PIGEONHOLE PRINCIPLE

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Pigeonhole Principle

Assume we have 365 people.

Does anyone share a birthday? Are we able to say for certain?

Is it possible that no one shares a birthday?

Jan 1	Jan 2	…	Sep 27	Sep 28	Sep 29	Sep 30	…	Dec 30	Dec 31

Pigeonhole Principle

Assume we have 366 people (and no one was born on Feb. 29th).

Does anyone share a birthday? Are we able to say for certain?

There are at least 2 people who share a birthday. Why?

There are 365 days in a year (not in leap year).

366 people. Assume no one shares a birthday.

Jan 1	Jan 2	…	Sep 27	Sep 28	Sep 29	Sep 30	…	Dec 30	Dec 31

Pigeonhole Principle

Country where everyone's last name is 1 Upper Case letter and 1 lower case letter.

Xe, Gt, Po, etc.

In a country with ≥ 677 people, at least two must have the same last name. Why?

How many last names are there in total? Can we determine using the Product Rule?

Task 1: Choose an Upper Case letter – 26 ways to choose

Task 2: Choose a lower case letter – 26 ways to choose

$$26 \cdot 26 = 676$$

Therefore there are 676 possible last names.

Aa	Ab	Ac	…	Zx	Zy	Zz

Pigeonhole Principle

k holes ("boxes")
$\geq k+1$ pigeons ("objects")

Then $\exists$ hole with ≥ 2 pigeons.

Proof by contradiction. Assume the negation:

$\neg(\exists$ hole with ≥ 2 pigeons)
$\forall$ holes, $\neg(\geq 2$ pigeons)
$\forall$ holes, ≤ 1 pigeons

Since there are at most k holes with ≤ 1 pigeon, there are $\leq k$ pigeons.

But we have $k+1$ pigeons, which is a contradiction.

Box 1	Box 2	Box 3	…	Box $k-2$	Box $k-1$	Box k

Pigeonhole Principle

k holes ("boxes")
$\geq k + 1$ pigeons ("objects")

Then ∃ hole with ≥ 2 pigeons.

Or we can try to construct a counter-example.

Task 1: Place pigeon 1 in an empty box

Task 2: Place pigeon 2 in an empty box

…

Task k : Place pigeon k in last empty box.

Task $k + 1$: There are no empty boxes. Place pigeon $k + 1$ in a box with another pigeon.

Box 1	Box 2	Box 3	…	Box $k - 2$	Box $k - 1$	Box k

Exercise 3.84

$S \subseteq \{1, 2, \ldots, 2n\}$

$|S| = n + 1$

Claim: $\exists a, b \in S$:

$$a - b = 1$$

First we will try some examples to gain some insight.

$n = 4$

$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

(Try and construct a subset where the claim is not true).

We cannot choose 2 consecutive numbers:

$$S = \{1, 3, 5, 7, \quad \}$$

Exercise 3.84

$S \subseteq \{1, 2, \dots, 2n\}$

$|S| = n + 1$

Claim: $\exists a, b \in S$:

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First we will try some examples to gain some insight.

$n = 4$

$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

(Try and construct a subset where the claim is not true)

We cannot choose 2 consecutive numbers:

$$S = \{1, 3, 5, 7, \quad\quad \}$$

Or

$$S = \{2, 4, 6, 8, \quad\quad \}$$

Exercise 3.84

$S \subseteq \{1, 2, \dots, 2n\}$

$|S| = n + 1$

Claim: $\exists a, b \in S$:

$$a - b = 1$$

First we will try some examples to gain some insight.

$n = 4$

$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$S = \{1, 3, 5, 7, 8\}, |S| = 5$

4 Boxes:

1, 2

3, 4

5, 6

7, 8

We have 5 elements in S and 4 boxes.

Every element from the original set is represented in a box.

At least one box contains 2 elements of S.

Exercise 3.84

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a - b = 1$$

How can we prove this using the Pigeonhole principle?

It comes down to making the right boxes…there should be n boxes.
Then start putting elements of S into their boxes… Since $|S| = n + 1$, there must be a box with 2 elements

n Boxes:

1, 2

3, 4

5, 6

⋮

$\mathbf{2n - 1, 2n}$

We have $n + 1$ elements and n boxes.

Every element is represented in a box.

By the pigeonhole principle, one box has ≥ 2 elements from the subset S.

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1\}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1, 2\}$

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1, 2, 3\}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1, 2, 3, 4\}$

At this point we cannot add anything else.

It seems to be true. To use the pigeonhole principle we need to figure out what our boxes will be.

Since $|S| = n + 1$ we want n or fewer boxes.

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

We will use n boxes.

The elements in each box sums to $2n + 1$.

If we can do this using all elements, we can prove the claim using the pigeonhole principle.

n Boxes:

$\mathbf{1, 2n}$
$\mathbf{2, 2n - 1}$
$\mathbf{3, 2n - 2}$
$\vdots$
$\mathbf{n - 1, n + 2}$
$\mathbf{n, n + 1}$

We have n boxes and $n + 1$ things to place in these boxes.

By the pigeonhole principle, one box must have 2 or more items.

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1.

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3\}$

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3, 5\}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3, 5, 7\}$

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3, 5, 7, 4\}$

Out of options…

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

Seems to be true, but what the boxes should be is less clear.

$$S = \{a_1, a_2, a_3, \ldots, a_{n+1}\}$$

for $i = 1, \ldots, n + 1$, we express each term a_i as follows:

$a_i = 2^{k_i} \cdot q_i$ where $k_i \geq 0$ and q_i is an odd number.

$$48 = 2^4 \cdot 3$$
$$45 = 2^0 \cdot 45$$
$$7 = 2^0 \cdot 7$$
$$5 = 2^0 \cdot 5$$
$$4 = 2^2 \cdot 1$$
$$3 = 2^0 \cdot 3$$
$$2 = 2^1 \cdot 1$$
$$1 = 2^0 \cdot 1$$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

Seems to be true, but what the boxes should be is less clear.

$S = \{a_1, a_2, a_3, \dots, a_{n+1}\}$

$a_i = 2^{k_i} \cdot q_i$ where $k_i \geq 0$ and q_i is an odd number.

$O = \{q_1, q_2, \dots, q_{n+1}\}$ are

$n + 1$ odd integers

that belong to

$\{1, 3, 5, \dots, 2n - 1\}$ (a set of size n).

Thus by the pigeonhole principle in the set O there are two elements that are equal.

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

Seems to be true, but what the boxes should be is less clear.

$S = \{a_1, a_2, a_3, \ldots, a_{n+1}\}$

$a_i = 2^{k_i} \cdot q_i$ where $k_i \geq 0$ and q_i is an odd number.

$O = \{q_1, q_2, \ldots, q_{n+1}\}$

By pigeonhole principle,
$\exists\ i \neq j$: $q_i = q_j$

Without loss of generality, assume $2^{k_i} \geq 2^{k_j}$. Then:

$\frac{a_i}{a_j} = \frac{2^{k_i} \cdot q_i}{2^{k_j} \cdot q_j} = 2^{k_i - k_j}$ an integer,
therefore a_i is a multiple of a_j.

TA: Amber Lager

During September, a TA (to remain nameless) drinks 45 bottles of beer and $\geq$ 1 bottle per day.

Claim: $\exists$ consecutive days during which the TA drinks exactly 14 bottles.

TA: Amber Lager

During September, a TA (to remain nameless) drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which the TA drinks exactly 14 bottles.

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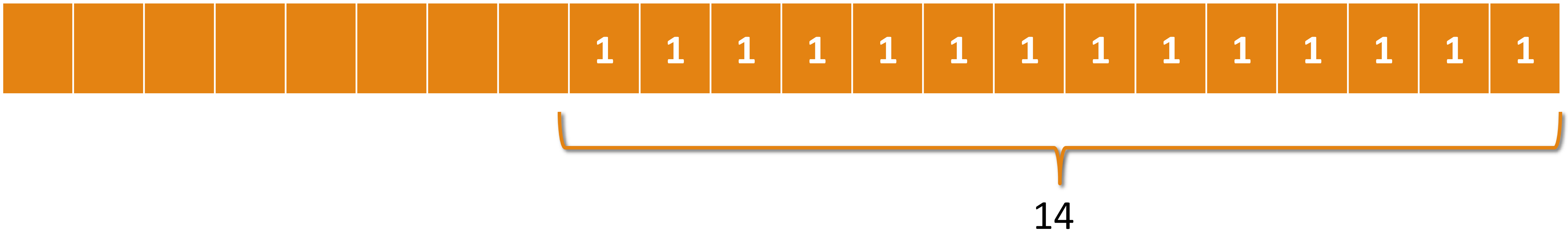
Claim: $\exists$ consecutive days during which the TA drinks exactly 14 bottles.



TA: Amber Lager

During September, a TA drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

For $i = 1, \ldots, 30, b_i =$ number of bottles drank on September $i\text{th}, \mathrm{b_i} \geq 1$

$$b_1 + b_2 + b_3 + \ \ldots + b_{30} = 45$$

We want to find a subsum $b_i + \ \ldots + b_j = 14.$

$a_i = b_1 + b_2 + \cdots + b_i$ = total bottles drank from Sept 1st to Sept i^{th}

Since each day at least one bottle is drank, $a_1, a_2, a_3, \ldots, a_{30}$ are all distinct.

We must find two values, a_i and a_j, $a_i < a_j$, such that $a_j - a_i = 14.$

Recall by our definition of a_i and a_j that

$$a_j - a_i = b_{i+1} + b_{i+2} + \cdots + b_{j-1} + b_j$$

Which means $b_{i+1} + b_{i+2} + \cdots + b_{j-1} + b_j = 14$ and we are done.

TA: Amber Lager

During September, a TA (to remain nameless) drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: $\exists$ consecutive days during which the TA drinks exactly 14 bottles.

Recall by our definition of a_i and a_j that

$$a_j - a_i = b_{i+1} + b_{i+2} + \cdots + b_{j-1} + b_j$$

Which means

$$b_{i+1} + b_{i+2} + \cdots + b_{j-1} + b_j = 14$$

and we are done.

								b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}								
								1	3	1	5	1	1	2								

a_{14}

$a_{21} = a_{14} + 14$

TA: Amber Lager

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Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

For $i = 1, \ldots, 30, b_i =$ number of bottles drank on September ith, $\mathrm{b_i} \geq 1$

$$b_1 + b_2 + b_3 + \ \ldots + b_{30} = 45$$

We want to find a subsum
$b_i + \ \ldots + b_j = 14.$

$a_i = b_1 + b_2 + \cdots + b_i$ = total bottles drank from Sept 1st to Sept i^{th}

Since each day at least one bottle is drank, $a_1, a_2, a_3, \ldots, a_{30}$ are all distinct.
Now add 14 to each a_i:

$$a_1 + 14, a_2 + 14, a_3 + 14, \ldots, a_{30} + 14$$

Take the union of the set above with:

$$a_1, a_2, a_3, \ldots, a_{30}$$

How many numbers are in the set:

$$\{1, 2, \ldots, a_{30} + 14\}$$
$$= \{1, 2, \ldots, 45 + 14\}$$
$$= \{1, 2, \ldots, 59\}$$

TA: Amber Lager

During September, TA drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

For $i = 1, \dots, 30, b_i =$ number of bottles drank on September $i\text{th}, \mathrm{b_i} \geq 1$

$$b_1 + b_2 + b_3 + \ \dots + b_{30} = 45$$

We want to find a subsum that $= 14$.

$S_1 = a_1, a_2, a_3, \dots, a_{30}$
$S_2 = a_1 + 14, a_2 + 14, a_3 + 14, \dots ,$
$a_{30} + 14$

$S_1 \cup S_2$ are 60 numbers belonging to set:

$$= \{1, 2, \dots, 59\}$$

By the pigeonhole principle, there must be 2 numbers that are equal.

Can the numbers that are equal both be from the same sequence?

All numbers within each sequence are distinct.

TA: Amber Lager

During September, TA drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

For $i = 1, \ldots, 30, b_i =$ number of bottles drank on September $i\mathrm{th}, \mathrm{b_i} \geq 1$

$$b_1 + b_2 + b_3 + \ \ldots + b_{30} = 45$$

We want to find a subsum that $= 14$.

$S_1 = a_1, a_2, a_3, \ldots, a_{30}$
$S_2 = a_1 + 14, a_2 + 14, a_3 + 14, \ldots,$
$a_{30} + 14$

$S_1 \cup S_2$ are 60 numbers belonging to set:
$$= \{1, 2, \ldots, 59\}$$

There must be a number in S_1 and a number in S_2 that match.

$\exists i, j{:}\, a_i = a_j + 14$
$14 = a_i - a_j$
$= b_{j+1} + b_{j+1} + \cdots + b_i$

Therefore from Sept j+1 to Sept i the TA drank 14 bottles.

Generalized Pigeonhole Principle

Consider a week (7 days).

How many people would I need to guarantee that at least 5 people we born on the same day of the week?

What is the most number of people we could have and still not have 5 people born on the same day?

Mon	Tues	Wed	Thu	Fri	Sat	Sun

Generalize Pigeonhole Principle:

If we place n objects into k boxes, there is at least one box with $\left\lceil \frac{n}{k} \right\rceil$ objects.

In our example:

If we have 7 days and 29 people, at least one day will have $\left\lceil \frac{29}{7} \right\rceil = 5$ people born on that day.