Math 232
Homework #5: Practice Midterm #2

Instructor: Sean Hart

Spring 2024

Name: ______________________________

1. Determine (with justification) if the following statements are true or false.
   (a) (5 points) If $p_1, \ldots, p_n$ are prime, then $p_1 \cdots p_n + 1$ is prime.

   (b) (5 points) If $a \equiv b \pmod{n}$ then $a^k \equiv b^k \pmod{n}$ for any natural number k.

   (c) (5 points) For any integer a and prime p, we have $a^p \equiv a \pmod{p}$.

   (d) (5 points) For any $a, x, y \in \mathbb{Z}$, if $ax \equiv ay \pmod{n}$ then $x \equiv y \pmod{n}$.

2. (20 points) Prove that $\gcd(ac, bc) = c \cdot \gcd(a, b)$ for any $a, b, c \in \mathbb{N}$.

3. (20 points) Determine all integers x which satisfy the linear congruence

$$18x \equiv 42 \pmod{33}.$$

Be sure to show your work, and describe the method that you are using to solve the congruence.

4. (20 points) Prove that for each integer $n \geq 12$, there exist nonnegative integers a and b such that $n = 3a + 7b$.

   [**Hint:** Use induction! You may need multiple base cases.]

5. Define a sequence a_n recursively so that

$$a_1 = a_2 = a_3 = 1 \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n \text{ for } n \geq 1.$$

(a) (10 points) Write down the first ten terms of the given sequence.

n	1	2	3	4	5	6	7	8	9	10
a_n										

(b) (10 points) Use induction to prove that $a_n \leq 2^{n-2}$ for all $n \geq 2$.

This page is intentionally left blank.

Additional problems.

6. Consider the following statements:

   I. Every integer $n \geq 6$ can be written as a sum of exactly three primes.

   II. Every even integer $n \geq 4$ can be written as a sum of exactly two primes.

   (a). The mathematician Christian Goldbach conjectured, in a letter to Euler in 1742, that (I) was true (or, to be more precise, he conjectured something that is equivalent to (I) in modern conventions). Euler responded to this correspondence, noting **Goldbach's conjecture** was equivalent to (II).

   Check Euler's work by proving yourself that (I) and (II) are equivalent.

   (b). While Euler was certain that Goldbach's conjecture was true, sadly it remains an open problem. To gain evidence in favor of the conjecture, however, it has been verified (using computers) that the conjecture is true for $n \leq 4 \times 10^{18}$.

   Verify yourself that there are no counterexamples to (II) for $40 \leq n \leq 60$ by explicitly writing each even number in this range as a sum of two primes.

7. For each $n \in \mathbb{N}$, the n**-th Mersenne number** is the number $M_n = 2^n - 1$. A **Mersenne prime** is a Mersenne number that is also prime.

   (a). Write out the first eight Mersenne numbers. Which of these are Mersenne primes?

   (b). Prove that if $2^n - 1$ is prime, then n is prime.

   (c). Find a counter-example to the converse of the statement in (b).

   A **perfect number** is a number that is equal to the sum of all its (positive) divisors, excluding itself. For example:

   - the divisors of 6 are $\{1, 2, 3, 6\}$ so 6 is perfect since $1 + 2 + 3 = 6$, while
   - the divisors of 45 are $\{1, 3, 5, 9, 15, 45\}$ so 45 is not perfect since $1+3+5+9+15 = 33 \neq 45$.

   There is a nice relationship between Mersenne primes and perfect numbers:

   (d). Prove that if $2^n - 1$ is prime then $2^{n-1}(2^n - 1)$ is perfect.

8. For each natural number n, let p_n be the nth prime number and define the n**th prime gap** by

$$g_n = p_{n+1} - p_n.$$

   (a) A consequence of the **prime number theorem** is the estimate $p_n \sim n \ln n$. Using this, prove that

$$\lim_{n\to\infty} \frac{g_n}{p_n} = 0.$$

   (b) Recall a **twin prime pair** is a set consisting of two prime numbers which differ by 2, and the **twin prime conjecture** claims that there are infinitely many twin prime pairs.

   Formulate a statement about the sequence g_n equivalent to the twin prime conjecture.

   (c) Given any natural number $M \geq 2$, prove that every integer in the interval $[M!+2, M!+M]$ is is composite, and conclude from this that there exists some n such that $g_n \geq M$.