## Geometry of surfaces - Solutions

**39.** Let $Ldu^2 + 2Mdudv + Ndv^2$ be the second fundamental form of $\sigma$. We have $\tilde{\sigma}_u = \lambda\sigma_u$ and $\tilde{\sigma}_v = \lambda\sigma_v$. This implies $\tilde{\sigma}_{uu} = \lambda\sigma_{uu}$, $\tilde{\sigma}_{uv} = \lambda\sigma_{uv}$ and $\tilde{\sigma}_{vv} = \lambda\sigma_{vv}$. Moreover, the unit normal $\tilde{\mathbf{N}}$ of $\tilde{\sigma}$ satisfies $\tilde{\mathbf{N}} = \frac{\tilde{\sigma}_u\times\tilde{\sigma}_v}{\|\tilde{\sigma}_u\times\tilde{\sigma}_v\|} = \frac{\lambda^2(\sigma_u\times\sigma_v)}{\lambda^2\|\sigma_u\times\sigma_v\|} = \frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|} = \mathbf{N}$. This implies $\tilde{L} = \tilde{\sigma}_{uu}\cdot\tilde{\mathbf{N}} = \lambda\sigma_{uu}\cdot\mathbf{N} = \lambda L$, $\tilde{M} = \tilde{\sigma}_{uv}\cdot\tilde{\mathbf{N}} = \lambda\sigma_{uv}\cdot\mathbf{N} = \lambda M$ and $\tilde{N} = \tilde{\sigma}_{vv}\cdot\tilde{\mathbf{N}} = \lambda\sigma_{vv}\cdot\mathbf{N} = \lambda N$. Thus the second fundamental form of $\tilde{\sigma}$ is $\lambda(Ldu^2 + 2Mdudv + Ndv^2)$.

**40.** We have $\sigma_u(u,v) = (2u, v, 1)$ and $\sigma_v(u,v) = (0, u, -1)$. Thus $\sigma_u(1,1) = (2,1,1)$ and $\sigma_v(1,1) = (0,1,-1)$, which implies $(\sigma_u\times\sigma_v)(1,1) = (-2,2,2)$. Thus the unit normal at $\sigma(1,1)$ is $\mathbf{N}(1,1) = \frac{1}{\sqrt{3}}(-1,1,1)$. Next, we have $\sigma_{uu}(u,v) = (2,0,0)$, $\sigma_{uv}(u,v) = (0,1,0)$ and $\sigma_{vv}(u,v) = (0,0,0)$. Thus $\sigma_{uu}(1,1) = (2,0,0)$, $\sigma_{uv}(1,1) = (0,1,0)$ and $\sigma_{vv}(1,1) = (0,0,0)$. This implies $L(1,1) = \sigma_{uu}(1,1)\cdot\mathbf{N}(1,1) = -\frac{2}{\sqrt{3}}$, $M(1,1) = \sigma_{uv}(1,1)\cdot\mathbf{N}(1,1) = \frac{1}{\sqrt{3}}$ and $N(1,1) = \sigma_{vv}(1,1)\cdot\mathbf{N}(1,1) = 0$. Thus the second fundamental form of the surface at $\sigma(1,1)$ is $-\frac{2}{\sqrt{3}}du^2 + \frac{2}{\sqrt{3}}dudv$.

**41.** We have $\kappa_n' = \ddot{\gamma}\cdot\mathbf{N}' = \ddot{\gamma}\cdot(-\mathbf{N}) = -\ddot{\gamma}\cdot\mathbf{N} = -\kappa_n$.

**42.** We have $\dot{\tilde{\gamma}}(s) = -\dot{\gamma}(-s)$ and $\ddot{\tilde{\gamma}}(s) = \ddot{\gamma}(-s)$; thus $\dot{\tilde{\gamma}}(0) = -\dot{\gamma}(0)$ and $\ddot{\tilde{\gamma}}(0) = \ddot{\gamma}(0)$. This gives $\tilde{\kappa}_g(0) = (\ddot{\tilde{\gamma}}\cdot(\mathbf{N}\times\dot{\tilde{\gamma}}))(0) = (\ddot{\gamma}\cdot(\mathbf{N}\times(-\dot{\gamma})))(0) = -(\ddot{\gamma}\cdot(\mathbf{N}\times\dot{\gamma}))(0) = -\kappa_g(0)$ and $\tilde{\kappa}_n(0) = (\ddot{\tilde{\gamma}}\cdot\mathbf{N})(0) = (\ddot{\gamma}\cdot\mathbf{N})(0) = \kappa_n(0)$.

**43.** At $O$ we have $\kappa_g = \ddot{\gamma}(0)\cdot(\mathbf{N}\times\dot{\gamma}(0)) = (1,-1,2)\cdot\frac{1}{\sqrt{2}}(-1,1,0) = -\sqrt{2}$ and $\kappa_n = \ddot{\gamma}(0)\cdot\mathbf{N} = (1,-1,2)\cdot(0,0,1) = 2$.

**44.** At $O$ we have $\kappa_g = \ddot{\gamma}(0)\cdot(\mathbf{N}\times\dot{\gamma}(0)) = (0,2,1)\cdot(0,1,0) = 2$ and $\kappa_n = \ddot{\gamma}(0)\cdot\mathbf{N} = (0,2,1)\cdot(0,0,1) = 1$.

**45.** It follows from the assumption that the unit normal $\mathbf{N}$ of $\mathcal{S}$ is constant along $\gamma$ and therefore $0 = \frac{d}{dt}(\mathbf{N}\cdot\dot{\gamma}) = \mathbf{N}\cdot\ddot{\gamma} = \kappa_n$ along $\gamma$. Since the curvature $\kappa$ of $\gamma$ satisfies $\kappa^2 = \kappa_n^2 + \kappa_g^2$, we obtain $\kappa_g = \pm\kappa$ along $\gamma$.

**46.** Let $\gamma$ be a normal section with $\gamma(t_0) = p$. Because a normal section is a curve on $\sigma$, the distance $\|\gamma(t)\|$ from the origin $O$ to $\gamma(t)$ obtains a maximum at $t_0$ and we can apply Exercise 11, using the fact that $\kappa(t_0) = |k_n|$.

**47.** From Exercise 31 we know that $\tilde{\mathcal{F}}_I = \lambda^2\mathcal{F}_I$ and from Exercise 39 we know that $\tilde{\mathcal{F}}_{II} = \lambda\mathcal{F}_{II}$. Therefore

$$\det(\tilde{\mathcal{F}}_{II} - \kappa\tilde{\mathcal{F}}_I) = \det(\lambda\mathcal{F}_{II} - \kappa\lambda^2\mathcal{F}_I) = \det(\lambda(\mathcal{F}_{II} - \kappa\lambda\mathcal{F}_I)) = \lambda^2\det(\mathcal{F}_{II} - \kappa\lambda\mathcal{F}_I)$$

The principal curvatures $\kappa_1$ and $\kappa_2$ of $\sigma$ are the roots of the equation $\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0$. The above equation then tells us that the roots $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ of the equation $\det(\tilde{\mathcal{F}}_{II} - \kappa\tilde{\mathcal{F}}_I) = 0$ are $\frac{\kappa_1}{\lambda}$ and $\frac{\kappa_2}{\lambda}$. In other words, the principal curvatures of $\tilde{\sigma}$ are $\frac{\kappa_1}{\lambda}$ and $\frac{\kappa_2}{\lambda}$.