1 CURVES

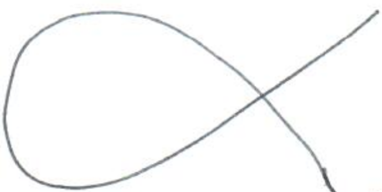
1.1 WHAT IS A CURVE?

IN PLANE

$$y = mx + c$$

$$x^2 + y^2 = 1$$

$$y = x^2$$



$$f(x, y) = 0$$

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$$

IN SPACE

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$$

CURVE $\Leftrightarrow$ PATH TRACED OUT BY MOVING POINT

DEF 1.1.1 A CURVE IN $\mathbb{R}^n$ IS A MAP

$$\gamma : (\alpha, \beta) \to \mathbb{R}^n, \quad -\infty \leq \alpha < \beta \leq \infty$$

IMAGE $\gamma((\alpha,\beta))$ IS A CURVE IN "SET OF POINTS" SENSE

EXAMPLE 1.1.2: PARABOLA

WRITE $\gamma(t) = (\gamma_1(t), \gamma_2(t))$

$$\Rightarrow \quad \gamma_2(t) = \gamma_1(t)^2$$

OBVIOUS SOLUTION:

$$\gamma_1(t) = t, \ \gamma_2(t) = t^2$$

$$\Rightarrow \gamma : (-\infty, \infty) \to \mathbb{R}^2, \ t \mapsto (t, t^2)$$

ANOTHER SOLUTION

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2, \ t \mapsto (t^3, t^6)$$

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2, \ t \mapsto (2t, 4t^2)$$

EXAMPLE 1.1.3 CIRCLE $x^2+y^2=1$

FIRST ATTEMPT:

$x = t \Rightarrow y = \sqrt{1-t^2}$ (OR $-\sqrt{1-t^2}$)

$\Rightarrow \gamma(t) = (t, \sqrt{1-t^2})$, $t \in (-1,1)$

PARAMETRIZES UPPER SEMICIRCLE

$\gamma((-1,1))$

γ MUST SATISFY

$$\gamma_1(t)^2 + \gamma_2(t)^2 = 1$$

OBVIOUS SOLUTION:

$\gamma(t) = (\cos(t), \sin(t))$

$-\infty < t < \infty$, OR $\alpha < t < \alpha + 3\pi$

AIM: STUDY CURVES (AND SURFACES)

USING METHODS OF CALCULUS.

WRITE

$$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$$

THEN

$$\dot{\gamma}(t) = \frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \dots, \frac{d\gamma_n}{dt}\right) = (\gamma_1'(t), \dots, \gamma_n'(t))$$

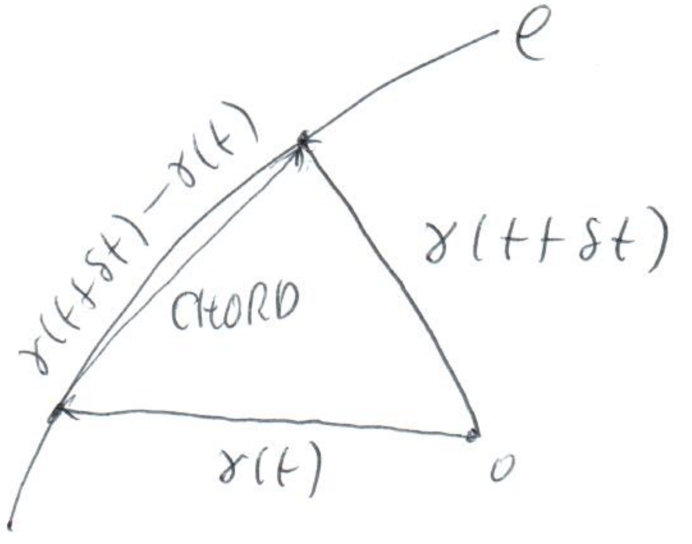
$$\ddot{\gamma}(t) = \frac{d^2\gamma}{dt^2} = \left(\frac{d^2\gamma_1}{dt^2}, \dots, \frac{d^2\gamma_n}{dt^2}\right) = (\gamma_1''(t), \dots, \gamma_n''(t))$$

AND SO ON.

γ SMOOTH $\iff \gamma_1, \dots, \gamma_n$ SMOOTH

$\iff \forall i = 1, \dots, n : \frac{d\gamma_i}{dt}, \frac{d^2\gamma_i}{dt^2}, \frac{d^3\gamma_i}{dt^3}, \dots \cancel{\frac{d^n\gamma_i}{dt^n}}$ EXIST

$\dot{\gamma}(t)$ TANGENT VECTOR OF γ AT $\gamma(t)$



IF $\delta t \to 0$, THEN CHORD BECOMES PARALLEL TO TANGENT OF ℓ AT $\gamma(t)$.

$$\lim_{\delta t \to 0} \frac{\gamma(t+\delta t) - \gamma(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \left(\frac{\gamma_1(t+\delta t) - \gamma_1(t)}{\delta t}, \ldots, \frac{\gamma_n(t+\delta t) - \gamma_n(t)}{\delta t} \right)$$

$$= \left(\lim_{\delta t \to 0} \frac{\gamma_1(t+\delta t) - \gamma_1(t)}{\delta t}, \ldots, \lim_{\delta t \to 0} \frac{\gamma_n(t+\delta t) - \gamma_n(t)}{\delta t} \right)$$

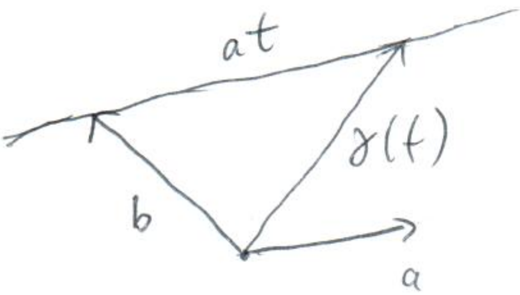
$$= \left(\frac{d\gamma_1}{dt}(t), \ldots, \frac{d\gamma_n}{dt}(t) \right) = \frac{d\gamma}{dt}(t) = \dot{\gamma}(t)$$

PROPOSITION 1.14

$\dot{\gamma}$ CONSTANT $\Rightarrow$ $\gamma((\alpha, \beta))$ PART OF STRAIGHT LINE

PROOF ASSUME $\forall t : \dot{\gamma}(t) = a \in \mathbb{R}^n$

$$\Rightarrow \gamma(t) = \int \frac{d\gamma}{dt}\, dt = \int a\, dt = at + b$$



$a \neq 0$: LINE PARALLEL TO a

$a = 0$: γ CONSTANT $\forall t : \gamma(t) = b$

$\square$

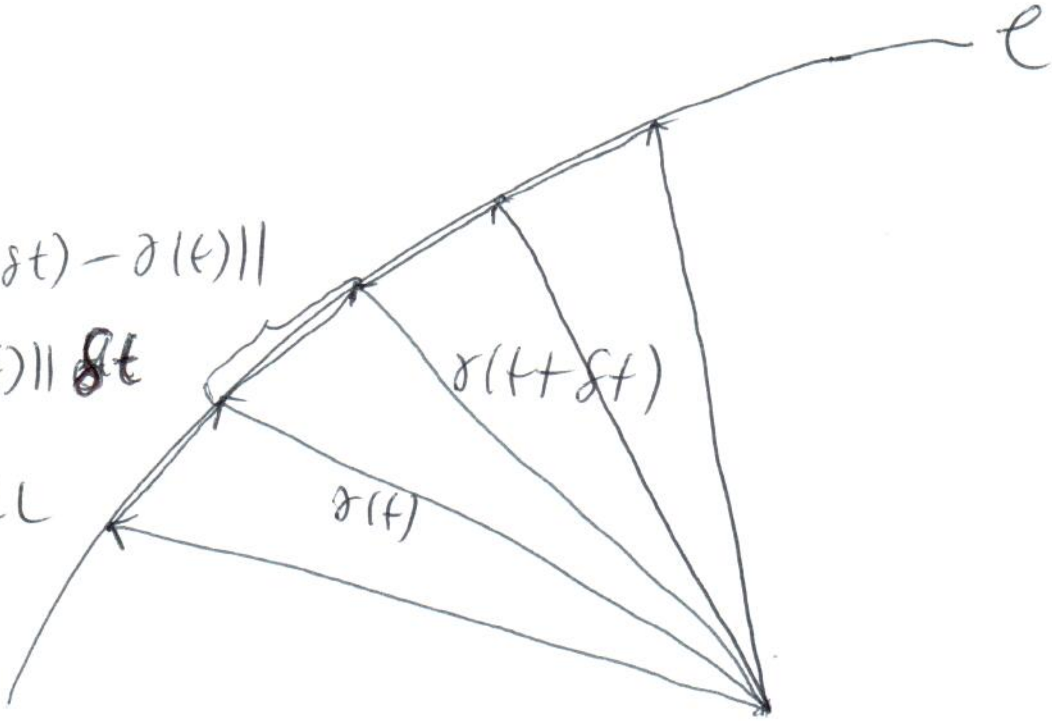
1.2 ARC LENGTH

$\|\gamma(t+\delta t) - \gamma(t)\|$

$\sim \|\dot\gamma(t)\| \delta t$

$\uparrow$

δt SMALL



LENGTH OF $\mathcal{C}$ $\sim$ SUM OF SUCH SEGMENTS

$\delta t \to 0 \leadsto$ EXACT LENGTH

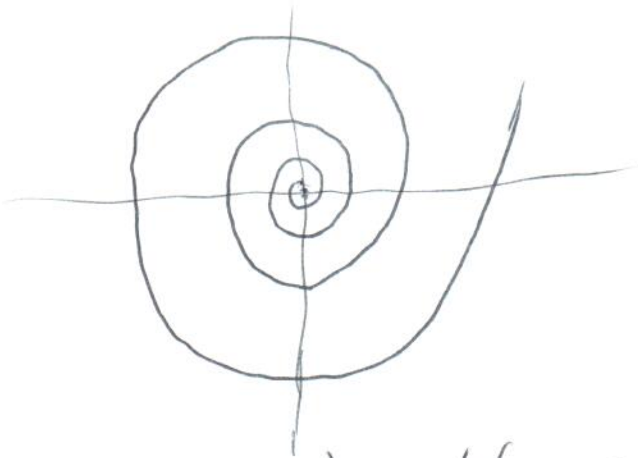
<u>DEF 1.2.1</u> THE ARC-LENGTH OF A CURVE γ STARTING AT $\gamma(t_0)$ IS

$$s(t) = \int_{t_0}^{t} \|\dot\gamma(u)\| \, du$$

NOTE: $s(t) \begin{cases} <0 & \text{IF } t < t_0 \\ =0 & \text{IF } t = t_0 \\ >0 & \text{IF } t > t_0 \end{cases}$ $\qquad \dot\gamma \neq 0$

EXAMPLE 1.2.2 LOGARITHMIC SPIRAL

$$\gamma(t) = (e^t \cos(t), e^t \sin(t)) = e^t (\cos(t), \sin(t))$$



$$\dot{\gamma}(t) = e^t(\cos(t), \sin(t)) + e^t(-\sin(t), \cos(t))$$

$$= e^t(\cos(t) - \sin(t), \sin(t) + \cos(t))$$

$$\Rightarrow \|\dot{\gamma}(t)\|^2 = e^{2t}\left((\cos(t) - \sin(t))^2 + (\sin(t) + \cos(t))^2\right)$$

$$= e^{2t}(2\cos^2(t) + 2\sin^2(t))$$

$$= 2e^{2t}$$

TAKE $t_0 = 0$

$$\Rightarrow s(t) = \int_0^t \sqrt{2e^{2u}}\, du = \sqrt{2}\int_0^t e^u\, du = \sqrt{2}(e^t - 1)$$

REMARKS $\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du = \|\dot{\gamma}(t)\|$ SPEED OF γ

RATE OF CHANGE OF DISTANCE ALONG γ.

1.3 UNIT SPEED REPARAMETRIZATION

RECALL: CURVE MAY HAVE MANY PARAMETRIZATIONS.

CAN WE FIND A DISTINGUISHED ONE?

DEF 1.3.1 A CURVE $\bar{\gamma}$ IS A REPARAMETRIZATION OF A CURVE $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ IF THERE EXISTS A SMOOTH FUNCTION $\phi : (\alpha, \beta) \to \mathbb{R}$ (THE SO-CALLED REPARAMETRIZATION MAP) SO THAT

(i) $\forall t \in (\alpha, \beta) : \phi'(t) = \frac{d\phi}{dt} \neq 0$.

(ii) $\forall t \in (\alpha, \beta) : \bar{\gamma}(\phi(t)) = \gamma(t)$

REMARK: (i) TELLS US THAT WE CAN APPLY THE INVERSE FUNCTION THM:

$\exists \bar{\alpha}, \bar{\beta} \in \mathbb{R} : \phi((\alpha, \beta)) = (\bar{\alpha}, \bar{\beta})$,

$\phi : (\alpha, \beta) \to (\bar{\alpha}, \bar{\beta})$ BIJECTION

$\phi^{-1} : (\bar{\alpha}, \bar{\beta}) \to (\alpha, \beta)$ SMOOTH

$(\phi^{-1} \circ \Phi)(t) = t$

$\underset{\text{CHAIN RULE}}{\Rightarrow} (\Phi^{-1})'(\phi(t)) \cdot \phi'(t) = 1$

$\Rightarrow (\phi^{-1})'(\phi(t)) = \dfrac{1}{\phi'(t)} \neq 0$

$\Rightarrow$ ϕ^{-1} REPARAMETRIZATION MAP

γ REPARAMETRIZATION OF $\bar{\gamma}$:

$\forall \bar{t} \in (\bar{\alpha}, \bar{\beta}): \gamma(\phi^{-1}(\bar{t})) = \bar{\gamma}(\phi(\phi^{-1}(\bar{t}))) = \bar{\gamma}(\bar{t})$

<u>EXAMPLE 1.3.2</u> CIRCLE $\gamma(t) = (\cos(t), \sin(t))$

CAN ALSO WRITE $\bar{\gamma}(t) = (\sin(t), \cos(t))$

CLAIM: $\bar{\gamma}$ REPARAMETRIZATION OF γ

HAVE TO FIND ϕ WITH $\bar{\gamma}(\phi(t)) = \gamma(t)$:

$(\sin(\phi(t)), \cos(\phi(t))) = (\cos(t), \sin(t))$

A SOLUTION IS $\phi(t) = \frac{\pi}{2} - t$

NOTE $\phi'(t) = -1 \neq 0$.

UNIT SPEED $\|\dot\gamma\| = 1$ IS CONVENIENT!

REASON:

PROPOSITION 1.3.3 LET $\gamma(t)$ BE UNIT SPEED. THEN

$$\dot\gamma(t) \cdot \ddot\gamma(t) = 0$$

PROOF FOLLOWS FROM PRODUCT FORMULA:

$$\|\dot\gamma\| = 1 \Rightarrow \dot\gamma \cdot \dot\gamma = \|\dot\gamma\|^2 = 1$$

$$\Rightarrow \ddot\gamma \cdot \dot\gamma + \dot\gamma \cdot \ddot\gamma = 0 \quad \Rightarrow \dot\gamma \cdot \ddot\gamma = 0 \quad \square.$$

WHICH CURVES DO HAVE UNIT SPEED REPARAMETRIZATIONS?

PROP 1.3.4 CURVE γ HAS UNIT SPEED REPARAMETRIZATION $\Leftrightarrow \forall t: \dot\gamma(t) \neq 0$.

PROOF: „$\Rightarrow$" ASSUME $\exists \overbrace{\phi}^{\text{REPARA MAP}} : \cancel{\bar\gamma = \gamma\circ\phi}$ $\bar\gamma\circ\phi = \gamma$, $\|\dot{\bar\gamma}\| = 1$

PUT $u = \phi(t)$. THEN $\bar\gamma(u) = \gamma(t)$

$$\underset{\text{CHAIN RULE}}{\Longrightarrow} \quad \frac{d\bar\gamma}{du}\frac{du}{dt} = \frac{d\gamma}{dt}$$

$$\Longrightarrow \left\|\frac{d\bar\gamma}{du}\frac{du}{dt}\right\| = \left\|\frac{d\gamma}{dt}\right\|$$

$$= \quad \underbrace{\left\|\frac{d\bar\gamma}{du}\right\|}_{=1}\cdot\left|\frac{du}{dt}\right| = \left|\frac{du}{dt}\right|$$

$$\Rightarrow \underset{\neq 0}{\frac{du}{dt}} = \pm\overbrace{\left\|\frac{d\gamma}{dt}\right\|}^{=\|\dot\gamma(t)\|}$$

BY ASSUMPTION

„$\Leftarrow$" ASSUME $\forall t : \dot\gamma(t) \neq 0$.

$$\Rightarrow \frac{ds}{dt} = \|\dot\gamma(t)\| > 0 \qquad (s \text{ arc length})$$

$\Rightarrow$ s REPARAMETRIZATION MAP:

WRITE $\bar\gamma\circ s = \gamma$ $\quad(\bar\gamma = \gamma\circ s^{-1})$

$$\Rightarrow \frac{d\bar\gamma}{ds}\frac{ds}{dt} = \frac{d\gamma}{dt}$$

$$\Rightarrow \left\|\frac{d\bar\gamma}{ds}\right\|\frac{ds}{dt} = \left\|\frac{d\gamma}{dt}\right\| = \frac{ds}{dt}$$

$$\Rightarrow \left\|\frac{d\bar\gamma}{ds}\right\| = 1 \qquad \square.$$

THIS MOTIVATES

DEF 1.3.5 CURVE γ REGULAR IF $\forall t: \dot{\gamma}(t) \neq 0$

COR 1.3.6 γ REGULAR WITH $\bar{\gamma}$ UNIT SPEED REPARAMETRIZATION:

$$\forall t: \bar{\gamma}(u(t)) = \gamma(t)$$

THEN

$$\exists c \in \mathbb{R}: u = \pm s + c \quad (s \text{ arc length})$$

CONVERSELY, IF u IS AS ABOVE, THEN $\bar{\gamma}$ IS A UNIT SPEED REPARAMETRIZATION OF γ

PROOF "AS SHOWN IN PROOF OF PROP 1.3.4:

u UNIT SPEED REPARA OF γ

$$\underset{\text{PROOF OF PROP 1.3.4}}{\iff} \frac{du}{dt} = \pm\left\|\frac{d\gamma}{dt}\right\| = \pm\frac{ds}{dt}$$

$$\iff u = \pm s + c \qquad \square$$

EXAMPLE 1.3.7 LOGARITHMIC SPIRAL

$$\gamma(t) = e^t(\cos(t), \sin(t))$$

THEN $\|\dot{\gamma}(t)\|^2 = 2e^{2t}$, SO γ REGULAR

ARC LENGTH $s = \sqrt{2}(e^t - 1)$ (SEE EX 1.2.2)

$$\Rightarrow t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$\Rightarrow$ UNIT SPEED REPARA OF γ IS

$$\tilde{\gamma}(s) = \left(\frac{s}{\sqrt{2}}+1\right)\left(\cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\right).$$

EXAMPLE 1.3.8 TWISTED CUBIC

$$\gamma(t) = (t, t^2, t^3) \qquad t \in (-\infty, \infty)$$

$$\dot{\gamma}(t) = (1, 2t, 3t^2)$$

$\|\dot{\gamma}(t)\| = \sqrt{1 + 4t^2 + 9t^4} \neq 0 \to \gamma$ REGULAR

ARC LENGTH (STARTING AT 0)

$$s(t) = \int_0^t \sqrt{1 + 4u^2 + 9u^4}\, du$$ ELLIPTIC INTEGRAL

UNIT SPEED REPARA CANNOT BE WRITTEN DOWN EXPLICITLY.

EXAMPLE 1.3.9 PARABOLA

$\gamma(t) = (t, t^2)$ IS PARAMETRIZATION OF PARABOLA

$\dot{\gamma}(t) = (1, 2t)$

$\|\dot{\gamma}(t)\| = \sqrt{1+4t^2} \neq 0 \Rightarrow \gamma$ REGULAR

$\gamma_1(t) = (t^3, t^6)$ IS PARAMETRIZATION OF PARABOLA

$\dot{\gamma}_1(t) = (3t^2, 6t^5)$

$\|\dot{\gamma}_1(t)\| = \sqrt{9t^4 + 36t^{10}} = 0$ FOR $t = 0$

$\Rightarrow \gamma_1$ NOT REGULAR

THUS A CURVE CAN HAVE REGULAR AND NON-REGULAR PARAMETRIZATIONS.