Number theory: Final Exam

Due on June 24, 2023 at 3:10pm

Professor J Section A

V **U**

Problem 1

Calculate $\mathbf{Y}$. Provide all steps you used to get the result.

$$(\mathbf{I} - \mathbf{A})\mathbf{Y} = \mathbf{X}^T \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -1 \\ -6 & -4 & -4 \\ -13 & -10 & 1 \end{bmatrix}$$

X $= [2, 1, 3]$, and $\mathbf{I}$ is the identity matrix.

Solution

To calculate $\mathbf{Y}$ in the equation $(\mathbf{I} - \mathbf{A})\mathbf{Y} = \mathbf{X}^T$, we can follow these steps:

1. Define the matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -1 \\ -6 & -4 & -4 \\ -13 & -10 & 1 \end{bmatrix}, \quad \mathbf{X}^T = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Calculate the matrix $(\mathbf{I} - \mathbf{A})$ by subtracting $\mathbf{A}$ from $\mathbf{I}$:

$$\mathbf{I} - \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 & -1 \\ -6 & -4 & -4 \\ -13 & -10 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 5 & 4 \\ 13 & 10 & 0 \end{bmatrix}$$

3. Rewrite the equation as a matrix equation:

$$(\mathbf{I} - \mathbf{A})\mathbf{Y} = \mathbf{X}^T$$

becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & 5 & 4 \\ 13 & 10 & 0 \end{bmatrix} \mathbf{Y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Let's solve the equation using Gaussian elimination:

1. Set up the augmented matrix by combining the matrix $(\mathbf{I} - \mathbf{A})$ and the vector $\mathbf{X}^T$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 6 & 5 & 4 & 1 \\ 13 & 10 & 0 & 3 \end{array}\right]$$

2. Perform row operations to transform the augmented matrix into row-echelon form:

R2 = R2 - 6R1

R3 = R3 - 13R1

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -11 \\ 0 & -3 & -13 & -23 \end{array}\right]$$

R3 = R3 - 3R2

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -11 \\ 0 & 0 & -7 & 10 \end{array}\right]$$

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R3 = (-1/7)R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -11 \\ 0 & 0 & 1 & -10/7 \end{array}\right]$$

R2 = R2 + 2R3
R1 = R1 - R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 24/7 \\ 0 & -1 & 0 & -97/7 \\ 0 & 0 & 1 & -10/7 \end{array}\right]$$

R1 = R1 + R2

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -73/7 \\ 0 & -1 & 0 & -97/7 \\ 0 & 0 & 1 & -10/7 \end{array}\right]$$

3. The row-echelon form of the augmented matrix gives us the solution to the system of equations. The values in the rightmost column correspond to the elements of $\mathbf{Y}$.
Therefore, the solution is:

$$\mathbf{Y} = \begin{bmatrix} -73/7 \\ 97/7 \\ -10/7 \end{bmatrix}$$

Thus, $\mathbf{Y} = \left[-\frac{73}{7}, \frac{97}{7}, -\frac{10}{7}\right]$.

Problem 2

Find the solution of the following system:

$$\begin{cases} x + 2y + 3z = 143 \\ x + 2y + z = 103 \\ x + y + 2z = 11 \end{cases} \tag{2}$$

Solution

First off, we set up the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 143 \\ 1 & 2 & 1 & 103 \\ 1 & 1 & 2 & 11 \end{array}\right]$$

Then, we perform row operations to transform the augmented matrix into row-echelon form:
R3 = R1 - R2
R2 = R2 - R3

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 143 \\ 0 & 1 & -1 & 92 \\ 0 & 0 & 2 & 40 \end{array}\right]$$

R3 = (1/2)R3

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$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 143 \\ 0 & 1 & -1 & 92 \\ 0 & 0 & 1 & 20 \end{array}\right]$$

R2 = R2 + R3
R1 = R1 - 3R3

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 83 \\ 0 & 1 & 0 & 112 \\ 0 & 0 & 1 & 20 \end{array}\right]$$

R1 = R1 - 2R2

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -141 \\ 0 & 1 & 0 & 112 \\ 0 & 0 & 1 & 20 \end{array}\right]$$

Therefore, the solution is:

$$\begin{cases} x = -141 \\ y = 112 \\ z = 20 \end{cases}$$