1: Definition of Injection

Let $f : X \to Y, x_1, x_2 \in X$, then $f(x_1) = f(x_2)$ implies $x_1 = x_2$

2: Definition of Surjection(on-to)

Let $f : X \to Y$, $\forall y \in Y \exists x \in X f(x) = y$.

3: Homework Problem

Prove that if f is bijection then f_{cube} is bijection as well.

Proof of injectivity Let $x_1, x_2 \in \mathbb{R}$. We need to show $f_{\text{cube}}(x_1) = f_{\text{cube}}(x_2) \implies x_1 = x_2$. By definition of f_{cube}, we have

$$f_{\text{cube}}(x_1) = f_{\text{cube}}(x_2) \implies (f(x_1))^3 = (f(x_2))^3$$

. Applying the fact in the textbook, we have

$$(f(x_1))^3 = (f(x_2))^3 \implies ((f(x_1))^3)^{\frac{1}{3}} = ((f(x_2))^3)^{\frac{1}{3}} \implies f(x_1) = f(x_2)$$

(Need to show $f(x_1) = f(x_2) \implies x_1 = x_2$)

Applying the fact that f is bijection (hence injective), we have that

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

Proof of surjectivity

We need to show $\forall y \exists x f_{\text{cube}}(x) = y$.

Let $y \in \mathbb{R}$. (Need to show $\exists f_{\text{cube}}(x) = y \iff f(x)^3 = y \iff f(x) = y^{\frac{1}{3}}$).

Since $y \in \mathbb{R}$, $y^{\frac{1}{3}}$ is also $\in \mathbb{R}$.

Since f is bijection (hence on-to), we have $\exists x \in \mathbb{R}$ such that

$$f(x) = y^{\frac{1}{3}}$$

, then

$$f_{\text{cube}}(x) = \left(y^{\frac{1}{3}}\right)^3 = y$$

This proves the surjectivity of f_{cube}.