Problem 1

Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Solution:
Suppose the parts are of sizes k and $v - k$. Then the maximum number of edges the graph may have is $k(v - k)$ (an edge between each pair of vertices in different parts). We know that the function $f(k) = k(v - k)$ achieves its maximum when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

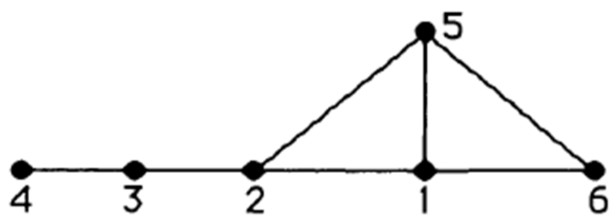
Problem 2

Radio stations broadcast their signal at certain frequencies. However, there are a limited number of frequencies to choose from, so nationwide many stations use the same frequency. This works because the stations are far enough apart that their signals will not interfere; no one radio could pick them up at the same time.
Suppose six new radio stations are to be set up in a currently unpopulated (by radio stations) region. The distances among stations are recorded in the table below. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

Table 1: Distances in miles among stations

	1	2	3	4	5	6
1	——	85	175	200	50	100
2	85	——	125	175	100	160
3	175	125	——	100	200	250
4	200	175	100	——	210	220
5	50	100	200	210	——	100
6	100	160	250	220	100	——

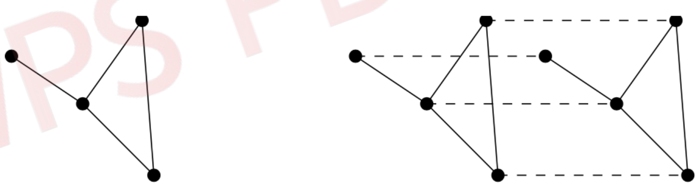
Solution:
We use vertices to represent stations. We draw the graph in which two vertices (representing locations) are adjacent if the locations are within 150 miles of each other.



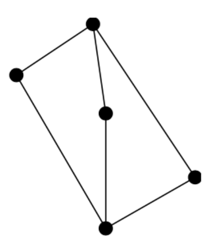
Since this graph contains K_3, it can be colored by at least 3 colors. We can easily find 3 colors to color this graph: say red for vertices 4, 2, and 6; blue for 3 and 5; and yellow for 1. Thus three channels are necessary and sufficient.

Problem 3

The double of a graph G consists of two copies of G with edges joining corresponding vertices. For example, a graph appears below on the left and its double appears on the right. Some edges in the graph on the right are dashed to clarify its structure.

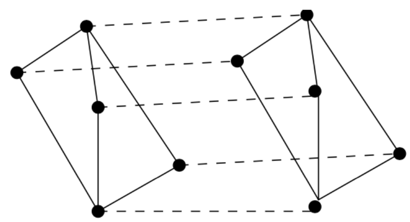


(a) Draw the double of the graph shown below.

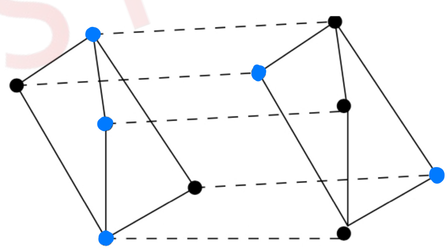


(b) Suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so forth. Use induction on n to prove that G_n is bipartite for all $n \geq 1$.

Solution:
(a)



(b)
We use induction: Let $P(n)$ be the proposition that G_n is bipartite.
Base case: $P(1)$ is true because G_1 is bipartite by assumption.
Inductive step: For $n \geq 1$, we assume $P(n)$ in order to prove $P(n+1)$. The graph G_{n+1} consists of two subgraphs isomorphic to G_n with edges joining corresponding vertices. Remove these extra edges. By the assumption $P(n)$, G_n is bipartite, so it has chromatic number 2. We can color each vertex of one subgraph by black and blue so that adjacent vertices get different colors. If we color the corresponding vertices in the other subgraph oppositely, then adjacent vertices get different colors within that subgraph as well. And now if we add back the extra edges, each of these joins a white vertex and a black vertex. Therefore, G_{n+1} is bipartite. An illustration of graph in (a) is as follows.



Problem 4

Let m, n, and r be nonnegative integers with $r \leq m$ and $r \leq n$. Prove the following formula by a combinatorial proof.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}.$$

Solution:

Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$.

Another way to pick r elements from the union is to pick k elements from the second set and then $r-k$ elements from the first set, where k is an integer with $0 \leq k \leq r$. Because there are $\binom{n}{k}$ ways to choose k elements from the second set and $\binom{m}{r-k}$ ways to choose $r-k$ elements from the first set, the product rule tells us that this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Hence, the total number of ways to pick r elements from the union also equals $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$.

We have found two expressions for the number of ways to pick r elements from the union of a set with m items and a set with n items. Equating them gives us the formula, which is called Vandermonde's identity.

Problem 5

Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2}+\binom{3}{2}\binom{n-1}{2}+\binom{4}{2}\binom{n-2}{2}+\cdots+\binom{n}{2}\binom{2}{2}=\binom{n+3}{5}.$$

Solution:

Question: How many 5-element subsets are there of the set $\{1,2,\ldots,n+3\}$.

Answer 1: We choose 5 out of the $n+3$ elements, so $\binom{n+3}{5}$ subsets.

Answer 2: Break this up into cases by what the "middle" (third smallest) element of the 5 element subset is. The smallest this could be is a 3. In that case, we have $\binom{2}{2}$ choices for the numbers below it, and $\binom{n}{2}$ choices for the numbers above it. Alternatively, the middle number could be a 4. In this case there are $\binom{3}{2}$ choices for the bottom two numbers and $\binom{n-1}{2}$ choices for the top two numbers. If the middle number is 5, then there are $\binom{4}{2}$ choices for the bottom two numbers and $\binom{n-2}{2}$ choices for the top two numbers. And so on, all the way up to the largest the middle number could be, which is $n+1$. In that case there are $\binom{n}{2}$ choices for the bottom two numbers and $\binom{2}{2}$ choices for the top number. Thus the number of 5 element subset is:

$$\binom{2}{2}\binom{n}{2}+\binom{3}{2}\binom{n-1}{2}+\binom{4}{2}\binom{n-2}{2}+\cdots+\binom{n}{2}\binom{2}{2}.$$

Since the two answers correctly answer the same question, we have:

$$\binom{2}{2}\binom{n}{2}+\binom{3}{2}\binom{n-1}{2}+\binom{4}{2}\binom{n-2}{2}+\cdots+\binom{n}{2}\binom{2}{2}=\binom{n+3}{5}.$$

Problem 6

Find the number of solutions of the equation $x_1+x_2+x_3=11$, where x_1, x_2, x_3 are non-negative integers with $x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6$.

Solution:
To apply the principle of inclusion-exclusion, let a solution have property P_1 is $x_1 > 3$, property P_2 is $x_2 > 4$, and property P_3 is $x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ is

$$N(\overline{P_1}\cap\overline{P_2}\cap\overline{P_3}) = N-N(P_1)-N(P_2)-N(P_3)+N(P_1\cap P_2)+N(P_1\cap P_3)+N(P_2\cap P_3)-N(P_1\cap P_2\cap P_3)$$

It follows that:

- N = total number of solutions $=\binom{3+11-1}{11}$=78,
- $N(P_1)$ = (number of solutions with $x_1 \geq 4$) $=\binom{3+7-1}{7}=\binom{9}{7}$=36,
- $N(P_2)$ = (number of solutions with $x_2 \geq 5$) $=\binom{3+6-1}{6}=\binom{8}{6}$=28,
- $N(P_3)$ = (number of solutions with $x_3 \geq 7$) $=\binom{3+4-1}{4}=\binom{6}{4}$=15,
- $N(P_1 \cap P_2)$ = (number of solutions with $x_1 \geq 4$ and $x_2 \geq 5$) $=\binom{3+2-1}{2}=\binom{4}{2}$=6,
- $N(P_1 \cap P_3)$ = (number of solutions with $x_1 \geq 4$ and $x_3 \geq 7$) $=\binom{3+0-1}{0}$=1,
- $N(P_2 \cap P_3)$ = (number of solutions with $x_2 \geq 5$ and $x_3 \geq 7$) = 0,
- $N(P_1 \cap P_2 \cap P_3)$ = (number of solutions with $x_1 \geq 4$, $x_2 \geq 5$, and $x_3 \geq 7$) = 0.

Inserting these quantities into the formula for $N(\overline{P_1}\cap\overline{P_2}\cap\overline{P_3})$ shows that the number of solutions with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ equals

$$N(\overline{P_1}\cap\overline{P_2}\cap\overline{P_3}) = 78-36-28-15+6+1+0-0=6.$$

Problem 7

Show that in any set of $n+1$ positive integers not exceeding $2n$ there must be two that are relatively prime.

Solution:
Consider the n pairs of the form $(2k-1, 2k)$ for $k = 1, 2, \ldots, n$. Note that these pairs cover the positive integers less than or equal to n. So by the pigeonhole principle, if we are to choose $n+1$ positive integers less than or equal to $2n$, we must choose both integers from at least one of the pairs. So we must have chosen $2k-1$ and $2k$ for some k with $1 \leq k \leq n$. But those numbers are relatively prime; for if m divides both $2k-1$ and $2k$ then it divides their difference, which is 1, and thus $m = 1$ implying the greatest common divisor of $2k-1$ and $2k$ is 1, or in other words , $2k-1$ and $2k$ are relatively prime.

Problem 8

A 0-1 sequence a_n with $2m$ terms is said to be normal if the following two conditions are satisfied.

- There exist m terms equal to 0 and the other m terms equal to 1 in a_n.
- For arbitrary $k \leq 2m$, the number of terms equal to 0 is not less than that of terms equal to 1 in the first k terms $a_1, a_2, \ldots, a_k$.

Please complete the following questions.
(a) Show that the number of abnormal 0-1 sequences a_n with $2m$ terms equals that of sequences a_n of which $(m+1)$ terms are 0s and $(m-1)$ terms are 1s.
(b) For $m = 4$, determine the number of different normal 0-1 sequences a_n. Note: An abnormal 0-1 sequence a_n is a 0-1 sequence that does not satisfy the properties of normal 0-1 sequences.

Solution:
(a) Consider 0-1 sequence with $2m$ terms. Prove that the number of normal 0-1 sequence $=$ the number of 0-1 sequence $-$ the number of abnormal 0-1 sequence. We can consider the following lemma.

Lemma: the number of abnormal 0-1 sequences $=$ the number of sequences with $(m+1)$ terms equal to 0, and $(m-1)$ terms equal to 1.

General steps to prove this lemma:
(1) Each abnormal 0-1 sequence corresponds to a sequence with $(m+1)$ 0s and $(m-1)$ 1s.
(2) Inversely, each sequence with $(m+1)$ 0s and $(m-1)$ 1s also corresponds to an abnormal 0-1 sequence.

Prove (1):
If a sequence is not abnormal 0-1 sequence that there exists k as small as possible, such that for the first k terms of this sequence: $a_1, a_2, \ldots, a_k$, the k-th term is exactly equal to 1, and the number of 0s and number of 1s are equal for the first $(k-1)$ terms.
As a result, for the first k terms, the number of 0s is equal to the number of 1s−1; for the remaining terms, the number of 0s is equal to the number of 1s+1.
Now, shift the first k terms by transforming all 0s into 1s, all 1s into 0s. After shifting, the number of 0s is equal to the number of 1s+2 for the shifted sequence.
This shifted sequence has $(m+1)$ 0s and $(m-1)$ 1s.
Prove (2):
If a sequence has $(m+1)$ 0s and $(m-1)$ 1s. By following the above proof. Similarly, there exists the first k such that for the first k terms, the number of 0s exactly exceeds the number of 1s. Shifting this first k terms by the same rule, we can derive an abnormal 0-1 sequence with m 0s and m 1s.

(b) The number of sequences which satisfy condition (1): $\binom{8}{4}$=70; The number of abnormal sequences: $\binom{8}{5}$=56.
Therefore, the number of normal sequences is: $70 - 56 = 14$.