Geometry of Surfaces - Exercises

Exercises marked with $*$ are to be answered (partially) in the online quiz for this week on the Keats page for this module.

64. Explain why the plane, the sphere and the hyperbolic paraboloid cannot be isometric to each other.

65.* Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two surfaces with surface patches $\sigma_i : (-1,1) \times (-1,1) \to \mathbb{R}^3$. Assume that the first fundamental forms with respect to these surface patches are identical. Suppose that the second fundamental form of $\mathcal{S}_1$ at $\sigma_1(0,0)$ is $L_1 = 1$, $M_1 = 0$ and $N_1 = 0$. Can the second fundamental form of $\mathcal{S}_2$ at $\sigma_2(0,0)$ be $L_2 = 2$, $M_2 = 1$ and $N_2 = 2$? Justify your answer!

66.* Consider the cone with surface patch $\sigma : U \to \mathbb{R}^3$, $(u,v) \mapsto v(\cos(u), \sin(u), 1)$, $U = (0, 2\pi) \times (0, \infty)$. Let γ be the positively oriented unit speed curve parametrizing the image under σ of the circle $\{(u,v) \in U : (u-\pi)^2 + (v-2)^2 = 1\}$. Compute $\int_\gamma \kappa_g ds$.

67.* Consider the cylinder with surface patch $\sigma : U \to \mathbb{R}^3$, $(u,v) \mapsto (\cos(u), \sin(u), v)$, $U = (0, 2\pi) \times \mathbb{R}$. Let γ be the positively oriented unit speed curve parametrizing the image under σ of the circle $\{(u,v) \in U : (u-\pi)^2 + (v-2)^2 = 1\}$. Compute $\int_\gamma \kappa_g ds$.

68. Show that a simple closed curve γ on the unit sphere S^2 with $\int_\gamma \kappa_g ds = 0$ bounds two regions of equal area.

69.* Consider the paraboloid with surface patch $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $(u,v) \mapsto (u, v, u^2 + v^2)$ and the curve $\gamma : [0, 2\pi] \to \mathbb{R}^3$, $t \mapsto (\cos(t), \sin(t), 1)$ in the paraboloid. Compute the geodesic curvature of γ and use the local version of the Gauss-Bonnet Theorem to compute the value of

$$\iint_{\mathrm{int}(\gamma)} K d\mathcal{A}_\sigma$$

70.* Let $\gamma(s)$ be a unit speed simple closed curve on a surface σ with Gaussian curvature $K \leq 0$ and assume that γ is positively oriented. Can γ be a geodesic?

71.* Consider the curvilinear polygon

$$\Gamma : \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to S^2,\ t \mapsto \begin{cases} \gamma_1(t) & \text{if } t \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \gamma_2(t) & \text{if } t \in [\frac{\pi}{2}, \frac{3\pi}{2}] \end{cases}$$

on the unit sphere S^2, where

$$\gamma_1(t) = (\cos(t), 0, -\sin(t)),$$
$$\gamma_2(t) = (\cos(t-\pi)\cos(\phi), \cos(t-\pi)\sin(\phi), \sin(t-\pi)),$$

and $\phi \in (0, 2\pi)$ is a constant. Suppose Γ with the orientation given is positively oriented. Calculate the area of $\mathrm{int}(\Gamma)$.

72. Let γ_1 and γ_2 be two geodesics on a surface σ with negative Gaussian curvature emanating from the same point. Show that γ_1 and γ_2 cannot meet again on σ.