Geometry of surfaces - Solutions

17. We have

$$\frac{d\gamma}{dt}(t) = \left(-\frac{1}{\sqrt{2}}\sin(t) + \frac{1}{\sqrt{3}}\cos(t), -\frac{1}{\sqrt{3}}\cos(t), \frac{1}{\sqrt{2}}\sin(t) + \frac{1}{\sqrt{3}}\cos(t)\right),$$

which implies that $\gamma(t)$ is unit speed. Differentiating again gives

$$\ddot{\gamma}(t) = \left(-\frac{1}{\sqrt{2}}\cos(t) - \frac{1}{\sqrt{3}}\sin(t), \frac{1}{\sqrt{3}}\sin(t), \frac{1}{\sqrt{2}}\cos(t) - \frac{1}{\sqrt{3}}\sin(t)\right)$$

and thus $\kappa(t) = \|\ddot{\gamma}(t)\| = 1$. Note that $\ddot{\gamma} = -\gamma$. Hence, $\mathbf{n} = \frac{1}{\kappa}\dot{\mathbf{t}} = \ddot{\gamma} = -\gamma$ and $\mathbf{b} = \mathbf{t} \times \mathbf{n} = -\dot{\gamma} \times \gamma$. Thus $\dot{\mathbf{b}} = -\ddot{\gamma} \times \gamma - \dot{\gamma} \times \dot{\gamma} = \gamma \times \gamma - \dot{\gamma} \times \dot{\gamma} = 0$, which gives $\tau = 0$. Since γ has vanishing torsion, it is a planar curve (by Proposition 2.3.3). Since $\kappa = 1$, γ is a circle of radius 1 in that plane (use Example 2.2.3).

18. Curvature κ and torsion τ of the circular helix are constant and equal to $\kappa = \frac{|a|}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$ (see Example 2.1.3 and Example 2.3.2).

Consider a curve γ in $\mathbb{R}^3$ with constant curvature κ and constant torsion τ.

(i) If $\kappa = 0$, then γ is (part of) a straight line (Proposition 1.1.4).

(ii) If $\kappa \neq 0$ and $\tau = 0$, then γ is a planar curve and (part of) a circle (Proposition 2.3.3 and Example 2.2.3).

(iii) If $\kappa \neq 0$ and $\tau \neq 0$, then γ is a non-planar curve and (part of) a helix (Theorem 2.3.6, Example 2.1.3 and Example 2.3.2).

19. Taylor's Theorem tells us that

$$\gamma(s) = \gamma(0) + s\frac{d\gamma}{ds}(0) + \frac{s^2}{2}\frac{d^2\gamma}{ds^2}(0) + \frac{s^3}{6}\frac{d^3\gamma}{ds^3}(0) + O(s^4).$$

Using the Frenet-Serret equations we obtain

$$\begin{aligned}
\frac{d\gamma}{ds} &= \mathbf{t}, \\
\frac{d^2\gamma}{ds^2} &= \dot{\mathbf{t}} = \kappa\mathbf{n}, \\
\frac{d^3\gamma}{ds^3} &= \dot{\kappa}\mathbf{n} + \kappa\dot{\mathbf{n}} = \dot{\kappa}\mathbf{n} + \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) = -\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \kappa\tau\mathbf{b}.
\end{aligned}$$

Inserting this into the above equation gives the local canonical form of γ at p.

20. The local canonical form of γ at p shows that γ crosses its osculating plane at p if and only if $\tau(0) \neq 0$.

21. Differentiating $\gamma(s) + \alpha(s)\mathbf{t}(s) = v$ and using $\mathbf{t} = \dot{\gamma}$ gives

$$0 = \dot{\gamma} + \dot{\alpha}\dot{\gamma} + \alpha\ddot{\gamma} = (1 + \dot{\alpha})\dot{\gamma} + \alpha\ddot{\gamma}.$$

Since $\dot{\gamma} \neq 0$ and $\dot{\gamma} \cdot \ddot{\gamma} = 0$ everywhere, this gives $1 + \dot{\alpha} = 0$ and thus $\alpha(s) = -s + c$ with some $c \in \mathbb{R}$. This implies $0 = (-s + c)\ddot{\gamma}(s)$ for all s and hence $\ddot{\gamma} = 0$. Thus the curvature of γ vanishes, which implies that γ is (part of) a straight line.

22. Differentiating $\gamma(s) + \alpha(s)\mathbf{n}(s) = v$ and using the Frenet-Serret equation for $\dot{\mathbf{n}}$ gives

$$0 = \dot{\gamma} + \dot{\alpha}\mathbf{n} + \alpha\dot{\mathbf{n}} = (1 - \alpha\kappa)\mathbf{t} + \dot{\alpha}\mathbf{n} + \alpha\tau\mathbf{b}.$$

Since $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are linearly independent, this gives $1 - \alpha\kappa = 0$, $\dot{\alpha} = 0$ and $\alpha\tau = 0$. From $\dot{\alpha} = 0$ we see that α is constant, and from $1 - \alpha\kappa = 0$ we then deduce that $\alpha \neq 0$ and the curvature κ is constant (and nonzero). Using $\alpha\tau = 0$ we see that the torsion τ vanishes and hence γ lies in a plane. Altogether this implies that γ is (part of) a circle (see Exercise 18(ii) above).

23. Differentiating $\mathbf{t} \cdot a = \alpha$ implies $\kappa\mathbf{n} \cdot a = 0$. Since $\kappa \neq 0$ everywhere, this implies $\mathbf{n} \cdot a = 0$. Therefore there exists a function β such that $a = \alpha\mathbf{t}(s) + \beta(s)\mathbf{b}(s)$. From $\|a\|^2 = \alpha^2 + \beta(s)^2$ we see that β is constant and therefore $a = \alpha\mathbf{t}(s) + \beta\mathbf{b}(s)$. Differentiating this equation and using the Frenet-Serret equations gives $0 = \alpha\dot{\mathbf{t}}(s) + \beta\dot{\mathbf{b}}(s) = (\alpha\kappa - \beta\tau)\mathbf{n}(s)$. Thus $0 = \alpha\kappa - \beta\tau$, which implies that $\frac{\kappa}{\tau} = \frac{\beta}{\alpha}$ is constant.