Discrete Math: PSet

Due on December 15, 2022 at 3:10pm

Professor Isaac Newton Section A

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Problem 1

Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Solution

Suppose $G = ((V_1, V_2), E)$, where there is no edges in set V_1, and V_2. Let $v := |V|, k = |V_1|$, then $|V_2| = v - k$. Then in maximally, there are $v(k - v)$ edges. Thus by AM-GM inequality,

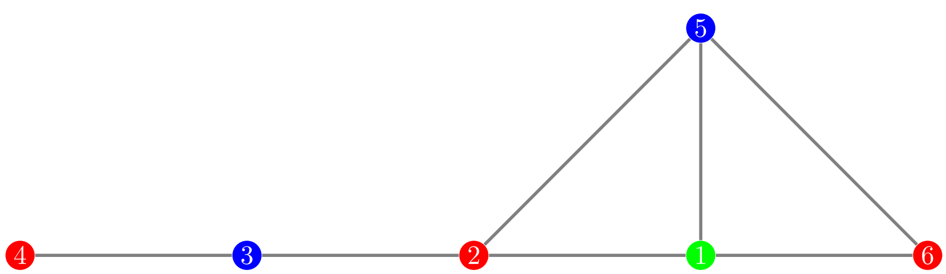
$$\begin{aligned} e &\leq v(k-v) \\ &\leq \frac{(v+k-v)^2}{4} \\ &= \frac{v^2}{4} \end{aligned}$$

Problem 2

Radio stations broadcast their signal at certain frequencies. However, there are a limited number of frequencies to choose from, so nationwide many stations use the same frequency. This works because the stations are far enough apart that their signals will not interfere; no one radio could pick them up at the same time. Suppose six new radio stations are to be set up in a currently unpopulated (by radio stations) region. The distances among stations are recorded in the table below. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

Solution

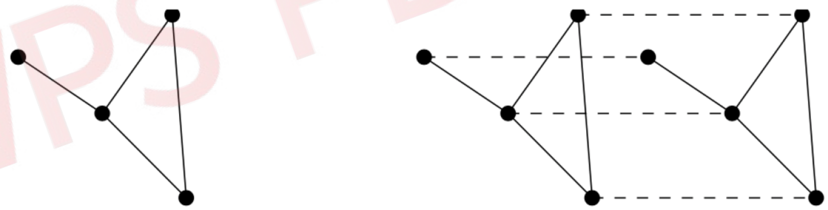
We regard each radio station as vertex, and two vertices connects together if their distance is below 150 miles. And the graph can be colored in 3 colors in the following way.



Problem 3

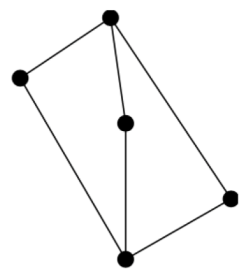
The double of a graph G consists of two copies of G with edges joining corresponding vertices. For example, a graph appears below on the left and its double appears on the right. Some edges in the graph on the right are dashed to clarify its structure.

Problem 3 continued on next page. . .



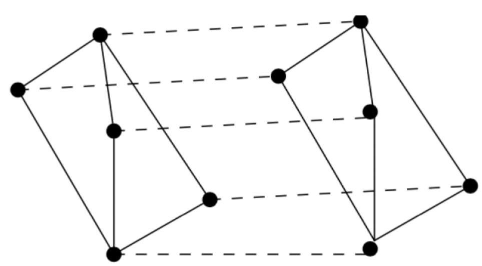
(a)

Draw the double of the graph shown below.



Solution

(a)



(b) Suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so forth. Use induction on n to prove that G_n is bipartite for all $n \geq 1$.

Solution

Base case is obvious , since we have G_1 is bipartite by definition.

Assume that G_n is bipartite. We will construct 2-coloring for G_{n+1}. First off we are able to construct a 2-coloring for the two subgraphs(G'_n and G''_n) of G_{n+1} which are isomorphic to G_n, for the fact that G_n is bipartite.

Now we examine the edges that connect G'_n and G''_n. Let $v \in G'_n$, then $\forall w \in N_v$ we have $w \in G'_n$ xor $w \in G''_n$. If $w \in G'_n$, and color(v) is black, then by the bipartite graph property, color(w) is white. If $w \in G''_n$, we set color(w) = white. We do the opposite operation of color(v) is black. We then claim that G''_n is then constructed with 2-color.

Assume the contrary that there are two vertices x'', y'' in G''_n such that x'', y'' are of the same color and $\{x'', y''\} \in E$. By definition of the color construction, we have the corresponding vertices $x', y' \in G'_n$ are also the same color but different from $\{x'', y''\}$. Then the path from x'' to y'' has even length. but that is impossible since x'' to y'' has odd length.

Thus G_{n+1} must be bipartite as well. By induction principle, $\forall n \in \mathbb{N}.G_n$ is bipartite. $\square$

Problem 4

Let m, n, and r be nonnegative integers with $r \leq m$ and $r \leq n$. Prove the following formula by a combinatorial proof.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} \tag{*}$$

Solution

Let $S = \{1, \cdots, m+n\}$, $A = \{1, \cdots, n\}$, $B = \{n+1, \cdots, m\}$. We immediately have $S = A \cup B$, and $A \cap B = \{\}$.

On the one hand, S has

$$\binom{m+n}{r}$$

subsets with cardinality r. On the other hand, let $U_k \subseteq S$, with $|U_k| = r$. Then $U_k = S \cup T$, where $S \subseteq A$, and $T \subseteq B$, Thus there are

$$\binom{m}{k}\binom{n}{r-k}$$

such U_k if $|S| = k$. Therefore the number of subsets of S with cardinality r can also be expressed as

$$\sum_{k=0}^{r} |U_k| = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Thus (*) has been proven.

□

Problem 5

Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}$$

Solution

We rewrite the identity as

$$\sum_{k=0}^{n-1} \binom{k+2}{2}\binom{n-k}{2} = \binom{n+3}{5}$$

Consider $S = \{1, \cdots, n+3\}$. Let $T_i \subseteq S$, with $|T_i| = 5$. T_i has pattern $T_i = A_i \cup \{i\} \cup B_i$, where $A_i \subseteq \{1, \cdots, i-1\}$, and $B_i \subseteq \{i+1, \cdots, n+3\}$, with $|A_i| = |B_i| = 2$, and $i \in \{3, \cdots, n+1\}$ Thus, we have set $k = i - 2$

$$\begin{aligned}
\binom{n+3}{5} &= |\{T|T \subseteq S, |T| = 5\}| \\
&= \sum_{i=2}^{n+1} |T_i| \\
&= \sum_{i=2}^{n+1} \binom{i-1}{2} \cdot 1 \cdot \binom{(n+3)-(i-1)-1}{2} \\
&= \sum_{i=2}^{n+1} \binom{i-1}{2}\binom{n+3-i}{2} \\
&= \sum_{k=0}^{n-1} \binom{k+2}{2}\binom{n-k}{2}
\end{aligned}$$

□

Problem 6

Find the number of solutions of the equation $x_1 + x_2 + x_3 = 11$, where x_1, x_2, x_3 are nonnegative integers with $x_1 \leq 3, x_2 \leq 4, x_3 \leq 6$.

Solution

The generating function for this problem is

Problem 6 continued on next page. . .

$$\begin{aligned}
G(x) &= \left(1+x+x^2+x^3\right)\left(1+x+x^2+x^3+x^4\right)\left(1+x+x^2+x^3+x^4+x^5+x^6\right) \\
&= \left(\frac{1}{1-x}-\frac{x^4}{1-x}\right)\left(\frac{1}{1-x}-\frac{x^5}{1-x}\right)\left(\frac{1}{1-x}-\frac{x^7}{1-x}\right) \\
&= \frac{(1-x^4)(1-x^5)(1-x^7)}{(1-x)^3} \\
&= (1-x^4)(1-x^5)(1-x^7)\sum_{k\geq 0}\binom{2+k}{k}x^k \\
&= (1+x^9+x^{12}+x^{11}-x^{16}-x^4-x^5-x^7)\sum_{k\geq 0}\binom{2+k}{k}x^k
\end{aligned}$$

Thus, the coefficient of x^{11} in this function is

$$\binom{2+11}{11}+\binom{2+2}{2}+\binom{2+0}{0}-\binom{2+7}{7}-\binom{2+6}{6}-\binom{2+4}{4}=6$$

□

Problem 7

Show that in any set of $n+1$ positive integers not exceeding $2n$ there must be two that are relatively prime.

Solution

Let S be a set of positive integers of $n+1$ elements with the maximum element $m \leq 2n$. Let m be the minimum element of S. We claim that there are two elements in S, such that they differ by one.
Assume the contrary, there are no such elements. Then the maximum element of S is at least

$$M \geq m + \sum_{k=1}^{n} 2 = m + 2n \geq 1 + 2n > 2n$$

We have a contradiction then.

□

Problem 8

A 0-1 sequence an with $2m$ terms is said to be normal if the following two conditions are satisfied.

- There exist m terms equal to 0 and the other m terms equal to 1 in an.
- For arbitrary $k \leq 2m$, the number of terms equal to 0 is not less than that of terms equal to 1 in the first k terms $a_1, a_2, \cdots, a_k$.

Please complete the following questions.

(a) Show that the number of abnormal 0-1 sequences an with $2m$ terms equals that of sequences an of which $(m+1)$ terms are 0s and $(m-1)$ terms are 1s.

(b) For $m = 4$, determine the number of different normal 0-1 sequences an. Note: An abnormal 0-1 sequence an is a 0-1 sequence that does not satisfy the properties of normal 0-1 sequences.

Solution

(a) Let $f : \{0, 1\} \to \{-1, 1\}$ be a function such that:

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ -1 & \text{if } n = 0 \end{cases} \tag{1}$$

The condition 2 for a normal sequence become

$$\forall n \in \{1, \cdots, 2m\} \sum_{k=1}^{n} f(a_k) \geq 0$$

We consider a arbitrary sequence $\{b_1, \cdots, b_n\}$ with $m+1$ 1s, and $m-1$ zeros. Let A be the set of all abnormal sequences, and let B be the set of all $(m+1)$ 1s sequences We will construct a bijection from A to B We have

$$\sum_{k=0}^{2m} f(b_k) = 2$$

Thus there must be an minimal m_0 such that

$$\sum_{k=0}^{m_0} f(b_k) = 1$$

And deduced that

$$\sum_{k=m_0+1}^{2m} f(b_k) = 1$$

We construct σ given by

$$b_n \mapsto \begin{cases} \text{flip}(b_n) & \text{for } n \leq m_0 \\ b_n & \text{otherwise} \end{cases} \tag{2}$$

where $\text{flip}(0) = 1, \text{flip}(1) = 0$ In this case we have

$$\sum_{k=0}^{2m} f(\sigma(b_n)) = \sum_{k \leq m_0} -f(b_n) + \sum_{k > m_0} f(b_n) = -1 + 1 = 0$$

Problem 8 continued on next page. . .

Hence $\sigma(b_n)$ is a abnormal sequence. Note that it is easy to check that the flip function is 1-1, then so is σ. We now need to show that σ is also on-to. Let $\{a'_n\}$ be an abnormal sequence, by its definition there is a minimal n_0 such that

$$\sum_{k=0}^{n_0} f(a'_k) = -1$$

We choose a sequence $\{b'_n\}_1^{2m} = \{\text{flip}(a'_1), \cdots, \text{flip}(a'_{n_0}), \cdots, a'_{n_0+1}, \cdots, a'_{2m}\}$ We compute

$$\sum_{b \in \{b'_n\}_1^{2m}} f(b) = \sum_{k=0}^{n_0} \text{flip}(a'_k) + \sum_{k > n_0} f(a'_k) = 1 + 1 = 2$$

Thus $\{b'_n\}_1^{2m} \in B$, and we have the bijection constructed. □

(b) The number of normal sequence is obtained by the number of sequences minus the number of abnormal sequences, that is .

$$\binom{2n}{n} - \binom{2n}{n+1}$$

Thus the answer to this question is

$$\binom{2 \times 4}{4} - \binom{2 \times 4}{4+1} = 70 - 56 = 14$$

□