PROP 2.3.3 γ REGULAR CURVE IN $\mathbb{R}^3$ WITH $\varkappa \neq 0$ EVERYWHERE. THEN

$\gamma \subset$ PLANE $\iff \tau = 0$.

PROOF „$\Leftarrow$": ASSUME $\tau = 0$. CAN ASSUME $\|\dot\gamma\| = 1$

$\Rightarrow \dot b = -\tau n = 0 \Rightarrow b$ CONSTANT

$\Rightarrow \frac{d}{ds}(\gamma \cdot b) = \dot\gamma \cdot b = t \cdot b = 0$

$\Rightarrow \exists d \in \mathbb{R} : \gamma \cdot b = d$

$\Rightarrow \gamma$ CONTAINED IN PLANE GIVEN BY

$r \cdot b = d$, $r \in \mathbb{R}^3$.

„$\Rightarrow$" ASSUME γ CONTAINED IN PLANE

$r \cdot a = d$ WITH $a \in \mathbb{R}^3$, $d \in \mathbb{R}$, $\|a\| = 1$.

$\Rightarrow \gamma \cdot a = d \Rightarrow t \cdot a = 0 \Rightarrow \dot t \cdot a = 0$

$\underset{\dot t = \varkappa n}{\Rightarrow} \varkappa n \cdot a = 0 \underset{\varkappa \neq 0}{\Rightarrow} n \cdot a = 0$

$\Rightarrow t, n \perp a \Rightarrow t, n$ PARALLEL TO $r \cdot a = d$

$\Rightarrow b = t \times n$ PERPENDICULAR TO $r \cdot a = d$

$\Rightarrow b \parallel a \underset{\|a\| = 1 = \|b\|}{\Longrightarrow} b = \pm a \Rightarrow b$ CONSTANT

$\underset{b \text{ CONT}}{\Rightarrow} \dot b = 0 \Rightarrow \tau = 0$ $\square$

WE KNOW $\dot{t} = \varkappa n$, $\dot{b} = -\tau n$.

$$\dot{n} = ?$$

WRITE

$$\dot{n} = \lambda t + \mu n + \nu b$$

$$\Rightarrow \quad \dot{n}\cdot t = \lambda \underbrace{t\cdot t}_{=1} + \mu \underbrace{n\cdot t}_{=0} + \nu \underbrace{b\cdot t}_{=0} = \lambda$$

$$0 \underset{\substack{\uparrow \\ n\cdot n = 1}}{=} \dot{n}\cdot n = \lambda \underbrace{t\cdot n}_{=0} + \mu \underbrace{n\cdot n}_{=1} + \nu \underbrace{b\cdot n}_{=0} = \mu$$

$$\dot{n}\cdot b = \lambda \underbrace{t\cdot b}_{=0} + \mu \underbrace{n\cdot b}_{=0} + \nu \underbrace{b\cdot b}_{=1} = \nu$$

$$0 = t\cdot n \Rightarrow 0 = \dot{t}\cdot n + t\cdot\dot{n} = \varkappa \underbrace{n\cdot n}_{=1} + \underbrace{t\cdot\dot{n}}_{=\lambda}$$

$$0 = b\cdot n \Rightarrow 0 = \dot{b}\cdot n + b\cdot\dot{n} = -\tau \underbrace{n\cdot n}_{=1} + \underbrace{b\cdot\dot{n}}_{=\nu}$$

$$\Rightarrow \quad \dot{n} = -\varkappa t + \tau b$$

THUS WE HAVE PROVED

THEOREM 2.3.4 (FRENET-SERRET EQUATIONS)
γ UNIT SPEED CURVE IN $\mathbb{R}^3$ WITH $\varkappa \neq 0$ EVERYWHERE. THEN

$$\begin{aligned} \dot{t} &= \quad \varkappa n \\ \dot{n} &= -\varkappa t \qquad\quad + \tau b \\ \dot{b} &= \quad -\tau n \end{aligned}$$

AN APPLICATION:

PROP 2.3.5 γ UNIT SPEED CURVE IN $\mathbb{R}^3$ WITH CONSTANT CURVATURE $\neq 0$ AND ZERO TORSION. THEN γ IS (PART OF) A CIRCLE.

PROOF: $\tau = 0 \Rightarrow \dot{b} = -\tau n = 0 \underset{2.3.3}{\Rightarrow} b$ CONSTANT AND $\gamma \subset$ PLANE PERPENDICULAR TO b.

$$\frac{d}{ds}\left(\gamma + \frac{1}{\varkappa} n\right) = t + \frac{1}{\varkappa}\dot{n} \underset{FSE}{=} 0$$

$$\Rightarrow \exists\, a \in \mathbb{R}^3 : \gamma + \frac{1}{\varkappa} n = a$$

$$\Rightarrow \quad \|\gamma - a\| = \frac{1}{\varkappa}\|n\| = \frac{1}{\varkappa}$$

$\Rightarrow \gamma \subset$ SPHERE WITH CENTRE a AND RADIUS $\frac{1}{\varkappa}$.

SINCE PLANE $\cap$ SPHERE = CIRCLE, THE ASSERTION FOLLOWS

$\square$

THEOREM 2.36. (UNIQUENESS OF CURVES IN $\mathbb{R}^3$ WITH GIVEN CURVATURE & TORSION)

LET $\gamma(s)$, $\gamma_1(s)$ UNIT SPEED CURVES IN $\mathbb{R}^3$ WITH SAME CURVATURE $\kappa(s)$ AND SAME TORSION $\tau(s)$. THEN THERE EXISTS A RIGID MOTION M OF $\mathbb{R}^3$ (ROTATION FOLLOWED BY TRANSLATION) SUCH THAT

$$\forall s: \gamma_1(s) = M(\gamma(s))$$

MOREOVER, IF k AND t ARE SMOOTH FUNCTIONS WITH $k > 0$ EVERYWHERE, THERE IS A UNIT SPEED CURVE IN $\mathbb{R}^3$ WHOSE CURVATURE IS k AND WHOSE TORSION IS τ.

[NOT TRUE WITHOUT $k > 0$].

PROOF: SEE TEXTBOOK

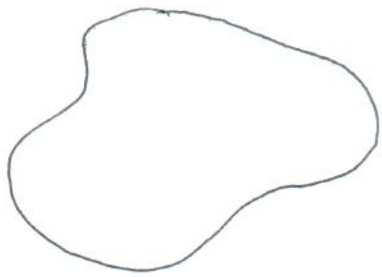
3 THE ISOPERIMETRIC INEQUALITY

3.1 THE ISOPERIMETRIC INEQUALITY

SO FAR: "LOCAL" GEOMETRY OF CURVES
NOW: "GLOBAL" GEOMETRY OF CURVES

DEF 3.1.1 $0 < a \in \mathbb{R}$. A SIMPLE CLOSED CURVE IN $\mathbb{R}^2$ WITH PERIOD a IS A REGULAR CURVE $\gamma : \mathbb{R} \to \mathbb{R}^2$, PARAMETRIZED BY A MULTIPLE OF ARC LENGTH, SUCH THAT

$$\forall t, t' \in \mathbb{R} : \gamma(t) = \gamma(t') \iff t' - t \in \mathbb{Z}a$$



SIMPLE CLOSED CURVE

NON-SIMPLE CLOSED CURVE

EX 3.1.2 $\gamma(t) = \left(\cos\left(\frac{2\pi t}{a}\right), \sin\left(\frac{2\pi t}{a}\right) \right)$

SIMPLE CLOSED CURVE WITH PERIOD a.

DEFINE $\ell(\gamma)$, LENGTH OF γ, BY

$$\ell(\gamma) = \int_0^a \|\gamma'(t)\| \, dt \qquad {}' = \frac{d}{dt}$$

[NOTE: DEF OF $\ell(\gamma)$ IS INDEPENDENT OF PARAMETRIZATION.]

$\mathrm{int}(\gamma)$, INTERIOR OF γ,

[NOTE: WELL-DEFINED BY JORDAN CURVE THEOREM]

$A(\mathrm{int}(\gamma))$, AREA OF $\mathrm{int}(\gamma)$,

$$A(\mathrm{int}(\gamma)) = \iint_{\mathrm{int}(\gamma)} dx\,dy$$

$$= \iint_{\mathrm{int}(\gamma)} \left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right) dx\,dy$$

$$f(x,y) = -\tfrac{1}{2} y$$
$$g(x,y) = \tfrac{1}{2} x$$

$$= \iint_{\mathrm{int}(\gamma)} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$\underset{\text{GREEN'S THEOREM}}{=} \int_\gamma \left(f(x,y)\,dx + g(x,y)\,dy\right)$$

$$= \frac{1}{2}\int_\gamma (x\,dy - y\,dx) = \frac{1}{2}\int_0^a (xy' - yx')\,dt$$

THM 3.1.3 (ISOPERIMETRIC INEQUALITY)

γ SIMPLE CLOSED CURVE. THEN

$$A(\mathrm{int}(\gamma)) \leq \frac{1}{4\pi} \ell(\gamma)^2$$

„=" $\Leftrightarrow$ γ CIRCLE

CLASSICAL PROBLEM: AMONG CLOSED CURVES γ IN $\mathbb{R}^2$ WITH FIXED PERIMETER, WHICH CURVE (IF EXISTS) MAXIMIZES AREA OF ENCLOSED REGION?

FOR PROOF USE RESULT FROM ANALYSIS:

PROP 3.1.4 LET $F : [0,\pi] \to \mathbb{R}$ SMOOTH WITH $F(0) = 0 = F(\pi)$. THEN

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \geq \int_0^\pi F(t)^2 \, dt$$

„=" $\Leftrightarrow$ $F(t) = A\sin(t)$ $\forall t \in [0,\pi]$ FOR SOME $A \in \mathbb{R}$.

WIRTINGER'S INEQUALITY

PROOF OF ISOPERIMETRIC INEQUALITY

REPARAMETRIZE γ SO THAT

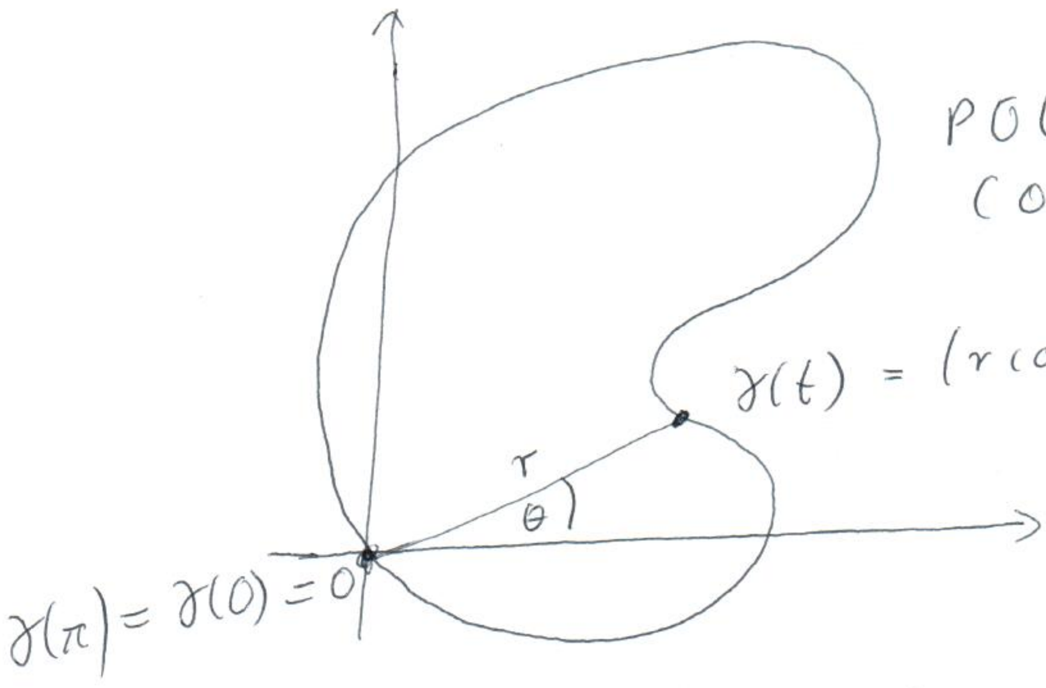
$$t = \frac{\pi s}{\ell(\gamma)} \qquad s = \text{ARC LENGTH}$$

[$\ell(\gamma)$, $\mathrm{int}(\gamma)$ REMAIN UNCHANGED]

SO $\gamma : [0, \pi] \to \mathbb{R}$, $\gamma(0) = \gamma(\pi)$

PERIOD π

CAN ALSO ASSUME THAT $\gamma(0) = 0 \; (= \gamma(\pi))$
SINCE TRANSLATION LEAVES
$\ell(\gamma)$, $A(\mathrm{int}(\gamma))$ UNCHANGED



POLAR COORDINATES

$$\gamma(t) = (r\cos(\theta), r\sin(\theta)) = (x, y)$$

$$x = r\cos(\theta) \Rightarrow x' = \cos(\theta) r' - r\sin(\theta)\theta'$$

$$y = r\sin(\theta) \Rightarrow y' = \sin(\theta) r' + r\cos(\theta)\theta'$$

$$\Rightarrow x'^2 + y'^2 = r'^2 + r^2\theta'^2$$

$$xy' - yx' = r^2\theta'$$

$$r'^2 + r^2\theta'^2 = x'^2 + y'^2 \underset{t = \frac{\pi s}{\ell(\gamma)}}{=} \frac{\ell(\gamma)^2}{\pi^2} \underbrace{(\dot{x}^2 + \dot{y}^2)}_{= \|\dot{\gamma}\|^2 = 1} = \frac{\ell(\gamma)^2}{\pi^2}$$

$$A(\operatorname{int}(\gamma)) = \frac{1}{2}\int_0^\pi (xy' - yx')\,dt = \frac{1}{2}\int_0^\pi r^2\theta'\,dt$$

$$\frac{\ell(\gamma)^2}{4\pi} - A(\operatorname{int}(\gamma)) = \frac{1}{4}\int_0^\pi (r'^2 + r^2\theta'^2)\,dt - \frac{1}{2}\int_0^\pi r^2\theta'\,dt$$

$$= \frac{1}{4}\int_0^\pi (r'^2 + r^2\theta'^2 - 2r^2\theta')\,dt$$

$$= \frac{1}{4}\left(\underbrace{\int_0^\pi r^2(\theta' - 1)^2\,dt}_{\geq 0} + \underbrace{\int_0^\pi (r'^2 - r^2)\,dt}_{\geq 0 \text{ BY PROP 3.1.4 WIRTINGER'S INEQUALITY}} \right)$$

"$=0$" $\Leftrightarrow \theta' = 1$

$\Leftrightarrow \theta(t) = t + \alpha$

"$=0$" $\Leftrightarrow r = A\sin(t)$

≥ 0 AND "$=0$" $\Leftrightarrow r = A\sin(\theta - \alpha)$

POLAR EQUATION OF CIRCLE

$\square$.

3.2 PROOF OF WIRTINGER'S INEQUALITY

$$\int_0^\pi F'(t)^2\,dt \geq \int_0^\pi F(t)^2\,dt$$ F SMOOTH $F(0)=0=F(\pi)$

"$=0$" $\Leftrightarrow$ $F(t) = A\sin(t)$

PUT $G(t) = \dfrac{F(t)}{\sin(t)}$ NOT DEFINED FOR $t=0, t=\pi$ USE L'HOSPITAL RULE TO MAKE SENSE OF IT.

$$\Rightarrow \int_0^\pi F'^2\,dt = \int_0^\pi (G'\sin(t) + G\cos(t))^2\,dt$$

$$= \int_0^\pi G'^2\sin^2(t)\,dt + \underbrace{2\int_0^\pi GG'\sin(t)\cos(t)\,dt}_{\text{INT BY PARTS}} + \int_0^\pi G^2\cos^2(t)\,dt$$

$$\underbrace{G^2\sin(t)\cos(t)\Big|_0^\pi}_{=0} - \int_0^\pi G^2(\cos^2(t)-\sin^2(t))\,dt$$

$$= \int_0^\pi (G^2+G'^2)\sin^2(t)\,dt$$

$$= \int_0^\pi F^2\,dt + \int_0^\pi G'^2\sin^2(t)\,dt$$

$$\Rightarrow \int_0^\pi (F'^2 - F^2)\,dt = \int_0^\pi G'^2\sin^2(t)\,dt \geq 0$$

"$=0$" $\Leftrightarrow$ $G'=0$ $\Leftrightarrow$ $\exists A\in\mathbb{R}: G(t)=A$

$\Leftrightarrow F(t) = A\sin(t)$ $\square$.

4 - SURFACES IN 3 DIMENSIONS

SURFACE $\approx \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$

WITH SUITABLE $f : \mathcal{U} \to \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^3$

PLANE $= \{(x,y,z) \mid ax + by + cz - d = 0\}$

$(a,b,c) \neq (0,0,0)$

UNIT SPHERE $= \{(x,y,z) \mid x^2 + y^2 + z^2 - 1 = 0\}$

ALTERNATIVE APPROACH:

<u>DEF 4.1.1</u> $\mathcal{U} \subset \mathbb{R}^2$ OPEN

$\sigma : \mathcal{U} \to \mathbb{R}^3$ SMOOTH INJECTIVE MAP

CALLED <u>SURFACE PATCH</u>

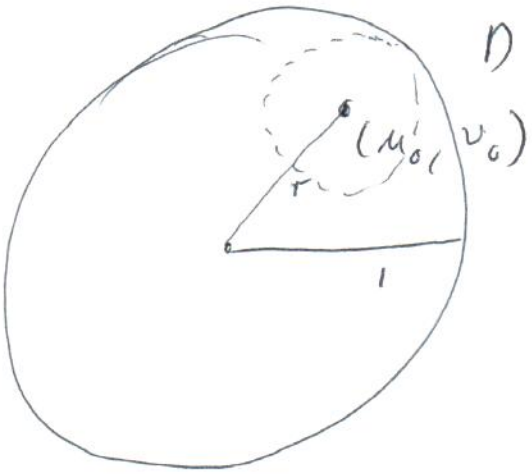
$\mathcal{U} \subset \mathbb{R}^2$ <u>OPEN</u>

$\Leftrightarrow \forall (u_0, v_0) \in \mathcal{U}\ \exists \varepsilon > 0\ \forall (u,v) \in \mathbb{R}^2$:

$\underbrace{\|(u,v) - (u_0,v_0)\|}_{\text{DISTANCE BETWEEN } (u,v),\,(u_0,v_0)} < \varepsilon \Rightarrow (u,v) \in \mathcal{U}$

EXAMPLES 1) $\mathbb{R}^2 = \mathcal{U}$ OPEN IN $\mathbb{R}^2$

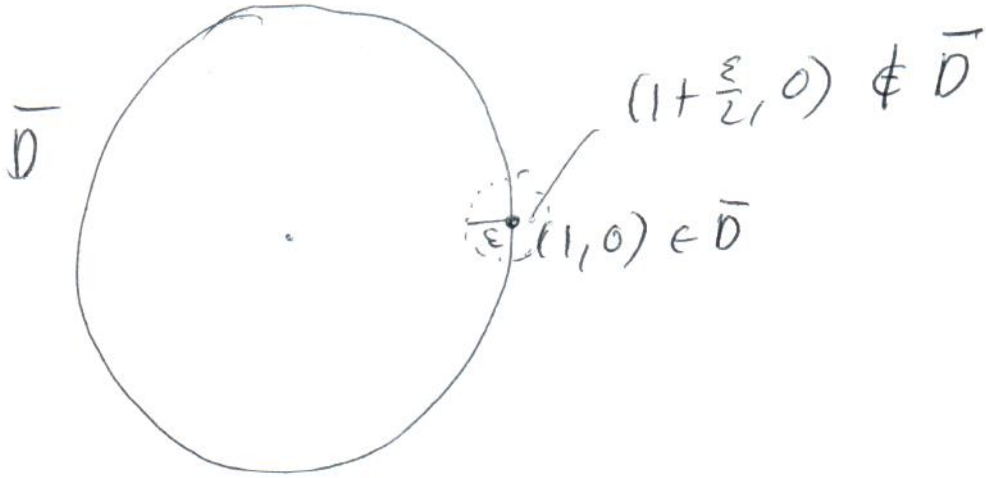
2) $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ OPEN IN $\mathbb{R}^2$



PUT

$\varepsilon = 1 - r$

3) $\bar{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ NOT OPEN IN $\mathbb{R}^2$



σ SMOOTH $\Longleftrightarrow$ $\sigma_1, \sigma_2, \sigma_3$ HAVE CONTINUOUS PARTIAL DERIVATIVES OF ALL ORDER

$\parallel$

$(\sigma_1, \sigma_2, \sigma_3)$

(FOR THIS TO MAKE SENSE WE NEED $\mathcal{U}$ OPEN!)

NOTATION:

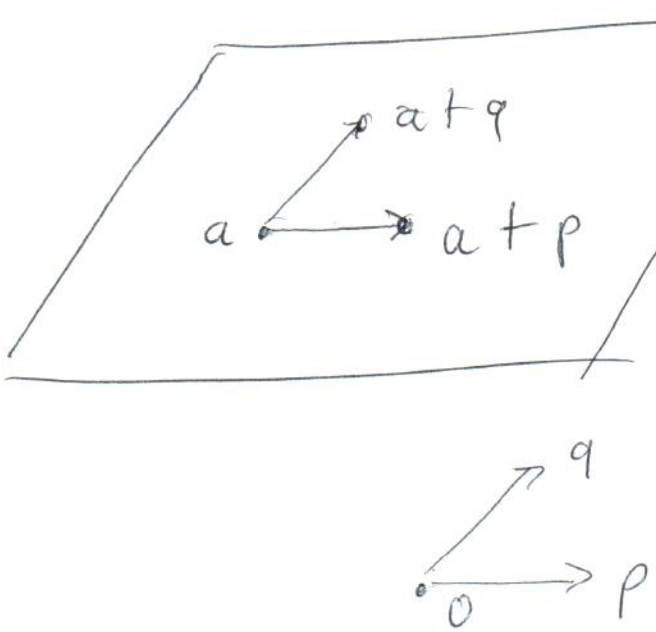
$$\sigma_u = \frac{\partial \sigma}{\partial u}, \quad \sigma_v = \frac{\partial \sigma}{\partial v}$$

$$\frac{\partial^2 \sigma}{\partial u^2} = \sigma_{uu}, \quad \frac{\partial^2 \sigma}{\partial u \partial v} = \sigma_{uv}, \quad \frac{\partial^2 \sigma}{\partial v^2} = \sigma_{vv}$$

AND SO ON...

NOTE: $\sigma_{uv} = \sigma_{vu}$ (SCHWARZ'S THM)

<u>EXAMPLE 4.1.2</u> PLANE $\subset \mathbb{R}^3$



PLANE π

$a \in \pi$

$0 \neq p, q \in \mathbb{R}^3$

$p \parallel \pi$, $q \parallel \pi$

$p \nparallel q$

$$\pi = \{ r = a + up + vq \mid u, v \in \mathbb{R} \}$$

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \ (u, v) \mapsto a + up + vq$$

σ SMOOTH? $\frac{\partial \sigma}{\partial u} = p$, $\frac{\partial \sigma}{\partial v} = q$, $\frac{\partial^2 \sigma}{\partial u^2} = 0, \ldots$

ALL HIGHER DERIVATIVES 0

$\Rightarrow \sigma$ SMOOTH

σ INJECTIVE?

$$\sigma(u, v) = \sigma(u', v')$$

$$\begin{array}{ccc} \sigma(u,v) & = & \sigma(u',v') \\ \shortparallel & & \shortparallel \\ a + up + vq & & a + u'p + v'q \end{array}$$

$\Leftrightarrow \quad (u - u')p = (v' - v)q$

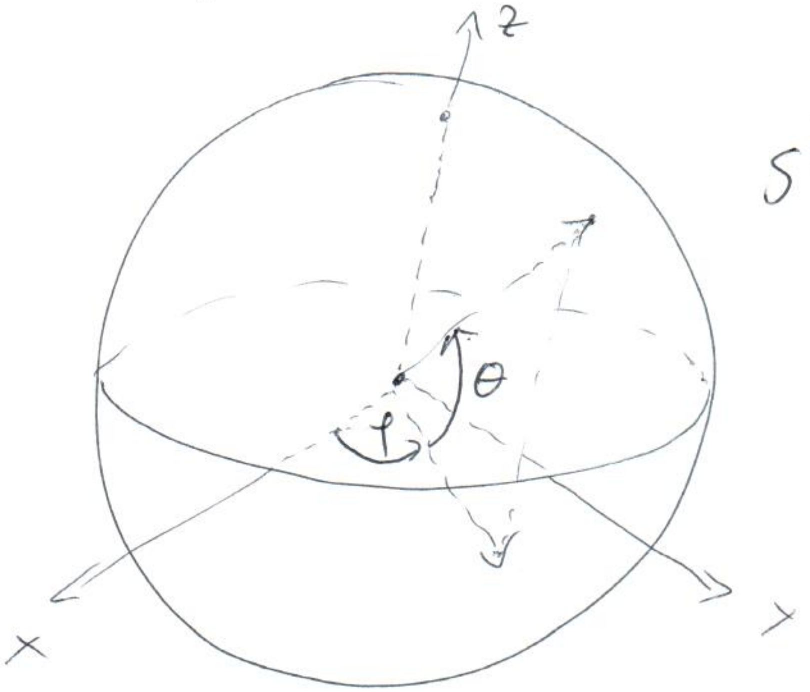
$\underset{p \nparallel q}{\Longleftrightarrow} \quad u = u' \ \& \ v = v'$

WHY "PATCH"? CONSIDER SPHERE

$$\sigma(\theta, \varphi) = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$$

$\theta \sim$ LATITUDE

$\varphi \sim$ LONGITUDE



S^2 UNIT SPHERE

$$\|\sigma(\theta,\varphi)\|^2 = \cos^2(\theta)\cos^2(\varphi) + \cos^2(\theta)\sin^2(\varphi) + \sin^2(\theta)$$

$$= 1$$

$$\Rightarrow \forall \theta, \varphi : \sigma(\theta, \varphi) \in S^2$$

HOWEVER: $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ NOT INJECTIVE
(cos, sin 2π – PERIODIC!)

CLEARLY:

$$S^2 = \left\{ \sigma(\theta,\varphi) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le \varphi \le 2\pi \right\}.$$

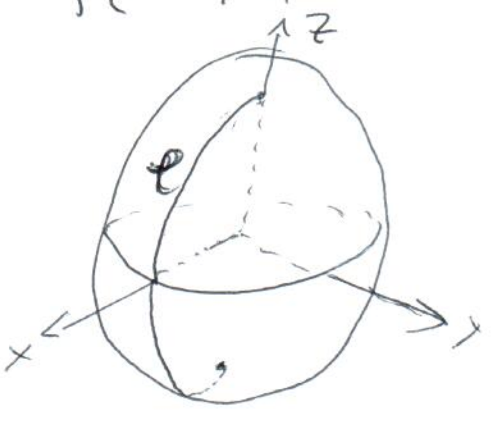
$$\mathcal{U} = \left\{ (\theta,\varphi) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le \varphi \le 2\pi \right\}$$

IS NOT OPEN.

PUT

$$\mathcal{U} = \left\{ (\theta,\varphi) : -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \ 0 < \varphi < 2\pi \right\}$$

$$\sigma(\mathcal{U}) = S^2 \setminus \overbrace{\{(x,0,z) \mid x \ge 0\}}^{=\ell}$$



COVERS ONLY "PATCH" OF SPHERE.

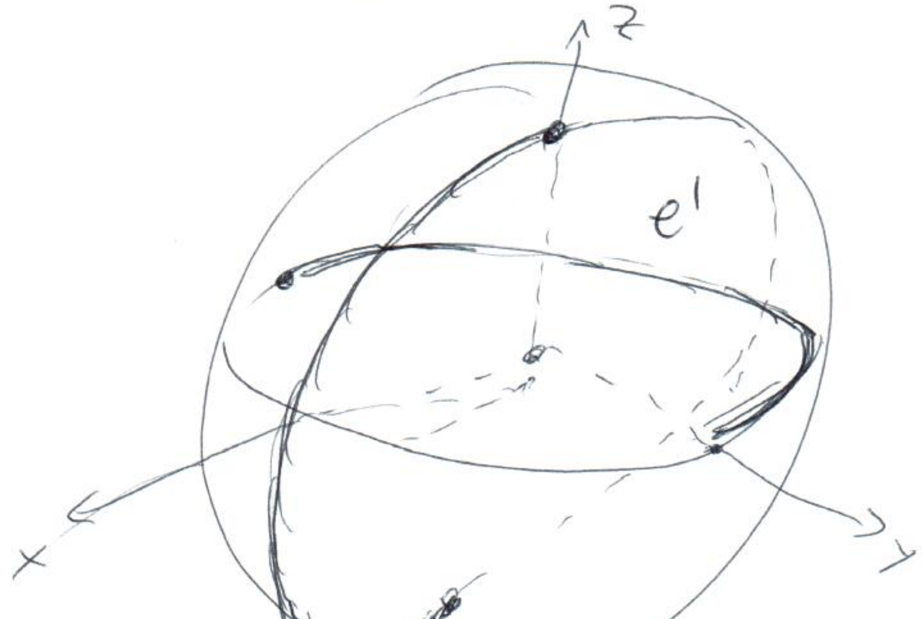
INTUITIVELY CLEAR, BUT NOT EASY TO PROVE:

S^2 CANNOT BE COVERED BY ONE SURFACE PATCH.

CAN BE COVERED BY 2 PATCHES: $\sigma, \tilde{\sigma}$ WITH

$$\tilde{\sigma}: \mathcal{U} \to \mathbb{R}^3, \ \sigma(\theta, \varphi) = \big(-\cos(\theta)\cos(\varphi), -\sin(\theta), -\cos(\theta)\sin(\varphi)\big)$$

$\tilde{\sigma}$ OBTAINED FROM σ BY FIRST ROTATE σ BY π ABOUT z-AXIS THEN BY $\frac{\pi}{2}$ ABOUT x-AXIS



$$\tilde{\sigma}(\theta, \varphi) = S^2 \setminus \{\underbrace{(x, y, 0) \mid x \leq 0}_{= e'}\}$$

SURFACE PATCHES ARE SUFFICIENT TO STUDY "LOCAL" GEOMETRY.

SHOULD BE INDEPENDENT OF CHOICE OF PATCH

DEF 4.1.3 SURFACE PATCH $\tilde{\sigma} : \tilde{\mathcal{U}} \to \mathbb{R}^3$ IS REPARAMETRIZATION OF SURFACE PATCH $\sigma : \mathcal{U} \to \mathbb{R}^3$ IF THERE EXISTS A SMOOTH BIJECTIVE MAP $\Phi : \mathcal{U} \to \tilde{\mathcal{U}}$ (REPARAMETRIZATION MAP) WHOSE INVERSE MAP $\Phi^{-1} : \tilde{\mathcal{U}} \to \mathcal{U}$ IS SMOOTH AND $\forall u, v \in \mathcal{U} : \tilde{\sigma}(\Phi(u,v)) = \sigma(u,v)$