Geometry of Surfaces - Exercises

9. True or False? Justify your answer!

(a) The curvature of a curve in $\mathbb{R}^3$ is always ≥ 0.
(b) The torsion of a curve in $\mathbb{R}^3$ is always ≥ 0.
(c) A curve in $\mathbb{R}^3$ with zero curvature everywhere is (part of) a straight line.
(d) A curve in $\mathbb{R}^3$ with constant positive curvature is (part of) a circle.

10. Let $\mathcal{C}$ be the curve obtained by intersecting the sphere

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4\}$$

and the cylinder

$$\{(x, y, z) \in \mathbb{R}^3 \mid (x-1)^2 + y^2 = 1\}.$$

Verify that

$$\gamma : \mathbb{R} \to \mathbb{R}^3 \ , \ t \mapsto (1 + \cos(2t), \sin(2t), 2\sin(t))$$

is a parametrization of $\mathcal{C}$ and that $p = (1, 1, \sqrt{2})$ is a point in $\mathcal{C}$. Calculate the curvature of $\mathcal{C}$ at p.

11. Let $\gamma : (a, b) \to \mathbb{R}^n$ be a unit speed curve and assume that there exists $t_0 \in (a, b)$ such that the distance function $\|\gamma(t)\|$, which measures the distance from 0 to $\gamma(t)$, has a maximum at t_0. Prove that $\gamma'(t_0) \cdot \gamma(t_0) = 0$ and $\kappa(t_0) \geq \frac{1}{\|\gamma(t_0)\|}$. [Hint: Use Cauchy-Schwarz inequality.]

12. In the curve below, roughly mark where the signed curvature is negative and where it is positive. Explain your answer.

13. Consider the curve

$$\gamma : (-1, 1) \to \mathbb{R}^2 \ , \ s \mapsto \int_0^s \left(\cos\left(\frac{t^5}{5}\right), \sin\left(\frac{t^5}{5}\right) \right) dt.$$

Show that γ is a unit speed curve with signed curvature s^4. Find the unit speed curve $\tilde{\gamma} : (-1, 1) \to \mathbb{R}^2$ with signed curvature s^4, $\tilde{\gamma}(0) = (1, 2)$ and $\tilde{\mathbf{t}}(0) = (0, 1)$.

14. Find all unit speed curves $\gamma : \mathbb{R} \to \mathbb{R}^2$ with signed curvature e^s and $\dot{\gamma}(0) = (0, 1)$.

15. Let $\gamma(s) = \left(\cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right)$. Compute $\mathbf{t}, \mathbf{n}, \mathbf{b}$. Note that γ is unit speed.

16. A rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ of $\mathbb{R}^3$ is a rotation followed by a translation, that is, $Mv = Rv + a$ for all $v \in \mathbb{R}^3$, where R is a rotation in $\mathbb{R}^3$ and $a \in \mathbb{R}^3$. Recall that a rotation satisfy the property $Rv \cdot Rw = v \cdot w$ for all $v, w \in \mathbb{R}^3$. Show that the curvature and torsion of a unit speed curve in $\mathbb{R}^3$ are invariant under rigid motions.