1: The Lotka-Volterra predator-prey model revisited.

The equation is as presented:

$$\frac{dx}{dt} = \gamma(xy - x)$$
$$\frac{dy}{dt} = \frac{1}{\gamma}(y - xy)$$

where

$$E(x,y) = \frac{1}{\gamma^2}(\ln x - x) + \ln y - y = T(x) + V(y)$$

We verify that

$$\begin{aligned}\frac{dE}{dt} &= \frac{1}{\gamma^2}\left(\frac{\dot{x}}{x} - \dot{x}\right) + \left(\frac{\dot{y}}{y} - \dot{y}\right) \\ &= \frac{1}{\gamma^2}\left(\gamma(y-1) - \gamma(xy - x)\right) + \left(\frac{1}{\gamma}(1-x) - \frac{1}{\gamma}(y - xy)\right) \\ &= \frac{1}{\gamma}(y - 1 + x - xy + 1 - x + xy - y) \\ &= 0\end{aligned}$$

We make the substitution $p = \ln(x), q = \ln(y)$. Then

$$H(p,q) = E(x,y) = \frac{1}{\gamma^2}(p - e^p) + (q - e^q)$$

We compute the Hessian Matrix.

$$\begin{aligned}H &= \begin{bmatrix} \frac{\partial^2 E}{\partial x^2} & \frac{\partial^2 E}{\partial x \partial y} \\ \frac{\partial^2 E}{\partial x \partial y} & \frac{\partial^2 E}{\partial y^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{x^2\gamma^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}\end{aligned}$$

Noted that H is symmetric and all its eigenvalues are positive, H is postive definite.

3:

Suppose

$$\dot{x} = f(x)$$

with a graph. Show that it has no *conserved quantity.*

proof. Assume the contrary, that there is a $E(x)$ such that $\frac{dE}{dt} = 0$. Then

$$0 = \frac{dE}{dt} = \dot{x}\frac{dE}{dx} = f(x)\frac{dE}{dx}$$

From aboved, we can see that whenever $f(x) \neq 0$, $dE/dx = 0$. Since there is a continous interval in which $f(x) \neq 0$, in this area, $E(x)$ is a constant. This creates a contradiction.

4: Competing species

Consider the system

$$\frac{dx}{dt} = x(1-y)$$
$$\frac{dy}{dt} = y(1-x)$$

(a) Find two fixed points,and classify them by computing the Jacobian matrix at each of them and analyzing its eigenvalues.

solution. The fixed points are $(0,0)$ and $(1,1)$. The Jacobian matrix is

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 1-y & -x \\ -y & 1-x \end{bmatrix} \end{aligned}$$

At $(0,0)$, $J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which has eigenvalues 1 and 1. So it is a unstable node. At $(1,1)$, $J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, which has eigenvalues 1 and -1. So it is a saddle point.

(b) Show that there is no conserved quantity for this system.

solution.