UNIFORM PROBABILITY SPACES

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

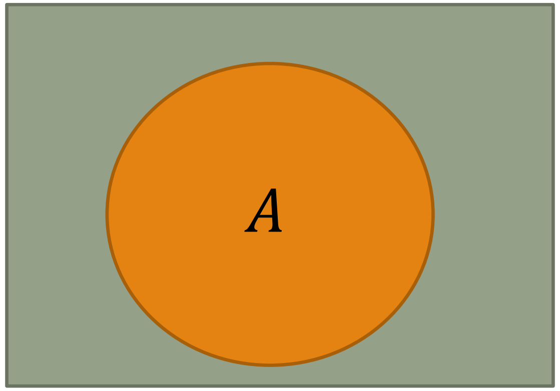
DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

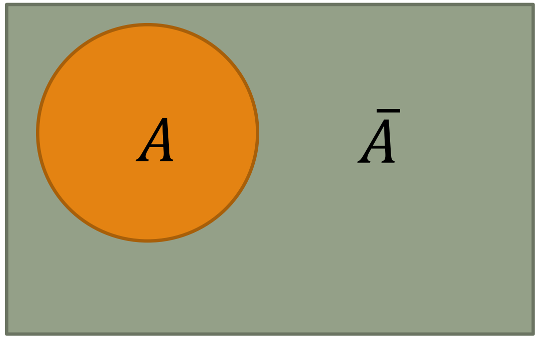
Probability

Sample space S.
Outcome: an element of S.
$\Pr: S \rightarrow \mathbb{R}$
$0 \leq \Pr(w) \leq 1$
$\sum_{w \in S} \Pr(w) = 1$

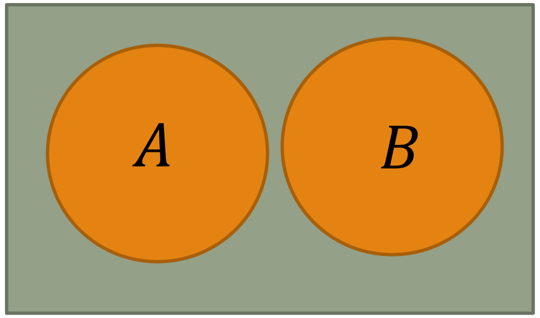
Event: subset A of S.
$\Pr(A) = \sum_{w \in A} \Pr(w)$



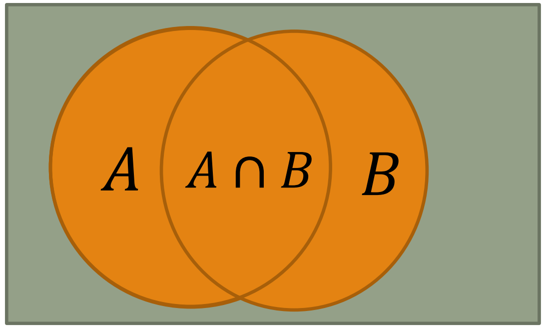
$\Pr(S) = 1$
$\Pr(\emptyset) = 0$
$\Pr(A) = 1 - \Pr(\bar{A})$



If $A \cap B = \emptyset$ then
$\Pr(A \cup B) = \Pr(A) + \Pr(B)$



If $A \cap B \neq \emptyset$ then $\Pr(A \cup B)$
$= \Pr(A) + \Pr(B) - \Pr(A \cap B)$



Uniform Probability

Probability function: A function $\Pr: S \to \mathbb{R}$ such that

$$\sum_{w \in S} \Pr(w) = 1$$

Uniform probability: Every outcome in S has the same probability.

Since they must sum to 1, $\forall w \in S, \Pr(w) = \frac{1}{|S|}$

Now we can define the probability of an **Event** A using **counting** and **sets**

Event A: $\Pr(A) = \sum_{w \in A} \Pr(w)$

$$\Pr(A) = \sum_{w \in A} \frac{1}{|S|}$$
$$= \frac{|A|}{|S|}$$

Some Examples of Uniform Probability:

Rolling a die
Flipping a coin
Drawing a card
Playing the lottery?

Uniform Probability

$$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$$

Lotto 6/49: 6-element subset of $\{1,2,\ldots,49\}$

OLG: uniformly random subset of size 6

You choose: $\{1,2,3,4,5,6\}$

Is this a good idea? Is this better:

$$\{15, 17, 29, 33, 42, 48\}$$

(These numbers won in 2014)

Lotto 6/49 has a uniform sample space.

That means each possibility comes up with equal probability.

Our sample space is defined by:

$S =$ "All subsets of size 6 chosen from $\{1,2,\ldots,49\}$"

How big is our **Sample Space**?

Uniform Probability

$$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$$

Lotto 6/49: 6-element subset of $\{1,2,\ldots,49\}$

OLG: uniformly random subset of size 6

How big is the **Sample Space**?

Assume these numbers are drawn:

$$\{15, 17, 29, 33, 42, 48\}$$

What is a procedure for drawing these numbers?

And in particular, does the order they are drawn in matter? Is there a “first number”?

Order does not matter for Lotto6/49.

If you choose $(15, 17, 29, 33, 42, 48)$ and the numbers are drawn in order: $(48, 17, 29, 33, 42, 15)$ you win.

Thus the size of the **Sample Space** is

$$|S| = \binom{49}{6}$$

Uniform Probability

$$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$$

Lotto 6/49: 6-element subset of $\{1,2,\ldots,49\}$

What is $\Pr(\{1,2,3,4,5,6\})$?

What is $\Pr(\{15,17,29,33,42,48\})$?

$S =$ "Set of all subsets of size 6“

$|S| = \binom{49}{6}$

Thus $\Pr(\{1,2,3,4,5,6\})$

$$= \frac{1}{\binom{49}{6}}$$

$$= 0.00000072$$

and

$$\Pr(\{15,17,29,33,42,48\}) = \frac{1}{\binom{49}{6}}$$

$$= 0.00000072$$

Both have the same probability of being the winning numbers.

(I wonder if $\{1,2,3,4,5,6\}$ has ever been drawn)

Uniform Probability

$$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$$

Deck of 52 cards – suit ♠ ♣ ♥ ♦

rank: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Poker: Hand consisting of 5 cards.

Each hand in poker consists of 5 cards drawn uniformly at random from whatever cards remain in the deck.

For our purposes we will assume a full deck of 52 cards.

Our sample space is then:

$S =$ "all hands of 5 cards from a deck of 52 cards"

And the size of our sample space is:

$$|S| = \binom{52}{5}$$

Uniform Probability

$$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$$

Deck of 52 cards – suit ♠ ♣ ♥ ♦

rank: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Full house: 3 of some rank, 2 of a different rank

Any 5 card hand that we deal from a (uniformly randomly shuffled deck) has a uniform probability.

$S =$ "all hands of 5 cards"

$$|S| = \binom{52}{5}$$

What is the probability that we are dealt a full house? For example:

7H, 7D, 3S, 3C, 3D (two 7's, three 3's)

Uniform Probability

$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$

Deck of 52 cards – suit ♠ ♣ ♥ ♦

rank: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Full house: 3 of some rank, 2 of a different rank

$|S| = \binom{52}{5}$

Define the **Event** $A =$ "we get a full house"

$\Pr(A) = |A|/|S|$

Procedure of being dealt a **Full House** from a deck of 52 cards – a **Sequence** of **Tasks**.

Task 1: We require 3 cards of the same rank. Choose a rank for 3 of a kind

There are 13 ways to do this

Task 2: There are 4 cards of that rank. Choose 3 of them.

There are $\binom{4}{3}$ ways to do this.

Uniform Probability

$$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$$

Deck of 52 cards – suit ♠ ♣ ♥ ♦

rank: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Full house: 3 of some rank, 2 of a different rank

$|S| = \binom{52}{5}$

Define the **Event** $A =$ "we get a full house"

$\Pr(A) = |A|/|S|$

Procedure of being dealt a **Full House** from a deck of 52 cards – a **Sequence** of **Tasks**.

Task 3: We require 2 cards of the same rank.
Choose a rank for 2 of a kind
- must be a different rank from 3 of a kind.

There are 12 ways to do this

Task 4: there are 4 cards of that rank.
Choose 2 of them.

There are $\binom{4}{2}$ ways to do this.

Uniform Probability

$\Pr(A) = \sum_{w \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$

Deck of 52 cards – suit ♠ ♣ ♥ ♦

rank: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Full house: 3 of some rank, 2 of a different rank

$|S| = \binom{52}{5}$

Define the **Event** A = "we get a full house"

$\Pr(A) = |A|/|S|$

Sequence of Tasks:

1. choose rank for 3 cards 13
2. choose 3 suits $\binom{4}{3}$
3. choose rank for 2 cards 12
4. choose 2 suits $\binom{4}{2}$

$|A| = 13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$

$|S| = \binom{52}{5}$

$$\Pr(A) = \frac{|A|}{|S|} = \frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}}$$

$$= 0.00144$$

A *hand of cards* is a subset consisting of five cards. A hand of cards is called a *straight*, if the ranks of these five cards are consecutive and the cards are not all of the same suit.

An Ace and a 2 are considered to be consecutive, whereas a King and an Ace are also considered to be consecutive. For example, each of the three hands below is a straight:

$$8\spadesuit, 9\heartsuit, 10\diamondsuit, J\spadesuit, Q\clubsuit$$

$$A\diamondsuit, 2\heartsuit, 3\spadesuit, 4\spadesuit, 5\clubsuit$$

$$10\diamondsuit, J\heartsuit, Q\spadesuit, K\spadesuit, A\clubsuit$$

- Assume you get a uniformly random hand of cards. Determine the probability that this hand is a straight.

Probability of a flush (all cards have same suit).

$|S| = \binom{52}{5}$

$$Pr(\text{flush}) = \frac{|\text{flush}|}{|S|}$$

$$= \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}}$$

$$= \frac{33}{16660}$$

Procedure for a flush:

① Choose a suit. – 4 choices

③ Choose 5 out of 13 cards $\binom{13}{5}$

$|\text{flush}| = 4 \cdot \binom{13}{5}$

What if we compute Pr(Flush) but we consider the order the cards are dealt?

Flush is still defined the same.

How many hands of 5 cards (where order matters)?

$$|S| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!}$$

How many of these are flushes?

(1) Choose suit : 4 ways

(2) Choose 1^{st} card (of suit) : 13

(3) Choose 2^{nd} Card : 12

11

10

9

$$|\text{Flush}| = 4 \cdot \frac{13!}{8!}$$

$$\Pr(\text{Flush}) = \frac{|\text{Flush}|}{|S|}$$

$$= \frac{\dfrac{4 \cdot 13!}{8!}}{\dfrac{52!}{47!}} = \frac{33}{16\,660}$$

The probabilities are the same (the permutations cancel).

Probability of a pair (P)

$$Pr(P) = \frac{|P|}{|S|}$$

$$|S| = \binom{52}{5}$$

$$= 13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3$$

Procedure to count P:

1. Choose a rank: 13 ways
2. Choose 2 of 4 cards from that rank: $\binom{4}{2}$
3. Choose 3 cards – cannot be a pair, cannot be 3 of a kind.
   $\binom{12}{3}$
4. Choose a suit for each card $4 \cdot 4 \cdot 4$

Probability of 2 Pair:

$|2P| =$

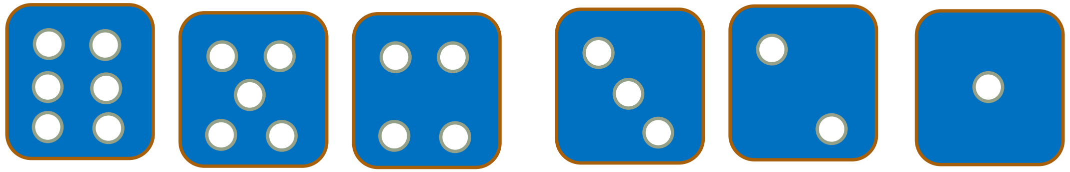
1. Choose 2 ranks: $\binom{13}{2}$
2. Choose 2 of 4 suits (x2): $\binom{4}{2}^2$
3. Choose final card rank: $\binom{11}{1}$
4. Choose final card suit: $\binom{4}{1}$

Newton – Pepys Problem

1693 – Pepys poses problem to Newton
- no notion of probability, sample spaces

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times: B = "at least two 6's"
3. roll a die 18 times: C = "at least three 6's"

Which of these is more likely?
Pepys thought 3 was most likely, 2 was second and 1 was least likely.



Newton answered correctly.

More impressive since there was no notion of sets, sample spaces, outcomes, events.

We will solve it using the framework we have been building.

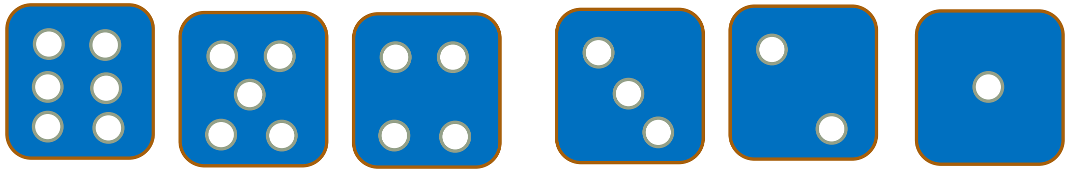
Newton – Pepys Problem

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times: B = "at least two 6's"
3. roll a die 18 times: C = "at least three 6's"

We will start by computing
A = "at least one 6 in six rolls"

What is our **Sample Space?**

All possible rolls made with 6 dice.



$$S = \{r_1, r_2, \dots, r_6\}, 1 \le r_i \le 6$$

What is the size of the sample space?

We can use the **Product Rule** to count.

We define a sequence of tasks:

Task 1: Roll die 1	1. 6 possible outcomes
Task 2: Roll die 2	2. 6 possible outcomes
Task 3: Roll die 3	3. 6 possible outcomes
Task 4: Roll die 4	4. 6 possible outcomes
Task 5: Roll die 5	5. “”
Task 6: Roll die 6	6. “”

Newton – Pepys Problem

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times: B = "at least two 6's"
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We will start by computing
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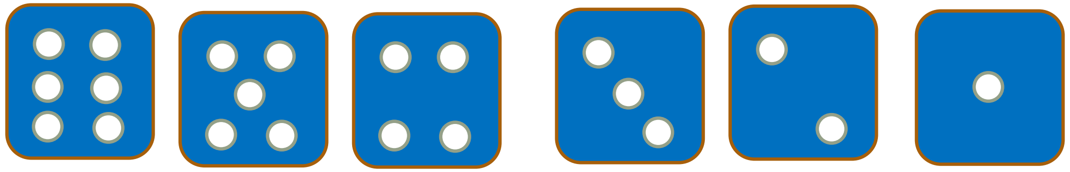
All possible rolls made with 6 dice.



Applying the **Product Rule** over our six **Tasks** we have:

$$|S| = 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6$$
$$= 6^6$$

Event A = “rolled at least one 6”

We know that $\Pr(A) = \frac{|A|}{|S|}$

How do we determine $|A|$?

Newton – Pepys Problem

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times: B = "at least two 6's"
3. roll a die 18 times: C = "at least three 6’s”

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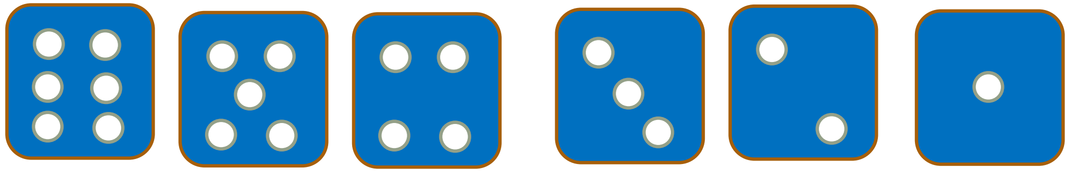


How do we determine $|A|$?

Can break it into subsets:

A_1 = exactly one 6 was rolled
A_2 = exactly two 6′s were rolled
…
A_6 = exactly six 6′s were rolled

This would work, but is tedious.

If we can’t count A directly, we can perhaps count the complement…

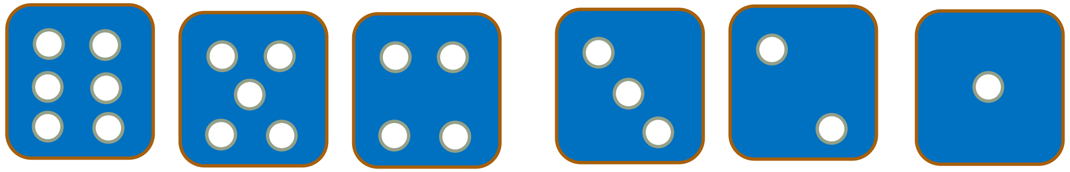
Newton – Pepys Problem

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times: B = "at least two 6's"
3. roll a die 18 times: C = "at least three 6's"

We will start by computing
A = "at least one 6 in six rolls"

What is our **Sample Space?**

All possible rolls made with 6 dice.



Event $\bar{A}$ = "did not roll a six"
= "rolled 1-5 on six dice"

How many ways can we roll 1-5 on six dice?

Task 1: roll 1-5
Task 2: roll 1-5
…
Task 6: roll 1-5

$$\begin{aligned}|\bar{A}| &= 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \\ &= 5^6\end{aligned}$$

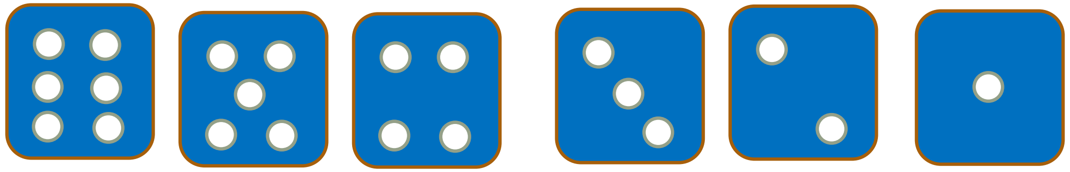
Newton – Pepys Problem

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times: B = "at least two 6's"
3. roll a die 18 times: C = "at least three 6’s”

We will start by computing
A = "at least one 6 in six rolls"

What is our **Sample Space?**

All possible rolls made with 6 dice.



Now two ways to compute $\Pr(A)$. We can use sets directly…

$$|A| = |S| - |\bar{A}|$$
$$|\bar{A}| = 5^6$$
$$|S| = 6^6$$

$$\Pr(A) = \frac{|A|}{|S|}$$
$$= \frac{|S| - |\bar{A}|}{|S|}$$
$$= \frac{6^6 - 5^6}{6^6}$$
$$= 1 - \frac{5^6}{6^6}$$

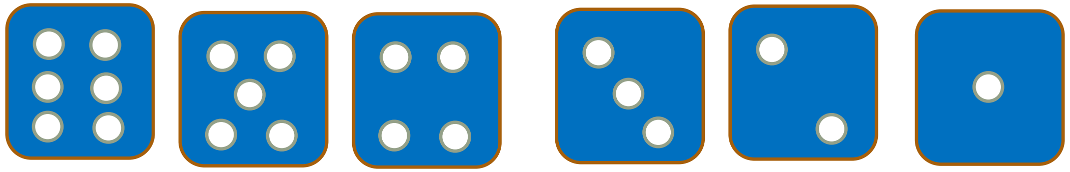
Newton – Pepys Problem

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times: B = "at least two 6's"
3. roll a die 18 times: C = "at least three 6's"

We will start by computing
A = "at least one 6 in six rolls"

What is our **Sample Space?**

All possible rolls made with 6 dice.



Or remember the complement rule of probabilities:

$$\Pr(A) = 1 - \Pr(\bar{A})$$

$$= 1 - \frac{|\bar{A}|}{|S|}$$

$$= 1 - \frac{5^6}{6^6}$$

$$= 0.6651$$

Newton – Pepys Problem

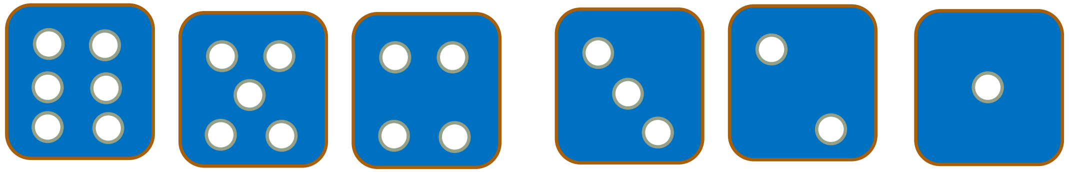
2. roll a die 12 times, roll at least two 6's

Event B = "rolled at least two 6's out of twelve dice"

Sample space S:

$$|S| = \{r_1, r_2, \dots, r_{12}\}, 1 \le r_i \le 6$$
$$|S| = 6^{12}$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$



$$|\bar{B}| = |B_1| + |B_2|$$

Count $|\bar{B}|$:

1. Exactly zero 6's: Event B_1
2. Exactly one 6: Event B_2

Count B_1:

Task 1: roll 1-5 on die 1
Task 2: roll 1-5 on die 2
Task 3: roll 1-5 on die 3
…
Task 12: roll 1-5 on die 12

Product Rule tells us $|B_1| = 5 \cdot 5 \cdot \ldots \cdot 5$
$= 5^{12}$

Newton – Pepys Problem

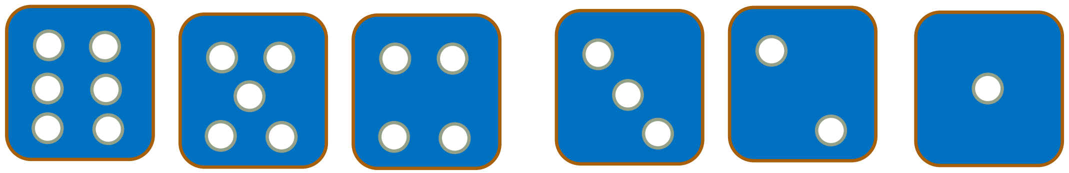
2. roll a die 12 times, roll at least two 6’s

Event B = “rolled at least two 6’s out of twelve dice”

Sample space S:

$$|S| = \{r_1, r_2, \ldots, r_{12}\}, 1 \leq r_i \leq 6$$
$$|S| = 6^{12}$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$



$$|\bar{B}| = |B_1| + |B_2|$$

Count $|\bar{B}|$:

1. Exactly zero 6’s: Event B_1
2. Exactly one 6: Event B_2

Count B_2:

Task 1: choose 1 die to roll a 6

Task 2: roll 1-5 on die 2

Task 3: roll 1-5 on die 3

…

Task 12: roll 1-5 on die 12

Product Rule tells us $|B_2| = 12 \cdot 5 \cdot \ldots \cdot 5$
$= 12 \cdot 5^{11}$

Newton – Pepys Problem

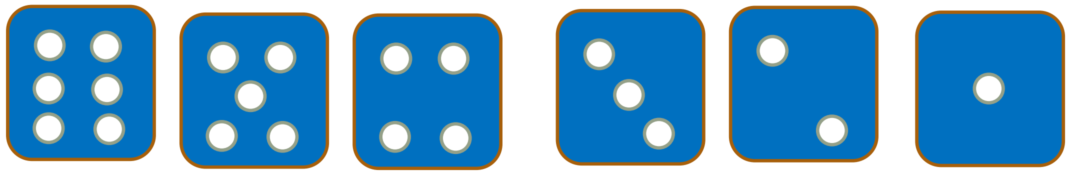
2. roll a die 12 times, roll at least two 6's

Event B = "rolled at least two 6's out of twelve dice"

Sample space S:

$$|S| = \{r_1, r_2, \dots, r_{12}\}, 1 \le r_i \le 6$$
$$|S| = 6^{12}$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$



$$|\bar{B}| = |B_1| + |B_2|$$

Count $|\bar{B}|$:

1. Exactly zero 6's: Event B_1
2. Exactly one 6: Event B_2

$$|B_1| = 5^{12}$$
$$|B_2| = 12 \cdot 5^{11}$$

Since these are disjoint sets we can use the sum rule:

$$|\bar{B}| = 5^{12} + 12 \cdot 5^{11}$$

Newton – Pepys Problem

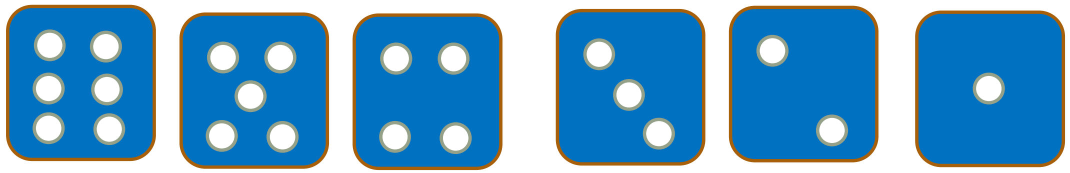
2. roll a die 12 times, roll at least two 6's

Event B = "rolled at least two 6's out of twelve dice"

Sample space S:

$$|S| = \{r_1, r_2, \dots, r_{12}\}, 1 \leq r_i \leq 6$$
$$|S| = 6^{12}$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$



Count $|\bar{B}|$:

$$|\bar{B}| = |B_1| + |B_2|$$

$$= 5^{12} + 12 \cdot 5^{11}$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$

$$= 1 - \frac{5^{12} + 12 \cdot 5^{11}}{6^{12}}$$

$$= 0.6187$$

Newton – Pepys Problem

1. roll a die 6 times: A = "at least one 6"
2. roll a die 12 times B = "at least two 6's"
3. roll a die 18 times C = "at least three 6's"

Event A = "rolled at least one 6 out of six dice"
Event B = "rolled at least two 6's out of twelve dice"

Sample space S_1:

$$|S_1| = \{r_1, r_2, \dots, r_6\}, 1 \le r_i \le 6$$
$$|S_1| = 6^6$$

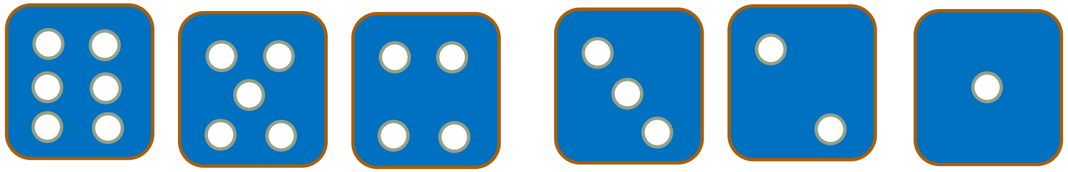
Sample space S_2:

$$|S_2| = \{r_1, r_2, \dots, r_{12}\}, 1 \le r_i \le 6$$
$$|S_2| = 6^{12}$$

$$\Pr(A) = 1 - \Pr(\bar{A})$$
$$= 1 - \frac{|\bar{A}|}{|S_1|} = 1 - \frac{5^6}{6^6} = 0.6651$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$
$$= 1 - \frac{|\bar{B}|}{|S_2|}$$

$$= 1 - \frac{5^{12} + 12 \cdot 5^{11}}{6^{12}} = 0.6187$$



Newton – Pepys Problem

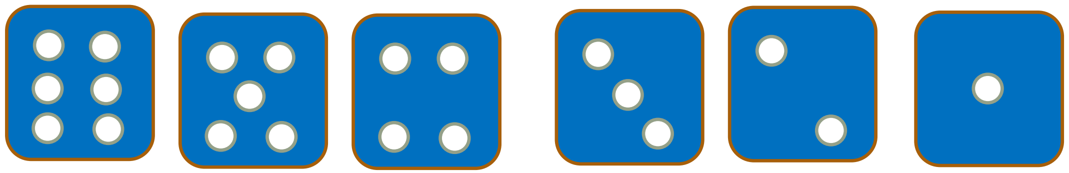
3. roll a die 18 times C = "at least three 6's"

Event C = "rolled at least three 6's out of 18 dice"

Sample space S:

$$|S| = \{r_1, r_2, \ldots, r_{18}\}, 1 \leq r_i \leq 6$$
$$|S| = 6^{18}$$

$$\Pr(C) = 1 - \Pr(\bar{C})$$



$$|\bar{C}| = |C_1| + |C_2| + |C_3|$$

Count $|\bar{C}|$:

1. Exactly zero 6's: Event C_1
2. Exactly one 6: Event C_2
3. Exactly two 6's: Event C_3

Count C_1:

Task 1: roll 1-5 on die 1

Task 2: roll 1-5 on die 2

…

Task 18: roll 1-5 on die 18

Product Rule tells us $|C_1| = 5 \cdot 5 \cdot \ldots \cdot 5$
$= 5^{18}$

Newton – Pepys Problem

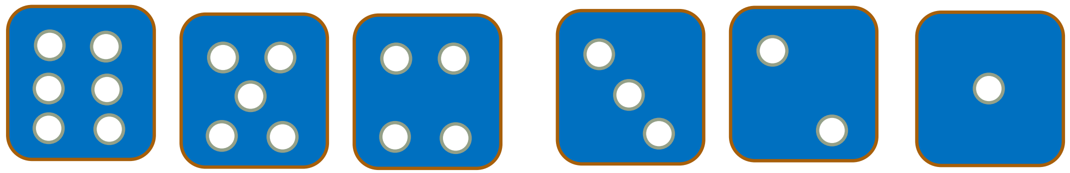
3. roll a die 18 times C = "at least three 6's"

Event C = "rolled at least three 6's out of 18 dice"

Sample space S:

$$|S| = \{r_1, r_2, \ldots, r_{18}\}, 1 \le r_i \le 6$$
$$|S| = 6^{18}$$

$$\Pr(C) = 1 - \Pr(\bar{C})$$



$$|\bar{C}| = |C_1| + |C_2| + |C_3|$$

Count $|\bar{C}|$:

1. Exactly zero 6's: Event C_1
2. Exactly one 6: Event C_2
3. Exactly two 6's: Event C_3

Count C_2:

Task 1: choose 1 die to show 6
Task 2: roll 1-5 on die 2
Task 3: roll 1-5 on die 3
…
Task 18: roll 1-5 on die 18

Product Rule tells us $|C_2| = 18 \cdot 5 \cdot \ldots \cdot 5$
$= 18 \cdot 5^{17}$

Newton – Pepys Problem

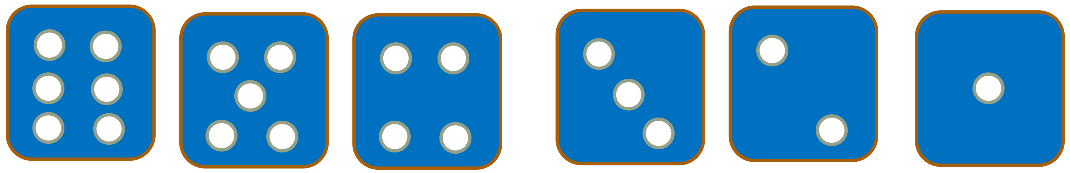
3. roll a die 18 times C = "at least three 6's"

Event C = "rolled at least three 6's out of 18 dice"

Sample space S:

$|S| = \{r_1, r_2, \dots, r_{18}\}, 1 \le r_i \le 6$
$|S| = 6^{18}$

$\Pr(C) = 1 - \Pr(\bar{C})$



$|\bar{C}| = |C_1| + |C_2| + |C_3|$

Count $|\bar{C}|$:

1. Exactly zero 6's: Event C_1
2. Exactly one 6: Event C_2
3. Exactly two 6's: Event C_3

Count C_3:

Task 1: choose 2 dice to show 6
Task 2: roll 1-5 on die 3
…
Task 17: roll 1-5 on die 18

Product Rule tells us $|C_3| = \binom{18}{2} \cdot 5 \cdot \dots \cdot 5$
$= \binom{18}{2} \cdot 5^{16}$

Newton – Pepys Problem

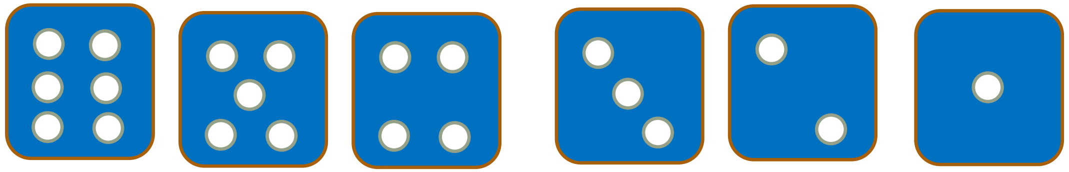
3. roll a die 18 times C = "at least three 6's"

Event C = "rolled at least three 6's out of 18 dice"

Sample space S:

$$|S| = \{r_1, r_2, \ldots, r_{18}\}, 1 \le r_i \le 6$$
$$|S| = 6^{18}$$

$$\Pr(C) = 1 - \Pr(\bar{C})$$



Count $|\bar{C}|$:

$$|\bar{C}| = |C_1| + |C_2| + |C_3|$$

$$= 5^{18} + 18 \cdot 5^{17} + \binom{18}{2} \cdot 5^{16}$$

$$\Pr(C) = 1 - \Pr(\bar{C})$$

$$= 1 - \frac{|\bar{C}|}{|S|}$$

$$= 1 - \frac{5^{18} + 18 \cdot 5^{17} + \binom{18}{2} \cdot 5^{16}}{6^{18}}$$

$$= 0.5973$$

Newton – Pepys Problem

Event A = "rolled at least one 6 out of six dice"
Event B = "rolled at least two 6's out of twelve dice"
Event C = "rolled at least three 6's out of 18 dice"

Sample space S_1, $|S_1| = 6^6$

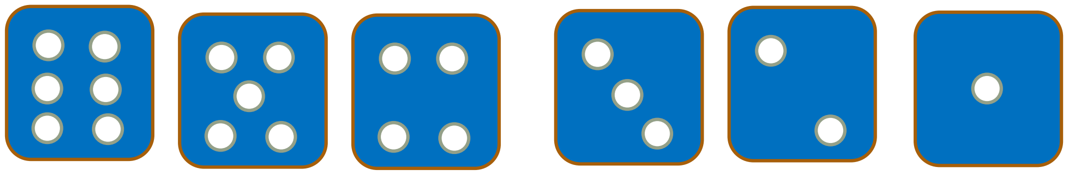
Sample space S_2, $|S_2| = 6^{12}$

Sample space S_3, $|S_3| = 6^{18}$

$$\Pr(A) = 1 - \Pr(\bar{A}) = 1 - \frac{|\bar{A}|}{|S_1|} = 1 - \frac{5^6}{6^6} = 0.6651$$

$$\Pr(B) = 1 - \Pr(\bar{B}) = 1 - \frac{|\bar{B}|}{|S_2|} = 1 - \frac{5^{12}+12\cdot 5^{11}}{6^{12}} = 0.6187$$

$$\Pr(C) = 1 - \Pr(\bar{C}) = 1 - \frac{|\bar{C}|}{|S_3|} = 1 - \frac{5^{18}+18\cdot 5^{17}+\binom{18}{2}\cdot 5^{16}}{6^{18}} = 0.5973$$



Newton – Pepys Problem

This is true generally for any k-sided die. That is, in order of likelihood, we have:

- Roll ≥ 1 k out of k dice
- Roll ≥ 2 k's out of $2k$ dice
- Roll ≥ 3 k's out of $3k$ dice

We can change the odds by rolling more dice. Consider:

- Roll ≥ 1 6 out of 8 dice
- Roll ≥ 2 6's out of 16 dice
- Roll ≥ 3 6's out of 24 dice

That is because in this case, the expected number of $6's$ (proportionally) goes up.

We will see this when we take expected value.
(It is more complex than this, but that is one influence.)

Newton – Pepys Problem

1. roll a die 7 times: A = "at least one 6"
2. roll a die 14 times B = "at least two 6's"
3. ~~roll a die 21 times C = "at least three 6's"~~

Event A = "rolled at least one 6 out of 7 dice"
Event B = "rolled at least two 6's out of 14 dice"

Sample space S_1:

$$|S_1| = \{r_1, r_2, \dots, r_7\}, 1 \le r_i \le 6$$
$$|S_1| = 6^7$$

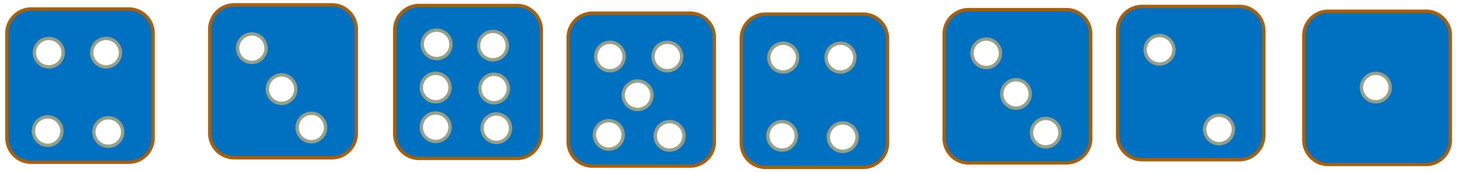
Sample space S_2:

$$|S_2| = \{r_1, r_2, \dots, r_{14}\}, 1 \le r_i \le 6$$
$$|S_2| = 6^{14}$$

$$\Pr(A) = 1 - \Pr(\bar{A})$$
$$= 1 - \frac{|\bar{A}|}{|S_1|} = 1 - \frac{5^7}{6^7} = 0.7209$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$
$$= 1 - \frac{|\bar{B}|}{|S_2|}$$

$$= 1 - \frac{5^{14} + 14 \cdot 5^{13}}{6^{14}} = 0.7040$$



Newton – Pepys Problem

1. roll a die 8 times: A = "at least one 6"
2. roll a die 16 times B = “at least two 6’s”
3. ~~roll a die 24 times C = “at least three 6’s”~~

Event A = “rolled at least one 6 out of 8 dice”
Event B = “rolled at least two 6’s out of 16 dice”

Sample space S_1:

$$|S_1| = \{r_1, r_2, \dots, r_8\}, 1 \le r_i \le 6$$
$$|S_1| = 6^8$$

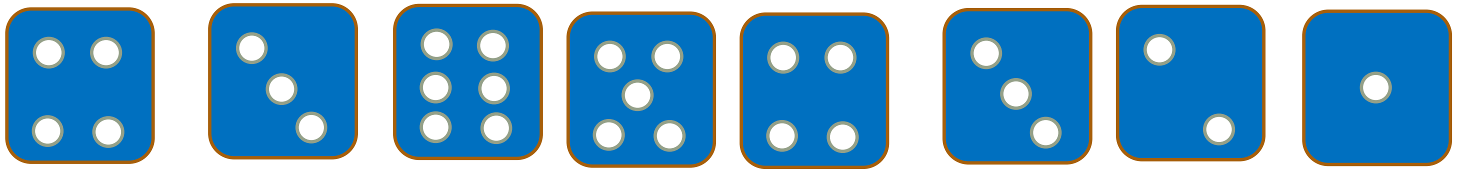
Sample space S_2:

$$|S_2| = \{r_1, r_2, \dots, r_{16}\}, 1 \le r_i \le 6$$
$$|S_2| = 6^{16}$$

$$\Pr(A) = 1 - \Pr(\bar{A})$$
$$= 1 - \frac{|\bar{A}|}{|S_1|} = 1 - \frac{5^8}{6^8} = 0.7674$$

$$\Pr(B) = 1 - \Pr(\bar{B})$$
$$= 1 - \frac{|\bar{B}|}{|S_2|}$$

$$= 1 - \frac{5^{16} + 16 \cdot 5^{15}}{6^{16}} = 0.7728$$



Birthday Paradox

365 days in a year, n people, uniformly random birthdays.

$P_n = \Pr(\geq 2$ people have the same birthday$)$

$n = 2$: $\Pr($2 people same birthday$)$

$$P_2 = \frac{|A|}{|S|} = \frac{365}{365^2} = \frac{1}{365}$$

$$P_{366} = ?$$
$$P_{366} = 1$$

When does the probability cross ½? For what value n?