## Problem 4:

(a)

$$\begin{aligned}
\frac{1}{Z}\frac{\partial Z}{\partial \beta} &= \frac{1}{Z}\frac{\partial \sum_i e^{-\beta E_i}}{\partial \beta} \\
&= \frac{1}{Z}\sum_i \frac{\partial e^{-\beta E_i}}{\partial \beta} \\
&= \frac{1}{Z}\sum_i -E_i e^{-\beta E_i} \\
&= -\left(E_i \sum_i \frac{1}{Z} e^{-\beta E_i}\right) \\
&= -\langle E \rangle
\end{aligned}$$

(b)

$$\begin{aligned}
S &= -k \sum_i p_i \log(p_i) \\
&= -k \sum_i p_i \log\left(\frac{1}{Z} e^{-\beta E_i}\right) \\
&= -k \sum_i p_i \log\left(e^{-\beta E_i}\right) - k \sum_i p_i \log(Z) \\
&= -k \sum_i -\beta p_i E_i - k \left(\sum_i p_i\right) \log(Z) \\
&= k\beta \langle E \rangle - k \log(Z)
\end{aligned}$$

## problem 3 b:

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E^2(X) \\
&= \sum_{n=0}^{\infty} n^2 p(n) - M^2 \\
&= \sum_{n=0}^{\infty} n^2 (1 - e^{-\alpha}) e^{-\alpha n} - M^2 \\
&= \frac{d^2}{d\alpha^2} t(\alpha) - M^2 \\
&= \frac{e^{\alpha}(e^{\alpha}+1)(1-e^{-\alpha})}{(e^{\alpha}-1)^3} - M^2 \\
&= \frac{e^{\alpha}(e^{\alpha}+1)(1-e^{-\alpha})}{(e^{\alpha}-1)^3} - \frac{e^{2\alpha}(1-e^{-\alpha})^2}{(e^{\alpha}-1)^4} \\
&= \frac{e^{\alpha}}{(e^{\alpha}-1)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sigma = \sqrt{\mathrm{Var}(X)} &= \frac{e^{\alpha/2}}{e^{\alpha} - 1} \\
&= \frac{1}{2}\frac{1}{\frac{e^{\alpha}-1}{2e^{\alpha/2}}} \\
&= \frac{1}{2}\frac{1}{e^{\alpha/2}/2 - e^{-\alpha/2}/2} \\
&= \frac{1}{2}\frac{1}{\sinh\frac{\alpha}{2}}
\end{aligned}$$