Equivalence Relations and Partial Orders

CS 2LC3

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Equivalence Relations

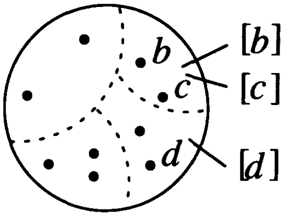
- **Definition**. A relation R is an *equivalence relation* iff it is reflexive, symmetric, and transitive.
- In other words, $R \subseteq B \times B$ is an *equivalence relation* iff
  for all $b, c, d \in B$,
  $bRb \wedge (bRc \Leftrightarrow cRb) \wedge (bRc \wedge cRd \Rightarrow bRd)$.
- For example, equality $=$ is an equivalence relation, while $<$ is not.
- **Alternative Definition**. A relation $R \subseteq B \times B$ is an *equivalence relation* iff there exists a set C and *function* $f_R : B \to C$ such that $bRc \iff f_R(b) = f_R(c)$.
- We will prove equivalence of these two definitions later.
- An equivalence relation R on a set B partitions B into non-empty disjoint subsets.
- Notation. The most popular symbol used to denote an equivalence relation is $\equiv$. However in this textbook, the symbol $\equiv$ is used to denote $\iff$.

Equivalence Classes

- **Definition**. Let R be an equivalence relation on B. Then $[b]_R$, the *equivalence class* of b, is the subset of elements of B that are equivalent (under R) to b:

$$x \in [b]_R \equiv xRb.$$

- In what follows, we eliminate the subscript R and write $[b]_R$ as $[b]$ when it is obvious from the context what relation is meant.



Theorem

Let R be an equivalence relation on B and let b, c be members of B. The following three predicates are equivalent:

(a) bRc,

(b) $[b] \cap [c] \neq \emptyset$, *and*

(c) $[b] = [c]$.

That is, $(bRc) = ([b] \cap [c] \neq \emptyset) = ([b] = [c])$.

Proof.

We can prove, in turn, (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a). Mutual implication and transitivity of $\Rightarrow$ then give the equivalence of all three.
(a) $\Rightarrow$ (b): $bRc \Rightarrow (c \in [b] \wedge b \in [c]) \Rightarrow [b] \cap [c] \neq \emptyset$. A very detailed proof is in the textbook.
(b) $\Rightarrow$ (c): Let $x \in [b] \cap [c]$. Then $bRxRc$, so bRc (note that we actually have proved (b) $\Rightarrow$ (a)!). Let $y \in [b]$. then $yRbRc$, so yRc, i.e. $y \in [c]$. Identically when starting with $y \in [c]$.
(c) $\Rightarrow$ (a): $[b] = [c] \Rightarrow c \in [b] \equiv cRb \equiv bRc$. ■

- Thus, an equivalence relation on B induces a partition of B , where each partition element consists of equivalent elements.

Theorem

Let P *be the set of sets of a partition of* B *. The following relation* R *on* B *is an equivalence relation:*

$$bRc = \exists(Q \mid p \in \mathrm{P} : b \in Q \wedge c \in Q).$$

Proof.

We must show that p is reflexive, symmetric, and transitive. Reflexivity follows from the fact that each element is in one of the sets in P . Symmetry follows from the definition of R in terms of $\wedge$, which is symmetri Transitivity follows from the definition of the composition of relations $\circ$. A detailed proof in the textbook.

■

Theorem

For every relation $R \subseteq B \times B$ the following two statements are equivalent

(a) *R is an equivalence relation, i.e., it is reflexive, symmetric and transitive;*

(b) *There exists a set C and* function *$f_R : B \to C$ such that $bRc \iff f_R(b) = f_R(c)$.*

Proof.

(b) $\Rightarrow$ (a) Since $=$ it is reflexive, symmetric and transitive.
(a) $\Rightarrow$ (b) Define $C = B/_R$ and $f_R(b) = [b]_R$, for all $b \in B$.
Then from the two last theorems we have that R is an equivalence relation. ∎

- An alternative definition from point (b) of the last theorem is usually used in applications.
- Quite often $C = \{true, false\}$ and f_R is just some predicate.
- The function f_R is often called *invariant under R*.

Some Properties of Equivalence Relation

- Let $R \subseteq B \times B$ is an equivalence relation. The set of all equivalence classes of$R$ is denoted by $B/_R$, i.e.
$$B/_R = \{[b] \mid b \in B\}$$
- For every equivalence relation R, $=$ is a subset of R, i.e. $= \subseteq R$. Hence $=$ is the *smallest* equivalence relation.
- For every x, $[x]_= = \{x\}$, so $|[x]_=| = 1$.
- Let R, Q be equivalence relations and $R \subseteq Q$. Then, for all $b \in B$, $[b]_R \subseteq [b]_Q$.
- Let R, Q be finite equivalence relations and $R \subseteq Q$. Then, $|B/_R| \geq |B/_Q|$.
- For finite B, $|B/_=| = |B|$.
- In a natural language, the word "equal" is very often used instead of more precise "equivalent", the most famous example is "all people are equal".

Order Relations

Definition

A binary relation R on a set B is called a *partial order* on B if it is reflexive, antisymmetric, and transitive.
In other words, R is a *partial order* if for all $b, c, d \in B$, we have bRb, $(bRc \wedge cRb) \Rightarrow b = c)$ and $(bRc \wedge cRd \Rightarrow bRd)$.
In this case, the pair (B, R) is called a *partially ordered set* or *poset*.

- We use the symbol $\preceq$ for an arbitrary partial order.

Examples of Partial Orders

(a) $\langle \mathbb{N}, \leq \rangle$ is a poset and $\leq$ is a partial order on $\mathbb{N}$.

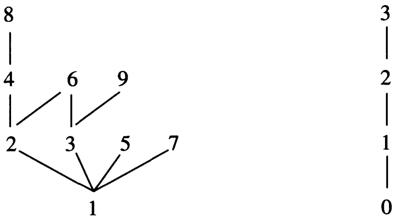
(b) Let B be a set. Then $\langle \mathcal{P}B, \subseteq \rangle$ is a poset and $\subseteq$ is a partial order on $\mathcal{P}B$, since $\subseteq$ is reflexive, antisymmetric, and transitive (Theorems (11.57)–(11.59) on page 207).

(c) Consider the set C of courses offered at Cornell University. Define the relation $\preceq$ by $c1 \preceq c2$ if courses $c1 = c2$ or if $c1$ is a prerequisite for $c2$. Then $\langle C, \preceq \rangle$ is a poset and $\preceq$ is a partial order on C.

(d) Let P be the set of loops in a Pascal program. Define $\preceq$ on P by $l1 \preceq l2$ if loops $l1$ and $l2$ are the same or if $l1$ is nested within $l2$. Then $\langle P, \preceq \rangle$ is a poset and $\preceq$ is a partial order on P.

(e) In constructing a house, certain jobs have to be done before other jobs. Let J be the set of jobs to be done, and let $\preceq$ on J be defined by $j1 \preceq j2$ if $j1$ and $j2$ are the same or if $j1$ has to be completed before $j2$ can be started. Then $\langle J, \preceq \rangle$ is a poset. The scheduling of jobs in such situations, including redefining and ordering jobs in order to reduce time to completion, is sometimes referred to as PERT (Program Evaluation and Review Technique). $\square$

- If B is finite, the relation $\preceq$ can be depicted as a *Hasse diagram*, as illustrated below, where $b|c \equiv$ "b divides c".



poset $\langle 1..9, | \rangle$ poset $\langle 0..3, < \rangle$

- In Hasse diagram, a line is drawn from b up to c iff $b \preceq c$ and no element d (other than b, c) satisfies $b \preceq d \preceq c$.
- The Hasse diagram for a partial order R actually presents the smallest relation R' such that $R = (R')^*$.
- Relation R' is called the *transitive reduction* of partial order R.

Strict/Sharp Partial Orders

Definition

Relation $\prec$ is a *quasi order* or *strict/sharp partial order* if $\prec$ is irreflexive and transitive, i.e. $b \not\prec b \wedge (b \prec c \prec d \Rightarrow b \prec d)$.

Theorem

(a) *If $\preceq$ is a partial order on B then $\preceq \setminus id_B$ is a sharp partial order on B.*

(b) *If $\prec$ is a sharp partial order on B then $\prec \cup id_B$ is a partial order on B.*

- In mathematics $\preceq$ is usually used, while in theoretical computer science and economy/management science (algorithms, concurrency, ranking, etc.) $\prec$ is usually used.

Total/Linear Orders

Definition

A partial order $\preceq$ over B is called a *total* or *linear* order if

$$\forall(b, c \mid: b \preceq c \vee c \preceq b),$$

i.e. iff $\preceq \cup \preceq^{-1} = B \times B$. In this case, the pair $(B, \preceq)$ is called a *linearly/totally ordered set* or a *chain*.

Examples of total orders and chains

(a) $\leq$ over the natural numbers is a total order, and $\langle \mathbb{N}, \leq \rangle$ is a chain.

(b) $\leq$ over the reals is a total order, and $\langle \mathbb{R}, \leq \rangle$ is a chain.

(c) Let set S contain more than one element. Then $\subseteq$ over $\mathcal{P}S$ is not a total order. For example, if b and c are distinct elements in S, then neither $\{b\} \subseteq \{c\}$ nor $\{c\} \subseteq \{b\}$ holds.

(d) Let C be the set of courses at Cornell. Let $b \preceq c$ mean that $b = c$ or b is a prerequisite for c. Relation $\preceq$ is a partial order but not a total order. $\square$

Stratified/Weak Orders

- **Definition**. Let $\preceq$ be a partial order on B. We define a relation $\frown_{\preceq}$ on B, called *incomparability*, as follows:

  $$b \frown_{\preceq} c \equiv b \not\preceq c \wedge c \not\preceq b.$$

  Note that $b \not\prec b$, so $b \frown_{\preceq} b$, i.e. id_B is included in $\frown_{\preceq}$.

Definition

A partial order $\preceq$ on B is *stratified* (or *weak*) iff $\frown_{\preceq}$ is an equivalence relation.

- If $\preceq$ is a total order, then $\frown_{\preceq} = id_B$, so every total order is stratified.

Theorem

Let $\preceq$ be a stratified order on B. Define a relation $\lhd_{\preceq}$ on $B/_{\frown_{\preceq}}$ as follows

$$[b]_{\frown_{\preceq}} \lhd_{\preceq} [c]_{\frown_{\preceq}} \equiv b \preceq c.$$

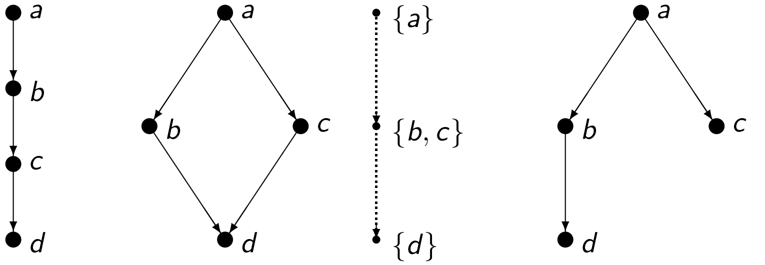
The relation $\lhd_{\preceq}$ is a total order.

Proof.

Since $\preceq$ is a partial order, $\lhd_{\preceq}$ is a partial order as well. We just need to show that it is total. Take arbitrary $[b]$ and $[c]$ from $B/_{\frown_{\preceq}}$. Clearly $b \preceq c \vee c \preceq b \vee b \frown_{\preceq} c \equiv \mathit{true}$. Moreover $b \preceq c \Rightarrow [b] \lhd [c]$, $c \preceq b \Rightarrow [c] \lhd [b]$ and $b \frown_{\preceq} c \Rightarrow [b] = [c]$, so $\lhd$ is total. ∎

- Stratified (or Weak) orders play important role in management science, especially in ranking theory.

Examples of Partial Orders



$\prec_1$ total | $\prec_2$ stratified | $\lhd_2$ total | $\prec_3$ not stratified

Well Founded Sets and Partial Orders

Recall:

- Element y is a *minimal* element of S iff $y \in$ and $\forall(x \mid x \prec y : x \notin S)$.
- $(U, \prec)$ is *well founded* if every nonempty subset of U has a minimal element.

Theorem

Every well founded set $(U, \prec)$ is a (sharp) partially ordered set.

Proof.

We have to prove that $\prec$ is (a) irreflexive and (b) transitive.
(a) Assume $b \prec b$ for some $b \in U$. Then the set $\{b\}$ does not have a minimum. Hence $b \not\prec b$.
(b) Assume $b \prec c \prec d$. Consider the set $B = \{b, c, d\}$. Neither c nor d can be minimal as $b \prec c$ and $c \prec d$. On the other hand $(U, \prec)$ is well founded, so $S = \{b, c, d\}$ has a minimal element. So, b must be minimal, which means $b \prec d$. ∎

Lower/Upper Bounds

Definition (Lower)

Let S be a nonempty subset of poset $(U, \prec)$.

(a) Element b of S is a *minimal* element of S if no element of S is smaller than b, i.e. if $b \in S \wedge \forall(c \mid c \prec b : c \notin S)$.

(b) Element b of S is the *least element* of S if $b \in S \wedge \forall(c \mid c \in S : b \preceq c)$.

(c) Element b is a *lower bound* of S if $\forall(c \mid c \in S : b \preceq c)$. (Element b need not be in S.)

(d) Element b is the *greatest lower bound* of S, written $glb(S)$, if b is a lower bound and if every lower bound c satisfies $c \preceq b$.

- A set may have more than one minimal element.
- However, a set has at most one least element.
- Minimal elements and least elements of a set belong to the set.
- Lower bounds need not belong to the set.

Definition (Upper)

Let S be a nonempty subset of poset $(U, \prec)$.

(a) Element b of S is a *maximal* element of S if no element of S is smaller than b, i.e. if $b \in S \wedge \forall(c \mid b \prec c : c \notin S)$.

(b) Element b of S is the *greatest element* of S if $b \in S \wedge \forall(c \mid c \in S : c \preceq b)$.

(c) Element b is an *upper bound* of S if $\forall(c \mid c \in S : c \preceq b)$. (Element b need not be in S.)

(d) Element b is the *least upper bound* of S, written $lub(S)$, if b is an upper bound and if every upper bound c satisfies $b \preceq c$.

- A set may have more than one maximal element.
- However, a set has at most one greatest element.
- Maximal elements and greatest elements of a set belong to the set.
- Upper bounds need not belong to the set.

Examples of minimal and least elements

(a) Set $\mathbb{N}$ of poset $\langle \mathbb{N}, \leq \rangle$ has minimal element 0 and least element 0.

(b) Set $\mathbb{R}$ of poset $\langle \mathbb{R}, \leq \rangle$ has no minimal or least element. But subset $\{x \mid 0 \leq x\}$ has 0 as its minimal and least element.

(c) Consider $\langle \mathbb{N}, | \rangle$, where $i \,|\, j$ means "i divides j". Subset $\{3, 5, 7, 15, 20\}$ has three minimal elements, 3, 5, and 7, but it has no least element. Subset $\{2, 4, 6, 8\}$ has minimal and least element 2.

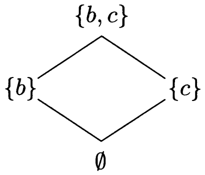
(d) Consider poset $\langle \mathcal{P}\{b, c\}, \subseteq \rangle$, with Hasse diagram in Fig. 14.4. The elements of $\mathcal{P}\{b, c\}$ are $\emptyset$, $\{b\}$, $\{c\}$, and $\{b, c\}$. Its minimal and least element is $\emptyset$. The minimal and least element of subset $\{\{b\}\}$ is $\{b\}$. Subset $\{\{b\}, \{c\}\}$; has two minimal elements, $\{b\}$ and $\{c\}$, and no least element. $\square$

Examples of lower bounds and greatest lower bounds

(a) Consider poset $\langle \mathbb{R}, \leq \rangle$. Subset $S = \{x \mid 0 < x < 1\}$ has 0 and all nonnegative numbers for lower bounds. Its greatest lower bound is 0. But S has no least element. On the other hand, subset $T = \{x \mid 0 \leq x < 1\}$ has the same lower bounds and greatest lower bounds, but it has a least element: 0.

(b) Consider poset $\langle \mathcal{P}\{b,c\}, \subseteq \rangle$, with Hasse diagram in Fig. 14.4. Subset $\{\{b\}\}$ has $\emptyset$ and $\{b\}$ for lower bounds, and its greatest lower bound is $\{b\}$. Subset $\{\{b,c\}\}$ has every element of $\mathcal{P}\{b,c\}$ as lower bound, and itself as its greatest lower bound. $\square$

FIGURE 14.4. Poset $\langle \mathcal{P}\{b,c\}, \subseteq \rangle$



Theorem

Every finite nonempty subset S of poset $(U, \preceq)$ has a minimal (maximal) element.

Proof.

For minimal element: Choose any element x_0 of S and construct a decreasing chain of elements of S: $x_n \prec \ldots \prec x_2 \prec x_1 \prec x_0$ (for some n) until no longer possible. Antisymmetry of $\preceq$ implies that all elements of the chain are distinct. Since S is finite, this chain is finite. Element x_n is a minimal element of S. Similarly for maximal element. ■

Theorem

Let B be a nonempty subset of poset $(U, \prec)$.

(a) *A least (greatest) element of B is also a minimal (maximal) element of B (but not necessarily vice versa).*

(b) *A least (greatest) element of B is also a greatest lower bound (least upper bound) of B (but not necessarily vice versa).*

(c) *A lower (upper) bound of B that belongs to B is also a least (greatest) element of B.*

Proof.

Follows immediately from the definitions. ■

- Upper and lower bounds create a foundations of a special class of partial partial orders called "*lattices*".
- Lattice theory is a foundation of the theory of "*fixed points*", i.e. finding solutions to the equations $F(x) = x$, where $F : X \to X$, and X is some set, not necessarily set on numbers, could be set of computations, languages, relations.
- For numbers, we prefer finding solution to $f(x_1, \ldots, x_n) = (0, \ldots, 0)$ (for example $ax^2 + bx + c = 0$, etc.), but *zero* has no equivalence in other non-numerical domains. For example *empty set* has some properties similar to zero properties, but not all.
- Theory of fixed points create a basis for *denotational semantics* of programming languages, one of the fundamental approaches to formal semantics of programming languages.

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  as *zero* does not have the same interpretation for sets or strings.
- When we can solve $\mathbf{x} = F(\mathbf{x})$?
- When the domain of $\mathbf{x}$ is a *complete lattice* and the function $F(\mathbf{x})$ is *monotone*.

Lattices

Definition

- A poset $(X, \preceq)$ is called a **lattice** iff for all $a, b \in X$ both $lub(\{a, b\})$ and $glb(\{a, b\})$ exist.
- A poset $(X, \preceq)$ is called a **complete lattice** iff for each $A \subseteq X$ both $lub(A)$ and $glb(A)$ exist.

- $(\mathbb{Z}, \leq)$ is a lattice $lub(\{i, j\}) = \max\{i, j\}$ and $glb(\{i, j\}) = \min\{i, j\}$, for all integers i, j.
- $(\mathbb{Z}, \leq)$ is not a complete lattice, since, for example, for the set $odd = \{i \mid: i \text{ is odd } \} \subseteq \mathbb{Z}$, neither $lub(odd)$ nor $glb(odd)$ exists.
- For every set X, a poset $(\mathcal{P}(X), \subseteq)$ is a complete lattice
- For every set X, a poset $(\mathcal{P}(X \times X), \subseteq) = (Rel(X), \subseteq)$ is a complete lattice

Monotone Functions

Definition

Let $(X, \preceq)$ be a poset.
A function $f : X \to X$ is **monotone** iff $x \preceq y \iff f(x) \preceq f(y)$.

Theorem

Let $(X, \preceq)$ be a complete lattice.
If $f : X \to X$ is monotone, then both the least fixed point and the greatest fixed point of $x = f(x)$ do exist.

Lattice of Relations

Let X be a set and $Rel(X) = \mathcal{P}(X \times X)$ be the set of all relations on X.
Consider the *complete lattice* $(\mathcal{P}(X \times X), \subseteq)$, and the function $f : \mathcal{P}(X \times X) \rightarrow \mathcal{P}(X \times X)$ such that for every $x \subseteq X \times X$ (i.e. $x \in \mathcal{P}(X \times X)$),

$$f(x) = R \circ x \cup Q,$$

where $R, Q \subseteq \mathcal{P}(X \times X)$ and $\circ$ is *composition of relations*.

- Note that $f(x) = R \circ x \cup Q$ is monotone, since if $x \subseteq y$ then $R \circ x \subseteq R \circ y$, so $f(x) \subseteq f(y)$.
- What is the fixed point of the equation $x = f(x)$, i.e. the fixed point of

$$x = R \circ x \cup Q.$$

Fixed Point of $x = R \circ x \cup Q$

Theorem

If $R, Q \subseteq X \times X$ then

1. $R^* \circ Q$ *is the least solution of* $x = R \circ x \cup Q$
2. *if* $id_x \not\subseteq R$, *i.e. the identity on* X *is not included in* R, *then* $R^* \circ Q$ *is the only solution of* $x = R \circ x \cup Q$.

Proof.

First note that $R^0 = id_X$.
(1) $R \circ (R^* \circ Q) \cup Q = (R \circ R^* \cup id_X) \circ Q = R^* \circ Q$, so $R^* Q$ is a solution. Let S be another solution, i.e. $S = R \circ S \cup Q$.
Hence $Q \subseteq S$ and $R \circ S \subseteq S$.
Note that $R \circ S \subseteq S \implies R \circ R \circ S \subseteq R \circ S \subseteq S \implies R^k \circ S \subseteq S$ for any $k \geq 1$, i.e. $R^* \circ S = (\bigcup_{i=0}^{\infty} R^k) \circ S = \bigcup_{i=0}^{\infty}(R^k \circ S) \subseteq S$.
But $Q \subseteq S$ and $R^* \circ S \subseteq S$ imply $R^* \circ Q \subseteq S$, so $R^* \circ Q$ is the least solution.
(2) Opposite assumption leads to a contradiction. ∎

The case of $id_X \subseteq R$

Consider the case $id_X \subseteq R$. Then
$x = R \circ x \cup Q = (R \setminus id_X) \circ x \cup id_X \circ x \cup Q = (R \setminus id_X) \circ x \cup x \cup Q$.
Let $R' = A \setminus id_X$. Hence we have:

$$\spadesuit \qquad x = x \cup R' \circ x \cup Q$$

Recall that $X \subseteq Y \iff Y = X \cup Y$, so if $R' \circ x \cup Q \subseteq x$ then the equation $\spadesuit$ holds!
In particular, $x = X \times X$ works, since we always have:

$$X \times X \cup R' \circ (X \times X) \cup Q = X \times X$$

no matter what R' and Q are.

Thus $X \times X$ is the **greatest fixed point** of the equation

$$x = R \circ x \cup Q,$$

if $id_X \subseteq R$.

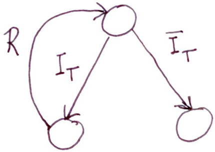
while T **do** S **od** - again

Consider the loop **while** T **do** S **od**

- Let D be the domain of all variables involved in this loop, $R \subseteq D \times D$ be a relation that model S, and $I_T, \bar{I}_T$ be tests corresponding to T.
- Recall that $I_T, \bar{I}_T$ are partial functions from D to D, and for each $x \in D$, we have

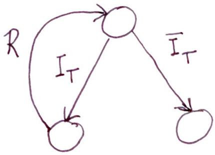
$$I_T(x) = \begin{cases} x & T \text{ is } \textit{true} \\ \perp & \text{otherwise} \end{cases} \qquad \bar{I}_T(x) = \begin{cases} (x & T \text{ is } \textit{false} \\ \perp & \text{otherwise} \end{cases}$$

- The meaning of the loop **while** T **do** S **od** can be expressed by the following *state machine diagram* :



Fixed Point model of **while** T **do** S **od**

- The meaning of the loop **while** T **do** S **od** can be expressed by the following *state machine diagram*(or *finite automaton*):



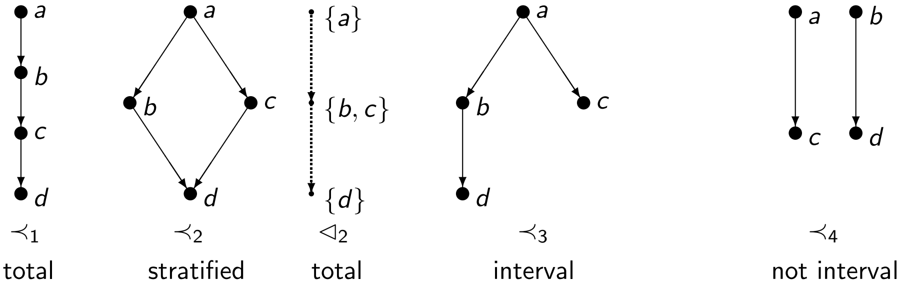
- This state machine can be modelled by the following *equation*:

$$x = (I_T \circ R) \circ x \cup \bar{I}_T.$$

- The above equation actually says that $I_T \circ R$ is a *loop invariant*.
- If $T \not\equiv true$, then $id_X \not\subseteq I_T$, so by Theorem from page 28, $(I_T \circ R)^* \circ \bar{I}_T$ is the only solution!
- Hence the loop **while** T **do** S **od** is represented by the relation $(I_T \circ R)^* \circ \bar{I}_T$.

Definition

A partial order $\preceq$ on B is *interval* iff $a \prec c$ and $b \prec d$ implies $a \prec d$ or $b \prec c$, for all $a, b, c, d \in B$.



- In other words, a partial order $\preceq$ is interval iff it does not contain $\prec_4$ as a suborder.
- Note that every stratified order is interval.

Theorem

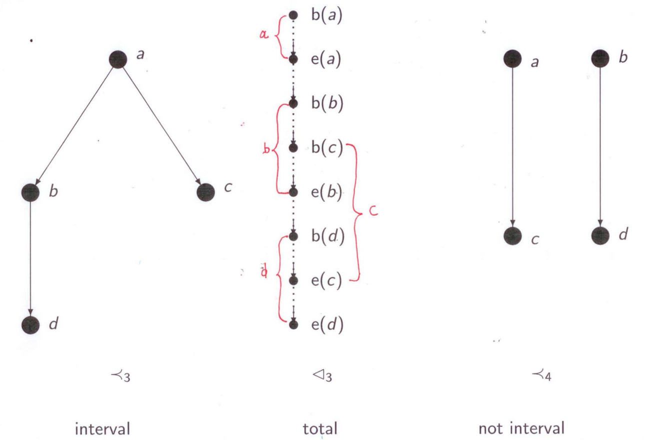
A partial order $(X, \prec)$ *is interval if and only if there is a total order* $(Y, \lhd)$ *and two mappings,* $\mathrm{b}, \mathrm{e} : X \to Y$*, such that, for all* $a, b \in X$*:*

1. $\mathrm{b}(a) \lhd \mathrm{e}(a)$,
2. $a \prec b \iff \mathrm{e}(a) \lhd \mathrm{b}(b)$.

- Usually, $\mathrm{b}(a)$ is interpreted as the 'beginning', and $\mathrm{e}(a)$ as the 'end', of *interval* a.
- If X has at least two elements and $\prec$ is not total, the mappings $\mathrm{b}, \mathrm{e} : X \to Y$ are not unique.

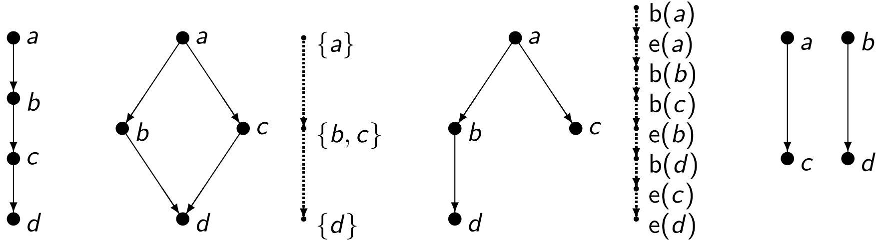
Theorem

A partial order $(X, \prec)$ is interval if and only if there is a total order $(Y, \lhd)$ and two mappings, $\mathrm{b}, \mathrm{e} : X \to Y$, such that, for all $a, b \in X$:
(a) $\mathrm{b}(a) \lhd \mathrm{e}(a)$, *(b)* $a \prec b \iff \mathrm{e}(a) \lhd \mathrm{b}(b)$



- For example $\lhd_3'$ that is derived from $\lhd_3$ by switching $\mathrm{b}(b)$ and $\mathrm{b}(c)$ also represents $\prec_3$.

An Example of All Kinds of Partial Orders



$\prec_1$	$\prec_2$	$\lhd_2$	$\prec_3$	$\lhd_3$	$\prec_4$
total	stratified	total	interval	total	not interval

- Interval orders play important role in concurrency theory and management science.
- It has been known for long time that any execution that can be observed by a single observer must be an interval order.