VANDERMONDE IDENTITY AND PASCAL'S TRIANGLE

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Vandermonde Identity

Continue our work on combinatorial proofs.

In a combinatorial proof, we can prove things by equating them to a counting problem.

Consider m, n, r where $m \geq 0, n \geq 0, r \geq 0$, and $r \leq m, r \leq n$

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

$$= \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r}\binom{n}{0}$$

Notice that we are always choosing r things (from two different sets)

Vandermonde Identity

Prove by counting

$m \geq 0, n \geq 0, r \geq 0$, and $r \leq m, r \leq n$

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

What does this mean? We know that $\binom{m+n}{r}$ represents the number of subsets of size r in a set of size $m + n$.

Can be two sets, or one set divided into two.

$\binom{m+n}{r}$ represents the number of ways to choose a subset of r animals from among these two groups.

m dogs

n cats

Vandermonde Identity

m dogs	n cats

We want to argue that

$$\binom{m+n}{r}$$

and

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Count the same thing.

If we are taking a subset of r animals, we can split it into cases (which would correspond to our summation). Perhaps based on number of dogs chosen and number of cats.

Product Rule!

Sum Rule!

	dogs chosen	cats chosen
$\binom{m}{0}\binom{n}{r}$	0	r
$\binom{m}{1}\binom{n}{r-1}$	1	r-1
$\binom{m}{2}\binom{n}{r-2}$	2	r-2
	…	…
$\binom{m}{r-1}\binom{n}{1}$	r-1	1
$\binom{m}{r}\binom{n}{0}$	r	0

Vandermonde Identity

We have shown combinatorially that for $m \geq 0, n \geq 0, r \geq 0$, and $r \leq m, r \leq n$

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Special Case: $m = n = r$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \binom{n+n}{n} = \binom{2n}{n}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \binom{2n}{n}$$

Recall:

$$\binom{n}{n-k} = \binom{n}{k}$$

Rewrite:

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

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$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\binom{0}{0}$$ Row 0

$$\binom{1}{0} \quad \binom{1}{1}$$ Row 1

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$ Row 2

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\binom{0}{0} \qquad (x+y)^0$$

$$\binom{1}{0} \quad \binom{1}{1} \qquad (x+y)^1$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \qquad (x+y)^2$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\boxed{\binom{n}{0} = \binom{n}{n} = 1}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\boxed{\binom{0}{0}}$$

$$\boxed{\binom{1}{0}} \quad \boxed{\binom{1}{1}}$$

$$\boxed{\binom{2}{0}} \quad \binom{2}{1} \quad \boxed{\binom{2}{2}}$$

$$\boxed{\binom{3}{0}} \quad \binom{3}{1} \quad \binom{3}{2} \quad \boxed{\binom{3}{3}}$$

$$\boxed{\binom{4}{0}} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \boxed{\binom{4}{4}}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Pascal's Triangle

$$\binom{0}{0} \quad \text{Row 0}$$

$$\binom{1}{0} \quad \binom{1}{1} \quad \text{Row 1}$$

$$\binom{2}{0} \quad \boxed{\binom{2}{1}} \quad \boxed{\binom{2}{2}} \quad \text{Row 2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \boxed{\binom{3}{2}} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

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Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

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$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Pascal's Triangle

$$1$$

$$1 \quad 1$$

$$1 \quad \boxed{2} \quad \boxed{1}$$

$$1 \quad 3 \quad \boxed{3} \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

Pascal's Triangle

Some rules we've learned:

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & \boxed{4} & & 6 & & \boxed{4} & & 1 \\ & & \binom{4}{1} & & & & \binom{4}{3} & & \end{array}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

row 3, $2^3 = 8 = \boxed{1 + 3 + 3 + 1}$

1 4 6 4 1

Sum of any row n is 2^n

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

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$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

row 3 $\boxed{1^2 + 3^2 + 3^2 + 1^2}$

1 4 6 4 1

Sum of the squares of any row n is $\binom{2n}{n}$

You can find as the middle element of row $2n$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}}$$

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$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

row 3 $\boxed{1^2 + 3^2 + 3^2 + 1^2}$

row 4 1 4 6 4 1

1 6 15 $\boxed{20}$ 15 6

row 6

Sum of the squares of any row n is $\binom{2n}{n}$

You can find as the middle element of row $2n$

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Some rules we've learned:

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$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

$$(x+y)^3 = \boxed{1 \cdot x^3 + 3 \cdot x^2 y + 3 \cdot x y^2 + 1 \cdot y^3}$$

1 4 6 4 1

You can find the coefficients of the n^{th} polynomial by looking a row n

Pascal's Triangle

Some rules we've learned:

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$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

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$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \boxed{\binom{3}{1}} & & \boxed{\binom{3}{2}} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \boxed{\binom{4}{2}} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

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$n \geq 2, 1 \leq k \leq n-1$:

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$$\binom{4}{0} \quad \binom{4}{1} \quad \boxed{\binom{4}{2}} \quad \binom{4}{3} \quad \binom{4}{4}$$

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$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

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$n \geq 2, 1 \leq k \leq n-1$:

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$$\binom{3}{0} \quad \boxed{\binom{3}{1}} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \boxed{\binom{4}{2}} \quad \binom{4}{3} \quad \binom{4}{4}$$

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$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

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$$\binom{3}{0} \quad \boxed{\binom{3}{1}} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \boxed{\binom{4}{2}} \quad \binom{4}{3} \quad \binom{4}{4}$$

Binomial Coefficient Example

MISSISSIPPI
→ SSIIMSSIPIP

How many ways can we rearrange the letters of the word MISSISSIPPI?

(We don't need meaningful words, simply arrangements).

Our first idea might be to take all permutations.

11 letters $= 11!$ permutations

One such permutation is to swap 3rd and 4th letters, which gives us:

MISSISSIPPI

Since we want distinct arrangements, this is no good.

Binomial Coefficient Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

M: 1
I: 4
S: 4
P: 2

11

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11
S	P	I	I	M	P	I	S	S	I	S

Task 1: Place 1 M in 11 possible locations. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's in 10 possible locations. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's in 6 possible locations. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 2 P's in 2 possible locations. There are $\binom{2}{2}$ ways to do this.

Binomial Coefficient Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

M: 1
I: 4
S: 4
P: 2
11

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11
S	P	I	I	M	P	I	S	S	I	S

Task 1: Place 1 M. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 2 P's. There are $\binom{2}{2}$ ways to do this.

$$\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2}$$

$$= \frac{11!}{1!10!} \cdot \frac{10!}{4!6!} \cdot \frac{6!}{4!2!} \cdot \frac{2!}{2!0!}$$

$$= \frac{11!}{4!4!2!} = 34650$$

Binomial Coefficient Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11

M: 1
I: 4
S: 4
P: 2
11

Task 1: Place 1 M. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 2 S's. There are $\binom{2}{2}$ ways to do this.

What if we change the order we place the letters?

Binomial Coefficient Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11

Task 1: Place 4 S's. There are $\binom{11}{4}$ ways to do this.

Task 2: Place 2 P's. There are $\binom{7}{2}$ ways to do this.

Task 3: Place 4 I's. There are $\binom{5}{4}$ ways to do this.

Task 3: Place 1 M. There are $\binom{1}{1}$ ways to do this.

M: 1
I: 4
S: 4
P: 2

11

$$\binom{11}{4} \cdot \binom{7}{2} \cdot \binom{5}{4} \cdot \binom{1}{1}$$

$$= \frac{11!}{4!7!} \cdot \frac{7!}{2!5!} \cdot \frac{5!}{4!1!} \cdot \frac{1!}{1!0!}$$

$$= \frac{11!}{4!4!2!} = 34650$$

Binomial Coefficient Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

M: 1
I: 4
S: 4
P: 2
11

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11

Task 1: Place 1 M. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{2}{2}$ ways to do this.

Can you think of another way to compute this?

Binomial Coefficient Example

MISSISSIPPI

There are $11!$ ways to arrange 11 letters.

How many are duplicates?

1	2	3	4	5	6	7	8	9	10	11
M	I	S	S	I	S	S	I	P	P	I

There are $4!$ permutations with the same arrangement of S

There are $4!$ permutations with the same arrangement of I

There are $2!$ permutations with the same arrangement of P

There is $1!$ permutations with the same arrangement of M

M: 1
I: 4
S: 4
P: 2
11

$$\frac{11!}{4!4!2!} = 34650$$

Binomial Coefficient Example

MISSISSIPPI

There are 11! ways to arrange 11 letters.

How many are duplicates?

1	2	3	4	5	6	7	8	9	10	11
M	I	S	S	I	S	S	I	P	P	I

$\frac{11!}{4!4!2!}$ is known as a *multinomial,* and we can write it like

$$\binom{11}{4,4,2,1}$$

M: 1
I: 4
S: 4
P: 2
11

$$\frac{11!}{4!4!2!} = 34650$$

Binomial Coefficient Example

1	2	3	4	5	6	7	8	9	10	11
M	I	S	S	I	S	S	I	P	P	I

For a set of n items, if we choose i, j, k items in turn, where

$$i + j + k = n$$

then the number of ways to do that is

$$\binom{n}{i, j, k} = \frac{n!}{i!\,j!\,k!},$$

and we can generalize to as many terms as we like.

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M: 1
I: 4
S: 4
P: 2
11

$$\frac{11!}{4!4!2!} = 34650$$

Binomial Coefficient Example

How many solutions are there to this problem?

$(2,1,4), (1,1,5), (5,1,1), (0,7,0)$, etc

We can try proving it is equivalent to a problem we know how to solve.

We can map it to bitstrings.

$$(2,1,4) \rightarrow 001010000$$
$$(1,4,2) \rightarrow 010000100$$
$$(0,7,0) \rightarrow 100000001$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 = 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

We have found a 1-to-1 function from this problem to bitstrings.

To prove that counting bitstrings also counts solutions to this problem, we also need a 1-to-1 function from bitstrings to solutions to a linear equation.

Binomial Coefficient Example

To prove a bijection, we must argue that for every bitstring of length 9 with exactly 2 1's there is a corresponding linear equation.

$$010000100 \rightarrow (1,4,2)$$

We count leading 0’s. There is 1 leading 0, so write a 1.

We count one 1, then count 0’s until next 1.

There are 4 0’s so write a 4.

Count one 1, then count the rest of the 0’s. There are 2, so write a 2.

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 = 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

There is a bijection between the two problems. That means they are the same size.

Binomial Coefficient Example

$$010000100 \rightarrow (1,4,2)$$

Instead of figuring out how to count the number of linear equations to

$$x_1 + x_2 + x_3 = 7$$

we can count the number of bitstrings of length 9 with 7 zeros instead.

Procedure: Write down nine 0's.

Choose 2 0's to flip into a 1

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 = 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

There are $\binom{9}{2}$ such solutions.

Binomial Coefficient Example

We've shown that for each linear equation with 3 terms summing to 7, there is a corresponding bitstring of length 9 with exactly 2 1's.

Then we showed that for every bitstring of length 9 with exactly 2 1's there is a corresponding linear equation.

$$010000100 \leftrightarrow (1,4,2)$$

Since there is a bijection between the sets, they are the same size.

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 = 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

Thus we can count the number of bitstrings, and see that there are $\binom{9}{2}$ solutions to

$$x_1 + x_2 + x_3 = 7$$

Bitstrings and Linear Equations

More generally:

$$x_1 + x_2 + \cdots + x_k = n$$

We can map this to a bitstring with $k - 1$ many 1's and n 0's.

Thus it is a bitstring with length $n + k - 1$, where we choose $k - 1$ of the bits to be 1's

There are $\binom{n+k-1}{k-1}$ solutions.

I like to think of it as a bitstring of length n plus the number of plus (+) signs (call this p).

Then you can choose either n 0's or p 1's.

There are $\binom{n+p}{p}$ solutions, where n is the total and p is the number of plus signs.

Bitstrings and Linear Equations

How many solutions are there to this problem?

There are the original solutions

$(2,1,4), (1,1,5), (5,1,1), (0,7,0)$, etc

And also solutions of the type

$(2,1,3), (1,1,1), (0,1,1), (0,0,0)$, etc.

$x_1 + x_2 + x_3 + x_4 = 7$

The claim is these are the same. We need to show a bijection.

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 \leq 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

But now possibly less than 7 0's.

Bitstrings and Linear Equations

Here are some solutions to $x_1 + x_2 + x_3 \leq 7$. We will map these to solutions to $x_1 + x_2 + x_3 + x_4 = 7$

$$\underbrace{(2,1,4)}_{(2,1,4,0)},\quad \underbrace{(2,1,3)}_{(2,1,3,1)},\quad \underbrace{(0,1,1)}_{(0,1,1,5)},\quad \underbrace{(0,7,0)}_{(0,7,0,0)},\quad \underbrace{(0,0,0)}_{(0,0,0,7)}$$

$x_1 + x_2 + x_3 \leq 7$

Let $x_4 = 7 - (x_1 + x_2 + x_3)$. Then

$x_1 + x_2 + x_3 + x_4 = 7$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

$x_1 + x_2 + x_3 \leq 7$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}1\underbrace{0 \ldots 0}_{x_4}$$

We have shown we can map solutions from

$$x_1 + x_2 + x_3 \leq 7$$

to

$$x_1 + x_2 + x_3 + x_4 = 7$$

Bitstrings and Linear Equations

Now we will show a mapping from

$x_1 + x_2 + x_3 + x_4 = 7$ to

$x_1 + x_2 + x_3 \ \leq 7$

$\underbrace{(2,1,4,0)}_{(2,1,4)},\ \underbrace{(2,1,3,1)}_{(2,1,3)},\ \underbrace{(0,1,1,5)}_{(0,1,1)},\ \underbrace{(0,7,0,0)}_{(0,7,0)},\ \underbrace{(0,0,0,7)}_{(0,0,0)}$

$$x_1 + x_2 + x_3 + x_4 = 7$$

$$x_1 + x_2 + x_3 \qquad\quad \leq 7$$

Thus the number of solutions is $\binom{10}{3}$ for both $x_1 + x_2 + x_3 \leq 7$ and $x_1 + x_2 + x_3 + x_4 = 7$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 \leq 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}1\underbrace{0 \ldots 0}_{x_4}$$

Subtract x_4 from both sides.

Bitstrings and Linear Equations

How could we solve this version?

Let $x_1' = x_1 - 1,\ x_2' = x_2 - 1, x_3' = x_3 - 1,$

$(2,1,4),\quad (2,2,3),\quad (1,1,5)$

$(1,0,3),\quad (1,1,2),\quad (0,0,4)$

$x_1 + x_2 + x_3 = 7, \qquad x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

$\leftrightarrow$

$x_1' + x_2' + x_3' = 4, \qquad x_1' \geq 0, x_2' \geq 0, x_3' \geq 0$

Thus the number of solutions is $\binom{6}{2}$

$x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

$x_1 + x_2 + x_3 = 7$

$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$

Can we map this problem to the old version?