§27: The Idea of a Formal Proof: Reasoning from Assumptions

Introductory Logic

University of Adelaide » PHIL 1111OL

Antony Eagle
antony.eagle@adelaide.edu.au antonyeagle.org

Formal Proof

- Any formal proof is some structure of sentences.
- A formal proof system is a set of rules governing what structures are permissible, and (often) how to extend an existing formal proof to an expanded formal proof.
- Some approaches to formal proofs will give explicit conditions on what counts as a formal proof, and what operations on formal proofs are permissible. That formal proofs can themselves be represented as precise mathematical objects is of use in computer science and in so-called 'automated reasoning'.
- We will not be aiming in this course at such precision, and we will characterise our natural deduction system in a less abstract way.

Basics of Natural Deduction: Assumptions

- A natural deduction system is fundamentally about putting forward a system of rules which provide a regimented framework for reasoning from assumptions.
- It provides a template for how to write down a chain of reasoning in a way that makes it clear what is happening, what steps are involved, what justifies the various steps, and what depends on what.
  - It makes available devicess for marking assumptions, introducing new assumptions, and getting rid of – or discharging – assumptions once we no longer need to make them.
  - Our proofs also help us keep track of which assumptions are 'active' at any point in the proof, and thus which assumptions a particular sentence depends on.

Anatomy of Natural Deduction Proofs

- Most natural deduction proofs start with an assumption, and all finish with a conclusion.
- We employ a graphical representation of our proofs, as follows:

$$\begin{array}{r|l l}
1 & \mathcal{A} & \text{This is an assumption} \\ \cline{2-2}
 & \vdots & \\
n & \mathcal{C} & \text{This is the conclusion}
\end{array}$$

- The horizontal line marks that the sentence(s) above it are assumptions, and not proved from earlier sentences in the proof.
- The vertical line – an assumption line – indicates the range of the assumption. Any sentence which we write which occurs to the right of this vertical line (or its subsequent continuation), has been proved under this assumption.
- The line numbers are for future reference.

Adding More Assumptions

$$
\begin{array}{r|l}
1 & (A \wedge B) \\
2 & C \to D \\
\hline
 & \vdots \\
n & A
\end{array}
\qquad\qquad
\begin{array}{r|l}
1 & (A \wedge B) \\
\hline
\end{array}
\qquad\qquad
\begin{array}{r|l}
1 & (A \wedge B) \\
\hline
2 & \begin{array}{|l} C \to D \\ \hline \vdots \\ A \end{array} \\
n &
\end{array}
$$

- A proof consisting of a single assumption is already a correctly formed proof.
  - No need to mark the conclusion in a special way: it is the last line of the proof.
- We will say this is a proof of the conclusion '$(A \wedge B)$' from the assumption '$A \wedge B$'. Not an interesting proof, but a proof nonetheless!
- We can easily add more assumptions to a proof, in a couple of ways.
  1. We can make them 'all together', within the range of the same assumption line...
  2. ...or we can give each their own assumption line – can be more flexible.
- A sentence like 'A' next to an assumption line is in the RANGE of the ACTIVE ASSUMPTION(S) attached to the assumption line.

Proofs and Arguments

- Formal proofs interest us because we can use them to construct arguments!
  - Natural deduction systems can be helpful in developing arguments – truth table methods are really only useful for evaluating arguments we already have.
- Take an argument like:

$$(F \lor G), ((F \lor \neg F) \to \neg G) \therefore F.$$

We will set up a proof like this, with the premises as assumptions, and the conclusion as the last line in range of those assumptions:

$$
\begin{array}{r|l|l}
1 & (F \lor G) & \\
\cline{2-2}
2 & & ((F \lor \neg F) \to \neg G) \\
\cline{3-3}
 & & \vdots \\
n & & F
\end{array}
$$

- What we don't yet know how to do is get from assumptions to conclusions, since I haven't yet told you the rules.

Arguments in English

- If we have a successful proof of an argument, that strongly suggests that a parallel argument in natural language (like English) will also succeed – an argument that uses the English analogs of the proof rules to argue from 'de-symbolised' analogs of the Sentential sentences we use.
- I give a simple example, using rules we'll see in the next section.

$$\begin{array}{r|l l}
1 & (P \to Q) & \\
\hline
2 & \quad P & \\
3 & \quad Q & \to\text{E}, 1, 2
\end{array}$$

1. If the Prince is late, then the Queen will be angry ('If P then Q').
2. The Prince is late ('P').

So: The Queen will be angry ('Q').

- This natural deduction proof uses the rule of '→E' or conditional elimination.
- It corresponds to the argument on the right, with the natural symbolisation key, which uses the English conditional rule *modus ponens*.