King's College London

University Of London

PLACE this paper and any answer booklets **in the EXAM ENVELOPE provided**

Candidate No: .... **Desk No:** ....

BSc and MSci Examination

6CCM223B Geometry of surfaces

Summer 2019

Time Allowed: Two Hours

This paper consists of two sections, Section A and Section B.
Section A contributes 45 percent of the total marks for the paper.
Answer all questions.

You are permitted to use a Calculator.
Only calculators from the Casio FX83 and FX85 range are allowed.

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TURN OVER WHEN INSTRUCTED

Part A

For each question in Part A there is exactly one correct answer.
All ten questions in Part A carry equal marks.
Write your answers to Part A in the answer grid provided.

A 1. Consider the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\cosh(t), \sinh(t)).$$

Which of the following statements is true?

(A) The curvature of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(0)$ is -1.

(B) The curvature of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(0)$ is 0.

(C) The curvature of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(0)$ is 1.

(D) The curvature of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(0)$ is not defined.

(E) None of the above.

A 2. Consider the curve

$$\gamma : (-1,1) \to \mathbb{R}^3,\ t \mapsto \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}}\right).$$

Which of the following statements is true?

(A) The torsion of γ at the point $\gamma(0)$ is $2\sqrt{2}$.

(B) The torsion of γ at the point $\gamma(0)$ is $-2\sqrt{2}$.

(C) The torsion of γ at the point $\gamma(0)$ is $-\frac{1}{2\sqrt{2}}$.

(D) The torsion of γ at the point $\gamma(0)$ is $\frac{1}{2\sqrt{2}}$.

(E) None of the above.

A 3. Let γ be a simple closed curve in $\mathbb{R}^2$ of length 1 and $\mathcal{A}(\mathrm{int}(\gamma))$ be the area of the interior $\mathrm{int}(\gamma)$ of γ. Which of the following statements is true for any such curve γ?

(A) $4\pi\mathcal{A}(\mathrm{int}(\gamma)) \leq 1$.

(B) $4\pi\mathcal{A}(\mathrm{int}(\gamma)) < 1$.

(C) $4\pi\mathcal{A}(\mathrm{int}(\gamma)) \geq 1$.

(D) $4\pi\mathcal{A}(\mathrm{int}(\gamma)) > 1$.

(E) None of the above.

A 4. Which of the following statements is true?

(A) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ is a hyperbolic paraboloid.

(B) $\{(x, y, z) \in \mathbb{R}^3 : x^2 - y^2 - z^2 = 1\}$ is a hyperbolic paraboloid.

(C) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z\}$ is a hyperbolic paraboloid.

(D) $\{(x, y, z) \in \mathbb{R}^3 : x^2 - y^2 = z\}$ is a hyperbolic paraboloid.

(E) None of the above.

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A 5. Consider the surface $\mathcal{S}$ given by the surface patch

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \left(u-3, 1-v, 6-\frac{1}{3}u^2-\frac{2}{3}v^2\right).$$

Which of the following statements is true?

(A) The tangent plane of $\mathcal{S}$ at $(0,0)$ is the xy-plane in $\mathbb{R}^3$.

(B) The tangent plane of $\mathcal{S}$ at $(0,0)$ is the xz-plane in $\mathbb{R}^3$.

(C) The tangent plane of $\mathcal{S}$ at $(0,0)$ is the yz-plane in $\mathbb{R}^3$.

(D) The tangent plane of $\mathcal{S}$ at $(0,0)$ is the plane in $\mathbb{R}^3$ given by the equation $x+y+z=0$.

(E) None of the above.

A 6. Consider the surface $\mathcal{S}$ given by the surface patch

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \left(u, v, u^2+v^2\right).$$

Let $\mathcal{G}(\mathbb{R}^2)$ be the image of the Gauss map $\mathcal{G}$ from $\mathbb{R}^2$ into the unit sphere S^2. Which of the following statements is true?

(A) $\mathcal{G}(\mathbb{R}^2) = \{(x,y,z) \in S^2 : z = 0\}$.

(B) $\mathcal{G}(\mathbb{R}^2) = \{(x,y,z) \in S^2 : z < 0\}$.

(C) $\mathcal{G}(\mathbb{R}^2) = \{(x,y,z) \in S^2 : z > 0\}$.

(D) $\mathcal{G}(\mathbb{R}^2) = S^2$.

(E) None of the above.

A 7. Let $\gamma : (-1,1) \to \mathbb{R}^3$ be a unit speed curve that is contained in a surface $\mathcal{S}$. Assume that $\gamma(0) = O = (0,0,0)$, $\dot{\gamma}(0) = \frac{1}{\sqrt{2}}(1,1,0)$ and $\ddot{\gamma}(0) = (1,-1,2)$, and that the unit normal $\mathbf{N}$ to $\mathcal{S}$ at O is $(0,0,1)$. Let κ_g be the geodesic curvature of γ at O. Which of the following statements is true?

(A) $\kappa_g = -\sqrt{2}$.

(B) $\kappa_g = \sqrt{2}$.

(C) $\kappa_g = 0$.

(D) $\kappa_g = \frac{1}{\sqrt{2}}$.

(E) None of the above.

A 8. Consider the surface $\mathcal{S}$ given by the surface patch

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \left(u - v, u + v, u^2 + v^2\right).$$

and put $O = \sigma(0,0) = (0,0,0)$. Which of the following statements is true?

(A) The principal curvatures of $\mathcal{S}$ at O are -1 and 1.

(B) The principal curvatures of $\mathcal{S}$ at O are 0 and 1.

(C) Both principal curvatures of $\mathcal{S}$ at O are equal to 0.

(D) Both principal curvatures of $\mathcal{S}$ at O are equal to 1.

(E) None of the above.

A 9. Which of the following statements is true?

(A) There exists no surface with mean curvature $H = 0$ and Gaussian curvature $K = 0$.

(B) There exists no surface with mean curvature $H = 0$ and Gaussian curvature $K > 0$.

(C) There exists no surface with mean curvature $H < 0$ and Gaussian curvature $K = 0$.

(D) There exists no surface with mean curvature $H > 0$ and Gaussian curvature $K = 0$.

(E) None of the above.

A 10. Let γ be a unit speed simple closed curve of length π on a surface σ and assume that γ is positively oriented and has constant geodesic curvature 1. Let K be the Gaussian curvature of σ and $d\mathcal{A}_\sigma$ be the area element on σ. Which of the following statements is true?

(A) $\iint_{\text{int}(\gamma)} K d\mathcal{A}_\sigma = -\pi$.

(B) $\iint_{\text{int}(\gamma)} K d\mathcal{A}_\sigma = 0$.

(C) $\iint_{\text{int}(\gamma)} K d\mathcal{A}_\sigma = \pi$.

(D) $\iint_{\text{int}(\gamma)} K d\mathcal{A}_\sigma = 2\pi$.

(E) None of the above.

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Part B

All four questions in Part B carry equal marks.

B 11. (i) Let γ be a unit speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. State the Frenet-Serret equations for γ.

(ii) Let γ be a unit speed curve in $\mathbb{R}^3$ with constant positive curvature and zero torsion. Prove that γ is a circle, or part of a circle.

B 12. Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve with nowhere vanishing curvature, $I \subseteq \mathbb{R}$ open interval. Put $U = I \times \mathbb{R}_+ \subseteq \mathbb{R}^2$ with $\mathbb{R}_+ = \{v \in \mathbb{R} : v > 0\}$ and define

$$\sigma : U \to \mathbb{R}^3 \ , \ (u, v) \mapsto \sigma(u, v) = \gamma(u) + v\dot{\gamma}(u).$$

Assume that σ is an injective map.

(i) Show that σ is a regular surface patch.

(ii) Calculate the first fundamental form of σ.

(iii) Calculate the second fundamental form of σ.

(iv) Prove that the Gaussian curvature K of σ satisfies $K = 0$.

B 13. Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$ with regular surface patch $\sigma : U \to \mathbb{R}^3$.

(i) State the definition of a geodesic on the surface.

(ii) Prove that a unit speed curve γ on σ is a geodesic if and only if its geodesic curvature κ_g is zero everywhere.

(iii) Let p and q be any points in $\mathcal{S}$. Does there always exist a geodesic in $\mathcal{S}$ passing through p and q? Justify your answer!

(iv) Every great circle in the unit sphere S^2 is a geodesic. Give a justification why this is true.

B 14. Consider the paraboloid with surface patch

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u, v) \mapsto (u, v, u^2 + v^2)$$

and the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (\cos(t), \sin(t), 1)$$

in the paraboloid. Compute the geodesic curvature of γ and use the local version of the Gauss-Bonnet Theorem to compute the value of

$$\iint_{\mathrm{int}(\gamma)} K d\mathcal{A}_\sigma,$$

where K is the Gaussian curvature of σ and $d\mathcal{A}_\sigma$ is the area element on σ.

Solutions

For each question I state one possible solution that is based on the material taught in the course. For some questions, in particular proofs, there are of course other solutions for which a student can get full marks.

A 1. We have $\gamma'(t) = (\sinh(t), \cosh(t)) \neq 0$ for all $t \in \mathbb{R}$. Thus γ is a regular curve and its curvature is well-defined everywhere. Furthermore, $\gamma''(t) = (\cosh(t), \sinh(t)) = \gamma(t)$. Thus $\gamma'(0) = (0,1)$ and $\gamma''(0) = (1,0)$. The curvature $\kappa(0)$ of γ at $\gamma(0)$ is

$$\kappa(0) = \frac{\|\gamma''(\gamma'\cdot\gamma') - \gamma'(\gamma'\cdot\gamma'')\|}{\|\gamma'\|^4}(0) = 1.$$

A 2. We have

$$\gamma'(t) = \left(\frac{1}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right), \text{ (implies } \gamma \text{ unit speed and regular)},$$
$$\gamma''(t) = \left(\frac{1}{4}(1+t)^{-\frac{1}{2}}, \frac{1}{4}(1-t)^{-\frac{1}{2}}, 0\right) \text{ (implies } \kappa \neq 0 \text{ and torsion well-defined)},$$
$$\gamma'''(t) = \left(-\frac{1}{8}(1+t)^{-\frac{3}{2}}, \frac{1}{8}(1-t)^{-\frac{3}{2}}, 0\right).$$

Thus $\gamma'(0) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, $\gamma''(0) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$, $\gamma'''(0) = \left(-\frac{1}{8}, \frac{1}{8}, 0\right)$ and hence $(\gamma' \times \gamma'')(0) = \left(-\frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4}\right)$. For the torsion $\tau(0)$ of γ at $\gamma(0)$ we then get

$$\tau(0) = \frac{(\gamma' \times \gamma'')\cdot\gamma'''}{\|\gamma' \times \gamma''\|^2}(0) = \frac{1}{2\sqrt{2}}.$$

A 3. The isoperimetric inequality states that $\mathcal{A}(\mathrm{int}(\gamma)) \leq \frac{1}{4\pi}\ell(\gamma)^2$ for any simple closed curve γ in $\mathbb{R}^2$ of length $\ell(\gamma)$, with equality holding if and only if γ is a circle. Since $\ell(\gamma) = 1$ by assumption, it follows that $4\pi\mathcal{A}(\mathrm{int}(\gamma)) \leq 1$ is the correct inequality.

A 4. Equation (D) describes a hyperbolic paraboloid. [(A) gives a hyperboloid of one sheet, (B) a hyperboloid of two sheets and (C) an elliptic paraboloid.]

A 5. We have $\sigma_u(u,v) = \left(1, 0, -\frac{2}{3}u\right)$ and $\sigma_v(u,v) = \left(0, -1, -\frac{4}{3}v\right)$. The tangent plane at $(0,0)$ is spanned by $\sigma_u(0,0) = (1,0,0)$ and $\sigma_v(0,0) = (0,-1,0)$ and therefore coincides with xy-plane in $\mathbb{R}^3$.

A 6. We have $\sigma_u(u,v) = (1,0,2u)$ and $\sigma_v(u,v) = (0,1,2v)$. Thus $(\sigma_u \times \sigma_v)(u,v) = (-2u,-2v,1)$ and hence the Gauss map is

$$\mathcal{G} : \mathbb{R}^2 \to S^2,\ (u,v) \mapsto \frac{1}{\sqrt{1+4(u^2+v^2)}}(-2u,-2v,1).$$

From this we easily see that $\mathcal{G}(\mathbb{R}^2) = \{(x,y,z) \in S^2 : z > 0\}$.

A 7. At O we have $\kappa_g = \ddot{\gamma}(0) \cdot (\mathbf{N} \times \dot{\gamma}(0)) = (1,-1,2) \cdot \frac{1}{\sqrt{2}}(-1,1,0) = -\sqrt{2}$.

A 8. We have $\sigma_u(u,v) = (1,1,2u)$, $\sigma_v(u,v) = (-1,1,2v)$, $\sigma_{uu}(u,v) = (0,0,2)$, $\sigma_{uv}(u,v) = (0,0,0)$ and $\sigma_{vv}(u,v) = (0,0,2)$. This gives $\mathbf{N}(0,0) = \frac{(\sigma_u \times \sigma_v)(0,0)}{\|(\sigma_u \times \sigma_v)(0,0)\|} = (0,0,1)$ for the normal vector at $(0,0)$. Thus, at $(0,0)$, we get $E = \|\sigma_u(0,0)\|^2 = 2$, $F = (\sigma_u \cdot \sigma_v)(0,0) = 0$, $G = \|\sigma_v(0,0)\|^2 = 2$, $L = (\sigma_{uu} \cdot \mathbf{N})(0,0) = 2$, $M = (\sigma_{uv} \cdot \mathbf{N})(0,0) = 0$ and $N = (\sigma_{vv} \cdot \mathbf{N})(0,0) = 2$. It follows that both the first and the second fundamental form have the matrix representation $\mathcal{F}_I = \mathcal{F}_{II} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $\mathcal{F}_I^{-1}\mathcal{F}_{II} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and thus both principal curvatures are equal to 1.

A 9. Let κ_1, κ_2 be the principal curvatures. Then, by definition, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K = \kappa_1\kappa_2$. If $H = 0$, then $0 = 4H^2 = (\kappa_1 + \kappa_2)^2 = 2K + \kappa_1^2 + \kappa_2^2$, which implies $K \leq 0$. Thus there exists no surface with $H = 0$ and $K > 0$. The other cases are possible. For example, the round cylinder has $K = 0$ and $H > 0$ (with inward unit normal) or $H < 0$ (with outward unit normal). The plane has $H = 0$ and $K = 0$.

A 10. The local version of the Gauss-Bonnet Theorem states

$$\int_\gamma \kappa_g ds = 2\pi - \iint_{\mathrm{int}(\gamma)} K d\mathcal{A}_\sigma.$$

By assumption, $\kappa_g = 1$ and $\ell(\gamma) = \pi$, thus $\int_\gamma \kappa_g ds = \int_0^\pi ds = \pi$. Inserting this into the above equation implies $\iint_{\mathrm{int}(\gamma)} K d\mathcal{A}_\sigma = \pi$.

B 11. (i) Let κ be the curvature and τ be the torsion of γ. Furthermore, define $\mathbf{t} = \dot{\gamma}$, $\mathbf{n} = \frac{1}{\kappa}\dot{\mathbf{t}}$ and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. The Frenet-Serret equations are

$$\dot{\mathbf{t}} = \kappa\mathbf{n}\ ,\ \dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b}\ ,\ \dot{\mathbf{b}} = -\tau\mathbf{n}.$$

(ii) Assume that $\kappa(s) = \kappa > 0$ is constant and $\tau(s) = 0$. The third Frenet-Serret equation gives $\dot{\mathbf{b}} = 0$ and thus $\mathbf{b}(s) = \mathbf{b}$ is a constant vector in $\mathbb{R}^3$. Thus $\frac{d}{ds}(\gamma \cdot \mathbf{b}) = \dot{\gamma} \cdot \mathbf{b} + \gamma \cdot \dot{\mathbf{b}} = \mathbf{t} \cdot \mathbf{b} = 0$ since $\mathbf{t}$ and $\mathbf{b}$ are perpendicular. Thus $\gamma \cdot \mathbf{b}$ is constant, say equal to $d \in \mathbb{R}$. This shows that γ lies in the plane in $\mathbb{R}^3$ given by the equation $(x, y, z) \cdot \mathbf{b} = d$.

Using the second Frenet-Serret equation and the assumptions that $\kappa(s) = \kappa > 0$ is constant and $\tau = 0$, we obtain $\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}\mathbf{n}\right) = \mathbf{t} + \frac{1}{\kappa}\dot{\mathbf{n}} = 0$. Thus there exists $\mathbf{a} \in \mathbb{R}^3$ so that $\gamma(s) + \frac{1}{\kappa}\mathbf{n}(s) = \mathbf{a}$ for all s. This implies $\|\gamma(s) - \mathbf{a}\| = \|-\frac{1}{\kappa}\mathbf{n}(s)\| = \frac{1}{\kappa}\|\mathbf{n}(s)\| = \frac{1}{\kappa}$ for all s, which shows that $\gamma(s)$ lies on the sphere with radius $\frac{1}{\kappa}$ and centre $\mathbf{a}$.

Since the intersection of a sphere with a plane is circle, the result follows.

B 12. (i) We have $\sigma_u = \dot\gamma + v\ddot\gamma$ and $\sigma_v = \dot\gamma$, and therefore $\sigma_u \times \sigma_v = v\ddot\gamma \times \dot\gamma$. Using the first Frenet-Serret equation, we obtain $\sigma_u \times \sigma_v = v(\dot{\mathbf{t}} \times \mathbf{t}) = v\kappa(\mathbf{n} \times \mathbf{t}) = -v\kappa\mathbf{b}$. Since both v and κ are non-zero by assumption, it follows that σ is a regular surface patch.

(ii) Since $\|\dot\gamma\| = 1$, we have $\dot\gamma \cdot \ddot\gamma = 0$. Using this we get

$$\begin{aligned} E &= \|\sigma_u\|^2 = (\dot\gamma + v\ddot\gamma)\cdot(\dot\gamma + v\ddot\gamma) = \|\dot\gamma\|^2 + v^2\|\ddot\gamma\|^2 = 1 + v^2\kappa^2, \\ F &= \sigma_u \cdot \sigma_v = (\dot\gamma + v\ddot\gamma)\cdot\dot\gamma = \|\dot\gamma\|^2 = 1, \\ G &= \|\sigma_v\|^2 = \|\dot\gamma\|^2 = 1. \end{aligned}$$

Thus the first fundamental form is $(1 + v^2\kappa(u)^2)du^2 + 2dudv + dv^2$.

(iii) Using the Frenet-Serret equations we calculate $\sigma_{uu} = \dot{\mathbf{t}} + v\dot\kappa\mathbf{n} + v\kappa\dot{\mathbf{n}} = (\kappa + v\dot\kappa)\mathbf{n} - v\kappa^2\mathbf{t} + v\kappa\tau\mathbf{b}$, $\sigma_{uv} = \kappa\mathbf{n}$ and $\sigma_{vv} = 0$. Moreover, for the unit normal we get $\mathbf{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = -\mathbf{b}$. This implies $L = \sigma_{uu} \cdot \mathbf{N} = -v\kappa\tau$, $M = \sigma_{uv} \cdot \mathbf{N} = 0$ and $N = \sigma_{vv} \cdot \mathbf{N} = 0$. Thus the second fundamental form is $-v\kappa\tau(u)du^2$.

(iv) $K = \frac{LN - M^2}{EG - F^2} = 0$.

B 13. (i) A unit speed curve γ on σ is a geodesic if $\ddot{\gamma}(s)$ is perpendicular to the surface at $\gamma(s)$ for all s, or equivalently, if $\ddot{\gamma}(s)$ is parallel to the unit normal $\mathbf{N}(s)$ for all s.

(ii) Let γ be a unit speed geodesic on σ. Then $\ddot{\gamma}(s)$ is parallel to $\mathbf{N}(s)$ and therefore perpendicular to $(\mathbf{N} \times \dot{\gamma})(s)$. So $\kappa_g(s) = \ddot{\gamma}(s) \cdot (\mathbf{N} \times \dot{\gamma})(s) = 0$.

Conversely, assume $\kappa_g = 0$. Then $0 = \kappa_g(s) = \ddot{\gamma}(s) \cdot (\mathbf{N} \times \dot{\gamma})(s)$ for all s. Since γ is a unit speed curve, we have $\dot{\gamma} \cdot \dot{\gamma} = 1$ and hence $\dot{\gamma} \cdot \ddot{\gamma} = 0$. Thus $\ddot{\gamma}(s)$ is perpendicular to $(\mathbf{N} \times \dot{\gamma})(s)$ and $\dot{\gamma}(s)$ for all s. Since $\dot{\gamma}(s)$, $\mathbf{N}(s)$ and $(\mathbf{N} \times \dot{\gamma})(s)$ are perpendicular unit vectors in $\mathbb{R}^3$, it follows that $\ddot{\gamma}(s)$ is parallel to $\mathbf{N}(s)$ for all s, which shows that γ is a geodesic on σ.

(iii) No! Consider for example the plane $\mathbb{R}^2$ with the origin $O = (0,0)$ removed, denoted by $\mathcal{S}$. The geodesics in the plane are the straight lines. The line passing through the points $p = (-1,0)$ and $q = (1,0)$ also passes through O, but since this point has been removed from $\mathbb{R}^2$ we cannot find a line in $\mathcal{S}$ passing through p and q.

(iv) Every great circle on S^2 is a normal section of S^2, that is, the intersection of S^2 with a plane perpendicular to the tangent plane at some point of S^2. Every normal section of a surface has vanishing geodesic curvature and hence is a geodesic.

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B 14. We have $\gamma(t) = \sigma(\rho(t))$ with $\rho : \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t))$. Since ρ is positively oriented, also γ is positively oriented. Clearly, γ is a unit speed, simple closed curve.

We have $\sigma_u(u,v) = (1,0,2u)$ and $\sigma_v(u,v) = (0,1,2v)$. Thus the unit normal $\mathbf{N}$ of σ is

$$\mathbf{N}(u,v) = \frac{1}{\sqrt{4u^2+4v^2+1}}(-2u, -2v, 1).$$

Moreover, $\dot\gamma(t) = (-\sin(t), \cos(t), 0)$ and $\ddot\gamma(t) = -(\cos(t), \sin(t), 0)$. Therefore,

$$\begin{aligned}\mathbf{N}(\rho(t)) \times \dot\gamma(t) &= \frac{1}{\sqrt{5}}(-2\cos(t), -2\sin(t), 1) \times (-\sin(t), \cos(t), 0)\\ &= -\frac{1}{\sqrt{5}}(\cos(t), \sin(t), 2).\end{aligned}$$

Altogether this implies

$$\begin{aligned}\kappa_g(t) &= \ddot\gamma(t) \cdot (\mathbf{N}(\rho(t)) \times \dot\gamma(t))\\ &= (\cos(t), \sin(t), 0) \cdot \frac{1}{\sqrt{5}}(\cos(t), \sin(t), 2) = \frac{1}{\sqrt{5}}.\end{aligned}$$

The local version of the Gauss-Bonnet Theorem then implies

$$\begin{aligned}\iint_{\mathrm{int}(\gamma)} K d\mathcal{A}_\sigma &= 2\pi - \int_\gamma \kappa_g ds = 2\pi - \frac{1}{\sqrt{5}}\int_\gamma ds\\ &= 2\pi - \frac{1}{\sqrt{5}}\ell(\gamma) = 2\pi - \frac{1}{\sqrt{5}}2\pi = 2\pi\left(1 - \frac{1}{\sqrt{5}}\right).\end{aligned}$$