RECURSION

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Recursion

Define an "object" in terms of itself.

The object can be a function, sequence, algorithm, set, etc.

Function $f: \mathbb{Z} \to \mathbb{Z}$

$f(0) = 5$

Base case is required,
otherwise recursion is infinite.

if $n \geq 1$, $f(n) = f(n-1) + 2n - 1$

$f(0) = 5$
$f(1) = ?$

Recursion

Define an "object" in terms of itself.

The object can be a function, sequence, algorithm, set, etc.

Function $f: \mathbb{Z} \to \mathbb{Z}$

$f(0) = 5$

if $n \geq 1$, $f(n) = f(n-1) + 2n - 1$

$f(0) = 5$
$f(1) = f(0) + 2(1) - 1 = 5 + 2 - 1 = 6$

Recursion

Define an "object" in terms of itself.

The object can be a function, sequence, algorithm, set, etc.

Function $f: \mathbb{Z} \rightarrow \mathbb{Z}$

$f(0) = 5$

if $n \geq 1$, $f(n) = f(n-1) + 2n - 1$

$f(0) = 5$
$f(1) = f(0) + 2(1) - 1 = 5 + 2 - 1 = 6$
$f(2) = f(1) + 2(2) - 1 = 6 + 4 - 1 = 9$

Recursion

Define an "object" in terms of itself.

The object can be a function, sequence, algorithm, set, etc.

Function $f: \mathbb{Z} \to \mathbb{Z}$

$f(0) = 5$

if $n \geq 1$, $f(n) = f(n-1) + 2n \ - 1$

$f(0) = 5$
$f(1) = f(0) + 2(1) - 1 = 5 + 2 \ - 1 = 6$
$f(2) = f(1) + 2(2) - 1 = 6 + 4 \ - 1 = 9$
$f(3) = f(2) + 2(3) - 1 = 9 + 6 \ - 1 = 14$

Recursion

Define an "object" in terms of itself.

The object can be a function, sequence, algorithm, set, etc.

Function $f: \mathbb{Z} \to \mathbb{Z}$

$f(0) = 5$

Argument on right hand side is smaller than argument on left!

if $n \geq 1$, $f(n) = f(n-1) + 2n - 1$

$f(0) = 5$
$f(1) = f(0) + 2(1) - 1 = 5 + 2 - 1 = 6$
$f(2) = f(1) + 2(2) - 1 = 6 + 4 - 1 = 9$
$f(3) = f(2) + 2(3) - 1 = 9 + 6 - 1 = 14$

Recursion

Define an "object" in terms of itself.

The object can be a function, sequence, algorithm, set, etc.

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How do we "solve" this recurrence?

$f(n) =$ some expression (not recursive)

AKA closed form.

A closed form is an expression without recursion or iteration.

Recursion

Define an "object" in terms of itself.

The object can be a function, sequence, algorithm, set, etc.

Function $f: \mathbb{Z} \to \mathbb{Z}$

$f(0) = 5$

if $n \geq 1$, $f(n) = f(n-1) + 2n - 1$

$f(0) = 5$
$f(1) = f(0) + 2(1) - 1 = 5 + 2 - 1 = 6$
$f(2) = f(1) + 2(2) - 1 = 6 + 4 - 1 = 9$
$f(3) = f(2) + 2(3) - 1 = 9 + 6 - 1 = 14$

How do we "solve" this recurrence?

$f(n) =$ some expression (not recursive)

AKA closed form.

1. Find a pattern
2. Guess a solution (sometimes tricky)
3. Verify by induction

Guess: $f(n) = n^2 + 5$

Recursion

Function $f: \mathbb{Z} \to \mathbb{Z}$
$f(0) = 5$
if $n \geq 1$, $\quad f(n) = f(n-1) + 2n - 1$

Claim: for $n \geq 0$, $\quad f(n) = n^2 + 5$

Proof: By induction.
Base case: $n = 0$
$f(0) = 0^2 + 5 = 5$ which is true

Inductive Step: Let $n \geq 1$
Assume claim is true for $n - 1$.
That is
$f(n-1) = (n-1)^2 + 5$ is true.

Show: $f(n) = n^2 + 5$
(using recursive definition).

$$\begin{aligned} f(n) &= f(n-1) + 2n - 1 \\ &= [(n-1)^2 + 5] + 2n - 1 \\ &= n^2 - 2n + 1 + 5 + 2n - 1 \\ &= n^2 + 5\,. \end{aligned}$$

We have shown it is true for the base case, and shown it is true in the inductive step, thus

for $n \geq 0, f(n) = n^2 + 5$.

Recursion

Function $g: \mathbb{Z} \to \mathbb{Z}$

$g(0) = 1$

if $n \geq 1$, $\quad g(n) = n \cdot g(n-1)$

$g(0) = 1$
$g(1) = 1 \cdot g(0) = 1 \cdot 1 = 1$
$g(2) = 2 \cdot g(1) = 2 \cdot 1 = 2$
$g(3) = 3 \cdot g(2) = 3 \cdot 2 = 6$
$g(4) = 4 \cdot g(3) = 4 \cdot 6 = 24$

Recursion

Function $g: \mathbb{Z} \to \mathbb{Z}$

$g(0) = 1$

if $n \geq 1, \qquad g(n) = n \cdot g(n-1)$

$g(0) = 1$
$g(1) = 1$
$g(2) = 1 \cdot 2$
$g(3) = 1 \cdot 2 \cdot 3$
$g(4) = 1 \cdot 2 \cdot 3 \cdot 4$

Claim: $\forall n \geq 0, g(n) = n!$

Base Case: $g(0) = 1 = 0!$ is true

Inductive Step: $n \geq 1, g(n-1) = (n-1)!$

$$g(n) = n \cdot g(n-1)$$
$$= n \cdot (n-1)!$$
$$= n!$$

Therefore $\forall n \geq 0, g(n) = n!$

Fibonacci Numbers

$f_0 = 0$
$f_1 = 1$

For $n \geq 2$:

$f_n = f_{n-1} + f_{n-2}$

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Can we solve this?
Yes, but we will give you the solution.

$x^2 = x + 1$ has two solutions:

$$\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

Claim: for $n \geq 0$, $\quad f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$

Prove this using induction.

($\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio)

Fibonacci Numbers

$f_0 = 0, f_1 = 1.$ For $n \geq 2$:

$$f_n = f_{n-1} + f_{n-2}$$

Claim: for $n \geq 0$, $\quad f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$

$x^2 = x + 1$ has two solutions:

$$\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

$$\varphi^2 = \varphi + 1$$
$$\psi^2 = \psi + 1$$

Proof by induction:

Base Case:

$$f(0) = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

$$f(1) = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2 \cdot \sqrt{5}}$$

$$= \frac{1+\sqrt{5} - 1 + \sqrt{5}}{2 \cdot \sqrt{5}}$$

$$= \frac{2 \cdot \sqrt{5}}{2 \cdot \sqrt{5}} = 1$$

So the base case holds.

Fibonacci Numbers

$f_0 = 0, f_1 = 1.$ For $n \geq 2$:

$f_n = f_{n-1} + f_{n-2}$

Claim: for $n \geq 0, f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$

$x^2 = x + 1$ has two solutions:

$$\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

$$\varphi^2 = \varphi + 1$$
$$\psi^2 = \psi + 1$$

Proof by induction:

Inductive Step: For $\mathrm{n} \geq 2$, assume

$f_{n-1} = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$ and

$f_{n-2} = \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}}$

$f_n = f_{n-1} + f_{n-2}$

$$= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}}$$
$$= \frac{\varphi^{n-1} - \psi^{n-1} + \varphi^{n-2} - \psi^{n-2}}{\sqrt{5}}$$
$$= \frac{\varphi^{n-1} + \varphi^{n-2} - (\psi^{n-1} + \psi^{n-2})}{\sqrt{5}}$$

Fibonacci Numbers

$f_0 = 0, f_1 = 1.$ For $n \geq 2$:

$$f_n = f_{n-1} + f_{n-2}$$

Claim: for $n \geq 0$, $f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$

$x^2 = x + 1$ has two solutions:

$$\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

$$\varphi^2 = \varphi + 1$$
$$\psi^2 = \psi + 1$$

Proof by induction:

$$f_n = f_{n-1} + f_{n-2}$$

$$= \frac{\varphi^{n-1} + \varphi^{n-2} - (\psi^{n-1} + \psi^{n-2})}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2}(\varphi + 1) - \psi^{n-2}(\psi + 1)}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2}(\varphi^2) - \psi^{n-2}(\psi^2)}{\sqrt{5}}$$

$$= \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

Fibonacci Numbers

$f_0 = 0, f_1 = 1.$ For $n \geq 2$:

$$f_n = f_{n-1} + f_{n-2}$$

Claim: for $n \geq 0$, $\boxed{f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}}$

$x^2 = x + 1$ has two solutions:

$$\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

$$\varphi^2 = \varphi + 1$$
$$\psi^2 = \psi + 1$$

Proof by induction:

$$\boxed{f_n} = f_{n-1} + f_{n-2}$$

$$= \frac{\varphi^{n-1} + \varphi^{n-2} - (\psi^{n-1} + \psi^{n-2})}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2}(\varphi + 1) - \psi^{n-2}(\psi + 1)}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2}(\varphi^2) - \psi^{n-2}(\psi^2)}{\sqrt{5}}$$

$$\boxed{= \frac{\varphi^n - \psi^n}{\sqrt{5}}}$$

00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$$n = 1: \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} \quad B_1 = 2$$

$$n = 2: \left.\begin{matrix} 01 \\ 10 \\ 11 \end{matrix}\right\} \quad B_2 = 3$$

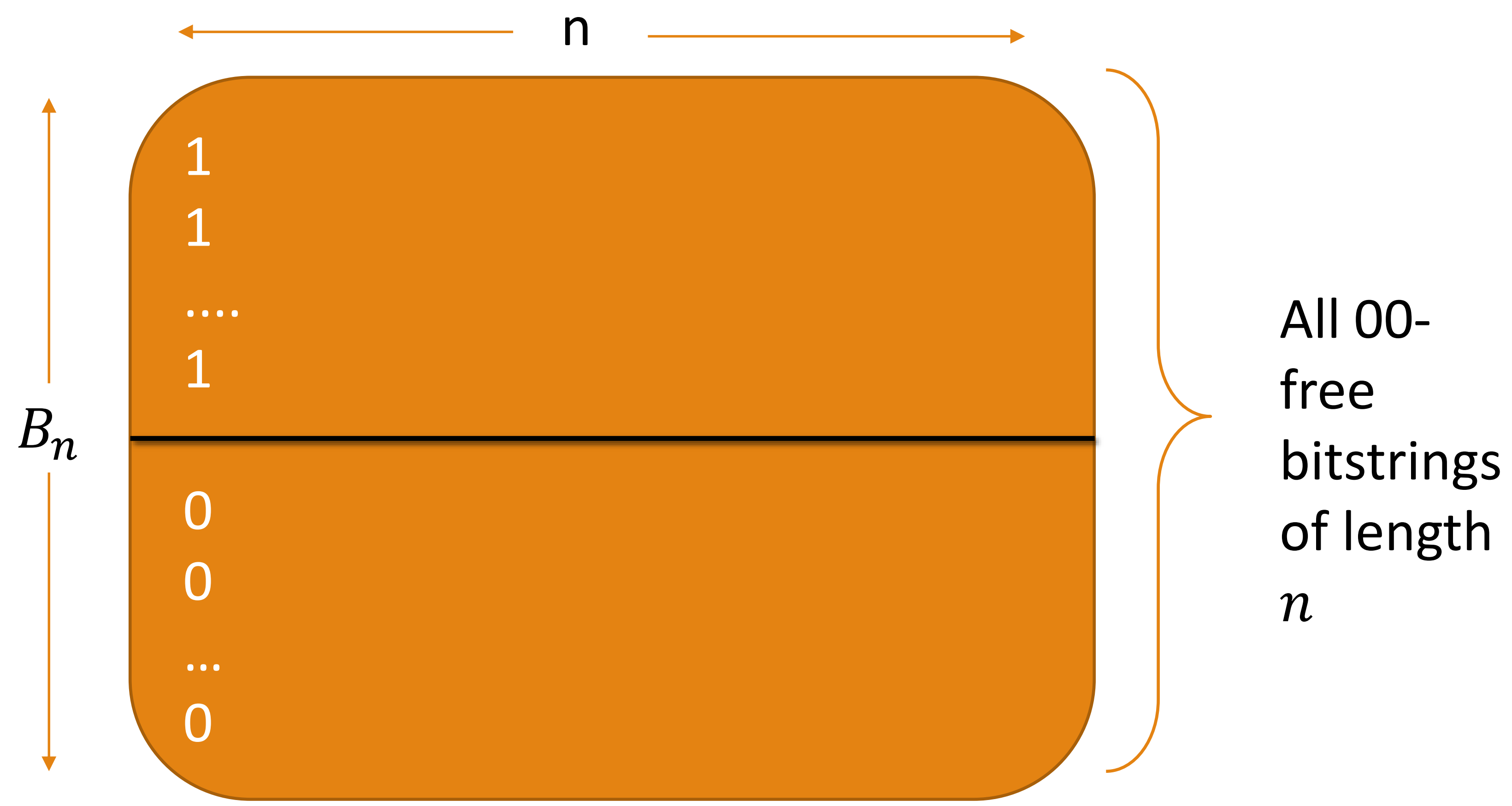
$$n = 3: \quad \left.\begin{matrix} 010 \\ 011 \\ 101 \\ 110 \\ 111 \end{matrix}\right\} \quad B_3 = 5$$

00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$B_1 = 2, B_2 = 3, B_3 = 5$

We want to derive a recurrence.



00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$B_1 = 2, B_2 = 3, B_3 = 5$

We want to derive a recurrence.

n

B_n

1
1
....
1

All 00-free bitstrings of length n that start with 1 =All 00-free bitstrings of length $n - 1$ i.e. B_{n-1}

0
0
...
0

All 00-free bitstrings of length n that start with 0

All 00-free bitstrings of length n

00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$B_1 = 2, B_2 = 3, B_3 = 5$

We want to derive a recurrence.

n

B_n

1 1 …. 1	All 00-free bitstrings of length n that start with 1 =All 00-free bitstrings of length $n-1$ i.e. B_{n-1}
01 01 … 01	All 00-free bitstrings of length n that start with 0 =All 00-free bitstrings of length $n-2$ i.e. B_{n-2}

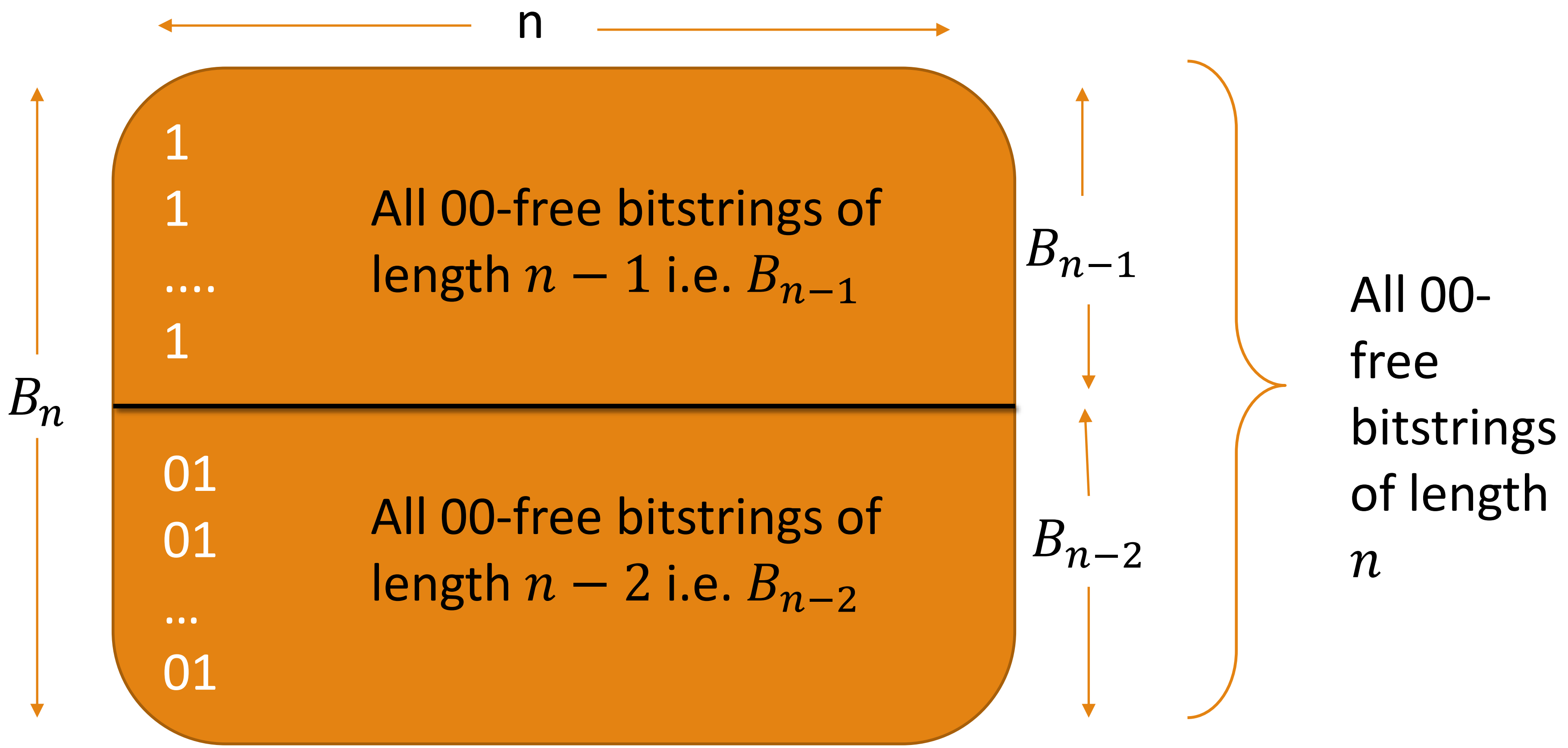
All 00-free bitstrings of length n

00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$B_1 = 2, B_2 = 3, B_3 = 5$

We want to derive a recurrence.



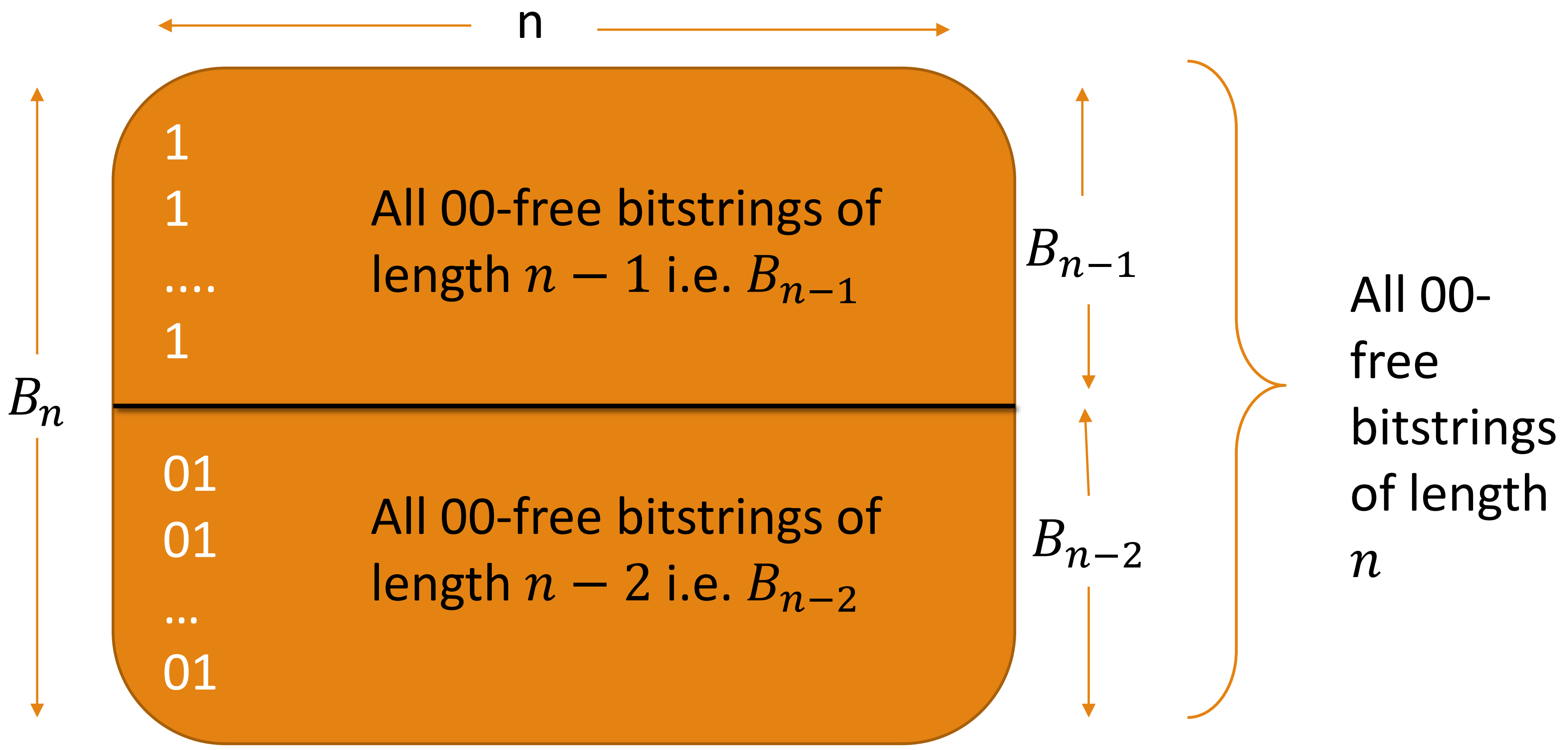
00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$$B_n = B_{n-1} + B_{n-2}$$

$$B_1 = 2, B_2 = 3, B_3 = 5$$

We want to derive a recurrence.



00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$$B_n = B_{n-1} + B_{n-2}$$
$$f_n = f_{n-1} + f_{n-2}$$

$B_1 = 2, B_2 = 3, B_3 = 5$

We want to derive a recurrence.

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7

00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$$B_1 = 2, B_2 = 3, B_3 = 5$$

We want to derive a recurrence.

$$B_n = B_{n-1} + B_{n-2}$$
$$f_n = f_{n-1} + f_{n-2}$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7
0	1	1	2	3	5	8	13

00-free Bitstrings

B_n = number of 00-free bitstrings of length n.

$$B_1 = 2, B_2 = 3, B_3 = 5$$

We want to derive a recurrence.

$$B_n = B_{n-1} + B_{n-2}$$
$$f_n = f_{n-1} + f_{n-2}$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7
0	1	1	2	3	5	8	13
			B_1	B_2	B_3	B_4	B_5

$$B_n = f_{n+2}$$

Sum of 1's and 2's

S_n = number of ways n can be written as a sum of 1's and 2's (order matters)

$1 = 1$
$S_1 = 1$

$2 = 1 + 1$
$2 = 2$
$S_2 = 2$

$3 = 1 + 1 + 1$
$3 = 1 + 2$
$3 = 2 + 1$
$S_3 = 3$

$4 = 1 + 1 + 1 + 1$
$4 = 1 + 1 + 2$
$4 = 1 + 2 + 1$
$4 = 2 + 1 + 1$
$4 = 2 + 2$

$S_4 = 5$

Sum of 1's and 2's

$S_1 = 1, S_2 = 2, S_3 = 3, S_4 = 5$

S_n = number of ways n can be written as a sum of 1's and 2's (order matters)

$n = 1 + \ \ldots$
$n = 2 + \ \ldots$

Sum of 1's and 2's

$S_1 = 1, S_2 = 2, S_3 = 3, S_4 = 5$

S_n = number of ways n can be written as a sum of 1's and 2's (order matters)

$$S_n \begin{cases} n = 1 + \ \dots \ (n-1 \text{ as a sum of 1's and 2's}) & S_{n-1} \\ n = 2 + \ \dots \ (n-2 \text{ as a sum of 1's and 2's}) & S_{n-2} \end{cases}$$

$S_n = S_{n-1} + S_{n-2}$

Sum of 1's and 2's

$S_1 = 1, S_2 = 2, S_3 = 3, S_4 = 5$

S_n = number of ways n can be written as a sum of 1's and 2's (order matters)

$$S_n \begin{cases} n = 1 + \ \dots \ (n-1 \text{ as a sum of 1's and 2's}) & S_{n-1} \\ n = 2 + \ \dots \ (n-2 \text{ as a sum of 1's and 2's}) & S_{n-2} \end{cases}$$

$S_n = S_{n-1} + S_{n-2}$
$f_n = f_{n-1} + f_{n-2}$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7
0	1	1	2	3	5	8	13

Sum of 1's and 2's

$S_1 = 1, S_2 = 2, S_3 = 3, S_4 = 5$

S_n = number of ways n can be written as a sum of 1's and 2's (order matters)

$$S_n \begin{cases} n = 1 + \ \dots\ (n-1 \text{ as a sum of 1's and 2's}) & S_{n-1} \\ n = 2 + \ \dots\ (n-2 \text{ as a sum of 1's and 2's}) & S_{n-2} \end{cases}$$

$$S_n = S_{n-1} + S_{n-2}$$
$$f_n = f_{n-1} + f_{n-2}$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7
0	1	1	2	3	5	8	13
		S_1	S_2	S_3	S_4	S_5	S_6

An Alternate Expression for 00-free bitstrings

Find B_n^i the number of 00-free bitstrings of length n with exactly i many 1's.

For example, let B_n be the set of 00-free bitstrings of length 9.

Consider all the bitstrings B_9^3 (length 9 with $3\ 1's$). Can we count them?

Using the Product Rule, what could be our procedure?

Task 1: Write down 3 1's in a row – there is 1 way to do this.

Task 2: Place 6 0's between the 1's such that no two 0's are next to one another.

An Alternate Expression for 00-free bitstrings

Find B_9^i the number of 00-free bitstrings of length n with exactly i many 1's.

For example, let B_9 be the set of 00-free bitstrings of length 9.

Consider all the bitstrings B_9^4 (length 9 with 4 1's). Can we count them?

Using the Product Rule, what could be our procedure?

Task 1: Write down 4 1's in a row – there is 1 way to do this.

Task 2: Place 5 0's between the 1's such that no two 0's are next to one another.

An Alternate Expression for 00-free bitstrings

	1		1		1		1	

Counting all the bitstrings in B_9^4 (length 9 with $4\ 1's$).

Task 1: Write down 4 1's – there is 1 way to do this.

Task 2: Place 5 0's between the 1's such that no two 0's are next to one another.

How many ways can we do Task 2?

An Alternate Expression for 00-free bitstrings

0	1	0	1	0	1	0	1	0

Counting all the bitstrings in B_9^4 (length 9 with $4\ 1'\!s$).

Task 1: Write down 4 1's – there is 1 way to do this.

Task 2: Place 5 0's between the 1's such that no two 0's are next to one another.

How many ways can we do Task 2?

Exactly 1 – there were 5 possible locations and 5 0's to place, so $\binom{5}{5} = 1$ ways to place them

An Alternate Expression for 00-free bitstrings

	1		1		1		1		1	

Counting all the bitstrings in B_9^5 (length 9 with $5\ 1's$).

Task 1: Write down 5 1's – there is 1 way to do this.

Task 2: Place 4 0's between the 1's such that no two 0's are next to one another.

How many ways can we do Task 2?

There are 6 possible locations and 4 0's to place, so $\binom{6}{4}$ ways to place them

An Alternate Expression for 00-free bitstrings

	1		1		1		1		1		1	

Counting all the bitstrings in B_9^6 (length 9 with 6 $1's$).

Task 1: Write down 6 1’s – there is 1 way to do this.

Task 2: Place 3 0’s between the 1’s such that no two 0’s are next to one another.

How many ways can we do Task 2?

There are 7 possible locations and 3 0’s to place, so
$\binom{7}{3}$ ways to place them

An Alternate Expression for 00-free bitstrings

Find B_n^i the number of 00-free bitstrings of length n with exactly i many 1's.

We can sum up all possibilities

$$B_9 = \sum_{i=0}^{9} \binom{i+1}{9-i}$$

$$B_n = \sum_{i=0}^{n} \binom{i+1}{n-i}$$

Alternatively the full expression is:

$$B_9 = \binom{1}{9} + \binom{2}{8} + \binom{3}{7} + \binom{4}{6} + \binom{5}{5} + \binom{6}{4} + \binom{7}{3} + \binom{8}{2} + \binom{9}{1} + \binom{10}{0} = 89$$

Where $\binom{1}{9} + \binom{2}{8} + \binom{3}{7} + \binom{4}{6} = 0$

And i is the number of 1's in the string.

An Alternate Expression for 00-free bitstrings

We know B_n = f_{n+2}, and

$$f_{n+2} = \frac{\varphi^{n+2} - \psi^{n+2}}{\sqrt{5}}$$

Thus:

$$f_{n+2} = \sum_{i=0}^{n} \binom{i+1}{n-i} = \frac{\varphi^{n+2} - \psi^{n+2}}{\sqrt{5}}$$

This gives us an alternate form for f_n where $n \geq 2$.