Problem 1:

Differentiate the following functions with respect to x:

$$y = \frac{8x}{9 - \cot x}$$

Solution. We use the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

where $f = 8x$ and $g = 9 - \cot x$. We compute the derivatives of f and g:

$$\begin{aligned}
y' &= \frac{(8x)'(9 - \cot x) - (8x)(9 - \cot x)'}{(9 - \cot x)^2} \\
&= \frac{8(9 - \cot x) - 8x \cot x'}{(9 - \cot x)^2} \\
&= \frac{8(9 - \cot x) + 8x \csc^2 x}{(9 - \cot x)^2} \\
&= \frac{72 - 8 \cot x + 8x \csc^2 x}{(9 - \cot x)^2}
\end{aligned}$$

Problem 2:

Calculate the area of one petal of the three-petal rose $r = \cos 3\theta$.

Solution.

We use the formula for the area of a polar curve:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

where α and β are the angles at which the curve intersects itself. In this case, we have $\alpha = \pi/2$ and $\beta = 5\pi/6$. The reason for which $\beta = \frac{5\pi}{6}$ is that the curve intersects itself at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, and the area of one petal is the area between these two points.

Thus, we have:

$$\begin{aligned}
A &= \frac{1}{2}\int_{\pi/2}^{5\pi/6} \cos^2 3\theta d\theta \\
&= \frac{1}{2}\int_{\pi/2}^{5\pi/6} \frac{1+\cos 6\theta}{2} d\theta \\
&= \frac{1}{4}\int_{\pi/2}^{5\pi/6} (1+\cos 6\theta) d\theta \\
&= \frac{1}{4}\left(\int_{\pi/2}^{5\pi/6} 1 d\theta + \int_{\pi/2}^{5\pi/6} \cos 6\theta d\theta\right) \\
&= \frac{1}{4}\left(\theta + \frac{\sin 6\theta}{6}\right)_{\pi/2}^{5\pi/6} \\
&= \frac{1}{4}\left(\frac{5\pi}{6} + \frac{\sin 5\pi}{6} - \frac{\pi}{2} - \frac{\sin \pi}{6}\right) \\
&= \frac{\pi}{12}
\end{aligned}$$

4:

Comparison test and ratio test

5:

By the power rule for integrals, we have:

$$\int_0^\infty \frac{1}{x^{3/4}} = \lim_{b\to\infty}\int_0^b \frac{1}{x^{3/4}} = \lim_{b\to\infty} \frac{4}{1/4} b^{1/4} = \infty$$

Thus (a) diverges.

$$\int_0^\infty \frac{1}{e^{-2x}} = \lim_{b\to\infty}\int_0^b \frac{1}{e^{-2x}} = \lim_{b\to\infty} \frac{1}{-1/2} e^{-2x} = -2$$

Thus (b) converges.

7:

Determine whether the following series converge or diverge:

$$\sum_{n=0}^{\infty} \frac{5e^{-2n^2}}{n^2-1}$$

Solution. We use the limit comparison test. We compare the series to $\sum_{n=0}^{\infty} \frac{1}{n^2}$, which converges. We have:

$$\lim_{n\to\infty} \frac{5e^{-2n^2}/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{5e^{-2n^2} n^2}{n^2-1} = \lim_{n\to\infty} \frac{5e^{-2n^2}}{1-1/n^2} = 5$$

3:

Find the volume of region created by rotating the region bounded by $y = -x^2 + 4x + 1$ and $y = 1$ about the $y = 1$-axis.

Solution. We use the formula for volume of a solid of revolution:

$$V = \int_a^b \pi(f(x) - g(x))^2 dx$$

where R is the outer radius and r is the inner radius. In this case, we have $R = 1$ and $r = 1 - x^2 + 4x + 1 = 2 - x^2 + 4x$, and $f(x) = -x^2 + 4x + 1$, $g(x) = 1$ We determine a, and b:

$$\begin{aligned}
-x^2 + 4x + 1 &= 1 \\
-x^2 + 4x &= 0 \\
x^2 - 4x &= 0 \\
x(x - 4) &= 0 \\
x &= 0, 4
\end{aligned}$$

Thus we have:

$$\begin{aligned}
V &= \int_0^4 \pi(f(x) - g(x))^2 dx \\
&= \int_0^4 \pi(x^2 + 4x)^2 dx \\
&= \pi \int_0^4 (x^4 + 8x^3 + 16x^2) dx \\
&= \pi \left(\frac{x^5}{5} + 2x^4 + \frac{16x^3}{3} \right)_0^4 \\
&= \pi \left(\frac{1024}{15} + 128 + \frac{2048}{3} \right) \\
&= \frac{15872\pi}{15}
\end{aligned}$$

Problem 4:

Solution. We use the formula for hydrostatic force:

$$F = \int_a^b w(y) y dA$$

where $w(y)$ is the weight density of the fluid at depth y and dA is the area of the plate at depth y. In this case, we have $w(y) = \gamma y$ and $A = (3 - y)^2/3 \implies dA = \frac{2(3-y)}{3} dy$.

Thus, we have:

$$
\begin{aligned}
F &= \int_0^3 \gamma y \frac{2(3-y)}{3} dy \\
&= \frac{2\gamma}{3} \int_0^3 (3y - y^2) dy \\
&= \frac{2\gamma}{3} \left(\frac{3y^2}{2} - \frac{y^3}{3} \right)_0^3 \\
&= \frac{2\gamma}{3} \left(\frac{27}{2} - 9 \right) \\
&= 3\gamma
\end{aligned}
$$

Put $\gamma = 90 \mathrm{lb}/\mathrm{ft}^3$ We have the force is $F = 270 \mathrm{lb}$.