9 GAUSS'S THEOREMA EGREGIUM

9.1 GAUSS'S REMARKABLE THEOREM

"EGREGIUM" = REMARKABLE

THEOREM 9.1.1 THE GAUSSIAN CURVATURE OF A SURFACE DEPENDS ONLY ON ITS FIRST FUNDAMENTAL FORM (THAT IS, IT IS PRESERVED BY ISOMETRIES)

FAR FROM OBVIOUS !!!

$$K = \frac{LN - M^2}{EG - F^2}$$

DEPENDS ON 2^{ND} FUNDAMENTAL FORM AS WELL !

GAUSSIAN CURVATURE IS INTRINSIC PROPERTY.

CHOOSE (LOCALLY) ORTHONORMAL FRAME FIELD $e', e'', \vec{N} = e' \times e''$;

e', e'' ARE ORTHONORMAL AND TANGENT TO SURFACE AT EACH POINT

NOTE: $1 = e' \cdot e' \Rightarrow 0 = e'_u \cdot e' = e'_v \cdot e'$

$1 = e'' \cdot e'' \Rightarrow 0 = e''_u \cdot e'' = e''_v \cdot e''$

THUS

$$e'_u = \alpha e'' + \lambda' \vec{N}$$

$$e'_v = \beta e'' + \mu' \vec{N}$$

$$e''_u = -\alpha' e' + \lambda'' \vec{N}$$

$$e''_v = -\beta' e' + \mu'' \vec{N}$$

WITH $\alpha, \beta, \alpha', \beta', \lambda', \mu', \lambda'', \mu''$ FUNCTIONS DEPENDING ON SURFACE PARAMETERS u, v.

$0 = e' \cdot e'' \Rightarrow 0 = e'_u \cdot e'' + e' \cdot e''_u = \alpha - \alpha'$

$0 = e'_v \cdot e'' + e' \cdot e''_v = \beta - \beta'$

$\Rightarrow \alpha = \alpha'$ & $\beta = \beta'$

THUS

$$\begin{aligned} e'_u &= \quad \alpha e'' + \lambda' \vec{N} \\ e'_v &= \quad \beta e'' + \mu' \vec{N} \\ e''_u &= -\alpha e' \qquad\quad + \lambda'' \vec{N} \\ e''_v &= -\beta e' \qquad\quad + \mu'' \vec{N} \end{aligned} \qquad (1)$$

LEMMA 9.1.2

$$e'_u \cdot e''_v - e''_u \cdot e'_v = \lambda' \mu'' - \lambda'' \mu' \qquad (2)$$

$$= \alpha_v - \beta_u \qquad (3)$$

$$= \frac{LN - M^2}{\sqrt{EG - F^2}} \qquad (4)$$

PROOF (1) $\Rightarrow$ (2) SINCE e', e'', N ORTHONORMAL

$$\alpha_v - \beta_u = \frac{\partial}{\partial u}(\underbrace{e' \cdot e''_v}_{=-\beta}) - \frac{\partial}{\partial v}(\underbrace{e' \cdot e''_u}_{=-\alpha})$$

$$= e'_u \cdot e''_v + \cancel{e' \cdot e''_{uv}} - e'_v \cdot e''_u - \cancel{e' \cdot e''_{uv}} \quad \Rightarrow (3).$$

~~# RECALL:~~ $\cancel{\vec{N}_u \times \vec{N}_v = K \sigma_u \times \sigma_v}$ ~~(7.2.2)~~

$\cancel{\vec{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}}$ $\cancel{\|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2}}$ ~~(5.3.12)~~

$$\vec{N}_u \times \vec{N}_v \underset{7.2.2.}{=} K\, \sigma_u \times \sigma_v$$

$$= \frac{LN - M^2}{EG - F^2} \underbrace{\frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}}_{= \vec{N}} \underbrace{\|\sigma_u \times \sigma_v\|}_{\underset{5.3.2}{=} \sqrt{EG - F^2}}$$

$$= \frac{LN - M^2}{\sqrt{EG - F^2}} \vec{N}$$

$$\Rightarrow (\vec{N}_u \times \vec{N}_v) \cdot \vec{N} = \frac{LN - M^2}{\sqrt{EG - F^2}}$$

#1

$$(\vec{N}_u \times \vec{N}_v) \cdot (e' \times e'')$$

$$= (\vec{N}_u \cdot e')(\vec{N}_v \cdot e'') - (\vec{N}_u \cdot e'')(\vec{N}_v \cdot e')$$

$$= (\vec{N} \cdot e'_u)(\vec{N} \cdot e''_v) - (\vec{N} \cdot e''_u)(\vec{N} \cdot e'_v)$$

$$= \lambda' \mu'' - \mu' \lambda''$$

$\square$

$$\vec{N} \cdot e' = 0 = \vec{N} \cdot e''$$

PROOF OF THEOREMA EGREGIUM:

(3) & (4) $\Rightarrow$ $$K = \frac{\alpha_v - \beta_u}{\sqrt{EG - F^2}}$$

IT SUFFICES TO PROVE THAT α, β DEPEND ONLY ON E, F, G FOR A SUITABLE CHOICE OF e', e''.

IDEA: APPLY GRAM-SCHMIDT PROCESS TO σ_u, σ_v:

$$e' = \frac{\sigma_u}{\|\sigma_u\|} = \varepsilon \sigma_u \quad , \quad \varepsilon = \frac{1}{\|\sigma_u\|} = \frac{1}{\sqrt{E}}$$

$$e'' = \gamma \sigma_u + \delta \sigma_v \quad , \quad e'' \cdot e'' = 1 \; , \; e' \cdot e'' = 0$$

~~$e' \cdot e'' = 0 \Rightarrow$~~

$$0 = e' \cdot e'' = \varepsilon \sigma_u \cdot (\gamma \sigma_u + \delta \sigma_v)$$

$$= \varepsilon \gamma \underbrace{\sigma_u \cdot \sigma_u}_{= E = \frac{1}{\varepsilon^2}} + \varepsilon \delta \underbrace{\sigma_u \cdot \sigma_v}_{= F}$$

$$= \varepsilon (\gamma E + \delta F)$$

$$\Rightarrow \gamma = - \frac{\delta F}{E}$$

$$1 = e'' \cdot e'' = \gamma^2 \underbrace{\sigma_u \cdot \sigma_u}_{=E} + 2\gamma\delta \underbrace{\sigma_u \cdot \sigma_v}_{=F} + \delta^2 \underbrace{\sigma_v \cdot \sigma_v}_{=G}$$

$$= \gamma^2 E + 2\gamma\delta F + \delta^2 G$$

$$= \left(-\frac{\delta F}{E}\right)^2 E + 2\left(-\frac{\delta F}{E}\right)\delta F + \delta^2 G$$

$$= \delta^2\left(\frac{F^2}{E} - 2\frac{F^2}{E} + G\right) = \delta^2\left(G - \frac{F^2}{E}\right)$$

$$= \delta^2\left(\frac{EG - F^2}{E}\right)$$

$$\Rightarrow \delta = \frac{\sqrt{E}}{\sqrt{EG - F^2}}, \quad \gamma = -\frac{F}{\sqrt{E}\sqrt{EG-F^2}}, \quad \varepsilon = \frac{1}{\sqrt{E}}$$

(CHOOSING "−" FOR δ DOES NOT MAKE A DIFFERENCE IN THE END)

THUS

$$e' = \varepsilon \sigma_u$$
$$e'' = \gamma \sigma_u + \delta \sigma_v$$

WITH $\varepsilon, \gamma, \delta$ DEPENDING ONLY ON E, F, G.

$$\alpha \underset{(I)}{=} e'_u \cdot e''$$

$$= (\varepsilon_u \sigma_u + \varepsilon \sigma_{uu}) \cdot (\gamma \sigma_u + \delta \sigma_v)$$

$$= \underbrace{\frac{\varepsilon_u}{\varepsilon} \underbrace{(\varepsilon \sigma_u)}_{= e'} \cdot \underbrace{(\gamma \sigma_u + \delta \sigma_v)}_{= e''}}_{= 0} + \varepsilon \gamma \underbrace{\sigma_{uu} \cdot \sigma_u}_{= \frac{1}{2}(\underbrace{\sigma_u \cdot \sigma_u}_{= E})_u} + \varepsilon \delta \underbrace{\sigma_{uu} \cdot \sigma_v}_{= (\sigma_u \cdot \sigma_v)_u - \sigma_u \cdot \sigma_{uv} = F_u - \frac{1}{2} E_v}$$

$$= \frac{1}{2} \varepsilon \gamma E_u + \varepsilon \delta \left(F_u - \frac{1}{2} E_v \right)$$

$$\beta \underset{(I)}{=} e'_v \cdot e''$$

$$= (\varepsilon_v \sigma_u + \varepsilon \sigma_{uv}) \cdot (\gamma \sigma_u + \delta \sigma_v)$$

$$= \underbrace{\frac{\varepsilon_v}{\varepsilon} \underbrace{(\varepsilon \sigma_u)}_{= e'} \cdot \underbrace{(\gamma \sigma_u + \delta \sigma_v)}_{= e''}}_{= 0} + \varepsilon \gamma \underbrace{\sigma_{uv} \cdot \sigma_u}_{= \frac{1}{2}(\underbrace{\sigma_u \cdot \sigma_u}_{= E})_v} + \varepsilon \delta \underbrace{\sigma_{uv} \cdot \sigma_v}_{= \frac{1}{2}(\underbrace{\sigma_v \cdot \sigma_v}_{= G})_u}$$

$$= \frac{1}{2} \varepsilon \gamma E_v + \frac{1}{2} \varepsilon \delta \cdot G_u$$

$\Rightarrow$ α, β DEPEND ONLY ON E, F, G. $\square$

COROLLARY 9.1.3. THERE EXISTS AN EXPLICIT EXPRESSION FOR K IN TERMS OF E, F, G (AND THEIR DERIVATIVES)
$\Rightarrow$ LECTURE NOTES FOR EXPLICIT EXPRESSION
PROOF IS TEDIOUS, FORMULA IN GENERAL NOT USEFUL. SPECIAL CASES:

COROLLARY 9.1.4

(a) $F=0 \Rightarrow K = -\frac{1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right\}$.

(b) $E=1, F=0 \Rightarrow K = -\frac{1}{\sqrt{G}} \frac{\partial^2}{\partial u^2}(\sqrt{G})$

PROOF (a) $F=0$

$\Rightarrow \delta = \frac{1}{\sqrt{G}}, \gamma = 0, \varepsilon = \frac{1}{\sqrt{E}}$

$\Rightarrow \alpha = -\frac{1}{2}\varepsilon\delta E_v = \frac{-E_v}{2\sqrt{EG}}, \beta = \frac{1}{2}\varepsilon\delta G_u = \frac{G_u}{2\sqrt{EG}}$

$\Rightarrow K = \frac{\alpha_v - \beta_u}{\sqrt{EG}} = \dots$

(b) $E=1, F=0 \Rightarrow \alpha = 0 \Rightarrow K = -\frac{\beta_u}{\sqrt{EG}} = \dots$

$\square$

EXAMPLE 9.1.5 (SURFACE OF REVOLUTION)

$$\sigma(u, v) = \big(f(u)\cos(v), f(u)\sin(v), g(u)\big)$$

$$f > 0, \quad \dot{f}^2 + \dot{g}^2 = 1 \quad \left(\dot{} = \frac{d}{du} \right)$$

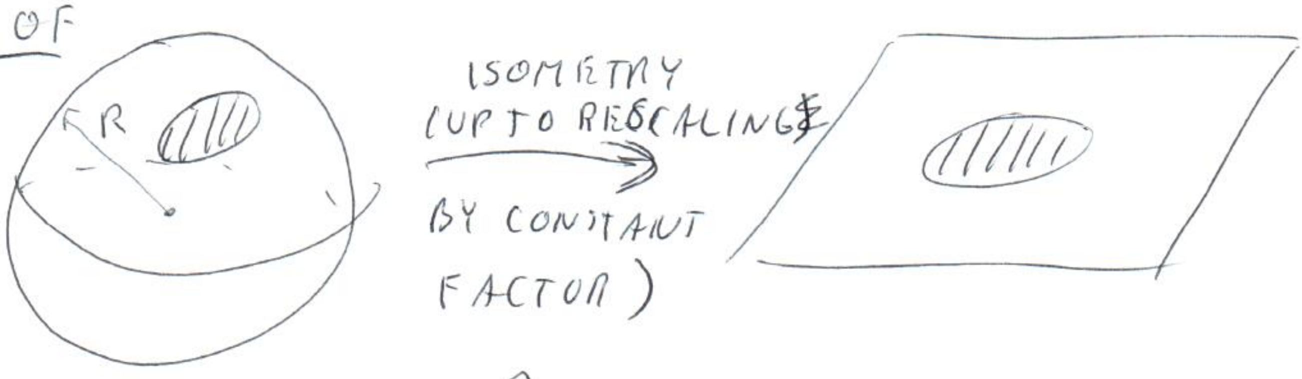
7.1.4: $E = 1, \ F = 0, \ G = f(u)^2$

$$\underset{9.1.4}{\Longrightarrow} \quad K = -\frac{1}{\sqrt{G}} \frac{\partial^2}{\partial u^2} \sqrt{G} = -\frac{1}{f} \cdot \ddot{f} = -\frac{\ddot{f}}{f}$$

9.2. APPLICATION TO CARTOGRAPHY

PROPOSITION 9.2.1 ANY MAP OF ANY REGION OF THE EARTH'S SURFACE MUST DISTORT DISTANCES.

PROOF



$K = \frac{1}{R^2}$ $\qquad\uparrow\qquad$ $K = 0$

CANNOT EXIST BY THEOREMA EGREGIUM. $\square$

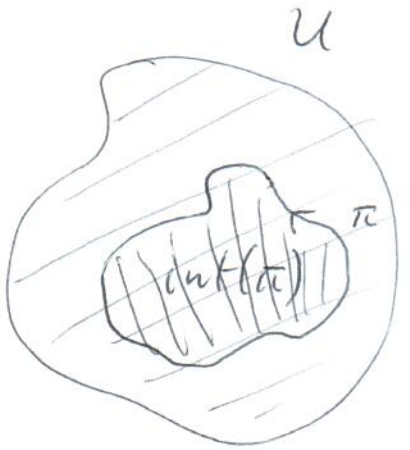
SAME STORY FOR SPHERE & CYLINDER.

10 GAUSS-BONNET THEOREM

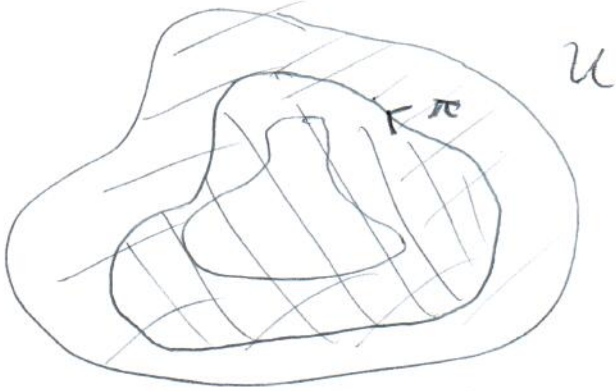
10.1 LOCAL VERSION

LET $\sigma : \mathcal{U} \to \mathbb{R}^3$ SURFACE

$\pi(s) = (u(s), v(s))$ SIMPLE CLOSED CURVE IN $\mathbb{R}^2$
WITH $\mathrm{int}(\pi) \subset \mathcal{U}$



OK



NOT OK

$$\gamma(s) = \sigma(\pi(s)) = \sigma(u(s), v(s))$$

WITH $\|\dot\gamma\| = 1$; $\mathrm{int}(\gamma) = \sigma(\mathrm{int}(\pi))$

γ POSITIVELY ORIENTED

$\iff$ π POSITIVELY ORIENTED

$\iff$ ~~UNIT~~ (ORIENTED) UNIT NORMAL n_s OF π
POINTS INTO $\mathrm{int}(\pi)$ EVERYWHERE

THEOREM 10.1.1.

$$\int_\gamma \varkappa_g \, ds = 2\pi - \iint_{\mathrm{int}(\pi)} K \, dA_\sigma$$

$\varkappa_g$ = GEODESIC CURVATURE OF γ

K = GAUSSIAN CURVATURE OF σ.

$dA_\sigma = (EG - F^2)^{\frac{1}{2}} du \, dv$ AREA ELEMENT ON σ.

PROOF. CHOOSE SMOOTH ORTHONORMAL BASIS $\{e', e''\}$ OF TANGENT PLANES OF SURFACE

$\vec{N} = e' \times e''$ UNIT NORMAL TO SURFACE.

$\{e', e'', \vec{N}\}$ RIGHT-HANDED ORTHONORMAL BASIS OF $\mathbb{R}^3$.

FIRST:

$$\int_\gamma e' \cdot \dot{e}'' \, ds$$

$$= \int_\gamma e' \cdot (e''_u \dot{u} + e''_v \dot{v}) \, ds$$

$$= \int_\pi \left((e' \cdot e''_u) \, du + (e' \cdot e''_v) \, dv \right)$$

$$\underset{\substack{\uparrow \\ \text{GREEN'S THM}}}{=} \iint_{int(\pi)} \left\{ (e' \cdot e''_v)_u - (e' \cdot e''_u)_v \right\} du\, dv$$

$$= \iint_{int(\pi)} \left\{ (e'_u \cdot e''_v) - (e'_v \cdot e''_u) \right\} du\, dv$$

$$\underset{\text{LEMMA 9.1.2}}{=} \iint_{int(\pi)} \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}}\, du\, dv$$

$$= \iint_{int(\pi)} \underbrace{\frac{LN - M^2}{EG - F^2}}_{\substack{= K \\ \text{PROP 7.1.2}}} \underbrace{(EG - F^2)^{\frac{1}{2}}\, du\, dv}_{\substack{= dA_\sigma \\ 5.3.}}$$

$$= \iint_{int(\pi)} K\, dA_\sigma$$

PUT $\theta(s) = \measuredangle\left(\mathring{\gamma}(s), e'(\gamma(s))\right)$.

THEN

$$\mathring{\gamma} = \cos(\theta)\, e' + \sin(\theta)\, e''$$

$$\Rightarrow \vec{N} \times \mathring{\gamma} = (e' \times e'') \times (\cos(\theta) e' + \sin(\theta) e'')$$

$$\underset{\substack{\uparrow \\ \text{USE } (a \times b) \times c \;=\; -(b \cdot c)a + (a \cdot c)b}}{=} -\sin(\theta)\, e' + \cos(\theta)\, e''$$

$$\ddot{\gamma} = \cos(\theta)\dot{e}' + \sin(\theta)\dot{e}'' + \dot{\theta}\left(-\sin(\theta)e' + \cos(\theta)e''\right)$$

THUS

$$\varkappa_g \underset{6.2}{=} (\vec{N} \times \dot{\gamma}) \cdot \ddot{\gamma}$$

$$= \dot{\theta} + \cos^2(\theta)\underbrace{(\dot{e}' \cdot e'')}_{= -e' \cdot \dot{e}'' \text{ SINCE } e' \cdot e'' = 0} - \sin^2(\theta)(\dot{e}'' \cdot e')$$

$$+ \sin(\theta)\cos(\theta)\Big(\underbrace{\dot{e}'' \cdot e''}_{=0} - \underbrace{\dot{e}' \cdot e'}_{=0}\Big) \quad \text{SINCE } e' \cdot e' = 1 = e'' \cdot e''$$

$$= \dot{\theta} - e' \cdot \dot{e}''$$

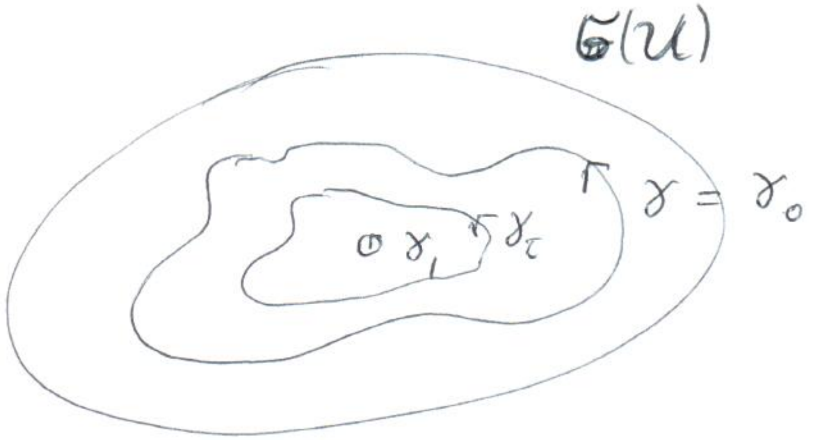
$$\Rightarrow \iint_{int(\pi)} K \, dA_6 = \int_\gamma e' \cdot \dot{e}'' \, ds = \int_\gamma (\dot{\theta} - \varkappa_g) \, ds$$

IT REMAINS TO PROVE:

$$\int_\gamma \dot{\theta} \, ds = 2\pi \qquad \boxed{\text{HOPF'S UMLAUFSATZ}}$$

PROOF REQUIRES TOPOLOGY, HENCE GIVE HEURISTIC ARGUMENT!

DEFORM γ TO "VERY SMALL" CIRCLE



USES $int(\gamma) \subseteq \mathfrak{S}(\mathcal{U})$

$(\gamma_\tau)_{\tau \in [0,1]}$

$\int_{\gamma_\tau} \dot\theta \, ds$ DEPENDS CONTINUOUSLY ON τ AND MUST BE IN $\mathbb{Z}(2\pi)$ SINCE γ_τ, $e^1 \circ \gamma_\tau$ RETURN TO ORIGINAL VALUE AS ONE GOES ONCE ROUND γ_τ.

THUS SUFFICES TO COMPUTE $\int_\gamma \dot\theta \, ds$ FOR γ_1.

(1) e^1 ESSENTIALLY CONSTANT ALONG γ_1

(2) $\dot\gamma_1$ ROTATES BY 2π ON GOING ONCE ROUND γ_1

$int(\gamma_1)$ ESSENTIALLY FLAT IF γ_1 VERY SMALL)

IN PLANE: TANGENT VECTOR OF SIMPLE CLOSED CURVE ROTATES BY 2π WHEN GOING ONCE ROUND THE CURVE.

$\square$.

10.2 GAUSS-BONNET THEOREM FOR CURVILINEAR POLYGONS

CONSIDER γ WITH FINITELY MANY CORNERS:

$\exists\ 0 = s_0 < s_1 < \dots < s_{n-1} < s_n = \text{LENGTH}(\gamma)$

(a) γ SMOOTH ON (s_{i-1}, s_i), $i \in \{1, \dots, n\}$

(b) $\dot\gamma^-(s_i) = \lim\limits_{s \uparrow s_i} \dfrac{\gamma(s) - \gamma(s_i)}{s - s_i}$ EXIST;

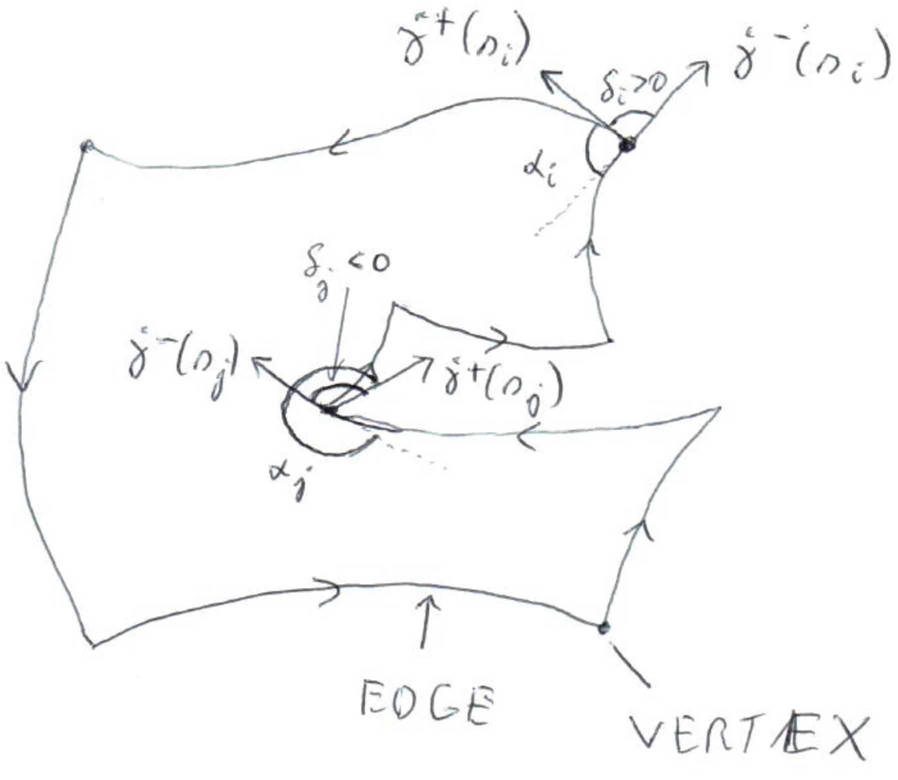
ARE $\neq 0$;

$\dot\gamma^+(s_i) = \lim\limits_{s \downarrow s_i} \dfrac{\gamma(s) - \gamma(s_i)}{s - s_i}$ ARE NOT PARALLEL.

SUCH γ IS CALLED <u>CURVILINEAR POLYGON</u>.

$\delta_i = \measuredangle(\dot\gamma^-(s_i), \dot\gamma^+(s_i)) \in (-\pi, \pi)$ <u>EXTERNAL ANGLE</u>

$\alpha_i = \pi - \delta_i \in (0, 2\pi)$ <u>INTERNAL ANGLE</u>



SAME ARGUMENT AS IN 10.1 GIVES

$$\iint_{\mathrm{int}(\gamma)} K \, dA_G = \underbrace{\int_\gamma \dot{\theta} \, ds}_{\parallel} - \int_\gamma \varkappa_g \, ds$$

$\parallel \leftarrow$ SEE BELOW

$$2\pi - \sum_{i=1}^{n} \underbrace{\delta_i}_{= \pi - \alpha_i}$$

$$= n\pi - \sum \alpha_i$$

THUS:

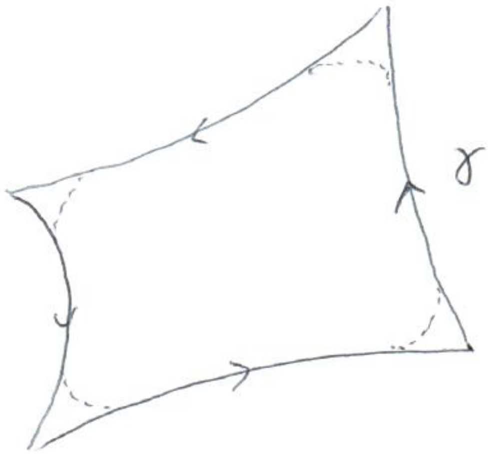
THEOREM 10.2.1

$$\int_\gamma \varkappa_g \, ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \iint_{\mathrm{int}(\gamma)} K \, dA_G$$

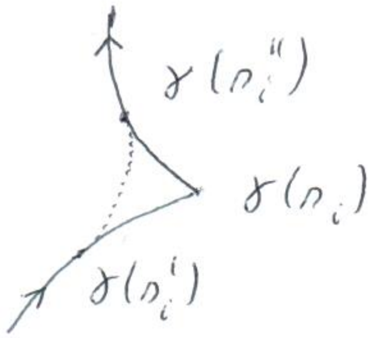
NEED TO PROVE

$$\int_\gamma \dot{\theta} \, ds = 2\pi - \sum_{i=1}^{n} \delta_i$$

MAKE SMOOTH EACH VERTEX



$\rightsquigarrow$ $\tilde{\gamma}$ SMOOTH CURVE



$$\int_{\tilde{\gamma}} \dot{\tilde{\theta}}\, ds = 2\pi$$

$$\int_{\tilde{\gamma}} \dot{\tilde{\theta}}\, ds - \int_{\gamma} \dot{\theta}\, ds = \sum_{i=1}^{n} \cdots ? \cdots$$

$$= \underbrace{\int_{s_i'}^{s_i''} \dot{\tilde{\theta}}\, ds} - \underbrace{\int_{s_i'}^{s_i} \dot{\theta}\, ds}_{\xrightarrow[s_i' \to s_i]{} 0} - \underbrace{\int_{s_i}^{s_i''} \dot{\theta}\, ds}_{\xrightarrow[s_i'' \to s_i]{} 0}$$

$= \measuredangle\left(\dot{\tilde{\gamma}}(s_i''), \dot{\tilde{\gamma}}(s_i')\right)$

SINCE γ SMOOTH ON (s_i', s_i) & (s_i, s_i'')

$= \measuredangle\left(\dot{\gamma}^-(s_i), \dot{\gamma}^+(s_i)\right)$

$= \delta_i$

AS $s_i' \to s_i$ & $s_i'' \to s_i$

THUS "?" $= \delta_i$

COROLLARY 10.2.2 LET γ BE A CURVILINEAR POLYGON WITH n EDGES EACH OF WHICH IS A GEODESIC. THEN

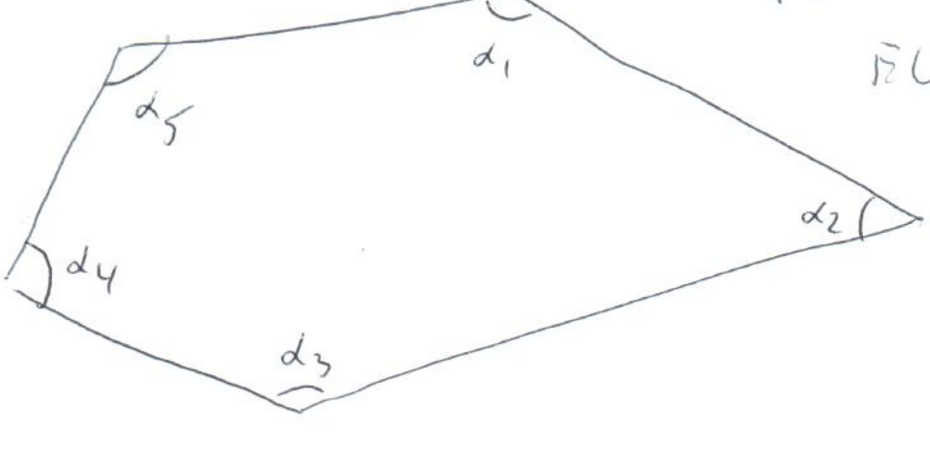
$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi + \iint_{int(\gamma)} K \, dA_G .$$

PROOF: FOLLOWS FROM THM 10.2.1 SINCE $\kappa_g = 0$ ALONG GEODESICS. $\square$.

SPERIAL CASES (1) ~~n~~ - GON IN PLANE WITH STRAIGHT EDGES. THEN $K=0$ AND

$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi$$

WELL-KNOWN FROM PLANAR EUCLIDEAN GEOMETRY



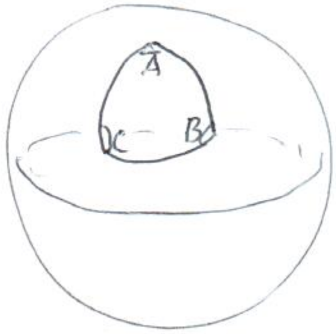
(2) UNIT SPHERE $(K=1)$

$$\sum \alpha_i = (n-2)\pi + (\text{AREA OF POLYGON})$$

$n=3$ (SPHERICAL TRIANGLE):

$$\mathcal{A}(ABC) = \angle A + \angle B + \angle C - \pi$$

(SEE THM 5.4.3)



(3) PSEUDOSPHERE $(K=-1)$

$$\sum \alpha_i = (n-2)\pi - (\text{AREA OF POLYGON})$$

$n=3$ (HYPERBOLIC TRIANGLE)

$$\mathcal{A}(ABC) = \pi - \angle A - \angle B - \angle C$$