Geometry of Surfaces - Exercises

Exercises marked with $*$ are to be answered (partially) in the online quiz for this week on the Keats page for this module.

39. Let σ be a surface and $Ldu^2+2Mdudv+Ndv^2$ its second fundamental form. Compute the second fundamental form of the surface $\tilde{\sigma} = \lambda\sigma$ with $0 \neq \lambda \in \mathbb{R}$.

40.* Consider the surface given by

$$\sigma : (0,3) \times (0,3) \to \mathbb{R}^3 \ , \ (u,v) \mapsto (u^2, uv, u-v).$$

Calculate the second fundamental form of σ at $\sigma(1,1) = (1,1,0)$.

41. Let $\mathcal{S}$ be a surface and let γ be a unit speed curve in $\mathcal{S}$. Let σ and σ' be two parametrizations of $\mathcal{S}$ such that $\mathbf{N}' = -\mathbf{N}$. Show that $\kappa_n' = -\kappa_n$ where κ_n' and κ_n are the normal curvatures of γ with respect to σ' and σ.

42. Let $\gamma : (-1,1) \to \mathbb{R}^3$ be a unit speed curve on a surface σ and let κ_n and κ_g be its normal and geodesic curvatures. Compute the normal and geodesic curvatures $\tilde{\kappa}_n$ and $\tilde{\kappa}_g$ of $\tilde{\gamma} : (-1,1) \to \mathbb{R}^3 \ , \ s \mapsto \gamma(-s)$ at $\tilde{\gamma}(0)$.

43.* Let $\gamma : (-1,1) \to \mathbb{R}^3$ be a unit speed curve that is contained in a surface $\mathcal{S}$. Assume that $\gamma(0) = O = (0,0,0)$, $\mathbf{t}(0) = \frac{1}{\sqrt{2}}(1,1,0)$ and $\dot{\mathbf{t}}(0) = (1,-1,2)$, and that the unit normal $\mathbf{N}$ to $\mathcal{S}$ at O is $(0,0,1)$. Compute the geodesic curvature and the normal curvature of γ at O.

44.* Let $\gamma : (-1,1) \to \mathbb{R}^3$ be a unit speed curve that is contained in a surface $\mathcal{S}$. Assume that $\gamma(0) = O = (0,0,0)$, $\mathbf{t}(0) = (1,0,0)$ and $\dot{\mathbf{t}}(0) = (0,2,1)$, and that the unit normal $\mathbf{N}$ to $\mathcal{S}$ at O is $(0,0,1)$. Compute the geodesic curvature and the normal curvature of γ at O.

45. Let $\mathcal{S}$ be a surface and suppose that there is a plane that is tangent to $\mathcal{S}$ along a unit speed curve γ. Show that the normal curvature of $\mathcal{S}$ along γ is identically zero. What is its geodesic curvature?

46. Let $\sigma : U \to \mathbb{R}^3$ be a regular surface and suppose that p is a point on the surface that is farthest away from the origin $O = (0,0,0)$. Let k_n be the normal curvature of σ at p with respect to a normal section. Prove that $|k_n| \geq \frac{1}{\|p\|}$. [Hint: Use Exercise 11.]

47.* Let κ_1 and κ_2 be the principal curvatures of a surface σ. What are the principal curvatures of the surface $\tilde{\sigma} = \lambda\sigma$ with $0 \neq \lambda \in \mathbb{R}$?