I. Areas

The Problem:
Let f and g be continuous on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$.
Find the area A of the region S where $S = \{(x, y)/a \leq x \leq b,\ g(x) \leq y \leq f(x)\}$.

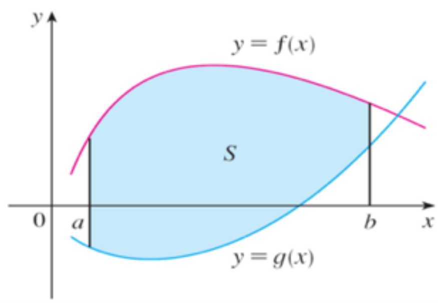

Fig.1

We divide S into n strips of equal width and then we approximate the i th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. ($x_i^* = x_i$ when the sample points are right endpoints.)

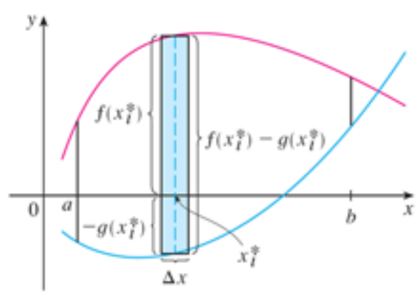

Fig.2

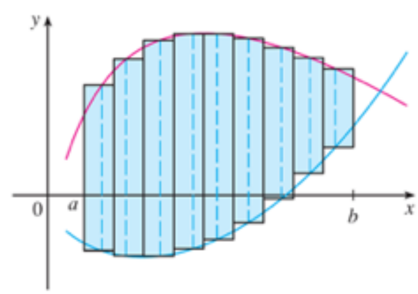

Fig.3

The area A is ***approximated*** by the Riemann sum $\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$.

When $n \to \infty$, we define A as:

D1: $A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$

D2: $A = \int_a^b [\underbrace{f(x)}_{y_{top}} - \underbrace{g(x)}_{y_{bottom}}]dx$

Notes: 1. If $g(x) = 0$, then $A = \int_a^b |f(x)|\,dx$.

2. If $f(x) \geq g(x)$ for some $x \in [a, b]$ and $f(x) \leq g(x)$ for other $x \in [a, b]$,

then $A = \int_a^b |f(x) - g(x)|\,dx$.

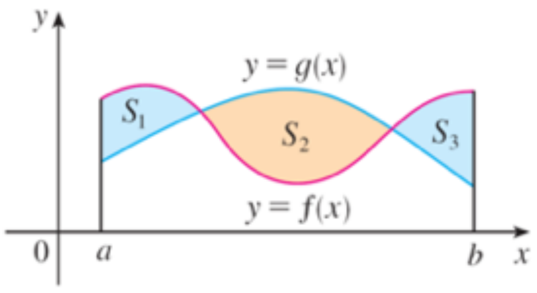

Fig.4

1. a) Find the area A of the region S enclosed by the curves $y = 2x$ and $y = x^2 - 2x$.

b) Find the area A of the region S enclosed by the curves $y = 2x$ and $y = x^2 - 2x$ for all $x \in [-1, 4]$.

2. Find the area A of the region S enclosed by the given curves.

(a) $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$, and the y-axis

(b) $y = \sin x$, $y = \cos x$, and the x-axis (just between 0 and $\frac{\pi}{2}$)

(c) $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{3}$, and the y-axis

3. Find the area A of the region S enclosed by the curves $y = x$ and $y = \sqrt[3]{x}$.

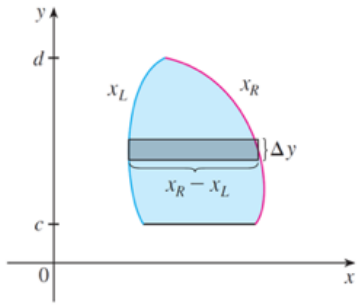

Fig.5

Let f and g be continuous on $[c, d]$ and $f(y) \geq g(y)$ for all $y \in [c, d]$.

If $S = \{(x, y)/g(y) \leq x \leq f(y), c \leq y \leq d\}$, then $A = \int_c^d [\underset{\substack{\downarrow \\ x_{right}}}{f(y)} - \underset{\substack{\downarrow \\ x_{left}}}{g(y)}]\, dy$.

4. Find the area A of the region S enclosed by the curves $x + y = 0$ and $x = y^2 + 3y$.

Area of a region enclosed by three curves

5. Find the area A of the region S enclosed by the curves $y = |x|$ and $y = x^2 - 2$.

6. Find the area A of the region S enclosed by the lines $y = x$, $y = -\frac{1}{2}x$, and $y = 3 - 2x$.

II. Volumes

In the next several sections, we will be finding the volumes of various solids.

We will use the following **three step process** (introduced for calculating area in section 6.1).

Step1

Cut the object into thin slices of width Δx (or some other convenient variable).

Find an approximation for the volume of a slice.

Step 2

Use geometry or some other known relationship between various quantities so that everything that varies among the various slices is expressed in terms of the eventual integration variable x (or whatever you decided was convenient in the previous step).

Step3

The total approximate volume is the sum of the approximate volumes of all the slices.

Take the limit of this sum to obtain a definite integral. The limits of integration will be the values of the integration variable that correspond to all the slices.

7. Use integration to calculate the volume of a cone with radius r meters and height h meters.

8. Find the volume of a pyramid of height 12 m whose base is a square with side length 4 m.

9. Consider the region in the xy-plane between the parabola $y = x^2$ and the line $y = 2$. Find the volume of the solid whose base is this region, and whose cross sections perpendicular to the x-axis are squares.

10. Consider the region in the xy-plane between the parabola $y = 2 - x^2$ and the x-axis. Find the volume of the solid whose base is this region, and whose cross sections perpendicular to the y-axis are a) semicircles; b) equilateral triangles.

11. Use integration to calculate the volume of a sphere of radius R.

(See Ex.3/ p.367 in the textbook)

Answers: 1a) $\frac{32}{3}$; 1b) 13; 2a) $2(\sqrt{2}-1)$; 2b) $2-\sqrt{2}$; 3. $\frac{1}{2}$; 4. $\frac{32}{3}$; 5. $\frac{20}{3}$; 6. $\frac{3}{2}$;
7. $\frac{1}{3}\pi r^2 h$; 8. $64m^3$; 9. $\frac{64\sqrt{2}}{15}$; 10a) π; 10b) $4\sqrt{3}$.