Geometry of Surfaces

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Video 29

Existence and uniqueness of geodesics

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Questions. Let $\mathcal{S}$ be a surface, $p \in \mathcal{S}$ and $X \in T_p\mathcal{S}$ with $\|X\| = 1$. Does there exist a geodesic $\gamma(t) = \sigma(u(t), v(t))$ on the surface through p and tangent to X, that is, $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = X$? If yes, how many such geodesics exist?

To answer these questions, we use existence and uniqueness results about ordinary differential equations.

Recall: The geodesic equations are of the form

$$\ddot{u} = f(u, v, \dot{u}, \dot{v}) \ , \ \ddot{v} = g(u, v, \dot{u}, \dot{v})$$

with f, g smooth functions. This is an explicit system of ordinary differential equations of order 2 and dimension 2.

Existence and uniqueness results about ordinary differential equations imply:

For all $a, b, c, d, t_0 \in \mathbb{R}$ there exists a unique solution $(u(t), v(t))$ of

$$\ddot{u} = f(u, v, \dot{u}, \dot{v}) \ , \ \ddot{v} = g(u, v, \dot{u}, \dot{v})$$

with

$$u(t_0) = a \ , \ v(t_0) = b \ , \ \dot{u}(t_0) = c \ , \ \dot{v}(t_0) = d$$

and $|t - t_0| < \epsilon$ for $\epsilon > 0$ sufficiently small

Proposition. *Let* $p \in \mathcal{S} = \sigma(U)$, $X \in T_p\mathcal{S}$ *with* $\|X\| = 1$, *and* $t_0 \in \mathbb{R}$. *Then there exists a unique (maximal) geodesic* $\gamma(t) = \sigma(u(t), v(t))$ *on the surface with* $\gamma(t_0) = p$ *and* $\dot{\gamma}(t_0) = X$

Proof. Let $(a, b) \in U$ with $p = \sigma(a, b)$ and $c, d \in \mathbb{R}$ with $X = c\sigma_u(a, b) + d\sigma_v(a, b)$.
Then $\gamma(t_0) = p$ if and only if $u(t_0) = a$ and $v(t_0) = b$.
Moreover, since $\dot{\gamma}(t_0) = \dot{u}(t_0)\sigma_u(a, b) + \dot{v}(t_0)\sigma_v(a, b)$ we have $\dot{\gamma}(t_0) = X$ if and only if $\dot{u}(t_0) = c$ and $\dot{v}(t_0) = d$.
Now apply the above existence and uniqueness result about ordinary differential equations.

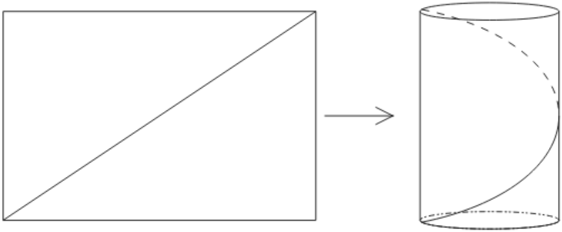
Conclusion: There is a unique (maximal) geodesic through any given point of a surface in any given direction

Example 1. The geodesics on the plane are the straight lines

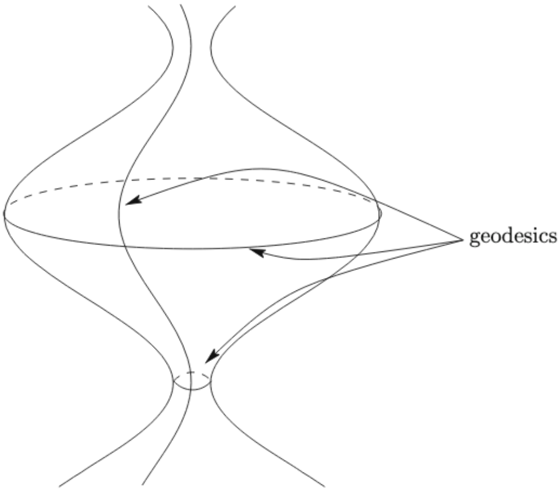
Example 2. The geodesics on the sphere are the great circles

Example 3. The geodesics on the circular cylinder are

1. the lines parallel to the axis
2. the circles perpendicular to the axis
3. circular helices



Question. What are the geodesics on a surface of revolution?



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Geodesics are critical points of the length functional

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Observations:

Lines in a plane give shortest paths between points in the plane (globally)

Great circles on the sphere give shortest paths between points on the sphere (locally but not globally)

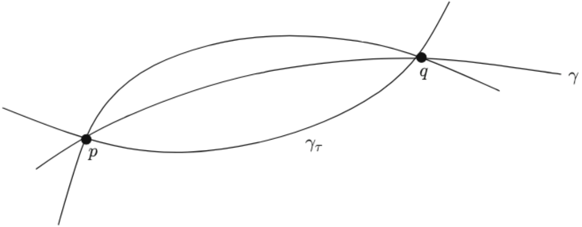
Problem. When are geodesics shortest paths?

To investigate this problem, we use methods from classical *calculus of variations*

Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $p, q \in \mathcal{S} = \sigma(U)$, $p \neq q$. Let $\delta, \epsilon > 0$ and $a, b \in \mathbb{R}$ with $[a, b] \subset (-\epsilon, \epsilon)$. Consider a smooth map

$$(-\delta, \delta) \times (-\epsilon, \epsilon) \to \mathcal{S} \ , \ (\tau, t) \mapsto \gamma_\tau(t) = \sigma(u(\tau, t), v(\tau, t))$$

Then each γ_τ is a smooth curve on the surface and the family $(\gamma_\tau)_{\tau \in (-\delta,\delta)}$ is a smooth variation of $\gamma = \gamma_0$. We consider variations where $\gamma = \gamma_0$ is a unit speed curve and $p = \gamma_\tau(a)$ and $q = \gamma_\tau(b)$ for all $\tau \in (-\delta, \delta)$



Define

$$\mathcal{L}(\gamma_\tau) = \int_a^b \|\dot{\gamma}_\tau(t)\| dt = \text{length of } \gamma_\tau\big|_{[a,b]}$$

Theorem. *Let γ be a unit speed geodesic on the surface and $(\gamma_\tau)_{\tau\in(-\delta,\delta)}$ be a smooth variation of γ with $p = \gamma_\tau(a)$ and $q = \gamma_\tau(b)$ for all $\tau \in (-\delta, \delta)$. Then*

$$\frac{d}{d\tau}\bigg|_{\tau=0} \mathcal{L}(\gamma_\tau) = 0$$

Proof. Assume that γ is a geodesic. We have to prove that

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \mathcal{L}(\gamma_\tau) = 0$$

for all smooth variations of γ. Put

$$g(\tau, t) = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \qquad \dot{} = \frac{d}{dt}$$

Then

$$\begin{aligned}\frac{d}{d\tau}\mathcal{L}(\gamma_\tau) &= \frac{d}{d\tau}\int_a^b \|\dot{\gamma}_\tau(t)\| dt = \frac{d}{d\tau}\int_a^b g(\tau, t)^{\frac{1}{2}} dt \\ &= \int_a^b \frac{d}{d\tau} g(\tau, t)^{\frac{1}{2}} dt = \frac{1}{2}\int_a^b \frac{d}{d\tau} g(\tau, t)^{-\frac{1}{2}} \frac{\partial g}{\partial \tau}(\tau, t) dt\end{aligned}$$

Recall: $g(\tau, t) = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$. Put $g_\tau = \frac{\partial g}{\partial \tau}$

$$\begin{aligned}
g_\tau &= E_\tau \dot{u}^2 + 2F_\tau \dot{u}\dot{v} + G_\tau \dot{v}^2 + 2E\dot{u}\dot{u}_\tau + 2F\left(\dot{u}_\tau \dot{v} + \dot{u}\dot{v}_\tau\right) + 2G\dot{v}\dot{v}_\tau \\
&= \left(E_u u_\tau + E_v v_\tau\right)\dot{u}^2 + 2\left(F_u u_\tau + F_v v_\tau\right)\dot{u}\dot{v} + \left(G_u u_\tau + G_v v_\tau\right)\dot{v}^2 \\
&\quad + 2E\dot{u}u_{\tau t} + 2F\left(u_{\tau t}\dot{v} + \dot{u}v_{\tau t}\right) + 2G\dot{v}v_{\tau t} \\
&= \left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2\right)u_\tau + \left(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2\right)v_\tau \\
&\quad + 2\left(E\dot{u} + F\dot{v}\right)u_{\tau t} + 2\left(F\dot{u} + G\dot{v}\right)v_{\tau t}
\end{aligned}$$

Insert this into

$$\frac{d}{d\tau}\mathcal{L}(\gamma_\tau) = \frac{1}{2}\int_a^b \frac{d}{d\tau} g(\tau, t)^{-\frac{1}{2}} g_\tau(\tau, t)\,dt$$

and use INTEGRATION BY PARTS (see notes for detailed computations) to conclude

$$\frac{d}{d\tau}\mathcal{L}(\gamma_\tau) = \int_a^b (Uu_\tau + Vv_\tau)\, dt$$

with

$$U = \frac{1}{2}g^{-\frac{1}{2}}\left(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right) - \frac{d}{dt}\left(g^{-\frac{1}{2}}\left(E\dot{u} + F\dot{v}\right)\right)$$
$$V = \frac{1}{2}g^{-\frac{1}{2}}\left(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right) - \frac{d}{dt}\left(g^{-\frac{1}{2}}\left(F\dot{u} + G\dot{v}\right)\right)$$

Observe that $U(0,t) = 0$ and $V(0,t) = 0$ are the geodesic equations. It follows that

$$\left.\frac{d}{d\tau}\right|_{\tau=0}\mathcal{L}(\gamma_\tau) = 0$$

if γ is a geodesic

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Critical points of the length functional are geodesics

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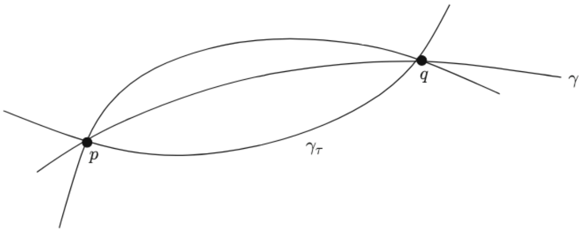
Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $p, q \in \mathcal{S} = \sigma(U)$, $p \neq q$. Let $\delta, \epsilon > 0$ and $a, b \in \mathbb{R}$ with $[a, b] \subset (-\epsilon, \epsilon)$. Consider smooth variations

$$(-\delta, \delta) \times (-\epsilon, \epsilon) \to \mathcal{S} \ , \ (\tau, t) \mapsto \gamma_\tau(t) = \sigma(u(\tau, t), v(\tau, t))$$

of the unit speed curve $\gamma = \gamma_0$ with $p = \gamma_\tau(a)$ and $q = \gamma_\tau(b)$ for all $\tau \in (-\delta, \delta)$



Define

$$\mathcal{L}(\gamma_\tau) = \int_a^b \|\dot{\gamma}_\tau(t)\| dt = \text{length of } \gamma_\tau|_{[a,b]}$$

Theorem. *Let γ be a unit speed curve on the surface. Assume that*

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \mathcal{L}(\gamma_\tau) = 0$$

for all smooth variations $(\gamma_\tau)_{\tau\in(-\delta,\delta)}$ of γ with $p = \gamma_\tau(a)$ and $q = \gamma_\tau(b)$ for all $\tau \in (-\delta, \delta)$. Then γ is a geodesic.

Proof. Assume that

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \mathcal{L}(\gamma_\tau) = 0$$

for all such smooth variations of γ. Recall that

$$\frac{d}{d\tau}\mathcal{L}(\gamma_\tau) = \int_a^b (Uu_\tau + Vv_\tau)\, dt$$

with

$$U = \frac{1}{2} g^{-\frac{1}{2}} \left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2\right) - \frac{d}{dt}\left(g^{-\frac{1}{2}} \left(E\dot{u} + F\dot{v}\right)\right)$$
$$V = \frac{1}{2} g^{-\frac{1}{2}} \left(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2\right) - \frac{d}{dt}\left(g^{-\frac{1}{2}} \left(F\dot{u} + G\dot{v}\right)\right)$$

We have to prove that the geodesic equations $U(0,t) = 0$ and $V(0,t) = 0$ are satisfied.

Assume that there exists $t_0 \in (a, b)$ with $U(0, t_0) > 0$ (for < 0 use similar argument). Since U is continuous, there exists $\eta > 0$ such that $U(0, t) > 0$ for all $t \in (t_0 - \eta, t_0 + \eta)$. Define

$$\phi : \mathbb{R} \to \mathbb{R} \ , \ t \mapsto \psi\left(\frac{t - t_0}{\eta}\right)$$

with

$$\psi(t) = \theta(1 + t)\theta(1 - t) \ , \ \theta(t) = \begin{cases} e^{-1/t^2} & \text{if } t > 0 \\ 0 & \text{if } t \leqslant 0 \end{cases}$$

Then ϕ is a smooth function with $\phi(t) > 0$ for $t \in (t_0 - \eta, t_0 + \eta)$ and $\phi(t) = 0$ for $t \notin (t_0 - \eta, t_0 + \eta)$

Write $\gamma(t) = \sigma(u(t), v(t))$ and consider the smooth variation

$$\gamma_\tau(t) = \sigma(u(t) + \tau\phi(t), v(t)) = \sigma(u(\tau, t), v(\tau, t))$$

Then $u_\tau = \phi$ and $v_\tau = 0$. Thus

$$0 = \frac{d}{d\tau}\bigg|_{\tau=0} \mathcal{L}(\gamma_\tau) = \int_a^b (Uu_\tau + Vv_\tau)\, dt\bigg|_{\tau=0} = \int_{t_0-\eta}^{t_0+\eta} U(0,t)\phi(t) > 0$$

which is a contradiction. Thus we must have $U(0, t) = 0$. Analogously we can prove that $V(0, t) = 0$.

Altogether we proved

Theorem. *Let γ be a unit speed curve on a surface $\mathcal{S}$. Then γ is a geodesic on the surface if and only if*

$$\frac{d}{d\tau}\bigg|_{\tau=0} \mathcal{L}(\gamma_\tau) = 0$$

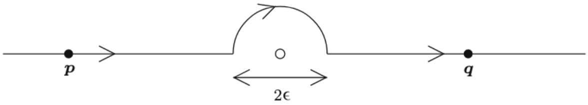
for all smooth variations $(\gamma_\tau)_{\tau\in(-\delta,\delta)}$ of γ with $p = \gamma_\tau(a)$ and $q = \gamma_\tau(b)$ for all $\tau \in (-\delta, \delta)$.

Corollary. *Shortest paths between points are geodesics*

Comments.

1. If $\frac{d}{d\tau}\Big|_{\tau=0} \mathcal{L}(\gamma_\tau) = 0$, then γ is not necessarily a shortest path. Consider for example a great circle on S^2 and distinct points p, q on the great circle. Then there are two arcs of the great circle connecting p and q, both of which are geodesics, but only one minimizes distance (if p, q are not antipodal).

2. Shortest paths may not exist



3. If a surface is a closed subset of $\mathbb{R}^3$, then there always exists a shortest path between two points on the surface