EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH2031-WE01

Title: Analysis in Many Variables II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 **1.1** Use index notation to show that $\boldsymbol{\nabla} r^2 = 2\boldsymbol{x}$, where $r = |\boldsymbol{x}|$, for $\boldsymbol{x}$ the position vector in $\mathbb{R}^3$.

1.2 Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field on $\mathbb{R}^3$ given by $f(\boldsymbol{x}) = \boldsymbol{\nabla} \cdot r^2\boldsymbol{x}$. Use index notation to show that $\boldsymbol{\nabla} f(\boldsymbol{x}) = 10\boldsymbol{x}$.

1.3 Find the directional derivative of f in the direction $\boldsymbol{n} = 2\,\boldsymbol{e}_1 + 3\,\boldsymbol{e}_2 + 6\,\boldsymbol{e}_3$, at the point $\boldsymbol{p} = 3\,\boldsymbol{e}_1 + \boldsymbol{e}_2 + 2\,\boldsymbol{e}_3$.

Q2 **2.1** State the divergence theorem.

2.2 Let V be the region contained between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$, that is

$$V = \{(x, y, z) \mid x^2 + y^2 \leq z \leq 4\}.$$

Calculate the volume of the region V.

2.3 Using the divergence theorem, calculate $\int_S \boldsymbol{F} \cdot d\boldsymbol{A}$, where S is the boundary of V taken with outwards-pointing normal, and where

$$\boldsymbol{F} = x\,\boldsymbol{e}_1 + 2y\,\boldsymbol{e}_2 + 3(x - y)\,\boldsymbol{e}_3.$$

Q3 Write the curve

$$(\,x^2 + y^2\,)^2 = a^2\,(\,x^2 - y^2\,), \qquad a \in \mathbb{R}_0^+,$$

in polar coordinates and sketch it. Using polar coordinates, evaluate the area of the region in the xy-plane bounded by the curve.

Q4 **4.1** Convert the following linear second order differential operator into Sturm-Liouville form:

$$L = x\,\frac{d^2}{dx^2} + (1 - x)\,\frac{d}{dx} + 1.$$

4.2 Denote by $\mathcal{L}$ the Sturm-Liouville operator obtained in part 4.1. Suppose u is a solution to the boundary value problem

$$\mathcal{L}u(x) = \sin(x) \qquad \text{with} \quad u(1) = 1 \text{ and } u(5) = 5$$

on $[1, 5]$. Find a linear function $g : [1, 5] \to \mathbb{R}$ such that $u = g + v$ where $v(1) = v(5) = 0$.

4.3 With $\mathcal{L}$ as in part 4.2, consider the boundary value problem

$$\mathcal{L}v(x) = h(x) \qquad \text{with} \quad v(1) = 0 \text{ and } v(5) = 0$$

on $[1, 5]$. Give an expression for the source term $h(x)$. Is this boundary value problem self-adjoint? Justify your answer fully.

SECTION B

Q5 Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(\boldsymbol{x}) = \begin{cases} \dfrac{x^2 + y^2}{y} & y \neq 0, \\ a & y = 0, \end{cases}$$

where $\boldsymbol{x} = x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2$, and for $a \in \mathbb{R}$ an unknown constant.

5.1 Find a such that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at the origin. What are the values of the partial derivatives at the origin for this value of a?

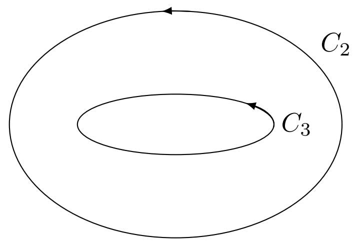
5.2 With the value of a as found in the previous part of this question, is f continuous at the origin? Comment on this in relation to the previous part of this question.

5.3 With the value of a as found in the first part of this question, show that f is not differentiable at the origin.

Q6 **6.1** Let C_1 be any simple closed curve which does not pass through the origin, taken with positive (anti-clockwise) orientation. Assuming C does not enclose the origin, show that

$$\oint_{C_1} \left(\frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy \right) = 0.$$

6.2 Let C_2 and C_3 be simple closed curves, taken with positive (anti-clockwise) orientation, with C_3 contained entirely within the interior of C_2, as in the diagram below:



Let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ be two continuously differentiable scalar fields on $\mathbb{R}^2$. Show that, if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, then

$$\oint_{C_2} (P\; dx + Q\; dy) = \oint_{C_3} (P\; dx + Q\; dy)\,.$$

Hint: Consider splitting the region between the two curves into two pieces, by adding two line segments joining the two curves.

6.3 Let C_4 be any simple closed curve which does not pass through the origin, taken with positive (anti-clockwise) orientation, and which encloses the origin. Calculate

$$\oint_{C_4} \left(\frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy \right).$$

Hint: Consider a small circle, centred on the origin.

Q7 **7.1** Let b and c be two nonzero real constants such that $b \neq \pm c$. Using the Green's function method based on the properties of the Dirac delta distribution, solve the differential equation

$$\frac{d^2u}{dt^2} + 2b\frac{du}{dt} + c^2 u = f(t) \qquad \text{on } [0, \infty),$$

subject to the initial conditions $u(0) = 0$, $\frac{du}{dt}(0) = 0$, and with $f(t) = 0$ when $t < 0$. Full justification of each step is required.

7.2 Once you have obtained the solution $u(t)$ as a function of the general source $f(t)$, evaluate your answer explicitly for $f(t) = \Theta(t)$, where Θ is the unit step function.

Q8 **8.1** Consider the two-dimensional domain

$$D = \mathring{D} \cup \partial D := \{\, (r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi/3 \,\},$$

where $\mathring{D} = \{(r, \theta) : 0 < r < 1,\ 0 < \theta < 2\pi/3\,\}$ is the interior of the domain and ∂D its boundary. Denote the origin of the plane by O and label P the point in $\mathring{D}$ with $\mathbf{OP} := \mathbf{x_0} = \frac{1}{2}\,\mathbf{e}_1 + \frac{1}{2}\,\mathbf{e}_2$. Use the method of images to construct the Green's function $G(\mathbf{x}, \mathbf{x_0})$ satisfying

$$\begin{aligned} &\boldsymbol{\nabla}^2\, G(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}) && \text{for } \mathbf{x} \in D, \\ &G(\mathbf{x}, \mathbf{x_0}) = 0 && \text{for } \mathbf{x} \in \partial D. \end{aligned}$$

You may use the fact that the fundamental solution of Laplace's equation, which is regular on $\mathbb{R}^2$-$\{\mathbf{x_0}\}$, is given by $G_0(\mathbf{x}, \mathbf{x_0}) = \frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x_0}|$.

Draw a rough sketch indicating the position of the point P and of its images to support your result for the Green's function $G(\mathbf{x}, \mathbf{x_0})$. Clearly mark the domain D, label your image points as P_i (with $\mathbf{OP_i} := \mathbf{x_i}$) and call Q the point such that $\mathbf{OQ} := \mathbf{x}$. Give your answer for the Green's function $G(\mathbf{x}, \mathbf{x_0})$ in terms of $\mathbf{x_0}, \mathbf{x}$ and $\mathbf{x_i}$.

8.2 Prove that the solution $G(\mathbf{x}, \mathbf{x_0})$ you obtained in part 8.1 satisfies $G(\mathbf{x}, \mathbf{x_0}) = 0$ for $Q \in (\,0, 1\,]$.

8.3 Give the polar coordinates of all image points P_i in terms of the polar coordinates (r_0, θ_0) of the point P.

1. 1.1. [SEEN]

$$\begin{aligned}[\boldsymbol{\nabla} r^2]_i &= \partial_i(x_j x_j) = 2\delta_{ij}x_j = 2x_i \\ &= [2\boldsymbol{x}]_i.\end{aligned}$$

Since this holds for an arbitrary component, we have $\boldsymbol{\nabla} r^2 = 2\boldsymbol{x}$ as required.

[2 marks]

1.2. [SEEN SIMILAR]

$$\begin{aligned}[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot r^2\boldsymbol{x}\right)]_i &= \partial_i\partial_j r^2 x_j \\ &= \partial_i x_j \partial_j r^2 + \partial_i r^2 \partial_j x_j \\ &= 2\partial_i x_j x_j + \partial_i r^2 \delta_{jj} \\ &= 4x_i + 6x_i \\ &= [10\boldsymbol{x}]_i.\end{aligned}$$

Since this holds for an arbitrary component, we have $\boldsymbol{\nabla} f = 10\boldsymbol{x}$ as required.

[4 marks]

1.3. [SEEN SIMILAR]

The direction derivative of f in the direction $\boldsymbol{n}$, at the point $\boldsymbol{p}$, is given by

$$\left.\frac{df}{d\hat{\boldsymbol{n}}}\right|_{\boldsymbol{p}} = \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla} f|_{\boldsymbol{p}},$$

where $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|}$.

We have $|\boldsymbol{n}| = \sqrt{4+9+36} = 7$, and so $\hat{\boldsymbol{n}} = \frac{1}{7}(2,3,6)$. At the point $\boldsymbol{p}$, $\boldsymbol{\nabla} f(\boldsymbol{p}) = 10(3,1,2)$ using the previous part of the question, and so

$$\left.\frac{df}{d\hat{\boldsymbol{n}}}\right|_{\boldsymbol{p}} = \frac{10}{7}(2,3,6)\cdot(3,1,2) = 30.$$

[4 marks]

2. 2.1. [BOOKWORK]

If $\boldsymbol{F}$ is a continuously differentiable vector field defined over the volume V with bounding surface S, then

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{A} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV,$$

where S is taken with outwards pointing normal.

[3 marks]

2.2. [SEEN SIMILAR]

We can compute the volume of V as $\text{Vol}(V) = \int_V dV$. Noticing that for each fixed z, the region V is a disk D_z of radius $r_z = \sqrt{z}$ and hence area πz, we have

$$\begin{aligned}
\text{Vol}(V) &= \int_0^4 \left(\int_{D_z} dA \right) \, dz \\
&= \int_0^4 \pi z \, dz \\
&= \left[\frac{\pi}{2} z^2 \right]_0^4 \\
&= 8\pi.
\end{aligned}$$

[4 marks]

2.3. [SEEN SIMILAR]

Since all the components of $\boldsymbol{F}$ are polynomials and hence continuously differentiable, $\boldsymbol{F}$ is a continuously differentiable vector field. Hence by the divergence theorem, $\int_S \boldsymbol{F} \cdot d\boldsymbol{A} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV$. We have $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 3$, and so

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{A} = \int_V 3 \, dV = 3\text{Vol}(V) = 24\pi.$$

[3 marks]

5. 5.1. [UNSEEN]

Using the limit definition of the partial derivatives,

$$\left.\frac{\partial f}{\partial x}\right|_{\boldsymbol{x}=\mathbf{0}} = \lim_{h\to 0}\frac{f(h\boldsymbol{e}_1)-f(\mathbf{0})}{h} = \lim_{h\to 0}\frac{a-a}{h} = \lim_{h\to 0} 0 = 0,$$

for all values of a. So the x partial derivative exists for all values of a and is equal to 0, $\frac{\partial f}{\partial x}\big|_{\boldsymbol{x}=\mathbf{0}} = 0$. Similarly,

$$\left.\frac{\partial f}{\partial y}\right|_{\boldsymbol{x}=\mathbf{0}} = \lim_{h\to 0}\frac{f(h\boldsymbol{e}_2)-f(\mathbf{0})}{h} = \lim_{h\to 0}\frac{h-a}{h} = \lim_{h\to 0}\left(1-\frac{a}{h}\right).$$

This limit, and hence the y partial derivative, exists if and only if $a = 0$, in which case we have $\frac{\partial f}{\partial y}\Big|_{\boldsymbol{x}=\mathbf{0}} = 1$.

[4 marks]

5.2. [UNSEEN]

f is continuous at the origin if $f(\mathbf{0})$ exists and $\lim_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) = f(\mathbf{0})$, and so f is continuous if $\lim_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) = 0$.

Taking the limit along the parabola $y = x^2$, the limit becomes

$$\lim_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) = \lim_{x\to 0}\frac{x^2+x^4}{x^2} = 1.$$

We therefore cannot hope to find a $\delta > 0$ such that for all $\epsilon > 0$, $|f(\boldsymbol{x}) - 0| < \epsilon$ for all $0 < |\boldsymbol{x} - \mathbf{0}| < \delta$. We therefore have $\lim_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) \neq 0$, and hence f is not continuous at the origin.

The partial derivatives exist at the origin even though the function f is not continuous at the origin. This is because the partial derivatives depend on the behaviour of the function as we move parallel to one of the coordinate axes, whereas continuity depends on the behaviour of the function in all directions.

[5 marks]

5.3. [SEEN SIMILAR]

In order for f to be differentiable at the origin, we must have

$$f(\mathbf{0}+\boldsymbol{h}) = f(\mathbf{0}) + \boldsymbol{h}\cdot\boldsymbol{\nabla} f|_{\boldsymbol{x}=\mathbf{0}} + R(\boldsymbol{h}),$$

with

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{R(\boldsymbol{h})}{|\boldsymbol{h}|} = 0.$$

Letting $\boldsymbol{h} = h_1\boldsymbol{e}_1 + h_2\boldsymbol{e}_2$, $R(\boldsymbol{h})$ is therefore given by

$$\begin{aligned} R(\boldsymbol{h}) &= f(\boldsymbol{h}) - f(\boldsymbol{0}) - \boldsymbol{h}\cdot\boldsymbol{\nabla} f|_{\boldsymbol{x}=\boldsymbol{0}} \\ &= \frac{h_1^2+h_2^2}{h_2} - 0 - \frac{h_2^2}{h_2} = \frac{h_1^2}{h_2}. \end{aligned}$$

So, taking the limit along the line $h_1 = h_2 = h$,

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{R(\boldsymbol{h})}{|\boldsymbol{h}|} = \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{h_1^2}{h_2|\boldsymbol{h}|} = \lim_{h\to 0} \frac{h^2}{\sqrt{2}h^2} = \frac{1}{\sqrt{2}}.$$

and hence $\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{R(\boldsymbol{h})}{|\boldsymbol{h}|} \neq 0$, and f is not differentiable at the origin.

[6 marks]

6. 6.1. [UNSEEN]

We let $P(x,y) = \frac{-y}{x^2+y^2}$, and $Q(x,y) = \frac{x}{x^2+y^2}$, defined everywhere on $\mathbb{R}^2$ aside from at the origin, whose partials are given by

$$\begin{aligned} \frac{\partial P}{\partial x} &= \frac{2xy}{(x^2+y^2)^2} & \qquad \frac{\partial P}{\partial y} &= \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial Q}{\partial x} &= \frac{y^2-x^2}{(x^2+y^2)^2} & \qquad \frac{\partial Q}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2}. \end{aligned}$$

(By Theorem 5.3, these partials are all continuous everywhere aside from at the origin, and hence by Theorem 5.10, P and Q are continuously differentiable, and so we may apply Green's Theorem.)

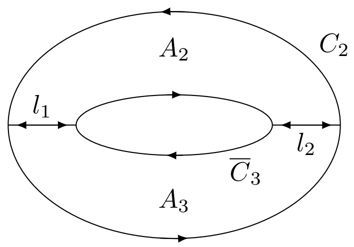
Letting the region bounded by C_1 be A_1, by Green's Theorem we therefore have

$$\oint_{C_1} \left(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy \right) = \int_{A_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = 0,$$

since the origin is not contained in A_1, and hence $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ everywhere in A_1.
[5 marks]

6.2. [UNSEEN CHALLENGING]

Join the two curves C_2 and C_3 with two line segments l_1 and l_2, so as to split the region between the two curves into an 'upper' and 'lower' region A_2 and A_3 respectively, as shown in the following diagram (where C_3 is shown with opposite orientation to as in the question):



Now, since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$,

$$\begin{aligned} 0 &= \int_{A_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy + \int_{A_3} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy \\ &= \oint_{\gamma_U} (P\, dx + Q\, dy) + \oint_{\gamma_L} (P\, dx + Q\, dy), \end{aligned}$$

where γ_U is the (piecewise smooth simple closed) curve bounding A_2 with positive orientation, and γ_L is the (piecewise smooth simple closed) curve bounding A_3 with positive orientation. Since the line segments l_1 and l_2 contribute with both orientations in this sum, their contributions cancel, and we are left with

$$\oint_{\gamma_U} (P\,dx + Q\,dy) + \oint_{\gamma_L} (P\,dx + Q\,dy) = \oint_{C_2} (P\,dx + Q\,dy) - \oint_{C_3} (P\,dx + Q\,dy)\,,$$

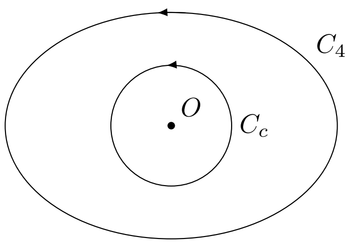
since C_3 is navigated with negative orientation in the above sum. We therefore have

$$0 = \oint_{C_2} (P\,dx + Q\,dy) - \oint_{C_3} (P\,dx + Q\,dy)\,,$$

and so $\oint_{C_2} (P\,dx + Q\,dy) = \oint_{C_3} (P\,dx + Q\,dy)$ as required. [6 marks]

6.3. [UNSEEN CHALLENGING]

Following the hint, we consider a small circular curve C_c of radius α around the origin with positive orientation, contained entirely within C_4, as shown in the following diagram:



By part two of this question, we then have $\oint_{C_4} (P\,dx + Q\,dy) = \oint_{C_c} (P\,dx + Q\,dy)$, since by the first part of this question, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ throughout the region between the two curves.

We can now parameterise C_c as points $\boldsymbol{x}(t) = (\alpha \cos t, \alpha \sin t)$, for $0 \leq t \leq 2\pi$. We then have $\boldsymbol{x}'(t) = (-\alpha \sin t, \alpha \cos t)$, $P(x(t), y(t)) = -\frac{1}{\alpha} \sin t$, and $Q(x(t), y(t)) = \frac{1}{\alpha} \cos t$. Putting this into our usual formula for the line integral gives

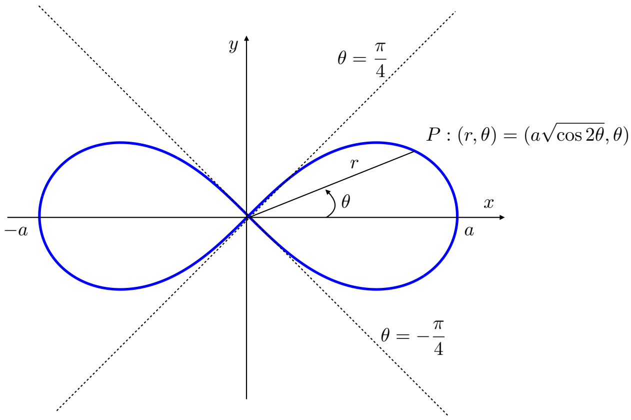
$$\oint_{C_c} (P\,dx + Q\,dy) = \int_0^{2\pi} \frac{\alpha}{\alpha}(-\sin t, \cos t) \cdot (-\sin t, \cos t)\;dt = \int_0^{2\pi} dt = 2\pi.$$

[4 marks]

3. [SEEN SIMILAR]

In polar coordinates, with $x = r\cos\theta$ and $y = r\sin\theta$, the curve is given by

$$r^2 = a^2\,(\cos^2\theta - \sin^2\theta) = a^2\,\cos 2\theta.$$ [1 mark]



[2 marks]

The region bounded by the curve is made of four subregions of equal areas, so one can first focus on one subregion. For instance the region

$$R_1 := \{0 \le r \le a\sqrt{\cos 2\theta}, 0 \le \theta \le \pi/4\}.$$ [2 marks]

Since the Jacobian in polar coordinates is $J = r$ and $r \ge 0$, [1 mark]

the area of region R_1 is

$$S_1 := \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}r^2\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \frac{a^2}{2}\int_0^{\frac{\pi}{4}} \cos 2\theta\,d\theta$$
$$\overset{(u=2\theta)}{=} \frac{a^2}{4}\,[\sin u]_0^{\frac{\pi}{2}} = \frac{a^2}{4}.$$ [3 marks]

Hence the total area of the region bounded by the curve is $4 \times \frac{a^2}{4} = a^2$.

[1 mark]

4. [SEEN SIMILAR]

4.1 With $p_0(x) = x$ and $p_1(x) = 1-x$, the multiplicative factor needed to transform L into a SL operator is

$$\rho(x) = \frac{1}{p_0(x)} \exp\left(\int^x \frac{p_1(u)}{p_0(u)}\,du\right) = \frac{1}{x}\exp\left(\int^x \frac{1-u}{u}\,du\right) = e^{-x}.$$

Hence $\mathcal{L} = \rho(x)L = xe^{-x}\frac{d^2}{dx^2} + (1-x)\,e^{-x}\frac{d}{dx} + e^{-x}$. [2 marks]

4.2 For boundary conditions of the type $u(a) = A$ and $u(b) = B$, the sought linear function is $g(x) = A\dfrac{b-x}{b-a} + B\dfrac{x-a}{b-a}$, so here $g(x) = \dfrac{5-x}{4} + 5\dfrac{x-1}{4} = x$ since $a = A = 1$ and $b = B = 5$. [2 marks]

4.3 Since $\mathcal{L}v = \mathcal{L}(u-g) = \sin x - \mathcal{L}x$, the source term is $h(x) = \sin x - e^{-x}$.

[2 marks]

First, $\mathcal{L}$ is formally self-adjoint, i.e. $\mathcal{L} = \mathcal{L}^*$ since $\mathcal{L}$ is of Sturm-Liouville type ($p_0(x) = xe^{-x}$ and $p_0'(x) = (1-x)e^{-x} = p_1(x)$). [2 marks]

Second, the adjoint boundary conditions are identical to the boundary conditions. Indeed, the Green's formula for functions $v, w \in \mathcal{C}^2([a,b])$ is

$$\langle \mathcal{L}v(x), w(x)\rangle - \langle v(x), \mathcal{L}w(x)\rangle = \left[\underbrace{\overline{p_0}(x)\left[\,w(x)\overline{v'}(x) - w'(x)\,\overline{v}(x)\,\right]}_{\text{boundary terms}}\right]_a^b.$$

Since $\overline{p_0}(x) = p_0(x) = xe^{-x}$, and since $v(5) = 0 \Rightarrow \overline{v}(5) = 0$ (and analogously for the boundary condition $v(1) = 0$), the boundary terms reduce to

$$5e^{-5}w(5)\overline{v'}(5) - e^{-1}w(1)\overline{v'}(1),$$

and hence vanish provided that $w(1) = w(5) = 0$, which are identical boundary conditions to those on v, hence the BVP for v is self-adjoint. [2 marks]

7. [Technique seen]

7.1
- First step: find a solution to

$$\partial_t^2 G(t,\tau)+2b\,\partial_t G(t,\tau)+c^2 G(t,\tau)=\delta(t-\tau), \qquad G(0,\tau)=0,\ \partial_t G(0,\tau)=0.$$

  The characteristic equation is $\lambda^2+2b\,\lambda+c^2=0$, hence $\lambda_\pm=-b\pm\sqrt{\Delta}$ for $\Delta:=b^2-c^2\neq 0$. If $t\neq\tau$, we set the general solution to be
  for $t<\tau$: $G(t,\tau):=G_1(t,\tau)=A_1(\tau)e^{\lambda_+ t}+A_2(\tau)e^{\lambda_- t}$
  for $\tau<t$: $G(t,\tau):=G_2(t,\tau)=B_1(\tau)e^{\lambda_+ t}+B_2(\tau)e^{\lambda_- t}$. [2 marks]

- Second step: impose the homogeneous boundary conditions $G_1(0,\tau)=0$ and $\partial_t G_1(0,\tau)=0$.
  This yields
  $G_1(0,\tau)=0\Rightarrow A_1(\tau)=-A_2(\tau)$
  $\partial_t G_1(0,\tau)=0\Rightarrow A_1(\tau)\,\lambda_+ + A_2(\tau)\lambda_-=0$. Hence, $A_1(\tau)(\lambda_+-\lambda_-)=2A_1(\tau)\Delta=0$ which requires $A_1(\tau)=0$. Therefore $G(t,\tau)=0$ for $t<\tau$.
  [2 marks]

- Third step: impose continuity at $t=\tau$.

$$\lim_{\epsilon\to 0^+}G_1(\tau-\epsilon,\tau)=\lim_{\epsilon\to 0^+}G_2(\tau+\epsilon,\tau)\Rightarrow 0=B_1(\tau)e^{\lambda_+\tau}+B_2(\tau)e^{\lambda_-\tau}.$$

  So $B_2(\tau)=-B_1(\tau)e^{2\Delta\,\tau}$. [2 marks]

- Fourth step: impose jump discontinuity of $\partial_t G$ at $t=\tau$, with jump $\dfrac{1}{p_0(x)}=1$.

$$1=\lim_{\epsilon\to 0^+}\left\{\partial_t G_2(\tau+\epsilon,\tau)-\underbrace{\partial_t G_1(\tau-\epsilon,\tau)}_{=0}\right\}=B_1(\tau)\lambda_+e^{\lambda_+\tau}+B_2(\tau)\lambda_-e^{\lambda_-\tau}.$$

  [2 marks]

- Solve for $B_1(\tau),B_2(\tau)$ using the equations obtained in steps 3 and 4. One gets

$$B_1(\tau)=\frac{1}{2\Delta}e^{-\lambda_+\tau}, \qquad B_2(\tau)=-\frac{1}{2\Delta}e^{-\lambda_-\tau},$$

and the Green's function is

$$G(t,\tau) = \begin{cases} G_1(t,\tau) = 0 & \text{for } 0 \le t < \tau, \\ G_2(t,\tau) = \frac{1}{2\Delta} e^{-b(t-\tau)} \sinh(\,\Delta(t-\tau)\,) & \text{for } 0 \le \tau < t. \end{cases}$$

The solution to this problem is thus $u(t) = \int_0^t G(t,\tau)\, f(\tau)\, d\tau$. [2 marks]

7.2 If $f(t) = \Theta(t)$, one gets

$$\begin{aligned} u(t) &= \int_0^t G(t,\tau)\,\Theta(\tau)\, d\tau = \int_0^t G(t,\tau)\, d\tau \\ &\overset{(\tau_1 = t-\tau)}{=} \frac{1}{2\Delta} \int_0^t \left(e^{\lambda_+ \tau_1} - e^{\lambda_- \tau_1} \right) d\tau_1 = \frac{1}{2\Delta} \left\{ \frac{1}{\lambda_+} [e^{\lambda_+ t} - 1] - \frac{1}{\lambda_-} [e^{\lambda_- t} - 1] \right\}. \end{aligned}$$

[5 marks]

8. Question 8 should read:

8.1 Consider the two-dimensional domain

$$D = \mathring{D} \cup \partial D := \{ (r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi/3 \},$$

where $\mathring{D} = \{(r, \theta) : 0 < r < 1,\ 0 < \theta < \pi/3 \}$ is the interior of the domain and ∂D its boundary. Denote the origin of the plane by O and label P the point in $\mathring{D}$ with $\mathbf{OP} := \mathbf{x_0} = \frac{1}{2}\,\mathbf{e}_1 + \frac{1}{2}\,\mathbf{e}_2$. Use the method of images to construct the Green's function $G(\mathbf{x}, \mathbf{x_0})$ satisfying

$$\begin{aligned} &\nabla^2 G(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}) && \text{for } \mathbf{x} \in D, \\ &G(\mathbf{x}, \mathbf{x_0}) = 0 && \text{for } \mathbf{x} \in \partial D. \end{aligned}$$

You may use the fact that the fundamental solution of Laplace's equation, which is regular on $\mathbb{R}^2$-$\{\mathbf{x_0}\}$, is given by

$$G_0(\mathbf{x}, \mathbf{x_0}) = \frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x_0}|.$$

Draw a rough sketch indicating the position of the point P and of its images to support your result for the Green's function $G(\mathbf{x}, \mathbf{x_0})$. Clearly mark the domain D, label your image points as P_i (with $\mathbf{OP_i} := \mathbf{x_i}$) and call Q the point such that $\mathbf{OQ} := \mathbf{x}$. Give your answer for the Green's function $G(\mathbf{x}, \mathbf{x_0})$ in terms of $\mathbf{x_0}, \mathbf{x}$ and $\mathbf{x_i}$.

8.2 Prove that the solution $G(\mathbf{x}, \mathbf{x_0})$ you obtained in part 8.1 satisfies $G(\mathbf{x}, \mathbf{x_0}) = 0$ for $Q \in (\,0, 1\,]$.

8.3 Give the polar coordinates of all image points P_i in terms of the polar coordinates (r_0, θ_0) of the point P.

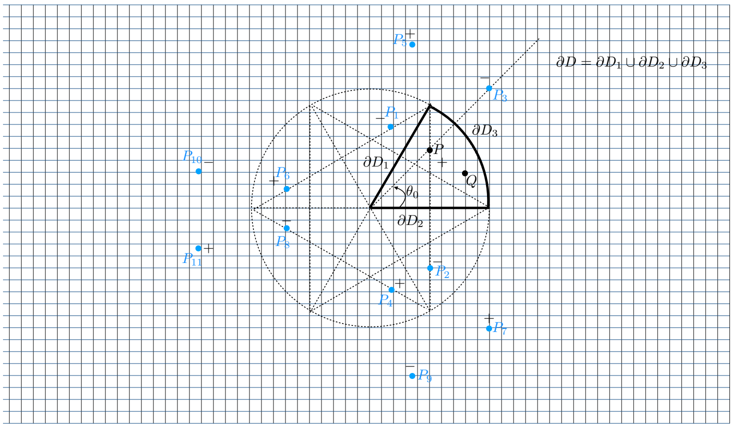
Solution

8.1 [method of images seen in the case of half plane and disk domains]

Let P be the point such that $\mathbf{OP} := \mathbf{x_0} = \frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$ and Q the point such that $\mathbf{OQ} := \mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$. The method of images, given the symmetries of the domain, dictates to take the points P_i such that $\mathbf{OP_i} := \mathbf{x_i}$ as images of P for $i \in \{1,2,3,4,5,6,7,8,9,10,11\}$. These image points are obtained by first reflecting P w.r.t. the boundaries ∂D_1 and ∂D_2, yielding the points P_1 and P_2, and by taking P_3, the inverse point of P w.r.t. the circle $r=1$. One then proceeds by reflecting P_1, P_2 and P_3 w.r.t. the lines supporting the boundaries ∂D_1 and ∂D_2 and by taking the inverse of P_1, P_2 w.r.t. the circle $r=1$, and so on (see Figure). The Green's function is therefore

$$G(\mathbf{x},\mathbf{x_0}) = \frac{1}{2\pi}\left\{\ln|QP| - \ln\frac{|QP_1|\,|QP_2|\,|QP_3|\,|QP_8|\,|QP_9|\,|QP_{10}|}{|QP_4|\,|QP_5|\,|QP_6|\,|QP_7|\,|QP_{11}|}\right\}$$

where $|QP| = |\,\mathbf{x}-\mathbf{x_0}\,|$ and $|QP_i| = |\,\mathbf{x}-\mathbf{x_i}\,|$. Full marks for any other labelling



consistent with the choice made here. [10 marks]

8.2 When $Q \in (\,0,1\,]$, we see from the sketch that $|QP| = |QP_2|$, $|QP_1| = |QP_4|$, $|QP_i| = |QP_{i+4}|$ for $i \in \{3,5\}$, $|QP_6| = |QP_8|$ and $|QP_{10}| = |QP_{11}|$. Hence $G(\mathbf{x},\mathbf{x_0}) = 0$ in that case. [3 marks]

8.3 The polar coordinates of the image points are

$$P_1 : (r_0, \frac{2\pi}{3} - \theta_0),\ P_2 : (r_0, -\theta_0),\ P_3 : (\frac{1}{r_0}, \theta_0)),\ P_4 : (r_0, \frac{4\pi}{3} + \theta_0),$$
$$P_5 : (\frac{1}{r_0}, \frac{2\pi}{3} - \theta_0),\ P_6 : (r_0, \frac{2\pi}{3} + \theta_0),\ P_7 : (\frac{1}{r_0}, -\theta_0),\ P_8 : (r_0, \frac{4\pi}{3} - \theta_0),$$
$$P_9 : (\frac{1}{r_0}, \frac{4\pi}{3} + \theta_0),\ P_{10} : (\frac{1}{r_0}, \frac{2\pi}{3} + \theta_0),\ P_{11} : (\frac{1}{r_0}, \frac{4\pi}{3} - \theta_0),$$

with $(r_0, \theta_0) = (\frac{1}{\sqrt{2}}, \frac{\pi}{4})$. [2 marks]