Discrete Mathematics

PSet

Problem 1

Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Problem 2

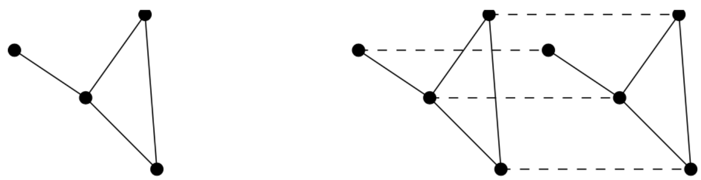
Radio stations broadcast their signal at certain frequencies. However, there are a limited number of frequencies to choose from, so nationwide many stations use the same frequency. This works because the stations are far enough apart that their signals will not interfere; no one radio could pick them up at the same time.
Suppose six new radio stations are to be set up in a currently unpopulated (by radio stations) region. The distances among stations are recorded in the table below. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

Table 1: Distances in miles among stations

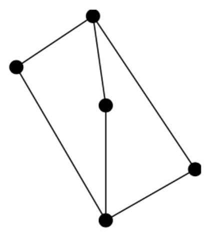
	1	2	3	4	5	6
1	——	85	175	200	50	100
2	85	——	125	175	100	160
3	175	125	——	100	200	250
4	200	175	100	——	210	220
5	50	100	200	210	——	100
6	100	160	250	220	100	——

Problem 3

The double of a graph G consists of two copies of G with edges joining corresponding vertices. For example, a graph appears below on the left and its double appears on the right. Some edges in the graph on the right are dashed to clarify its structure.



(a) Draw the double of the graph shown below.



(b) Suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so forth. Use induction on n to prove that G_n is bipartite for all $n \geq 1$.

Problem 4

Let m, n, and r be nonnegative integers with $r \leq m$ and $r \leq n$. Prove the following formula by a combinatorial proof.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}.$$

Problem 5

Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}.$$

Problem 6

Find the number of solutions of the equation $x_1 + x_2 + x_3 = 11$, where x_1, x_2, x_3 are nonnegative integers with $x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6$.

Problem 7

Show that in any set of $n + 1$ positive integers not exceeding $2n$ there must be two that are relatively prime.

Problem 8

A 0-1 sequence a_n with $2m$ terms is said to be normal if the following two conditions are satisfied.

- There exist m terms equal to 0 and the other m terms equal to 1 in a_n.
- For arbitrary $k \leq 2m$, the number of terms equal to 0 is not less than that of terms equal to 1 in the first k terms $a_1, a_2, \ldots, a_k$.

Please complete the following questions.
(a) Show that the number of abnormal 0-1 sequences a_n with $2m$ terms equals that of sequences a_n of which $(m + 1)$ terms are 0s and $(m - 1)$ terms are 1s.
(b) For $m = 4$, determine the number of different normal 0-1 sequences a_n. Note: An abnormal 0-1 sequence a_n is a 0-1 sequence that does not satisfy the properties of normal 0-1 sequences.