MAS381, Tutorial 8: Scalars, Vectors, Double Operators

1. An electric field is given by

$$\mathbf{E} = (2x + yz, 2y + xz, 2z + xy)$$

Show that $\mathbf{E}$ is conservative, and find the electric scalar potential, Φ, such that

$$\mathbf{E} = \nabla\Phi$$

2. Find the constants a, b and c such that the vector field

$$\mathbf{v} = (x + 2y + az, bx - 3y - z, 4x + cy + 2z)$$

satisfies the condition $curl(\mathbf{v}) = 0$. Using the values obtained, determine the scalar field, Φ, such that

$$\mathbf{v} = \nabla\Phi$$

3. Given that

$$\mathbf{A} = (x^2y, y^2z, z^2x)$$

calculate
(a) $\nabla \cdot \mathbf{A}$
(b) $\nabla(\nabla \cdot \mathbf{A})$
(c) $\nabla \times \mathbf{A}$
(d) $\nabla \times (\nabla \times \mathbf{A})$
(e) $\nabla^2\mathbf{A}$
where

$$\nabla^2\mathbf{A} = (\nabla^2 A_x)\mathbf{i} + (\nabla^2 A_y)\mathbf{j} + (\nabla^2 A_z)\mathbf{k}.$$

For the vector field $\mathbf{A}$ verify the vector identity

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2\mathbf{A}.$$

4. If $\mathbf{F}$ is a differentiable vector field, prove that

$$(i)\nabla \times (\Omega\mathbf{F}) = \Omega\nabla \times \mathbf{F} - \mathbf{F} \times \nabla(\Omega)$$

and

$$(ii)\nabla \times (\nabla\Omega) = 0$$

where $\Omega = \Omega(x, y, z)$ is a scalar field. A vector field $\mathbf{H}$ is such that it can be expressed in the form

$$\mathbf{H} = \phi\nabla(\psi)$$

where ϕ and ψ are differentiable scalar fields. Show that $\nabla \times \mathbf{H}$ is perpendicular to $\mathbf{H}$ at all points where neither of the vector field vanishes.
5. If $r^2 = x^2 + y^2 + z^2$, show that

$$\nabla^2 r^n = n(n + 1)r^{n-2}$$

Answers

1.

$$\Phi(x,y,z) = x^2 + xyz + y^2 + z^2 + C$$

2

$$\Phi(x,y,z) = \frac{x^2}{2} + 2yx + 4xz - 3\frac{y^2}{2} - zy + \frac{z^2}{2} + C$$

3.
(a) $2xy + 2yz + 2xz$; (b) $(2y + 2z, 2x + 2z, 2y + 2x)$; (c) $(-y^2, -z^2, -x^2)$; (d) $(2z, 2x, 2y)$; (e) $\nabla^2 \mathbf{A} = (2y, 2z, 2x)$