MATH 465 - INTRODUCTION TO COMBINATORICS
LECTURE 7

1. Solving linear recurrences using generating functions

Consider the linear recurrence

$$h_n = 5h_{n-1} - 6h_{n-2}, \quad (n \geq 2)$$

with $h_0 = 1, h_1 = -2$. Solving the recurrence, we obtain an infinite sequence $h_0, h_1, h_2, \dots$. We would like to find a formula for the numbers h_n. Let $h(x) = \sum_{n \geq 0} h_n x^n$ denote the generating function for this sequence. We have

$$\sum_{n \geq 2} h_n x^n = 5 \sum_{n \geq 2} h_{n-1} x^n - 6 \sum_{n \geq 2} h_{n-2} x^n. \tag{1.1}$$

We would like to express each of these sums in terms of $h(x)$. Notice that

$$\begin{aligned}
\sum_{n \geq 2} h_n x^n &= h(x) - h_1 x - h_0, \\
\sum_{n \geq 2} h_{n-1} x^n &= x \sum_{n \geq 2} h_{n-1} x^{n-1} \\
&= x \sum_{n \geq 1} h_n x^n \\
\sum_{n \geq 2} h_{n-2} x^n &= x^2 \sum_{n \geq 2} h_{n-2} x^{n-2} \\
&= \sum_{n \geq 0} h_n x^n.
\end{aligned}$$

Substituting these into (1.1) and using $h_0 = 1, h_1 = -2$, we get

$$h(x) + 2x - 1 = 5x(h(x) - 1) - 6x^2 h(x).$$

Solving for $h(x)$, we get

$$\begin{aligned}
h(x) &= \frac{1 - 7x}{1 - 5x + 6x^2} = \frac{1 - 7x}{(1 - 2x)(1 - 3x)} = \frac{5}{1 - 2x} - \frac{4}{1 - 3x} \\
&= 5(1 + 2x + 4x^2 + 8x^3 + \cdots) - 4(1 + 3x + 9x^2 + 27x^3 + \cdots).
\end{aligned}$$

Therefore $h_n = 5 \cdot 2^n - 4 \cdot 3^n$. Notice that the solution is a linear combination of geometric progressions.

2. Linear homogeneous recurrences

We aim to solve recurrences of the form

$$h_n + a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} = 0, \quad (n \geq k) \quad (*)$$

where

(1) The number k is called the *order* of the recurrence.

(2) $a_1, a_2, \ldots, a_k$ are called *coefficients*, with $a_k \neq 0$ without loss of generality.
(3) The values of $h_0, h_1, \ldots, h_{k-1}$, the *initial values* are given.

The theory of linear recurrences is very similar to the theory of linear ordinary differential equations.

Lemma 2.1. *For $q \neq 0$, the geometric progression $h_n = q^n$ is a solution of the recurrence $(*)$ if and only if q is a root of the characteristic equation*

$$q^k + a_1 q^{k-1} + a_2 q^{k-2} + \cdots + a_k = 0.$$

Proof. Substitute $h_n = q^n$ into $(*)$ and factor out q^{n-k}. □

Example 2.2. Consider the recurrence of order 2

$$h_n - 5h_{n-1} + 6h_{n-2} = 0, \quad (n \geq 2).$$

The characteristic equation is $q^2 - 5q + 6 = (q-2)(q-3)$, so $h_n = 2^n$ and $h_n = 3^n$ are solutions.

Lemma 2.3. *The solutions of the linear recurrence $(*)$ form a $\mathbb{C}$-vector space of dimension k.*

Proof. Let V denote the set of all solutions to $(*)$. It is easy to see that if $h_n, g_n \in V$, then so is $h_n + g_n$ and $c \cdot h_n$ for $c \in \mathbb{C}$, so V is a $\mathbb{C}$-vector space. Consider the linear transformation

$$T : V \to \mathbb{C}^k$$

sending a solution $(h_n)_{n\geq 0}$ to $(h_0, \ldots, h_{k-1})$, i.e., to the initial values. T is a bijection since (h_n) is completely determined by the initial conditions, so T is an isomorphism of vector spaces. Since isomorphic vector spaces have the same dimension, V is k-dimensional. □

The proof gives more: the solutions to the recurrences with initial conditions

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$$

form a basis for V.

Lemma 2.4. *Geometric progressions with distinct ratios are linearly independent.*

Proof. Suppose that k geometric progressions

$$1 \quad q_i \quad q_i^2 \quad q_i^3 \quad \cdots$$

(with $q_i, i = 1, 2, \ldots k$ distinct and nonzero) are linearly dependent. Then

$$\det \begin{pmatrix} 1 & q_1 & q_1^2 & \cdots & q_1^{k-1} \\ 1 & q_2 & q_2^2 & \cdots & q_2^{k-1} \\ & & \vdots & & \\ 1 & q_k & q_k^2 & \cdots & q_k^{k-1} \end{pmatrix} = 0 \;\Rightarrow\; \exists\, (c_0, c_1, \ldots, c_k) \neq 0 \quad \begin{cases} c_0 + c_1 q_1 + \cdots + c_{k-1} q_1^{k-1} = 0, \\ c_0 + c_1 q_2 + \cdots + c_{k-1} q_2^{k-1} = 0, \\ \qquad \vdots \\ c_0 + c_1 q_k + \cdots + c_{k-1} q_k^{k-1} = 0. \end{cases}$$

Thus, $c_0 + c_1 t + \cdots + c_{k-1} t^{k-1}$, a nonzero polynomial of degree $\leq k-1$, has k distinct roots, a contradiction. □

Putting the three lemmas together, we get:

Theorem 2.5. *Given a linear recurrence*

$$h_n + a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} = 0, \tag{$*$}$$

if the characteristic equation

$$q^k + a_1 q^{k-1} + a_2 q^{k-2} + \cdots + a_k = 0$$

has distinct roots $q_1, \dots, q_k$, then the general solution of ($*$) is given by

$$h_n = \sum_{i=1}^{k} c_i \, q_i^n.$$

Example 2.6. The general solution of the recurrence $h_n - 5h_{n-1} + 6h_{n-2} = 0$ is given by $h_n = c_1 \cdot 2^n + c_2 \cdot 3^n$.

3. The Fibonacci numbers

The Fibonacci numbers are defined by the recurrence

$$f_n = f_{n-1} + f_{n-2}, \quad (n \geq 2),$$

with the initial conditions $f_0 = 0, f_1 = 1$. The characteristic equation is

$$q^2 - q - 1 = 0,$$

which has roots

$$q_1 = \frac{1+\sqrt{5}}{2}, \quad q_2 = \frac{1-\sqrt{5}}{2}.$$

Therefore the general solution is

$$f_n = c_1 q_1^n + c_2 q_2^n,$$

for some constants $c_1, c_2 \in \mathbb{C}$. To find c_1 and c_2, we use the initial conditions

$$f_0 = 0 = c_1 + c_2, \quad f_1 = 1 = c_1 q_1 + c_2 q_2 = c_1(q_1 - q_2) = c_1\sqrt{5}.$$

Therefore we get $c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$. We have proved:

Theorem 3.1.

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Corollary 3.2.

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}.$$

The number $\dfrac{1+\sqrt{5}}{2}$ is known as the *golden ratio*.

Theorem 3.3. *The number of ways to represent n as an ordered sum of summands, each of which is equal to 1 or 2, is equal to f_{n+1}.*

Theorem 3.4. *The number of ways to tile a $2 \times n$ rectangular board by dominoes is equal to f_{n+1}.*

Theorem 3.5. *The number of binary strings of length n that do not contain a pair of consecutive 1's is f_{n+2}.*

4. Linear recurrences with multiple roots

We say that q_0 is a root of multiplicity m of a polynomial $P(q)$ if $P(q) = (q - q_0)^m R(q)$ where $R(q_0) \neq 0$.

Theorem 4.1. *Any polynomial of degree $d > 0$ with complex coefficients has d complex roots, counted with multiplicities.*

Lemma 4.2. *For a polynomial $P(q)$ and $q_0 \in \mathbb{C}$, the following are equivalent:*

(1) q_0 is a root of $P(q)$ with multiplicity m,
(2) q_0 is a root of $P(q), P'(q), \ldots, P^{(m-1)}(q)$ but not $P^{(m)}(q)$.

Proposition 4.3. *Suppose $q_0 \neq 0$ is a multiple root of the characteristic equation. Then $h_n = nq_0^n$ is a solution to the recurrence.*

Proof. Consider the characteristic equation

$$\sum_{i=0}^{k} a_i q^{k-i} = 0.$$

Multiply by q^{n-k} to get

$$\sum_{i=0}^{k} a_i q^{n-i} = 0.$$

Since q_0 is a root of multiplicity of $P(q) = \sum_{i=0}^{k} a_i q^{n-i} = 0$, by Lemma 4.2, we have $P'(q_0) = 0$. Multiplying by q_0, we get

$$\sum_{i=0}^{k} a_i (n-i) q_0^{n-i} = 0.$$

□

Lemma 4.4 (Exercise)**.** *Suppose q_0 is a root of multiplicity m of the characteristic equation. Then for $l \in \{0, 1, \ldots, m-1\}$, the sequence $h_n = n^l q_0^n$ satisfies the recurrence.*

Lemma 4.5. *These k solutions are linearly independent. Therefore, they form a basis for the vector space of solutions.*

We omit the proof of this lemma.

Theorem 4.6. *Consider the linear recurrence*

$$(*) \qquad \sum_{i=0}^{k} a_i \, h_{n-i} = 0.$$

Let $q_1, \ldots, q_r$ be the complex roots of its characteristic equation

$$(**) \qquad \sum_{i=0}^{k} a_i \, q^{k-i} = 0,$$

with respective multiplicities $m_1, \ldots, m_r$. Then the general solution of the recurrence $(*)$ is given by

$$h_n = C_1(n) q_1^n + \cdots + C_r(n) q_r^n$$

where each $C_i(n)$ is a polynomial in n of degree $\deg(C_i) < m_i$.

Problem 4.7. Solve the linear recurrence $h_n = 3h_{n-2} + 2h_{n-3}$ with the initial conditions $h_0 = 0, h_1 = 0, h_2 = 1$.

The characteristic equation

$$q^3 - 3q - 2 = (q+1)^2(q-2) = 0$$

has roots $-1, -1, 2$. Hence $h_n = (c_1 + c_2 n)(-1)^n + c_3 \cdot 2^n$. Using the initial conditions, we have the linear system

$$\begin{aligned} c_1 + c_3 &= 0 \\ -c_1 - c_2 + 2c_3 &= 0 \\ c_1 + 2c_2 + 4c_3 &= 1, \end{aligned}$$

solving which we get

$$c_1 = -\frac{1}{9}, \quad c_2 = \frac{1}{3}, \quad c_3 = \frac{1}{9}.$$

Therefore $h_n = \frac{(3n-1)(-1)^n + 2^n}{9}$.

Corollary 4.8. *The general solution (h_n) of the linear recurrence*

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} h_{n-i} = 0$$

is a polynomial in n of degree $\leq k-1$.

Proof. The characteristic equation is $(q-1)^k = 0$. □

Example 4.9. The solutions of

$$h_n - 2h_{n-1} + h_{n-2} = 0$$

are arithmetic progressions.