BINOMIAL COEFFICIENTS

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Problem

I want a password of length 5 consisting of lowercase letters and digits and there must be exactly 2 digits.

First we will look at the WRONG way to do it (using double counting, a common mistake).

Define a procedure:

- Choose the first digit.
- Choose a location for the first digit
- Choose a second digit
- Choose a location for the second digit
- Choose letters for the remaining three locations.

We will show how this double counts.

Problem

I want a password of length 5 consisting of lowercase letters and digits and there must be exactly 2 digits.

1	2	3	4	5

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- Choose the first digit.
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Choose 2 and location 5.
Choose 3 and location 1.
Choose all c’s

Choose 3 and location 1.
Choose 2 and location 5.
Choose all c’s

Problem

I want a password of length 5 consisting of lowercase letters and digits and there must be exactly 2 digits.

1	2	3	4	5
3	c	c	c	2

1	2	3	4	5
3	c	c	c	2

Procedure:

- Choose the first digit.
- Choose a location for the first digit
- Choose a second digit
- Choose a location for the second digit
- Choose letters for the remaining three locations.

Choose 2 and location 5.
Choose 3 and location 1.
Choose all c's

Choose 3 and location 1.
Choose 2 and location 5.
Choose all c's

I did the tasks differently but produced the same strings.

How do we solve this?

Binomial Coefficient

For a set $S, |S| = n$, the number of permutations is $n!$.

For a set $S, |S| = n$, the number of subsets of size $k, 0 \leq k \leq n$ is given by the notation:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

"n choose k" – binomial coefficient

Since these are sets, order does not matter

We will prove this by looking at how to choose all **sequences** of size k from a set of size n, and relating that to all **subsets**.

Binomial Coefficient

Example: $S = \{a, b, c\}$

All **subsets** of size 2
$= \{a, b\}, \{a, c\}, \{b, c\}$. There are 3 such **subsets**

All **sequences** of size 2:
$= (a, b), (b, a), (a, c), (c, a), (b, c), (c, b)$. There are 6 such **sequences.**

One way to find all **sequences,** if we know the **subsets**, is to find all permutations of all subsets.

We have shown that any **set** of size n has $n!$ permutations, or **sequences**.

For each **subset** of size **2**, there are **2!=2** ways to arrange them into sequences.

Informally we can see that, in this example, **#sequences = #subsets** $*$ **2!.**

Binomial Coefficient

Example: $S = \{a_1, a_2, \ldots, a_n\}$

Use the **Product Rule** to write a **sequence** of size k elements from a set S, $|S| = n$.

Procedure: write the k elements from left to right

Task 1: Choose one of n elements for position 1
Task 2: Choose one of remaining $n - 1$ elements for position 2
Task 3: Choose one of remaining $n - 2$ elements for position 3
…
Task $k - 1$: Choose one of remaining $n - (k - 2) = n - k + 2$ elements for position $k - 1$
Task k : Choose one of remaining $n - (k - 1) = n - k + 1$ elements for position k

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$: Choose a sequence of size 5.

Task 1: $(\, , \, , \, , \, , \,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, a_2, a_3, \quad , a_5, a_6, a_7\}$: Choose a sequence of size 5.

Task 1: $(a_4, , , ,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, \quad , a_3, \quad , a_5, a_6, a_7\}$: Choose a sequence of size 5.

Task 2: $(a_4, a_2, , ,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, \quad , \quad , \quad , a_5, a_6, a_7\}$: Choose a sequence of size 5.

Task 3: $(a_4, a_2, a_3, ,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, \quad , \quad , \quad , a_5, a_6, \quad \}$: Choose a sequence of size 5.

Task 4: $(a_4, a_2, a_3, a_7,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, \quad , \quad , \quad , \quad , a_6, \quad \}$: Choose a sequence of size 5.

Task 5: $(a_4, a_2, a_3, a_7, a_5)$

Binomial Coefficient

Product rule: Number of ways to write a **sequence** of size k elements from a set $S, |S| = n$, is

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+2) \cdot (n-k+1)$$

which is "sort of" a factorial:

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \cdot (n-k) \cdot \ldots \cdot 2 \cdot 1}{(n-k) \cdot \ldots \cdot 2 \cdot 1}$$

$$= \frac{n!}{(n-k)!}$$

Thus we have the $m = \frac{n!}{(n-k)!}$, where m is the number of sequences of size k in a set $S, |S| = n$.

Binomial Coefficient

Consider a set $S = \{a_1, a_2, \dots, a_n\}$ of size n

If we know the number of **subsets** of S of size k, then we can find all **sequences** of S of size k by finding all **permutations** of each **subset**.

Let's formalize this using the Product Rule.

Task 1: Choose a subset S_k of size k from a set $S, |S| = n$. There are $\binom{n}{k}$ ways to do this task

Task 2: Choose a permutation of S_k. There are $k!$ ways to do this task

Using the Product rule, there are $\binom{n}{k}\,k!$ **sequences** of size k in a set $S, |S| = n$.

Binomial Coefficient

The number of **sequences** of size k in a set $S, |S| = n$ is $\binom{n}{k} k!$, where $\binom{n}{k}$ is the number of **subsets** of size k.

Let the number of sequences of size k in a set S of size n be m.

So $m = \binom{n}{k} k!$

If we solve for $\binom{n}{k}$ we get

$$\binom{n}{k} = \frac{m}{k!}$$

We have an expression for $\binom{n}{k}$ in terms of m.

Binomial Coefficient

The number of **subsets** of size k in a set $S, |S| = n$ is given by

$$\binom{n}{k} = \frac{m}{k!}$$

The number of **sequences** of size k is given by

$$m = \frac{n!}{(n-k)!}$$

Thus

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

We can also get the number of **sequences** by multiplying both sides by $k!$:

$$\binom{n}{k} k! = \frac{n!}{(n-k)!}$$

The number of sequences of size k differs from the number of subsets of size k by a factor of $k!$.

$k!$ is the cost of putting these subsets in order, or counting all permutations as identical.

Binomial Coefficient

All subsets of size k from set S:

List all permutations of each subset. Note there are $k!$ such permutations

$$\binom{n}{k} \begin{cases} a = \{k \text{ elements}\} & a_1, a_2, \dots, a_{k!} \\ b = \{k \text{ elements}\} & b_1, b_2, \dots, b_{k!} \\ c = \{k \text{ elements}\} & c_1, c_2, \dots, c_{k!} \\ \dots & \dots \end{cases}$$

The total number of permutations listed is exactly the total number of permutations of size k, which is $\frac{n!}{(n-k)!}$.

Thus

$$\frac{n!}{(n-k)!} = \binom{n}{k} k!$$

or

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial Coefficient

The number of ways to choose a subset of size k from a set $S, |S| = n$ is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Given a deck of 52 cards, how many hands of 5 cards are there?

In this case we do not care about the order (though you might rearrange them, but the order you receive them does not matter).

Binomial Coefficient

The number of ways to choose a subset of size k from a set S, $|S| = n$ is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Given a deck of 52 cards, how many hands of 5 cards are there?

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} = \frac{52!}{5!\,47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2\,598\,960$$

Binomial Coefficient

The number of ways to choose a subset of size k from a set $S, |S| = n$ is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

How many bitstrings of length n with exactly k many 1's are there?

1	2	3	4	…	n-2	n-1	n
1	0	1	1	…	0	0	1

Choose k positions out of n possible positions
-chosen positions get a 1
-other positions get a 0
= $\binom{n}{k}$ bitstrings of length n with k 1's.

Binomial Coefficient

Choose k positions out of n possible positions

-chosen positions get a 1

-other positions get a 0

= $\binom{n}{k}$ bitstrings of length n with k 1's.

This is not surprising, since we have seen that we can encode subsets as bitstrings and vice versa

1	2	3	4	…	n-2	n-1	n
1	0	1	1	…	0	0	1

Binomial Coefficient

We can think of multiplying by a factorial, or dividing by a factorial, as ways of accounting for items in order or where order doesn't matter.

I have b beer bottles and c cider bottles. How many ways can I arrange them on a line? Start with

- all beer bottles are identical and
- all cider bottles are identical.

So the order of beer bottles doesn't matter and the order of the cider bottles doesn't matter.

All that matters is whether there is a beer bottle or cider bottle at location i.

Binomial Coefficient

I have b identical beer bottles and c identical cider bottles to arrange on a line.

There are $b + c$ possible locations on this line for any bottle. Our procedure:

- Choose b of these $b + c$ locations for beer. There are $\binom{b+c}{b}$ ways to do this.
- Choose the remaining c locations for cider. There is 1 way to do this.

Our answer is $\binom{b+c}{b}$. But what if every cider bottle was distinct? (Beer are still identical.) I need to add a third task, which is to permute the cider bottles. This counts all possible orders of cider bottles:

$$\binom{b+c}{b} \cdot c!$$

Binomial Coefficient

What if now the beer and cider are both distinct? I add a 4^{th} task to arrange the beer bottles:

$$\binom{b+c}{b} \cdot c! \cdot b!$$

$$= \frac{(b+c)!}{b!\,c!} b!\,c!$$

$$= (b+c)!$$

Which makes perfect sense, as now we have a set of $b + c$ elements that are all distinct and we are simply counting permutations of this set.

Binomial Coefficient

What if we had a string of length n, we could use any digit, but there has to be exactly k 0's.

Procedure:

- Choose k locations for 0.
- Each of the remaining $n-k$ locations gets one of 9 digits.

$$\binom{n}{k} \cdot 9^{n-k}$$

1	2	3	4	…	n-2	n-1	n
4	0	2	9	…	0	0	5

Binomial Coefficient

What if we had a string of length n, we could use any digit or lower case letter, but there has to be exactly k digits?

Procedure:

- Choose k locations for digits.
- Choose one of 10 digits for each of the k locations.
- Each of the remaining $n-k$ locations gets one of 26 lowercase letters.

$$\binom{n}{k} \cdot 10^k \cdot 26^{n-k}$$

1	2	3	4	…	n-2	n-1	n
4	a	2	x	…	t	p	5

Why is it a Binomial Coefficient?

$$(x+y)^2$$

$$= x^2 + 2xy + y^2$$

$$(x+y)^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^6 = (x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

Binomial Coefficient

$$(x+y)^6 = (x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

We can determine the coefficients of, for example, x^4y^2. Note that in each of the terms $(x+y)$, we choose either x or y. That is why the polynomials for each term sum to 6.

Binomial Coefficient

$$(x+y)^6 = (\boxed{x}+y)\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

For the term x^4y^2 we choose 2 y 's and the rest are x's. What are the number of ways to do that?

Binomial Coefficient

$$(x+y)^6 = (\boxed{x} + y)\cdot(\boxed{x} + y)\cdot(x + \boxed{y})\cdot(\boxed{x} + y)\cdot(x + \boxed{y})\cdot(\boxed{x} + y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

For the term x^4y^2 we choose 2 y 's and the rest are x's. What are the number of ways to do that?

Observe for each term there are two choices, x or y.

When choosing a subset there are two choices for each element, include or don’t include

Binomial Coefficient

$$(x+y)^6 = (\boxed{x} + y)\cdot(\boxed{x} + y)\cdot(x + \boxed{y})\cdot(\boxed{x} + y)\cdot(x + \boxed{y})\cdot(\boxed{x} + y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \binom{6}{2}x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

For the term x^4y^2 we choose 2 y 's and the rest are x's. What are the number of ways to do that?

Binomial Coefficient

$$(x+y)^6 = (\boxed{x}+y)\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)$$

$$(x+y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6$$

Notice that for the term x^4y^2 we can instead choose 4 x's and the rest are y's. What are the number of ways to do that? It should be the same and it is.

$$(x+y)^6 = \binom{6}{6}x^6 + \binom{6}{5}x^5y + \binom{6}{4}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{2}x^2y^4 + \binom{6}{1}xy^5 + \binom{6}{0}y^6$$

Newton's Binomial Theorem

In general:

$$(x+y)^n$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-2}x^2y^{n-2}$$

$$+ \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

Newton's Binomial Theorem

Example:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$(x+y)^{75}$, what is the coefficient of $x^{50}y^{25}$?

$$= \binom{75}{25}$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For $(3x + 2y)^{10}$, what is the coefficient of x^3y^7?

Let $x' = 3x$ and let $y' = 2y$. We can rewrite our expression as:

$$(3x + 2y)^{10} = (x' + y')^{10}$$

$$(x' + y')^{10} = \sum_{k=0}^{10} \binom{10}{k} x'^{10-k} y'^{k}$$

We want the coefficient of the term corresponding to $k = 7$

The term with x^3y^7 is equivalent to when $\mathrm{k} = 7$ which is:

$$\binom{10}{7} x'^3 y'^7 = \binom{10}{7} (3x)^3 (2y)^7 = \binom{10}{7} 3^3 x^3 2^7 y^7 = \binom{10}{7} 3^3 2^7 x^3 y^7$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For $(3x+2y)^{10}$, what is the coefficient of x^3y^7?

Let $x' = 3x$ and let $y' = 2y$. We can rewrite our expression as:

$$(3x+2y)^{10} = (x'+y')^{10}$$

$$(x'+y')^{10} = \sum_{k=0}^{10} \binom{10}{k} x'^{10-k} y'^k$$

The term of this series when $\mathrm{k} = 7$ is:

We want the coefficient of the term corresponding to $k = 7$

$$\binom{10}{7} x'^3 y'^7 = \binom{10}{7} (3x)^3 (2y)^7 = \binom{10}{7} 3^3 x^3 2^7 y^7 = \boxed{\binom{10}{7} 3^3 2^7} \; x^3 y^7$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$(7x - 13y)^{75}$, what is the coefficient of $x^{50}y^{25}$?

$$(7x - 13y)^{75} = [(7x) + (-13y)]^{75}$$

$$[(7x) + (-13y)]^{75} = \sum_{k=0}^{75} \binom{75}{k} (7x)^{75-k} (-13y)^k$$

We want the coefficient of the term corresponding to $k = 25$

$$\binom{75}{25} (7x)^{50} (-13y)^k \;= \binom{75}{25} 7^{50} x^{50} (-1)^{25} 13^{25} y^{25}$$

$$= -\binom{75}{25} 7^{50} \cdot 13^{25} x^{50} y^{25}$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

If we take Newton's Binomial Theorem and set $x = 1$ and $y = 1$ we get

$$(x+y)^n = (1+1)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 1^k = \binom{n}{0} 1^n \cdot 1^0 + \binom{n}{1} 1^{n-1} \cdot 1^1 + \binom{n}{2} 1^{n-2} \cdot 1^2 + \cdots + \binom{n}{n} 1^0 \cdot 1^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

If we take Newton's Binomial Theorem and set $x = 1$ and $y = 1$ then we get the expression above. Thus

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$$

Is there another way to argue this?

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$$

Let's use counting. Can we "map" it to another problem?

The number of subsets of a set of size $n = 2^n$. So let us argue that the left hand side of the above expression also represents the number of subsets of a set of size n.

How many subsets of size 0? $\binom{n}{0}$
How many subsets of size 1? $\binom{n}{1}$
How many subsets of size 2? $\binom{n}{2}$
…
How many subsets of size n? $\binom{n}{n}$

Combinatorial Proof: Show that we know what one side counts, and show the other side counts the same thing

Example

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Can we show the following using counting?

$$0 \le k \le n, \binom{n}{k} = \binom{n}{n-k}$$

We can use the formula of course.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,\big(n-(n-k)\big)!} = \frac{n!}{(n-k)!\,k!}$$

Example

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Can we show the following using counting?

$$0 \le k \le n, \binom{n}{k} = \binom{n}{n-k}$$

k elements

$n-k$ elements

Or we can use a combinatorial proof:

$$\binom{5}{3} = \binom{5}{2}$$

If we choose 3 elements, we are leaving 2 elements behind. So when we choose 2 elements we are also choosing 3 elements.

Likewise if we choose 2 elements, we are leaving 3 elements behind.

Example: Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$1 \le k \le n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

We can use the formula again. Can we use a combinatorial proof? What are we counting?

In $\binom{n+1}{k}$ we have a set of size $n+1$ and we are counting the number of subsets of size k.

$n+1$ elements

Example: Pascal's Triangle

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$$1 \le k \le n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- **1 special element**

n non-special elements

We can use the formula again. Can we use a combinatorial proof? What are we counting?

In $\binom{n+1}{k}$ we have a set of size $n+1$ and we are counting the number of subsets of size k.

Let us take one element, the special element.
How many subsets of size k include the special element?

How many subsets of size k do not include the special element?

Example: Pascal's Triangle

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Of the remaining n elements, take all subsets of size $k-1$

How many subsets of size k do not include the special element?

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$$1 \le k \le n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- **1 special element**

n non-special elements

We are counting two sets of subsets. In one set all the subsets contain the special element.

In the other set, all the subsets do not contain the special element.

Are they disjoint?

Combinatorial proofs are nice because you can understand why something is true.