COMP SCI 2LC3. Solution to the Assignment #1

Total = 127 pts,

The solutions below are very detailed on purpose. Such level of details is not required from students' solutions. Most questions have more than one solution, and the solution provided below may not be the simplest or more elegant one. Your solution might actually be better.

If you think your solution has been marked wrongly, write a short memo stating where marking in wrong and what you think is right, and resubmit to me during class, office hours, slip under the door to my office, or send via e-mail as pdf file.

1.[2] Perform the following textual substitutions. Be careful with parenthesization and remove unnecessary parentheses.

$(x+x\cdot y+x\cdot y\cdot z)[x,y:=y,x]$

Solution: $y+y\cdot x+y\cdot x\cdot z$

2.[2] Leibniz's definition of equality given just before inference rule Leibniz (1.5) says that $X=Y$ is true in every state iff $E[z:=X]=E[z:=Y]$ is true in every state. Inference rule Leibniz (1.5), however, gives only the "if"' part. Give an argument to show that the "only if" part follows from Leibniz (1.5). That is, suppose $E[z:=X]=E[z:=Y]$ is true in every state, for every expression E . Show that $X=Y$ is true in every state.

Solution: $E[z:=X]=E[z:=Y]$ holds for all expressions E. Choose E to be variable z, so $z[z:=X]=z[z:=Y]$, which is $X=Y$.

3.[2] Inference rule Leibniz (1.5) stands for an infinite number of inference rules, each of which is constructed by instantiatin E,X , and Y with different expressions. Below, are a number of instantiations of Leibniz, with parts missing. Fill in the missing parts and write down what expression E is. Do not simplify. This exercise has three answers; give them all.

$$\frac{z=y+1}{7\cdot x+7\cdot y=?}$$

Solution: $\dfrac{7=y+1}{7\cdot x+7\cdot y=(y+1)\cdot x+7\cdot y}$ (using $E=z\cdot x+7\cdot y$) or

$\dfrac{7=y+1}{7\cdot x+7\cdot y=7\cdot x+(y+1)\cdot y}$ (using $E=7\cdot x+z\cdot y$) or

$\dfrac{7=y+1}{7\cdot x+7\cdot y=(y+1)\cdot x+(y+1\cdot y}$ (using $E=z\cdot x+z\cdot y$)

4.[2] The purpose of this exercise is to reinforce your understanding of the use of Leibniz (1.5) along with a hint in proving two expressions equal. For each of the following pair of expressions $E[z := X]$ and $E[z := Y]$, identify a hint $X = Y$ that would show them to be equal and indicate what E is. Do it for

$X = (x+y)\cdot(x+y)$ and $Y = (x+y)\cdot(y+x)$

Solution: A trivial hint is $X = Y$ with $E = z$. A finer hint could be $x+y = y+x$ for $E = (x+y)\cdot z$.

5.[2] Using Definition (1.12) of the assignment statement on page 18, determine precondition for the following statement and postcondition:

Statement: $y := x+y$, Postcondition: $y = x+y$

Solution: $x+y = x+x+y$ - simplify to $x = 0$.

6.[$6 = 3\times 2$] Each line below contains an expression and two states $S0$ and $S1$ (using t for *true* and f for *false*). Evaluate the expression in both states.

	expression	state $S0$ m	n	p	q	state $S1$ m	n	p	q
(k)	$(m \equiv n \wedge p) \Rightarrow q$	f	t	f	t	t	t	f	f
(l)	$(m \Rightarrow n) \Rightarrow (p \Rightarrow q)$	f	f	f	f	t	t	t	t
(m)	$(m \Rightarrow (n \Rightarrow p)) \Rightarrow q$	f	f	f	f	t	t	t	t

Solution: (k) $S0 : true, S1 : true$
(l) $S0 : true, S1 : true$
(m) $S0 : false, S1 : true$

7.[$4 = 2\times 2$] Write truth tables to compute values for the following expressions in all states.
(g) $(\neg b \equiv c) \vee b$
(h) $(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Rightarrow b)$

Solution:

(g)

b	c	$\neg b$	$\neg b \equiv c$	$(\neg b \equiv c) \vee b$
t	t	f	f	t
t	f	f	t	t
f	t	t	t	t
f	f	t	f	f

(h)

b	c	$b \equiv c$	$b \Rightarrow c$	$c \Rightarrow b$	$(b \Rightarrow c) \wedge (c \Rightarrow b)$	$(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Rightarrow b)$
t	t	t	t	t	t	t
t	f	f	f	t	f	t
f	t	f	t	f	f	t
f	f	t	t	t	t	t

8.[$4 = 2 \times 2$] Write the duals P_D for each of the following expressions P.

(f) $\neg b \Leftarrow b \vee c$

(g) $(\neg b \equiv true) \vee b$

Solution:

(f) $\neg b \not\Rightarrow b \wedge c$

(g) $(\neg b \not\equiv false) \wedge b$

9.[$4 = 2 \times 2$] Translate the following English statements into boolean expressions.

(d) It's raining cats or dogs.

(e) If it rains cats and dogs I'll eat my hat, but I won't go swimming.

Solution: Associate identifiers with the primitive subexpressions as follows.

r : It's raining

s : I'm going swimming

sc : It's raining cats

sd : It's raining dogs

eh : I'll eat my hat

The translations are then

(d) $sc \vee sd$

(e) $sc \wedge sd \implies eh \wedge \neg s$

10.[$6 = 3 \times 2$] Give names to the primitive components (e.g. $x < y$ and $x = y$) of the following English sentences and translate the sentences into boolean expressions.

(h) When $x < y$, then $y < z$; when $x \geq y$, then $v = w$.

(i) When $x < y$, then $y < z$ means that $v = w$, but if $x \geq y$ then $y > z$ does not hold; however, if $v = w$ then $x < y$.

(j) If execution of program P is begun with $x < y$, then execution terminates with $y = 2^x$.

Solution: Associate identifiers with the primitive subexpressions as follows (not necessary for (h) and (i).

$xly : x < y$	$vew : v = w$
$xey : x = y$	ep : Execution of P is begun with $x < y$.
$xgy : x > y$	ty : Execution of P terminates with $y = 2^x$.
$ylz : y < z$	$ep1$: Execution of P is begun with $x < 0$.
$ygz : y > z$	ept : Execution of P terminates.

The translations are then:

(h) $(xly \Rightarrow ylz) \wedge (\neg xly \Rightarrow vew)$

(i) $(xly \Rightarrow (ylz \equiv vew)) \wedge (\neg xly \Rightarrow \neg ygz) \wedge (vew \Rightarrow xly)$

(j) $ep \Rightarrow ty$

11.[10] The Tardy Bus Problem has three assumptions:

1. If Bill takes the bus, then Bill misses his appointment if the bus is late.
2. Bill shouldn't go home if Bill misses his appointment and Bill feels downcast.
3. If Bill doesn't get the job, he feels downcast and shouldn't go home.

The problem has eight conjectures:

4. If Bill takes the bus, then Bill does get the job if the bus is late.
5. Bill gets the job, if Bill misses his appointment and he should go home.
6. If the bus is late and Bill feels downcast and he goes home, then he shouldn't take the bus.
7. Bill doesn't take the bus if, the bus is late and Bill doesn't get the job.
8. If Bill doesn't miss his appointment, then Bill shouldn't go home and Bill doesn't get the job.
9. Bill feels downcast if the bus is late or Bill misses his appointment.
10. If Bill takes the bus and the bus is late and he goes home, then he gets the job.
11. If Bill takes the bus but doesn't get the job, then either the bus is on time or he shouldn't go home.

Translate the assumptions and conjectures into boolean expressions. Write down a boolean expression that stands for "conjecture (11) follows from the three assumptions".

Solution: Let the identifiers associated with the primitive subpropositions be:

tb : Bill $t$akes the $b$us	gh : Bill should go $h$ome
ma : Bill $m$isses his $a$ppointment	fd : Bill $f$eels $d$owncast
bl : The $b$us is $l$ate	gj : Bill $g$ets the $j$ob

The premises are written as follows.

(a) $tb \Rightarrow (bl \Rightarrow ma)$
(b) $ma \wedge fd \Rightarrow \neg gh$
(c) $\neg gj \Rightarrow fd \wedge \neg gh$

The conjectures are written as follows.

(d) $tb \Rightarrow (bl \Rightarrow gj)$
(e) $ma \wedge gh \Rightarrow gj$
(f) $bl \wedge fd \wedge gh \Rightarrow \neg tb$
(g) $bl \wedge \neg gj \Rightarrow \neg tb$
(h) $\neg ma \Rightarrow \neg gh \wedge \neg gj$
(i) $bl \vee ma \Rightarrow fd$
(j) $tb \wedge bl \wedge gh \Rightarrow gj$
(k) $tb \wedge \neg gj \Rightarrow \neg bl \vee \neg gh$

The formula that expresses "conjecture (k) follows from the three assumptions" is $(a) \wedge (b) \wedge (c) \Rightarrow (k)$.

12.[6] Solve the following puzzle. A certain island is inhabited by people who either always tell the truth or always lie and who respond to questions with a yes or a no. A tourist comes to a fork in the road, where one branch leads to a restaurant and the other does not. There is no sign indicating which branch to take, but there is an islander standing at the fork. What single yes/no question can the tourist ask to find the way to the restaurant?

Hint: Let p stand for "the islander at the fork always tells the truth" and let q stand for "the left-hand branch leads to the restaurant". Let E stand for a boolean expression such that, whether the islander tells the truth or lies, the answer to the question "Is E true?" will be yes iff the left-hand branch leads to the restaurant. Construct the truth table that E must have, in terms of p and q, and then design an appropriate E according to the truth table.

Solution:

p	q	E
t	t	t
t	f	f
f	t	f
f	f	t

If the islander at the fork always tells the truth (p), then E should equivale q. If the islander always lies ($\neg p$), then E should equivale $\neg q$. Hence, $E \equiv (p \equiv q)$. You can see this in the truth table to the left. So the question "is E true?" is "is $p \equiv q$?" We translate this into English: "is it the case you tell the truth precisely when lefthand branch leads to the restaurant?"

13.[3] Show that in a Boolean algebra, every element x has a unique complement $\neg x$ such that $x \vee \neg x = true$ and $x \wedge \neg x = false$.

Solution: Assume there are two, say z_1, z_2 and $z_1 \neq z_2$. From the definition of complement we have: $x \vee z_1 = true$, $x \wedge z_1 = false$ and $x \vee z_2 = true$, $x \wedge z_2 = false$. Moreover we have $x = true$ or $x = false$.

Case $x = true$

$x = true$

$\langle (x \wedge z_1 = false) \implies z_1 = false$ and $(x \wedge z_2 = false) \implies z_2 = false \rangle$

$z_1 = false \wedge z_2 = false$

$\langle$ transitivity of $= \rangle$

$z_1 = z_2$

Case $x = false$

$x = false$

$\langle (x \vee z_1 = true) \implies z_1 = true$ and $(x \vee z_2 = true) \implies z_2 = true \rangle$

$z_1 = false \wedge z_2 = false$

$\langle$ transitivity of $= \rangle$

$z_1 = z_2$

Since $(x = true \vee x = false) \equiv true$, then $z_1 = z_2$, so there is only one complement of x.

14.[6] Any Boolean expression can be interpreted as a Boolean function. For example: $x \vee (\neg y \wedge z)$ is a Boolean function $f : \mathbb{B} \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$, $f(x,y,z) = x \vee (\neg y \wedge z)$.

How many different Boolean functions $f(x,y,z)$ are there so $f(\neg x, \neg y, \neg z) = f(x,y,z)$ for all values of the Boolean variables x,y,z?

Solution: How many different one dimensional boolean functions $f : \mathbb{B} \to \mathbb{B}$ we have? Let t denote *truth* and f denote *false*. We have:
$f_1(t) = f_1(f) = t$
$f_2(t) = f_2(f) = f$
$f_3(t) = t$ and $f_3(f) = f$
$f_4(t) = f$ and $f_4(f) = t$
There are no other one dimensional Boolean function, so we have 4 of them.

What about n variables, i.e. the functions $f(x_1, \cdots, x_n)$? More formally how many different functions $f : \mathbb{B}^n \to \mathbb{B}$ we have? Any variable x_i' can have 2 values i.e, t or $f1$. For n variables there are 2^n entries in the truth table. And each output of any particular row in the truth table can be t or f. Hence, we have 2^{2^n} different Boolean functions with n variables. Hence we have 4 Boolean functions with one variable, 16 Boolean functions with two variables and 256 Boolean functions with three variables.

However if $f(\neg x, \neg y, \neg z) = f(x,y,z)$, then sequences x,y,z are not distinguishable from $\neg x, \neg y, \neg z$, for example tft and ftf are consider equivalent. Hence we do not have $2^3 = 8$ distinguishable sequences but only $2^3/2 = 4$. So while the number of all 3 dimensional Boolean functions is $2^8 = 256$, the number of functions that have the property $f(\neg x, \neg y, \neg z) = f(x,y,z)$ is $2^4 = 16$.

15.[3] Prove the following metatheorem. $Q \equiv true$ is a theorem iff Q is a theorem.

Solution. Suppose we have a proof of Q that begins with a theorem and ends with Q (the other way around is similar). Use axiom Identity of $\equiv$ (3.3) to extend it to a proof of $Q \equiv true$. Suppose we have a proof of $Q \equiv true$ that begins with a theorem and ends with $Q \equiv true$ (the other way around is similar). Use axiom Identity of $\equiv$ (3.3) to extend it to a proof of Q.

16.[3] Assume that operator $\equiv$ is identified with operator $\equiv$ of Sec. 2.1 (see Exercise 3.1) and *true* is identified with the symbol *true* of Sec. 2.1. Prove that axioms (3.8) and (3.9) uniquely define operator $\neg$. That is, determine which of the four prefix operators $\circ$ defined in the truth table on page 26 satisfy $false \equiv \circ true$ and $\circ(p \equiv q) \equiv \circ p \equiv q$.

Solution. The first operator or function of the truth table on page 29, call it $\mathbf{t}$, does not satisfy $false \equiv \mathbf{t}.true$. Operator id does not satisfy $false \equiv id.true$. The third operator, call it $\mathbf{f}$, does not satisfy $\mathbf{f}(p \equiv q) \equiv \mathbf{f}.p \equiv q$ with the assignment $q := false$. That leaves only $\neg$, which satisfies both axioms. Hence, the two axioms uniquely determine $\neg$.

17.[3] Prove Associativity of $\not\equiv$ (3.17), $((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$, using the heuristic of Definition elimination (3.23) - by eliminating $\not\equiv$, using a property of $\equiv$, and reintroducing $\not\equiv$.

Solution.
Proof of Associativity of $\not\equiv$ (3.17), $((p \not\equiv q) \not\equiv r) \ \equiv \ (p \not\equiv (q \not\equiv r))$.

$$
\begin{array}{ll}
 & (p \not\equiv q) \not\equiv r \\
= & \quad \langle (3.14),\ (p \not\equiv q) \ \equiv \ \neg p \equiv q \ \rangle \\
 & (\neg p \equiv q) \not\equiv r \\
= & \quad \langle (3.14) \text{ —also uses Symmetry of } \not\equiv \ (3.16) \rangle \\
 & (\neg p \equiv q) \equiv \neg r \\
= & \quad \langle \text{Associativity of } \equiv \ (3.1) \rangle \\
 & \neg p \equiv (q \equiv \neg r) \\
= & \quad \langle (3.14) \rangle \\
 & p \not\equiv (q \equiv \neg r) \\
= & \quad \langle (3.14) \rangle \\
 & p \not\equiv (q \not\equiv r)
\end{array}
$$

18.[3] Prove Mutual associativity (3.18), $((p \not\equiv q) \equiv r) \equiv (p \not\equiv (qr))$, using the heuristic of Definition elimination (3.23) - by eliminating $\not\equiv$, using a property of $\equiv$, and reintroducing $\not\equiv$.

Solution.
Proof of (3.18), $((p \not\equiv q) \equiv r) \ \equiv \ (p \not\equiv (q \equiv r))$.

$$
\begin{array}{ll}
 & (p \not\equiv q) \equiv r \\
= & \quad \langle (3.14),\ (p \not\equiv q) \ \equiv \ \neg p \equiv q \ \rangle \\
 & (\neg p \equiv q) \equiv r \\
= & \quad \langle \text{Associativity of } \equiv \ (3.1) \rangle \\
 & \neg p \equiv (q \equiv r) \\
= & \quad \langle (3.14) \rangle \\
 & p \not\equiv (q \equiv r)
\end{array}
$$

19.[3] Prove Identity of $\vee$ (3.30), $p \vee \mathit{false} \equiv p$, by transforming its more structured side into its simpler side. Theorem (3.15) may be a suitable way to introduce an equivalence.

Solution.
Proof of (3.30), $p \ \vee \ \mathit{false} \ \equiv \ p$.

$$
\begin{array}{ll}
 & p \ \vee \ \mathit{false} \\
= & \quad \langle (3.15),\ \neg p \equiv p \equiv \mathit{false} \ \rangle \\
 & p \ \vee \ (\neg p \equiv p) \\
= & \quad \langle \vee \text{ distributes over } \equiv \ (3.27) \rangle
\end{array}
$$

$$
\begin{array}{cl}
 & p \vee \neg p \equiv p \vee p \\
= & \quad \langle \text{Excluded middle (3.28); Idempotency of } \vee \text{ (3.26)} \rangle \\
 & true \equiv p \\
= & \quad \langle \text{Identity of } \equiv \text{ (3.3), with } q := p \rangle \\
 & p
\end{array}
$$

20.[3] Prove Distributivity of $\vee$ over $\vee$ (3.31), $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$. The proof requires only the symmetry, associativity, and idempotency of $\vee$.

Solution.

Proof of Distributivity of $\vee$ over $\vee$, (3.31), $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$.

$$
\begin{array}{cl}
 & (p \vee q) \vee (p \vee r) \\
= & \quad \langle \text{Associativity of } \vee \text{ (3.25)} \rangle \\
 & p \vee (q \vee p) \vee r \\
= & \quad \langle \text{Symmetry of } \vee \text{ (3.24)} \rangle \\
 & p \vee (p \vee q) \vee r \\
= & \quad \langle \text{Associativity of } \vee \text{ (3.25)} \rangle \\
 & (p \vee p) \vee (q \vee r) \\
= & \quad \langle \text{Idempotency of } \vee \text{ (3.26)} \rangle \\
 & p \vee (q \vee r)
\end{array}
$$

21.[3] Prove Symmetry of $\wedge$ (3.36), $p \wedge q = q \wedge p$, using the heuristic of Definition elimination (3.23) - eliminate $\wedge$ (using its definition, the Golden rule), manipulate, and then reintroduce $\wedge$.

Solution.

Proof of Symmetry of $\wedge$ (3.36), $p \wedge q \equiv q \wedge p$.

$$
\begin{array}{cl}
 & p \wedge q \\
= & \quad \langle \text{Golden rule (3.35)} \rangle \\
 & p \equiv q \equiv p \vee q \\
= & \quad \langle \text{Symmetry of } \equiv \text{ (3.2); Symmetry of } \vee \text{ (3.24)} \rangle \\
 & q \equiv p \equiv q \vee p \\
= & \quad \langle \text{Golden rule (3.35), with } p, q := q, p \rangle \\
 & q \wedge p
\end{array}
$$

22.[3] Prove Zero of $\wedge$ (3.40), $p \wedge \mathit{false} \equiv \mathit{false}$, using the heuristic of Definition elimination (3.23) - eliminate $\wedge$ (using its definition, the Golden rule) and manipulate.

Solution.

Proof of (3.40), $p \wedge \mathit{false} \equiv \mathit{false}$.

$$
\begin{array}{cl}
 & p \wedge \mathit{false} \equiv \mathit{false} \\
= & \quad \langle \text{Golden rule (3.35), with } q := \mathit{false} \rangle \\
 & p \vee \mathit{false} \equiv p \\
= & \quad \langle \text{Identity of } \vee \text{ (3.30)} \rangle \\
 & p \equiv p \quad \text{—Reflexivity of } \equiv \text{ (3.5)}
\end{array}
$$

23.[3] Prove De Morgan (3.47a), $\neg(p \wedge q) \equiv \neg p \vee \neg q$. Start by using the Golden rule; (3.32) should come in handy.

Solution.

Proof of De Morgan's first law, (3.47a), beginning with the LHS.

$$
\begin{array}{ll}
 & \neg(p \wedge q) \\
= & \quad \langle \text{Golden rule (3.35)} \rangle \\
 & \neg(p \equiv q \equiv p \vee q) \\
= & \quad \langle (3.9),\ \neg(p \equiv q) \equiv \neg p \equiv q\,, \\
 & \quad\ \text{with } q := q \equiv p \vee q \text{ —to move } \neg \text{ to an equivalent} \rangle \\
 & \neg p \equiv q \equiv p \vee q \\
= & \quad \langle (3.32),\ p \vee q \equiv p \vee \neg q \equiv p\,, \text{ with } p, q := q, p \\
 & \neg p \equiv q \vee \neg p \\
= & \quad \langle (3.32) \text{ —again!— with } p := \neg p \rangle \\
 & \neg p \vee \neg q
\end{array}
$$

24.[3] Prove $p \wedge q) \vee (p \wedge \neg q) \equiv p$.

Solution.

Proof of $(p \wedge q) \vee (p \wedge \neg q) \equiv p$.

$$
\begin{array}{ll}
 & (p \wedge q) \vee (p \wedge \neg q) \\
= & \quad \langle \text{Distributivity of } \wedge \text{ over } \vee \text{ (3.46)} \rangle \\
 & p \wedge (q \vee \neg q) \\
= & \quad \langle \text{Excluded Middle (3.28)} \rangle \\
 & p \wedge \mathit{true} \\
= & \quad \langle \text{Identity of } \wedge \text{ (3.39)} \rangle \\
 & p
\end{array}
$$

25.[3] Prove Contrapositive (3.61), $p \implies q \equiv \neg q \implies \neg p$.

Solution.

Proof of the law of the Contrapositive (3.61), $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$.

$$
\begin{array}{ll}
 & \neg q \Rightarrow \neg p \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & \neg\neg q \vee \neg p \\
= & \quad \langle \text{Double negation (3.12)} \rangle \\
 & q \vee \neg p \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & p \Rightarrow q
\end{array}
$$

26.[3] Prove $p \implies q \equiv \neg p \vee \neg q \equiv \neg p$.

Solution.

Proof of $p \Rightarrow q \equiv \neg p \vee \neg q \equiv \neg p$.

$$
\begin{array}{ll}
 & p \Rightarrow q \\
= & \quad \langle \text{Contrapositive (3.61)} \rangle \\
 & \neg q \Rightarrow \neg p \\
= & \quad \langle \text{Implication (3.57)} \rangle \\
 & \neg p \vee \neg q \equiv \neg p
\end{array}
$$

27.[3] Prove theorem (3.64), $p \implies (q \implies r \equiv (p \implies q) \implies (p \implies r)$.

Solution.

Proof of theorem (3.64), $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$.

$$
\begin{array}{ll}
 & (p \Rightarrow q) \Rightarrow (p \Rightarrow r) \\
= & \quad \langle \text{Implication (3.59), twice} \rangle \\
 & \neg p \vee q \Rightarrow \neg p \vee r \\
= & \quad \langle \text{Definition of implication (3.57)} \rangle \\
 & \neg p \vee q \vee \neg p \vee r \equiv \neg p \vee r \\
= & \quad \langle \text{Idempotency of } \vee \text{ (3.26)} \rangle \\
 & \neg p \vee q \vee r \equiv \neg p \vee r \\
= & \quad \langle \text{Distributivity of } \vee \text{ over } \equiv \text{ (3.27)} \rangle \\
 & \neg p \vee (q \vee r \equiv r) \\
= & \quad \langle \text{Definition of implication (3.57)} \rangle \\
 & \neg p \vee (q \Rightarrow r) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & p \Rightarrow (q \Rightarrow r)
\end{array}
$$

28.[3] Prove theorem (3.67), $p \wedge (q \implies p) \equiv p$. Hint: Try to eliminate the implication in a manner that allows an Absorption law to be used.

Solution.

Proof of theorem (3.67), $p \wedge (q \Rightarrow p) \equiv p$.

$$
\begin{array}{ll}
 & p \wedge (q \Rightarrow p) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & p \wedge (\neg q \vee p) \\
= & \quad \langle \text{Absorption (3.43a)} \rangle \\
 & p
\end{array}
$$

29.[3] Prove Replace by *true* (3.85a), $p \implies E[z := p] \equiv p \implies E[z := true]$. In order to be able to use (3.84b), introduce the equivalent true into the antecedent.

Solution.

Proof of theorem (3.85a), $p \Rightarrow E[z := p] \equiv p \Rightarrow E[z := true]$.

$$
\begin{array}{ll}
 & p \Rightarrow E[z := p] \\
= & \quad \langle \text{Identity of } \equiv \text{ (3.3)} \rangle \\
 & p = true \Rightarrow E[z := p] \\
= & \quad \langle \text{Substitution (3.84b)} \rangle \\
 & p = true \Rightarrow E[z := true] \\
= & \quad \langle \text{Identity of } \equiv \text{ (3.3)} \rangle \\
 & p \Rightarrow E[z := true]
\end{array}
$$

30.[3] Prove Replace by *false* (3.86a), $E[z := p] \implies p = E[z := false] \implies p$.

Solution.

Proof of theorem (3.86a), $E[z := p] \Rightarrow p \equiv E[z := false] \Rightarrow$

$$
\begin{array}{ll}
 & E[z := p] \Rightarrow p \\
= & \quad \langle \text{Contrapositive (3.61)} \rangle \\
 & \neg p \Rightarrow \neg E[z := p] \\
= & \quad \langle \text{(3.15)}, \ \neg p \equiv p \equiv false \text{ ; property of textual subst.} \rangle \\
 & p = false \Rightarrow (\neg E)[z := p] \\
= & \quad \langle \text{Substitution (3.84b)} \rangle \\
 & p = false \Rightarrow (\neg E)[z := false] \\
= & \quad \langle \text{(3.15)}, \ \neg p \equiv p \equiv false \text{ ; property of textual subst.} \rangle \\
 & \neg p \Rightarrow \neg E[z := false] \\
= & \quad \langle \text{Contrapositive (3.61)} \rangle \\
 & E[z := false] \Rightarrow p
\end{array}
$$

31.[10] Suppose Portia puts a dagger in one of three caskets and places the following inscriptions on the caskets:

Gold casket: The dagger is in this casket.
Silver casket: The dagger is not in this casket.
Lead casket: At most one of the caskets has a true inscription.

Portia tells her suitor to pick a casket that does not contain the dagger. Which casket should the suitor choose? Formalize and calculate an answer.

Solution.
Calculation of a casket that does not contain Portia's dagger. Let G, S, and L denote that the dagger is in the gold, silver, and lead casket, respectively. Using ig, is, and il to denote the inscriptions on the gold, silver, and lead caskets, we formalize the inscriptions as follows.

Fact 1: $ig \equiv G$
Fact 2: $is \equiv \neg S$
Fact 3: $il \Rightarrow \neg ig \wedge \neg is$
Fact 4: $\neg il \Rightarrow ig \wedge is$

Given these facts, we want to calculate which casket the dagger is in. Because of the nature of facts 3 and 4, we proceed as follows.

$$
\begin{array}{ll}
& il \vee \neg il \qquad \text{—Excluded middle (3.28)} \\
\Rightarrow & \quad \langle \text{Fact 3; Fact 4} \rangle \\
& (\neg ig \wedge \neg is) \ \vee \ (ig \wedge is) \\
= & \quad \langle \text{Fact 1; Fact 2; Double negation (3.12)} \rangle \\
& (\neg G \wedge S) \ \vee \ (G \wedge \neg S) \\
\Rightarrow & \quad \langle \text{Strengthening (3.76b)} \rangle \\
& S \vee G
\end{array}
$$

Hence, either the gold or the silver casket contains the dagger (but we don't know which one), so the dagger is not in the lead casket.

32.[$10 = 5 + 5$] This set of questions concerns an island of knights and knaves. Knights always tell the truth and knaves always lie. In formalizing these questions, associate identifiers as follows:

b : B is a knight.
c : C is a knight.
d : *Disaknight*.

If B says a statement "X", this gives rise to the expression $b \equiv X$, since if b, then B is a knight and tells the truth, and if $\neg b$, B is a knave and lies.

(b) Inhabitant B says of inhabitant C, "If C is a knight, then I am a knave." What are B and C?
(c) It is rumored that gold is buried on the island. You ask B whether there is gold on the island. He replies, "There is gold on the island if and only if I am a knight." Can it be determined whether B is a knight or a knave? Can it be determined whether there is gold on the island?

Solution.

(b) In boolean expression form, B's statement is $c \Rightarrow \neg b$. So we take as *true* the expression $b \equiv (c \Rightarrow \neg b)$. We simplify it.

$$
\begin{array}{ll}
 & b \equiv (c \Rightarrow \neg b) \\
= & \quad \langle \text{Definition of implication (3.57)} \rangle \\
 & b \equiv c \vee \neg b \equiv \neg b \\
= & \quad \langle \text{(3.11)}, \ \neg p \equiv q \equiv p \equiv \neg q \text{, with } p, q := b, c \vee \neg b \text{;} \\
 & \quad\ \text{Double negation (3.12)} \rangle \\
 & \neg(c \vee \neg b) \\
= & \quad \langle \text{De Morgan (3.47b); Double negation (3.12)} \rangle \\
 & \neg c \wedge b
\end{array}
$$

Hence, C is a knave and B is a knight.

(c) Let G stand for "there is gold on the island". Then B's statement is equivalent to $G \equiv b$. So we take as *true* the expression $b \equiv G \equiv b$. We simplify it.

$$
\begin{array}{ll}
 & b \equiv G \equiv b \\
= & \quad \langle \text{Symmetry of } \equiv \text{ (3.2)} \rangle \\
 & G
\end{array}
$$

Hence, there is gold on the island. However, it cannot be determined whether B is a knight or a knave.