7.2 THE GAUSS MAP

$\kappa_s = \dfrac{d\varphi}{ds}$ SIGNED CURVATURE

MEASURES RATE OF CHANGE OF DIRECTION OF $\dot\gamma$ (PER UNIT LENGTH)

GENERALIZE THIS TO SURFACES IN $\mathbb{R}^3$.

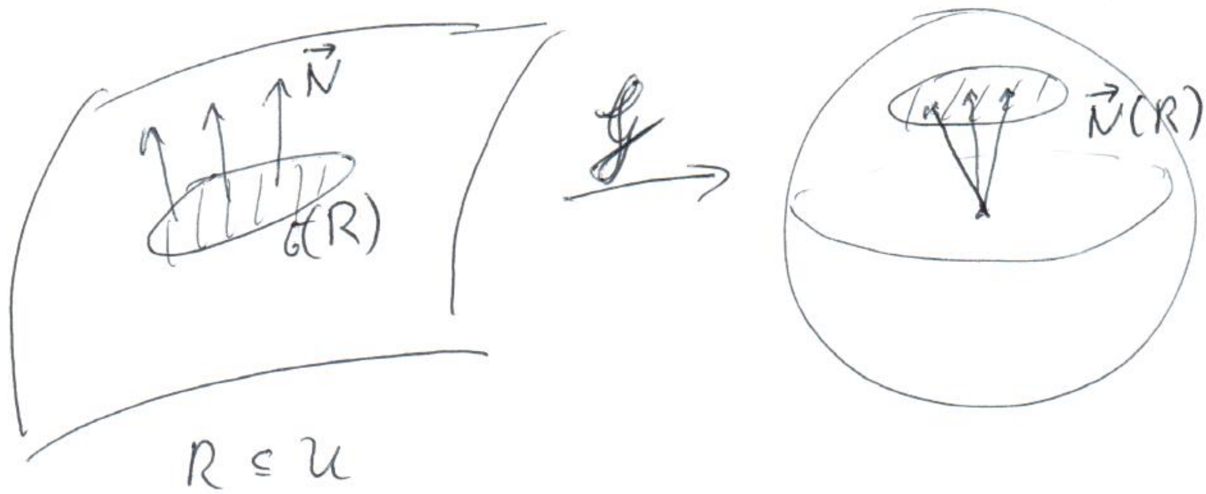
DIRECTION OF TANGENT PLANE MEASURED BY UNIT NORMAL $\vec{N}$

EXPECT: RATE OF CHANGE OF DIRECTION OF $\vec{N}$ MEASURES CURVATURE.

CONSIDER SURFACE $\sigma : \mathcal{U} \to \mathbb{R}^3$
WITH UNIT NORMAL $\vec{N}$, $S = \sigma(\mathcal{U})$

$$\mathcal{G} : S \to S^2,\ \sigma(u,v) \to \vec{N}(u,v)$$
$$\mathcal{U} \to S^2,\ (u,v) \mapsto \vec{N}(u,v)$$

IS CALLED GAUSS MAP OF SURFACE.



AREA$(\vec{N}(R))$ MEASURES AMOUNT BY WHICH DIRECTION OF $\vec{N}$ VARIES. THUS, APPROXIMATELY, RATE OF CHANGE OF DIRECTION PER UNIT AREA IS

$$\frac{\text{AREA}(\vec{N}(R))}{\text{AREA}(\sigma(R))} = \frac{\mathcal{A}_{\vec{N}}(R)}{\mathcal{A}_{\sigma}(R)}$$

(SEE 5.3 FOR SURFACE AREA)

THM 7.2.1 $\sigma : \mathcal{U} \to \mathbb{R}^3$ SURFACE,

$(u_0, v_0) \in \mathcal{U}$. CHOOSE $\delta > 0$ SO THAT

$$R_\delta = \{(u,v) \in \mathbb{R}^2 \mid (u-u_0)^2 + (v-v_0)^2 \le \delta^2\} \subset \mathcal{U}$$

THEN

$$\lim_{\delta \to 0} \frac{\mathcal{A}_{\vec{N}}(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} = |K|$$

GAUSSIAN CURVATURE OF σ AT $\sigma(u_0, v_0)$.

PROOF

$$\frac{\mathcal{A}_{\vec{N}}(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} \underset{5.3.1.}{\overset{\uparrow}{=}} \frac{\iint_{R_\delta} \|\vec{N}_u \times \vec{N}_v\| \, du \, dv}{\iint_{R_\delta} \|\sigma_u \times \sigma_v\| \, du \, dv}$$

NEED TO CALCULATE $\vec{N}_u \times \vec{N}_v$.

LEMMA 7.2.2.

$$\vec{N}_u = a\,\sigma_u + b\,\sigma_v\,,\quad \vec{N}_v = c\,\sigma_u + d\,\sigma_v$$

WITH $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_{I}^{-1}\,\mathcal{F}_{II}$

PROOF $\|\vec{N}\| = 1 \Rightarrow \vec{N}_u, \vec{N}_v \perp \vec{N}$

$$\Rightarrow \exists\, a,b,c,d \in \mathbb{R}: \quad \vec{N}_u = a\,\sigma_u + b\,\sigma_v$$
$$\vec{N}_v = c\,\sigma_u + d\,\sigma_v$$

$$0 = \vec{N}\cdot\sigma_u \Rightarrow 0 = \vec{N}_u\cdot\sigma_u + \underbrace{\vec{N}\cdot\sigma_{uu}}_{=L}$$

$$\Rightarrow \vec{N}_u\cdot\sigma_u = -L$$

SIMILARLY:

$$\vec{N}_u\cdot\sigma_v = \vec{N}_v\cdot\sigma_u = -M$$
$$\vec{N}_v\cdot\sigma_v = -N$$

$$-L = \vec{N}_u\cdot\sigma_u = a\,\sigma_u\cdot\sigma_u + b\,\sigma_v\cdot\sigma_u = aE + bF$$
$$-M = \vec{N}_u\cdot\sigma_v = a\,\sigma_u\cdot\sigma_v + b\,\sigma_v\cdot\sigma_v = aF + bG$$
$$-M = \vec{N}_v\cdot\sigma_u = c\,\sigma_u\cdot\sigma_u + d\,\sigma_u\cdot\sigma_v = cE + dF$$
$$-N = \vec{N}_v\cdot\sigma_v = c\,\sigma_v\cdot\sigma_u + d\,\sigma_v\cdot\sigma_v = cF + dG.$$

$$\Rightarrow -\underbrace{\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{-\mathcal{F}_{II}} = \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{\mathcal{F}_{I}} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_{I}^{-1} \mathcal{F}_{II}. \qquad \square$$

CONTINUE WITH PROOF OF 7.2.1. :

$$\vec{N}_u \times \vec{N}_v = (a\sigma_u + b\sigma_v) \times (c\sigma_u + d\sigma_v)$$

$$= (ad - bc)\, \sigma_u \times \sigma_v$$

$$= \det\left(-\mathcal{F}_{I}^{-1} \mathcal{F}_{II}\right) \sigma_u \times \sigma_v$$

$$= \frac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_{I})}\, \sigma_u \times \sigma_v$$

$$\mathcal{F}_{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$= \frac{LN - M^2}{EG - F^2}\, \sigma_u \times \sigma_v$$

$$\underset{7.1.2}{=} K\, \sigma_u \times \sigma_v$$

$$\Rightarrow \frac{A_{\vec{N}}(R_\delta)}{A_\sigma(R_\delta)} = \frac{\iint_{R_\delta} |k| \, \|\sigma_u \times \sigma_v\| \, du\, dv}{\iint_{R_\delta} \|\sigma_u \times \sigma_v\| \, du\, dv}$$

SINCE $|k|$ IS CONTINUOUS:

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall (u,v) \in R_\delta$:

$$-\varepsilon < |k(u,v)| - |k(u_0,v_0)| < \varepsilon$$

$$\Updownarrow$$

$$|k(u_0,v_0)| - \varepsilon < |k(u,v)| < |k(u_0,v_0)| + \varepsilon$$

$$\Rightarrow \left(|k(u_0,v_0)| - \varepsilon\right) \iint_{R_\delta} \|\sigma_u \times \sigma_v\| \, du\, dv$$

$$< \iint_{R_\delta} |k(u,v)| \, \|\sigma_u \times \sigma_v\| \, du\, dv$$

$$< \left(|k(u_0,v_0)| + \varepsilon\right) \iint_{R_\delta} \|\sigma_u \times \sigma_v\| \, du\, dv$$

$$\Rightarrow |k(u_0,v_0)| - \varepsilon < \frac{A_{\vec{N}}(R_\delta)}{A_\sigma(R_\delta)} < |k(u_0,v_0)| + \varepsilon$$

$$\underset{\varepsilon \to 0}{\Longrightarrow} \quad \frac{A_{\vec{N}}(R_\delta)}{A_\sigma(R_\delta)} = |k(u_0,v_0)| \ . \qquad \square$$

EXAMPLES 7.2.3

(i) PLANE $\Rightarrow \vec{N}$ CONSTANT

$$\Rightarrow \forall R \subseteq \mathcal{U} : \vec{N}(R) = \{point\}$$

$$\Rightarrow A_{\vec{N}}(R) = 0$$

$$\Rightarrow K = 0$$

7.2.1.

(ii) $\sigma(u,v) = (f(v), g(v), u)$ GENERALIZED CYLINDER

$$\vec{N}(u,v) = \frac{1}{\sqrt{\dot{f}^2 + \dot{g}^2}} (\dot{g}, -\dot{f}, 0)$$

$$\Rightarrow \vec{N}(\mathcal{U}) \subseteq \text{EQUATOR OF } S^2$$

$$\Rightarrow A_{\vec{N}}(R) = 0 \quad \forall R \subseteq \mathcal{U}$$

$$\Rightarrow K = 0.$$

~~(iii)~~

iiii) SPHERE

$$\sigma(\theta, \varphi) = \big(\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta)\big)$$

$$\vec{N}(\theta, \varphi) = \big(-\cos(\theta)\cos(\varphi), -\cos(\theta)\sin(\varphi), -\sin(\theta)\big)$$

$$= -\sigma(\theta, \varphi)$$

$\Rightarrow$ GAUSS MAP $\mathcal{G}$ IS ANTIPODAL MAP

$\Rightarrow \forall R \subseteq \mathcal{U}: \quad \mathcal{A}_{\sigma}(R) = \mathcal{A}_{\vec{N}}(R)$

$\Rightarrow \quad |K| = 1$.

8 GEODESICS

8.1 DEFINITION AND BASIC PROPERTIES

GEODESIC ~ SHORTEST PATH (LOCALLY)

DEF 8.1.1 γ UNIT SPEED CURVE ON SURFACE σ.

γ GEODESIC $\iff \forall t\ \ddot{\gamma}(t)$ PERPENDICULAR TO σ AT $\gamma(t)$

$$\iff \forall t: \ddot{\gamma}(t) \parallel \vec{N}(t)$$

NOTE: IF REPARAMETRIZE σ, THEN $\vec{N}$ MAY CHANGE SIGN, BUT THIS DOES NOT AFFECT DEF OF GEODESIC.

MECHANICAL INTERPRETATION:

PARTICLE MOVING ON σ SUBJECT TO NO FORCES EXCEPT A FORCE THAT KEEPS PARTICLE ON σ, MOVES ALONG GEODESIC

NEWTON'S 2nd LAW OF MOTION:

$m\ddot{\gamma}$ FORCE ON THE PARTICLE IS PERPENDICULAR TO σ.

RECALL:

$\varkappa_g = \ddot{\gamma} \cdot (\vec{N} \times \dot{\gamma})$ GEODESIC CURVATURE OF γ

PROP 8.1.2: γ GEODESIC $\iff \varkappa_g = 0$

PROOF: "$\Rightarrow$" γ GEODESIC $\Rightarrow \ddot{\gamma} \parallel \vec{N}$

$\Rightarrow \ddot{\gamma} \perp \vec{N} \times \dot{\gamma} \Rightarrow \varkappa_g = 0$

"$\Leftarrow$": $\left.\begin{array}{l} \varkappa_g = 0 \Rightarrow \ddot{\gamma} \perp \vec{N} \times \dot{\gamma} \\ \|\dot{\gamma}\| = 1 \Rightarrow \ddot{\gamma} \perp \dot{\gamma} \end{array}\right\} \Rightarrow \ddot{\gamma} \parallel \vec{N}$

$\uparrow$

$\dot{\gamma}, \vec{N}, \vec{N} \times \dot{\gamma}$ PERPENDICULAR UNIT VECTORS. $\square$

γ REGULAR CURVE ON σ. WE DEFINE

γ GEODESIC $\iff$ UNIT SPEED REPARAMETRIZ. OF γ IS GEODESIC

MAKES SENSE? YES:

$\|\dot{\gamma}\| = 1$, $\tilde{\gamma}(t) = \gamma(c \pm t) \Rightarrow \|\dot{\tilde{\gamma}}\| = 1$

SHOW: $\dot{\tilde{\gamma}} = \pm \dot{\gamma}$, $\ddot{\tilde{\gamma}} = \ddot{\gamma}$, $\tilde{\varkappa}_g = \pm \varkappa_g$

THUS: $\varkappa_g = 0 \Rightarrow \tilde{\varkappa}_g = 0$.

PROP 8.1.3 ANY (PART OF A) STRAIGHT LINE ON SURFACE IS A GEODESIC.

PROOF: $\gamma(t) = a + bt$, $a, b \in \mathbb{R}^3$

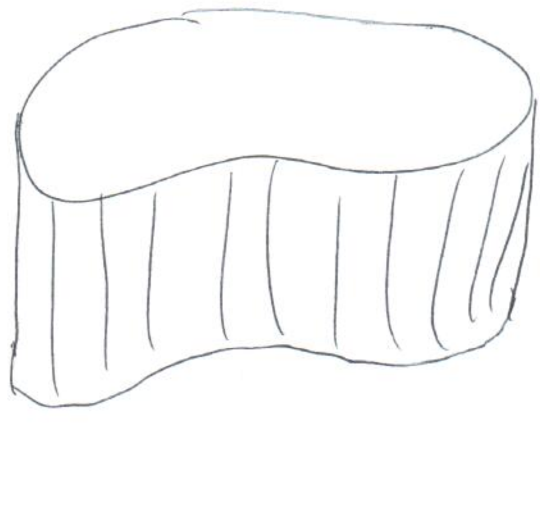
$\|b\| = \|\dot{\gamma}\| = 1$

$\Rightarrow \ddot{\gamma}(t) = 0 \Rightarrow \kappa_g = 0$.

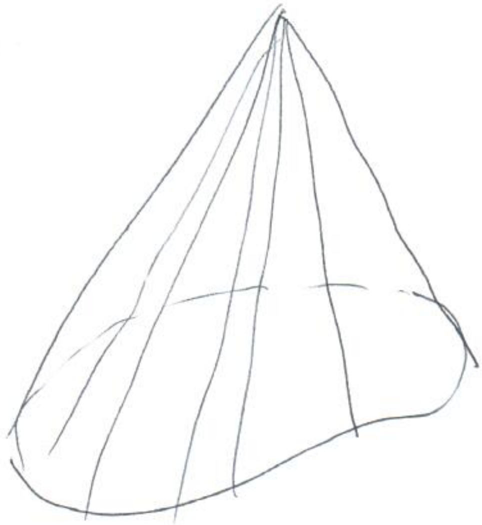
EXAMPLES 8.1.4

(1) STRAIGHT LINES IN PLANE ARE GEODESICS 

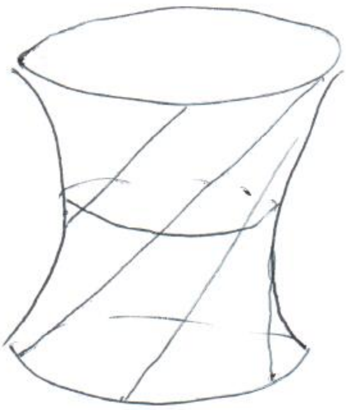
(2) RULINGS OF RULED SURFACES ARE GEODESICS, E.G. GENERATORS OF CYLINDER



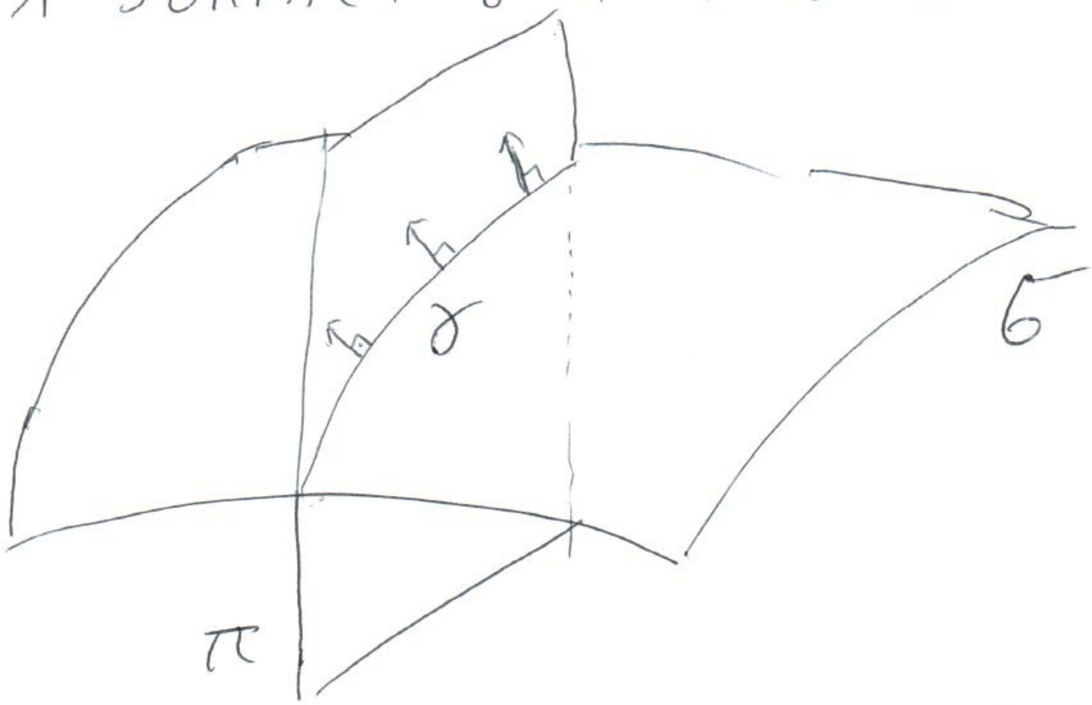
OR CONE



OR STRAIGHT LINES ON HYPERBOLOID OF ONE SHEET



PROP 8.1.5 ANY NORMAL SECTION OF A SURFACE σ IS A GEODESIC.



PROOF: 6.2: $\gamma = \sigma \cap \Pi$

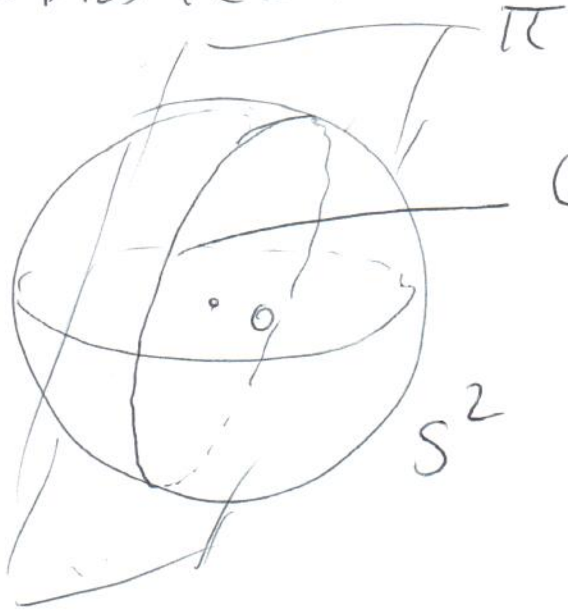
$\Pi \perp \sigma$ ALONG γ

$\varkappa_g = 0$ (SEE 6.2) $\square$.

EXAMPLE 8.1.6

GREAT CIRCLES ON SPHERES ARE GEODESICS.

PROOF.

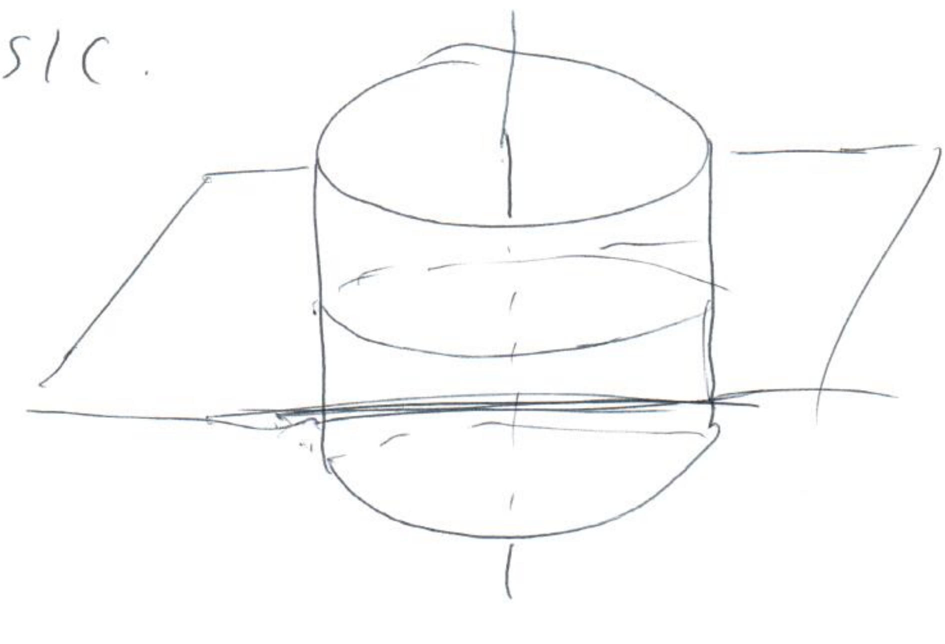


$$\text{GREAT CIRCLE} = \pi \cap S^2, \quad \pi \ni O$$

GREAT CIRCLE IS NORMAL SECTION.

EXAMPLE 8.1.7

INTERSECTION OF CONE OR CYLINDER WITH PLANE π PERPENDICULAR TO THE AXIS OF CONE OR CYLINDER IS A GEODESIC.



FOR CYLINDER:

$$\sigma(u,v) = (f(u), g(u), v)$$

$$\vec{N}(u,v) = \frac{1}{\sqrt{\dot{f}^2+\dot{g}^2}}(\dot{g}, -\dot{f}, 0)(u)$$

$\Rightarrow \vec{N} \perp$ z-AXIS

$\Rightarrow \vec{N} \parallel \pi$

$\Rightarrow \pi \perp$ TANGENT PLANE

FOR CONE: SIMILAR.

8.1.3 & 8.1.5 DO NOT SUFFICE TO DETERMINE ALL GEODESICS.

<u>THEOREM 8.1.8</u> UNIT SPEED CURVE $\gamma(t) = \sigma(u(t), v(t))$ ON SURFACE σ IS GEODESIC IF AND ONLY IF

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2)$$

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2)$$

"GEODESIC EQUATIONS".

PROOF $\{\sigma_u, \sigma_v\}$ BASIS OF TANGENT PLANE

$\downarrow$

SO γ GEODESIC $\iff \ddot{\gamma} \perp \sigma_u, \sigma_v$ ALONG γ

$$\iff \begin{cases} \dfrac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v) \cdot \sigma_u = 0 \\ \dfrac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v) \cdot \sigma_v = 0 \end{cases}$$

$\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$

$$0 = \frac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v) \cdot \sigma_u$$

$$= \frac{d}{dt}\Big(\underbrace{(\dot{u}\sigma_u + \dot{v}\sigma_v)\cdot\sigma_u}_{= \dot{u}\underbrace{\sigma_u\cdot\sigma_u}_{=E} + \dot{v}\underbrace{\sigma_v\cdot\sigma_u}_{=F}}\Big) - (\dot{u}\sigma_u + \dot{v}\sigma_v)\cdot \underbrace{\frac{d}{dt}\sigma_u}_{= \dot{u}\sigma_{uu} + \dot{v}\sigma_{uv}}$$

$$= \frac{d}{dt}(E\dot{u} + F\dot{v}) - \Big(\dot{u}^2 \underbrace{\sigma_u\cdot\sigma_{uu}}_{=\frac{1}{2}E_u} + \dot{v}^2 \underbrace{\sigma_v\cdot\sigma_{uv}}_{=\frac{1}{2}G_u} + \dot{u}\dot{v}\underbrace{(\sigma_u\cdot\sigma_{uv} + \sigma_v\cdot\sigma_{uu})}_{=F_u}\Big)$$

SIMILAR FOR OTHER EQUATION

$\square$.

COROLLARY 8.1.9 ISOMETRIES BETWEEN SURFACES MAP GEODESICS TO GEODESICS.

PROOF. $f : \sigma_1 \to \sigma_2$ ISOMETRY

CAN REPARAMETRIZE (BY 5.2.3) σ_1, σ_2 SO THAT THEIR FIRST FUNDAMENTAL FORMS COINCIDE.

REPARAMETRIZING SURFACES DOES NOT CHANGE GEODESICS.

COROLLARY FOLLOWS FROM 8.1.8 SINCE GEODESIC EQUATIONS ONLY INVOLVE COEFFICIENTS OF FIRST FUNDAMENTAL FORM. $\square$.