INTERPRETATION USING PARTIAL DERIVATIVES:

PUT $\Psi := \Phi^{-1}$, $(\tilde{u}, \tilde{v}) = \Phi(u, v)$

$(\Rightarrow \ (u, v) = \Psi(\tilde{u}, \tilde{v}))$

JACOBIAN MATRICES:

$$J\Phi = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$$

$$J\Psi = J\Phi^{-1} = (J\Phi)^{-1} = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

BY CHAIN RULE

CONVERSELY, BY INVERSE FCTN THM

$$\left.\begin{array}{l} \Phi : \mathcal{U} \to \tilde{\mathcal{U}} \text{ BIJECTIVE} \\ \qquad \text{SMOOTH} \\ J\Phi \text{ INVERTIBLE} \\ \qquad \text{EVERYWHERE} \end{array}\right\} \Rightarrow \Phi^{-1} \text{ SMOOTH.}$$

EXAMPLE 4.1.3.

$\hat{\mathcal{U}} = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$

$\tilde{\sigma} : \tilde{\mathcal{U}} \to \mathbb{R}^3, (u,v) \mapsto (u, \sqrt{1-u^2-v^2}, v)$

$\tilde{\sigma}(\tilde{\mathcal{U}}) \subset S^2$ AS $u^2+(1-u^2-v^2)+v^2 = 1$

$\tilde{\mathcal{U}}$ OPEN IN $\mathbb{R}^2$

$\tilde{\sigma}$ SMOOTH INJECTIVE

$\tilde{\sigma} : \hat{\mathcal{U}} \to \mathbb{R}^3$ IS SURFACE PATCH OF S^2

WITH $\tilde{\sigma}(\tilde{\mathcal{U}}) = \{(x,y,z) \in S^2 \mid y > 0\}$

ANOTHER PATCH:

$\sigma : \mathcal{U} \to \mathbb{R}^3, (\theta,\varphi) \mapsto (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$

$\mathcal{U} = \{(\theta,\varphi) \in \mathbb{R}^2 \mid -\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \varphi < \pi\}.$

CLAIM: $\tilde{\sigma}$ IS A REPARAMETRIZATION OF σ.

NEED TO FIND $\Phi : \mathcal{U} \to \tilde{\mathcal{U}}$ WITH SMOOTH INVERSE $\Phi^{-1} : \hat{\mathcal{U}} \to \mathcal{U}$ AND

$$\tilde{\sigma}(\underbrace{\Phi(\theta,\varphi)}_{=:(u,v)}) = \sigma(\theta,\varphi)$$

$$\Rightarrow \begin{cases} u = \cos(\theta)\cos(\varphi) \\ \sqrt{1-u^2-v^2} = \cos(\theta)\sin(\varphi) \\ v = \sin(\theta) \end{cases}$$

$$\Rightarrow \Phi(\theta,\varphi) = (\cos(\theta)\cos(\varphi), \sin(\theta)) \text{ SMOOTH}$$

Φ BIJECTION (EASY TO CHECK)

$$J(\Phi) = \begin{pmatrix} -\sin(\theta)\cos(\varphi) & -\cos(\theta)\sin(\varphi) \\ \cos(\theta) & 0 \end{pmatrix}$$

$$\Rightarrow \det(J(\Phi)) = \cos^2(\theta)\sin(\varphi) \underset{\uparrow}{\neq} 0$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2},\ 0 < \varphi < \pi$$

$\Rightarrow J(\Phi)$ INVERTIBLE

$\Rightarrow \Phi$ REPARAMETRIZATION MAP

(Φ^{-1} SMOOTH BY INVERSE FUNCTION THM).

4.2 REGULAR SURFACES AND THEIR TANGENT PLANES

STUDY SURFACES USING CURVES IN SURFACES (WHICH ARE CURVES IN $\mathbb{R}^3$)

LET $\sigma : \mathcal{U} \to \mathbb{R}^3$ SURFACE PATCH

CURVE ON σ IS

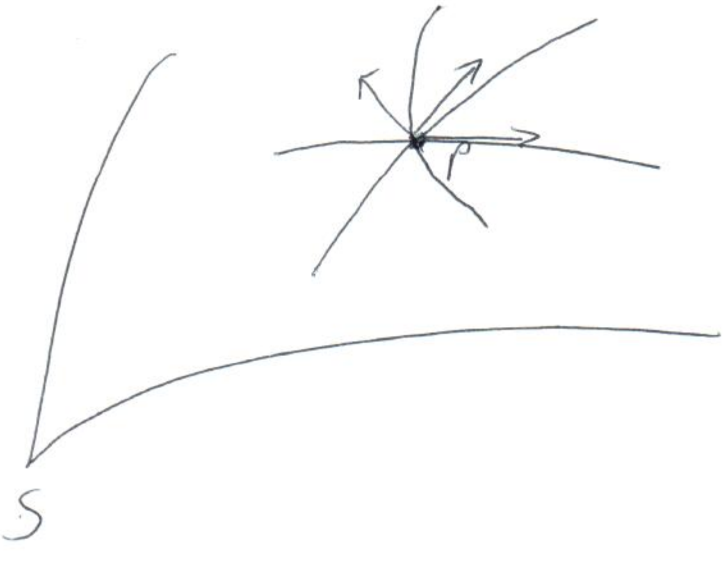
$$\gamma : (\alpha, \beta) \to \mathbb{R}^3$$
$$t \mapsto \gamma(t) = \sigma(u(t), v(t))$$

DEF 4.2.1 $\sigma : \mathcal{U} \to \mathbb{R}^3$ SURFACE PATCH

$p \in S = \sigma(\mathcal{U}) \subset \mathbb{R}^3$.

TANGENT SPACE TO S AT p

$= \{ \dot\gamma(t_0) \mid \gamma \text{ CURVE ON } \sigma, \gamma(t_0) = p \} = T_p S$



<u>PROP 4.2.2</u> IF $p = \sigma(u_0, v_0)$, THEN

$$T_pS = \mathrm{SPAN}\{\sigma_u(u_0,v_0), \sigma_v(u_0,v_0)\}$$

<u>PROOF</u> "$\subseteq$": $\gamma(t) = \sigma(u(t), v(t))$ SMOOTH CURVE IN S $\gamma(t_0) = p$

$\Rightarrow \dot\gamma = \sigma_u \dot u + \sigma_v \dot v$ $\quad \dot{} = \frac{d}{dt}$

$\Rightarrow \dot\gamma(t_0) \in \mathrm{SPAN}\{\sigma_u(u_0,v_0), \sigma_v(u_0,v_0)\}$.

"$\supseteq$": ~~LET~~ TAKE $(\xi\sigma_u + \eta\sigma_v)(u_0,v_0)$ AND DEFINE

$\gamma(t) = \sigma(u_0 + \xi t, v_0 + \eta t)$ ~~SMO~~

γ SMOOTH ($\checkmark$), $\gamma(0) = \sigma(u_0,v_0) = p$

$\dot\gamma(0) = (\xi\sigma_u + \eta\sigma_v)(u_0,v_0) \in T_pS$. $\square$

$\dim(S) = 2 \Longrightarrow \dim(T_pS)$ SHOULD BE 2

$\Updownarrow$

σ_u, σ_v LINEARLY INDEPENDENT AT (u_0,v_0)

$\Updownarrow$

$\sigma_u \times \sigma_v \neq 0$ AT (u_0,v_0)

DEF 4.2.3 $\sigma : U \to \mathbb{R}^3$ REGULAR

$$\Leftrightarrow \forall (u,v) \in U : (\sigma_u \times \sigma_v)(u,v) \neq 0.$$

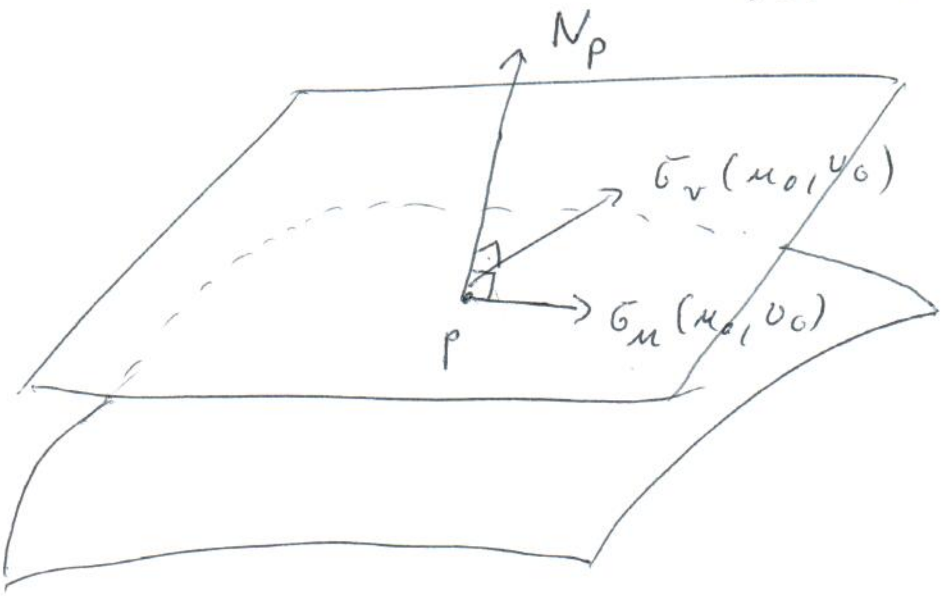
IN THIS CASE CALL T_pS THE TANGENT PLANE OF S AT p.

$\sigma_u \times \sigma_v \perp \sigma_u, \sigma_v$

$$\Rightarrow N_p = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}(u_0, v_0) \perp T_pS$$

UNIT NORMAL TO σ (OR S)

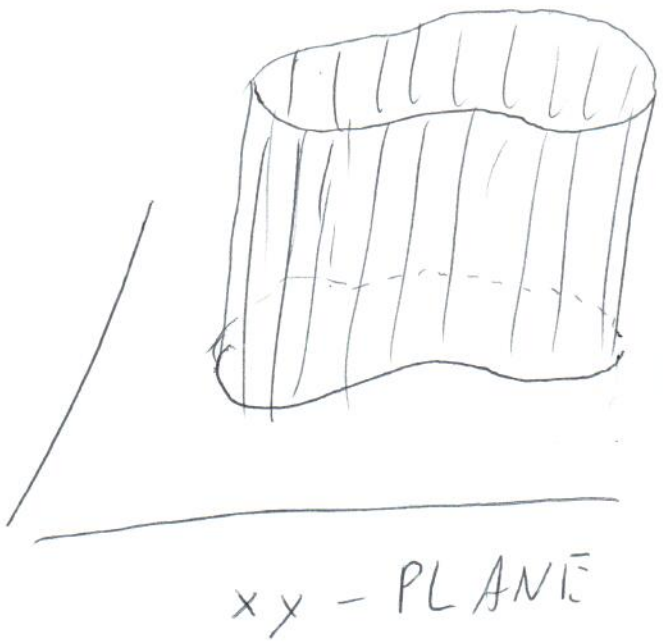
REMARK: CAN BE SHOWN TO BE INDEPENDENT OF PARAMETRIZATION. (UP TO SIGN)



4.3 EXAMPLES OF SURFACES

EX 4.3.1 (GENERALIZED CYLINDER)

TRANSLATE PLANE CURVE PERPENDICULAR TO PLANE



$\gamma(u) = (f(u), g(u), 0)$

SMOOTH CURVE IN $\mathbb{R}^2$, REGULAR

$$\sigma(u,v) = (f(u), g(u), v)$$

σ SMOOTH SINCE γ SMOOTH

σ INJECTIVE $\iff$ γ INJECTIVE

$\iff$ γ DOES NOT SELF-INTERSECT

$\sigma_u = (\dot{f}, \dot{g}, 0)$, $\sigma_v = (0,0,1)$ $\quad \dot{} = \frac{d}{du}$

$\sigma_u \times \sigma_v = (\dot{g}, -\dot{f}, 0) = 0 \iff \dot{f} = 0 = \dot{g}$

$\cancel{\iff}$ CANNOT HAPPEN

σ REGULAR $\iff$ γ REGULAR

EXAMPLE : γ CIRCLE

$$\gamma(u) = (\cos(u), \sin(u), 0)$$

$$\Rightarrow \sigma(u,v) = (\cos(u), \sin(u), v)$$

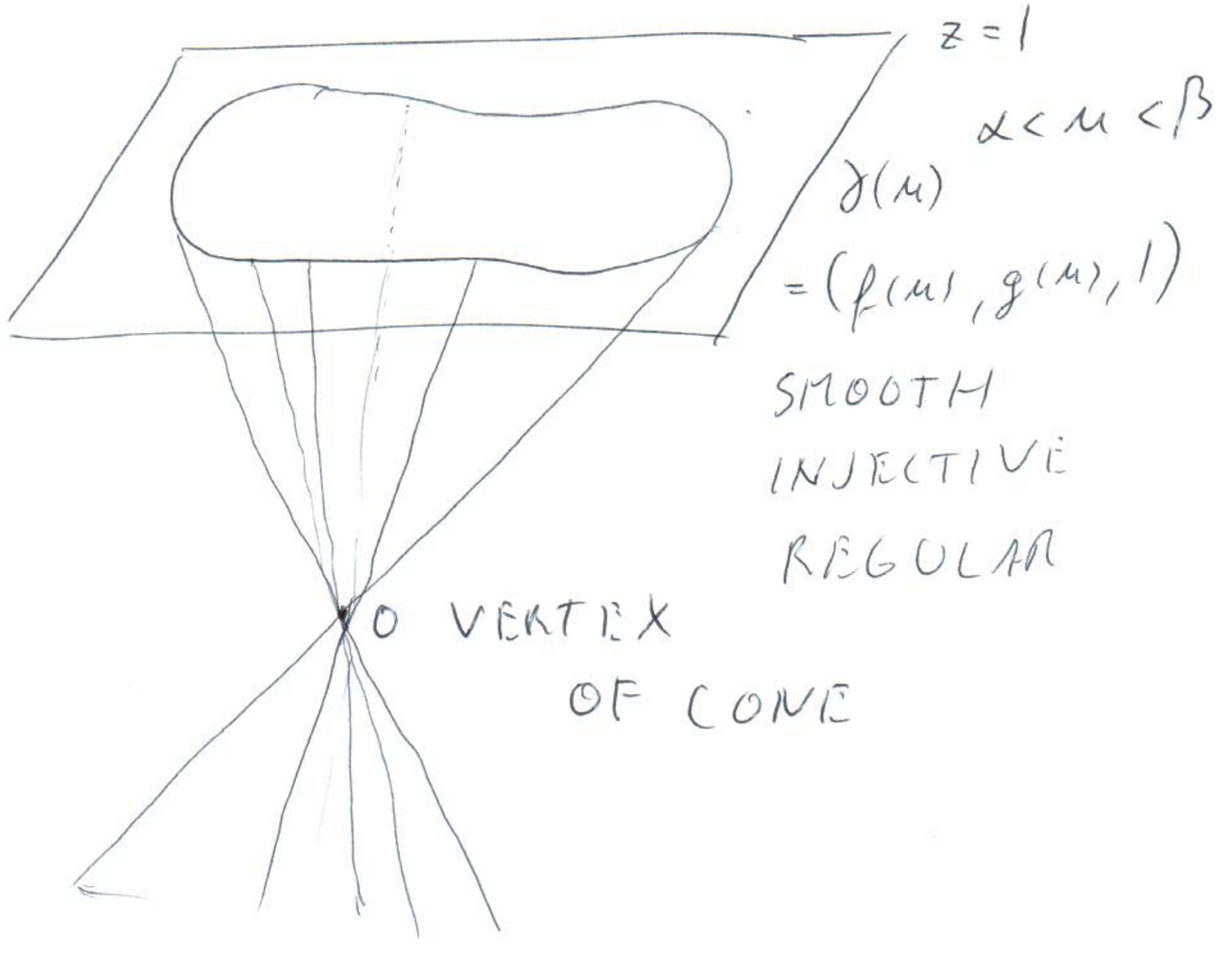
<u>CIRCULAR CYLINDER</u>

σ INJECTIVE REQUIRES u IN OPEN INTERVAL OF LENGTH $\leq 2\pi$, E.G.:

$$\mathcal{U} = \{(u,v) \in \mathbb{R}^2 \mid 0 < u < 2\pi\}$$

EXAMPLE 4.3.2 (GENERALIZED CONE)

UNION OF LINES PASSING THROUGH A FIXED POINT AND THE POINTS OF A PLANE CURVE NOT PASSING THROUGH POINT.



$$\sigma(u,v) = v\,(f(u), g(u), 1)$$
$$= (f(u)v, g(u)v, v)$$

$$\sigma_u = (\dot{f}(u)v, \dot{g}(u)v, 0)$$

$$\sigma_v = (f(u), g(u), 1)$$

$$(\sigma_u \times \sigma_v)(u,v) = \left(\dot{g}(u)v, -\dot{f}(u)v, v\dot{f}(u)g(u) - v\dot{g}(u)f(u)\right)$$

$$= v\left(\dot{g}(u), -\dot{f}(u), \dot{f}(u)g(u) - \dot{g}(u)f(u)\right)$$

$$(\sigma_u \times \sigma_v)(u,v) = 0 \underset{\gamma \text{ REGULAR}}{\iff} v = 0 \quad \text{VERTEX OF CONE}$$

PUT

$$\mathcal{U} = \{(u,v) \in \mathbb{R}^2 \mid \alpha < u < \beta, v \neq 0\}$$

$\Rightarrow$ σ REGULAR.

<u>EXAMPLE 4.3.3.</u> (QUADRIC SURFACE)

$v = (x,y,z)$, $A \in M_3(\mathbb{R})$ SYMMETRIC, $b \in \mathbb{R}^3$, $c \in \mathbb{R}$

$$vAv^T + bv^T + c = 0$$

$$Av \cdot v + b \cdot v + c = 0$$

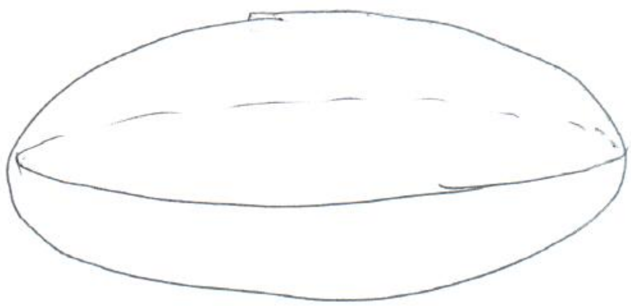
WRITE $A = \begin{pmatrix} a_1 & a_4 & a_6 \\ a_4 & a_2 & a_5 \\ a_6 & a_5 & a_3 \end{pmatrix}$, $b = (b_1, b_2, b_3)$

$$a_1x^2 + a_2y^2 + a_3z^2 + 2a_4xy + 2a_5yz + 2a_6xz$$

$$+ b_1x + b_2y + b_3z + c = 0$$

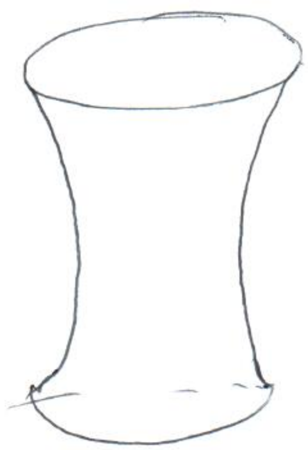
USING LINEAR ALGEBRA & GEOMETRY, THE FOLLOWING (NON-DEGENERATE) CASES CAN OCCUR

(i) ELLIPSOID: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$



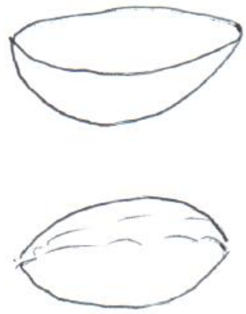
(ii) HYPERBOLOID OF ONE SHEET:

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$$



(iii) HYPERBOLOID OF TWO SHEETS

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$$

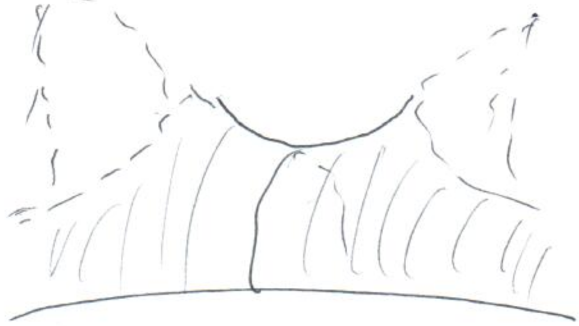


(iv) ELLIPTIC PARABOLOID

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$$

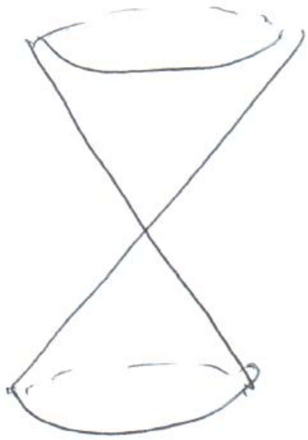
(v) HYPERBOLIC PARABOLOID

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$$



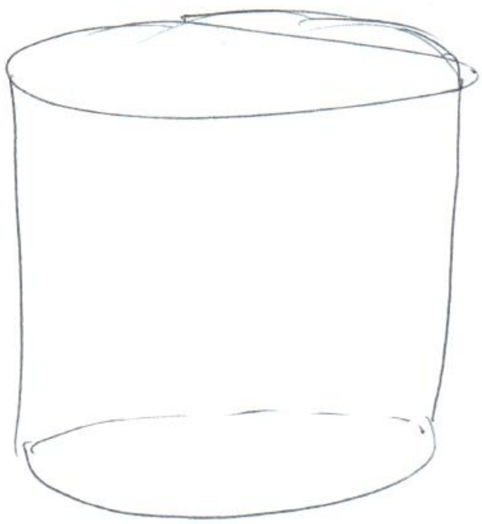
(vi) QUADRIC CONE

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$$



(vii) ELLIPTIC CYLINDER

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$$

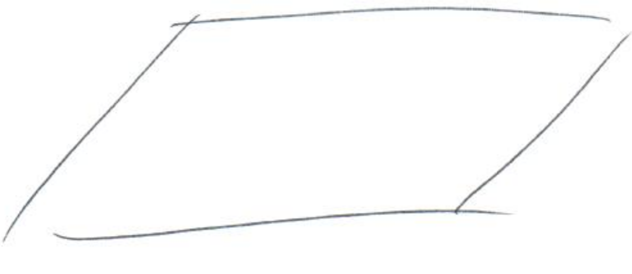


(viii) HYPERBOLIC CYLINDER

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$$

(ix) PARABOLIC CYLINDER $\frac{x^2}{p^2} = y$

(x) PLANE $z = 0$



MUST FIND PARAMETRIZATIONS.

NOT DIFFICULT, BUT TEDIOUS.

FOR ELLIPSOID:

$$\sigma(\theta, \varphi) = (p\cos(\theta)\cos(\varphi), q\cos(\theta)\sin(\varphi), r\sin(\theta))$$

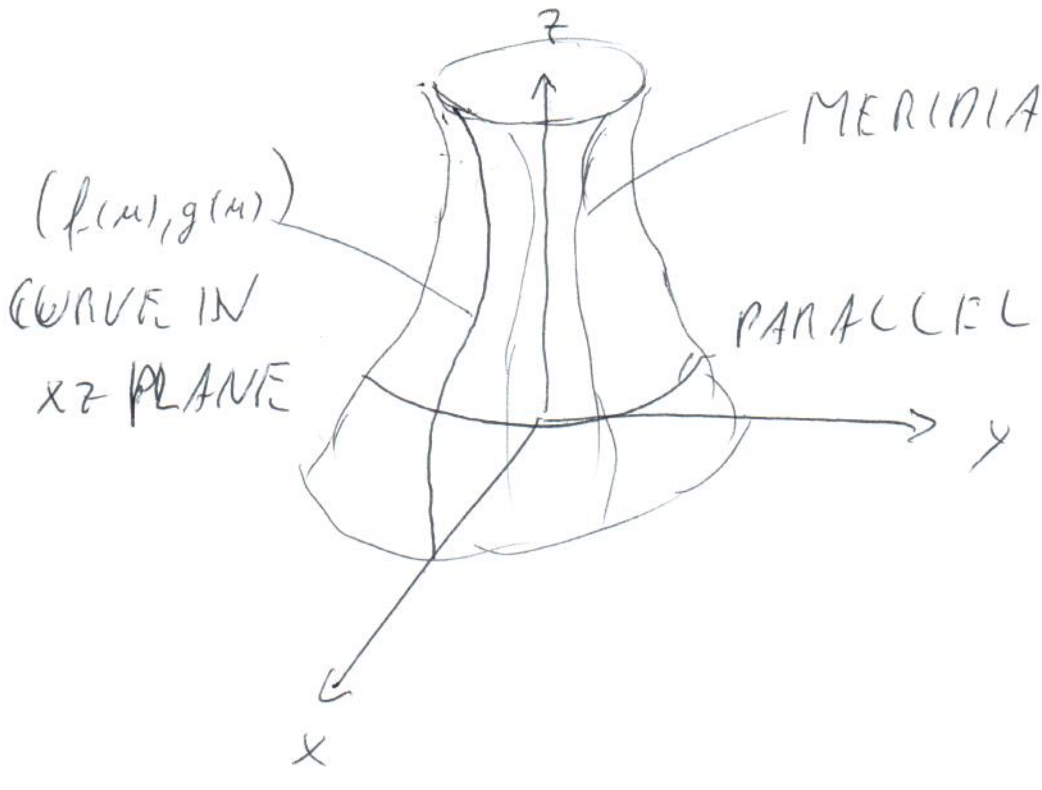
HYPERBOLOID OF ONE SHEET:

$$\sigma(\theta, \varphi) = (p\cos(\theta), q\sin(\theta)\cosh(\varphi), r\sin(\theta)\sinh(\varphi))$$

$\rightsquigarrow$ EXERCISES.

EXAMPLE 4.3.4 (SURFACE OF REVOLUTION)

ROTATE PLANE CURVE (PROFILE CURVE) AROUND STRAIGHT LINE IN PLANE



PROFILE CURVE $\gamma(u) = (f(u), 0, g(u))$

ROTATE BY ANGLE v ABOUT z-AXIS:

$$\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

$u \equiv \text{const}$ PARALLELS

$v \equiv \text{const}$ MERIDIANS

$$\sigma_u(u,v) = (\dot f(u)\cos(v), \dot f(u)\sin(v), \dot g(u))$$

$$\sigma_v(u,v) = (-f(u)\sin(v), f(u)\cos(v), 0)$$

$$(\sigma_u \times \sigma_v)(u,v) = (-f(u)\dot g(u)\cos(v), -f(u)\dot g(u)\sin(v), f(u)\dot f(u))$$

$$\|(\sigma_u \times \sigma_v)(u,v)\|^2 = \underbrace{f^2(u)}_{\neq 0 \text{ IF } \gamma \text{ DOES NOT INTERSECT } z\text{-AXIS}}\underbrace{(\dot f^2(u) + \dot g^2(u))}_{\neq 0 \text{ IF } \gamma \text{ REGULAR}}$$

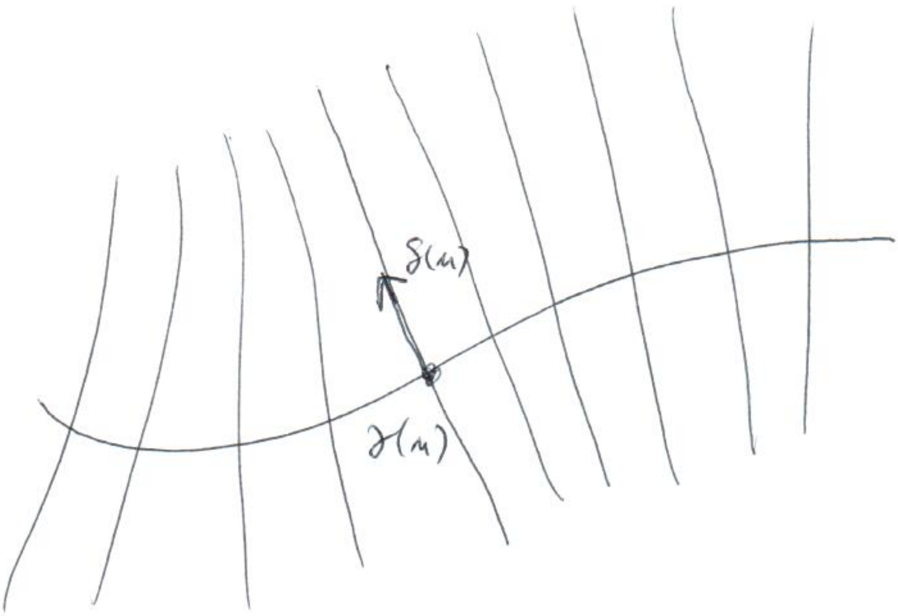
SPECIAL CASES: SPHERES,

CIRCULAR CYLINDERS

CIRCULAR CONES.

EXAMPLE 4.3.5 (RULED SURFACES)

SURFACE THAT IS UNION OF STRAIGHT LINES



$$\sigma(u,v) = \gamma(u) + v\,\delta(u)$$

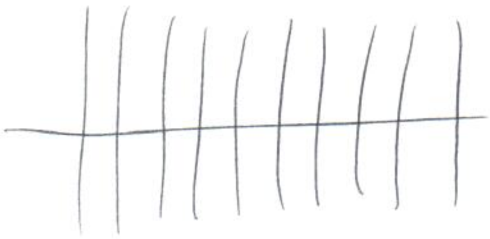
$$\sigma_u(u,v) = \dot{\gamma}(u) + v\,\dot{\delta}(u)$$

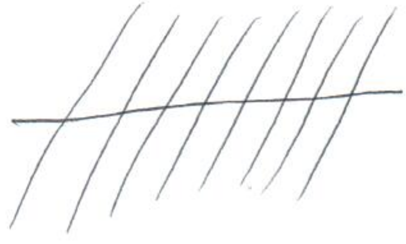
$$\sigma_v(u,v) = \delta(u)$$

σ REGULAR $\iff$ $\dot{\gamma}(u) + v\,\dot{\delta}(u)$, $\delta(u)$ LINEARLY INDEPENDENT

OK IF $\dot{\gamma}, \delta$ LINEARLY INDEPENDENT AND v SUFFICIENTLY SMALL.

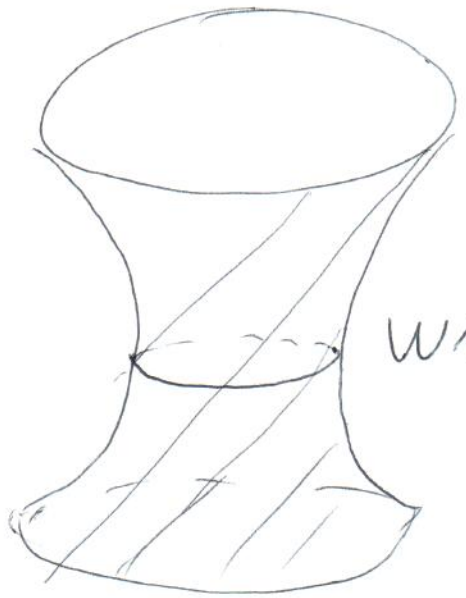
SPECIAL CASE: PLANE





GENERALIZED CYLINDERS } OBVIOUS
GENERALIZED CONES } FROM DEF

HYPERBOLOID OF ONE SHEET (LESS OBVIOUS)



DOUBLY RULED

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$$

$z=0$: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ "WAIST" OF HYPERBOLOID

PARAMETRIZED BY

$$\gamma(u) = (p\cos(u), q\sin(u), 0)$$

LET $\delta = (a, b, c) \neq 0 \in \mathbb{R}^3$

STRAIGHT LINE

$$\gamma(u) + v\delta = (av + p\cos(u), bv + q\sin(u), cv)$$

CONTAINED IN HYPERBOLOID IF

$$\frac{(av + p\cos(u))^2}{p^2} + \frac{(bv + q\sin(u))^2}{q^2} - \frac{c^2 v^2}{r^2} = 1$$

$$\shortparallel$$

$$\underbrace{\left(\frac{a^2}{p^2} + \frac{b^2}{q^2} - \frac{c^2}{r^2}\right)}v^2 + 2v\underbrace{\left(\frac{a}{p}\cos(u) + \frac{b}{q}\sin(u)\right)} + 1$$

MUST BE $= 0$

$a = \lambda p \sin(u)$

$b = -\lambda q \cos(u)$

$$0 = \lambda^2 \sin^2(u) + \lambda^2 \cos^2(u) - \frac{c^2}{r^2} = \lambda^2 - \frac{c^2}{r^2}$$

$$\Rightarrow \lambda = \pm \frac{c}{r}$$

$$\Rightarrow \delta(u) = \frac{c}{r}(p\sin(u), -q\cos(u), r).$$

$\lambda = \frac{c}{r}$

CAN TAKE $c = r$:

$$G(u,v) = \Big(p\big(\cos(u) + v\sin(u)\big),\ q\big(\sin(u) - v\cos(u)\big),\ rv \Big)$$

"RULED" PARAMETRIZATION OF HYPERBOLOID.

TAKE $\lambda = -\frac{c}{r}$:

$$G(u,v) = \Big(p\big(\cos(u) - v\sin(u)\big),\ q\big(\sin(u) + v\cos(u)\big),\ rv \Big)$$

ANOTHER "RULED" PARAMETRIZATION

REMARK: ONE CAN SHOW THAT EVERY DOUBLY RULED SURFACE IS A QUADRIC SURFACE.