Geometry of surfaces - Solutions

1. For each question we give a counterexample or a proof.

(a) False. For example, for $\gamma(t) = (2t, 2t)$ we have $\dot\gamma(t) = (2, 2)$ and $\|\dot\gamma\| = 2\sqrt{2} \neq 1$.

(b) True. If γ is a unit speed curve, then $\|\dot\gamma\| = 1$ and hence $\|\dot\gamma\| \neq 0$, which means that γ is regular.

(c) False. Take $\gamma_1(t) = \frac{1}{\sqrt{2}}(t,t)$ and $\gamma_2(t) = \frac{1}{\sqrt{2}}(t+1, t+1)$. These are two different unit speed parametrizations of the same curve $\{(x, y) \in \mathbb{R}^2 \mid x = y\}$.

(d) False. $\gamma(t) = (\cos(t^2), \sin(t^2))$ is not regular at $t = 0$.

2. $\text{Length}(\gamma) = \int_a^b \|\dot\gamma(t)\| dt = \int_a^b 1 = b - a$.

3. Let $\tilde\gamma$ be a reparametrization of γ with reparametrization map $\phi : (\alpha, \beta) \to \mathbb{R}$. Using integration by substitution with $u = \phi(t)$ we obtain

$$\begin{aligned}\text{Length}(\gamma) &= \int_a^b \|\dot\gamma(t)\|\, dt = \int_a^b \left\| \frac{d}{dt}(\tilde\gamma \circ \phi)(t) \right\| dt = \int_a^b \left\| \dot{\tilde\gamma}(\phi(t)) \right\| |\phi'(t)| dt \\ &= \int_{\phi(a)}^{\phi(b)} \|\dot{\tilde\gamma}(u)\| du = \text{Length}(\tilde\gamma).\end{aligned}$$

4. Using the fundamental theorem of calculus, we have

$$\|\gamma(c) - \gamma(d)\| = \left\| \int_c^d \frac{d\gamma}{dt} dt \right\| \leq \int_c^d \left\| \frac{d\gamma}{dt} \right\| dt = \int_c^d \|\dot\gamma(t)\|\, dt = \text{Length}(\gamma : (c, d) \to \mathbb{R}^n).$$

5. We have $\dot\gamma(t) = (\sin(t) + t\cos(t), \cos(t) - t\sin(t), \frac{\sqrt{8}}{2} t^{\frac{1}{2}})$ and thus $\|\dot\gamma(t)\| = \sqrt{(t+1)^2} = t + 1$. Thus

$$\text{Length}(\gamma) = \int_0^1 (t+1) dt = \frac{3}{2} \quad \text{and} \quad s(t) = \int_0^t (u+1) du = \frac{1}{2} t^2 + t.$$

Solving for t we get $t = -1 + \sqrt{1 + 2s}$ (recall that $t > 0$) and $\tilde\gamma(s) = \gamma(-1 + \sqrt{1 + 2s})$ with $s \in (0, \frac{3}{2})$ is a unit speed reparametrization of γ.

6. Let v be a vector perpendicular to ℓ and $p_0 \in \ell$. Without loss of generality, we can assume that $\frac{v \cdot (\gamma(t_0) - p_0)}{\|v\|}$ is positive, otherwise take $-v$ as the perpendicular vector. By hypothesis, the function $\frac{v \cdot (\gamma(t) - p_0)}{\|v\|}$ obtains a minimum at t_0 giving that t_0 is a critical point and so $0 = \frac{d}{dt} \frac{v \cdot (\gamma(t) - p_0)}{\|v\|}(t_0) = \frac{v \cdot \dot\gamma(t_0)}{\|v\|}$. This means that $\dot\gamma(t_0)$ is perpendicular to v and therefore parallel to ℓ.

7. We have $\dot\gamma(t) = (-\sin(t), \cos(t), 2)$ and $\dot\gamma$ lies in the plane $z = 2$. Furthermore, $\|\dot\gamma(t) - (0, 0, 2)\| = 1$, hence $\dot\gamma$ traces out a circle with radius 1 in the plane $z = 2$ centred at the point $(0, 0, 2)$. Next, $\|\dot\gamma\| = \sqrt{5}$ and $s = \int_0^t \sqrt{5} = \sqrt{5}t$. Thus, $t = \frac{s}{\sqrt{5}}$ and a unit-speed reparametrization is $\left(\cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}}\right)$.

8. By assumption, the function $\|\gamma(t)\|^2 = \gamma(t) \cdot \gamma(t)$ has a critical point, namely a minimum, at t_0. Thus $0 = \frac{d}{dt}(\gamma \cdot \gamma)(t_0) = 2\dot\gamma(t_0) \cdot \gamma(t_0)$.