Geometry of Surfaces

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The Gauss map of a surface

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Recall: The signed curvature of a planar curve measures the rate of change of the tangent lines to the curve

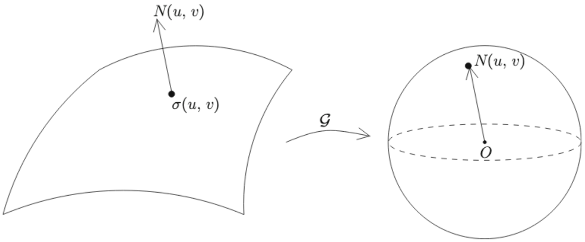
The Gauss map is used to develop an analogue for surfaces in 3-dimensional space

The direction of the tangent plane to a surface is measured by the unit normal $\mathbf{N}$ of the surface

Expectation: The rate of change of $\mathbf{N}$ measures curvature

Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch with unit normal $\mathbf{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ and $\mathcal{S} = \sigma(U)$. The Gauss map of $\mathcal{S}$ is the map

$$\mathcal{G} : \mathcal{S} \to S^2 \ , \ \sigma(u, v) \to \mathbf{N}(u, v)$$



Note that $\mathbf{N} = \mathcal{G} \circ \sigma$

Let $R \subseteq U$. The area $\mathcal{A}_{\mathbf{N}}(R)$ of $\mathbf{N}(R) \subseteq S^2$ measures the amount by which the direction of $\mathbf{N}$ varies. Thus, approximately, the rate of change of direction per unit area is

$$\frac{\mathcal{A}_{\mathbf{N}}(R)}{\mathcal{A}_{\sigma}(R)} = \frac{\iint_R \|\mathbf{N}_u \times \mathbf{N}_v\| du dv}{\iint_R \|\sigma_u \times \sigma_v\| du dv}$$

Need to compute $\mathbf{N}_u \times \mathbf{N}_v$ for a better understanding

Lemma. *$\mathbf{N}_u = a\sigma_u + b\sigma_v$ and $\mathbf{N}_v = c\sigma_u + d\sigma_v$ with*

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -(\mathcal{F}_I)^{-1}\mathcal{F}_{II}$$

Proof. Since $\|\mathbf{N}\| = 1$, we have $\mathbf{N}_u \cdot \mathbf{N} = 0 = \mathbf{N}_v \cdot \mathbf{N}$ and hence there exist $a, b, c, d \in \mathbb{R}$ so that

$$\mathbf{N}_u = a\sigma_u + b\sigma_v \ , \ \mathbf{N}_v = c\sigma_u + d\sigma_v$$

Differentiating $0 = \mathbf{N} \cdot \sigma_u$ gives

$$0 = \mathbf{N}_u \cdot \sigma_u + \mathbf{N} \cdot \sigma_{uu} = \mathbf{N}_u \cdot \sigma_u + L$$

and hence $\mathbf{N}_u \cdot \sigma_u = -L$. Analogously, $\mathbf{N}_v \cdot \sigma_v = -N$ and $\mathbf{N}_u \cdot \sigma_v = -M = \mathbf{N}_v \cdot \sigma_u$. So

$$\begin{aligned}
-L &= \mathbf{N}_u \cdot \sigma_u = a\sigma_u \cdot \sigma_u + b\sigma_v \cdot \sigma_u = aE + bF \\
-M &= \mathbf{N}_u \cdot \sigma_v = a\sigma_u \cdot \sigma_v + b\sigma_v \cdot \sigma_v = aF + bG \\
-M &= \mathbf{N}_v \cdot \sigma_u = c\sigma_u \cdot \sigma_u + d\sigma_v \cdot \sigma_u = cE + dF \\
-N &= \mathbf{N}_v \cdot \sigma_v = c\sigma_u \cdot \sigma_v + d\sigma_v \cdot \sigma_v = cF + dG
\end{aligned}$$

$$-L = aE + bF\ ,\ -M = aF + bG\ ,\ -M = cE + dF\ ,\ -N = cF + dG$$

Equivalently, in matrix form,

$$-\underbrace{\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{\mathcal{F}_{II}} = \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{\mathcal{F}_{I}} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Thus

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -(\mathcal{F}_I)^{-1}\mathcal{F}_{II}$$

Proposition.

$$\mathbf{N}_u \times \mathbf{N}_v = K\sigma_u \times \sigma_v$$

Proof.

$$\mathbf{N}_u \times \mathbf{N}_v = (a\sigma_u + b\sigma_v) \times (c\sigma_u + d\sigma_v) = \underbrace{(ad - bc)}_{\det(-(\mathcal{F}_I)^{-1}\mathcal{F}_{II})} \sigma_u \times \sigma_v$$

$$= \frac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_I)}\sigma_u \times \sigma_v = \frac{LN - M^2}{EG - F^2}\sigma_u \times \sigma_v = K\sigma_u \times \sigma_v$$

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The Gauss map and Gaussian curvature

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Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch with unit normal $\mathbf{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$. Recall: $\mathbf{N}_u \times \mathbf{N}_v = K \sigma_u \times \sigma_v$

Theorem. *Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $(u_0, v_0) \in U$. Choose $\delta > 0$ with*

$$R_\delta = \{(u, v) \in \mathbb{R}^2 : (u - u_0)^2 + (v - v_0)^2 \leqslant \delta^2\} \subset U.$$

Then

$$\lim_{\delta \to 0} \frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} = |K(u_0, v_0)|$$

Proof. We have

$$\frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} = \frac{\iint_{R_\delta} \|\mathbf{N}_u \times \mathbf{N}_v\| dudv}{\iint_{R_\delta} \|\sigma_u \times \sigma_v\| dudv} = \frac{\iint_{R_\delta} |K| \|\sigma_u \times \sigma_v\| dudv}{\iint_{R_\delta} \|\sigma_u \times \sigma_v\| dudv}$$

Let $\epsilon > 0$. Since $|K|$ is continuous, there exists $\delta > 0$ so that

$$|K(u_0, v_0)| - \epsilon < |K(u, v)| < |K(u_0, v_0)| + \epsilon$$

for all $(u, v) \in R_\delta$. Then

$$\begin{aligned}(|K(u_0, v_0)| - \epsilon) \iint_{R_\delta} \|\sigma_u \times \sigma_v\| &< \iint_{R_\delta} |K| \|\sigma_u \times \sigma_v\| \\ &< (|K(u_0, v_0)| + \epsilon) \iint_{R_\delta} \|\sigma_u \times \sigma_v\|\end{aligned}$$

$$
\begin{aligned}
(|K(u_0, v_0)| - \epsilon) \iint\limits_{R_\delta} \|\sigma_u \times \sigma_v\| &< \iint\limits_{R_\delta} |K| \|\sigma_u \times \sigma_v\| \\
&< (|K(u_0, v_0)| + \epsilon) \iint\limits_{R_\delta} \|\sigma_u \times \sigma_v\|
\end{aligned}
$$

We have

$$
\iint\limits_{R_\delta} \|\sigma_u \times \sigma_v\| = \mathcal{A}_\sigma(R_\delta)
$$

$$
\iint\limits_{R_\delta} |K| \|\sigma_u \times \sigma_v\| = \iint\limits_{R_\delta} \|\mathbf{N}_u \times \mathbf{N}_v\| = \mathcal{A}_\mathbf{N}(R_\delta)
$$

Thus

$$
(|K(u_0, v_0)| - \epsilon)\mathcal{A}_\sigma(R_\delta) < \mathcal{A}_\mathbf{N}(R_\delta) < (|K(u_0, v_0)| + \epsilon)\mathcal{A}_\sigma(R_\delta)
$$

$$(|K(u_0, v_0)| - \epsilon)\mathcal{A}_\sigma(R_\delta) < \mathcal{A}_{\mathbf{N}}(R_\delta) < (|K(u_0, v_0)| + \epsilon)\mathcal{A}_\sigma(R_\delta)$$

This implies

$$|K(u_0, v_0)| - \epsilon < \frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} < |K(u_0, v_0)| + \epsilon$$

Taking the limit $\epsilon \to 0$ finally gives

$$\lim_{\delta \to 0} \frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} = |K(u_0, v_0)|$$

Question: What does the Theorem tell us for a plane, for a generalized cylinder, and for a sphere?

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Geodesics

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Geodesics on surfaces generalize the concept of lines in planes

A unit speed curve $\gamma(t) = \sigma(u(t), v(t))$ on a surface $\sigma : U \to \mathbb{R}^3$ is a geodesic if for every t the vector $\ddot{\gamma}(t)$ is perpendicular to the surface at $\gamma(t)$, or equivalently, if $\ddot{\gamma}(t) \| \mathbf{N}_\gamma(t) = \mathbf{N}(u(t), v(t))$ of the surface at $\gamma(t)$ for all t.

Note: The concept of geodesic on a surface does not depend on the parametrization of the surface

Motivation: A particle moving on a surface subject to no forces except a force that keeps the particle on the surface moves along a geodesic. By Newton's second law of motion, the force on the particle is the product of mass and acceleration of the particle

Recall: The geodesic curvature of γ is $\kappa_g = \ddot{\gamma} \cdot (\mathbf{N}_\gamma \times \dot{\gamma})$

Proposition. γ is a geodesic if and only if $\kappa_g = 0$

Proof. Assume γ is a geodesic. Then $\ddot{\gamma} \| \mathbf{N}_\gamma$ and hence $\ddot{\gamma} \perp (\mathbf{N}_\gamma \times \dot{\gamma})$, which implies $\kappa_g = 0$.

Conversely, assume $\kappa_g = 0$. Then $\ddot{\gamma} \perp (\mathbf{N}_\gamma \times \dot{\gamma})$. Since $\|\dot{\gamma}\| = 1$, we also have $\ddot{\gamma} \perp \dot{\gamma}$. Since $\dot{\gamma}, \mathbf{N}_\gamma, \mathbf{N}_\gamma \times \dot{\gamma}$ are orthonormal, it follows that $\ddot{\gamma} \| \mathbf{N}_\gamma$, which means that γ is a geodesic.

Let $\tilde{\gamma}$ be a unit speed reparametrization of γ. We know that $\tilde{\gamma}(t) = \gamma(c \pm t)$ with $c \in \mathbb{R}$. Then

$$\begin{aligned}
\dot{\tilde{\gamma}}(t) &= \pm\dot{\gamma}(c \pm t) \\
\ddot{\tilde{\gamma}}(t) &= \ddot{\gamma}(c \pm t) \\
\tilde{\kappa}_g(t) &= \ddot{\tilde{\gamma}}(t) \cdot (\mathbf{N}_{\tilde{\gamma}}(t) \times \dot{\tilde{\gamma}}(t)) \\
&= \ddot{\gamma}(c \pm t) \cdot (\mathbf{N}_{\gamma}(c \pm t) \times \pm\dot{\gamma}(c \pm t)) \\
&= \pm\kappa_g(c \pm t)
\end{aligned}$$

Therefore $\tilde{\kappa}_g = 0 \iff \kappa_g = 0$. The following definition therefore makes sense: A regular curve γ on the surface is a geodesic if a unit speed reparametrization of γ is a geodesic.

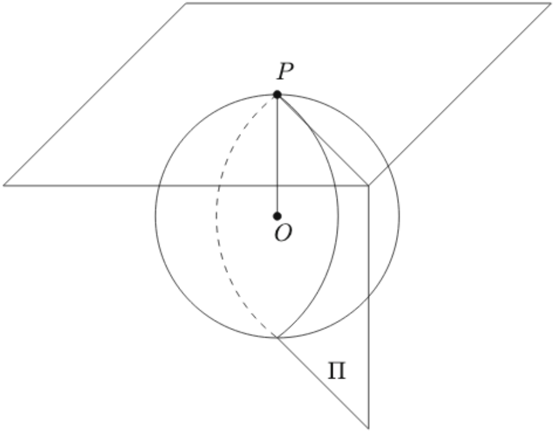
Examples of geodesics

Every part of a straight line on a surface

Proof. $\gamma(t) = at + b$ with $a, b \in \mathbb{R}^3$, $\|a\| = 1$. Then $\ddot{\gamma} = 0$

Every normal section of a surface is a geodesic

Proof. $\gamma(t)$ parametrizes (part of) intersection $\mathcal{S} \cap \Pi$ of surface $\mathcal{S}$ and plane Π with $\mathbf{N}_\gamma(t) \in \Pi$ for all t. Then $\dot{\gamma}(t)$ is tangent to $\mathcal{S}$ and Π since it is contained in both. Then $\ddot{\gamma}(t)$ is tangent to Π and perpendicular to $\dot{\gamma}(t)$ (since $\|\dot{\gamma}\| = 1$), hence parallel to $\mathbf{N}_\gamma(t)$. Thus γ is geodesic on the surface.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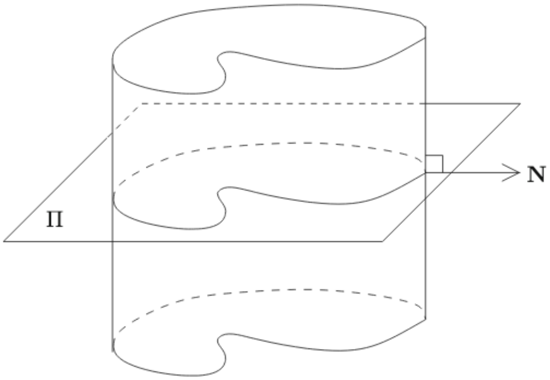
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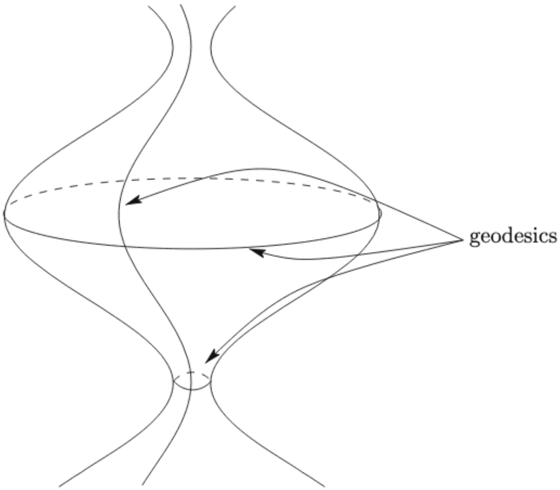
geodesics

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The geodesic equations

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Theorem. *Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit speed curve on a regular surface patch $\sigma : U \to \mathbb{R}^3$ and $Edu^2 + 2Fdudv + Gdv^2$ be the first fundamental form of the surface. Then γ is a geodesic on the surface if and only if*

$$\frac{d}{dt}\left(E\dot{u} + F\dot{v}\right) = \frac{1}{2}\left(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right)$$
$$\frac{d}{dt}\left(F\dot{u} + G\dot{v}\right) = \frac{1}{2}\left(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right)$$

These two equations are called geodesic equations

Proof. γ is a geodesic if and only if $\ddot{\gamma} \| \mathbf{N}_\gamma$. This is equivalent to $\ddot{\gamma} \perp \sigma_u, \sigma_v$ along γ. Since $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$, γ is a geodesic if and only if

$$0 = \frac{d}{dt}\left(\dot{u}\sigma_u + \dot{v}\sigma_v\right) \cdot \sigma_u$$
$$0 = \frac{d}{dt}\left(\dot{u}\sigma_u + \dot{v}\sigma_v\right) \cdot \sigma_v$$

$$
\begin{aligned}
0 &= \frac{d}{dt}\left(\dot{u}\sigma_u + \dot{v}\sigma_v\right)\cdot\sigma_u \\
&= \frac{d}{dt}\Big(\underbrace{(\dot{u}\sigma_u + \dot{v}\sigma_v)\cdot\sigma_u}_{\dot{u}E+\dot{v}F}\Big) - (\dot{u}\sigma_u + \dot{v}\sigma_v)\cdot \underbrace{\frac{d}{dt}\sigma_u}_{\dot{u}\sigma_{uu}+\dot{v}\sigma_{uv}} \\
&= \frac{d}{dt}\big(E\dot{u} + F\dot{v}\big) - \big(\dot{u}^2\underbrace{\sigma_u\cdot\sigma_{uu}}_{=\frac{1}{2}E_u} + \dot{v}^2\underbrace{\sigma_v\cdot\sigma_{uv}}_{=\frac{1}{2}G_u} + \dot{u}\dot{v}(\underbrace{\sigma_u\cdot\sigma_{uv} + \sigma_v\cdot\sigma_{uu}}_{=F_u})\big) \\
&= \frac{d}{dt}\big(E\dot{u} + F\dot{v}\big) - \frac{1}{2}\big(E_u\dot{u}^2 + G_u\dot{v}^2 + 2F_u\dot{u}\dot{v})\big)
\end{aligned}
$$

This is the first geodesic equation. The second geodesic equation can be proved analogously.

Corollary. *Isometries between surfaces map geodesics to geodesics*

Proof. Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be an isometry.
We can reparametrize σ_1, σ_2 so that their first fundamental forms coincide.
Reparametrizations do not change geodesics.
Since the geodesic equations involve only the coefficients of the first fundamental form, the statement follows from the previous Theorem.