TUTORIAL 9: Line and Surface integrals

1. Evaluate $\int_C x^2 \, |d\mathbf{r}|$, where C is the circle of radius a centred at the origin.

2. Consider the vector field $\mathbf{u} = (-z, 0, x)$ and the following three paths from $(0, 0, 0)$ to $(1, 1, 1)$.

- C_1 is just a straight line.
- C_2 is given by $(x, y, z) = (t, t^2, t^3)$ for $0 \le t \le 1$.
- C_3 is given by $(x, y, z) = (\sin(\theta), 2\theta/\pi, 1 - \cos(\theta))$ for $0 \le \theta \le \pi/2$.

Write $I_1 = \int_{C_1} \mathbf{u} \cdot d\mathbf{r}$, $I_2 = \int_{C_2} \mathbf{u} \cdot d\mathbf{r}$ and $I_3 = \int_{C_3} \mathbf{u} \cdot d\mathbf{r}$.

(a) Would you expect I_1, I_2 and I_3 to be the same? Why?

(b) Calculate I_1, I_2 and I_3, and check your answer to (a).

3. Let C be the curve given by

$$\mathbf{r} = (2^t \cos(10\pi t^2), 2^t \sin(10\pi t^2), 2\pi)$$

for $0 \le t \le 1$, and let $\mathbf{u}$ be the vector field

$$\mathbf{u} = (e^x \cos(y)\cos(z),\ -e^x \sin(y)\cos(z),\ -e^x \cos(y)\sin(z)).$$

Calculate $\int_C \mathbf{u} \cdot d\mathbf{r}$. Think carefully about the most efficient method before launching into calculations. *Hint: check whether* $\mathbf{u}$ *is conservative—–if yes, the value of the integral does not depend on how we go between the endpoints of* $\mathbf{r}$

4. In a simplified model the force due to air resistance inside a tornado has the form $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Calculate the work done by the tornado on a particle that moves in the counterclockwise direction along the circle with a centre at $(2, 0)$ and radius 1.

5. Find the flux of the vector $\mathbf{F} = (3xy, x - y)$ through the parabolic arc parametrised by $\mathbf{r} = (t, t^2)$ for $-1 \le t \le 4$.

6. Find the surface area of the right circular cone with height h for which the position vector is given in parametric form as $\mathbf{r}(u, v) = (2v\cos u, 2v\sin u, v)$, with $0 \le u \le 2\pi$ and $0 \le v \le h$.

7. The surface of a torus is given in parametric form by

$$\mathbf{r}(s, t) = (b + a\cos s)\cos t\mathbf{i} + (b + a\cos s)\sin t\mathbf{j} + a\sin s\mathbf{k},$$

with $0 \leq s \leq 2\pi, 0 \leq t \leq 2\pi$ and a and b are two positive constants describing the two radii.

Let S be defined as the cut in the torus corresponding to $s = 0$. In this plane we define the vector $\mathbf{F} = (x, 2x + y, 0)$. Calculate the flow of the vectorfield $\mathbf{F}$ through S.

Answers

1. $a^3\pi$
2.
b. $I(C_1) = 0$; $I(C_2) = 1/2$; $I(C_3) = \pi/2 - 1$
3. $e^2 - e$
4. 2π
5. $\frac{7465}{6}$
6. $2\pi\sqrt{5}h^2$
7. $2\pi(a + b)^2$