Sequences and Summations

1 Sequences

Unlike sets, sequences are *ordered* lists of elements. We can think of sequences as *lists* of elements from a set S. The sequence $(1, 2, 3) \neq (2, 1, 3)$, since the order of the elements is difference in the finite sequences.

Sequences may be infinite or finite. The sequence $(1, 1/2, 1/3, 1/4, \ldots)$ is an infinite sequence of real numbers, whereas $(1, 2, 3, 4)$ is a finite sequence. We can use variables to represent the terms in the sequence, and they are labelled with their position, as in:

$$(a_1, a_2, a_3, \ldots)$$

where the element in the nth position is referred to as a_n. For example, if $a_n = (a_1, a_2, a_3, \ldots) = (1, 1/2, 1/3, 1/4, \ldots)$ then $a_1 = 1$, $a_2 = 1/2$, etc. Note that repetitions are allowed, for example $(1, 1, 1, \ldots)$ is also a sequence.

When referring to the entire sequence, we often write a_n, or you may also see (a_n). Note that the notation for sequences uses *round* brackets, and that for sets uses *curly* brackets, to distinguish between the former (which has a order) and the latter (where order of the elements doesn't matter).

The formal definition of a sequence is given here:

Definition. *A sequence is a countable collection of objects, where order matters and repetitions are allowed. We denote the sequence as a_n, where the nth item is the value a_n.*

Some sequences start at a_1 and some start at a_0. The exact starting index of the sequence is usually given as part of the definition of the sequence. Often the sequence has a **closed formula**, which is simply an equation for determining each value of the sequence. For example, the sequence above, $(1, 1/2, 1/3, 1/4, \ldots)$ could be represented by the formula $a_n = 1/n$ where $n \geq 1$. Using this closed formula we can determine any element of the sequence. For example, the $100th$ item in the sequence is $\frac{1}{100}$. The sequence $(1, 1, 1, \ldots)$ is defined by the formula $a_n = 1$, which says that any nth term of the sequence is 1.

Functions can also be defined **inductively (or recursively)**. Such a definition involves a formula that *relates* the terms in the sequence to the previous terms. For example:

$$a_0 = 1, a_1 = 2, a_n = 2a_{n-1} - a_{n-2}$$

The equation above allows us to "*reconstruct*" the sequence element by element:

$$a_0 = 1$$
$$a_1 = 2$$
$$a_2 = 2a_1 - a_0 = 2 \cdot 2 - 1 = 3$$
$$a_3 = 2a_2 - a_1 = 2 \cdot 3 - 2 = 4$$
$$a_4 = 2a_3 - a_2 = 2 \cdot 4 - 3 = 5$$

etc. Finding formulas for sequences is not always trivial. In some cases, the sequence is of a *known type*, and can be related to a general formula for a specific type of sequence. We discuss two of those below:

> **Definition.** *A geometric sequence has the form $a_n = ar^n$ for constants a and r.*
>
> A geometric sequence can be recognized by the fact that the *ratio* between successive terms is constant, and is the value r
>
> **Definition.** *A arithmetic sequence has the form $a_n = a + nd$ for constants a and d.*
>
> An arithmetic sequence can be recognized by the fact that the *differentce* between successive terms is constant, and is the value d.

Example 1. *Determine a closed formula for the sequence:* $(1, 1/2, 1/4, 1/8, 1/16, \ldots)$

Solution:. Notice that the terms of this sequence seem to be powers of $1/2$. Thus it represents a geometric sequence, with $r = 1/2$. The formula for the sequence is then $a_n = (\frac{1}{2})^n$. If we let the sequence start at $n = 0$, then the first term is a 1, the second is $1/2$, etc. Thus the sequence is represented by the closed formula of a geometric sequence:

$$a_n = \left(\frac{1}{2}\right)^n, n \geq 0$$

Example 2. *Determine a closed formula for the sequence* $(3, 7, 11, 15, \ldots)$

Solution:. The difference between each term of the sequence is consistently 4. Thus it is an *arithmetic* sequence with $d = 4$. Since the first term is 3, then we can set $a = 3$ and represent the sequence as an arithmetic sequence starting at at $n = 0$:
$$a_n = 3 + 4n$$

Example 3. *Determine a closed formula for the sequence* $(2/3, 2/9, 2/27, 2/81, \ldots)$

Solution: The terms of the sequence are powers of $1/3$, multiplied by a common multiple of 2. Therefore it is a geometric sequence, with $r = 1/3$ and $a = 2$. Using the closed form of the geometric sequence, the formula is then $a_n = 2(\frac{1}{3})^n$. Notice that unlike Example 1, the first term does *not* correspond to $n = 0$. Therefore we specify that the closed formula of the sequence starts at $n = 1$.

$$a_n = 2\left(\frac{1}{3}\right)^n, n \geq 1$$

2 Summations

In this section we describe summation notation, which allows us to efficiently represent the summation of the terms of a sequence. This summation may be infinite or finite. Suppose that we wish to sum the first n terms of a sequence. Then this sum is represented as:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

The index below the summation term indicates the *first* item in the sum, and the index at the top of the summation represent the *last* index in the sum. The particular *variable* in the sum, either an n or an i is not important, and any variable name can be used. Below are two examples of finite sums with different starting variables:

$$\sum_{n=1}^{5} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$\sum_{n=3}^{10} n^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

2.1   Geometric sums

If a geometric *sequence* is used to construct a summation, it is called a geometric sum. The sum may be finite or infinite. We look first at the case where the sum is finite, and show the *exact expression* for the sum of the first n terms

> **Theorem 1.** *If a and r are real numbers and $r \neq 0, r \neq 1$ then*
> $$\sum_{i=0}^{n} ar^i = a \cdot \frac{r^{n+1} - 1}{r - 1}$$
> *In the case where $r = 1$ the summation is simply $a + a + a + ... + a = a(n+1)$.*

In the example below, $a = 5$ and $r = 1/2$:

$$\sum_{i=0}^{5} \frac{5}{2^i} = 5 \cdot \frac{(1/2^6) - 1}{(1/2) - 1}$$

Note that the summation does not *have* to start at $i = 0$. In this case we can use the formula above, and then *subtract* off the terms that are not part of the sum:

$$\sum_{i=2}^{4} ar^i = a \cdot \frac{r^5 - 1}{r - 1} - a - ar$$

where the $i = 0$ term is a and the $i = 1$ term is ar. Both of these terms were subtracted off because they were not included in the sum.

In other cases, some manipulation is required before applying the geometric series formula:

$$\sum_{i=0}^{10} (2^{i+1} - 3^{i-1}) = \sum_{i=0}^{10} 2(2^i) - \sum_{i=0}^{10} 3^{-1}(3^i)$$
$$= 2 \cdot \frac{(2^{11}) - 1}{1} - \frac{1}{3} \cdot \frac{(3^{11} - 1)}{2}$$

The geometric sum may also be *infinite*, as we shall see in section 3.

2.2   Other Summation Formulae

There are other formulae for many types of summations. The two formulae below are very common in computer science, and they will be applied often during the course. They represent cases where one sums up the first n integers, or the *square* of the first n integers:

Sum	Formula
$\sum_{k=1}^{n} a$	an
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$

We work through some examples of the above formulas here, (there are more examples in the chalkboard lecture). The examples below are from the Rosen text.

- Find a formula for $\sum_{k=1}^{n}(2k-1)$
  Solution:
  $$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n}(k) - \sum_{k=1}^{n}(1) = 2n(n+1)/2 - n$$

- Find a formula for $\sum_{k=1}^{n}(1+(-1)^k)$
  Solution: Notice that when k is odd, the term in the sum is 0. Thus we only need to consider the cases where k is even. If n is an even number, then
  $$\sum_{k=1}^{n}(1+(-1)^k) = 0+2+0+2+\ldots+2 = 2 \cdot (n/2) = n$$

If n is odd, then
$$\sum_{k=1}^{n}(1+(-1)^k) = 0+2+0+2+\ldots+0 = 2 \cdot (n-1)/2 = n-1$$

3   Infinite Series

Summations can be **infinite**, in which case we refer to them as **infinite series**. The notation for an infinite series is:
$$\sum_{i=1}^{\infty} a_n$$
The definition of what it means to sum an *infinite* number of terms is rather delicate. There is technically no such concept as *adding terms forever*. Thus $\sum_{i=1}^{\infty} a_n$ is not equivalent to simply adding $a_1+a_2+a_3+a_4\ldots$ forever, because the addition of an infinite number of terms is not defined. It is not as if at "infinity", you can hit the equal sign to see what the sum is.

The infinite series is instead defined using the concept of *partial sums*, which refers to the value of the sum if we stopped at a certain index. The definition is given below, but it will be made clearer in the next examples.

Definition. *If (a_n) is a sequence of elements in $\mathbb{R}$, then the* infinite series, *written $\sum_{n=1}^{\infty} a_n$,is the new sequence:*

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

*etc. The numbers a_n are the **terms** of the series and the numbers s_n are the **partial sums**.*

The value of $\sum_{n=0}^{\infty} a_n$ corresponds to the the behaviour of these partial sums, s_n. If the partial sums get closer and closer to a final result, then this is the value of the infinite sums. If they don't, then the infinite sum does not exist.

In this course, proving when these sums exist is *way* beyond the scope of the material. It is important to simply know how an infinite sum is defined and when it exists, and to understand how to apply certain well-known formulas for certain infinite sums, such as the geometric sum. We look at an example of an infinite sum that does not exist.

Example 4. *Show that the infinite series $\sum_{i=0}^{\infty}(-1)^i = 1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots$ does not exist.*

Solution: It is very common to look at this infinite sum and assume that the answer is 0. It may seem like overall the 1's are cancelling with the -1's and we are left with nothing. However it is important to determine the existence of the sum based on the definition of what it means to have an infinite sum: and it does *not* mean to average the result of adding together the numbers forever.

The partial sums for this sum are as follows:

$$s_1 = 1$$
$$s_2 = 1 - 1 = 0$$
$$s_3 = 1 - 1 + 1 = 1$$
$$s_4 = 1 - 1 + 1 - 1 = 0$$
$$s_5 = 1 - 1 + 1 - 1 + 1 = 1$$

Notice that the partial sums are alternating between 1 and 0. Thus it is impossible that the partial sums are getting *closer and closer* to a final number. In this case then, there is *no final sum* and we say that the infinite sum *does not exist*.

3.1 Infinite geometric series

In Theorem 2, we studied geometric sums, in the case where the summation was finite. Geometric sums can also be infinite, in which case we refer to them as *infinite geometric series*. There is an exact value for this sum also, as long as $|r| < 1$. However, if $|r| \geq 1$, the the summation doesn't exist. The two points are given in the theorem below:

Theorem 2. *If a and r are real numbers and $|r| < 1$ then*

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

If $|r| \geq 1$, then the sum doesn't have a finite value. We say that the infinite sum "doesn't exist".

Example 5. *Find the exact value of the infinite sum $3 + 3/2 + 3/4 + 3/8 + 3/16 + \ldots$.*

Solution: This is the geometric sum

$$\sum_{i=0}^{\infty} 3\left(\frac{1}{2}\right)^i = \frac{3}{1 - \frac{1}{2}} = 6$$

Example 6. *Evaluate the sum $\sum_{i=0}^{\infty} 2(3)^n$ if it exists.*

Since $r = 3$ in this case, the infinite sum does not exist. This makes sense intuitively, because we can see that the partial sums are getting bigger and bigger:

$$\sum_{i=0}^{4} 2(3)^n = 2 + 2(3) + 2(9) + 2(27) + 2(81) = 242$$

Example 7. *Evaluate the sum $\sum_{k=0}^{\infty} \frac{4}{3^{k+1}}$ if it exists.*

Solution: In some cases, the formula may at first not seem as though it matches up with the formula in Theorem 2. With some simply manipulation, we can reorganize the inside of the summation so that we have one term that is raised to the proper exponent.

$$\sum_{k=0}^{\infty} \frac{4}{3^{k+1}} = \sum_{k=0}^{\infty} \frac{4}{3}(1/3)^k = \frac{4}{3}\left(\frac{1}{1-(1/3)}\right) = \frac{4}{3} \cdot \frac{3}{2} = 2$$

Example 8. *Evaluate $\sum_{k=1}^{\infty} x^{2k}$ for $|x| < 1$.*

Solution: In this solution, we simply need to rewrite the exponent, so that we have a term that is raised to k. Secondly, since the sum starts at 1 and not 0, we need to use the formula and then subtract the first term.

$$\sum_{k=1}^{\infty} x^{2k} = \sum_{k=1}^{\infty} (x^2)^k$$
$$= \frac{1}{1-x^2} - (x^2)^0$$
$$= \frac{1}{1-x^2} - 1$$

4   Notation for double summation

Double summations occur when two summations signs are nested. Let's look at a double summation where the inner summation depends on the value i of the outer summation. In this example, the inner summation is executed for $j = 0$ to $j = i$. The red term on the right corresponds to $i = 0$, the next term (orange) corresponds to $i = 1$ (and we sum $j = 0, j = 1$), the next blue term corresponds to $i = 2$ (and we sum from $j = 0, 1, 2$), etc.

$$\sum_{i=0}^{3}\sum_{j=0}^{i}(i+j) = [0] + [(1+0)+(1+1)] + [(2+0)+(2+1)+(2+2)] + [(3+0)+(3+1)+(3+2)+(3+3)]$$