Geometry of Surfaces

5CCM223A/6CCM223B

Video 1

The concept of a curve

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What is a curve?

Two viewpoints: analytic and geometric

$y = mx + c$ line

$x^2 + y^2 = 1$ circle

$y = x^2$ parabola

$f(x, y) = 0$ $\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\} \subset$ plane

path traced out by moving point

A curve in $\mathbb{R}^n$ is a map

$$\gamma : (\alpha, \beta) \to \mathbb{R}^n$$

with $0 < n \in \mathbb{Z}$ and $-\infty \leqslant \alpha < \beta \leqslant \infty$

The image $\gamma((\alpha, \beta))$ is a curve in the *set of points* sense

Parabola $y = x^2$ Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Then

$$\gamma_2(t) = \gamma_1(t)^2$$

Obvious solution: $\gamma_1(t) = t$, $\gamma_2(t) = t^2$, so

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2 \ , \ t \mapsto (t, t^2)$$

Other solutions:

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2 \ , \ t \mapsto (t^3, t^6)$$

and

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2 \ , \ t \mapsto (2t, 4t^2)$$

Circle $x^2 + y^2 = 1$ First attempt: Put $x = t$
Then $y = \sqrt{1-t^2}$ or $y = -\sqrt{1-t^2}$

$$\gamma : (-1, 1) \to \mathbb{R}^2 \ , \ \ t \mapsto (t, \sqrt{1-t^2})$$

Parametrizes upper semicircle but not entire circle

Second attempt: γ must satisfy

$$\gamma_1(t)^2 + \gamma_2(t)^2 = 1$$

Obvious solution is $\gamma_1(t) = \cos(t)$ and $\gamma_2(t) = \sin(t)$, so

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2 \ , \ \ t \mapsto (\cos(t), \sin(t))$$

parametrizes the circle

Consider curve

$$\gamma : (\alpha, \beta) \to \mathbb{R}^n \ , \ \ t \mapsto (\gamma_1(t), \ldots, \gamma_n(t))$$

Define derivatives

$$\dot{\gamma}(t) = \frac{d\gamma}{dt}(t) = \left(\frac{d\gamma_1}{dt}(t), \ldots, \frac{d\gamma_n}{dt}(t) \right) = (\gamma_1'(t), \ldots, \gamma_n'(t))$$

$$\ddot{\gamma}(t) = \frac{d^2\gamma}{dt^2}(t) = \left(\frac{d^2\gamma_1}{dt^2}(t), \ldots, \frac{d^2\gamma_n}{dt^2}(t) \right) = (\gamma_1''(t), \ldots, \gamma_n''(t))$$

and so on...

γ smooth

$\iff \gamma_1, \ldots, \gamma_n$ smooth

$\iff$ derivatives $\frac{d^k\gamma_i}{dt^k}$ exist for all $i \in \{1, \ldots, n\}$, $0 < k \in \mathbb{Z}$

$\dot\gamma(t)$ tangent vector of γ at $\gamma(t)$

$$\begin{aligned}
\dot\gamma(t) &= \frac{d\gamma}{dt}(t) = \left(\frac{d\gamma_1}{dt}(t), \ldots, \frac{d\gamma_n}{dt}(t)\right) \\
&= \left(\lim_{\delta t \to 0} \frac{\gamma_1(t+\delta t) - \gamma_1(t)}{\delta t}, \ldots, \lim_{\delta t \to 0} \frac{\gamma_n(t+\delta t) - \gamma_n(t)}{\delta t}\right) \\
&= \lim_{\delta t \to 0} \left(\frac{\gamma_1(t+\delta t) - \gamma_1(t)}{\delta t}, \ldots, \frac{\gamma_n(t+\delta t) - \gamma_n(t)}{\delta t}\right) \\
&= \lim_{\delta t \to 0} \frac{\gamma(t+\delta t) - \gamma(t)}{\delta t}
\end{aligned}$$

Proposition. *If $\dot{\gamma}$ is constant, then $\gamma((\alpha, \beta))$ is part of a straight line*

Proof. Assume that $\dot{\gamma}(t) = a \in \mathbb{R}^n$ for all t. Then

$$\gamma(t) - \gamma(t_0) = \int_{t_0}^{t} \dot{\gamma}(u) du = \int_{t_0}^{t} a du = (t - t_0)a$$

Thus

$$\gamma(t) = ta + b \quad \text{with} \quad b = \gamma(t_0) - t_0 a$$

If $a \neq 0$, then $\gamma((\alpha, \beta))$ is contained in the line parallel to the vector a containing b

If $a = 0$, then γ is constant and hence contained in any line containing b

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Video 2

Arc length

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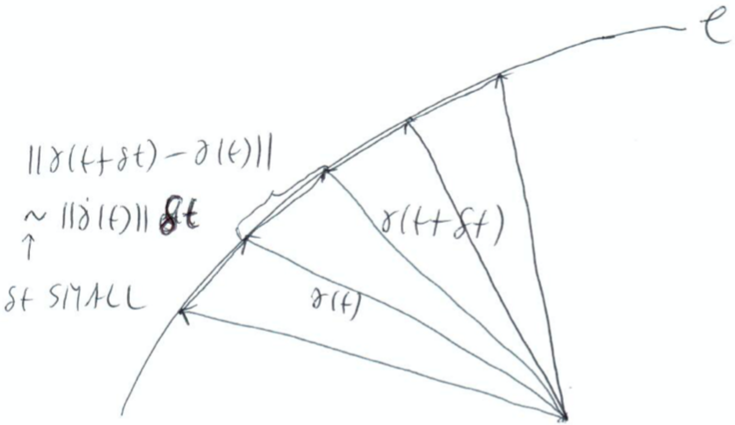


length of $\mathcal{C} \sim$ sum of such line segments

$\delta t \to 0$ leads to exact length

The arc length of a curve γ starting at $\gamma(t_0)$ is

$$\boxed{s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du}$$

If $\dot{\gamma}(t_0) \neq 0$, then

$$s(t) \begin{cases} < 0 & \text{if } t < t_0 \\ = 0 & \text{if } t = t_0 \\ > 0 & \text{if } t > t_0 \end{cases}$$

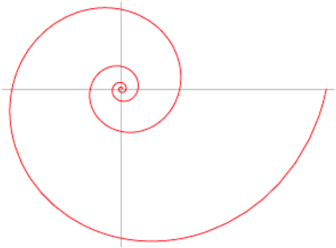
Note:

$$\frac{ds}{dt}(t) = \frac{d}{dt} \int_{t_0}^{t} \|\dot{\gamma}(u)\| du = \|\dot{\gamma}(t)\|$$

is called the speed of γ

Example. The logarithmic spiral

$$\gamma : \mathbb{R} \to \mathbb{R}^2 \ , \ t \mapsto e^t(\cos(t), \sin(t)) = (e^t \cos(t), e^t \sin(t))$$



$$
\begin{aligned}
\gamma(t) &= \left(e^t \cos(t), e^t \sin(t)\right) \\
\dot{\gamma}(t) &= \left(e^t \cos(t) - e^t \sin(t), e^t \sin(t) + e^t \cos(t)\right) \\
&= e^t \left(\cos(t) - \sin(t), \sin(t) + \cos(t)\right) \\
\|\dot{\gamma}(t)\|^2 &= e^{2t} \left((\cos(t) - \sin(t))^2 + (\sin(t) + \cos(t))^2\right) \\
&= e^{2t} \left(2\cos^2(t) + 2\sin^2(t)\right) = 2e^{2t}
\end{aligned}
$$

Calculate the arc length $s(t)$ at $t_0 = 0$

$$
s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du = \int_0^t \sqrt{2e^{2u}} du = \sqrt{2} \int_0^t e^u du = \sqrt{2}(e^t - 1)
$$

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Video 3

Unit speed reparametrizations

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A curve $\bar{\gamma}$ is a reparametrization of a curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if there exists a smooth function $\phi : (\alpha, \beta) \to \mathbb{R}$ (the so-called reparametrization map) so that

(i) $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$

(ii) $\bar{\gamma}(\phi(t)) = \gamma(t)$ for all $t \in (\alpha, \beta)$

By (i) we can apply the *Inverse Function Theorem*: There exist $\bar{\alpha}, \bar{\beta}$ such that $\phi : (\alpha, \beta) \to (\bar{\alpha}, \bar{\beta})$ is a bijection. Moreover, $\phi^{-1} : (\bar{\alpha}, \bar{\beta}) \to (\alpha, \beta)$ is smooth and

$$(\phi^{-1})'(\phi(t)) = \frac{1}{\phi'(t)} \neq 0$$

Thus ϕ^{-1} is a reparametrization map and γ is a reparametrization of $\bar{\gamma}$: $\gamma(\phi^{-1}(\bar{t})) = \bar{\gamma}(\phi(\phi^{-1}(\bar{t}))) = \bar{\gamma}(\bar{t})$ for all $\bar{t} \in (\bar{\alpha}, \bar{\beta})$

Example. Consider circle parametrizations

$$\gamma : \mathbb{R} \to \mathbb{R}^2 \ , \ \ t \mapsto (\cos(t), \sin(t))$$

and

$$\bar{\gamma} : \mathbb{R} \to \mathbb{R}^2 \ , \ \ t \mapsto (\sin(t), \cos(t))$$

Claim: $\bar{\gamma}$ is reparametrization of γ.

Need to find reparametrization map $\phi : \mathbb{R} \to \mathbb{R}$ with

$$(\sin(\phi(t)), \cos(\phi(t))) = \bar{\gamma}(\phi(t)) = \gamma(t) = (\cos(t), \sin(t))$$

A solution is

$$\phi : \mathbb{R} \to \mathbb{R} \ , \ \ t \mapsto \frac{\pi}{2} - t$$

What are *good* parametrizations? Unit speed $\|\dot{\gamma}\| = 1$ is convenient!

Proposition. *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a unit speed curve. Then*

$$\dot{\gamma}(t) \cdot \ddot{\gamma}(t) = 0 \text{ for all } t \in (\alpha, \beta)$$

Proof. This follows from the Product Rule:

$$\begin{aligned} \|\dot{\gamma}\| = 1 \quad &\Longrightarrow \quad \dot{\gamma} \cdot \dot{\gamma} = \|\dot{\gamma}\|^2 = 1 \\ &\Longrightarrow \quad \ddot{\gamma} \cdot \dot{\gamma} + \dot{\gamma} \cdot \ddot{\gamma} = 0 \\ &\Longrightarrow \quad \dot{\gamma} \cdot \ddot{\gamma} = 0 \end{aligned}$$

A curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is regular if $\dot{\gamma}(t) \neq 0$ for all $t \in (\alpha, \beta)$

Which curves do admit unit speed reparametrizations?

Proposition. *A curve* $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ *has a unit speed reparametrization if and only if it is a regular curve*

Proof. Assume that γ has a unit speed reparametrization $\bar{\gamma}$ with reparametrization map ϕ, so $\bar{\gamma} \circ \phi = \gamma$. The Chain Rule implies

$$\dot{\bar{\gamma}}(\phi(t))\phi'(t) = \dot{\gamma}(t)$$

Taking the norm implies

$$\|\dot{\gamma}(t)\| = \|\dot{\bar{\gamma}}(\phi(t))\phi'(t)\| = \underbrace{\|\dot{\bar{\gamma}}(\phi(t))\|}_{=1} |\phi'(t)| = |\phi'(t)|$$

Thus

$$\|\dot{\gamma}(t)\| = \pm\phi'(t) \neq 0 \qquad (*)$$

since ϕ is a reparametrization map

Conversely, assume that γ is a regular curve. Let $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du$ be the arc length function. Then $s'(t) = \|\dot{\gamma}(t)\| > 0$ and thus s is a reparametrization map with $\bar{\gamma} = \gamma \circ s^{-1}$, thus $\bar{\gamma} \circ s = \gamma$. The Chain Rule implies

$$\dot{\bar{\gamma}}(s(t))s'(t) = \dot{\gamma}(t)$$

This implies

$$\|\dot{\gamma}(t)\| = \|\dot{\bar{\gamma}}(s(t))s'(t)\| = \|\dot{\bar{\gamma}}(s(t))\||s'(t)| = \|\dot{\bar{\gamma}}(s(t))\|\|\dot{\gamma}(t)\|$$

and thus $\|\dot{\bar{\gamma}}(s(t))\| = 1$ since γ is regular. Hence $\bar{\gamma}$ is a unit speed reparametrization of γ and the arc-length function s is a reparametrization map

How many unit speed reparametrizations does a regular curve have?

Corollary *Let γ be a regular curve and $\bar{\gamma}$ be a unit speed reparametrization of γ with reparametrization map ϕ, thus $\bar{\gamma} \circ \phi = \gamma$. Then there exists $c \in \mathbb{R}$ such that*

$$\phi(t) = \pm s(t) + c,$$

where $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du$ is the arc length function. Conversely, if $\phi(t) = \pm s(t) + c$, then $\bar{\gamma} = \gamma \circ \phi^{-1}$ is a unit speed reparametrization of γ.

Proof. Using (*) in proof of previous proposition, we see that ϕ is a unit speed reparametrization map of γ if and only if $\phi'(t) = \pm\|\dot{\gamma}(t)\| = \pm s'(t)$, which holds if and only if $\phi(t) = \pm s(t) + c$ for some $c \in \mathbb{R}$.