TUTORIAL 10: The two-dimensional divergence theorem and Green's theorem

1. Let D be the disc of radius a centred at $(0,0)$, and let $\mathbf{u}$ be the vector field $(xy^2, 0)$. Let C be the boundary curve of D. Verify the divergence theorem $\iint_D div(\mathbf{u})\, dA = \int_C \mathbf{u} \cdot d\mathbf{n}$ in this case.

2. Consider the region D as shown (see Fig. 1), and the vector field $\mathbf{u} = (0, x^4 + x^2y^2 - x^2)$. Check the divergence theorem in this case.

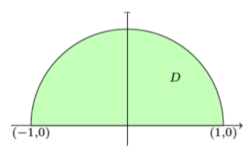

Figure 1: The domain of integration for Question 2

3. The following picture (Fig. 2) shows a hypocycloid curve C, which can be parametrised as

$$\mathbf{r} : (x, y) = (5\cos(t) + \cos(5t),\ 5\sin(t) - \sin(5t)).$$

Use the divergence theorem to find the area of the region D enclosed by C.

4. Consider the vector $\mathbf{u} = (3y^2, 2x^2)$ defined over the disk centered in the origin and radius 1. Verify Green's theorem.

5. Consider the triangular region shown in Fig. 3. Use the Green's theorem to calculate the area of the domain D.

Useful identities to be used in this tutorial

$$\cos^2(\theta)\sin(\theta) = \frac{1}{4}[\sin(3\theta) + \sin(\theta)]$$

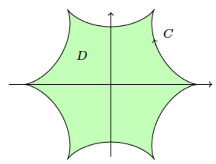

Figure 2: The domain of interest for Question 3

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\beta+\alpha) + \cos(\beta-\alpha)]$$

$$\sin^3\theta = \frac{1}{4}\left(3\sin\theta) - \sin 3\theta\right), \quad \cos^3\theta = \frac{1}{4}\left(3\cos\theta) + \cos 3\theta\right)$$

Unit area element in polar coordinate system

$$dA = rdrd\theta, \quad d\mathbf{A} = rdrd\theta\mathbf{k}$$

Unit area element in Cartesian coordinate system

$$dA = dxdy$$

Answers

1. $\pi a^4/4$
2. $4/15$
3. 20π
4. $2/3$ 5. $1/2$

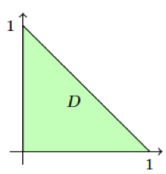

Figure 3: The domain of interest for Question 5. The coordinates of the bottom left corner is (0,0)