6 Green's functions for PDEs

Learning outcomes: *Ability to construct Green's functions for the Poisson equation with Dirichlet boundary conditions using the method of images.*

Relevance to the course: *PDEs are the mathematical cornerstone when attempting to solve real-life problems, and this part of the course touches upon one major class of PDEs: the Laplace/Poisson equation.*

6.1 Poisson Dirichlet problem in 2 dimensions - Unit 6 Lect 1

6.1.1 Similarities to Green's functions for ODEs

The Laplace operator plays a central role in the theory of PDEs. In particular, the Laplace equation

$$\boldsymbol{\nabla}^2 u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \mathring{\Omega}, \qquad \nabla^2 = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots \partial_{x_n}^2$$

where $\mathring{\Omega}$ is the interior of closed domain $\Omega \subseteq \mathbb{R}^n$, i.e. a nonempty, connected and open subset in $\mathbb{R}^n$, is the prototype for elliptic equations. We shall use the notation $\Omega = \mathring{\Omega} \cup \partial\Omega$ where $\partial\Omega$ is the boundary of Ω if it is bounded. The Laplace equation is a special case of a more general equation, called *Poisson's equation*

poisson's equation: $$\boldsymbol{\nabla}^2 u(\boldsymbol{x}) = f(\boldsymbol{x}),$$

where $f : \mathring{\Omega} \mapsto \mathbb{C}$ is a given function.

The function u may represent the gravitational potential, the electrostatic potential, the steady-state temperature (i.e. the temperature not changing with time) in a region $\mathring{\Omega}$ containing mass, electric charge or sources of heat or fluid respectively. In a heat transfer problem, f is the rate of heat production in Ω, while in electricity, the source f is proportional to the density of electric charge in $\mathring{\Omega}$.

Because there is no time dependence in the Laplace/Poisson equation, there are no initial conditions to be satisfied by the solutions u. They must however satisfy boundary conditions on the bounding curve ($n = 2$) or surface ($n = 3$) of the region $\mathring{\Omega}$ (we shall use the notation $\Omega = \mathring{\Omega} \cup \partial\Omega$ where $\partial\Omega$ is the boundary of Ω). Since the Laplace/Poisson equation is second-order, one might expect two boundary conditions when $n \geq 2$, since in one-dimensional problems, for $\mathring{\Omega} = (a, b)$, we impose boundary conditions at the two extremities (boundaries) of the interval. In generalising to higher dimension, what we need are for solutions to satisfy conditions at every point of the boundary $\partial\Omega$. For instance, if the value of the function u is imposed at each point of $\partial\Omega$, we have a Dirichlet problem; if instead the value $\partial_n u$ of the derivative (rate of change) of u in the direction of the normal to $\partial\Omega$ is given at each point of the boundary, we have a Neumann problem.

It is common in physical applications (in particular, but certainly not exclusively) to take advantage of any symmetry the problem might have, as this usually makes life much simpler. It turns out that the Laplace equation is invariant under

	translations	$T_{\boldsymbol{x_0}} : \mathbb{R}^n \to \mathbb{R}^n : \boldsymbol{x} \mapsto \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{x_0},\ \boldsymbol{x_0}$ constant,
	rotations	$R : \mathbb{R}^n \to \mathbb{R}^n : \boldsymbol{x} \mapsto \boldsymbol{x}' = R\boldsymbol{x},\ R^T R = RR^T = 1,\ R^T$ transpose matrix of R,
扩张	dilations	$D : \mathbb{R}^n \to \mathbb{R}^n : \boldsymbol{x} \mapsto \boldsymbol{x}' = \mu\boldsymbol{x},\ \mu \in \mathbb{R}$.

Remark 6.1 For the inquisitive mind (not examinable): it pays to convince oneself of the above symmetry (or invariance) statements, and by far the fastest (and coolest) method is to use *index notation* and the Einstein summation convention. With $\boldsymbol{x} = \sum_{i=1}^n x_i \mathbf{e}_i$ for $i \in \{1, \ldots, n\}$, the Laplacian may be written as $\nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} \stackrel{(1)}{=} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$, where equality (1) implements Einstein's summation convention, where repeated indices are meant to be summed from 1 to n. Let us now show:

- invariance under translations $T_{\boldsymbol{x_0}}$. In terms of vector components, translations by a constant vector $\boldsymbol{x_0}$ are given by $x_i' = x_i + (x_0)_i$, where the symbol $'$ here does not mean 'derivative', but new variable. We thus have
$$\frac{\partial u}{\partial x_i} \overset{\text{(chain rule)}}{=} \sum_{j=1}^{n} \frac{\partial x_j'}{\partial x_i}\frac{\partial u}{\partial x_j'} \overset{(1)}{=} \sum_{j=1}^{n} \delta_{ij}\frac{\partial u}{\partial x_j'} = \frac{\partial u}{\partial x_i'} \qquad \forall i \in \{1, \ldots, n\},$$
where the equality (1) follows from the fact that all the coordinates x_i are independent. Thus under translations $T_{\boldsymbol{x_0}}$, $\dfrac{\partial}{\partial x_i} = \dfrac{\partial}{\partial x_i'}$ and $\nabla^2 u = \dfrac{\partial}{\partial x_i}\dfrac{\partial}{\partial x_i}u = \dfrac{\partial}{\partial x_i'}\dfrac{\partial}{\partial x_i'}u$, yielding the result.

- invariance under rotations R, where R is an $n \times n$ orthogonal matrix, i.e. a matrix with constant entries and satisfying the relation $R^T R = RR^T = 1$, where 1 is the $n \times n$ identity matrix. In index notations, the rotation $\boldsymbol{x}' = R\boldsymbol{x}$ is given by $x_j' = \sum_{k=1}^n R_{jk}x_k = R_{jk}x_k$ and the ik entry of the orthogonality relation $1 = RR^T$ reads
$$(1 = RR^T)_{ik}: \quad \delta_{ik} = (RR^T)_{ik} = \sum_{j=1}^{n} R_{ij}(R^T)_{jk} = \sum_{j=1}^{n} R_{ij}R_{kj} = R_{ij}R_{kj}.$$
So we have
$$\frac{\partial u}{\partial x_i} \overset{\text{(chain rule)}}{=} \sum_{j=1}^{n} \frac{\partial x_j'}{\partial x_i}\frac{\partial u}{\partial x_j'} \overset{(1)}{=} \sum_{j=1}^{n}\left(\sum_{k=1}^{n} R_{jk}\,\delta_{ik}\right)\frac{\partial u}{\partial x_j'} = \sum_{j=1}^{n} R_{ji}\frac{\partial u}{\partial x_j'} = R_{ji}\frac{\partial u}{\partial x_j'}$$
where equality (1) comes from $\dfrac{\partial x_j'}{\partial x_i} = \dfrac{\partial}{\partial x_i}(\sum_{k=1}^{n} R_{jk}x_k) = \sum_{k=1}^{n} R_{jk}\dfrac{\partial x_k}{\partial x_i} = \sum_{k=1}^{n} R_{jk}\,\delta_{ik} = R_{ji}.$
Therefore, under rotations,
$$\frac{\partial}{\partial x_i} = R_{ji}\frac{\partial}{\partial x_j'},$$
and
$$\nabla^2 u = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}u = \frac{\partial}{\partial x_i}\left(R_{ji}\frac{\partial u}{\partial x_j'}\right) = R_{ji}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j'}u = \underbrace{R_{ji}R_{ki}}_{=\delta_{jk}}\frac{\partial}{\partial x_k'}\frac{\partial}{\partial x_j'}u = \frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_j'}u,$$
confirming the invariance under rotations of the Laplacian.

- Invariance under dilations D. If $\boldsymbol{x}' = \mu\boldsymbol{x}, \mu \in \mathbb{R}$, then $x_i' = \mu x_i$ for all $i \in \{1, \ldots, n\}$. Since
$$\frac{\partial}{\partial x_i'} = \frac{\partial}{\partial(\mu x_i)} = \frac{1}{\mu}\frac{\partial}{\partial x_i},$$
we immediately see that
$$\nabla^2 u = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}u = \mu^2\frac{\partial}{\partial x_i'}\frac{\partial}{\partial x_i'}u,$$
and $\dfrac{\partial}{\partial x_i}\dfrac{\partial}{\partial x_i}u = 0$ implies $\dfrac{\partial}{\partial x_i'}\dfrac{\partial}{\partial' x_i}u = 0.$ $\square$

Real-life applications usually restrict $n \leq 3$. **We limit our study to the Laplace and Poisson equations in two dimensions**, and use the notation $\boldsymbol{x} = x\mathbf{e}_1 + y\mathbf{e}_2$ rather than $\boldsymbol{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, namely

$$\nabla^2 u := u_{xx} + u_{yy} = 0, \qquad (x, y) \in \mathring{D}, \qquad \textbf{Laplace's equation in 2d,} \tag{6.1}$$

where $\mathring{D}$ is a planar domain (i.e. a nonempty connected and open set in $\mathbb{R}^2$). A function u satisfying (6.1) is called a *harmonic function*.

 $\Omega = \mathring{\Omega} \cup \partial\Omega$ → boundary. ↳ interior.

The corresponding *Poisson's equation* is,

$$\nabla^2 u(x,y) = f(x,y), \qquad (x,y) \in \mathring{D}, \qquad \textbf{Poisson's equation in 2d}, \tag{6.2}$$

where $f : \mathring{D} \to \mathbb{C}$ is a given function.

The uniqueness of the solution (when it exists) is linked to the conditions put on u at the boundary ∂D of the domain of definition D. We shall only study one type of boundary condition here.

Definition 6.2 ***Dirichlet problem in 2 dimensions***. *The problem defined by Poisson's equation and the following boundary condition*

$$u(\boldsymbol{x}) = g(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial D,$$

for a given function g is called the Dirichlet problem in 2 dimensions. In other words, the Dirichlet problem is the IN/IN BVP

$$\nabla^2 u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathring{D}, \quad u(\boldsymbol{x})\big|_{\partial D} = g(\boldsymbol{x}). \tag{6.3}$$

Proving the existence of a solution (let alone its uniqueness and its stability) is a hard mathematical problem. When the domain $\mathring{D}$ is bounded and 'sufficiently smooth', then the Dirichlet problem does have a solution. We shall accept this result (its proof is beyond the scope of this course).

To solve this problem with the method of Green's functions, we proceed as in the case of ODEs, and consider the Green's function for a system described by Poisson's equation as the response of the system to a unit point source. **If we consider the IN/HOM BVP first**, i.e. if we solve

$$\nabla^2 u_p(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathring{D}, \qquad u_p(\boldsymbol{x})\big|_{\partial D} = 0,$$

we expect a solution

$x_0 = x_0 e_1 + y_0 e_2$

$x = x e_1 + y e_2$

$$u_p(\boldsymbol{x}) = \int_{\mathring{D}} G(\boldsymbol{x}, \boldsymbol{x_0})\, f(\boldsymbol{x_0})\, d\boldsymbol{x_0}, \tag{6.4}$$

where the Green's function satisfies the equation $\nabla^2 G(\boldsymbol{x}, \boldsymbol{x_0}) = \delta(\boldsymbol{x} - \boldsymbol{x_0}) = \delta(x - x_0)\delta(y - y_0)$ with $\boldsymbol{x_0} \in \mathring{D}$ and $G(\boldsymbol{x}, \boldsymbol{x_0})\big|_{\partial D} = 0$.

Remark 6.3 In many applications, domains have 'corners', and near a corner the boundary is not differentiable, so the solutions may not be as smooth as one would want. □

To solve the Dirichlet IN/IN BVP (6.3), we first recall Green's identities.

6.1.2 Green's identities

Recall the 2-dimensional Divergence Theorem:

Theorem 6.4 *Let $\mathring{D}$ be any bounded piecewise smooth planar domain and let $\mathbf{F} \in \mathcal{C}^1(\mathring{D}) \cap \mathcal{C}^0(D)$ be a vector field. Then*

$$\int_{\mathring{D}} \boldsymbol{\nabla} \cdot \boldsymbol{F}\, dA = \int_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n}\, ds, \tag{6.5}$$

where $\boldsymbol{n}$ is the outward normal to ∂D, ds is a line element along the boundary ∂D and the surface element is $dA = dx\, dy$ in Cartesian coordinates.

Let ϕ, ψ be two arbitrary functions in $\mathcal{C}^2(\mathring{D}) \cap \mathcal{C}^1(D)$. Then

1. **Green's first identity** is obtained by choosing $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ in the Divergence Theorem (6.5):

$$\int_{\mathring{D}} \boldsymbol{\nabla}^2 \phi\, dx\, dy = \int_{\partial D} \boldsymbol{\nabla}\phi \cdot \boldsymbol{n}\, ds = \int_{\partial D} \boldsymbol{n} \cdot \boldsymbol{\nabla}\phi\, ds = \int_{\partial D} \partial_n \phi\, ds.$$

$\therefore \int_{D} \nabla^2 \phi\, dx dy = \int_{\partial D} \partial_n \phi\, ds.$

$\int_{\mathring{D}} \nabla\cdot F$

2. **Green's second identity** is obtained by choosing $\mathbf{F} = \phi\boldsymbol{\nabla}\psi$ in the Divergence Theorem:

$$\int_{\mathring{D}} \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\psi)\,dx\,dy = \int_{\mathring{D}} \Big(\phi\boldsymbol{\nabla}^2\psi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi\Big)\,dxdy$$

$\nabla\cdot F.$

$$= \int_{\partial D} \phi\boldsymbol{\nabla}\psi\cdot\boldsymbol{n}\,ds = \int_{\partial D} \phi\,(\boldsymbol{n}\cdot\boldsymbol{\nabla}\psi)\,ds = \int_{\partial D} \phi\,\partial_n\psi\,ds.$$

$F \cdot n.$

3. **Green's third identity** is obtained by first interchanging the roles of ϕ and ψ in Green's second identity, and then subtract one from the other,

$\int \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi = \int \psi\,\partial_n\phi$

$$\int_{\mathring{D}} \Big(\phi\nabla^2\psi - \psi\nabla^2\phi\Big)\,dx\,dy = \int_{\partial D} (\phi\,\partial_n\psi - \psi\,\partial_n\phi)\,ds. \tag{6.6}$$

End of Unit 6 Lect 1

6.1.3 The solution to the Poisson Dirichlet IN/IN BVP - Unit 6 Lect 2

As we already mentioned, the Green's function for Poisson's equation must satisfy

$$\nabla^2 G(\boldsymbol{x},\boldsymbol{x_0}) = \delta(\boldsymbol{x}-\boldsymbol{x_0}), \qquad \boldsymbol{x_0}\in\mathring{D}. \tag{6.7}$$

Suppose we do not impose boundary conditions on $G(\boldsymbol{x},\boldsymbol{x_0})$ at first. Letting $\phi = u(\boldsymbol{x})$ and $\psi = G(\boldsymbol{x},\boldsymbol{x_0})$ in Green's third identity (6.6), we get

$$\int_{\mathring{D}} \Bigg(u(\boldsymbol{x})\,\underbrace{\nabla^2 G(\boldsymbol{x},\boldsymbol{x_0})}_{=\delta(\boldsymbol{x}-\boldsymbol{x_0})} - G(\boldsymbol{x},\boldsymbol{x_0})\,\underbrace{\nabla^2 u(\boldsymbol{x})}_{=f(\boldsymbol{x})} \Bigg)\,d\boldsymbol{x} = \int_{\partial D} \{\, u(\boldsymbol{x})\,\partial_n G(\boldsymbol{x},\boldsymbol{x_0}) - G(\boldsymbol{x},\boldsymbol{x_0})\,\partial_n u(\boldsymbol{x}) \,\}\,ds, \tag{6.8}$$

or again, using the sifting property of the delta distribution,

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$$u(\boldsymbol{x_0}) = \int_{\mathring{D}} G(\boldsymbol{x},\boldsymbol{x_0})f(\boldsymbol{x})\,d\boldsymbol{x} + \int_{\partial D} \{\, u(\boldsymbol{x})\,\partial_n G(\boldsymbol{x},\boldsymbol{x_0}) - G(\boldsymbol{x},\boldsymbol{x_0})\,\partial_n u(\boldsymbol{x}) \,\}\,ds. \tag{6.9}$$

This is the solution to Poisson's equation before we impose boundary conditions on $G(\boldsymbol{x},\boldsymbol{x_0})$.

Remark 6.5 Note that the roles of $\boldsymbol{x}$ and $\boldsymbol{x_0}$ in the solution (6.9), and also that, when the Green's function for the Poisson problem is real, $G(\boldsymbol{x},\boldsymbol{x_0}) = G(\boldsymbol{x_0},\boldsymbol{x})$ i.e. it is symmetric. This is because the Laplacian operator ∇^2 is self-adjoint. □

Therefore, the solution to (6.3) may also be written as

$$u(\boldsymbol{x}) = \int_{\mathring{D}} G(\boldsymbol{x},\boldsymbol{x_0})f(\boldsymbol{x_0})\,d\boldsymbol{x_0} + \int_{\partial D} \{\, u(\boldsymbol{x_0})\,\partial_n G(\boldsymbol{x},\boldsymbol{x_0}) - G(\boldsymbol{x},\boldsymbol{x_0})\,\partial_n u(\boldsymbol{x_0}) \,\}\,ds.$$

Remark 6.6 If we restrict to solutions satisfying $u(\boldsymbol{x}) = 0$ on ∂D, and demand $G(\boldsymbol{x},\boldsymbol{x_0}) = 0$ on ∂D as well, then we recover the solution (6.4) as can be immediately seen from (6.9). □

If we choose homogeneous boundary conditions for the Green's function, i.e. $G(\boldsymbol{x},\boldsymbol{x_0}) = 0$ for $\boldsymbol{x}\in\partial D$, and consider inhomogeneous boundary conditions for the solution, i.e. $u(\boldsymbol{x}) = g(\boldsymbol{x})$ for $\boldsymbol{x}\in\partial D$, then the solution takes the form

$$u(\boldsymbol{x_0}) = \int_{\mathring{D}} G(\boldsymbol{x},\boldsymbol{x_0})f(\boldsymbol{x})\,d\boldsymbol{x} + \int_{\partial D} g(\boldsymbol{x})\,\partial_n G(\boldsymbol{x},\boldsymbol{x_0})\,ds. \tag{6.10}$$

Thus we want to find the Dirichlet Green's function $G(\boldsymbol{x},\boldsymbol{x_0})$ that

- *Condition 1*: satisfies (6.7), and hence is singular at $\boldsymbol{x} = \boldsymbol{x_0}$ $\quad \nabla^2 G(x, x_0) = \delta(x - x_0)$
- *Condition 2*: satisfies $G(\boldsymbol{x}, \boldsymbol{x_0}) = 0$ for $\boldsymbol{x} \in \partial D$. $\quad G(x, x_0)\big|_{\partial D} = 0.$

In general it is difficult to obtain this Green's function G directly, so instead we look for a solution to (6.7) of the form

$\nabla^2 G_2(x, x_0) \qquad \nabla^2 G_{reg}(x, x_1, \cdots x_N)$

$$G(\boldsymbol{x}, \boldsymbol{x_0}) = G_2(\boldsymbol{x}, \boldsymbol{x_0}) - G_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_N}), \tag{6.11}$$

where $G_2(\boldsymbol{x}, \boldsymbol{x_0})$ satisfies (6.7) and is singular at $\boldsymbol{x} = \boldsymbol{x_0}$, but does not necessarily satisfy the homogeneous boundary condition, i.e. $G_2(\boldsymbol{x}, \boldsymbol{x_0}) \neq 0$ for $\boldsymbol{x} \in \partial D$, while $G_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_N})$ satisfies Laplace's equation,

$$\nabla^2_{\boldsymbol{x}} G_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_N}) = 0, \quad \boldsymbol{x} \in \mathring{D}$$

and $G_{\text{reg}} = G_2$ on the boundary ∂D, so that conditions 1 and 2 above are satisfied.

The Green's function $G(\boldsymbol{x}, \boldsymbol{x_0})$ is therefore a solution to (6.7) since

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x_0}) = \nabla^2 G_2(\boldsymbol{x}, \boldsymbol{x_0}) - \nabla^2 G_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots, \boldsymbol{x}_N) = \delta(\boldsymbol{x} - \boldsymbol{x_0}) - 0 = \delta(\boldsymbol{x} - \boldsymbol{x_0})$$

AND $G(\boldsymbol{x}, \boldsymbol{x_0})$ vanishes on the boundary ∂D. The points corresponding to the vectors $\boldsymbol{x_i}$, $i \in \{1, \ldots N\}$ will be specified shortly.

Definition 6.7 *The function $G_2(\boldsymbol{x}, \boldsymbol{x_0})$ is called the* ***fundamental solution*** *to Poisson's equation.*

6.1.4 Calculation of the fundamental solution

The most important solution of Laplace's equation $\nabla^2 u(\boldsymbol{x}) = 0$ over the plane is the solution that is symmetric about the origin (i.e. the solution with radial symmetry). Let us construct it. We seek a solution u that only depends on the distance between the point $\boldsymbol{x}$ and the origin, which in polar coordinates is $|\boldsymbol{x}| = \sqrt{x^2 + y^2} = r$. Since the Laplacian in polar coordinates is

$$\boldsymbol{\nabla}^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2,$$

the function $u(\boldsymbol{x}) = u(|\boldsymbol{x}|) = u(r)$ actually satisfies the simpler equation

$$\frac{d^2}{dr^2}u + \frac{1}{r}\frac{d}{dr}u = 0, \qquad r^2 \partial_r u(r) + r\, \partial_r u(r) = 0, \tag{6.12}$$

which is of Cauchy-Euler type, i.e. of type $r^2 y'' + a r y' + b y = 0$, with y a function of r and a, b constants, as can be seen by multiplying (6.12) by r^2. To solve it, use the change of variable $r = e^t$ and set $u(r) = v(t)$ so that $\frac{du}{dr} = \frac{dt}{dr}\frac{dv}{dt} = e^{-t}\frac{dv}{dt}$ and $\frac{d^2u}{dr^2} = e^{-t}(-e^{-t}\frac{dv}{dt} + e^{-t}\frac{d^2v}{dt^2}) = e^{-2t}(-\frac{dv}{dt} + \frac{d^2v}{dt^2})$. Inserting this in (6.12), we get the following second order ODE with *constant* coefficient

$t = \ln(r)$

$$\frac{d^2v}{dt^2} = 0,$$

with general solution $v(t) = c_1 t + c_2$, which means that the radially symmetric Laplace equation over the plane has general solution

$$u(r) = c_1 \ln r + c_2, \qquad c_1, c_2 \text{ constants.}$$

This solution can be seen as the linear combination of two linearly independent solutions $u_1(r) = 1$ and $u_2(r) = \ln r$.

$c_1 = \frac{1}{2\pi}$

Definition 6.8 *The solution*

c_1

$$\boxed{G_2(\boldsymbol{x}) := \frac{1}{2\pi}\ln|\boldsymbol{x}| = \frac{1}{2\pi}\ln r = \frac{1}{4\pi}\ln(x^2+y^2), \qquad \forall \boldsymbol{x} \in \mathbb{R}^2 - \{\boldsymbol{0}\},}$$

is the ***fundamental solution*** *to Laplace's equation, and the choice of $c_1 = 1/2\pi$ is made for later convenience.*

The fundamental solution $G_2(\boldsymbol{x})$ is radially symmetric with a singularity (pole) at the origin $\boldsymbol{x} = 0$: it describes the electric potential due to a point-like electric charge at the origin for instance. We say that $G_2(\boldsymbol{x})$ is *harmonic in the punctured plane*.

Consequences of the invariance properties of Laplace's equation in 2 dimensions:

- its invariance under coordinate shifts $\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{x_0}$ for any constant $\boldsymbol{x_0}$ implies that, if $u(\boldsymbol{x})$ is a solution (i.e. if u is *harmonic*), then $u(\boldsymbol{x} - \boldsymbol{x_0})$ is also solution (harmonic). Therefore

$$\boxed{G_2(\boldsymbol{x}, \boldsymbol{x_0}) := G_2(\boldsymbol{x} - \boldsymbol{x_0}) = \frac{1}{2\pi}\ln(|\boldsymbol{x} - \boldsymbol{x_0}|)} \;.= G(x,x_0) \qquad (6.13)$$

  is also a solution to Laplace's equation, called the **fundamental solution to Laplace's equation in the plane with a pole at $\boldsymbol{x} = \boldsymbol{x_0}$.** Our Cartesian coordinates notations for $\boldsymbol{x}$ are (x, y) and for $\boldsymbol{x_0}$ are (x_0, y_0).

- its invariance under rotations of the coordinate system means that, if $u(r, \theta)$ is harmonic, then $u(r, \theta+\gamma)$ for γ constant (angle of rotation) is also harmonic. Recall that Cartesian coordinates (x, y) are expressed in polar coordinates (r, θ) as $x = r\cos\theta$ and $y = r\sin\theta$ (with $r > 0$ and $\theta \in [0, 2\pi)$). The point P' obtained from the point P of polar coordinates (r, θ) by a rotation of angle γ has polar coordinates $(r, \theta + \gamma)$.

- its invariance under dilation of the coordinate system means that if $u(x, y)$ is harmonic, then $u(\mu x, \mu y)$ is also harmonic for μ real constant.

Remark 6.9 The fundamental solution to Laplace's equation depends on the dimensionality n of space. For instance, if $n = 3$, the fundamental solution is given by

$$G_3(\boldsymbol{x}, \boldsymbol{x_0}) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x_0}|}.$$

while in one dimension, of course, $G_1(x, x_0) = K|x - x_0|$ and the constant is chosen to be $K = 1$.□

End of Unit 6 Lect 2

6.2 Examples of Green's functions for the Poisson Dirichlet IN/IN BVP - Unit 6 Lect 3

6.2.1 The method of images

The Green's function $G(\boldsymbol{x}, \boldsymbol{x_0})$ which enters in the solution (6.10) is constructed by adding to the fundamental solution $G_2(\boldsymbol{x}, \boldsymbol{x_0})$ 'copies' of itself that represent 'image' sources at different locations **outside** the domain D. One follows the steps below:

(i) To the unique source $\delta(\boldsymbol{x} - \boldsymbol{x_0})$ inside D, add N image sources outside D

$$\sum_{i=1}^{N} q_i \delta(\boldsymbol{x} - \boldsymbol{x_i}) =: -G_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots \boldsymbol{x_N}), \qquad \boldsymbol{x_i} \notin D \; \forall i \in \{1, \ldots, N\}, \quad \boldsymbol{x_0} \in \mathring{D},$$

number of strength.

ii. $G_2(x, x_0) = \frac{1}{2\pi} \ln|x - x_0| \longleftrightarrow \delta(x - x_0)$ Point source P: $OP = x_0 = x_0 e_1 + y_0 e_2$.
$G_2(x, x_i) = \frac{1}{2\pi} \ln|x - x_i| \longleftrightarrow \delta(x - x_i)$ P_i: $OP_i = x_i = x_i e_1 + y_i e_2$.

where the positions $\boldsymbol{x_i}$ and the strengths q_i of the image sources are to be determined as in step (iii) below. The number N of image sources depends on the geometry of the domain D considered.

(ii) Since all the image sources are **outside** D, the fundamental solution corresponding to each image source satisfies Laplace's equation **inside** D. Thus we may add the fundamental solutions $G_2(\boldsymbol{x}, \boldsymbol{x_n})$ corresponding to each image source to that corresponding to the single source inside D, obtaining

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$$G(\boldsymbol{x}, \boldsymbol{x_0}) = G_2(\boldsymbol{x}, \boldsymbol{x_0}) - G_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_N}) = G_2(\boldsymbol{x}, \boldsymbol{x_0}) + \sum_{i=1}^{N} q_i G_2(\boldsymbol{x}, \boldsymbol{x_i}).$$

$\nabla_x^2 G_2(x, x_0) = \delta(x - x_0)$. $x_0 \in \mathring{D}$, $x \in D$. singular at $x = x_0$ $\nabla_x^2 G_2(x, x_i) = 0$, $x \in D$, $x_i \notin D$.

(iii) Adjust the positions $\boldsymbol{x_i}$ and the strengths q_i so that the required boundary condition $G(\boldsymbol{x}, \boldsymbol{x_0}) = 0$ on ∂D is satisfied.

so, no singularity in D.

(iv) The solution to Poisson's Dirichlet IN/IN BVP is then (6.10).

$G(x, x_0) = G_2(x, x_0) - G_{reg}(x, x_1, \ldots, x_n) = G_2(x, x_0) + \sum_{i=1}^{N} q_i G_2(x, x_i)$

It is in general tricky to find the correct positions $\boldsymbol{x_i}$ and strengths q_i to ensure the Green's function boundary conditions are satisfied. However, if the domain $\mathring{D}$ has enough symmetry, the positions of the image sources can be found by imagining the boundary lines of the domain D to be mirrors in which the single source inside D (located at $\boldsymbol{x_0}$) reflects itself. We illustrate this in several examples, again in two dimensions.

6.2.2 Dirichlet problem on the half-plane

Solve the following problem on the half-plane $\Pi_+ := \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$,

$$\partial_x^2 u + \partial_y^2 u = 0, \quad (x, y) \in \mathring{\Pi}_+, \quad u(x, y)\Big|_{\partial\Pi_+} = g(y) \text{ and } \lim_{x \to \infty} u(x, y) = 0.$$

Here the domain is $\Pi_+ = \mathring{\Pi}_+ \cup \partial\Pi_+$ with $\mathring{\Pi}_+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and the boundary is the y-axis, i.e. $\partial\Pi_+ = \{x = 0\}$. The solution is given by (6.10) with $f(\boldsymbol{x}) = 0$ and the Green's function $G(\boldsymbol{x}, \boldsymbol{x_0})$ to be constructed by the method of images.

1. **Step 1**: Obtain the Green's function $G(\boldsymbol{x}, \boldsymbol{x_0}) = G_2(\boldsymbol{x}, \boldsymbol{x_0}) - G_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_N})$, where $G_2 = G_{\text{reg}}$ on the boundary, which is the y-axis, so that $G = 0$ along the y-axis.

   In order to construct G, we need G_{reg} (the fundamental solution G_2 is unique). The method of images provides an expression for G_{reg} when domains $\mathring{D}$ have enough symmetry. It uses *reflections* of a given point P of coordinates (x_0, y_0) in the domain $\mathring{D}$ w.r.t to its boundary or boundaries ∂D. In the present case, there is only one boundary and once P of coordinates (x_0, y_0) is fixed, the only reflection point of P is w.r.t. the y-axis and it is the point P_1 with coordinates $(-x_0, y_0)$. Let Q be an arbitrary point in Π_+ with coordinates (x, y) (see Fig. 12). Then

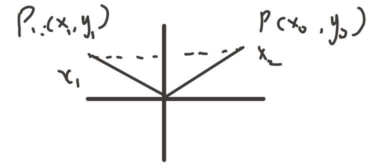
   $$|\mathbf{QP}| = |\boldsymbol{x} - \boldsymbol{x_0}| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \quad \text{and} \quad |\mathbf{QP_1}| = \sqrt{(x + x_0)^2 + (y - y_0)^2}.$$

   The fundamental solution on Π_+ is given by

   $$G_2(\boldsymbol{x}, \boldsymbol{x_0}) = \frac{1}{2\pi} \ln|\mathbf{QP}|, \quad \boldsymbol{x} \neq \boldsymbol{x_0}$$

   where $\boldsymbol{\nabla}^2 G_2(\boldsymbol{x}, \boldsymbol{x_0}) = \delta(\boldsymbol{x} - \boldsymbol{x_0})$, but G_2 does not satisfy homogeneous conditions on the boundary of Π_+. Indeed, when $x = 0$,

   $$G_2(0, y; x_0, y_0) = \frac{1}{2\pi} \ln\sqrt{(0 - x_0)^2 + (y - y_0)^2} \neq 0.$$



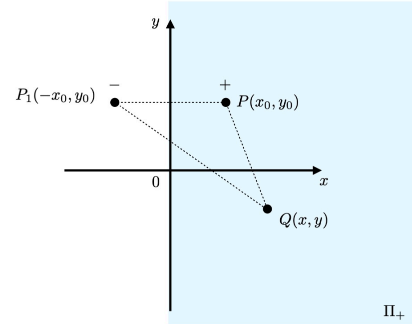


FIGURE 12: *The method of images when the domain is the half-plane Π_+. The point P_1 corresponding to $\boldsymbol{x_1}$ has coordinates $(-x_0, y_0)$.*

So a candidate function for G_{reg} could be

$$G_{\text{reg}}(\boldsymbol{x},\boldsymbol{x_1}) = G_2(\boldsymbol{x},\boldsymbol{x_1}) = \frac{1}{2\pi}\ln|\mathbf{QP_1}|, \quad \text{where } \boldsymbol{x} \in \mathring{\Pi}_+,\ \boldsymbol{x_1} \notin \mathring{\Pi}_+,$$

i.e. G_2 evaluated at the pair of points $\{\boldsymbol{x},\boldsymbol{x_1}\}$, where $\boldsymbol{x_1}$ is the vector corresponding to the point P_1, which is the reflection of the point P in the y-axis.

2. **Step 2**: Does G_{reg} satisfy the conditions required?
   - Since $P_1 \notin \mathring{\Pi}_+$, $G_{\text{reg}}(\boldsymbol{x},\boldsymbol{x_1})$ viewed as a function of $\boldsymbol{x}$ is regular on Π_+, and so satisfies the Laplace equation everywhere on Π_+. Indeed one can check that $\boldsymbol{\nabla}^2_{\boldsymbol{x}} G_{\text{reg}}(\boldsymbol{x},\boldsymbol{x_1}) = 0\ \forall \boldsymbol{x} \in \mathring{\Pi}_+$ by direct computation.
   - Does

$$G(\boldsymbol{x},\boldsymbol{x_0}) := G_2(\boldsymbol{x},\boldsymbol{x_0}) - G_{\text{reg}}(\boldsymbol{x},\boldsymbol{x_1}) = \frac{1}{2\pi}\ln\frac{|\mathbf{QP}|}{|\mathbf{QP_1}|} = \frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x+x_0)^2+(y-y_0)^2}$$

   fulfil the two requirements that $\boldsymbol{\nabla}^2_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{x_0}) = \delta(\boldsymbol{x}-\boldsymbol{x_0})$ and that G on the y-axis vanishes, i.e. $G(0,y;x_0,y_0) = 0$? Yes. We have

$$\boldsymbol{\nabla}^2_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{x_0}) = \boldsymbol{\nabla}^2_{\boldsymbol{x}} G_2(\boldsymbol{x},\boldsymbol{x_0}) - \boldsymbol{\nabla}^2_{\boldsymbol{x}} G_{\text{reg}}(\boldsymbol{x},\boldsymbol{x_1}) = \boldsymbol{\nabla}^2_{\boldsymbol{x}} G_2(\boldsymbol{x},\boldsymbol{x_0}) = \delta(\boldsymbol{x}-\boldsymbol{x_0}),$$

   and on the boundary $\partial\Pi_+$ we have $|\mathbf{QP}| = |\mathbf{QP_1}|$ since Q has coordinates $(0,y)$ (it is also immediate geometrically), and therefore $G(0,y;x_0,y_0) = 0$ as required.
   - **Step 3**: Since the solution is given by (6.10), we need to calculate $\partial_n G(\boldsymbol{x},\boldsymbol{x_0})$ on the boundary. In fact we have

$$\partial_n G(\boldsymbol{x},\boldsymbol{x_0})\big|_{x=0} = \frac{x_0}{\pi(x_0^2+(y-y_0)^2)},$$

   since the outward normal to $\partial\Pi_+$ is $\boldsymbol{n} = -\mathbf{e}_1$, and

$$\begin{aligned}\partial_n G(\boldsymbol{x},\boldsymbol{x_0})\big|_{x=0} &= \boldsymbol{n}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} G\big|_{x=0} = -\mathbf{e}_1\cdot(\mathbf{e}_1\partial_x + \mathbf{e}_2\partial_y)G(\boldsymbol{x},\boldsymbol{x_0})\big|_{x=0} \\ &= -\mathbf{e}_1\cdot\mathbf{e}_1\,\partial_x G(\boldsymbol{x},\boldsymbol{x_0})\big|_{x=0} = -\partial_x\left(\frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x+x_0)^2+(y-y_0)^2}\right)\Big|_{x=0}.\end{aligned}$$

   - **Step 4**: Write the solution. Since $f = 0$, the solution is

$$u(\boldsymbol{x_0}) = \frac{x_0}{\pi}\int_{-\infty}^{\infty}\frac{g(y)}{x_0^2+(y-y_0)^2}\,dy \quad \text{or} \quad u(\boldsymbol{x}) = \frac{x}{\pi}\int_{-\infty}^{\infty}\frac{g(y_0)}{x^2+(y-y_0)^2}\,dy_0. \qquad \square$$

End of Unit 6 Lect 3