Geometry of Surfaces

5CCM223A/6CCM223B

Video 35

Global surfaces

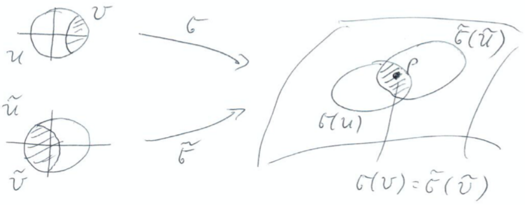
Jürgen Berndt

King's College London

Roughly, a global surface is a surface that is obtained by gluing together surface patches in a smooth manner
A sphere is obtained by gluing together a couple of hemispheres

Let $\sigma : U \to \mathbb{R}^3$ and $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ be two regular surface patches. Then σ and $\tilde{\sigma}$ are said to be compatible if

(a) $\sigma(U) \cap \tilde{\sigma}(\tilde{U}) = \emptyset$, or
(b) $\sigma(U) \cap \tilde{\sigma}(\tilde{U}) \neq \emptyset$ and for every $p \in \sigma(U) \cap \tilde{\sigma}(\tilde{U})$ there exist open subsets $V \subseteq U$, $\tilde{V} \subseteq \tilde{U}$ such that $p \in \sigma(V) = \tilde{\sigma}(\tilde{V})$ and $\tilde{\sigma}^{-1} \circ \sigma|_V : V \to \tilde{V}$, $\sigma^{-1} \circ \tilde{\sigma}|_{\tilde{V}} : \tilde{V} \to V$ are smooth maps



Let $S \subseteq \mathbb{R}^3$. An **atlas** for S is a collection of surface patches $\sigma_i : U_i \to \mathbb{R}^3$ ($i \in I$, I non-empty set) such that

(a) $S = \bigcup_{i \in I} \sigma_i(U_i)$;

(b) $\sigma_i : U_i \to \mathbb{R}^3$ and $\sigma_j : U_j \to \mathbb{R}^3$ are compatible for all $i, j \in I$;

(c) For every $p \in S$ and every surface patch $\sigma_i : U_i \to \mathbb{R}^3$ with $p \in \sigma_i(U_i)$ there exists an open subset $V_i \subseteq U_i$ such that $p \in \sigma_i(V_i)$ and $\sigma_i(V_i) = S \cap W_i$ with some open subset $W_i \subseteq \mathbb{R}^3$.

A **global surface** is a subset S of $\mathbb{R}^3$ together with an atlas for S.

Example. The following six regular surface patches form an atlas for the unit sphere S^2 in $\mathbb{R}^3$ and hence S^2 is a global surface:

$$\sigma_1^+ : U_1(0) \to \mathbb{R}^3,\ (u, v) \mapsto \left(\sqrt{1 - u^2 - v^2}, u, v\right)$$

$$\sigma_1^- : U_1(0) \to \mathbb{R}^3,\ (u, v) \mapsto \left(-\sqrt{1 - u^2 - v^2}, u, v\right)$$

$$\sigma_2^+ : U_1(0) \to \mathbb{R}^3,\ (u, v) \mapsto \left(u, \sqrt{1 - u^2 - v^2}, v\right)$$

$$\sigma_2^- : U_1(0) \to \mathbb{R}^3,\ (u, v) \mapsto \left(u, -\sqrt{1 - u^2 - v^2}, v\right)$$

$$\sigma_3^+ : U_1(0) \to \mathbb{R}^3,\ (u, v) \mapsto \left(u, v, \sqrt{1 - u^2 - v^2}\right)$$

$$\sigma_3^- : U_1(0) \to \mathbb{R}^3,\ (u, v) \mapsto \left(u, v, -\sqrt{1 - u^2 - v^2}\right)$$

where $U_1(0) = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$

Two atlases on S are equivalent if their union is an atlas for S. This gives an equivalence relation among atlases for S.

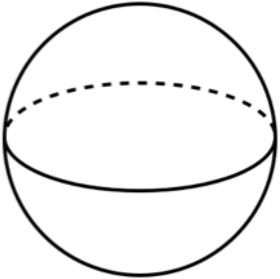
Problem. How many non-equivalent atlases are there for given S?

Theorem. *Any two atlases for a compact global surface S are equivalent.*

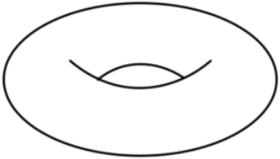
HEINE-BOREL THEOREM. *A subset S of $\mathbb{R}^3$ is compact if and only if S is closed and bounded*

S closed if $\mathbb{R}^3 \backslash S$ is open
S bounded if there exists $r > 0$ such that $S \subseteq U_r(0)$

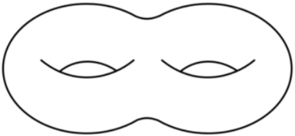
T_0 sphere



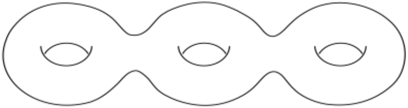
T_1 torus



T_2 double torus



T_3 triple torus



$T_0, T_1, T_2, T_3, \ldots, T_g, \ldots$; g is the genus of the surface

Theorem. *T_g, $g \geqslant 0$, can be equipped with an atlas making it a compact global surface. Every compact global surface is diffeomorphic to T_g for some $g \geqslant 0$*

Geometry of Surfaces

5CCM223A/6CCM223B

Video 36

Triangulations and the Euler number

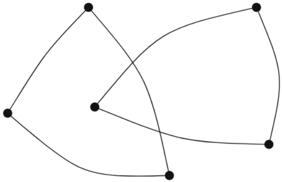
Jürgen Berndt
King's College London

Let S be a global surface with atlas $\{\sigma_i : U_i \to \mathbb{R}^3\}$. A triangulation of S is a collection of polygons each of which is contained in one of the sets $\sigma_i(U_i)$ such that

(i) Every point of S is in at least one of the polygons;

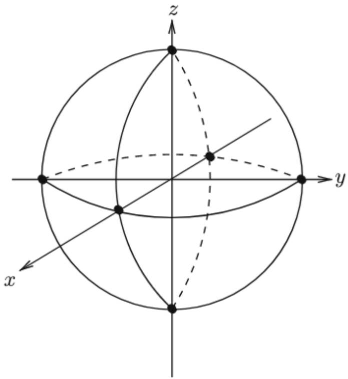
(ii) Two polygons are either disjoint, or their intersection is a common edge or a common vertex;

(iii) Each edge is an edge of exactly two polygons

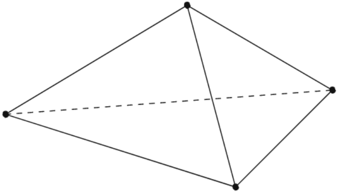


is not allowed

A triangulation of S^2 by 8 polygons



A triangulation of S^2 by 4 polygons



inflate tetrahedron

Theorem. *Every compact global surface has a triangulation with finitely many polygons*

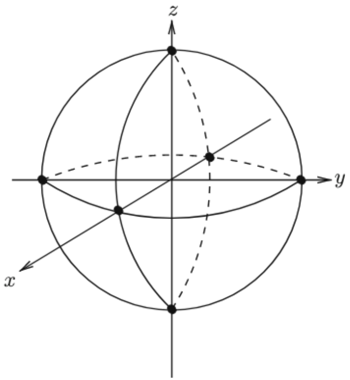
The Euler number χ of a triangulation of a compact surface is

$$\chi = V - E + F$$

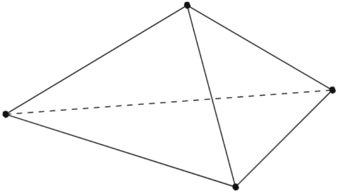
where

$$\begin{aligned} V &= \text{total number of vertices} \\ E &= \text{total number of edges} \\ F &= \text{total number of faces} \end{aligned}$$

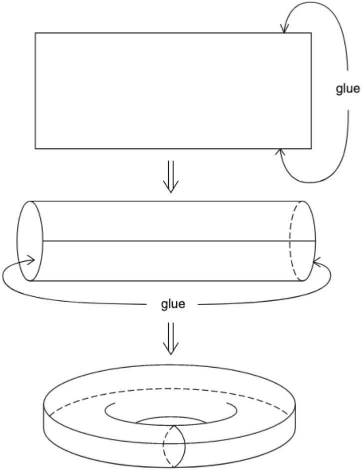
of the triangulation

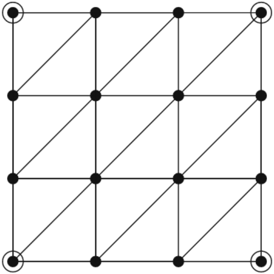


$$V = 6,\ E = 12,\ F = 8 \Longrightarrow \chi = 6 - 12 + 8 = 2$$



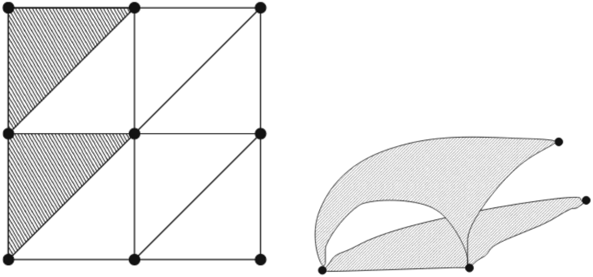
$$V = 4,\ E = 6,\ F = 4 \Longrightarrow \chi = 4 - 6 + 4 = 2$$





$$V = 9,\ E = 27,\ F = 18 \Longrightarrow \chi = 9 - 27 + 18 = 0$$

Need to be careful with choice of polygons
This is not a triangulation of the torus
shaded triangles intersect in two vertices



Question. For given S, does the Euler number depend on the triangulation?

Geometry of Surfaces

5CCM223A/6CCM223B

Video 37

Gauss-Bonnet Theorem (global version)

Jürgen Berndt
King's College London

Let S be a compact global surface. Fix a triangulation of S with polygons P_i. For each polygon P_i there exists a surface patch $\sigma_i : U_i \to \mathbb{R}^3$ in the atlas of S so that $P_i = \sigma_i(R_i)$ for some $R_i \subseteq U_i$.
Define the total Gaussian curvature of S by

$$\iint_S K d\mathcal{A} = \sum_i \iint_{R_i} K d\mathcal{A}_{\sigma_i}$$

Need to show that this definition is independent of choice of surface patches (or atlas) and choice of triangulation.

Assume $\sigma : U \to \mathbb{R}^3$ and $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ are compatible surface patches so that $P_j = \sigma(R_j) = \tilde{\sigma}(\tilde{R}_j)$. Then

$$\iint\limits_{R_j} K d\mathcal{A}_\sigma = \iint\limits_{\tilde{R}_j} K d\mathcal{A}_{\tilde{\sigma}}$$

because reparametrizations do not change area (see 5.3.3) and Gaussian curvature (by Theorema egregium)

Let $\{P_i\}$ and $\{P'_j\}$ be two triangulations of S. Refine the two triangulations to a triangulation $\{P''_k\}$ such that each P_i and each P'_j is the union of some polygons P''_k. The integral over the union of polygons is the sum of the integrals over the polygons (here we use that fact that different polygons in triangulations are either disjoint or intersect in a common edge or vertex). Then

$$\sum_i \iint_{R_i} K d\mathcal{A}_{\sigma_i} = \sum_k \iint_{R''_k} K d\mathcal{A}_{\sigma_k} = \sum_j \iint_{R'_j} K d\mathcal{A}_{\sigma_j}$$

Altogether this shows that the total Gaussian curvature is well-defined.

Gauss-Bonnet Theorem (global version). *Let S be a compact global surface. Then*

$$\boxed{\iint_S K d\mathcal{A} = 2\pi\chi}$$

where χ is the Euler number of any triangulation of S

Corollary. *The Euler number χ of a triangulation of a compact global surface S depends only on S and not on the choice of the triangulation.*

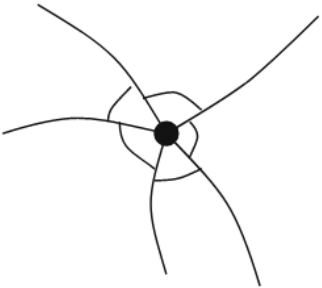
Therefore the Euler number of a compact global surface is well-defined. We denote this number by $\chi(S)$. From earlier calculations we can now deduce that $\chi(S^2) = 2$ for the sphere S^2 and $\chi(T^2) = 0$ for the torus T^2.

Proof. Fix triangulation $\{P_i\}$ of S with corresponding surface patches $\sigma_i : U_i \to \mathbb{R}^3$, $R_i \subseteq U_i$, $\sigma_i(R_i) = P_i$. Denote by γ_i the curvilinear polygon parametrizing the boundary of the polygon P_i. The Gauss-Bonnet Theorem for curvilinear polygons implies

$$\iint\limits_{R_i} K d\mathcal{A}_{\sigma_i} = \angle_i - (n_i - 2)\pi - \int_{\gamma_i} \kappa_g ds$$

where $\angle_i$ is the sum of the interior angles of P_i and n_i is the number of vertices of P_i. Need to compute the sums $\sum_i$ of these terms.

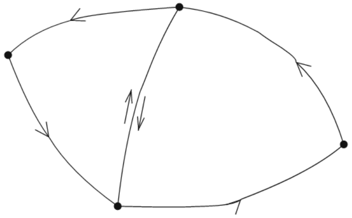
at each vertex we have the following picture



Therefore,

$$\sum_i \angle_i = 2\pi V$$

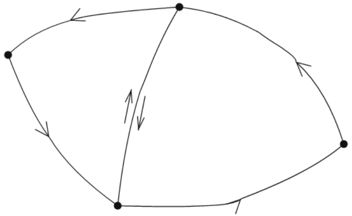
Each edge is counted twice, as each edge is an edge of exactly two polygons



Therefore,

$$\sum_i (n_i - 2)\pi = \pi \underbrace{\sum_i n_i}_{=2E} - 2\pi F = 2\pi E - 2\pi F$$

Integrate twice along each edge



κ_g changes sign when reversing direction of curve. Therefore corresponding pairs in $\sum_i \int_{\gamma_i} \kappa_g ds$ cancel out each other and hence

$$\sum_i \int_{\gamma_i} \kappa_g ds = 0$$

Altogether,

$$
\begin{aligned}
\iint\limits_S K d\mathcal{A} &= \sum_i \iint\limits_{R_i} K d\mathcal{A}_{\sigma_i} \\
&= \sum_i \angle_i - \sum_i (n_i - 2)\pi - \sum_i \int_{\gamma_i} \kappa_g ds \\
&= 2\pi V - (2\pi E - 2\pi F) - 0 \\
&= 2\pi\chi
\end{aligned}
$$

Theorem.

$$\chi(T_g) = 2 - 2g$$

Corollary.

$$\iint_{T_g} K d\mathcal{A} = 4\pi(1 - g)$$