluate the limit $\lim_{x \to 0} x^2 \tanh\left(\frac{1}{x}\right)$

(c) Let

$$f(x) = \begin{cases} \cos\left(\dfrac{\cosh(x) - 1}{x}\right) & x < 0 \\ a & x = 0 \\ x^2 \tanh\left(\dfrac{1}{x}\right) & x > 0 \end{cases}$$

where $a \in \mathbb{R}$ is a constant. For what value(s) of a is f continuous at $x = 0$? Justify your answer with reference to the definition of continuity.

Question 2 (5 marks)

Determine the value(s) of $c \in \mathbb{R}$ such that the series $\sum_{n=1}^{\infty} \arctan(cn)$ converges.

Question 3 (11 marks)

Determine if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + \cos^2(n) + 2n}{4n^5 + n^2 - 1}$

(b) $\sum_{n=1}^{\infty} n^{-n} n!$

Question 4 (8 marks)

(a) Sketch the graphs of $y = \operatorname{arctanh}(x)$ and $y = \operatorname{sech}(x)$.

(b) Show that

$$\operatorname{arcsech}(x) = \operatorname{arctanh}\big(\sqrt{1-x^2}\big)$$

where $\operatorname{arcsech}(x)$ is the inverse of $f : [0, \infty) \to \mathbb{R}, f(x) = \operatorname{sech}(x)$.

Question 5 (5 marks)

(a) Find all $x \in \mathbb{R}$ such that

$$\cosh(x) + \sinh(x) = -2022$$

or show that there are no such $x \in \mathbb{R}$.

(b) Find all $z \in \mathbb{C}$ such that

$$\cosh(z) + \sinh(z) = -2022$$

or show that there are no such $z \in \mathbb{C}$.

Question 6 (12 marks)

(a) Evaluate the following integral using the complex exponential: $\int e^{-2x} \sin(5x)\, dx$

(b) Evaluate $\int \sqrt{9+x^2}\,dx$

(c) Evaluate $\int x^2 \log(x^2)\, dx$

Question 7 (9 marks)

Consider the ODE

$$\frac{dy}{dx} = \frac{\operatorname{arctanh}(x)}{x^2\, e^y} - \frac{2}{x} \qquad (0 < x < 1)$$

(a) Make the substitution $u = e^y$ and show that the ODE reduces to

$$\frac{du}{dx} = \frac{\operatorname{arctanh}(x)}{x^2} - \frac{2u}{x}$$

(b) Solve the ODE for $\frac{du}{dx}$, and hence find $y(x)$.

More space for answering question 7 (b)

Question 8 (13 marks)

Consider a beehive of 100,000 bees. Suppose that an outbreak of a virus occurs in the beehive. Once a bee becomes infected, the bee remains infected and does not recover. The virus does not kill the bees.

Let $I(t)$ be the number (in thousands) of bees that are infected at time t days after the start of the outbreak. The rate of increase of I at time t is proportional to the product of the number of bees which are infected at time t, and the number of bees which are not infected at time t.

(a) $I(t)$ satisfies the ODE

$$\frac{dI}{dt} = \beta I(100 - I)$$

where $\beta > 0$ is a constant. Explain why, with reference to the information given above.

(b) Find the general solution of the ODE.

More space for answering question 8 (b)

(c) The outbreak begins with 100 infected bees at time $t = 0$. 3 days later there are 1000 infected bees. Find $I(t)$ in terms of t.

(d) What happens to $I(t)$ as $t \to \infty$? Justify your answer. Interpret this in terms of the virus's spread through the beehive.

Question 9 (6 marks)
Consider the ODE

$$\frac{dy}{dx} = \sin(y), \qquad x \geq 0, \quad 0 \leq y \leq 4\pi.$$

Without solving the ODE, sketch the family of solutions of this ODE.

Question 10 (11 marks)
Solve the ODE

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x + 18e^{5x}$$

subject to the initial conditions $y(0) = \dfrac{1}{2}$ and $y'(0) = 6$.

More space for answering question 10

Question 11 (6 marks)

An object of mass 4 kg is suspended vertically from a spring, with spring constant k N m^{-1} where k is a constant. Air resistance acts on the object with damping constant $\beta = 12$ N s m^{-1}. Gravity acts on the object with gravitational constant $g = 9.8$ m s^{-2}. No other forces act on the object. At equilibrium, the spring is stretched by s metres from its natural length. Let $y(t)$ be the distance in metres of the object below its equilibrium position at time t seconds.

(a) Use Newton's second law to derive the equation of motion of the system.

(b) Three springs are available to use in the system: one with spring constant $k = 4$, one with $k = 8$, and one with $k = 16$. Which of these springs, if any, would result in the object moving with oscillations? Justify your answer.

(c) Assume that $k = 4$. Give an example of a possible external force $f(t)$ which, if it were applied to the object, would result in the system oscillating with constant amplitude in the long term, or explain why it is not possible.

Question 12 (7 marks)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = 6 - \sqrt{x^2 + y^2}$$

and let S be the surface given by $z = f(x, y)$.

(a) Find the equation of the level curve corresponding to $z = 2$, and sketch the level curve.

(b) Find the equation of the level curve which passes through the point $(x, y) = (3, 4)$.

(c) Sketch the cross-section of the surface in the xz plane.

(d) Sketch the surface S.

Question 13 (7 marks)
Find the stationary points of $f(x, y) = \frac{1}{2}y^2 + \frac{1}{3}x^3y - xy + 2$, and classify them as local maxima, local minima or saddle points.

Question 14 (5 marks)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function.

(a) At the point $(x_0, y_0) = (1, 5)$, it is known that the equation of the tangent plane to the surface $z = f(x, y)$ is

$$-2x + 3y - z = 17.$$

Find the directional derivative of f at the point $(1, 5)$ in the direction towards the origin.

(b) At the point $(x, y) = (2022, -7)$, it is known that the directional derivative of f is zero in the direction $\theta = \pi/4$, and the directional derivative of f in the direction $\phi = -\pi/4$ is -8, where θ and ϕ are angles measured anticlockwise from the positive x axis.

Find $\nabla f\big|_{(2022,-7)}$.

End of Exam — Total Available Marks = 118

Turn the page for appended material

MAST10006 Calculus 2 Formulae Sheet

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\cot^2 x + 1 = \operatorname{cosec}^2 x \qquad \coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \cosh(2x) = \cosh^2 x + \sinh^2 x$$
$$\cos(2x) = 2\cos^2 x - 1 \qquad \cosh(2x) = 2\cosh^2 x - 1$$
$$\cos(2x) = 1 - 2\sin^2 x \qquad \cosh(2x) = 1 + 2\sinh^2 x$$
$$\sin(2x) = 2\sin x\cos x \qquad \sinh(2x) = 2\sinh x\cosh x$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \qquad \cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$
$$\sin(x+y) = \sin x\cos y + \cos x\sin y \qquad \sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$e^{ix} = \cos x + i\sin x \qquad \operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\operatorname{arcsinh} x = \log(x + \sqrt{x^2+1}) \qquad \operatorname{arccosh} x = \log(x + \sqrt{x^2-1})$$

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0) \qquad \lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$$

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0) \qquad \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (p \in \mathbb{R},\ a > 1)$$

$$\lim_{n\to\infty} \arctan(cn) = \frac{\pi}{2} \quad (c > 0)$$