MATH 465 - INTRODUCTION TO COMBINATORICS
LECTURE 5

1. Weight generating functions

Let $\alpha : A \to \mathbb{Z}_{\geq 0}$ be a *weight function* on a set A. For $k \in \mathbb{Z}_{\geq 0}$, let $h_k := \#\{a \in A \mid \alpha(a) = k\}$. We assume that h_k is finite for all k. Observe that

$$h(x) := \sum_{k=0}^{\infty} h_k x^k = \sum_{a \in A} x^{\alpha(a)}.$$

The generating function $h(x)$ can be viewed as a refinement of $\#A$. Indeed $h(1) = \#A$.

Example 1.1. Let $A = 2^{[n]}$ and let $\alpha : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ be the weight function that assigns to a subset S of A to its cardinality $\#S$. For instance, when $n = 3$, the weights and weight generating function are given by

$$\begin{array}{ccccccccc} \varnothing & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & \\ 1 & +x & +x & +x & +x^2 & +x^2 & +x^2 & +x^3 & = 1 + 3x + 3x^2 + x^3. \end{array}$$

In general, $h_k = \#\{S \in 2^{[n]} \mid \#S = k\} = \binom{[n]}{k}$ and the weight generating function $h(x) = \sum_{S \subseteq [n]} x^{\#S} = \sum_{k \geq 0} \binom{n}{k} x^k$.

2. The multiplication principle for generating functions

Proposition 2.1 (Multiplication principle for generating functions)**.** *Suppose we have a bijection of the form*

$$A \longleftrightarrow B \times C \times D \times \cdots$$
$$a \longleftrightarrow (b, c, d, \dots),$$

and that

$$\alpha : A \to \mathbb{Z}_{\geq 0}$$
$$\beta : B \to \mathbb{Z}_{\geq 0}$$
$$\gamma : C \to \mathbb{Z}_{\geq 0}$$
$$\delta : D \to \mathbb{Z}_{\geq 0}$$
$$\cdots\cdots\cdots\cdots$$

are weight functions as above satisfying the additivity condition

$$\alpha(a) = \beta(b) + \gamma(c) + \delta(d) + \cdots.$$

Then,

$$\sum_{a \in A} x^{\alpha(a)} = \left(\sum_{b \in B} x^{\beta(b)}\right) \left(\sum_{c \in C} x^{\gamma(c)}\right) \left(\sum_{d \in D} x^{\delta(d)}\right) \cdots$$

Note that setting $x = 1$ recovers the usual multiplication principle.

Proof. Expanding the right hand side, we have

$$\sum_{(b,c,d,\dots)\in B\times C\times D\times\cdots} x^{\beta(b)+\gamma(c)+\delta(d)+\cdots} = \sum_{a\in A} x^{\alpha(a)}.$$

$\square$

Example 2.2. Let h_k = number of ways to select a k-card hand from a double deck of cards. Let

$$\begin{aligned} A &= \{\text{hands (with any number of cards)}\} \leftrightarrow \{0,1,2\}^{52} \\ B_i &= \{0,1,2\} \text{ for } i = 1,2,\dots,52. \end{aligned}$$

Define the weight functions:

$$\begin{aligned} \alpha : A &\to \mathbb{Z}_{\geq 0} \\ \text{hand} &\mapsto \text{ number of cards in the hand} \\ \beta_i : B_i &\to \mathbb{Z}_{\geq 0} \\ k &\mapsto k. \end{aligned}$$

We have a bijection

$$\begin{aligned} A &\longleftrightarrow \prod_{i=1}^{52} B_i \\ \text{hand} &\mapsto (b_1, b_2, \dots, b_{52}) \end{aligned}$$

where b_i = number of times card i appears in the hand. By construction, we have $\alpha(\text{hand}) = \sum_{i=1}^{52} \beta_i(b_i)$, so by the multiplication principle for generating functions, we get

$$\sum_k h_k x^k = (1+x+x^2)^{52}.$$

Example 2.3. Let h_k = number of ways to change k cents into coins. We set

$$\begin{aligned} A &= \{\text{distinct collections of several coins}\}, \\ B &= \{\text{collections of several pennies}\}, \\ C &= \{\text{collections of several nickels}\}, \text{etc.} \end{aligned}$$

Each weight function $\alpha, \beta, \gamma, \dots$ assigns to a collection of coins its value in cents. The additivity condition clearly holds. Therefore,

$$\begin{aligned} \sum_k h_k x^k &= (1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)(1+x^{10}+x^{20}+\cdots)\cdots \\ &= (1-x)^{-1}(1-x^5)^{-1}(1-x^{10})^{-1}\cdots(1-x^{100})^{-1}. \end{aligned}$$

Example 2.4. Let $A = \mathbb{Z}^n_{\geq 0}$ with weight function $\alpha : \mathbb{Z}^n_{\geq 0} \to \mathbb{Z}$ sending $(x_1,\dots,x_n)$ to $x_1+\cdots+x_n$. Then, $a_n = \#\{(x_1,\dots,x_n) \mid \alpha(x_1,\dots,x_n) = k\}$ is the number of weak compositions of k with n parts.

Let $B_1,\dots,B_n$ be $\mathbb{Z}_{\geq 0}$ with weight function $\beta_i : B_i \to \mathbb{Z}$ sending x to x. Then, the identity map is a bijection between A and $B_1\times\cdots\times B_n$ and the additivity property of weights is satisfied. By the multiplication principle for generating functions,

$$\sum_n a_n x^n = (1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots) = (1-x)^{-n}.$$

3. Permutation statistics

Let S_n denote the set of permutations of $[n]$. A weight function on S_n is called a *permutation statistic.*

3.1. **Inversions.**

A pair (w_i, w_j) is called an *inversion* of the permutation $w_1 w_2 \cdots w_n$ if $i < j$ and $w_i > w_j$. In other words, an inversion is a pair of numbers that are out of order. Let $\mathrm{inv} : S_n \to \mathbb{Z}_{\geq 0}$ denote the permutation statistic assigning to a permutation the number of inversions in it.

Example 3.1. If $w \in S_5$ is the permutation 35142, then $\mathrm{inv}(w) = 6$. The inversions are $(3,1), (3,2), (5,1), (5,4), (5,2), (4,2)$.

Example 3.2.

w	123	132	213	231	312	321
$\mathrm{inv}(w)$	0	1	1	2	2	3

The corresponding generating function is

$$\sum_{w \in S_3} x^{\mathrm{inv}(w)} = 1 + 2x + 2x^2 + x^3 = (1+x)(1+x+x^2).$$

Theorem 3.3. *The generating function for counting permutations in S_n with respect to the number of inversions is given by*

$$\begin{aligned}\sum_{w \in S_n} x^{\mathrm{inv}(w)} &= 1 \cdot (1+x) \cdot (1+x+x^2) \cdots (1 + \cdots + x^{n-1}) \\ &= \prod_{k=1}^{n} \frac{1-x^k}{1-x}.\end{aligned}$$

The *code* of a permutation $w \in S_n$ is the sequence $(c_1, c_2, \ldots, c_n)$ where

$$c_k = \#\{\text{inversions of the form } (k, *)\}.$$

Example 3.4.

w	inversions	$\mathrm{inv}(w)$	$\mathrm{code}(w)$
123		0	$(0,0,0)$
132	$(3,2)$	1	$(0,0,1)$
213	$(2,1)$	1	$(0,1,0)$
231	$(2,1),(3,1)$	2	$(0,1,1)$
312	$(3,1),(3,2)$	2	$(0,0,2)$
321	$(3,1),(3,2),(2,1)$	3	$(0,1,2)$

Lemma 3.5. *We have:*

(1) $\mathrm{inv}(w) = c_1 + \cdots + c_n$ *;*
(2) Each c_k *takes values* $0, 1, \ldots, k-1$.

Proof. (1) c_k is the number of inversions of the form (k, w_i).
(2) For an inversion of the form (k, w_i), we must have $w_i \in \{1, \ldots, k-1\}$.

□

Proof of Theorem 3.3. We make the following observations:

(1) c_n is just the number of numbers to the right of n in w, so the position of n in w is $n - c_n$, i.e., $w_{n-c_n} = n$. For example, $c_5 = 3$ in 35142 and there are 3 numbers $1, 2, 4$ to the right of 5.
(2) If $(k, *)$ is an inversion, then $* < k$. Therefore, the inversions contributing to $c_1, \ldots, c_{n-1}$ only involve numbers $*$ that are less than n, and so $(c_1, \ldots, c_{n-1})$ does not change if we omit n from w. For example, the code of 35142 is $(0, 0, 2, 1, 3)$ and the code of the permutation 3142 obtained from 35142 by omitting 5 is $(0, 0, 2, 1)$.

We claim that the function

$$\begin{aligned} f : S_n &\to \{0\} \times \{0,1\} \times \{0,1,2\} \times \cdots \times \{0,1,\ldots,n-1\} \\ w &\mapsto (c_1, c_2, \ldots, c_n) \end{aligned}$$

is a bijection. The proof is based on the following procedure to construct a permutation w from a code $(c_1, \ldots, c_n)$:

(1) Write a 1,
(2) Suppose we have created a permutation of $[k-1]$ so far, where $k > 1$. We insert k into the permutation so that there are c_k numbers to its right to get a permutation of $[k]$. Do this until $k = n$.

The observations (1) and (2) above imply that the permutation w has code $(c_1, \ldots, c_n)$, so f is surjective. Let's show injectivity by induction. If $n = 1$, this is clear. If $v, w \in S_n$ such that $f(v) = f(w) = (c_1, \ldots, c_n)$, then the permutations v' and w' in S_{n-1} obtained from v and w respectively by omitting n have the same code. By induction, v' and w' are equal, and since c_n is the same for v and w, n is in the same position as well.

If we define that

- the weight of w is $\mathrm{inv}(w)$,
- the weight of each entry c_i is c_i,

then the additivity condition is satisfied. Now we apply the multiplication principle for generating functions. □

Example 3.6. From the code $(0, 0, 2, 1, 3)$, the steps to reconstruct the permutation are as follows:

i	c_i	permutation
1	0	1
2	0	12
3	2	312
4	1	3142
5	3	35142

3.2. **Cycles.** If we regard a permutation w as a bijection $w : [n] \to [n]$, then for each $x \in [n]$, the sequence $x, w(x), w^2(x), \ldots$ must eventually return to x. Consider the smallest $l \geq 1$ such that $w^l(x) = x$. The sequence $(x, w(x), \ldots, w^{l-1}(x))$ is called a cycle of length l of w. We regard w as a product of its distinct cycles $C_1, C_2, \ldots, C_k$ and write $w = C_1 C_2 \cdots C_k$. For example if $w = 5276134$, we have $w = (15)(2)(3746)$. Let $c(w)$ denote the number of cycles in w.

Example 3.7. When $n = 3$, we have:

w	cycle notation	$c(w)$
123	(1)(2)(3)	3
132	(1)(23)	2
213	(12)(3)	2
231	(123)	1
312	(132)	1
321	(13)(2)	2

$$\sum_{w \in S_3} x^{c(w)} = x^3 + 3x^2 + 2x = x(x+1)(x+2)$$

Suppose we have a tuple $(b_1, b_2, \dots, b_n) \in \{0\} \times \{0,1\} \times \cdots \times \{0, 1, 2, \dots, n-1\}$. We describe a procedure to build a permutation by inserting entries into cycles. Start by inserting the entry 1 to form the cycle (1). Now suppose the first $j-1$ numbers have been inserted, so that we have a permutation $C_1 C_2 \cdots C_k$.

(1) If $b_j = 0$, we create a new cycle (j) to get the permutation $C_1 C_2 \cdots C_k (j)$.
(2) If $b_j = i$, we take the cycle containing i and add j to the left of i.

The tuple $(b_1, b_2, \dots, b_n)$ is called the *encoding* of the permutation. This process is reversible: successively remove the highest number j from the permutation let $b_j = 0$ if (j) is a cycle and let $b_j = i$ if i is the element to the right of j in the cycle containing j. Therefore, the encoding gives us a bijection between S_n and tuples $\{0\} \times \{0,1\} \times \cdots \times \{0, 1, 2, \dots, n-1\}$. Note that $c(w)$ is the number of zeroes in the encoding of w. Let

$$\beta(b) := \begin{cases} 1 \text{ if } b = 0; \\ 0 \text{ otherwise.} \end{cases}$$

Then $c(w) = \beta(b_1) + \beta(b_2) + \cdots + \beta(b_n)$.

Example 3.8. Consider the permutation (15)(2)(3746). To compute its encoding, we have the following steps:

j	permutation	b_j
7	(15)(2)(3746)	3
6	(15)(2)(346)	4
5	(15)(2)(34)	1
4	(1)(2)(34)	3
3	(1)(2)(3)	0
2	(1)(2)	0
1	(1)	0

,

so its encoding is $(0, 0, 0, 3, 1, 4, 3)$.

Theorem 3.9. *The generating function for counting permutations in S_n with respect to the number of cycles is given by*

$$\sum_{w \in S_n} x^{c(w)} = x(x+1)(x+2) \cdots (x+n-1).$$

Proof. The generating function for b_k is $x + k - 1$. Now use the multiplication principle for generating functions. □