# 2 HOW MUCH DOES A CURVE CURVE?

## 2.1 CURVATURE

INTUITION:

1 - CURVATURE OF LINE IS 0

2 - CURVATURE OF LARGE CIRCLE IS SMALLER THAN CURVATURE OF SMALL CIRCLE

3 - CURVATURE SHOULD NOT DEPEND ON PARAMETRIZATION

RECALL 1.1.4: $\ddot{\gamma} = 0 \Rightarrow \gamma$ LINE

FIRST ATTEMPT: CURVATURE $= \|\ddot{\gamma}\|$

DEPENDS ON PARAMETRIZATION OF $\gamma$

IMPOSE $\|\dot{\gamma}\| = 1$ ($\gamma$ UNIT SPEED)

**DEFINITION 2.1.1** LET $\gamma(s)$ BE A UNIT SPEED CURVE. THE CURVATURE $\kappa(s)$ OF $\gamma$ AT $\gamma(s)$ IS

$$\kappa(s) := \|\ddot{\gamma}(s)\|$$

NOTE:

1- $\kappa = 0 \Rightarrow \|\ddot{\gamma}\| = 0 \Rightarrow \ddot{\gamma} = 0$

$\Longrightarrow \dot{\gamma}$ CONSTANT

$\Longrightarrow \gamma$ (PART OF) STRAIGHT LINE

1.1.4

2- CIRCLE CENTERED AT $(x_0, y_0)$ WITH RADIUS $R$:

$$\gamma(s) = \left(x_0 + R\cos\left(\frac{s}{R}\right), y_0 + R\sin\left(\frac{s}{R}\right)\right)$$

$$\Rightarrow \dot{\gamma}(s) = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\right)$$

$$\Longrightarrow \|\dot{\gamma}(s)\| = 1, \quad \gamma \text{ UNIT SPEED}$$

$$\ddot{\gamma}(s) = \left(-\frac{1}{R}\cos\left(\frac{s}{R}\right), -\frac{1}{R}\sin\left(\frac{s}{R}\right)\right)$$

$$\Longrightarrow \|\ddot{\gamma}(s)\| = \frac{1}{R}$$

$$\Rightarrow \text{CURVATURE } \kappa(s) \equiv \frac{1}{R}$$

3- UNIT SPEED REPARAMETRIZATION

$$\gamma(u), \quad u = \pm s + c.$$

CHAIN RULE

$$\Longrightarrow \frac{d\gamma}{ds} = \frac{d\gamma}{du}\frac{du}{ds} = \pm\frac{d\gamma}{du}$$

$$\Longrightarrow \frac{d^2\gamma}{ds^2} = \frac{d}{du}\left(\frac{d\gamma}{ds}\right)\frac{du}{ds} = \frac{d}{du}\left(\pm\frac{d\gamma}{du}\right)(\pm 1) = \frac{d^2\gamma}{du^2}.$$

PROBLEM: FOR GIVEN $\gamma$, NOT ALWAYS (17)

POSSIBLE TO FIND UNIT SPEED REPARAMET. EXPLICITLY. SO HOW TO CALCULATE CURVATURE THEN?

PROP 2.1.2 $\gamma(t)$ REGULAR CURVE IN $\mathbb{R}^n$.

THEN
$$\kappa = \frac{\| \gamma''(\gamma' \cdot \gamma') - \gamma'(\gamma' \cdot \gamma'') \|}{\|\gamma'\|^4} \qquad ' = \frac{d}{dt}$$

$n=3$: $$\kappa = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}. \qquad \times = \text{VECTOR PRODUCT}$$

PROOF LET $\tilde{\gamma}(s)$ BE UNIT SPEED REPARA OF $\gamma(t)$. NOTATION: $\dot{} = \frac{d}{ds}$, $' = \frac{d}{dt}$

CHAIN RULE $\gamma' = (\tilde{\gamma}(s))' = \dot{\tilde{\gamma}} s'$

$$\Rightarrow \kappa = \|\ddot{\tilde{\gamma}}\| = \|(\dot{\tilde{\gamma}})^{\cdot}\| = \left\| \frac{d}{ds}\left(\frac{\gamma'}{s'}\right)\right\| = \left\| \frac{\frac{d}{dt}\left(\frac{\gamma'}{s'}\right)}{s'} \right\|$$

$$\frac{d}{ds} = \frac{d}{dt}\frac{dt}{ds} = \frac{1}{s'}\frac{d}{dt}$$

$$\underset{\text{QUOTIENT RULE}}{=} \left\| \frac{1}{s'}\left(\frac{\gamma'' s' - \gamma' s''}{s'^2}\right)\right\| = \left\| \frac{\gamma'' s' - \gamma' s''}{s'^3} \right\|$$

WE HAVE $\|\gamma'\| = \|\dot{\tilde{\gamma}} s'\| = \|\dot{\tilde{\gamma}}\| |s'| = |s'|$

$\Rightarrow s'^2 = \|\gamma'\|^2 = \gamma' \cdot \gamma'$

$\Rightarrow \cancel{2} s' s'' = \cancel{2} \gamma' \cdot \gamma''$

$$\Rightarrow \varkappa = \left\| \frac{\gamma'' s' - \gamma' s''}{s'^3} \right\| = \left\| \frac{\gamma'' s'^2 - \gamma' s' s''}{s'^4} \right\|$$

$$= \frac{\| \gamma''(\gamma' \cdot \gamma') - \gamma'(\gamma' \cdot \gamma'') \|}{\|\gamma'\|^4}$$

NOW ASSUME $n = 3$. RECALL

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c \qquad a, b, c \in \mathbb{R}^3$$

$\Rightarrow \gamma' \times (\gamma'' \times \gamma') = \gamma''(\gamma' \cdot \gamma') - \gamma'(\gamma' \cdot \gamma'')$

$\|\gamma' \times (\gamma'' \times \gamma')\| = \|\gamma'\| \cdot \|\gamma'' \times \gamma'\|$

$\gamma' \cdot (\gamma'' \times \gamma') = 0$

$$\Rightarrow \varkappa = \frac{\| \gamma''(\gamma' \cdot \gamma') - \gamma'(\gamma' \cdot \gamma'') \|}{\|\gamma'\|^4} = \frac{\| \gamma' \times (\gamma'' \times \gamma') \|}{\|\gamma'\|^4}$$

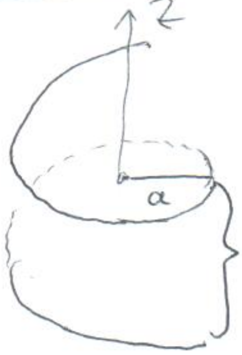
$$= \frac{\|\gamma'\| \|\gamma'' \times \gamma'\|}{\|\gamma'\|^4} = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3} \qquad \square$$

NOTE:

$\gamma(t)$ REGULAR POINT OF $\gamma$ $\Leftrightarrow$ $\gamma'(t) \neq 0$

THUS CURVATURE IS WELL-DEFINED AT REGULAR POINTS OF CURVES.

EXAMPLE (CIRCULAR HELIX)




$2\pi|b|$ PITCH

$$\gamma(\theta) = (a\cos(\theta), a\sin(\theta), b\theta)$$

$a, b \in \mathbb{R}$ $\quad -\infty < \theta < \infty$

$\gamma$ LIES ON CYLINDER
$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2\}$$

PUT $' = \frac{d}{d\theta}$

$$\gamma'(\theta) = (-a\sin(\theta), a\cos(\theta), b)$$

$$\Rightarrow \|\gamma'(\theta)\| = \sqrt{a^2 + b^2}$$

THUS $\gamma$ REGULAR (UNLESS $a = 0 = b$)

$$\gamma''(\theta) = (-a\cos(\theta), -a\sin(\theta), 0)$$

$$\gamma'' \times \gamma' = (-ab\sin(\theta), ab\cos(\theta), -a^2)$$

$$\Rightarrow \varkappa = \frac{\sqrt{a^2b^2+a^4}}{\sqrt{(a^2+b^2)^3}} = \frac{|a|}{a^2+b^2} \quad \text{CONSTANT}$$

LIMITING CASES:

$b=0$ $(a \neq 0)$. $\Rightarrow \gamma$ CIRCLE AND

$$\varkappa = \frac{1}{|a|}$$

$a=0$ $(b \neq 0)$ $\Rightarrow \gamma$ STRAIGHT LINE (z-AXIS)

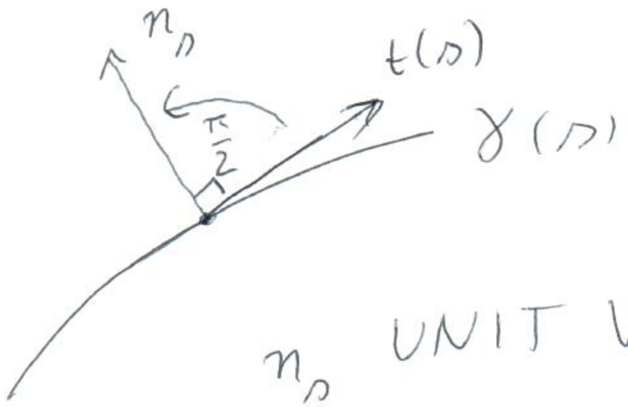
$$\varkappa = 0$$

# PLANE CURVES

AIM: GEOMETRIC INTERPRETATION OF PLANE CURVES

LET $\gamma(s)$ UNIT SPEED CURVE IN $\mathbb{R}^2$.

AS USUAL $\dot\gamma = \frac{d\gamma}{ds}$. PUT $t = \dot\gamma$ TANGENT VECTOR




$n_s$ UNIT VECTOR OBTAINED BY ROTATING $t$ ANTI-CLOCKWISE BY RIGHT ANGLE.

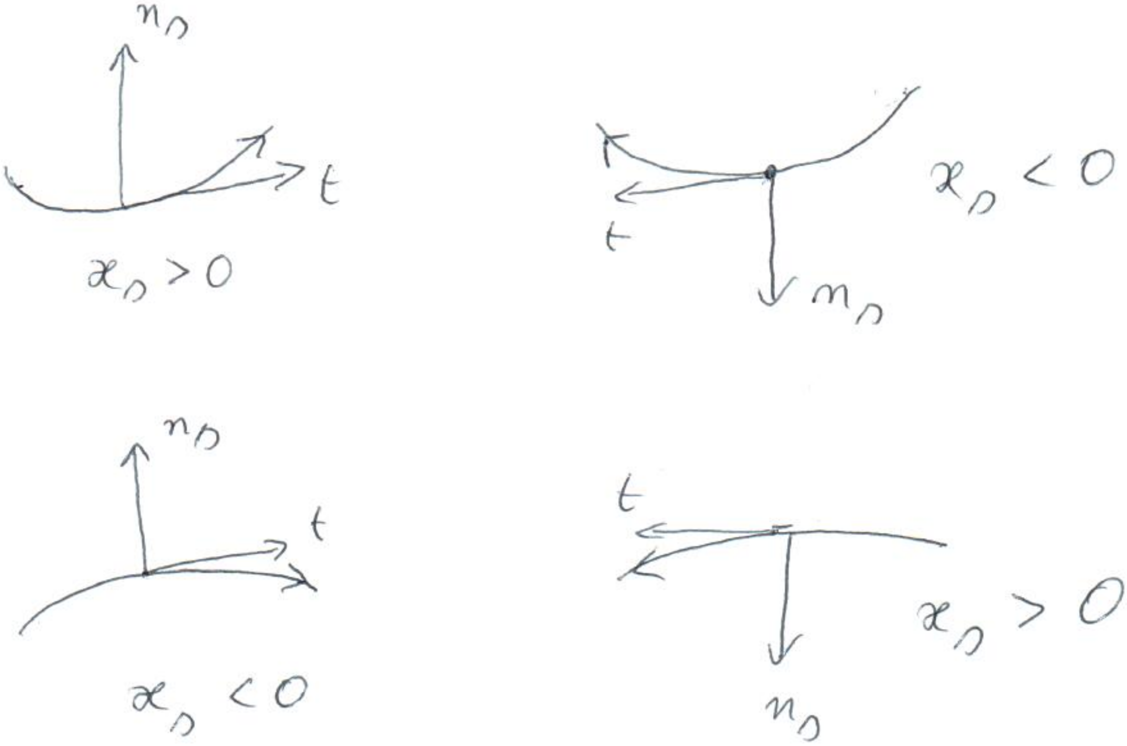
$\|\dot\gamma\| = 1 \implies \ddot\gamma \perp \dot\gamma \quad (\dot\gamma \cdot \ddot\gamma = 0)$

$\implies \exists\, \varkappa_s \in \mathbb{R} : \ddot\gamma = \varkappa_s n_s$

SIGNED CURVATURE OF $\gamma$

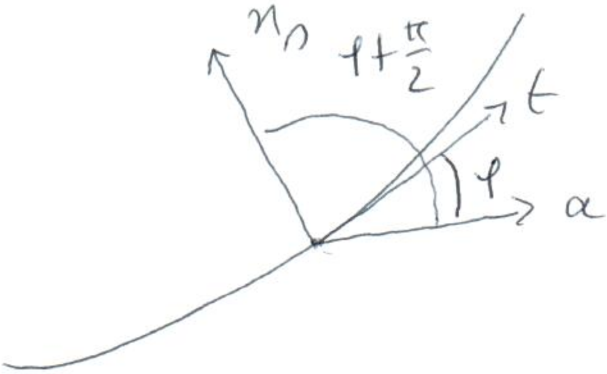
NOTE: $\varkappa = \|\ddot\gamma\| = \|\varkappa_s n_s\| = |\varkappa_s| \underbrace{\|n_s\|}_{=1} = |\varkappa_s|$

GEOMETRIC PICTURE OF SIGN:




<u>PROP 2.2.1</u> $\gamma(s)$ UNIT SPEED CURVE IN $\mathbb{R}^2$; $\varphi(s)$ = ANGLE THROUGH WHICH A FIXED UNIT VECTOR $\mathbf{a}$ MUST BE ROTATED ANTI-CLOCKWISE TO BRING IT INTO COINCIDENCE WITH $t = \dot{\gamma}$. THEN

$$\kappa_s = \frac{d\varphi}{ds}$$




THUS, $\kappa_s$ MEASURES ROTATION OF $t$ ALONG $\gamma$

PROOF BY ASSUMPTION:

$$t \cdot a = \cos(\varphi)$$

$$\underset{\frac{d}{ds}}{\Longrightarrow} \quad \underbrace{\dot{t}}_{\ddot{\gamma}} \cdot a + t \cdot \underbrace{\dot{a}}_{0} = -\sin(\varphi)\frac{d\varphi}{ds}$$

$$\underset{\ddot{\gamma} = \kappa_s n_s}{\Longrightarrow} \quad \kappa_s \underbrace{(n_s \cdot a)}_{\cos(\varphi + \frac{\pi}{2})\, =\, -\sin(\varphi)} = -\sin(\varphi)\frac{d\varphi}{ds}$$

$$\Longrightarrow \quad \kappa_s = \frac{d\varphi}{ds} \qquad \square$$

AIM: SHOW THAT SIGNED CURVATURE DETERMINES SHAPE OF CURVE (UP TO RIGID MOTION OF $\mathbb{R}^2$)

RIGID MOTIONS OF $\mathbb{R}^2$ ~~ARE~~ OF THE FORM

$$M = T_a \circ R_\alpha$$

$T_a : \mathbb{R}^2 \to \mathbb{R}^2,\ v \mapsto v + a$ TRANSLATION

$R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2,\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $0 \le \alpha < 2\pi$

ROTATION.

THM 2.2.2 LET

$$k : (a,b) \to \mathbb{R} \text{ SMOOTH}$$

THEN THERE EXISTS A UNIT SPEED CURVE $\gamma : (a,b) \longrightarrow \mathbb{R}^2$ WITH SIGNED CURVATURE $k$.

MOREOVER, IF $\gamma_1 : (a,b) \to \mathbb{R}^2$ IS A UNIT SPEED CURVE WITH SIGNED CURVATURE $k$, THEN THERE EXISTS A RIGID MOTION $M$ OF $\mathbb{R}^2$ SO THAT

$$\forall s \in (a,b) : \gamma_1(s) = M(\gamma(s)).$$

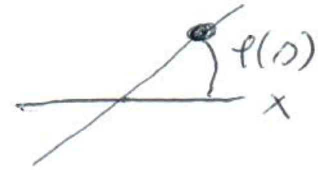
PROOF EXISTENCE LET $s_0 \in (a,b)$. DEFINE

$$\varphi(s) = \int_{s_0}^{s} k(u)\, du$$

$$\gamma(s) = \left( \int_{s_0}^{s} \cos(\varphi(t))\, dt , \int_{s_0}^{s} \sin(\varphi(t))\, dt \right)$$

THEN

$$\dot\gamma(s) = \left( \cos(\varphi(s)) , \sin(\varphi(s)) \right)$$

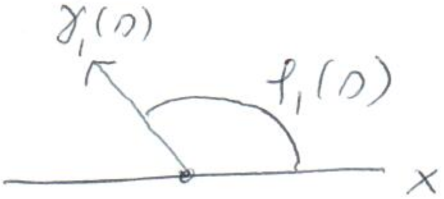



$\Rightarrow \|\dot\gamma\| = 1$ AND

$$\varkappa_s = \frac{d\varphi}{ds} = k(s)$$

$\Rightarrow \gamma$ UNIT SPEED CURVE WITH SIGNED CURVATURE $k$.

UNIQUENESS

DEFINE 


$$\Rightarrow \dot{\gamma}_1(s) = \left(\cos(\varphi_1(s)), \sin(\varphi_1(s))\right)$$

$$\Rightarrow \gamma_1(s) = \left(\int_{s_0}^{s} \cos(\varphi_1(t))\,dt, \int_{s_0}^{s} \sin(\varphi_1(t))\,dt\right) + \gamma_1(s_0)$$

SINCE $\frac{d\varphi_1}{ds} = k(s)$, WE HAVE

$$\varphi_1(s) = \int_{s_0}^{s} k(u)\,du + \varphi_1(s_0) = \varphi(s) + \varphi_1(s_0)$$

~~THUS~~ PUT $\mathbf{a} = \gamma_1(s_0)$, $\alpha = \varphi_1(s_0)$

$$\gamma_1(s) = T_a\left(\int_{s_0}^{s} \underbrace{\cos(\varphi(t)+\alpha)}_{\cos(\alpha)\cos(\varphi(t)) - \sin(\alpha)\sin(\varphi(t))}\,dt, \int_{s_0}^{s} \underbrace{\sin(\varphi(t)+\alpha)}_{\sin(\alpha)\cos(\varphi(t)) + \cos(\alpha)\sin(\varphi(t))}\,dt\right)$$

$$= T_a R_\alpha\left(\int_{s_0}^{s} \cos(\varphi(t))\,dt, \int_{s_0}^{s} \sin(\varphi(t))\,dt\right)$$

$$= T_a R_\alpha(\gamma(s)).$$

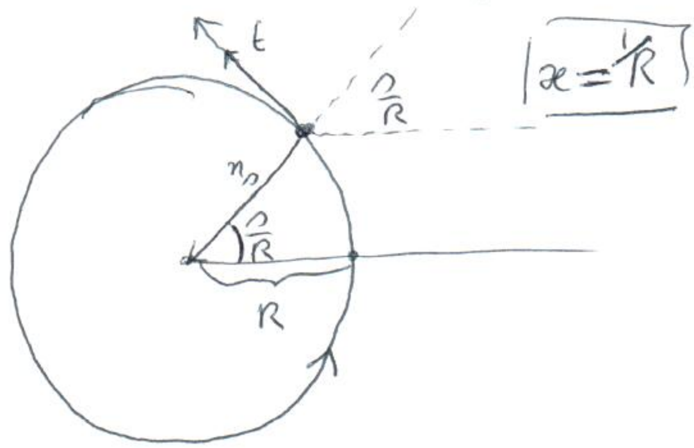
$\square$

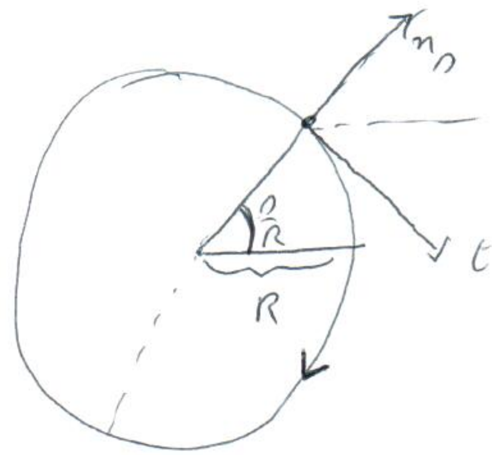
EXAMPLE 2.2.3 WHAT ARE THE REGULAR PLANE CURVES WITH CONSTANT CURVATURE $\varkappa > 0$?

LET $\gamma$ BE SUCH CURVE WITH SIGNED CURVATURE $\varkappa_s$. THEN, ~~BY~~ SINCE $\varkappa = \pm \varkappa_s$, WE HAVE $\varkappa_s = \varkappa > 0$ OR $\varkappa_s = -\varkappa < 0$ EVERYWHERE (BY CONTINUITY).

CONSIDER CIRCLE GIVEN BY

$$\left(R\cos\left(\tfrac{s}{R}\right), R\sin\left(\tfrac{s}{R}\right)\right) \qquad \left(R\cos\left(\tfrac{s}{R}\right), -R\sin\left(\tfrac{s}{R}\right)\right)$$




$\cancel{\varkappa_s > 0}$

$\varkappa_s = \frac{1}{R} > 0$ $\qquad$ $\varkappa_s = -\frac{1}{R} < 0$

FROM THM 2.2.2. WE CONCLUDE:

$\gamma$ IS PART OF CIRCLE WITH RADIUS R (UP TO RIGID MOTION OF $\mathbb{R}^2$)

EXAMPLE 2.2.4 ASSUME $\varkappa_s(s) = s$

USE CONSTRUCTION OF CURVE AS GIVEN IN PROOF OF THM 2.2.2.:

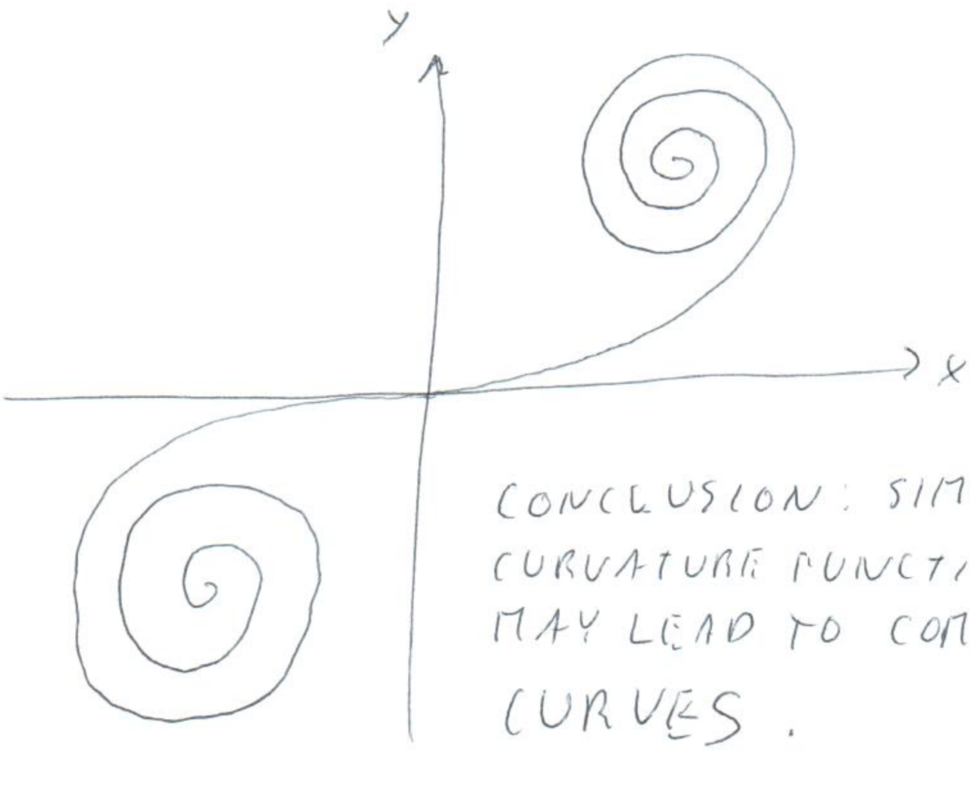
$$\varphi(s) = \int_0^s u\, du = \frac{1}{2} s^2$$

$$\gamma(s) = \left( \int_0^s \cos\left(\tfrac{1}{2}t^2\right) dt,\ \int_0^s \sin\left(\tfrac{1}{2}t^2\right) dt \right)$$

CANNOT BE EVALUATED IN TERMS OF ELEMENTARY FUNCTIONS (HERE: FRESNEL'S INTEGRALS)

$\gamma$ "CORNU'S SPIRAL" (OR "EULER SPIRAL")

FOR VISUALIZATION USE NUMERICAL CALCULATION:

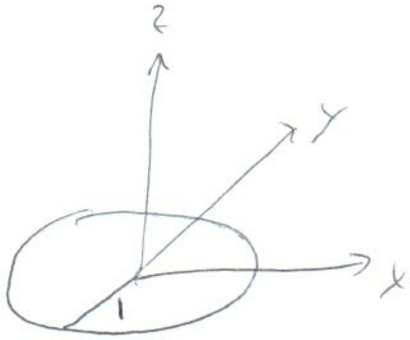



CONCLUSION: SIMPLE CURVATURE FUNCTIONS MAY LEAD TO COMPLICATED CURVES.

## 2.3 SPACE CURVES

AIM: STUDY CURVES IN $\mathbb{R}^3$:

OBSERVATION: CURVATURE DOES NOT DETERMINE CURVE:




CIRCLE IN xy-PLANE

HELIX WITH RADIUS $a = \frac{1}{2}$ AND PITCH $b = \frac{1}{2}$

BOTH HAVE CURVATURE 1, BUT CANNOT BE TRANSFORMED INTO EACH OTHER BY RIGID MOTIONS OF $\mathbb{R}^3$.

NEED NEW CONCEPT!

TORSION

[MEASURES IN HOW FAR A CURVE IS NOT CONTAINED IN A PLANE].

$\gamma(s)$ UNIT SPEED CURVE IN $\mathbb{R}^3$

$t = \dot{\gamma}$ UNIT TANGENT VECTOR.

DEF: IF $\varkappa(s) \neq 0$, DEFINE PRINCIPAL NORMAL OF $\gamma$ AT $\gamma(s)$ BY

$$n(s) = \frac{1}{\varkappa(s)} \dot{t}(s)$$

$$\|n(s)\| = \left\| \frac{1}{\varkappa(s)} \dot{t}(s) \right\| = \frac{1}{\varkappa(s)} \underbrace{\|\ddot{\gamma}(s)\|}_{=\varkappa(s)} = 1$$ UNIT VECTOR

$1 = t \cdot t \Rightarrow 0 = t \cdot \dot{t} \Rightarrow t, n$ PERPENDICULAR

$\Rightarrow b = t \times n$ UNIT VECTOR PERPENDICULAR TO $t$ AND $n$.

$b(s)$ = BINORMAL VECTOR OF $\gamma$ AT $\gamma(s)$

$t, n, b$ RIGHT-HANDED ORTHONORMAL BASIS OF $\mathbb{R}^3$, WELL ADAPTED TO STUDY CURVES

$b = t \times n$, $n = b \times t$, $t = n \times b$.

$$1 = b \cdot b \Rightarrow 0 = b \cdot \dot{b}$$

$$b = t \times n \Rightarrow \dot{b} = \underbrace{\dot{t} \times n}_{= \varkappa n \times n = 0} + t \times \dot{n} = t \times \dot{n} \Rightarrow \dot{b} \cdot t = 0$$

$$0 = b \cdot \dot{b} = t \cdot \dot{b}$$

$$\Rightarrow \exists \tau : \quad \dot{b} = -\tau n$$

$\tau$ <u>TORSION</u> OF $\gamma$ (ONLY DEFINED WHEN $\varkappa \neq 0$ !)

NOTE: FOR AN ARBITRARY REGULAR CURVE $\gamma$ WE DEFINE ITS TORSION VIA A UNIT SPEED REPARAMETRIZATION.

DOES THIS MAKE SENSE?

CHANGE UNIT SPEED PARAMETER

$$u = \pm s + c$$

CAN CHECK THAT

$t \to \pm t$, $\dot{t} \to \dot{t}$, $n \to n$, $b \mapsto \pm b$, $\dot{b} \to \dot{b}$

SINCE $\dot{b} = -\tau n$, CHANGE DOES NOT AFFECT TORSION.

PROP 2.3.1 $\gamma(t)$ REGULAR CURVE IN $\mathbb{R}^3$

WITH $\kappa(t) \neq 0$ EVERYWHERE. THEN

$$\tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2} \qquad ' = \frac{d}{dt}$$

PROOF OMITTED (SEE TEXTBOOK)

NOTE THAT $\gamma' \times \gamma'' \neq 0$ BY PROP 2.1.2.

EXAMPLE 2.3.2 CIRCULAR HELIX

$$\gamma(\theta) = (a\cos(\theta), a\sin(\theta), b\theta) \qquad a > 0,\ b \geq 0$$

WE KNOW $\kappa = \dfrac{a}{a^2+b^2}$ CONSTANT (SO $\tau$ DEFINED)

PUT $' = \dfrac{d}{d\theta}$.

$$\gamma'(\theta) = (-a\sin(\theta), a\cos(\theta), b)$$

$$\Rightarrow \|\gamma'(\theta)\| = \sqrt{a^2+b^2}$$

ARC-LENGTH $s$ OF $\gamma$

$$s = \int_0^\theta \|\gamma'(u)\|\,du = \sqrt{a^2+b^2}\,\theta$$

$$\Rightarrow \theta = cs, \quad c := \frac{1}{\sqrt{a^2+b^2}}.$$

UNIT SPEED REPARAMETRIZATION:

$$\tilde{\gamma}(s) = \gamma(cs) = (a\cos(cs), a\sin(cs), bcs)$$

PUT $\; \mathring{} = \dfrac{d}{ds}$.

$$t = \mathring{\tilde{\gamma}} = (-ac \overset{\sin}{\cancel{\cos}}(cs), ac\cos(cs), bc)$$

$$\mathring{t} = (-ac^2\cos(cs), -ac^2\sin(cs), 0)$$

$$\varkappa = \|\mathring{t}\| = ac^2$$

$$n = \frac{1}{\varkappa}\mathring{t} = (-\cos(cs), -\sin(cs), 0)$$

$$b = t \times n = (bc\sin(cs), -bc\cos(cs), ac)$$

$$\mathring{b} = (bc^2\cos(cs), bc^2\sin(cs), 0)$$

$$= -bc^2 n$$

$$\Rightarrow \quad \tau = bc^2 = \frac{b}{a^2+b^2}$$ TORSION (IS CONSTANT)

LIMIT CASE $b=0$ (CIRCLE IN $xy$-PLANE)

$$\longrightarrow \tau = 0.$$