Geometry of Surfaces

5CCM223A/6CCM223B

Video 11

Surface reparametrizations

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A surface patch $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ is a reparametrization of a surface patch $\sigma : U \to \mathbb{R}^3$ if there exists a smooth bijective map $\phi : U \to \tilde{U}$, the so-called reparametrization map, whose inverse map $\phi^{-1} : \tilde{U} \to U$ is smooth and $\tilde{\sigma} \circ \phi = \sigma$

Interpretation using partial derivatives:

Put $(\tilde{u}, \tilde{v}) = \phi(u, v)$, thus $(u, v) = \phi^{-1}(\tilde{u}, \tilde{v})$. Denote by $J\phi$ the Jacobi matrix of ϕ

$$J\phi = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$$

By Inverse Function Theorem, if

1. $\phi : U \to \tilde{U}$ is bijective and smooth
2. $J\phi$ is invertible everywhere

then $\phi^{-1} : \tilde{U} \to U$ is smooth and

$$\begin{aligned}
\begin{pmatrix} \frac{\partial u}{\partial \tilde{u}}(\tilde{u}, \tilde{v}) & \frac{\partial u}{\partial \tilde{v}}(\tilde{u}, \tilde{v}) \\ \frac{\partial v}{\partial \tilde{u}}(\tilde{u}, \tilde{v}) & \frac{\partial v}{\partial \tilde{v}}(\tilde{u}, \tilde{v}) \end{pmatrix} &= (J\phi^{-1})(\tilde{u}, \tilde{v}) \\
&= (J\phi^{-1})(\phi(u, v)) \\
&= (J\phi(u, v))^{-1} \\
&= \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u}(u, v) & \frac{\partial \tilde{u}}{\partial v}(u, v) \\ \frac{\partial \tilde{v}}{\partial u}(u, v) & \frac{\partial \tilde{v}}{\partial v}(u, v) \end{pmatrix}^{-1}
\end{aligned}$$

Example. Consider the surface patch $\sigma : U \to \mathbb{R}^3$ of the unit sphere S^2 given by spherical coordinates

$$U = \left\{(\theta, \varphi) \in \mathbb{R}^2 : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\ ,\ 0 < \varphi < 2\pi\right\}$$

$$\sigma : U \to \mathbb{R}^3\ ,\ (\theta, \varphi) \mapsto (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$$

Define

$$\tilde{U} = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\} = U_1((0, 0))$$

$$\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3\ ,\ (u, v) \mapsto (u, \sqrt{1 - u^2 - v^2}, v)$$

$\tilde{U}$ is open in $\mathbb{R}^2$ and $\tilde{\sigma}$ is smooth and injective. Thus $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ is a surface patch with $\tilde{\sigma}(\tilde{U}) = \{(x, y, z) \in S^2 : y > 0\}$

Claim: $\tilde{\sigma}$ is a reparametrization of σ restricted to $\left\{(\theta, \varphi) \in \mathbb{R}^2 : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\ ,\ 0 < \varphi < \pi\right\}$

Need to find smooth bijective map $\phi : U \to \tilde{U}$ with $\tilde{\sigma} \circ \phi = \sigma$ and ϕ^{-1} smooth. Writing $(u, v) = \phi(\theta, \varphi)$, we obtain

$$u = \cos(\theta)\cos(\varphi)\ ,\ \sqrt{1 - u^2 - v^2} = \cos(\theta)\sin(\varphi)\ ,\ v = \sin(\theta)$$

Thus

$$\phi(\theta, \varphi) = (u, v) = (\cos(\theta)\cos(\varphi), \sin(\theta))$$

It is easy to verify that ϕ is bijective on U. For the Jacobi matrix $J\phi$ we obtain

$$J\phi = \begin{pmatrix} -\sin(\theta)\cos(\varphi) & -\cos(\theta)\sin(\varphi) \\ \cos(\theta) & 0 \end{pmatrix}$$

Then $\det(J\phi) = \cos(\theta)^2 \sin(\varphi) \neq 0$ since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $0 < \varphi < \pi$. Thus $J\phi$ invertible everywhere and hence ϕ is a reparametrization map

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Video 12

Regular surfaces and their tangent planes

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Let $\sigma : U \to \mathbb{R}^3$ be a surface patch. A curve on σ is a smooth curve

$$\gamma : (\alpha, \beta) \to \mathbb{R}^3 \ , \ \ t \mapsto \gamma(t) = \sigma(u(t), v(t))$$

with $(u(t), v(t)) \in U \subset \mathbb{R}^2$ for all $t \in (\alpha, \beta)$

Let $p \in \mathcal{S} = \sigma(U)$. The tangent space to $\mathcal{S}$ at p is

$$T_p\mathcal{S} = \{\dot{\gamma}(t_0) : \gamma \text{ curve on } \sigma \text{ with } \gamma(t_0) = p\}$$

Proposition. *If $p = \sigma(u_0, v_0)$, then*

$$\begin{aligned} T_p\mathcal{S} &= \operatorname{span}\{\sigma_u(u_0, v_0), \sigma_v(u_0, v_0)\} \\ &= \{\xi\sigma_u(u_0, v_0) + \eta\sigma_v(u_0, v_0) \in \mathbb{R}^3 : \xi, \eta \in \mathbb{R}\} \end{aligned}$$

Proof. Let $\gamma(t) = \sigma(u(t), v(t))$ be a curve on σ with $\sigma(u(t_0), v(t_0)) = \gamma(t_0) = p = \sigma(u_0, v_0)$. Using the CHAIN RULE for $\gamma(t) = \sigma(u(t), v(t))$ we get

$$\dot{\gamma} = \sigma_u \dot{u} + \sigma_v \dot{v}$$

and thus

$$\begin{aligned}\dot{\gamma}(t_0) &= \dot{u}(t_0)\sigma_u(u(t_0), v(t_0)) + \dot{v}(t_0)\sigma_v(u(t_0), v(t_0)) \\ &= \dot{u}(t_0)\sigma_u(u_0, v_0) + \dot{v}(t_0)\sigma_v(u_0, v_0)\end{aligned}$$

This shows

$$T_p\mathcal{S} \subseteq \mathrm{span}\{\sigma_u(u_0, v_0), \sigma_v(u_0, v_0)\}$$

Conversely, let $\xi, \eta \in \mathbb{R}$ and define

$$\gamma(t) = \sigma(u_0 + \xi t, v_0 + \eta t)$$

Then γ is a smooth curve on σ ($|t|$ sufficiently small) with $\gamma(0) = \sigma(u_0, v_0) = p$ and, using the CHAIN RULE,

$$\dot{\gamma}(0) = \xi \sigma_u(u_0, v_0) + \eta \sigma_v(u_0, v_0)$$

This shows

$$\mathrm{span}\{\sigma_u(u_0, v_0), \sigma_v(u_0, v_0)\} \subseteq T_p\mathcal{S}$$

Altogether this finishes the proof of the proposition

A surface patch $\sigma : U \to \mathbb{R}^3$ is regular if

$$\forall (u, v) \in U : (\sigma_u \times \sigma_v)(u, v) \neq 0$$

In this case $T_p\mathcal{S}$ is called the tangent plane to $\mathcal{S}$ at $p = \sigma(u_0, v_0)$. Note that

$$(\sigma_u \times \sigma_v)(u_0, v_0) \perp \sigma_u(u_0, v_0), \sigma_v(u_0, v_0)$$

The vector

$$\mathbf{N}_p = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}(u_0, v_0) \perp T_p\mathcal{S}$$

is called the unit normal to σ (or $\mathcal{S}$) at p.

It can be shown that the definition of the unit normal is independent of the parametrization of the surface up to sign

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Video 13

Generalized cylinders, generalized cones, quadric surfaces

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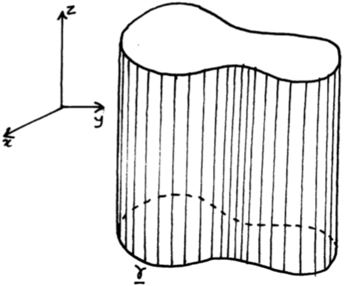
A generalized cylinder is obtained by translating a smooth plane curve perpendicular to the plane.



Parametrization:

$$\gamma(u) = (f(u), g(u), 0) \ , \ \sigma(u, v) = (f(u), g(u), v)$$

σ is smooth when γ is smooth. σ is injective if and only if γ is injective, which geometrically means that γ has no self-intersection.

$$\sigma_u(u, v) = (\dot{f}(u), \dot{g}(u), 0) \ , \ \sigma_v(u, v) = (0, 0, 1) \qquad \dot{} = \frac{d}{du}$$

$$(\sigma_u \times \sigma_v)(u, v) = (\dot{g}(u), -\dot{f}(u), 0)$$

Thus σ regular if and only if γ regular.

Example. A circular cylinder is the generalized cylinder generated by a circle

$$\gamma(u) = (\cos(u), \sin(u), 0)$$

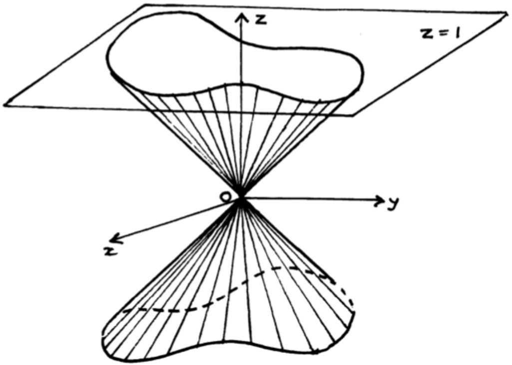
Then

$$\sigma(u, v) = (\cos(u), \sin(u), v)$$

Injectivity of σ requires u to be in open interval of length $< 2\pi$, e.g.

$$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 2\pi, \ v \in \mathbb{R}\}$$

A generalized cone is the union of lines passing through $0 \in \mathbb{R}^3$ and the points of a smooth curve in the plane $\{(x, y, 1) : x, y \in \mathbb{R}\} \subset \mathbb{R}^3$



Parametrization: Start with smooth curve

$$\gamma : (\alpha, \beta) \to \mathbb{R}^3 \ , \ u \mapsto (f(u), g(u), 1)$$

and define

$$\sigma(u, v) = (f(u)v, g(u)v, v) = v(f(u), g(u), 1)$$

σ is smooth since γ is smooth. σ is injective if and only if γ is injective (γ has no self-intersection).

$$\sigma_u(u, v) = v(\dot{f}(u), \dot{g}(u), 0) \ , \ \sigma_v(u, v) = (f(u), g(u), 1) \qquad \dot{} = \frac{d}{du}$$

$$(\sigma_u \times \sigma_v)(u, v) = v(\dot{g}(u), -\dot{f}(u), (\dot{f}g - f\dot{g})(u))$$

Thus σ regular if and only if γ regular and $v \neq 0$. Define σ on $U = (\alpha, \beta) \times \mathbb{R}_+$

Example. A circular cone is the generalized cone generated by a circle

$$\gamma(u) = (\cos(u), \sin(u), 1)$$

Then

$$\sigma(u, v) = v(\cos(u), \sin(u), 1)$$

Injectivity of σ requires u to be in open interval of length $< 2\pi$, e.g.

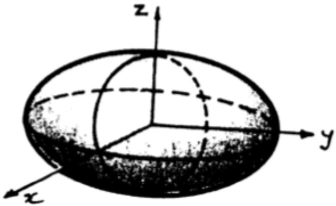
$$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 2\pi,\ \ v \in \mathbb{R}_+\}$$

A quadric surface is a surface whose points satisfy a quadratic equation of the form

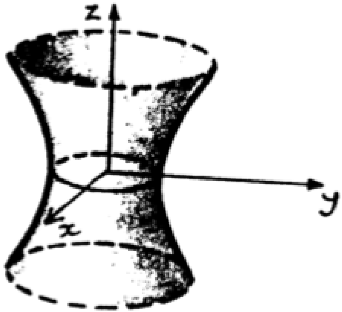
$$
\begin{aligned}
0 &= (x, y, z)\begin{pmatrix} a_1 & a_4 & a_6 \\ a_4 & a_2 & a_5 \\ a_6 & a_5 & a_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + (b_1, b_2, b_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + c \\
&= a_1x^2 + a_2y^2 + a_3z^2 + 2a_4xy + 2a_5yz + 2a_6xz \\
&\qquad + b_1x + b_2y + b_3z + c
\end{aligned}
$$

where $a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, c \in \mathbb{R}$. Using linear algebra and geometry, the following cases can occur:

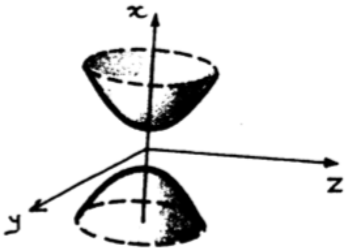
Ellipsoid $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$



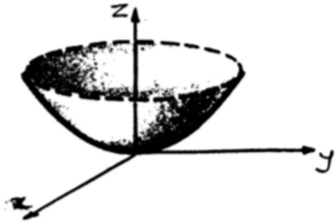
Hyperboloid of one sheet $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$



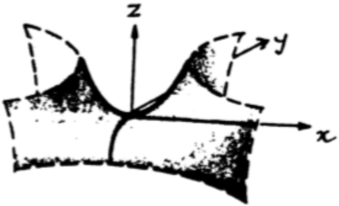
Hyperboloid of two sheets $\frac{x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$



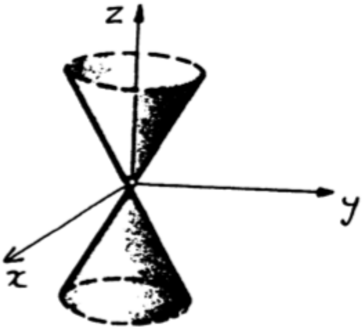
Elliptic paraboloid $\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$



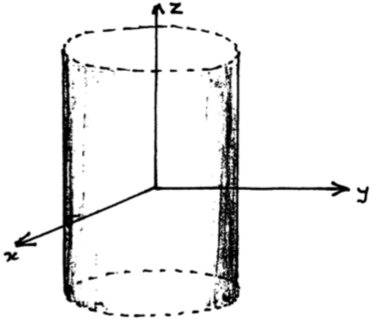
Hyperbolic paraboloid $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$



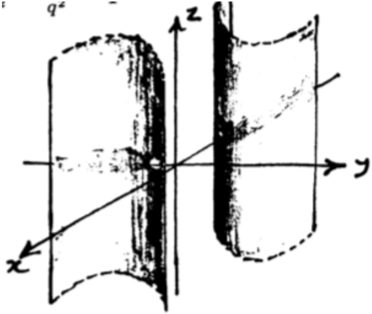
Quadric cone $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$



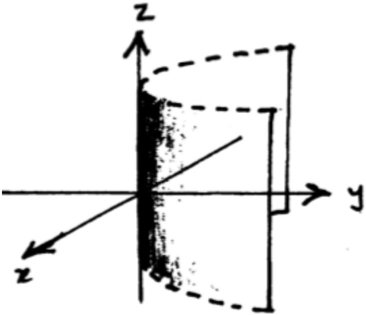
Elliptic cylinder $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$



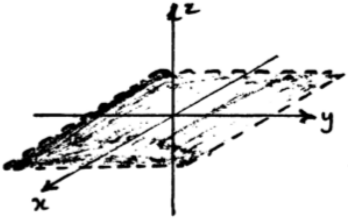
Hyperbolic cylinder $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$



Parabolic cylinder $\frac{x^2}{p^2} = y$



Plane $z = 0$



Parametrizations are not difficult to find, for example:

For ellipsoid $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$

$$\sigma(\theta, \varphi) = (p\cos(\theta)\cos(\varphi), q\cos(\theta)\sin(\varphi), r\sin(\theta))$$

For hyperboloid of one sheet $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$

$$\sigma(\theta, \varphi) = (p\cos(\theta), q\sin(\theta)\cosh(\varphi), r\sin(\theta)\sinh(\varphi))$$

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Video 14

Surfaces of revolution, ruled surfaces

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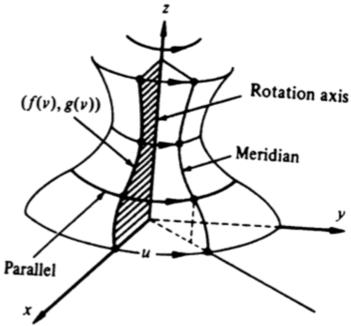
A surface of revolution is obtained by rotating a smooth plane curve (the so-called profile curve) in the xz-plane about the z-axis



Examples: spheres; circular cylinders; circular cones

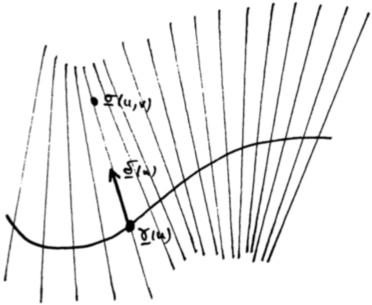
$$\gamma : (\alpha, \beta) \to \mathbb{R}^3, \;\; u \mapsto (f(u), 0, g(u))$$
$$\sigma(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

σ is smooth since γ is smooth. The curve $\sigma(c, v)$ is called a parallel and the curve $\sigma(u, c)$ is called a meridian of the surface

$$\sigma_u(u, v) = (\dot{f}(u)\cos(v), \dot{f}(u)\sin(v), \dot{g}(u)) \qquad \dot{} = \frac{d}{du}$$
$$\sigma_v(u, v) = (-f(u)\sin(v), f(u)\cos(v), 0)$$
$$(\sigma_u \times \sigma_v)(u, v) = f(u)(-\dot{g}(u)\cos(v), -\dot{g}(u)\sin(v), \dot{f}(u))$$
$$\|(\sigma_u \times \sigma_v)(u, v)\|^2 = f(u)^2(\dot{f}(u)^2 + \dot{g}(u)^2)$$

Thus σ is a regular surface patch if and only if γ is regular and does not intersect the z-axis or itself. Can choose $U = (\alpha, \beta) \times (0, 2\pi)$

A ruled surface is a surface that is the union of (parts of) straight lines



$$\sigma(u, v) = \gamma(u) + v\delta(u)$$

Examples: planes; generalized cylinders; generalized cones

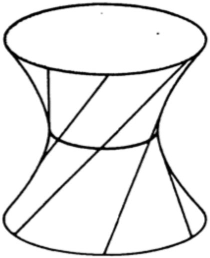
$$\sigma(u,v) = \gamma(u) + v\delta(u)$$

$$\sigma_u(u,v) = \dot{\gamma}(u) + v\dot{\delta}(u) \qquad \dot{} = \frac{d}{du}$$

$$\sigma_v(u,v) = \delta(u)$$

σ is a regular surface patch if and only if $\dot{\gamma}(u) + v\dot{\delta}(u)$ and $\delta(u)$ are linearly independent. This holds if $\dot{\gamma}(u)$ and $\delta(u)$ are linearly independent and v is sufficiently small

Less obvious example of a ruled surface is the hyperboloid of one sheet given by $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$



$$\gamma(u) = (p\cos(u), q\sin(u), 0)\ ,\ \delta(u) = (p\sin(u), -q\cos(u), r)$$

Can also take $\delta(u) = (-p\sin(u), q\cos(u), r)$. Thus hyperboloid of one sheet is *doubly ruled*