Geometry of Surfaces

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Video 7

Torsion-free space curves and the Frenet-Serret equations

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Proposition. *Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ with non-zero curvature $\kappa \neq 0$ everywhere. Then the torsion τ of γ is equal to 0 everywhere if and only if γ is contained in a plane.*

Proof.

$$\begin{aligned} \tau = 0 \quad &\Longrightarrow \quad \dot{\mathbf{b}} = -\tau \mathbf{n} = 0 \\ &\Longrightarrow \quad \mathbf{b} \text{ constant} \\ &\Longrightarrow \quad \frac{d}{ds}(\gamma \cdot \mathbf{b}) = \dot{\gamma} \cdot \mathbf{b} = \mathbf{t} \cdot \mathbf{b} = 0 \\ &\Longrightarrow \quad \exists\ d \in \mathbb{R} : \gamma \cdot \mathbf{b} = d \end{aligned}$$

Thus γ is contained in the plane $\{x \in \mathbb{R}^3 : x \cdot \mathbf{b} = d\} \subset \mathbb{R}^3$

Assume that γ is contained in a plane. Then there exist $a \in \mathbb{R}^3$, $\|a\| = 1$, $d \in \mathbb{R}$ so that γ is contained in $P = \{x \in \mathbb{R}^3 : x \cdot a = d\}$. Then

$$\begin{aligned} \gamma \cdot a = d \quad &\Longrightarrow \quad 0 = \mathbf{t} \cdot a \\ &\Longrightarrow \quad 0 = \dot{\mathbf{t}} \cdot a = \kappa \mathbf{n} \cdot a \\ &\Longrightarrow \quad 0 = \mathbf{n} \cdot a \end{aligned}$$

Hence $\mathbf{t}, \mathbf{n}$ are perpendicular to a and thus parallel to plane P. Then $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is perpendicular to plane P and hence parallel to a. Since $\|a\| = 1 = \|\mathbf{b}\|$ and $\mathbf{b}$ is continuous, we conclude $\mathbf{b} = a$ or $\mathbf{b} = -a$. Thus $\mathbf{b}$ is constant, so $\dot{\mathbf{b}} = 0$ and thus $\tau = 0$.

We know $\dot{\mathbf{t}} = \kappa \mathbf{n}$ and $\dot{\mathbf{b}} = -\tau \mathbf{n}$. Question: $\dot{\mathbf{n}} = ?$ Write

$$\dot{\mathbf{n}} = \lambda \mathbf{t} + \mu \mathbf{n} + \nu \mathbf{b} \text{ with } \lambda = \dot{\mathbf{n}} \cdot \mathbf{t},\ \mu = \dot{\mathbf{n}} \cdot \mathbf{n},\ \nu = \dot{\mathbf{n}} \cdot \mathbf{b}$$

Then

$$\begin{aligned}
0 = \mathbf{t} \cdot \mathbf{n} \quad &\Longrightarrow \quad 0 = \dot{\mathbf{t}} \cdot \mathbf{n} + \mathbf{t} \cdot \dot{\mathbf{n}} = \kappa \underbrace{\mathbf{n} \cdot \mathbf{n}}_{=1} + \lambda \\
1 = \mathbf{n} \cdot \mathbf{n} \quad &\Longrightarrow \quad 0 = \dot{\mathbf{n}} \cdot \mathbf{n} = \mu \\
0 = \mathbf{b} \cdot \mathbf{n} \quad &\Longrightarrow \quad 0 = \dot{\mathbf{b}} \cdot \mathbf{n} + \mathbf{b} \cdot \dot{\mathbf{n}} = -\tau \underbrace{\mathbf{n} \cdot \mathbf{n}}_{=1} + \nu
\end{aligned}$$

Thus

$$\dot{\mathbf{n}} = -\kappa \mathbf{t} + \tau \mathbf{b}$$

Theorem. [FRENET-SERRET EQUATIONS] *Let γ be a unit speed curve in $\mathbb{R}^3$ with $\kappa \neq 0$ everywhere. Then*

$$\dot{\mathbf{t}} = \kappa \mathbf{n}\ ,\ \ \dot{\mathbf{n}} = -\kappa \mathbf{t} + \tau \mathbf{b}\ ,\ \ \dot{\mathbf{b}} = -\tau \mathbf{n}$$

Note: We can write above equations in matrix form

$$\begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

Proposition. *Let γ be a unit speed curve in $\mathbb{R}^3$ with constant curvature $\kappa \neq 0$ and torsion $\tau = 0$. Then γ is (part of) a circle.*

Proof. Since $\tau = 0$, the curve γ is contained in a plane. The Frenet-Serret equations imply

$$\begin{aligned} &\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}\mathbf{n}\right) = \mathbf{t} + \frac{1}{\kappa}\dot{\mathbf{n}} = \mathbf{t} + \frac{1}{\kappa}(-\kappa\mathbf{t} + \tau\mathbf{b}) = 0 \\ &\implies \quad \exists a \in \mathbb{R}^3 : \gamma + \frac{1}{\kappa}\mathbf{n} = a \quad \implies \quad \gamma - a = \frac{1}{\kappa}\mathbf{n} \\ &\implies \quad \|\gamma - a\| = \left\|\frac{1}{\kappa}\mathbf{n}\right\| = \frac{1}{\kappa} \end{aligned}$$

Thus γ lies on a sphere with radius $\frac{1}{\kappa}$ and centre a. Since intersection of plane and sphere is a circle (or a point, not possible since $\dot{\gamma} \neq 0$), the assertion follows.

Theorem. *Let* $\gamma(s), \gamma_1(s)$ *be unit speed curves in* $\mathbb{R}^3$ *with same curvature* $\kappa(s)$ *and same torsion* $\tau(s)$*. Then there exists a rigid motion* M *of* $\mathbb{R}^3$ *with* $\gamma_1 = M \circ \gamma$*.*

Moreover, if k *and* t *are smooth functions with* $k > 0$ *everywhere, then there exists a unit speed curve in* $\mathbb{R}^3$ *whose curvature is* k *and torsion is* t*.*

Proof. See [Pressley, Theorem 2.3.6.]

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Wirtinger's inequality

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Theorem. (Wirtinger's Inequality) *Let $F : [0, \pi] \to \mathbb{R}$ be a smooth function with $F(0) = 0 = F(\pi)$. Then*

$$\int_0^{\pi} F'(t)^2 dt \geqslant \int_0^{\pi} F(t)^2 dt$$

and equality holds if and only if there exists $A \in \mathbb{R}$ so that $F(t) = A \sin(t)$ holds for all $t \in [0, \pi]$

Proof. Define $G(t) = \frac{F(t)}{\sin(t)}$, which is well-defined for $t \in \{0, \pi\}$ by L'Hôpital's Rule. Then

$$\int_0^\pi F'^2 = \int_0^\pi (G \sin)'^2 = \int_0^\pi (G' \sin + G \cos)^2$$
$$= \int_0^\pi G'^2 \sin^2 + 2 \int_0^\pi GG' \sin \cos + \int_0^\pi G^2 \cos^2$$

Integration by Parts gives

$$2 \int_0^\pi GG' \sin \cos = \left. G^2 \sin \cos \right|_{t=0}^{t=\pi} - \int_0^\pi G^2 (cos^2 - sin^2)$$

Altogether,

$$\int_0^\pi F'^2 = \int_0^\pi (G^2 + G'^2) \sin^2 = \int_0^\pi F^2 + \int_0^\pi G'^2 \sin^2$$

$$\int_0^\pi F'^2 = \int_0^\pi F^2 + \int_0^\pi G'^2 \sin^2$$

This gives

$$\int_0^\pi F'^2 \geqslant \int_0^\pi F^2$$

and equality holds if and only if

$$\begin{aligned}\int_0^\pi G'^2 \sin^2 = 0 &\iff G' = 0\\ &\iff \exists\ A \in \mathbb{R} : G(t) = A \text{ for all } t \in [0, \pi]\\ &\iff \exists\ A \in \mathbb{R} : F(t) = A \sin(t) \text{ for all } t \in [0, \pi]\end{aligned}$$

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Isoperimetric inequality

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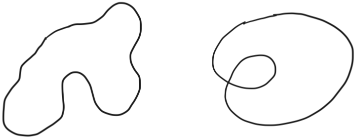
Let $0 < a \in \mathbb{R}$. A simple closed curve in $\mathbb{R}^2$ with period a is a regular curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, parametrized by a multiple of arc length, such that

$$\gamma(t) = \gamma(t') \quad \Longleftrightarrow \quad t' - t \in \mathbb{Z}a \qquad (\forall t, t' \in \mathbb{R}).$$



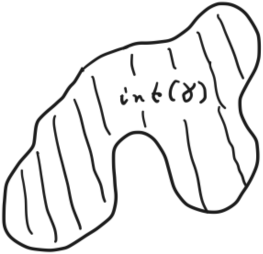
Example of a simple closed curve with period a:

$$\gamma : \mathbb{R} \to \mathbb{R}^2 \ , \ \ t \mapsto \left(\cos\left(\frac{2\pi t}{a}\right), \sin\left(\frac{2\pi t}{a}\right) \right)$$

Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve with period a. The length of γ is defined as

$$\ell(\gamma) = \int_0^a \|\gamma'(t)\| dt \qquad ' = \frac{d}{dt}$$

We denote by $\mathrm{int}(\gamma)$ the interior of $\gamma(\mathbb{R})$ (well-defined by Jordan Curve Theorem) and by $\mathcal{A}(\mathrm{int}(\gamma))$ the area of $\mathrm{int}(\gamma)$



How to compute $\mathcal{A}(\text{int}(\gamma))$? Put $f(x,y) = -\frac{1}{2}y$, $g(x,y) = \frac{1}{2}x$

$$
\begin{aligned}
\mathcal{A}(\text{int}(\gamma)) &= \iint\limits_{\text{int}(\gamma)} dxdy = \iint\limits_{\text{int}(\gamma)} \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) dxdy \\
&= \iint\limits_{\text{int}(\gamma)} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy \\
&= \int_\gamma (f(x,y)dx + g(x,y)dy) \quad (\text{by } \textsc{Green's Thm}) \\
&= \frac{1}{2}\int_\gamma (xdy - ydx) = \frac{1}{2}\int_0^a (xy' - yx')dt
\end{aligned}
$$

Problem. Among all simple closed curves γ in $\mathbb{R}^2$ of fixed perimeter, which curve maximizes the area of its interior $\mathrm{int}(\gamma)$?

Theorem. (ISOPERIMETRIC INEQUALITY) *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve. Then*

$$\mathcal{A}(\mathrm{int}(\gamma)) \leqslant \frac{1}{4\pi} \ell(\gamma)^2$$

and equality holds if and only if $\gamma(\mathbb{R})$ is a circle.

Proof. Reparametrize γ so that $t = \frac{\pi s}{\ell(\gamma)}$ with s arc length. Then $\ell(\gamma)$ and $\mathrm{int}(\gamma)$ remain unchanged and $\gamma : [0, \pi] \to \mathbb{R}^2$ with $\gamma(0) = \gamma(\pi)$. Since translations leave length and area unchanged, we can also assume that $\gamma(0) = (0, 0) \in \mathbb{R}^2$. Write $\gamma(t) = (x(t), y(t))$ and put $\dot{} = \frac{d}{ds}$ and $' = \frac{d}{dt}$. Then $\frac{dt}{ds} = \frac{\pi}{\ell(\gamma)}$ and $\frac{ds}{dt} = \frac{\ell(\gamma)}{\pi}$. Thus

$$x'(t) = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \dot{x}\frac{\ell(\gamma)}{\pi}\ , \ \ y'(t) = \dot{y}\frac{\ell(\gamma)}{\pi}$$

$$x'^2 + y'^2 = (\dot{x}^2 + \dot{y}^2)\frac{\ell(\gamma)^2}{\pi^2} = \frac{\ell(\gamma)^2}{\pi^2}$$

Use polar coordinates $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$:

$$x' = r'\cos(\theta) - r\sin(\theta)\theta' \ , \ y' = r'\sin(\theta) + r\cos(\theta)\theta'$$

$$\frac{\ell(\gamma)^2}{\pi^2} = x'^2 + y'^2 = r'^2 + r^2\theta'^2$$

$$\frac{\ell(\gamma)^2}{4\pi} = \frac{1}{4}\pi\frac{\ell(\gamma)^2}{\pi^2} = \frac{1}{4}\int_0^\pi \frac{\ell(\gamma)^2}{\pi^2} = \frac{1}{4}\int_0^\pi (r'^2 + r^2\theta'^2)$$

$$\mathcal{A}(\mathrm{int}(\gamma)) = \frac{1}{2}\int_0^\pi (xy' - yx') = \frac{1}{2}\int_0^\pi r^2\theta'$$

$$\begin{aligned}
\frac{\ell(\gamma)^2}{4\pi} - \mathcal{A}(\mathrm{int}(\gamma)) &= \frac{1}{4}\int_0^\pi (r'^2 + r^2\theta'^2) - \frac{1}{2}\int_0^\pi r^2\theta' \\
&= \frac{1}{4}\int_0^\pi (r'^2 + r^2\theta'^2 - 2r^2\theta') \\
&= \frac{1}{4}\left(\int_0^\pi r^2(\theta' - 1)^2 + \int_0^\pi (r'^2 - r^2)\right)
\end{aligned}$$

$$\frac{\ell(\gamma)^2}{4\pi} - \mathcal{A}(\mathrm{int}(\gamma)) = \frac{1}{4}\left(\int_0^\pi \underbrace{r^2(\theta' - 1)^2}_{\geqslant 0} + \int_0^\pi \underbrace{(r'^2 - r^2)}_{\geqslant 0}\right) \geqslant 0$$

Equality holds if and only if

1. $\theta' = 1$, which means $\theta(t) = t + \alpha$ for $\alpha \in \mathbb{R}$
2. $r(t) = A\sin(t)$ with $A \in \mathbb{R}$ by Wirtinger's Inequality

Altogether, equality holds if and only if $r = A\sin(\theta - \alpha)$, which is the polar equation of a circle

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The concept of a surface

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A subset U of $\mathbb{R}^2$ is open if for every point $p_0 \in U$ there exists $0 < \epsilon \in \mathbb{R}$ such that

$$U_\epsilon(p_0) = \{p \in \mathbb{R}^2 : \|p - p_0\| < \epsilon\} \subseteq U$$

$U_\epsilon(p_0)$ is the open disk with radius ϵ and centre p_0.

Examples.

1. $\mathbb{R}^2$ is open in $\mathbb{R}^2$
2. $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is open in $\mathbb{R}^2$
3. $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ is not open in $\mathbb{R}^2$

A surface patch is a smooth injective map

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) : U \to \mathbb{R}^3$$

with $U \subset \mathbb{R}^2$ open. The map σ is smooth if and only if $\sigma_1, \sigma_2, \sigma_3 : U \to \mathbb{R}$ have continuous partial derivatives of all orders

Notation:

$$\sigma_u = \frac{\partial \sigma}{\partial u}\ ,\ \sigma_v = \frac{\partial \sigma}{\partial v}\ ,\ \sigma_{uu} = \frac{\partial^2 \sigma}{\partial u^2}\ ,\ \sigma_{vv} = \frac{\partial^2 \sigma}{\partial v^2}\ ,\ \sigma_{uv} = \frac{\partial^2 \sigma}{\partial u \partial v}\ ,\ \cdots$$

Note that $\sigma_{uv} = \sigma_{vu}$ by SCHWARZ'S THEOREM

Example. Let Π be a plane in $\mathbb{R}^3$, $a \in \Pi$, $p, q \in \mathbb{R}^3$ linearly independent and parallel to Π. Then

$$\Pi = \{a + up + vq : u, v \in \mathbb{R}\}$$

Define

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3 \ , \ (u, v) \mapsto a + up + vq$$

σ is injective and smooth and hence a surface patch with $\sigma(\mathbb{R}^2) = \Pi$

Example. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Spherical coordinates:

$$x = \cos(\theta)\cos(\varphi) \ , \ \ y = \cos(\theta)\sin(\varphi) \ , \ \ z = \sin(\theta)$$

$\theta \sim$ latitude and $\varphi \sim$ longitude. Define

$$\sigma : U \to \mathbb{R}^3 \ , \ \ (\theta, \varphi) \to (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$$

$U =$? Need $U \subset \mathbb{R}^2$ open and σ injective. Can take

$$U = \left\{(\theta, \varphi) \in \mathbb{R}^2 : -\frac{\pi}{2} < \theta < \frac{\pi}{2} \ , \ \ 0 < \varphi < 2\pi\right\}$$

Then

$$\sigma(U) = S^2 \backslash \{(x, 0, z) \in S^2 : x \geqslant 0\}$$

σ covers only *patch* of S^2

One can show that S^2 cannot be covered by one surface patch. Define second surface patch by

$$\tilde{\sigma} : U \to \mathbb{R}^3 \ , \ (\theta, \varphi) \to (-\cos(\theta)\cos(\varphi), -\sin(\theta), -\cos(\theta)\sin(\varphi))$$

Then

$$\tilde{\sigma}(U) = S^2 \backslash \{(x, y, 0) \in S^2 : x \leqslant 0\}$$

$\tilde{\sigma}$ is obtained from σ by first rotating σ by angle π about z-axis and then by angle $\frac{\pi}{2}$ about x-axis

The two surface patches $\sigma, \tilde{\sigma}$ cover the entire sphere S^2

Surface patches are sufficient for studying *local* geometry of surfaces. Local geometry should be independent of choice of surface patch.

A surface patch $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ is a reparametrization of a surface patch $\sigma : U \to \mathbb{R}^3$ if there exists a smooth bijective map $\phi : U \to \tilde{U}$, the so-called reparametrization map, whose inverse map $\phi^{-1} : \tilde{U} \to U$ is smooth and $\tilde{\sigma} \circ \phi = \sigma$.