Geometry of surfaces - Solutions

62. The first fundamental form is $d\theta^2 + \cos^2(\theta)d\varphi^2$ (see Example 5.1.2). Thus we have

$$E(\theta,\varphi) = 1\ ,\ F(\theta,\varphi) = 0\ ,\ G(\theta,\varphi) = \cos^2(\theta).$$

Let $\gamma(t) = \sigma(\theta(t),\varphi(t))$ be a unit speed geodesic on S^2. Then

$$1 = \|\dot\gamma\|^2 = \dot\theta^2 + \cos^2(\theta)\dot\varphi^2.$$

The two geodesic equations (see Theorem 8.1.8) are in this situation:

$$\ddot\theta = -\sin(\theta)\cos(\theta)\dot\varphi^2 \text{ and } \frac{d}{dt}(\cos^2(\theta)\dot\varphi) = 0.$$

From the last equation we get

$$\cos^2(\theta)\dot\varphi = \Omega$$

for some $\Omega \in \mathbb{R}$.

If $\Omega = 0$, then $\dot\varphi = 0$ as $\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and hence $\cos(\theta) \neq 0$. Thus φ is constant, which means that γ is part of a meridian (great circle through $(0,0,1)$ and $(0,0,-1)$).

Assume $\Omega \neq 0$ from now on. Using the unit speed condition we get

$$1 = \dot\theta^2 + \cos^2(\theta)\dot\varphi^2 = \dot\theta^2 + \frac{\Omega^2}{\cos^2(\theta)}$$

and therefore

$$\dot\theta^2 = 1 - \frac{\Omega^2}{\cos^2(\theta)}.$$

Since $\Omega \neq 0$, we also have $\frac{d\varphi}{dt} = \dot\varphi \neq 0$ everywhere. Then

$$\left(\frac{d\theta}{d\varphi}\right)^2 = \left(\frac{d\theta}{dt}\frac{dt}{d\varphi}\right)^2 = \left(\frac{\dot\theta}{\dot\varphi}\right)^2 = \frac{\dot\theta^2}{\dot\varphi^2} = \frac{1-\frac{\Omega^2}{\cos^2(\theta)}}{\frac{\Omega^2}{\cos^4(\theta)}} = \cos^2(\theta)\left(\frac{\cos^2(\theta)}{\Omega^2} - 1\right)$$

and hence

$$d\varphi = \left(\cos^2(\theta)\left(\frac{\cos^2(\theta)}{\Omega^2} - 1\right)\right)^{-\frac{1}{2}} d\theta = \frac{1}{\cos(\theta)}\left(\frac{\cos^2(\theta)}{\Omega^2} - 1\right)^{-\frac{1}{2}} d\theta.$$

By substituting $u = \tan(\theta)$ we get $\theta = \arctan(u)$, $\cos(\theta) = \frac{1}{\sqrt{1+u^2}}$ and $d\theta = \frac{1}{1+u^2}du$. Inserting this into the previous equation gives

$$d\varphi = \frac{1}{\frac{1}{\sqrt{1+u^2}}}\left(\frac{\frac{1}{1+u^2}}{\Omega^2} - 1\right)^{-\frac{1}{2}} \frac{1}{1+u^2}du = \frac{1}{\sqrt{\lambda - u^2}}du$$

with $\lambda = \frac{1}{\Omega^2} - 1$. Integrating this expression leads to

$$\pm(\varphi - \varphi_0) = \int \frac{1}{\sqrt{\lambda - u^2}}du = \arcsin\left(\frac{u}{\sqrt{\lambda}}\right).$$

with some $\varphi_0 \in \mathbb{R}$. This implies

$$\tan(\theta) = u = \pm\sqrt{\lambda}\sin(\varphi - \varphi_0) = \pm\sqrt{\lambda}(\sin(\varphi)\cos(\varphi_0) - \cos(\varphi)\sin(\varphi_0)).$$

and hence $z = ax + by$ with $a = \mp\sqrt{\lambda}\sin(\varphi_0$ and $b = \pm\sqrt{\lambda}\cos(\varphi_0)$, where $(x,y,z) = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta)) = \gamma$. This shows that γ lies in a plane containing $(0,0,0)$ and hence γ is part of a great circle on S^2.

63. (i): We have

$$\sigma_u(u,v) = (\dot{f}(u)\cos(v), \dot{f}(u)\sin(v), \dot{g}(u)),$$
$$\sigma_v(u,v) = (-f(u)\sin(v), f(u)\cos(v), 0).$$

Then $\sigma_u(u,v)$ and $\frac{1}{\rho(u,v)}\sigma_v(u,v)$ are unit vectors that are tangent to the meridian and parallel through $\sigma(u,v)$, respectively, and perpendicular to each other. Then

$$\dot{\gamma}(t) = \cos(\psi(t))\sigma_u(u(t),v(t)) + \frac{\sin(\psi(t))}{\rho(u(t),v(t))}\sigma_v(u(t),v(t)).$$

Using $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$, this implies

$$\dot{v}(t)(\sigma_u \times \sigma_v)(u(t),v(t)) = \sigma_u(u(t),v(t)) \times \dot{\gamma}(t) = \frac{\sin(\psi(t))}{\rho(u(t),v(t))}(\sigma_u \times \sigma_v)(u(t),v(t))$$

and hence $\rho(u(t),v(t))\dot{v}(t) = \sin(\psi(t))$. This gives

$$\rho(u(t),v(t))\sin(\psi(t)) = \rho(u(t),v(t))^2\dot{v}(t) = f(u(t))^2\dot{v}(t),$$

which is constant by the second geodesic equation for γ (see Example 8.1.14).

(ii): If $\rho\sin(\psi)$ is constant along γ, say $\rho\sin(\psi) = \Omega \in \mathbb{R}$, then the second geodesic equation for γ is satisfied (see Example 8.1.14). We therefore need to check only the first geodesic equation:

$$\ddot{u}(t) = f(u(t))\dot{f}(u(t))\dot{v}(t)^2.$$

First, we have

$$\dot{v}(t)\rho(u(t),v(t))^2 = \dot{v}(t)\|\sigma_v(u(t),v(t))\|^2 = \dot{\gamma}(t)\cdot\sigma_v(u(t),v(t)) = \rho(u(t),v(t))\sin(\psi(t))$$

and hence

$$\dot{v}(t) = \frac{\sin(\psi(t))}{\rho(u(t),v(t))} = \frac{\Omega}{\rho(u(t),v(t))^2}.$$

Using $\|\dot{\gamma}\| = 1$ we then get

$$\begin{aligned} 1 &= \dot{u}(t)^2\|\sigma_u(u(t),v(t))\|^2 + \dot{v}(t)^2\|\sigma_v(u(t),v(t))\|^2 = \dot{u}(t)^2 + f(u(t))^2\dot{v}(t)^2 \\ &= \dot{u}(t)^2 + \rho(u(t),v(t))^2\dot{v}(t)^2 = \dot{u}(t)^2 + \rho(u(t),v(t))^2\frac{\sin(\psi(t))^2}{\rho(u(t),v(t))^2} \\ &= \dot{u}(t)^2 + \frac{\Omega^2}{\rho(u(t),v(t))^2} = \dot{u}(t)^2 + \frac{\Omega^2}{f(u(t))^2} \end{aligned}$$

Differentiating this equation leads to

$$\begin{aligned} 0 &= 2\dot{u}(t)\ddot{u}(t) - 2\frac{\Omega^2\dot{f}(u(t))\dot{u}(t)}{f(u(t))^3} = 2\dot{u}(t)\left(\ddot{u}(t) - \frac{\Omega^2\dot{f}(u(t))}{f(u(t))^3}\right) \\ &= 2\dot{u}(t)\left(\ddot{u}(t) - f(u(t))\dot{f}(u(t))\dot{v}(t)^2\right) \end{aligned}$$

Assume that there exists t_0 with $\ddot{u}(t_0) - f(u(t_0))\dot{f}(u(t_0))\dot{v}(t_0)^2 \neq 0$. By continuity, we then must have $\ddot{u}(t) - f(u(t))\dot{f}(u(t))\dot{v}(t)^2 \neq 0$ for t sufficiently close to t_0 and hence $\dot{u}(t) = 0$ for such t. But this means that $\gamma(t)$ is part of a parallel for such t, which contradicts our assumption. It follows that $\ddot{u}(t) - f(u(t))\dot{f}(u(t))\dot{v}(t)^2 = 0$ for all t, and thus the second geodesic equation is satisfied as well. Altogether we conclude now that γ is a geodesic.