5 FIRST FUNDAMENTAL FORM

"SURFACE" = REGULAR SURFACE PATCH

5.1 LENGTH OF CURVES ON SURFACES

σ SURFACE

$\gamma(t) = \sigma(u(t), v(t))$ CURVE ON σ

ARC LENGTH $\quad s = \int_{t_0}^{t} \|\dot\gamma(x)\| \, dx$

CHAIN RULE: $\dot\gamma = \sigma_u \dot u + \sigma_v \dot v$

$$\Rightarrow \|\dot\gamma\|^2 = \underbrace{\|\sigma_u\|^2}_{=:E} \dot u^2 + 2 \underbrace{\sigma_u \cdot \sigma_v}_{=:F} \dot u \dot v + \underbrace{\|\sigma_v\|^2}_{=:G} \dot v^2$$

INTRODUCE NOTATION

$$\Rightarrow s = \int_{t_0}^{t} \left(E \dot u^2 + 2F \dot u \dot v + G \dot v^2 \right)^{\frac{1}{2}} dx$$

$$= \int_{t_0}^{t} \left(E \frac{du^2}{dx^2} dx^2 + 2F \frac{du}{dx}\frac{dv}{dx} dx^2 + G \frac{dv^2}{dx^2} dx^2 \right)^{\frac{1}{2}}$$

$$= \int_{t_0}^{t} \Big(\underbrace{E\, du^2 + 2F\, du\, dv + G\, dv^2}_{\text{FIRST FUNDAMENTAL FORM OF } \sigma} \Big)^{\frac{1}{2}}$$

SINCE $s = \int \sqrt{ds^2}$, WE WRITE

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

DEPENDS ON σ AND ITS PARAMETRIZATION BUT NOT ON CURVE γ.

EXAMPLE 5.1.1. (PLANE)

$$\sigma(u, v) = a + up + vq$$

$a, p, q \in \mathbb{R}^3$; p, q LINEARLY INDEPENDENT

$\sigma_u = p$, $\sigma_v = q$

$\Rightarrow E = \|p\|^2$, $F = p \cdot q$, $G = \|q\|^2$

$\Rightarrow ds^2 = \|p\|^2 du^2 + 2p\cdot q\, du\, dv + \|q\|^2 dv^2$

CAN CHOOSE $\|p\| = 1 = \|q\|$, $p \cdot q = 0$

$\Rightarrow ds^2 = du^2 + dv^2$

EXAMPLE 5.1.2 (SPHERE)

$$\sigma(\theta,\varphi) = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$$

$$\sigma_\theta = (-\sin(\theta)\cos(\varphi), -\sin(\theta)\sin(\varphi), \cos(\theta))$$

$$\sigma_\varphi = (-\cos(\theta)\sin(\varphi), \cos(\theta)\cos(\varphi), 0)$$

$$E = \|\sigma_\theta\|^2 = 1$$

$$F = \sigma_\theta \cdot \sigma_\varphi = 0$$

$$G = \|\sigma_\varphi\|^2 = \cos^2(\theta)$$

$$\Rightarrow \quad ds^2 = d\theta^2 + \cos^2(\theta)\, d\varphi^2 .$$

EXAMPLE 5.1.3 (GENERALISED CYLINDER)

$$\sigma(u,v) = (f(u), g(u), v)$$

$$\sigma_u = (\dot{f}, \dot{g}, 0), \quad \sigma_v = (0,0,1) \qquad \dot{} = \frac{d}{du}$$

$$E = \dot{f}^2 + \dot{g}^2, \; F = 0, \; G = 1$$

σ REGULAR $\Rightarrow$ γ REGULAR

(CAN ASSUME $\|\dot{\gamma}\| = 1$, SO $E = 1$

$$\Rightarrow ds^2 = du^2 + dv^2$$

SAME AS FOR PLANE

CHANGING PARAMETRIZATION OF σ CHANGES E, F, G BUT NOT ds^2 (EXERCISE!).

EVERY SURFACE HAS REPARAMETRISATION WITH $E = G$ & $F = 0$ (DIFFICULT!)

CONFORMAL PARAMETRIZATION.

PROP 5.1.4 SURFACE PAREMETRISATION

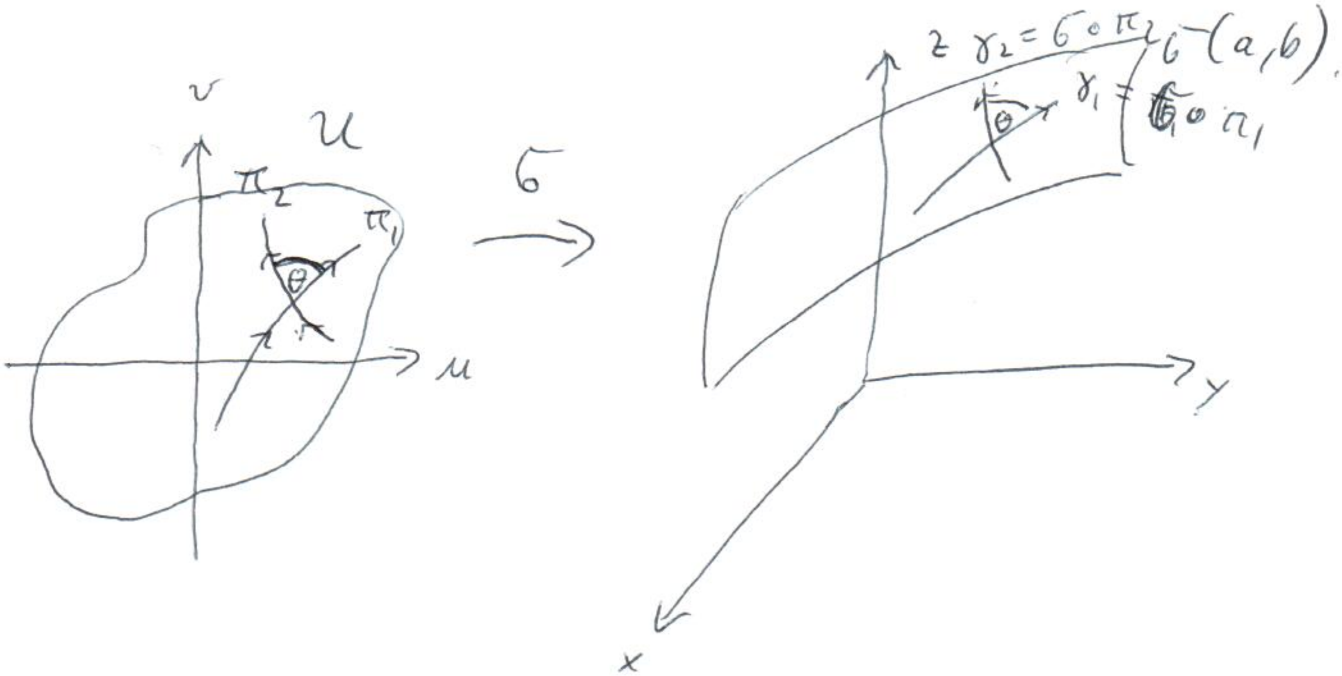
$\sigma : \mathcal{U} \to \mathbb{R}^3$ CONFORMAL

$\iff \forall\ \pi_1 = (u_1, v_1)$, $\pi_2 = (u_2, v_2)$ CURVES IN $\mathcal{U}$

WITH $\pi_1(t_0) = (a, b) = \pi_2(t_0)$:

ANGLE OF INTERSECTION OF π_1, π_2 AT t_0

= ANGLE OF INTERSECTION OF $\sigma \circ \pi_1, \sigma \circ \pi_2$ AT $\sigma(a,b)$.



PROOF "⟹"

$$\cos(\theta) = \frac{\dot{\gamma}_1 \cdot \dot{\gamma}_2}{\|\dot{\gamma}_1\| \|\dot{\gamma}_2\|} \qquad \text{AT } t_0$$

$$\dot{\gamma}_i = \sigma_u \dot{u}_i + \sigma_v \dot{v}_i \qquad i = 1, 2$$

$$\cos(\theta) = \frac{E \dot{u}_1 \dot{u}_2 + F(\dot{u}_1 \dot{v}_2 + \dot{v}_1 \dot{u}_2) + G \dot{v}_1 \dot{v}_2}{(E\dot{u}_1^2 + 2F\dot{u}_1\dot{v}_1 + G\dot{v}_1^2)^{\frac{1}{2}} (E\dot{u}_2^2 + 2F\dot{u}_2\dot{v}_2 + G\dot{v}_2^2)^{\frac{1}{2}}} \quad (*)$$

IF σ CONFORMAL, THEN $E=G$, $F=0$:

$$\cos(\theta) = \frac{\dot{u}_1 \dot{u}_2 + \dot{v}_1 \dot{v}_2}{(\dot{u}_1^2 + \dot{v}_1^2)^{\frac{1}{2}} (\dot{u}_2^2 + \dot{v}_2^2)^{\frac{1}{2}}} \quad (**)$$

$= \cos$ (ANGLE BETWEEN π_1, π_2 AT t_0)

"$\Leftarrow$": ASSUME $(*) = (**)$ $\forall \pi_1, \pi_2 \cdots$

TAKE $\pi_1(t) = (\underset{u_1}{a+t}, \underset{v_1}{b})$, $\pi_2(t) = (\underset{u_2}{a}, \underset{v_2}{b+t})$, $t_0 = 0$

$$\Rightarrow \quad 0 = \frac{F}{(EG)^{\frac{1}{2}}} \quad \Rightarrow \quad 0 = F$$

TAKE

$\pi_1(t) = (\underset{u_1}{a+t}, \underset{v_1}{b+t})$, $\pi_2(t) = (\underset{u_2}{a+t}, \underset{v_2}{b-t})$, $t_0 = 0$

$$\Rightarrow \quad 0 = \frac{E-G}{(E+\underbrace{\cancel{2F}}_{=0}+G)(E-\underbrace{\cancel{2F}}_{=0}+G)} \quad \Rightarrow \quad E = G$$

$\square$

5.2 ISOMETRIES OF SURFACES

RECALL: PLANE, CYLINDER HAVE SAME FIRST FUNDAMENTAL FORM.

PIECE OF PAPER $\sim$ CYLINDER

LENGTH OF CURVES REMAINS UNCHANGED

PIECE OF PAPER $\not\sim$ SPHERE

EXPECT DIFFERENT FIRST FUNDAMENTAL FORM

$\sigma_1 : \mathcal{U}_1 \to \mathbb{R}^3$
$\sigma_2 : \mathcal{U}_2 \to \mathbb{R}^3$ SURFACES

$S_1 = \sigma_1(\mathcal{U}_1)$
$S_2 = \sigma_2(\mathcal{U}_2)$

$f : S_1 \to S_2$ SMOOTH

$\iff \exists F : \mathcal{U}_1 \to \mathcal{U}_2$ SMOOTH : $f \circ \sigma_1 = \sigma_2 \circ F$

$$\begin{tikzcd}
\mathcal{U}_1 \arrow[r, "F"] \arrow[d, "\sigma_1"'] & \mathcal{U}_2 \arrow[d, "\sigma_2"] \\
S_1 \arrow[r, "f"'] & S_2
\end{tikzcd}$$

$F = \sigma_2^{-1} \circ f \circ \sigma_1$

σ_2 BIJECTION.

$f: S_1 \to S_2$ DIFFEOMORPHISM

$\iff$ f BIJECTIVE, f, f^{-1} SMOOTH

$\iff$ F BIJECTIVE, F, F^{-1} SMOOTH

~~EXAMPLE~~ RECALL: REPARAMETRIZATION MAPS Φ

$$\begin{tikzcd} \mathcal{U}_1 \arrow[r, "\Phi"] \arrow[d, "\sigma_1"'] & \mathcal{U}_2 \arrow[d, "\sigma_2"] \\ S \arrow[r, "f = id"'] & S \end{tikzcd}$$

$S_1 = S_2 = S$

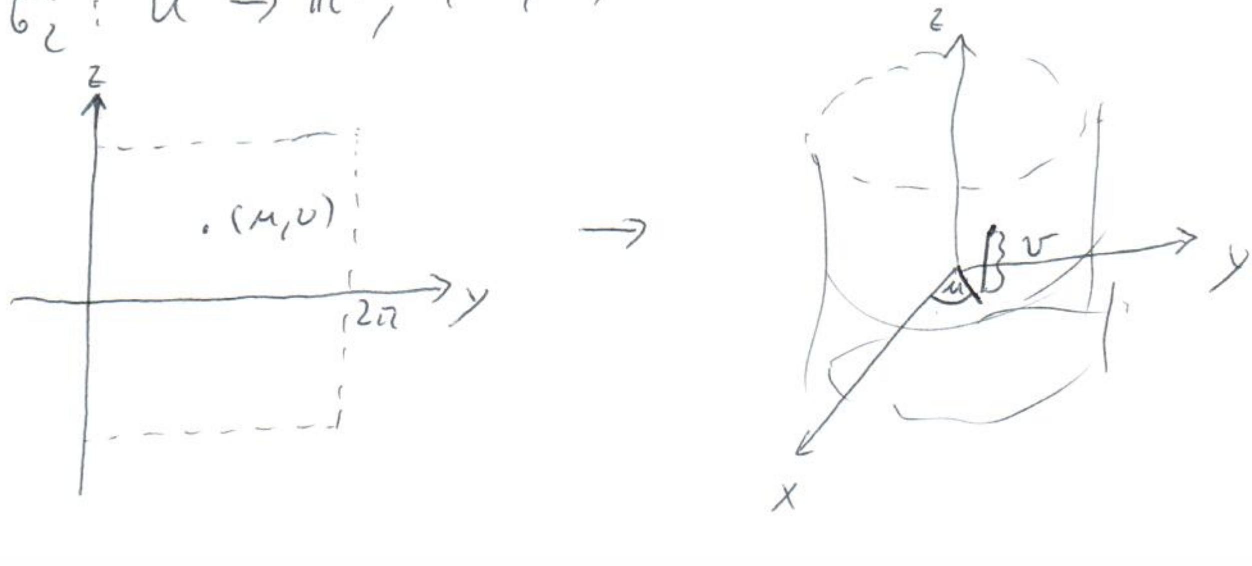
$\sigma_2 \circ \Phi = \sigma_1$

EXAMPLE 5.2.1 (PLANE $\sim$ CYLINDER)

$\mathcal{U}_\Phi = \{ (u, v) \in \mathbb{R}^2 : 0 < u < 2\pi \}$

$\sigma_1 : \mathcal{U}_\Phi \to \mathbb{R}^3, (u, v) \mapsto (0, u, v)$

$\sigma_2 : \mathcal{U} \to \mathbb{R}^3, (u, v) \mapsto (\cos(u), \sin(u), v)$



$F : \mathcal{U} \to \mathcal{U}$, $(u,v) \mapsto (u,v)$ DIFFEOM.

$f : S_1$ (PLANE) $\to S_2$ (CYLINDER)

$(0, u, v) \longmapsto (\cos(u), \sin(u), v)$

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & \mathcal{U} \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ S_1 & \longrightarrow & S_2 \end{array}$$

<u>DEF 5.2.2.</u> $\sigma_i : \mathcal{U}_i \to \mathbb{R}^3$ SURFACES, $S_i = \sigma_i(\mathcal{U}_i)$,

$f : S_1 \to S_2$ DIFFEOMORPHISM.

f <u>ISOMETRY</u> $\Leftrightarrow$ f PRESERVES LENGTHS OF CURVES

S_1, S_2 <u>ISOMETRIC</u> $\Leftrightarrow$ $\exists f : S_1 \to S_2$ ISOMETRY.

(OR σ_1, σ_2 ISOMETRIC) WRITE : $S_1 \cong S_2$

NOTE : CONCEPT IS INDEPENDENT OF REPARAMETRIZATIONS.

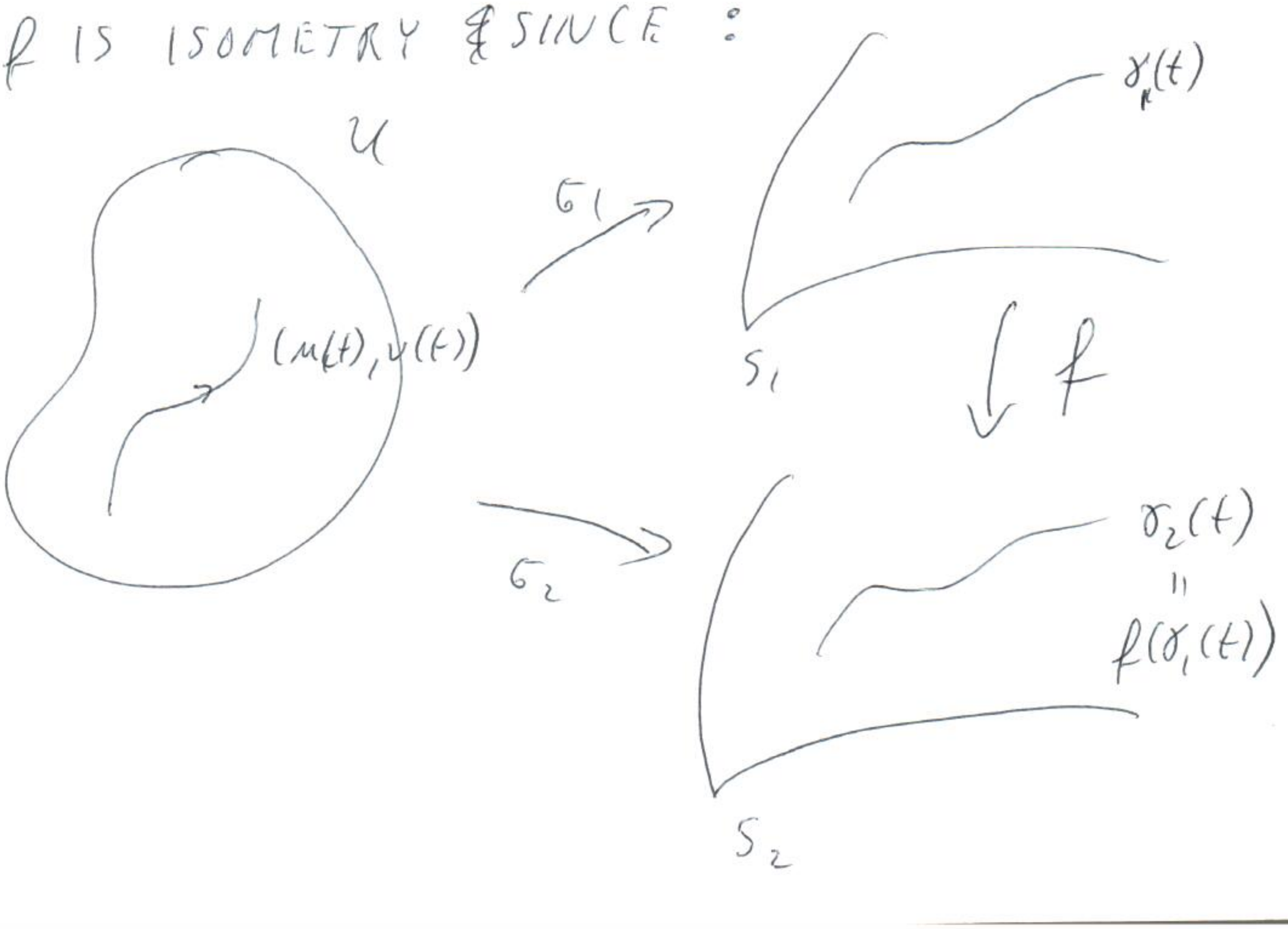
THM 5.2.3 TWO SURFACES ARE ISOMETRIC IFF THEY HAVE REPARAMETRIZATIONS $\sigma_1 : \mathcal{U} \to \mathbb{R}^3$, $\sigma_2 : \mathcal{U} \to \mathbb{R}^3$ WITH THE SAME FIRST FUNDAMENTAL FORM

PROOF: "$\Leftarrow$": DEFINE $f : S_1 \to S_2$ BY

$$\forall (u,v) \in \mathcal{U} : \ f(\sigma_1(u,v)) = \sigma_2(u,v)$$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "id"] \arrow[d, "\sigma_1"'] & \mathcal{U} \arrow[d, "\sigma_2"] \\ S_1 \arrow[r, "f"'] & S_2 \end{tikzcd}$$

f IS ISOMETRY & SINCE :



LENGTH OF γ_1, γ_2 OBTAINED BY INTEGRATING

$$(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{\frac{1}{2}},$$

WHICH IS SAME FOR BOTH SURFACES BY ASSUMPTION. SO f ISOMETRY

"$\Rightarrow$": ASSUME $\tilde{\sigma}_i : U_i \to \mathbb{R}^3$ SURFACES,

$S_i = \tilde{\sigma}_i(U_i)$, $f : S_1 \to S_2$ ISOMETRY.

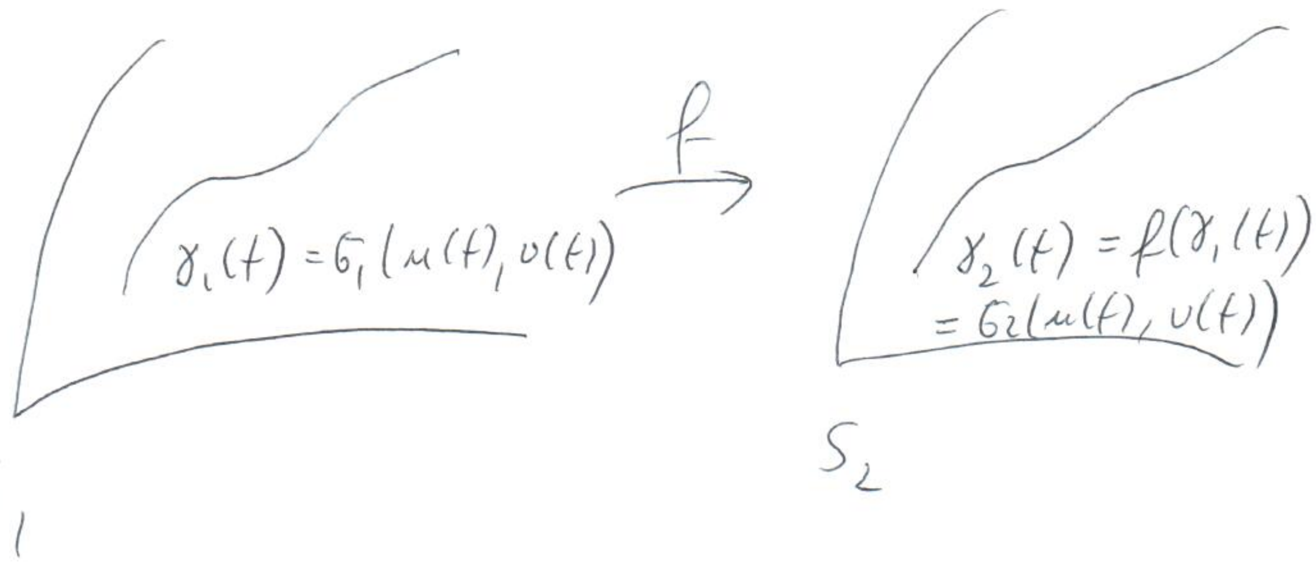
$$\begin{tikzcd} U = U_1 \arrow[r, "F"] \arrow[d, "\sigma_1 = \tilde{\sigma}_1"'] \arrow[dr, "\sigma_2"] & U_2 \arrow[d, "\tilde{\sigma}_2"] \\ S_1 \arrow[r, "f"'] & S_2 \end{tikzcd}$$

F BIJECTIVE SMOOTH

F^{-1} SMOOTH

$\sigma_2 = \tilde{\sigma}_2 \circ F : U_1 \to \mathbb{R}^3$ REPARAM. OF $\tilde{\sigma}_2$

PUT $U := U_1$, $\sigma_1 = \tilde{\sigma}_1$.



BY ASSUMPTION, γ_1, γ_2 HAVE SAME LENGTH

$$\Rightarrow \forall t_0, t_1: \int_{t_0}^{t_1} (E_1 \dot{u}^2 + 2F_1 \dot{u}\dot{v} + G_1 \dot{v}^2)^{\frac{1}{2}} dt$$

$$= \int_{t_0}^{t_1} (E_2 \dot{u}^2 + 2F_2 \dot{u}\dot{v} + G_2 \dot{v}^2)^{\frac{1}{2}} dt$$

E_i, F_i, G_i COEFFICIENTS OF 1^{st} FF OF σ_i

$\Rightarrow$ ~~BOTH FUNDAMENTAL FORMS~~ INTEGRANDS COINCIDE.

PUT $u_0 = u(t_0)$, $v_0 = v(t_0)$

(1) CHOOSE $(u(t), v(t)) = (u_0 + t - t_0, v_0)$

$$\Rightarrow \quad E_1 = E_2$$

(2) CHOOSE $(u(t), v(t)) = (u_0, v_0 + t - t_0)$

$$\Rightarrow \quad G_1 = G_2$$

(3) CHOOSE $(u(t), v(t)) = (u_0 + t - t_0, v_0 + t - t_0)$

$$\Rightarrow \quad E_1 + 2F_1 + G_1 = E_2 + 2F_2 + G_2$$

$$\underset{(1),(2)}{\Rightarrow} \quad 2F_1 = 2F_2 \quad \Rightarrow \quad F_1 = F_2 \qquad \square$$

PLANE $\cong$ CYLINDER

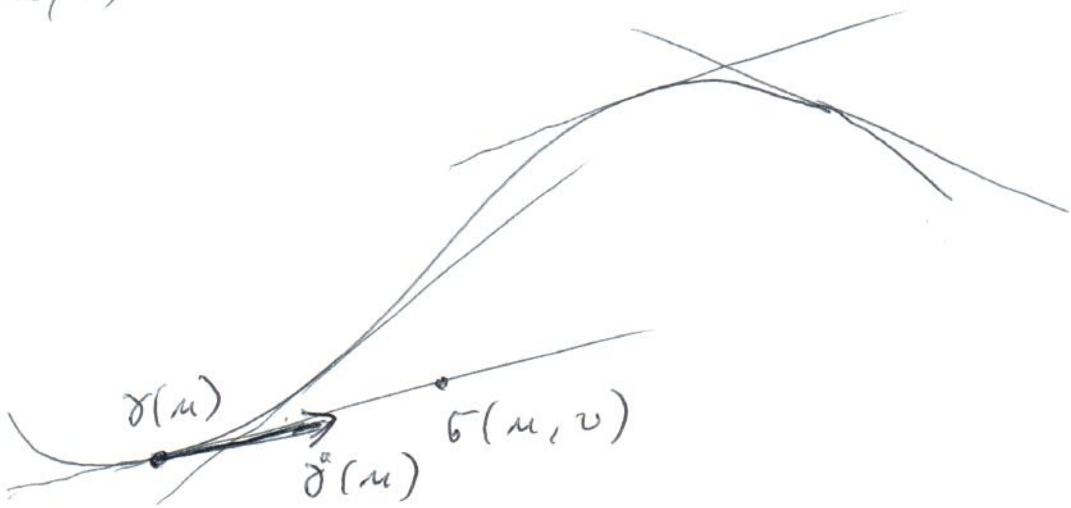
ALSO: PLANE $\cong$ CONE (EXERCISE)

PLANE $\cong$ TANGENT DEVELOPABLES

= UNION OF TANGENT LINES TO $\underbrace{\text{A CURVE}}_{\text{REGULAR}}$ γ IN $\mathbb{R}^3$.

ASSUME $\|\dot\gamma\| = 1$

$$\sigma(u,v) = \gamma(u) + v\,\dot\gamma(u)$$



σ REGULAR ?

$$\left.\begin{aligned} \sigma_u &= \dot\gamma + v\ddot\gamma \\ \sigma_v &= \dot\gamma \end{aligned}\right\} \Rightarrow \sigma_u \times \sigma_v = v\,\ddot\gamma \times \dot\gamma$$

$$= v\,\dot t \times t = \varkappa v\, n \times t$$

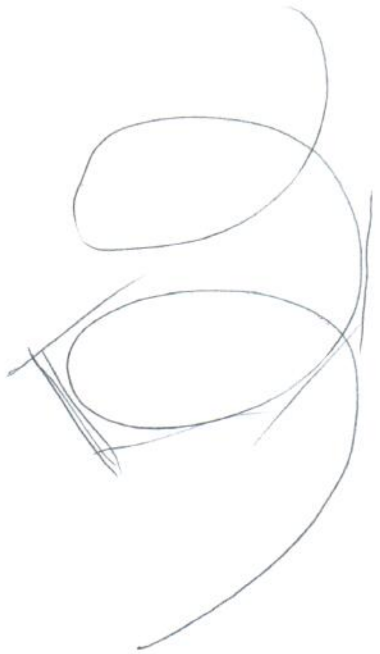
$\dot\gamma = t$ $\qquad$ $\dot t = \varkappa n$

$$= -\varkappa v\, b$$

$b = t \times n$ BINORMAL

σ REGULAR $\Leftrightarrow$ $\kappa > 0$, $v \neq 0$

(THUS EXCLUDE γ FROM SURFACE)



HELIX

SURFACE HAS 2 SHEETS MEETING ALONG CURVE.

PROP 5.24 ANY TANGENT DEVELOPABLE IS ISOMETRIC TO (PART OF) PLANE

PROOF

$$E = \|\sigma_u\|^2 = \|\dot\gamma + v\ddot\gamma\|^2 = 1 + v^2\kappa^2$$

$$F = \sigma_u \cdot \sigma_v = \dot\gamma + v\ddot\gamma \cdot \dot\gamma = \|\dot\gamma\|^2 = 1$$

$$G = \|\sigma_v\|^2 = \|\dot\gamma\|^2 = 1$$

$$\Rightarrow ds^2 = (1 + v^2\kappa^2)\, du^2 + 2\, du\, dv + dv^2 \quad (*)$$

CHOOSE A PLANE UNIT SPEED CURVE $\tilde{\gamma}$
WITH CURVATURE $\varkappa$ (SEE THM 2.2.2)
ABOVE CALCULATIONS SHOW THAT
1^{st} FF OF THE TANGENT DEVELOPABLE
OF $\tilde{\gamma}$ IS ALSO GIVEN BY (*).

$\tilde{\gamma}$ PLANE CURVE $\Rightarrow$ TANGENTS FILL OUT (PART OF PLANE).

$\square$

REMARK: SURFACE ISOMETRIC TO (PART OF) PLANE IS (PART OF)

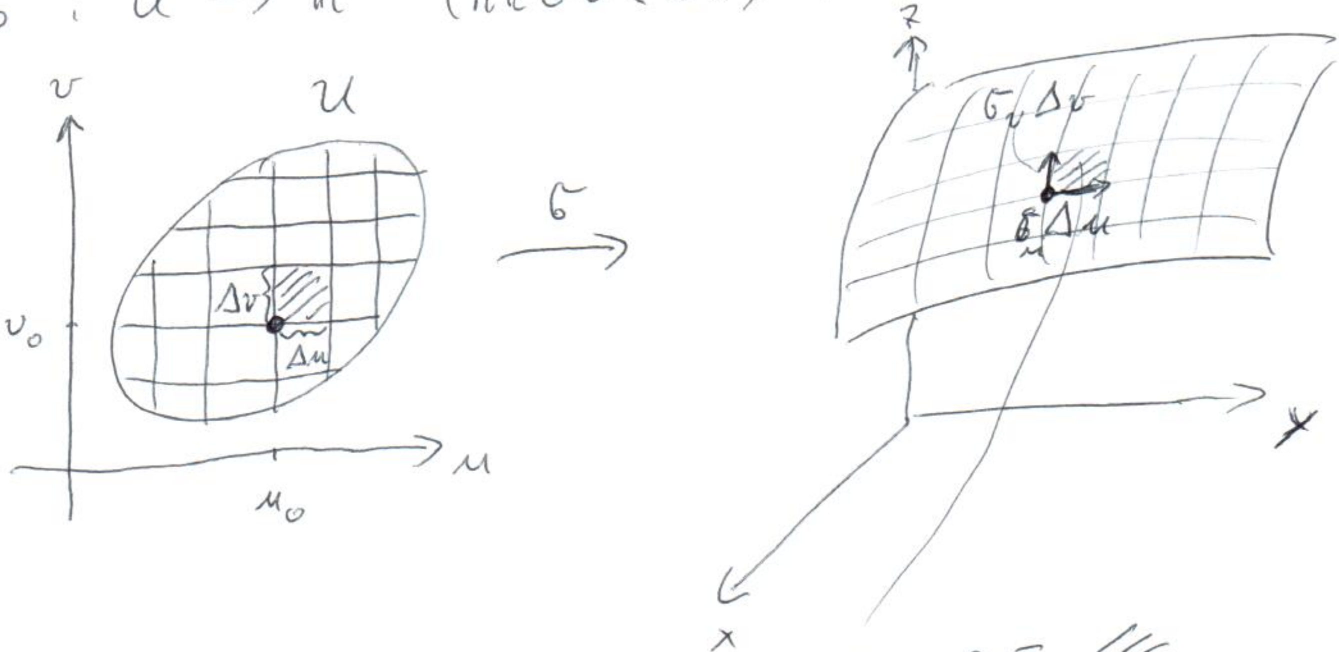
- PLANE
- GENERALISED CYLINDER
- GENERALISED CONE
- TANGENT DEVELOPABLE.

(PROOF OMITTED HERE)

5.3 SURFACE AREA

1^{st} FF $\rightsquigarrow$ LENGTHS OF CURVES
& AREA OF SURFACE

$\sigma : \mathcal{U} \to \mathbb{R}^3$ (REGULAR) SURFACE



AREA OF ////

$\approx$ AREA OF PARALLELOGRAM WITH SIDES $\sigma_u \Delta u, \sigma_v \Delta v$

$= \| \sigma_u \Delta u \times \sigma_v \Delta v \|$

THIS SUGGESTS:

$= \| \sigma_u \times \sigma_v \| \Delta u \Delta v$

DEF 5.3.1 $\sigma : \mathcal{U} \to \mathbb{R}^3$ SURFACE, $R \subseteq \mathcal{U}$

AREA OF $\sigma(R)$ IS

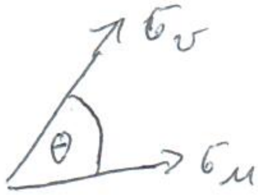
$$\mathcal{A}_\sigma (R) = \iint_R \| \sigma_u \times \sigma_v \| \, du \, dv .$$

NOTE: $\iint_R \|\sigma_u \times \sigma_v\| = \infty$ POSSIBLE (E.G. PLANE)

BUT $< \infty$ IF $R \subseteq [a,b] \times [c,d]$ BOUNDED.

PROP 5.3.2 $\|\sigma_u \times \sigma_v\| = \sqrt{(EG - F^2)}$

PROOF:



$$\|\sigma_u \times \sigma_v\|^2$$

$$= \|\sigma_u\|^2 \|\sigma_v\|^2 \underbrace{\sin^2(\theta)}_{= 1 - \cos^2(\theta)}$$

$$= \|\sigma_u\|^2 \|\sigma_v\|^2 - \underbrace{\|\sigma_u\|^2 \|\sigma_v\|^2 \cos^2(\theta)}_{= (\sigma_u \cdot \sigma_v)^2}$$

$$= EG - F^2 \qquad \square.$$

THEREFORE

$$\boxed{\mathcal{A}_\sigma(R) = \iint_R (EG - F^2)^{\frac{1}{2}} \, du \, dv.}$$

$A_\sigma(R)$ WELL-DEFINED? E, F, G CHANGE UNDER REPARAMETRIZATION !!

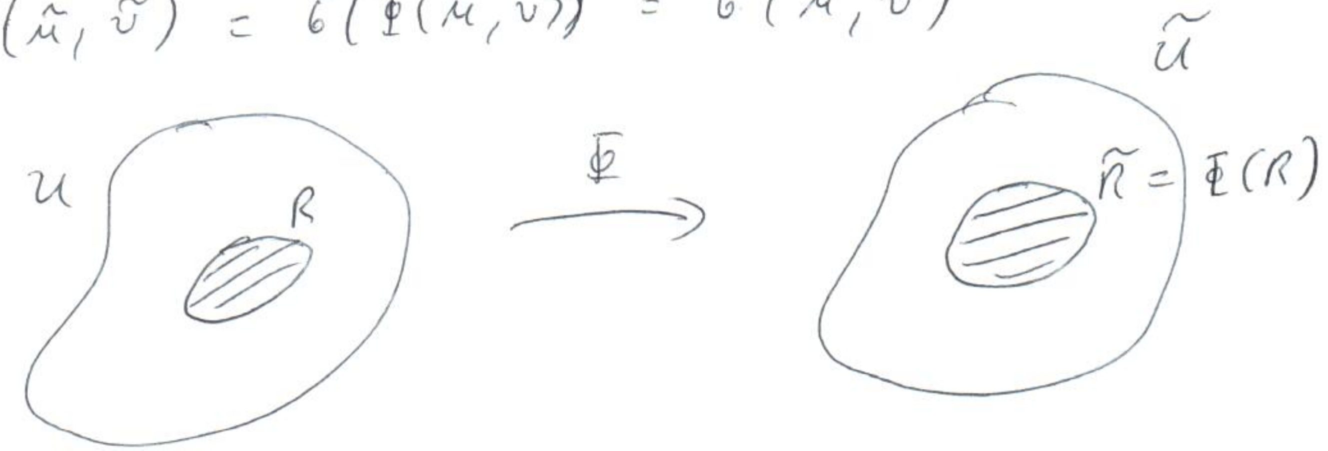
PROP 5.3.3 AREA IS UNCHANGED BY REPARAMETRIZATION.

PROOF: $\sigma : U \to \mathbb{R}^3$ SURFACE

$\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ REPARAMETRIZATION OF σ

WITH $\Phi : U \to \tilde{U}$ REPARAMETRIZATION MAP

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \tilde{\sigma}(\Phi(u,v)) = \sigma(u,v)$$



TO PROVE:

$$\iint_R \| \sigma_u \times \sigma_v \| \, du \, dv = \iint_{\tilde{R}} \| \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} \| \, d\tilde{u} \, d\tilde{v}.$$

CHAIN RULE:

$$\sigma_u = \tilde{\sigma}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial u} + \tilde{\sigma}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u}, \quad \sigma_v = \tilde{\sigma}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial v} + \tilde{\sigma}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}$$

$$\Rightarrow \sigma_u \times \sigma_v = \underbrace{\left(\frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v} \frac{\partial \tilde{v}}{\partial u} \right)}_{= \det(J(\Phi))} \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}$$

$$J(\Phi) = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$$ JACOBIAN MATRIX OF Φ

$$\Rightarrow \iint_R \|\sigma_u \times \sigma_v\| \, du \, dv = \iint_R |\det(J(\Phi))| \, \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| \, du \, dv$$

$$= \iint_{\tilde{R}} \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| \, d\tilde{u} \, d\tilde{v}$$

CHANGE OF VARIABLES FORMULA $\square$.

<u>REMARK</u>: $N, \tilde{N}$ UNIT NORMALS OF $\sigma, \tilde{\sigma}$

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{\det(J(\Phi)) \, \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}}{|\det(J(\Phi))| \, \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\|} = \pm \tilde{N}$$