Geometry of Surfaces

5CCM223A/6CCM223B

Video 18

A theorem of Archimedes

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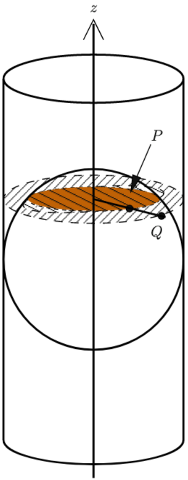
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$\mathcal{Z} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

$$f : S^2 \backslash \{(0, 0, \pm 1)\} \to \mathcal{Z},\ (x, y, z) \mapsto \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z \right)$$


z
P
Q

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$
$$\mathcal{Z} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

$$f : S^2 \backslash \{(0,0,\pm 1)\} \to \mathcal{Z},\ (x,y,z) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z \right)$$

Theorem of Archimedes**.** *The map f is area-preserving.*

Proof. Put $U = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0, 2\pi) \subset \mathbb{R}^2$ and define parametrizations $\sigma : U \to \mathbb{R}^3$ of S^2 and $\tilde{\sigma} : U \to \mathbb{R}^3$ of $\mathcal{Z}$ by

$$\sigma(\theta, \varphi) = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$$
$$\tilde{\sigma}(\theta, \varphi) = (\cos(\varphi), \sin(\varphi), \sin(\theta))$$

Let $R \subseteq U$. Claim: $\mathcal{A}_\sigma(R) = \mathcal{A}_{\tilde{\sigma}}(R)$

For σ we already know: $E = 1,\ F = 0,\ G = \cos(\theta)^2$

For $\tilde{\sigma}$: $\tilde{\sigma}_\theta = (0, 0, \cos(\theta))$ and $\tilde{\sigma}_\varphi = (-\sin(\varphi), \cos(\varphi), 0)$
Thus $\tilde{E} = \cos(\theta)^2,\ \tilde{F} = 0,\ \tilde{G} = 1$

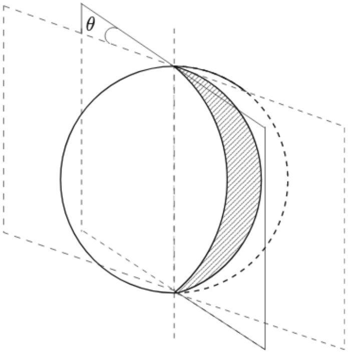
This implies $EG - F^2 = \cos(\theta)^2 = \tilde{E}\tilde{G} - \tilde{F}^2$ and hence

$$\mathcal{A}_\sigma(R) = \iint\limits_R (EG - F^2)^{\frac{1}{2}} d\theta d\varphi = \iint\limits_R (\tilde{E}\tilde{G} - \tilde{F}^2)^{\frac{1}{2}} d\theta d\varphi = \mathcal{A}_{\tilde{\sigma}}(R)$$

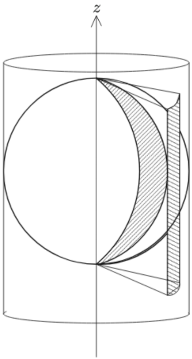
$$f : \mathcal{S}_1 \to \mathcal{S}_2 \text{ isometry} \Longrightarrow f : \mathcal{S}_1 \to \mathcal{S}_2 \text{ area-preserving}$$

The converse is not true; the map $f : S^2 \backslash \{(0,0,\pm 1)\} \to \mathcal{Z}$ in the Theorem of Archimedes is a counterexample

Application 1: Area of lune



Apply Theorem of ARCHIMEDES: area of lune $= 2\theta$

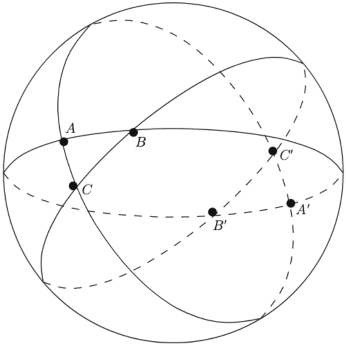


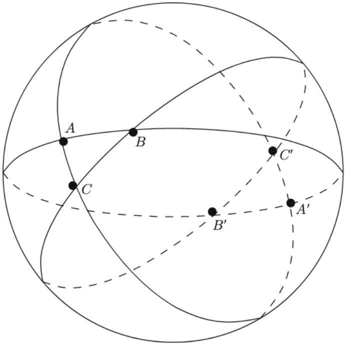
Take limit $\theta \to 2\pi$: lune $\to S^2$; $\mathcal{A}(S^2) = 4\pi$

Application 2: Area of spherical triangles

Theorem. Let ΔABC be a triangle on S^2. Then

$$\mathcal{A}(\Delta ABC) = \angle A + \angle B + \angle C - \pi$$





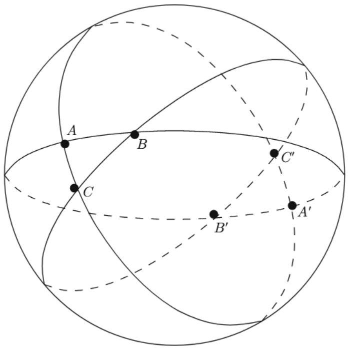
$$\mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta A'BC) = 2\angle A$$
$$\mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta AB'C) = 2\angle B$$
$$\mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta ABC') = 2\angle C$$

$$\begin{aligned}\mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta A'BC) &= 2\angle A\\ \mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta AB'C) &= 2\angle B\\ \mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta ABC') &= 2\angle C\end{aligned}$$

Summing up gives

$$\begin{aligned}&2\angle A + 2\angle B + 2\angle C\\ &= 3\mathcal{A}(\Delta ABC) + \underbrace{\mathcal{A}(\Delta A'BC)}_{=\mathcal{A}(\Delta AB'C')} + \mathcal{A}(\Delta AB'C) + \mathcal{A}(\Delta ABC')\\ &= 2\mathcal{A}(\Delta ABC)\\ &\qquad + \big(\mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta AB'C') + \mathcal{A}(\Delta AB'C) + \mathcal{A}(\Delta ABC')\big)\end{aligned}$$

$$
\begin{aligned}
&2\angle A + 2\angle B + 2\angle C \\
&= 2\mathcal{A}(\Delta ABC) \\
&\qquad + \underbrace{\big(\mathcal{A}(\Delta ABC) + \mathcal{A}(\Delta AB'C') + \mathcal{A}(\Delta AB'C) + \mathcal{A}(\Delta ABC')\big)}_{=2\pi \text{ as triangles form a hemisphere}}
\end{aligned}
$$



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The second fundamental form

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Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve. TAYLOR'S THEOREM:

$$\begin{aligned}\gamma(s+\Delta s) &= \gamma(s) + \dot{\gamma}(s)\Delta s + \tfrac{1}{2}\ddot{\gamma}(s)(\Delta s)^2 + \dots \\ &= \gamma(s) + \mathbf{t}(s)\Delta s + \tfrac{1}{2}\dot{\mathbf{t}}(s)(\Delta s)^2 + \dots \\ &= \gamma(s) + \mathbf{t}(s)\Delta s + \tfrac{1}{2}\kappa(s)\mathbf{n}(s)(\Delta s)^2 + \dots\end{aligned}$$

This implies

$$(\gamma(s+\Delta s) - \gamma(s)) \cdot \mathbf{n}(s) = \tfrac{1}{2}\kappa(s)(\Delta s)^2 + \dots$$

Left-hand side measures deviation of γ from tangent line at $\gamma(s)$

Generalize this to surfaces...

Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch with unit normal $\mathbf{N}$.
Apply Taylor's Theorem:

$$\begin{aligned}
&\sigma(u + \Delta u, v + \Delta v) \\
&= \sigma(u, v) + \sigma_u(u, v)\Delta u + \sigma_v(u, v)\Delta v \\
&\quad + \tfrac{1}{2}\left(\sigma_{uu}(u, v)(\Delta u)^2 + 2\sigma_{uv}(u, v)\Delta u\Delta v + \sigma_{vv}(u, v)(\Delta v)^2\right) + \ldots
\end{aligned}$$

Taking dot product with $\mathbf{N}$ gives

$$\begin{aligned}
&(\sigma(u + \Delta u, v + \Delta v) - \sigma(u, v)) \cdot \mathbf{N}(u, v) \\
&= \tfrac{1}{2}\left(L(u, v)(\Delta u)^2 + 2M(u, v)\Delta u\Delta v + N(u, v)(\Delta v)^2\right) + \ldots
\end{aligned}$$

with

$$\mathrm{L} = \sigma_{uu} \cdot \mathbf{N}\ , \quad \mathrm{M} = \sigma_{uv} \cdot \mathbf{N}\ , \quad \mathrm{N} = \sigma_{vv} \cdot \mathbf{N}$$

$Ldu^2 + 2Mdudv + Ndv^2$ is the second fundamental form of σ

The second fundamental form of a plane

Consider standard parametrization $\sigma(u, v) = a + up + vq$ with $a, p, q \in \mathbb{R}^3$ and p, q linearly independent. Then

$$\sigma_u = p,\ \sigma_v = q,\ \sigma_{uu} = 0,\ \sigma_{uv} = 0,\ \sigma_{vv} = 0$$

Thus the second fundamental form of σ is 0

The second fundamental form of a surface of revolution

Consider standard parametrization

$$\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

with $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$. Compute $\mathbf{N}, L, M, N$...
The second fundamental form of σ is

$$\left(\dot{f}\ddot{g} - \ddot{f}\dot{g}\right)(u)du^2 + \left(f\dot{g}\right)(u)dv^2$$

Unit sphere ($f(u) = \cos(u)$, $g(u) = \sin(u)$): $du^2 + \cos(u)^2 dv^2$

Round cylinder of radius 1 ($f(u) = 1$, $g(u) = u$): dv^2

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Curvature of curves on a surface

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Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch with unit normal $\mathbf{N}$ and $\gamma : I \to \mathbb{R}^3$ be a curve on the surface with $\|\dot{\gamma}\| = 1$. Write $\gamma(s) = \sigma(u(s), v(s))$ with $(u(s), v(s)) \in U$ for all $s \in I$. CHAIN RULE:

$$\dot{\gamma}(s) = \sigma_u(u(s), v(s))\dot{u}(s) + \sigma_v(u(s), v(s))\dot{v}(s)$$

Write $\mathbf{N}_\gamma = \mathbf{N} \circ \gamma$. The vectors

$$\dot{\gamma}(s)\ ,\ \mathbf{N}_\gamma(s)\ ,\ (\mathbf{N}_\gamma \times \dot{\gamma})(s)$$

form an orthonormal basis of $\mathbb{R}^3$ for each $s \in I$. Since $\|\dot{\gamma}\| = 1$, we have $\dot{\gamma} \cdot \ddot{\gamma} = 0$ and hence

$$\ddot{\gamma} = \kappa_n \mathbf{N}_\gamma + \kappa_g (\mathbf{N}_\gamma \times \dot{\gamma})$$

κ_n is called the normal curvature of γ

κ_g is called the geodesic curvature of γ

If γ is not unit speed, then define κ_n, κ_g using a unit speed reparametrization of γ

By definition, we have $\ddot{\gamma} = \kappa_n \mathbf{N}_\gamma + \kappa_g \mathbf{N}_\gamma \times \dot{\gamma}$. Thus

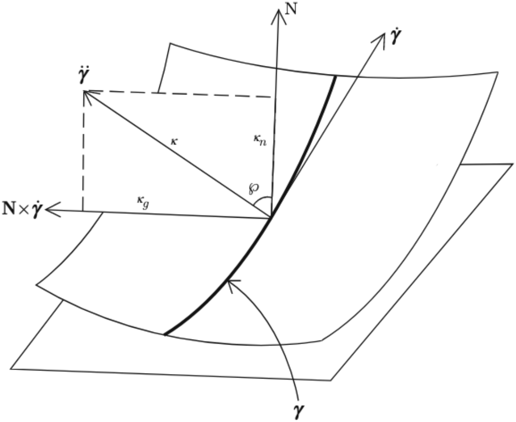
$$\kappa_n = \ddot{\gamma} \cdot \mathbf{N}_\gamma \ , \ \kappa_g = \ddot{\gamma} \cdot \mathbf{N}_\gamma \times \dot{\gamma} \ , \ \kappa^2 = \|\ddot{\gamma}\|^2 = (\kappa_n)^2 + (\kappa_g)^2$$

where κ is the curvature of γ. Since $\ddot{\gamma} = \kappa \mathbf{n}$, we get

$$\kappa_n = \kappa(\mathbf{n} \cdot \mathbf{N}_\gamma) = \kappa \cos(\psi) \text{ with } \psi = \angle(\mathbf{n}, \mathbf{N}_\gamma)$$

Then

$$\kappa_g = \pm\kappa \sin(\psi)$$


N
$\dot{\gamma}$
$\ddot{\gamma}$
κ
κ_n
$\wp$
$\mathbf{N}\times\dot{\gamma}$
κ_g
γ

Let $p \in \mathcal{S} = \sigma(U)$. A normal section at p is the intersection of the surface with a plane Π, $p \in \Pi$, perpendicular to $T_p\mathcal{S}$. Assume γ parametrizes a normal section with $\gamma(0) = p$. Since γ is contained in Π, we have $\mathbf{n}(0)\|\Pi$. By construction, we also have $\mathbf{N}_\gamma(0)\|\Pi$. Since both $\mathbf{n}(0)$ and $\mathbf{N}_\gamma(0)$ are perpendicular to $\dot{\gamma}(0)$, we have $\mathbf{n}(0)\|\mathbf{N}_\gamma(0)$ and hence $\psi \in \{0, \pi\}$. It follows that

$$\kappa_n(0) = \pm\kappa(0) \ , \ \kappa_g(0) = 0$$

Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and
$\gamma(s) = \sigma(u(s), v(s))$ be a curve on the surface with $\|\dot{\gamma}\| = 1$

Proposition. $\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$

Proof.

$$\begin{aligned}
\kappa_n &= \mathbf{N}_\gamma \cdot \ddot{\gamma} = \mathbf{N}_\gamma \cdot \frac{d\dot{\gamma}}{ds} = \mathbf{N}_\gamma \cdot \frac{d}{ds}\left(\sigma_u \dot{u} + \sigma_v \dot{v}\right) \\
&= \mathbf{N}_\gamma \cdot \left(\sigma_u \ddot{u} + \sigma_v \ddot{v} + \left(\sigma_{uu}\dot{u} + \sigma_{uv}\dot{v}\right)\dot{u} + \left(\sigma_{uv}\dot{u} + \sigma_{vv}\dot{v}\right)\dot{v}\right) \\
&= L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2
\end{aligned}$$

since $\mathbf{N} \cdot \sigma_u = 0 = \mathbf{N} \cdot \sigma_v$, $L = \mathbf{N} \cdot \sigma_{uu}$, $M = \mathbf{N} \cdot \sigma_{uv}$, $N = \mathbf{N} \cdot \sigma_{vv}$

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Principal curvatures

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Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch with first and second fundamental forms

$$Edu^2 + 2Fdudv + Gdv^2 \ , \ Ldu^2 + 2Mdudv + Ndv^2$$

Introduce matrix notation

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \ , \ \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Every tangent vector of the surface is of the form

$$\mathbf{t} = \xi\sigma_u + \eta\sigma_v$$

with $\xi, \eta \in \mathbb{R}$. We then write

$$\mathbf{T} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

If $\mathbf{t}_1, \mathbf{t}_2$ are two tangent vectors at $p \in \mathcal{S} = \sigma(U)$, then

$$\mathbf{t}_1 \cdot \mathbf{t}_2 = \begin{pmatrix} \xi_1 & \eta_1 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = T_1^\top \mathcal{F}_I T_2$$

Similarly, if $\gamma(s) = \sigma(u(s), v(s))$ and $\dot{\gamma} = \xi \sigma_u + \eta \sigma_v$, then the normal curvature κ_n is given by

$$\kappa_n = \begin{pmatrix} \xi & \eta \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = T^\top \mathcal{F}_{II} T$$

The principal curvatures of a surface are the roots of

$$0 = \det\left(\mathcal{F}_{II} - \kappa\mathcal{F}_{I}\right) = \det\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix}$$

If $0 \neq \mathbf{T} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ satisfies

$$\left(\mathcal{F}_{II} - \kappa\mathcal{F}_{I}\right)\mathbf{T} = 0$$

then

$$\mathbf{t} = \xi\sigma_u + \eta\sigma_v$$

is called a principal vector of the surface corresponding to the principal curvature κ

Let $\mathbf{t}_i = \xi_i \sigma_u + \eta_i \sigma_v$ be two unit tangent vectors at p with $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$. Put

$$\mathbf{T}_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \;,\; A = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{pmatrix}$$

Then

$$A^\top \mathcal{F}_I A = \begin{pmatrix} T_1^\top \mathcal{F}_I T_1 & T_1^\top \mathcal{F}_I T_2 \\ T_2^\top \mathcal{F}_I T_1 & T_2^\top \mathcal{F}_I T_2 \end{pmatrix} = \begin{pmatrix} \mathbf{t}_1 \cdot \mathbf{t}_1 & \mathbf{t}_1 \cdot \mathbf{t}_2 \\ \mathbf{t}_2 \cdot \mathbf{t}_1 & \mathbf{t}_2 \cdot \mathbf{t}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Define $\mathcal{G}_{II} = A^\top \mathcal{F}_{II} A$. Then

$$\mathcal{G}_{II}^\top = (A^\top \mathcal{F}_{II} A)^\top = A^\top \mathcal{F}_{II} A = \mathcal{G}_{II}$$

$$\Longrightarrow \exists B \in O_2 : B^\top \mathcal{G}_{II} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Define $C = AB \in GL_2(\mathbb{R})$. Then

$$C^\top \mathcal{F}_I C = B^\top \underbrace{A^\top \mathcal{F}_I A}_{=I_2} B = B^\top B = I_2$$

$$C^\top \mathcal{F}_{II} C = B^\top \underbrace{A^\top \mathcal{F}_{II} A}_{=\mathcal{G}_{II}} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Thus

$$\begin{aligned} 0 = \det\left(\mathcal{F}_{II} - \kappa \mathcal{F}_I\right) &\iff 0 = \det\left(C^\top (\mathcal{F}_{II} - \kappa \mathcal{F}_I) C\right) \\ &\iff 0 = \det\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \kappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \end{aligned}$$

Thus λ_1, λ_2 are the principal curvatures

Proposition. *Let κ_1, κ_2 be the principal curvatures of σ at $p \in \sigma(U)$.*

(i) *If $\kappa_1 \neq \kappa_2$, then principal vectors $\mathbf{t}_1, \mathbf{t}_2$ corresponding to κ_1, κ_2 are perpendicular*

(ii) *If $\kappa_1 = \kappa_2$, then every non-zero tangent vector at p is a principal vector*

Proof. (i): By assumption, $\mathcal{F}_{II} T_i = \kappa_i \mathcal{F}_I T_i$.

$$\begin{aligned}
\kappa_2(\mathbf{t}_1 \cdot \mathbf{t}_2) &= \kappa_2 T_1^\top \mathcal{F}_I T_2 = \mathbf{T}_1^\top \mathcal{F}_{II} \mathbf{T}_2 = (\mathbf{T}_1^\top \mathcal{F}_{II} \mathbf{T}_2)^\top \\
&= \mathbf{T}_2^\top \mathcal{F}_{II}^\top (\mathbf{T}_1^\top)^\top = \mathbf{T}_2^\top \mathcal{F}_{II} \mathbf{T}_1 = \kappa_1 T_2^\top \mathcal{F}_I T_1 \\
&= \kappa_1(\mathbf{t}_2 \cdot \mathbf{t}_1) = \kappa_1(\mathbf{t}_1 \cdot \mathbf{t}_2)
\end{aligned}$$

This implies $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$ since $\kappa_1 \neq \kappa_2$

(ii): Now assume $\kappa = \lambda_1 = \lambda_2$. We proved above that there exists $C \in GL_2(\mathbb{R})$ such that

$$C^\top \mathcal{F}_I C = I_2 \ , \ C^\top \mathcal{F}_{II} C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Then

$$\begin{aligned} & C^\top \mathcal{F}_I C = I_2 \ , \ C^\top \mathcal{F}_{II} C = \kappa I_2 \\ \Longrightarrow\ & C^\top (\mathcal{F}_{II} - \kappa \mathcal{F}_I) C = 0 \\ \Longrightarrow\ & \mathcal{F}_{II} - \kappa \mathcal{F}_I = 0 \\ \Longrightarrow\ & \forall T : (\mathcal{F}_{II} - \kappa \mathcal{F}_I) T = 0 \end{aligned}$$

All non-zero tangent vectors at p are principal vectors

$$0 = \det\left(\mathcal{F}_{II} - \kappa\mathcal{F}_{I}\right) \Longleftrightarrow 0 = \det\left(\mathcal{F}_{I}^{-1}\mathcal{F}_{II} - \kappa I_2\right)$$

Principal curvatures are eigenvalues of $\mathcal{F}_{I}^{-1}\mathcal{F}_{II}$ and principal vectors are eigenvectors of $\mathcal{F}_{I}^{-1}\mathcal{F}_{II}$. However, $\mathcal{F}_{I}^{-1}\mathcal{F}_{II}$ is **not** a symmetric matrix. Thus previous proposition does not follow from standard Linear Algebra.