Geometry of surfaces - Solutions

64. The Gaussian curvatures are $K = 0$ for the plane, $K = 1$ for the sphere and $K = -1$ for the hyperbolic paraboloid. It follows from the Theorema Egregium that no two of these surfaces can be isometric to each other.

65. Since the first fundamental forms coincide, the two surfaces are isometric to each other. By assumption, the Gaussian curvature of $\mathcal{S}_1$ at $\sigma_1(0,0)$ is equal to 0. The Theorema Egregium then implies that the Gaussian curvature of $\mathcal{S}_2$ at $\sigma_2(0,0)$ must be equal to 0. However, if $L_2 = 2$, $M_2 = 1$ and $N_2 = 2$, the Gaussian curvature is non-zero. It follows that such a surface cannot exist.

66. The cone is isometric to (an open part of) the plane and hence its Gaussian curvature K vanishes. The curve γ is unit speed, simple closed and positively oriented. The local version of the Gauss-Bonnet Theorem therefore implies $\int_\gamma \kappa_g ds = 2\pi$.

67. The cylinder is isometric to (an open part of) the plane and hence its Gaussian curvature K vanishes. The curve γ is unit speed, simple closed and positively oriented. The local version of the Gauss-Bonnet Theorem therefore implies $\int_\gamma \kappa_g ds = 2\pi$.

68. We can assume that γ is positively oriented. Otherwise, reverse the orientation of γ, which changes the sign of the geodesic curvature κ_g, but the integral over the geodesic curvature still remains zero. Let Ω be one of the two regions on S^2 and pick a surface patch $\sigma : U \to \mathbb{R}^3$ of S^2 with $\Omega \subset \sigma(U)$ (for example, define σ via stereographic projection from a point in the second region). Let $R \subset U$ with $\sigma(R) = \Omega$. The Gaussian curvature K of S^2 satisfies $K = 1$. Since $\int_\gamma \kappa_g = 0$ by assumption, the local version of the Gauss-Bonnet Theorem then implies $2\pi = \int_{\text{int}(\gamma)} d\mathcal{A}_\sigma = \mathcal{A}_\sigma(R)$. Thus each of the two regions on S^2 bounded by γ has area equal to 2π.

69. We have $\gamma(t) = \sigma(\rho(t))$ with $\rho : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t))$. Since ρ is positively oriented, also γ is positively oriented. Clearly, γ is a unit speed, simple closed curve. The unit normal $\mathbf{N}$ of σ is $\mathbf{N}(u,v) = \frac{1}{\sqrt{4u^2+4v^2+1}}(-2u, -2v, 1)$. Moreover, $\dot\gamma(t) = (-\sin(t), \cos(t), 0)$ and $\ddot\gamma(t) = (-\cos(t), -\sin(t), 0)$. Therefore,

$$\begin{aligned}\mathbf{N}(\rho(t)) \times \dot\gamma(t) &= \frac{1}{\sqrt{5}}(-2\cos(t), -2\sin(t), 1) \times (-\sin(t), \cos(t), 0)\\ &= \frac{1}{\sqrt{5}}(-\cos(t), -\sin(t), -2).\end{aligned}$$

Altogether this implies

$$\begin{aligned}\kappa_g(t) &= \ddot\gamma(t) \cdot (\mathbf{N}(\rho(t)) \times \dot\gamma(t))\\ &= (-\cos(t), -\sin(t), 0) \cdot \frac{1}{\sqrt{5}}(-\cos(t), -\sin(t), -2) = \frac{1}{\sqrt{5}}.\end{aligned}$$

The local version of the Gauss-Bonnet Theorem then implies

$$\iint_{\text{int}(\gamma)} K d\mathcal{A}_\sigma = 2\pi - \int_\gamma \kappa_g ds = 2\pi - \frac{1}{\sqrt{5}} \int_\gamma ds$$

$$= 2\pi - \frac{1}{\sqrt{5}}\text{Length}(\gamma) = 2\pi - \frac{1}{\sqrt{5}}2\pi = 2\pi\left(1 - \frac{1}{\sqrt{5}}\right).$$

70. If γ is a geodesic, then its geodesic curvature κ_g vanishes and the local Gauss-Bonnet Theorem implies

$$0 = \int_\gamma \kappa_g ds = 2\pi - \iint_{\text{int}(\gamma)} K d\mathcal{A}_\sigma \geq 2\pi$$

because of $K \leq 0$, which is a contradiction. Thus γ cannot be a geodesic.

71. Both γ_1 and γ_2 parametrize great circles in S^2 and hence their geodesic curvatures vanish. This gives $\int_\Gamma \kappa_g = 0$.

The polygon has two vertices, namely at $\Gamma(\frac{\pi}{2})$ and at $\Gamma(\frac{3\pi}{2}) = \Gamma(-\frac{\pi}{2})$. Denote the corresponding interior angles by α_1 and α_2. Since the number of edges of Γ is also equal to 2 and the Gaussian curvature of S^2 satisfies $K = 1$, the Gauss-Bonnet Theorem for curvilinear polygons implies

$$\mathcal{A}_\sigma(\text{int}(\Gamma)) = \iint_{\text{int}(\Gamma)} d\mathcal{A}_\sigma = \alpha_1 + \alpha_2.$$

It remains to compute the two interior angles:

$$\dot{\Gamma}^-\left(\frac{\pi}{2}\right) = \dot{\gamma}_1\left(\frac{\pi}{2}\right) = (-1, 0, 0),$$
$$\dot{\Gamma}^+\left(\frac{\pi}{2}\right) = \dot{\gamma}_2\left(\frac{\pi}{2}\right) = (\cos(\phi), \sin(\phi), 0),$$

which gives

$$\cos(\alpha_1) = (\cos(\phi), \sin(\phi), 0) \cdot (1, 0, 0) = \cos(\phi)$$

and hence $\alpha_1 = \phi$. Next,

$$\dot{\Gamma}^-\left(\frac{3\pi}{2}\right) = \dot{\gamma}_2\left(\frac{3\pi}{2}\right) = (-\cos(\phi), -\sin(\phi), 0),$$
$$\dot{\Gamma}^+\left(-\frac{\pi}{2}\right) = \dot{\gamma}_1\left(-\frac{\pi}{2}\right) = (1, 0, 0),$$

which gives

$$\cos(\alpha_2) = (\cos(\phi), \sin(\phi), 0) \cdot (1, 0, 0) = \cos(\phi)$$

and hence $\alpha_2 = \phi$. Altogether we now get

$$\mathcal{A}_\sigma(\text{int}(\Gamma)) = \alpha_1 + \alpha_2 = 2\phi.$$

72. Assume γ_1 and γ_2 meet again. Then we can construct from these two geodesics a curvilinear polygon γ on σ with two vertices and two edges. If α_1 and α_2 are the two interior angles of this polygon, then the Gauss-Bonnet Theorem for curvilinear polygons implies

$$0 \leq \alpha_1 + \alpha_2 = \iint_{\text{int}(\gamma)} K d\mathcal{A}_\sigma < 0,$$

which is a contradiction.