Formal Logic

CS 2LC3

Ryszard Janicki

Department of Computing and Software, McMaster University, Hamilton, Ontario, Canada

Acknowledgments: Material based on *A Logical Approach to Discrete Math* by David Gries and Fred B. Schneider (Chapter 7, Chapter 6.2 and Chapter 12.3).

Introduction

Informal logic or non-formal logic

- is the study of arguments as presented in ordinary language
- is often defined as "a branch of logic whose task is to develop non-formal standards, criteria, procedures for the analysis, interpretation, evaluation, criticism and construction of argumentation in everyday discourse."
- is used to reason about events in the human and social sciences
- Most reasoning from known facts to unknown facts that uses natural language can be regarded as an application of informal logic so long as it does not rely on additional empirical evidence
- It is not the subject of our study

- We study the general notion of a formal logical system and its interpretations
  - We discuss both syntax (proof theory) and semantics (models) for logics
  - We study constructive logic in a propositional setting

Formal logical systems

A formal logical system, or logic, is a set of rules defined in terms of

- a set of symbols,
- a set of formulas constructed from the symbols,
- a set of distinguished formulas called axioms, and
- a set of inference rules.

The set of formulas is called the language of the logic.

The language is defined syntactically; there is no notion of meaning or semantics in a logic per se.

- Inference rules allow formulas to be derived from other formulas

- Inference rules have the form

$$\frac{H_1,\ H_2,\ \cdots,\ H_n}{C}$$

where

  - formulas $H_1, H_2, \cdots, H_n$ are the premises (or hypotheses) of the inference rule

  - formula C is its conclusion

Formal logical systems: Proof Theory

- A formula is a theorem of the logic, if it is one of the following:
  - an axiom
  - can be generated from the axioms and already proved theorems using the inference rules
- A proof that a formula is a theorem is an argument that shows how the inference rules are used to generate the formula
- For the propositional logic, the symbols are $(,), =, \neq, \iff, \not\iff, \neg, \vee, \wedge, \implies, \impliedby$, the constants true and false, and boolean variables p, q, etc.
- The formulas are the boolean expressions constructed using these symbols.

Example

The system E (see chapter 1 of the textbook) has 15 axioms, starting with Associativity of $==$, (3.1). Its inference rules are Leibniz (1.5), Transitivity of equality (1.4), and Substitution (1.1). Its theorems are the formulas that can be shown to be equal to an axiom using these inference rules.

Formal logical systems: Proof Theory

- A logic can only be useful if it makes some distinction between formulas
- A logic is consistent if at least one of its formulas is a theorem and at least one is not; otherwise, the logic is inconsistent
- The propositional logic is consistent, because true is a theorem and false is not
- Adding false $\iff$ true as an axiom would make it inconsistent

Example of a "Toy" Logic (PQ-L)

- Symbols: $P, Q, *$
- Formulas: sequences of the form $aPbQc$ where a, b, and c denote finite sequences of zero or more $*$
- Axioms:

  Axiom 0: $*P * Q * *$

  Axiom 1: $* * P * Q * **$
- Inference Rule:

$$\frac{aPbQc, \quad dPeQf}{adPbeQcf}$$

PQ-L uses the Hilbert style of proof

Formal logical systems: Proof Theory

Claim: $* * * * P * * * Q * * * * * * **$ is a theorem.

Proof

1. $*P * Q * *$ Axiom 0
2. $* * P * Q * **$ Axiom 1
3. $* * *P * *Q * * * * **$ Inf. rule, 1, 2
4. $* * * * P * * * Q * * * * * * **$ Inf. rule, 1, 3

- PQ-L illustrates the view that a logic is a system for manipulating symbols, independent of meaning

- We do not say what the formulas, axioms, and inference rules mean

- How can we give a meaning to a formula?

Formal logical systems: Models

- Typically, the formulas of a logic are intended to be statements about some area of interest

- We give the formulas a meaning with respect to this domain
  1. by defining which formulas are true statements about the domain
  2. by defining which formulas are false statements about the domain

- An interpretation assigns meaning to the
  1. operators of a logic
  2. constants of a logic
  3. and variables of a logic

Formal logical systems: Models

Addition-equality Interpretation

we can give formulas of PQ-L meaning by providing the following interpretation

- A formula $aPbQc$ is mapped to $\#a + \#b = \#c$, where $\#x$ denotes the number of stars ($*$) in sequence x

Example

- $*PQ*$ is mapped to $1 + 0 = 1$ (Which is true)
- $*P**Q***$ is mapped to $1 + 2 = 3$ (Which is true)
- $*P*Q*$ is mapped to $1 + 1 = 1$ (Which is false)
- $*P*Q***$ is mapped to $1 + 1 = 3$ (Which is false)
- $*P*Q**$ is mapped to $1 + 1 = 2$ Axiom 0
- $**P*Q***$ is mapped to $2 + 1 = 3$ Axiom 1

Formal logical systems: Models

Because a logic is purely a syntactic object, it may have more than one interpretation

Addition-inequality Interpretation

- A formula $aPbQc$ is mapped to true iff $\#a + \#b \leq \#c$, where $\#x$ denotes the number of stars ($*$) in sequence x

Example

- $*PQ*$ is mapped to $1+0 \leq 1$ (Which is true)
- $*P**Q***$ is mapped to $1+2 \leq 3$ (Which is true)
- $*P*Q*$ is mapped to $1+1 \leq 1$ (Which is false)
- $*P*Q***$ is mapped to $1+1 \leq 3$ (Which is <u>TRUE</u>)
- $*P*Q**$ is mapped to $1+1 \leq 2$ Axiom 0
- $**P*Q***$ is mapped to $2+1 \leq 3$ Axiom 1

Formal logical systems: Models

- In a logic in which formulas have variables, an interpretation associates a value with each variable
- Each interpretation gives the meaning of formulas with a different variable-value association
- Conventionally, we split such an interpretation into two parts:
  - one gives a fixed meaning to the operators and constants
  - the other supplies values for variables (denotes a state)

Standard interpretation of expressions of (a) propositional logic

- For an expression P without variables, let $\text{eval}(P)$ be the value of P
- Let Q be any expression, and let s be a state that gives values to all the variables of Q
- Define $Q(s)$ to be a copy of Q in which all its variables are replaced by their corresponding values in state s
- Then function f given by $f(Q) = \text{eval}(Q(s))$ is an interpretation for Q

Satisfiability and validity of a formula with respect to any logic and interpretation

Definition

Let S be a set of interpretations for a logic and F be a formula of the logic. F is satisfiable (under S) iff at least one interpretation of S maps F to true.

Definition

F is valid (under S) iff every interpretation in S maps F to true.

Formal logical systems: Models

Definition

An interpretation is a model for a logic iff every theorem is mapped to true by the interpretation.

Definition

A logic is sound iff every theorem is valid.

Definition

A logic is complete iff every valid formula is a theorem.

- Soundness means that the theorems are true statements about the domain of discourse
- Completeness means that every valid formula can be proved

Formal logical systems: Models

Logic PQ-L

- Symbols: $P, Q, *$
- Formulas: sequences of the form $aPbQc$ where a, b, and c denote finite sequences of zero or more $*$
- Axioms:

  Axiom 0: $*P*Q**$
  Axiom 1: $**P*Q***$

- Inference Rule:

$$\frac{aPbQc, \quad dPeQf}{adPbeQcf}$$

Addition-equality Interpretation

- A formula $aPbQc$ is mapped to $\#a + \#b = \#c$, where $\#x$ denotes the number of stars (*) in sequence x

Logic PQ-L is sound with respect this interpretation but not complete (because the valid formula, PQ, is not a theorem of PQ-L.)

Problem

Is $*P**Q***$ *a theorem of PQ-L?*

Answer: No

Proof.

It has only one $*$ *before* P*, but it is not an axiom as the only axiom with one* $*$ *before* P *is* $*P*Q**$*. Since the is no axiom with zero* $*$ *before* P*, we cannot derive* $*P**Q***$*.* ∎

Formal logical systems: Models

Problem

Give a finite set of axioms that can be added to PQ-L to make it sound and complete under Addition-Equality Interpretation.

Solution

1. PQ
2. $P * Q *$
3. $*PQ*$

The above three axioms suffice. We do not need the previous axioms!

Problem

Prove $*P**Q***$ in the obtained logic (i.e. Axioms 1, 2 and 3 from the previous page).

Proof.

- $P * Q*$ - *Axiom 2*
- $$\frac{P * Q*, \quad P * Q*}{P * *Q * *}$$
- $*PQ*$ *(i.e. Axiom 3) and* $P * *Q * *$
- $$\frac{*PQ*, \quad P * *Q * *}{*P * *Q * **}$$
- *Hence* $*P * *Q * **$

∎

Formal logical systems: Models

- A sound and complete logic allows exactly the valid formulas to be proved
- Failure to prove that a formula is a theorem in such a logic cannot be attributed to weakness of the logic
- Unfortunately, many domains of discourse of concern to us do not have sound and complete axiomatizations (Peano arithmetic –not complete–)
- This is a consequence of Godel's incompleteness theorem, which states that no formal logical system that axiomatizes arithmetic can be both sound and complete
- This incompleteness is not a problem in practice

Peano Arithmetic:

- Symbols: $S, o, +, \cdot, <, =$
- Formulas: φ
- Axioms: The axioms of PA are:
  1. $\forall x(Sx \neq 0)$.
  2. $\forall x, y((Sx = Sy) \Rightarrow x = y)$.
  3. $(\varphi[0] \wedge \forall x(\varphi[x] \Rightarrow \varphi[Sx])) \Rightarrow \forall x(\varphi[x])$, for any formula φ in the language of PA.
  4. $\forall x(x + 0 = x)$.
  5. $\forall x, y(x + Sy = S(x + y))$.
  6. $\forall x(x \cdot 0 = 0)$.
  7. $\forall x, y(x \cdot Sy = (x \cdot y) + x)$.
- Each natural number is then equal to the set of natural numbers less than it, so that
  - $0 := \{\}$
  - $1 := S(0) = \{0\}$
  - $2 := S(1) = \{0, 1\} = \{\{\}, \{0\}\}$
  - $3 := S(2) = \{0, 1, 2\} = \{\{\}, \{0\}, \{0, \{0\}\}\}$

  and so on. This construction is due to John von Neumann.

Gödel's Incompleteness Theorems

- Gödel's incompleteness theorems are two celebrated theorems about the limitations of formal systems, proved by Kurt Gödel in 1931.
- These theorems show that there is no complete, consistent formal system that correctly describes the natural numbers, and that no sufficiently strong system describing the natural numbers can prove its own consistency.
- These theorems are widely regarded as showing that Hilbert's program to find a complete and consistent set of axioms for all of mathematics is impossible.
- The theorems have also been interpreted in philosophy and popular culture.

Goodstein's Theorem

- Goodstein's theorem is a statement about the natural numbers that is unprovable in Peano arithmetic but can be proven to be true using the stronger axiom system of set theory, in particular using the axiom of infinity.
- The theorem states that **every Goodstein sequence eventually terminates at 0**.
- It stands as an example that not all undecidable theorems are peculiar or contrived, as those constructed by Gödel's incompleteness theorem are sometimes considered.

- Hereditary base-n notation.
  - For example, 35 in ordinary base-2 notation is $2^5 + 2^1 + 2^0$, and in hereditary base-2 notation is $2^{2^2+1} + 2 + 1$.
  - 100 in ordinary base-3 notation is $3^4 + 2 \cdot 3^2 + 1$ and in hereditary base-3 notation is $3^{3+1} + 2 \cdot 3^2 + 1$
- The Goodstein sequence on a number m, notated $G(m)$, is defined as follows: the first element of the sequence is m. To get the next element, write m in hereditary base 2 notation, change all the 2's to 3's, and then subtract 1 from the result; this is the second element of $G(m)$. To get the third element of $G(m)$, write the previous number in hereditary base 3 notation, change all 3's to 4's, and subtract 1 again.

Example ($G(3)$)

$3 = 2^1 + 1 \to 3^1 + 1 - 1 = 3 \to 3 = 4^1 - 1 \to 3 - 1 = 2 \to$
$2 = 5^0 + 2 \to 2 - 1 = 1 \to 1 = 6^0 + 1 \to 1 - 1 = 0$

Example ($G(4)$)

$4 = 2^2 \to 3^3 - 1 = 2 \cdot 3^2 + 2 \cdot 3 + 2 = 26 \to^*$
$2 \cdot 4^2 + 2 \cdot 4 + 1 = 41 \to^* 2 \cdot 5^2 + 2 \cdot 5 = 60 \to^*$
$2 \cdot 6^2 + 2 \cdot 6 - 1 = 2 \cdot 6^2 + 6 + 5 = 83 \to^* 2 \cdot 7^2 + 7 + 4 = 109 \to^*$
$2 \cdot 11^2 + 11 = 253 \to^* 2 \cdot 12^2 + 12 - 1 = 2 \cdot 12^2 + 11 = 299 \to^*$
$3 \cdot 2^{402653210} - 1$ (Maximum) $\to^* 0$

- In spite of frequent rapid growth, Goodstein's theorem states that **every Goodstein sequence eventually terminates at 0**, no matter what the starting value is.

Formal logical systems: Models

- In order to isolate sources of incompleteness in a logic, the logic can be defined in a hierarchical fashion
- A logic L^+ is an extension of logic L if the symbols, formulas, axioms, and inference rules of L are included in L^+.

Example

We obtain a predicate logic by extending a propositional logic with variables that may be associated with other types of variables (e.g., the integers) and by introducing predicates on those variables (e.g., less(x, y))

- In propositional logic, there is a simple way to determine whether a formula is a theorem: just check its validity
- Compute the interpretation of a formula F in every possible state of the state space defined by the boolean variables in F
- F is a theorem of a propositional logic, iff it is mapped to true in every state

Example

1. $p \vee \text{true}$ — a theorem
2. $p \vee \neg p$ — a theorem
3. $(p \vee \text{true}) \ \wedge \ q$ — NOT a theorem

A Decision Procedure for Propositional Logic

- Determining whether a boolean expression involving n boolean variables is valid requires checking 2^n cases
- This decision procedure is time-consuming for formulas involving a large number of variables

- Not all logics have decision procedures

- For some logics, there is no algorithm to tell whether an arbitrary formula is a theorem or not
- In fact, most logics that deal with interesting domains of discourse, like the integers, do not have decision procedures

- Let P be any mathematical statement whose truth is not known

Example

P is "there are an infinite number of twin primes"

(twin primes are prime numbers that differ by 2; 11 and 13 are twin primes)

Given such a P, we can define a variable x as follows:

$$x = \begin{cases} 0 & \text{if } P \text{ is true} \\ 1 & \text{if } P \text{ is false} \end{cases}$$

- This definition defines x unambiguously
- We cannot compute the value of x
- We have given a non-constructive definition of x
- A constructive definition would tell us how to calculate the value of x

Natural Deduction and $\vdash$

- **Natural Deduction** is a version of Propositional Logic often better suited for formal proofs.
- Logicians express this relationship between a theorem and the formulas assumed for its proof as the **sequent**:

$$A_0, \ldots, A_n \vdash Q \text{ or } \vdash_L Q,$$

  where L is the name of the logic with axioms $A_0, \ldots, A_n$.
- Symbol $\vdash$ is called the "turnstile", and the A_i are called the *premises* of the sequent.
- The sequent $A_0, \ldots, A_n \vdash Q$ is read as "Q is **provable** from $A_0, \ldots, A_n$" (The order of the A_i is immaterial.)
- The sequent $\vdash_L Q$ is read as "Q is provable in logic L" - i.e. using the axioms of L.
- Often, when the logic is unambiguous from the context, the subscript L is omitted.
- Thus, $\vdash Q$ means that Q is a theorem in the logic at hand. In Lecture Notes 3 (and Chap. 3 of the textbook), we could have placed the turnstile before each theorem.

Difference between $\vdash$ and $\implies$

- The difference between the *sequent* $A_0, \ldots, A_n \vdash Q$ and the *formula* $A_0 \wedge \ldots A_n \implies Q$ is that the sequent is not a boolean expression.
- The sequent asserts that Q can be **proved** from $A_0, \ldots, A_n$.
- Formula $A_0 \wedge \ldots \wedge A_n \implies Q$, on the other hand, is a boolean expression (but it need not be a theorem).
- In propositional logic (logic E of the textbook), the sequent and the formula are related by *Deduction theorem* (4.4), which can now be rephrased using sequents as "If $P_0, \ldots, P_n f \vdash Q$, then $\vdash P_0 \wedge \ldots \wedge P_n \implies Q$".

Inference Rules for Natural Deduction

Introduction rules

$$\wedge\text{-I}: \quad \frac{P,\ Q}{P \wedge Q}$$

$$\vee\text{-I}: \quad \frac{P}{P \vee Q}, \quad \frac{P}{Q \vee P}$$

$$\Rightarrow\text{-I}: \quad \frac{P_1, \ldots, P_n \vdash Q}{P_1 \wedge \cdots \wedge P_n \ \Rightarrow \ Q}$$

$$\equiv\text{-I}: \quad \frac{P \Rightarrow Q,\ Q \Rightarrow P}{P \equiv Q}$$

$$\neg\text{-I}: \quad \frac{P \vdash Q \wedge \neg Q}{\neg P}$$

$$\mathit{true}\text{-I}: \quad \frac{P \ \equiv \ P}{\mathit{true}}$$

$$\mathit{false}\text{-I}: \quad \frac{\neg \mathit{true}}{\mathit{false}}$$

Elimination rules

$$\wedge\text{-E}: \quad \frac{P \wedge Q}{P}, \quad \frac{P \wedge Q}{Q}$$

$$\vee\text{-E}: \quad \frac{P \vee Q,\ P \Rightarrow R,\ Q \Rightarrow R}{R}$$

$$\Rightarrow\text{-E}: \quad \frac{P,\ P \Rightarrow Q}{Q}$$

$$\equiv\text{-E}: \quad \frac{P \equiv Q}{P \Rightarrow Q}, \quad \frac{P \equiv Q}{Q \Rightarrow P}$$

$$\neg\text{-E}: \quad \frac{\neg P \vdash Q \wedge \neg Q}{P}$$

$$\mathit{true}\text{-E}: \quad \frac{\mathit{true}}{P \ \equiv \ P}$$

$$\mathit{false}\text{-E}: \quad \frac{\neg \mathit{false}}{\mathit{true}}$$

- We can also prove things in a non-constructive way

Example

- A real number is rational if it can be written in the form b/c for two integers b and c $(c \neq 0)$; otherwise it is irrational
- The number $1/3$ is rational, while $\sqrt{2}$ and the number π are irrational.

Claim: There exist two irrational numbers b and c such that b^c is rational.

Give a non-constructive proof.

Proof.

The proof is by case analysis: $(\sqrt{2})^{\sqrt{2}}$ is either rational or irrational

- Case $(\sqrt{2})^{\sqrt{2}}$ is rational. Choose $b = c = \sqrt{2}$
- Case $(\sqrt{2})^{\sqrt{2}}$ is irrational. Choose $b = (\sqrt{2})^{\sqrt{2}}$ and $c = \sqrt{2}$. Since 2 is rational, we can show that b^c is rational:

$$((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} \qquad \text{(this is } b^c\text{)}$$

$$= \qquad \langle \text{ Arithmetic } \rangle$$

$$(\sqrt{2})^{\sqrt{2}\sqrt{2}}$$

$$= \qquad \langle \text{ Arithmetic } \rangle$$

$$2$$

This proof of the existence of rational b^c does not show us how to construct b^c, since we do not know whether $(\sqrt{2})^{\sqrt{2}}$ is rational. It is a nonconstructive proof. ∎

Constructive Logics

- Constructive mathematics is the branch of mathematics in which each definition or proof of existence of an object provides an algorithm for computing it

- There are several versions of constructive mathematics
- In some versions, it is enough to provide an algorithm to construct as close an approximation to an object as we desire, even if the object cannot be computed exactly

- This kind of constructive mathematics would allow as objects the irrational numbers (stricter form would not)

- Constructive mathematics has increased in popularity with the advent of computers

A Constructive Propositinal Logic

- This logic is based on the following principles:
  1. A proof of $p \wedge q$ is given by presenting a proof of p and a proof of q
  2. A proof of $p \vee q$ is given by presenting either a proof of p or a proof of q
  3. A proof of $p \implies q$ is a procedure that permits us to transform a proof of p into a proof of q
  4. The constant false, which is a contradiction, has no proof
  5. A proof of $\neg p$ is a procedure that transforms any hypothetical proof of p into a proof of a contradiction ($p \vdash$ false i.e., false is provable from p)

The fifth principle can be explained as follows. The constant false is not true, so there exists no proof for it. Hence, if we show that false follows from a hypothetical proof of p, then p itself is false. We regard the proof of $p \vdash$ false as a proof of $\neg p$.

- In constructive logic, the law of the Excluded Middle, $p \vee \neg p$, is not a theorem
- There are many statements in mathematics that no one has been able to prove or disprove

Example

"There are an infinite number of twin primes"
No one has been able to prove or disprove

- We cannot accept the law of the Excluded Middle in a constructive system

Rules for Constructive Natural Deduction

Introduction rules

$\wedge$-I : $\dfrac{\vdash P, \ \vdash Q}{\vdash P \wedge Q}$

$\vee$-I$_l$: $\dfrac{\vdash P}{\vdash P \vee Q}$, $\vee$-I$_r$: $\dfrac{\vdash Q}{\vdash P \vee Q}$

$\Rightarrow$-I : $\dfrac{P_1, \ldots, P_n \vdash Q}{\vdash P_1 \wedge \cdots \wedge P_n \ \Rightarrow \ Q}$

false-I : (none)

Elimination rules

$\wedge$-E$_l$: $\dfrac{\vdash P \wedge Q}{\vdash P}$, $\wedge$-E$_r$: $\dfrac{\vdash P \wedge Q}{\vdash Q}$

$\vee$-E : $\dfrac{\vdash P \vee Q, \ P \vdash R, \ Q \vdash R}{\vdash R}$

$\Rightarrow$-E : $\dfrac{\vdash P, \vdash P \Rightarrow Q}{\vdash Q}$

false-E : $\dfrac{\vdash \mathit{false}}{\vdash P}$

$P \equiv Q$	denotes	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
$\neg P$	denotes	$P \Rightarrow \mathit{false}$
true	denotes	$\neg \mathit{false}$

Constructive Propositional Logics

- $\neg\neg p \implies p$ is not a theorem
- $p \implies \neg\neg p$ is a theorem
- $p \vee \neg p$ is NOT a theorem
- $\neg\neg(p \vee \neg p)$ <u>is</u> a theorem

The proofs will be given in the tutorial

Problem

Two possible definitions for soundness of an inference rule are:

1. *Theorem-Soundness: An inference rule is considered sound if a formula derived using it is valid whenever the premises used in the inference are theorems.*
2. *Model-Soundness: An inference rule is considered sound if a formula derived using it is valid whenever the premises used in the inference are valid.*

What are the advantages/disadvantages of considering axiomatizations in which all inference rules satisfy Theorem-Soundness versus Model-Soundness?

- Making valid formulas into axioms in a logic where the inference rules are Theorem-Sound may cause some of the inference rules to no longer be Theorem-sound. Thus such a logic cannot be extended by adding axioms.
- Adding such formulas as axioms to a logic where inference rules are Model-Sound does not have this difficulty and therefore cannot cause invalid formulas to become provable.
- Thus, Model-Soundness is more sensible requirement.