King's College London

University Of London

FOLLOW THE INSTRUCTIONS YOU HAVE BEEN GIVEN ON HOW TO UPLOAD YOUR SOLUTIONS

BSc and MSci Examination

6CCM223B Geometry of surfaces

Summer 2021

Time Allowed: Two Hours

This paper consists of two sections, Section A and Section B.
Section A contributes 45% of the total marks for the paper.
Answer all questions in Section A.
All questions in Section B carry equal marks, but if more than TWO questions are attempted, then only the best two will count.

You may consult lecture notes and use a calculator.

Section A

All ten questions in Section A carry equal marks.
Answer all questions for full marks.

A 1. Calculate the arc length of $\gamma : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (3t-2, 4t+5)$ starting at $\gamma(0)$.

A 2. Calculate the curvature of $\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (t, t^2, \sin(t))$ at $\gamma(0)$.

A 3. Calculate the torsion of $\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (\sin(t), \cos(t), e^t)$ at $\gamma(0)$.

A 4. Calculate the tangent plane of the surface $\sigma(u, v) = (u\cos(v), u\sin(v), v)$ at $\sigma(0, 0)$.

A 5. Calculate the geodesic curvature of the curve $\gamma(s) = (-\frac{1}{2}\sin(2s), \frac{1}{2}, \frac{1}{2}\cos(2s))$ on the surface $\sigma(u, v) = (u, 2(u^2+v^2), v)$.

A 6. Calculate the first fundamental form of the surface $\sigma(u, v) = (e^u v, e^{-u} v, u)$.

A 7. Calculate the second fundamental form of the surface $\sigma(u, v) = (u+v, u-v, uv)$ at $\sigma(0, 0)$.

A 8. Calculate the principal curvatures of the surface $\mathcal{S}$ at $p \in \mathcal{S}$ whose coefficients of the first and second fundamental form at p are given by $E = 2$, $F = 2$, $G = 4$, $L = 4$, $M = 1$, $N = 2$.

A 9. Calculate the Gaussian curvature of the surface $\sigma(u, v) = (u^3 v, v, u - v^2)$ at $\sigma(1, 0)$.

A 10. Calculate the mean curvature of the surface $\sigma(u, v) = (e^u v, \sin(u)v, u\cos(v))$ at $\sigma(0, 0)$.

Section B

All three questions in Section B carry equal marks.
Answer TWO questions for full marks.

B 11. (i) Show that the smooth curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (t^3, t^6, t^9)$$

is not regular. Find a regular curve with the same image as γ.

(ii) Find a unit speed reparametrization of the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (4t, 4\cosh(t), 3\sinh(t)).$$

(iii) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with constant curvature $\kappa > 0$. Assume that $\gamma(\mathbb{R})$ is contained in a plane. Use the FRENET-SERRET EQUATIONS to prove that $\gamma(\mathbb{R})$ lies on a circle.

B 12. (i) Show that

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u + \cosh(v), u + \sinh(v), 1 + 2ue^{-v})$$

is a regular surface patch parametrizing a part of a hyperbolic paraboloid. Determine this part explicitly.

(ii) Let $U = \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and

$$\sigma : U \to \mathbb{R}^3,\ (u,v) \mapsto (u\cos(v), u\sin(v), \ln(\cos(v)) + u).$$

Let $v_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $a, b \in \mathbb{R}$ with $a < b$, and define the curve

$$\gamma_{v_0} : (a,b) \to \mathbb{R}^3,\ t \mapsto \sigma(t, v_0).$$

Prove that the length of γ_{v_0} does not depend on v_0.

(iii) Let

$$\sigma : U \to \mathbb{R}^3,\ (u,v) \to \sigma(u,v)$$

be a regular surface patch and assume that the first fundamental form of σ is of the form

$$ds^2 = du^2 + g(u)dv^2$$

with some smooth function $g(u)$. Write down the geodesic equations for a unit speed curve

$$\gamma(t) = \sigma(u(t), v(t))$$

on σ and use them to determine the geodesic $\gamma(t)$ for the initial values

$$\gamma(0) = (u_0, v_0) \in U \text{ and } \dot{\gamma}(0) = \frac{\sigma_u(u_0,v_0)}{\|\sigma_u(u_0,v_0)\|}.$$

B 13. (i) Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a unit speed curve, $U = (\alpha, \beta) \times (-\pi, \pi)$, $r \in \mathbb{R}^+$. Assume that the curvature κ of γ satisfies $0 < \kappa < \frac{1}{r}$. Define

$$\sigma : U \to \mathbb{R}^3 \ , \ (u, v) \mapsto \gamma(u) + r(\cos(v)\mathbf{n}(u) + \sin(v)\mathbf{b}(u)),$$

where $\mathbf{n}$ is the principal normal and $\mathbf{b}$ is the binormal of γ. Assume that σ is a regular surface patch. Compute the Gauss map of the surface and show that the curve

$$\alpha(t) = \sigma(u_0, \tfrac{t}{r}) \qquad ((u_0, \tfrac{t}{r}) \in U)$$

is a geodesic on the surface.

(ii) Let $\sigma_i : U_i \to \mathbb{R}^3$, $i \in \{1, 2\}$, be two regular surface patches with Gaussian curvature K_i and mean curvature H_i. Are the follow statements true or false? In each case, justify your answer.

(a) If the two surfaces are isometric to each other, then $K_1 = K_2$;

(b) If the two surfaces are isometric to each other, then $H_1 = H_2$.

(iii) Consider a triangulation of the sphere S^2 with 12 vertices and 20 faces. How many edges are there in this triangulation? Justify your answer.

Solutions

For each question I state one possible solution that is based on the material taught in the course. For some questions, in particular proofs, there are of course other solutions for which a student can get full marks.

A 1. To get the arc length function $s(t)$ starting at $\gamma(0)$, we calculate

$$\begin{aligned}
\gamma(t) &= (3t-2, 4t+5) \\
\gamma'(t) &= (3,4) \\
\|\gamma'(t)\| &= \sqrt{3^2+4^2} = \sqrt{25} = 5 \\
s(t) &= \int_0^t \|\gamma'(u)\| du = \int_0^t 5du = 5t
\end{aligned}$$

A 2. To get the curvature $\kappa(0)$ of γ at $\gamma(0)$, we calculate

$$\begin{aligned}
\gamma(t) &= (t, t^2, \sin(t)) \\
\gamma'(t) &= (1, 2t, \cos(t)) \\
\gamma''(t) &= (0, 2, -\sin(t)) \\
\gamma'(0) \times \gamma''(0) &= (1,0,1) \times (0,2,0) = (-2,0,2) \\
\kappa(0) &= \frac{\|\gamma'(0) \times \gamma''(0)\|}{\|\gamma'(0)\|^3} = \frac{\|(-2,0,2)\|}{\|(1,0,1)\|^3} = \frac{\sqrt{8}}{\sqrt{2}^3} = 1
\end{aligned}$$

A 3. To get the torsion $\tau(0)$ of γ at $\gamma(0)$, we calculate

$$\begin{aligned}
\gamma(t) &= (\sin(t), \cos(t), e^t) \\
\gamma'(t) &= (\cos(t), -\sin(t), e^t) \\
\gamma''(t) &= (-\sin(t), -\cos(t), e^t) \\
\gamma'''(t) &= (-\cos(t), \sin(t), e^t) \\
\gamma'(0) \times \gamma''(0) &= (1,0,1) \times (0,-1,1) = (1,-1,-1) \\
\gamma'''(0) &= (-1,0,1) \\
\tau(0) &= \frac{(\gamma'(0) \times \gamma''(0)) \cdot \gamma'''(0)}{\|\gamma'(0) \times \gamma''(0)\|^2} = \frac{-2}{3} = -\frac{2}{3}
\end{aligned}$$

A 4. To get the tangent plane $T_{\sigma(0,0)}\mathcal{S}$ of the surface $\mathcal{S}$ given by $\sigma(u,v) = (u\cos(v), u\sin(v), v)$ at $\sigma(0,0)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (\cos(v), \sin(v), 0)\\
\sigma_v(u,v) &= (-u\sin(v), u\cos(v), 1)\\
\sigma_u(0,0) &= (1,0,0)\\
\sigma_v(0,0) &= (0,0,1)\\
T_{\sigma(0,0)}\mathcal{S} &= \{(x,0,z) \in \mathbb{R}^3 : x, z \in \mathbb{R}\}
\end{aligned}$$

A 5. To get the geodesic curvature $\kappa_g(s)$ of $\gamma(s) = (-\frac{1}{2}\sin(2s), \frac{1}{2}, \frac{1}{2}\cos(2s))$ on the surface $\sigma(u,v) = (u, 2(u^2+v^2), v)$, we observe that $\gamma(s) = \sigma(-\frac{1}{2}\sin(2s), \frac{1}{2}\cos(2s))$ and calculate

$$\begin{aligned}
\dot\gamma(s) &= (-\cos(2s), 0, -\sin(2s)) \text{ (thus } \gamma \text{ is unit speed)}\\
\ddot\gamma(s) &= (2\sin(2s), 0, -2\cos(2s))\\
\sigma_u(-\tfrac{1}{2}\sin(2s), \tfrac{1}{2}\cos(2s)) &= (1, -2\sin(2s), 0)\\
\sigma_v(-\tfrac{1}{2}\sin(2s), \tfrac{1}{2}\cos(2s)) &= (0, 2\cos(2s), 1)\\
(\sigma_u \times \sigma_v)(-\tfrac{1}{2}\sin(2s), \tfrac{1}{2}\cos(2s)) &= (-2\sin(2s), -1, 2\cos(2s))\\
\mathbf{N}(-\tfrac{1}{2}\sin(2s), \tfrac{1}{2}\cos(2s)) &= \tfrac{1}{\sqrt{5}}(-2\sin(2s), -1, 2\cos(2s))\\
\mathbf{N}(-\tfrac{1}{2}\sin(2s), \tfrac{1}{2}\cos(2s)) \times \dot\gamma(s) &= \tfrac{1}{\sqrt{5}}(\sin(2s), -2, -\cos(2s))\\
\kappa_g(s) &= \ddot\gamma(s) \cdot (\mathbf{N}(-\tfrac{1}{2}\sin(2s), \tfrac{1}{2}\cos(2s)) \times \dot\gamma(s)) = \tfrac{2}{\sqrt{5}}
\end{aligned}$$

A 6. To get the first fundamental form ds^2 of the surface $\sigma(u,v) = (e^u v, e^{-u} v, u)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (e^u v, -e^{-u} v, 1)\\
\sigma_v(u,v) &= (e^u, e^{-u}, 0)\\
E(u,v) &= \sigma_u(u,v)\cdot\sigma_u(u,v) = e^{2u}v^2 + e^{-2u}v^2 + 1 = 1 + 2\cosh(2u)v^2\\
F(u,v) &= \sigma_u(u,v)\cdot\sigma_v(u,v) = e^{2u}v - e^{-2u}v = 2\sinh(2u)v\\
G(u,v) &= \sigma_v(u,v)\cdot\sigma_v(u,v) = e^{2u} + e^{-2u} = 2\cosh(2u)\\
ds^2 &= (1 + 2\cosh(2u)v^2)du^2 + 4\sinh(2u)v\,du\,dv + 2\cosh(2u)dv^2
\end{aligned}$$

A 7. To get the second fundamental form $II(0,0)$ of $\sigma(u,v) = (u+v, u-v, uv)$ at $\sigma(0,0)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (1,1,v) \\
\sigma_v(u,v) &= (1,-1,u) \\
\sigma_u(0,0) \times \sigma_v(0,0) &= (1,1,0) \times (1,-1,0) = (0,0,-2) \\
\mathbf{N}(0,0) &= (0,0,-1) \\
\sigma_{uu}(u,v) &= (0,0,0) \\
\sigma_{uv}(u,v) &= (0,0,1) \\
\sigma_{vv}(u,v) &= (0,0,0) \\
L(0,0) &= \sigma_{uu}(0,0) \cdot \mathbf{N}(0,0) = 0 \\
M(0,0) &= \sigma_{uv}(0,0) \cdot \mathbf{N}(0,0) = -1 \\
N(0,0) &= \sigma_{vv}(0,0) \cdot \mathbf{N}(0,0) = 0 \\
II(0,0) &= 2M(0,0)dudv = -2dudv
\end{aligned}$$

A 8. To get the principal curvatures of $\mathcal{S}$ at p, we calculate

$$\begin{aligned}
0 &= \det\begin{pmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{pmatrix} = \det\begin{pmatrix} 4-2\kappa & 1-2\kappa \\ 1-2\kappa & 2-4\kappa \end{pmatrix} \\
&= (4-2\kappa)(2-4\kappa) - (1-2\kappa)^2 = 8 - 16\kappa - 4\kappa + 8\kappa^2 - 1 + 4\kappa - 4\kappa^2 \\
&= 4\kappa^2 - 16\kappa + 7 = 4\left(\kappa^2 - 4\kappa + \frac{7}{4}\right) = 4\left((\kappa-2)^2 - \frac{9}{4}\right)
\end{aligned}$$

which has solutions $2 \pm \frac{3}{2}$. Thus the principal curvatures of $\mathcal{S}$ at p are $\frac{1}{2}$ and $\frac{7}{2}$.

A 9. To get the Gaussian curvature $K(1,0)$ of the surface $\sigma(u,v) = (u^3v, v, u - v^2)$ at $\sigma(1,0)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (3u^2v, 0, 1)\\
\sigma_v(u,v) &= (u^3, 1, -2v)\\
(\sigma_u \times \sigma_v)(1,0) &= (0,0,1) \times (1,1,0) = (-1,1,0)\\
\mathbf{N}(1,0) &= \tfrac{1}{\sqrt{2}}(-1,1,0)\\
\sigma_{uu}(u,v) &= (6uv, 0, 0)\\
\sigma_{uv}(u,v) &= (3u^2, 0, 0)\\
\sigma_{vv}(u,v) &= (0, 0, -2)\\
E(1,0) &= \sigma_u(1,0) \cdot \sigma_u(1,0) = 1\\
F(1,0) &= \sigma_u(1,0) \cdot \sigma_v(1,0) = 0\\
G(1,0) &= \sigma_v(1,0) \cdot \sigma_v(1,0) = 2\\
L(1,0) &= \sigma_{uu}(1,0) \cdot \mathbf{N}(1,0) = 0\\
M(1,0) &= \sigma_{uv}(1,0) \cdot \mathbf{N}(1,0) = -\tfrac{3}{\sqrt{2}}\\
N(1,0) &= \sigma_{vv}(1,0) \cdot \mathbf{N}(1,0) = 0\\
K(1,0) &= \tfrac{LN-M^2}{EG-F^2}(1,0) = -\tfrac{9}{4}
\end{aligned}$$

A 10. To get the mean curvature $H(0,0)$ of the surface $\sigma(u,v) = (e^u v, \sin(u)v, u\cos(v))$ at $\sigma(0,0)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (e^u v, \cos(u)v, \cos(v))\\
\sigma_v(u,v) &= (e^u, \sin(u), -u\sin(v))\\
(\sigma_u \times \sigma_v)(0,0) &= (0,0,1) \times (1,0,0) = (0,1,0)\\
\mathbf{N}(0,0) &= (0,1,0)\\
\sigma_{uu}(u,v) &= (e^u v, -\sin(u)v, 0)\\
\sigma_{uv}(u,v) &= (e^u, \cos(u), -\sin(v))\\
\sigma_{vv}(u,v) &= (0, 0, -u\cos(v))\\
E(0,0) &= \sigma_u(0,0) \cdot \sigma_u(0,0) = 1\\
F(0,0) &= \sigma_u(0,0) \cdot \sigma_v(0,0) = 0\\
G(0,0) &= \sigma_v(0,0) \cdot \sigma_v(0,0) = 1\\
L(0,0) &= \sigma_{uu}(0,0) \cdot \mathbf{N}(0,0) = 0\\
M(0,0) &= \sigma_{uv}(0,0) \cdot \mathbf{N}(0,0) = 1\\
N(0,0) &= \sigma_{vv}(0,0) \cdot \mathbf{N}(0,0) = 0\\
H(0,0) &= \tfrac{LG-2MF+NE}{2(EG-F^2)}(0,0) = \tfrac{0}{2} = 0
\end{aligned}$$

B 11. (i) We have $\gamma(t) = (t^3, t^6, t^9)$ and $\gamma'(t) = (3t^2, 6t^5, 9t^8)$. Then $\gamma'(0) = (0,0,0)$ and hence γ is not regular. Consider the curve

$$\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (t, t^2, t^3)$$

Then $\tilde{\gamma}'(t) = (1, 2t, 3t^2) \neq 0$ and hence $\tilde{\gamma}$ is regular [4 marks]. Furthermore, we have $\tilde{\gamma}(\tilde{t}) = \gamma(t)$ if and only if $(\tilde{t}, \tilde{t}^2, \tilde{t}^3) = (t^3, t^6, t^9)$, that is, if and only if, $\tilde{t} = t^3$, $\tilde{t}^2 = t^6$ and $\tilde{t}^3 = t^9$. Since the map $\mathbb{R} \to \mathbb{R},\ t \mapsto t^3$ is bijective, it follows that $\tilde{\gamma}$ has the same image as γ [4 marks].

(ii) We have $\gamma(t) = (4t, 4\cosh(t), 3\sinh(t))$ and $\gamma'(t) = (4, 4\sinh(t), 3\cosh(t))$. Thus

$$\|\gamma'(t)\|^2 = 16 + 16\sinh(t)^2 + 9\cosh(t)^2 = 25\cosh(t)^2,$$

where we used $\cosh(t)^2 - \sinh(t)^2 = 1$. The arc length function is

$$s(t) = \int_0^t \|\gamma'(u)\| du = 5\int_0^t \cosh(u) du = 5(\sinh(t) - \sinh(0)) = 5\sinh(t)$$

[4 marks]. This implies $t = \sinh^{-1}(\frac{1}{5}s)$ and hence

$$\begin{aligned}\mathbb{R} \to \mathbb{R}^3,\ s \mapsto & \left(4\sinh^{-1}(\tfrac{1}{5}s), 4\cosh(\sinh^{-1}(\tfrac{1}{5}s)), 3\sinh(\sinh^{-1}(\tfrac{1}{5}s))\right) \\ &= \left(4\sinh^{-1}(\tfrac{1}{5}s), 4\sqrt{1 + \tfrac{1}{25}s^2}, \tfrac{3}{5}s\right)\end{aligned}$$

is a unit speed parametrization of γ [4 marks].

(iii) Assume that γ lies in the plane $\Pi = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z)\cdot a = d\}$ with $a \in \mathbb{R}^3$, $\|a\| = 1$, $d \in \mathbb{R}$. Consider the curve $\tilde{\gamma} = \gamma + \frac{1}{\kappa}\mathbf{n}$. Using the Frenet-Serret equation $\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b}$, we get

$$\dot{\tilde{\gamma}} = \dot{\gamma} + \frac{1}{\kappa}\dot{\mathbf{n}} = \mathbf{t} + \frac{1}{\kappa}(-\kappa\mathbf{t} + \tau\mathbf{b}) = \frac{\tau}{\kappa}\mathbf{b} = 0$$

since γ is a planar curve and the torsion of planar curves vanishes. It follows that there exists $x_0 \in \mathbb{R}^3$ so that $\tilde{\gamma} = x_0$ and thus $\gamma + \frac{1}{\kappa}\mathbf{n} = x_0$. This tells us that γ lies on the sphere in $\mathbb{R}^3$ with center x_0 and radius $\frac{1}{\kappa}$ [6 marks]. Since γ is contained in Π, γ lies on the (non-empty) intersection of the sphere and the plane Π, which is a circle, or a point (which cannot happen since γ is unit speed and hence non-constant) [3 marks].

B 12. (i) Clearly, σ is smooth [1 mark]. Assume that $\sigma(u,v) = \sigma(u',v')$. Then $u+\cosh(v) = u'+\cosh(v')$ and $u+\sinh(v) = u'+\sinh(v')$. Subtracting these two equations gives $e^{-v} = \cosh(v) - \sinh(v) = \cosh(v') - \sinh(v') = e^{-v'}$ and thus $v = v'$. Inserting this into $u + \cosh(v) = u' + \cosh(v')$ implies $u = u'$. Thus σ is injective [2 marks]. We have

$$\sigma_u(u,v) = (1, 1, 2e^{-v}),$$
$$\sigma_v(u,v) = (\sinh(v), \cosh(v), -2ue^{-v}).$$

Since $\sinh(v) \neq \cosh(v)$ for all $v \in \mathbb{R}$, $\sigma_u(u,v)$ and $\sigma_v(u,v)$ are linearly independent for all $(u,v) \in \mathbb{R}^2$ [2 marks]. It follows that σ is a regular surface patch. Putting $x = u+\cosh(v)$, $y = u+\sinh(v)$ and $z = 1+2ue^{-v}$, we see that $z = x^2 - y^2$. Thus σ parametrizes part of the hyperbolic paraboloid $z = x^2 - y^2$ [2 marks]. We have $x - y = e^{-v}$ and $x + y = 2u + e^v$. Thus $x - y > 0$ and $v = -\ln(x-y) = \ln(\frac{1}{x-y})$. Then $u = \frac{1}{2}(x + y - e^v) = \frac{1}{2}(x + y - \frac{1}{x-y})$. Altogether we see that σ parametrizes the part of the hyperbolic paraboloid $z = x^2 - y^2$ given by $x > y$ [3 marks].

(ii) We have $\gamma_{v_0}'(t) = \sigma_u(t, v_0) = (\cos(v_0), \sin(v_0), 1)$ and hence $\|\gamma_{v_0}'(t)\|^2 = 2$ [2 marks]. The length of γ_{v_0} is $\int_a^b \|\gamma_{v_0}'(t)\|dt = \sqrt{2}(b-a)$ [2 marks] and independent of v_0 [1 mark].

(iii) The general geodesic equations are

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2),$$
$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

[2 marks]. By assumption, we have $E = 1$, $F = 0$ and $G = g(u)$. Inserting this into the geodesic equations gives

$$\ddot{u} = \tfrac{1}{2}g'(u)\dot{v}^2 \ , \ g'(u)\dot{u}\dot{v} + g(u)\ddot{v} = 0. \quad (*)$$

[2 marks]. We have $\dot{\gamma}(0) = \sigma_u(u_0, v_0)\dot{u}(0) + \sigma_v(u_0, v_0)\dot{v}(0)$. From the initial condition $\dot{\gamma}(0) = \frac{\sigma_u(u_0,v_0)}{\|\sigma_u(u_0,v_0)\|}$ we therefore get $\dot{u}(0) = \frac{1}{\|\sigma_u(u_0,v_0)\|} = a$ and $\dot{v}(0) = 0$ [2 marks]. We thus have to solve $(*)$ subject to the initial conditions $u(0) = u_0$, $v(0) = v_0$, $\dot{u}(0) = a$, $\dot{v}(0) = 0$ [1 mark]. The obvious solution is $u(t) = at + u_0$ and $v(t) = v_0$ [2 marks]. The geodesic $\gamma(t) = \sigma(u(t), v(t))$ with $\gamma(0) = \sigma(u_0, v_0)$ and $\dot{\gamma}(0) = a\sigma_u(u_0, v_0)$ is therefore $\gamma(t) = \sigma(at + u_0, v_0)$ [1 mark].

B 13. (i) We have $\sigma(u,v) = \gamma(u) + r(\cos(v)\mathbf{n}(u) + \sin(v)\mathbf{b}(u))$. Using the Frenet-Serret equations we get

$$\begin{aligned}\sigma_u(u,v) &= \mathbf{t}(u) + r(\cos(v)\dot{\mathbf{n}}(u) + \sin(v)\dot{\mathbf{b}}(u))\\ &= \mathbf{t}(u) + r(\cos(v)(-\kappa(u)\mathbf{t}(u) + \tau(u)\mathbf{b}(u)) + \sin(v)(-\tau(u)\mathbf{n}(u)))\\ &= (1 - r\kappa(u)\cos(v))\mathbf{t}(u) - r\tau(u)\sin(v)\mathbf{n}(u) + r\tau(u)\cos(v)\mathbf{b}(u),\\ \sigma_v(u,v) &= r(-\sin(v)\mathbf{n}(u) + \cos(v)\mathbf{b}(u)) = -r\sin(v)\mathbf{n}(u) + r\cos(v)\mathbf{b}(u)\end{aligned}$$

[2 marks]. Using the fact that $\mathbf{t}\times\mathbf{n} = \mathbf{b}$, $\mathbf{n}\times\mathbf{b} = \mathbf{t}$ and $\mathbf{b}\times\mathbf{t} = \mathbf{n}$, we get

$$(\sigma_u \times \sigma_v)(u,v) = r(r\kappa(u)\cos(v) - 1)(\cos(v)\mathbf{n}(u) + \sin(v)\mathbf{b}(u)).$$

Since $0 < \kappa(u) < \frac{1}{r}$, we have $r\kappa(u)\cos(v) - 1 < 0$. Hence the unit normal $\mathbf{N}$ of σ is

$$\mathbf{N}(u,v) = \frac{(\sigma_u \times \sigma_v)(u,v)}{\|(\sigma_u \times \sigma_v)(u,v)\|} = -\cos(v)\mathbf{n}(u) - \sin(v)\mathbf{b}(u)$$

[3 marks]. The Gauss map is the map

$$\sigma(U) \to S^2,\ \sigma(u,v) \mapsto \mathbf{N}(u,v) = -\cos(v)\mathbf{n}(u) - \sin(v)\mathbf{b}(u)$$

[1 mark]. For $\alpha(t) = \sigma(u_0, \frac{t}{r}) = \gamma(u_0) + r(\cos(\frac{t}{r})\mathbf{n}(u_0) + \sin(\frac{t}{r})\mathbf{b}(u_0))$ we get

$$\begin{aligned}\dot\alpha(t) &= -\sin(\tfrac{t}{r})\mathbf{n}(u_0) + \cos(\tfrac{t}{r})\mathbf{b}(u_0),\\ \ddot\alpha(t) &= \tfrac{1}{r}(-\cos(\tfrac{t}{r})\mathbf{n}(u_0) - \sin(\tfrac{t}{r})\mathbf{b}(u_0)) = \tfrac{1}{r}\mathbf{N}(u_0, \tfrac{t}{r}).\end{aligned}$$

Thus α is unit speed and $\ddot\alpha(t)$ is parallel to $\mathbf{N}(u_0, \frac{t}{r})$ for all t, which shows that α is a geodesic. [4 marks]

(ii) (a) True [1 mark]; follows from Theorema Egregium [4 marks].
(b) False [1 mark]; consider for example corresponding parts of a plane and a round cylinder which are isometric to each other. The plane has $H_1 = 0$ and the round cylinder has $H_2 \neq 0$ [4 marks].

(iii) The Euler number χ of S^2 is equal to 2 [1 mark] and satisfies $\chi = V - E + F = 12 - E + 20 = 32 - E$ [2 marks]. It follows that there are 30 edges in the triangulation [2 marks].