TUTORIAL 1: complex numbers and functions

NB. For Questions 1-5 it is advisable if you read **Additional material 1**

1. Express the following numbers in the form $x + iy$, where $x, y \in \mathbb{R}$

$$(i)\ \ (-1+3i)^{-1}, \quad (ii)\ \ (1+i)i(2-i), \quad (iii)\ \ (\sqrt{2}i)(\pi+3i),$$
$$(iv)\ \ (1+i)(i-2)(i+3), \quad (v)\ \ (1+i)^{-1}, \quad (vi)\ \ \frac{2i}{3-i}$$

2. Let $z \neq 0$ be a complex number. What is the absolute value of $z/\overline{z}$?
3. If $z_1 = 2+i$, $z_2 = 3-2i$ and $z_3 = -1/2 + i\sqrt{3}/2$, evaluate each of the following expressions

$$(i)\ \ |3z_1 - 4z_2| \quad (ii)\ \ z_1^3 - 3z_1^2 + 4z_1 - 8, \quad (iii)\ \ (\overline{z}_3)^4, \quad (iv)\ \ \left|\frac{2z_2 + z_1 - 5 - i}{2z_1 - z_2 + 3 - i}\right|^2$$

4. Find the real and imaginary part of $(1+i)^{100}$.
5. Use complex numbers to prove the identities (i) $\cos 3\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta)$ and (ii) $\sin^3\theta = (3/4)\sin\theta - (1/4)\sin 3\theta$
6. Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. Using the Cauchy-Riemann equation, find v such that $f(z) = u + iv$ is analytic.
7. Find an analytic function $f(z) = u(x,y) + iv(x,y)$ for $u(x,y) = y^3 - 3x^2y$.
8. Use the chain rule to show that the Cauchy-Riemann equations in polar form can be written as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

considering that the transformation from a Cartesian coordinate system to a polar coordinate system is made using the transformations

$$x = r\cos\theta, \quad y = r\sin\theta \Longrightarrow r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$$

Answers:

1.
(i) $(-1-3i)/10$; (ii) $3i-1$; (iii) $-3\sqrt{2} + \pi\sqrt{2}i$; (iv) $-8-6i$; (v) $(1-i)/2$; (vi) $(-1+3i)/5$

2.

$$\left|\frac{z}{\overline{z}}\right| = 1$$

3.
(i) $\sqrt{157}$; (ii) $-7+3i$; (iii) $-(1+i\sqrt{3})/2$; (iv) 1

4.
-2^{50}

6.
(b)

$$v(x,y) = ye^{-x}(\sin y + x\cos y) + c$$

7.

$$v(x,y) = -3xy^2 + x^3 + c$$