Geometry of Surfaces

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Theorema Egregium

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Recall: The Gaussian curvature K of a surface satisfies

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}$$

κ_1, κ_2 principal curvatures

$L du^2 + 2M du dv + N dv^2$ second fundamental form

These are extrinsic invariants and it appears that K depends on the extrinsic geometry of the surface. Carl Friedrich Gauss proved

Theorema egregium. *The Gaussian curvature K of a surface depends only on the first fundamental form of the surface*

Conclusion: The Gaussian curvature is an intrinsic invariant!

To prove the THEOREMA EGREGIUM, we use the METHOD OF MOVING FRAMES. Choose locally an orthonormal frame field $\mathbf{e}', \mathbf{e}'', \mathbf{N} = \mathbf{e}' \times \mathbf{e}''$. At each point the vectors $\mathbf{e}', \mathbf{e}''$ are tangent to the surface and orthonormal. The partial derivatives $\mathbf{e}'_u, \mathbf{e}'_v, \mathbf{e}''_u, \mathbf{e}''_v$ satisfy

$$\mathbf{e}'_u = \alpha\mathbf{e}'' + \lambda'\mathbf{N},\ \mathbf{e}'_v = \beta\mathbf{e}'' + \mu'\mathbf{N},\ \mathbf{e}''_u = -\alpha\mathbf{e}' + \lambda''\mathbf{N},\ \mathbf{e}''_v = -\beta\mathbf{e}' + \mu''\mathbf{N}$$

Lemma. *We have*

$$\mathbf{e}'_u \cdot \mathbf{e}''_v - \mathbf{e}''_u \cdot \mathbf{e}'_v = \lambda'\mu'' - \lambda''\mu' = \alpha_v - \beta_u = \frac{LN - M^2}{\sqrt{EG - F^2}} = K\sqrt{EG - F^2}$$

Proof. Straightforward computations, see lecture notes for details

The Lemma implies $K = \frac{\alpha_v - \beta_u}{\sqrt{EG-F^2}}$. It therefore suffices to prove that α, β depend only on E, F, G for a suitable choice of $\mathbf{e}', \mathbf{e}''$. Apply Gram-Schmidt process to σ_u, σ_v:

$$\mathbf{e}' = \epsilon\sigma_u,\ \ \epsilon = \frac{1}{\|\sigma_u\|} = \frac{1}{\sqrt{E}}$$

Write $\mathbf{e}'' = \gamma\sigma_u + \delta\sigma_v$. From $\mathbf{e}' \cdot \mathbf{e}'' = 0$ and $\mathbf{e}'' \cdot \mathbf{e}'' = 1$ we get

$$\mathbf{e}'' = \gamma\sigma_u + \delta\sigma_v,\ \ \gamma = -\frac{F}{\sqrt{E}\sqrt{EG-F^2}},\ \ \delta = \frac{\sqrt{E}}{\sqrt{EG-F^2}}$$

Thus ϵ, γ, δ depend on E, F, G only

$$
\begin{aligned}
\alpha = \mathbf{e}'_u \cdot \mathbf{e}'' &= (\epsilon\sigma_u)_u \cdot (\gamma\sigma_u + \delta\sigma_v) \\
&= (\epsilon_u\sigma_u + \epsilon\sigma_{uu}) \cdot (\gamma\sigma_u + \delta\sigma_v) \\
&= \cdots = \tfrac{1}{2}\epsilon\gamma E_u + \epsilon\delta(F_u - \tfrac{1}{2}E_v) \\
\beta = \mathbf{e}'_v \cdot \mathbf{e}'' &= (\epsilon\sigma_u)_v \cdot (\gamma\sigma_u + \delta\sigma_v) \\
&= (\epsilon_v\sigma_u + \epsilon\sigma_{uv}) \cdot (\gamma\sigma_u + \delta\sigma_v) \\
&= \cdots = \tfrac{1}{2}\epsilon\gamma E_v + \tfrac{1}{2}\epsilon\delta G_u
\end{aligned}
$$

Thus α, β depend on E, F, G only. This finishes the proof of the THEOREMA EGREGIUM

We see from the proof of the THEOREMA EGREGIUM that there exists an explicit expression for the Gaussian curvature K in terms of E, F, G and their derivatives. This involves tedious calculations and the formula is not practical in general. Special cases:

If $F = 0$, then

$$K = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) \right)$$

If $E = 1$ and $F = 0$, then

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}$$

Apply this to surface of revolution

Application of THEOREMA EGREGIUM to cartography:

Proposition. *Any planar map of any region of the earth's surface must distort distances.*

Proof. A sphere of radius r has constant Gaussian curvature $\frac{1}{r^2}$ and a plane has constant Gaussian curvature 0. A planar map of any region of the earth's surface not distorting distances is an isometry (up to rescaling by a constant factor) and hence preserves the first fundamental forms of sphere and plane. Such an isometry cannot exist by THEOREMA EGREGIUM.

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Gauss-Bonnet Theorem (local version)

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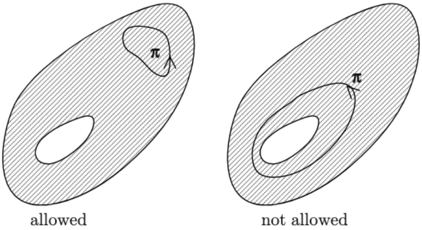
Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $\pi(s) = (u(s), v(s))$ be a simple closed curve in $\mathbb{R}^2$ whose interior $\text{int}(\pi)$ is contained in U.



Define $\gamma(s) = \sigma(\pi(s)) = \sigma(u(s), v(s))$ and assume $\|\dot{\gamma}\| = 1$. We say that γ is positively oriented if π is positively oriented, that is, if the oriented unit normal $\mathbf{n}_s$ of π points into $\text{int}(\pi)$ everywhere.

Gauss-Bonnet Theorem (local version). *Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $\pi(s) = (u(s), v(s))$ be a simple closed curve in $\mathbb{R}^2$ with int$(\pi) \subset U$. Let $\gamma(s) = \sigma(\pi(s))$ and assume that γ is positively oriented and $\|\dot{\gamma}\| = 1$. Then*

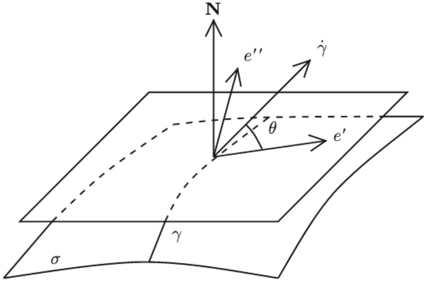
$$\boxed{\int_\gamma \kappa_g ds = 2\pi - \iint\limits_{\mathrm{int}(\pi)} K \, d\mathcal{A}_\sigma}$$

where κ_g is the geodesic curvature of γ,
K is the Gaussian curvature of σ,
$d\mathcal{A}_\sigma = \sqrt{EG - F^2} dudv$ is the area element on σ.

Choose moving frame $\mathbf{e}', \mathbf{e}'', \mathbf{N} = \mathbf{e}' \times \mathbf{e}''$. Then

$$\begin{aligned}
\int_\gamma \mathbf{e}' \cdot \dot{\mathbf{e}}'' ds &= \int_\gamma \mathbf{e}' \cdot (\mathbf{e}''_u \dot{u} + \mathbf{e}''_v \dot{v}) ds = \int_\pi \left((\mathbf{e}' \cdot \mathbf{e}''_u) du + (\mathbf{e}' \cdot \mathbf{e}''_v) dv \right) \\
&= \iint\limits_{\mathrm{int}(\pi)} \left((\mathbf{e}' \cdot \mathbf{e}''_v)_u - (\mathbf{e}' \cdot \mathbf{e}''_u)_v \right) dudv \quad (\textsc{Green's Thm}) \\
&= \iint\limits_{\mathrm{int}(\pi)} \left((\mathbf{e}'_u \cdot \mathbf{e}''_v) - (\mathbf{e}'_v \cdot \mathbf{e}''_u) \right) dudv \\
&= \iint\limits_{\mathrm{int}(\pi)} \frac{LN - M^2}{\sqrt{EG - F^2}} dudv \quad \text{(Lemma on moving frames)} \\
&= \iint\limits_{\mathrm{int}(\pi)} \frac{LN - M^2}{EG - F^2} \sqrt{EG - F^2} dudv = \iint\limits_{\mathrm{int}(\pi)} K \, d\mathcal{A}_\sigma
\end{aligned}$$

Put $\theta(s) = \angle(\mathbf{e}'(\gamma(s)), \dot{\gamma}(s))$



Then $\dot{\gamma} = \cos(\theta)\mathbf{e}' + \sin(\theta)\mathbf{e}''$ and

$$\mathbf{N} \times \dot{\gamma} = (\mathbf{e}' \times \mathbf{e}'') \times (\cos(\theta)\mathbf{e}' + \sin(\theta)\mathbf{e}'') = -\sin(\theta)\mathbf{e}' + \cos(\theta)\mathbf{e}''$$

$$\ddot{\gamma} = \cos(\theta)\dot{\mathbf{e}}' + \sin(\theta)\dot{\mathbf{e}}'' + \dot{\theta}(-\sin(\theta)\mathbf{e}' + \cos(\theta)\mathbf{e}'')$$

$$\mathbf{N} \times \dot{\gamma} = (\mathbf{e}' \times \mathbf{e}'') \times (\cos(\theta)\mathbf{e}' + \sin(\theta)\mathbf{e}'') = -\sin(\theta)\mathbf{e}' + \cos(\theta)\mathbf{e}''$$
$$\ddot{\gamma} = \cos(\theta)\dot{\mathbf{e}}' + \sin(\theta)\dot{\mathbf{e}}'' + \dot{\theta}(-\sin(\theta)\mathbf{e}' + \cos(\theta)\mathbf{e}'')$$

Then

$$\kappa_g = (\mathbf{N} \times \dot{\gamma}) \cdot \ddot{\gamma} = \cdots = \dot{\theta} - \mathbf{e}' \cdot \dot{\mathbf{e}}''$$

Altogether

$$\iint\limits_{\mathrm{int}(\pi)} K \, d\mathcal{A}_\sigma = \int_\gamma \mathbf{e}' \cdot \dot{\mathbf{e}}'' ds = \int_\gamma (\dot{\theta} - \kappa_g) ds$$

It remains to prove that $\int_\gamma \dot{\theta} ds = 2\pi$. This is the so-called HOPF UMLAUFSATZ (Hopf rotation angle theorem) and requires topological arguments for its proof (see notes for heuristic argument)

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Gauss-Bonnet Theorem for curvilinear polygons

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A curvilinear polygon is a simple closed curve γ for which there exist real numbers

$$0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = \ell = L(\gamma)$$

so that $\gamma(0) = \gamma(\ell)$, γ is smooth on the intervals (s_{i-1}, s_i) and the two one-sided derivatives at the corners s_i,

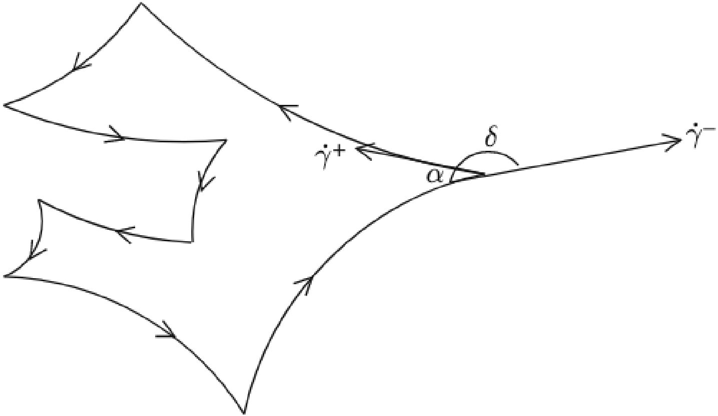
$$\dot{\gamma}^-(s_i) = \lim_{s \uparrow s_i} \frac{\gamma(s) - \gamma(s_i)}{s - s_i}, \quad \dot{\gamma}^+(s_i) = \lim_{s \downarrow s_i} \frac{\gamma(s) - \gamma(s_i)}{s - s_i},$$

exist, are non-zero and non-parallel. The angle

$$\delta_i = \angle(\dot{\gamma}^-(s_i), \dot{\gamma}^+(s_i)) \in (-\pi, \pi)$$

is called the external angle of γ at s_i

and $\alpha_i = \pi - \delta_i \in (0, 2\pi)$ is called the internal angle of γ at s_i


$\dot\gamma^+$
δ
α
$\dot\gamma^-$

Gauss-Bonnet Theorem for curvilinear polygons. *Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $\gamma(s) = \sigma(u(s), v(s))$ be a curvilinear polygon on σ for which the interior int(γ) of the curve $(u(s), v(s))$ is contained in U. Assume that γ is positively oriented and parametrized by arc length. Then*

$$\boxed{\int_\gamma \kappa_g ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \iint_{\mathrm{int}(\gamma)} K \, d\mathcal{A}_\sigma}$$

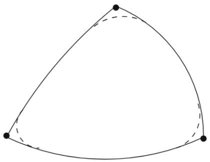
where κ_g is the geodesic curvature of γ,
K is the Gaussian curvature of σ,
$d\mathcal{A}_\sigma = \sqrt{EG - F^2} dudv$ is the area element on σ.

Proof. The same arguments as in the proof for the local version of the Gauss-Bonnet Theorem give

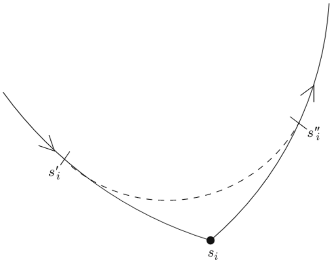
$$\iint_{\mathrm{int}(\gamma)} K\, d\mathcal{A}_\sigma = \int_\gamma \dot{\theta} ds - \int_\gamma \kappa_g ds$$

with $\theta(s) = \angle(\mathbf{e}'(\gamma(s)), \dot{\gamma}(s))$. Need to compute $\int_\gamma \dot{\theta} ds$.

We first smoothen each vertex of the polygon to get a smooth curve $\tilde{\gamma}$



Hopf's Umlaufsatz gives $\int_{\tilde{\gamma}} \dot{\tilde{\theta}} ds = 2\pi$ with $\tilde{\theta}(s) = \angle(\mathbf{e}'(\tilde{\gamma}(s)), \dot{\tilde{\gamma}}(s))$. Choose points s_i', s_i'' as illustrated here



Then

$$\int_{\tilde{\gamma}} \dot{\tilde{\theta}} ds - \int_{\gamma} \dot{\theta} ds = \sum_{i=1}^{n} \left(\int_{s_i'}^{s_i''} \dot{\tilde{\theta}} ds - \int_{s_i'}^{s_i} \dot{\theta} ds - \int_{s_i}^{s_i''} \dot{\theta} ds \right)$$

Since γ is smooth on (s_i', s_i) and (s_i, s_i''), we have

$$\lim_{s_i' \to s_i} \int_{s_i'}^{s_i} \dot{\theta} ds = 0 \ , \ \lim_{s_i'' \to s_i} \int_{s_i}^{s_i''} \dot{\theta} ds = 0$$

Furthermore,

$$\int_{s_i'}^{s_i''} \dot{\tilde{\theta}} ds = \angle(\dot{\tilde{\gamma}}(s_i''), \dot{\tilde{\gamma}}(s_i')) \underset{s_i', s_i'' \to s_i}{\longrightarrow} \angle(\dot{\gamma}^-(s_i), \dot{\gamma}^+(s_i)) = \delta_i$$

Altogether,

$$\int_{\tilde{\gamma}} \dot{\tilde{\theta}} ds - \int_{\gamma} \dot{\theta} ds = \sum_{i=1}^{n} \left(\int_{s_i'}^{s_i''} \dot{\tilde{\theta}} ds - \int_{s_i'}^{s_i} \dot{\theta} ds - \int_{s_i}^{s_i''} \dot{\theta} ds \right) = \sum_{i=1}^{n} \delta_i$$

Thus,

$$\int_{\gamma} \dot{\theta} ds = \int_{\tilde{\gamma}} \dot{\tilde{\theta}} ds - \sum_{i=1}^{n} \delta_i = 2\pi - \sum_{i=1}^{n} \delta_i = 2\pi - \sum_{i=1}^{n} (\pi - \alpha_i)$$

$$= 2\pi - n\pi + \sum_{i=1}^{n} \alpha_i$$

Altogether,

$$\int_{\gamma} \kappa_g ds = \int_{\gamma} \dot{\theta} ds - \iint_{\mathrm{int}(\gamma)} K \, d\mathcal{A}_{\sigma} = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \iint_{\mathrm{int}(\gamma)} K \, d\mathcal{A}_{\sigma}$$

Corollary. *If γ is a curvilinear polygon with n edges each of which is a geodesic, then*

$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi + \iint_{\mathrm{int}(\gamma)} K \, d\mathcal{A}_\sigma$$

Question: What does this tell us about polygons in the plane ($K = 0$) and about triangles in the unit sphere ($K = 1$) and in the pseudosphere ($K = -1$)?