automata,positioning

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Problem 1

Disrete Math: Assignment #3

Due on December 15, 2022 at 3:10pm

Professor J Section A

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Problem 1

Problem 1 continued on next page... Problem 1 (continued)Problem 1 continued on next page... Let $G = (V, E)$ be a simple graph where V is the set of vertices and E is the set of edges. We denote the number of vertices by $n := |V|$ and the number of edges by $m := |E|$. For a vertex $a \in V$, we denote the degree of vertex a by $d(a)$. Furthermore, denote the minimum degree of all vertex degrees by σ and the maximum degree by δ. Please prove the statements below

(a) The number of vertices that have odd degrees must be even;

(b) $m \leq n2$

(c) $\sigma \leq \frac{2m}{n} \leq \delta$

(d) The length of a path is defined to be the number of edges in the path. Assume $n > 0$. Let $k \geq 0$ be an integer. If $\sigma \geq k$, then G has a path with length k.

Solution.

(a)

Recall the Handshake Theorem,

$$\sum_{v \in V} d(v) = 2|E|$$

The RHS is even, implying that the LHS is also even, thus $\#\{v \in V | d(v) isodd\}$ must be even.

(b)

A simple fact is that $d(v) \leq |V| - 1, \forall v \in V$, which means each vertex has at most $|V| - 1$ vertices to connect. Thus by Handshake theorem,

m = —E— = $\frac{\sum_{v \in V} d(v)}{2 \leq \frac{\sum_{v \in V}(|V|-1)}{2} = \frac{|V|(|V|-1)}{2} = |V|2 = n2}$

(c) We calculate the average number of neibourhoods a vertex has, whcih is

$$\frac{\sum_{v\ inV} d(v)}{|V|} = \frac{2|E|}{|V|} = \frac{2m}{n}$$

The average value should be larger than the minimum value whilst less than the maximum value. Therefore

$$\sigma \leq \frac{2m}{n} \leq \delta$$

(d)

We use mathematical induction on this one.

If $k = 0$ it is trivial.

If $k = 1$, since $\sigma \geq k$, a vertex called v_0 which σ neibourhoods has at least one neibourhood, which is also a path with length 1.

Now assume that for $1 \leq k < \sigma$, there is a path issuing from v_0, with a sequence of vertices $v_0 v_1 \cdots v_{k-1}$.

Now we examine the situation of v_{k-1}. Let $D_{v_{k-1}}$ be the set of neibourhood of v_{k-1}. Clearly, $|D_{v_{k-1}}| = d(v_{k-1}) \geq \sigma$. Now set $R = D_{v_{k-1}} - \{v | v \in v_0, \cdots, v_{k-2}\}$.

Then $|R| \geq \sigma - (k - 1) = \sigma - k + 1 \geq 1$ Thus we can extend $v_0 v_{k-1}$ by at least one edge so we have a path $v0 \cdots v_k$ with length k.

Problem 1 (continued)Problem 1 continued on next page... Problem 2

Problem 2

Problem 2 continued on next page... Problem 2 (continued)Problem 2 continued on next page... Let G be a graph with $n-1$ edges where n is the number of vertices of G. Show the following three statements are equivalent:
(a) G is connected; (b) G has no cycles; (c) G is a tree.
Solution.
We first prove two lemmas.

Lemma 1

A connected graph G with n and $n-1$ edges removed by any edge is not connected.
proof.
If $n=2$, there are two vertices, and one edge. Removing the only edge results in a two disconnected vertices.
Assume that we have a connected graph with k vertices and $k-1$ edges. We now consider a connected graph with $|V|=k+1$, and $|E|=k$
Recall the Handshake Theorem,

$$\sum_{v\in V} d(v) = 2|E|$$

Assume that $\forall v.d(v) \geq 2$, we have

$$2k = 2|E| = \sum_{v\in V} d(v) \geq \sum_{v\in V} 2 = 2|V| = 2(k+1)$$

Which is $2k \geq 2k+2$, a contradiction.
Thus with the connectedness of G, $\exists v \in V.d(v)=1$. We pick $v_0 \in V.d(v_0)=1$. Remove v_0, and the edge that connects v_0.
We then obtain a connected graph $G' = (V', E')$, with $|V'| = k, |E'| = k-1$. According to the inductive hypothesis, Removing any edge of G' results in a disconnected graph.
Now return v_0, and v_0's edge back to G', Now removing any edge of G, if it is the v_0's edge, G then is not connected, if it is other edge, then it is also disconnected, by the induction.

$(a)(b)$

Assume the contrary, G is connected and has a cycle. We remove one edge e on any G's cycle. On the one hand, since e is on the cycle, it does not affect the connectivity of G.
On the other hand, by lemma 1, G is not connected then. We have a contradiction. Therefore $(a)(b)$.

$(b)(c)$

Recall the definition of a Tree. A tree is a connected, and cycle-free graph. So we need to only show that G is connected, if G has no cycle.
We again prove using induction. If there are two vertices and one edge, it has no cycle and connected for sure. Now assume any cycle-free graph G with k vertices and $k-1$ edges is connected.
Consider a cycle-free graph with $k+1$ vertices and k edges. Again it has a vertex v_0 $d(v_0)=1$. Now hide v_0 along with its edge. We obtain a graph with $|V|=k$, $|E|=k-1$.
By induction, it is also cycle-free and thus connected. Add v_0 and its edge back we conclude that G is also connected. .

$(c)(a)$

The definition of Tree tells us that a tree is connected. Problem 2 (continued)Problem 2 continued on next page... Problem 3

Problem 3

Problem 3 continued on next page... Problem 3 (continued)Problem 3 continued on next page... A walk in a graph G is a sequence $W := v_0e_1v_1...v_{l-1}e_lv_l$, whose terms are alternately vertices and edges of G (not necessarily distinct), such that $v_i - 1$ and v_i are the ends of $e_i, 1 \leq i \leq l$. A closed walk is a walk that starts from and ends on the same vertex. So an Eulerian cycle is a closed walk that traversed each edge exactly once and we also call it Eulerian tour. (Be cautious that an Eulerian cycle is not a cycle. Recall that, by definition, a cycle should be a connected (sub)graph whose vertices are all of degree 2.) We say that a graph is Eulerian if it contains an Eulerian tour. So we know an Eulerian graph has no vertices of odd degree. Let G be Eulerian.

(a) Prove that G contains a cycle;

(b) For two cycles with no edges in common, we call they are edgedisjoint. Please show that the edge set of G can be partitioned into edge sets corresponding to edge-disjoint cycles in the graph.

Solution for (a).

First off we note that G is connected. If it has no cycle, then it is a tree. Apparently, we can't traverse each edge only once and get back to the starting point in a tree.

Solution for (b).

First off, we prove a lemma: for any $v \in v.d(v)$ is even. Assume $v0$ is the starting point of a Eulerian tour. To commence a Eulerian tour, one must leave v and then come back to v repeatly. Thus v needs even number of neibourhood to achieve this goal, otherwise the Eulerian tour will never end at v. Same for other vertex w, one must come into w and then finally will leave. Without $d(w)$ being even, the Eulerian tour will finally end at w which is impossible.

Now let C_0 be a cycle of G. Removing C_0 from G we obtain G_0 which has every vertex v, $d(v)$ is even since a cycle is of degree 2 for each vertex. Repeating this process, removing any cycle C_k from G_{k-1} obtaining G_k. The invariant is that $d(v)$ is even for any vertex in G_{k-1}. If there is a k for which G_k ends up having a connected component being a tree which doesn't have a cycle, then in this tree there is a vertex whose degree is one, an odd number. This is a contradiction.

Problem 3 (continued)Problem 3 continued on next page... Problem 4

Problem 4

Problem 4 continued on next page... Problem 4 (continued)Problem 4 continued on next page... A k-regular graph is a graph in which every vertex is of degree k. The girth of a graph is the length of the shortest cycle of the graph. Prove that

(a) A k-regular graph of girth four has at least $2k$ vertices;

(b) A k-regular graph of girth five has at least $k^2 + 1$ vertices.

Solution for (a)

Pick $v_0 \in V$. Let $N_{v_0} = \{v_1, v_2, \cdots, v_k\}$ be the set of neibourhood of v_0. Since there is no triangle in G with all 3 edges in E, for any i, j in $\{1, \cdots, k\}$, $\{v_i, v_j\} \notin E$ $\forall w \in N_{v_j} - \{v_0\}, w, v_0 \notin E$. Let $N'_{v_i} := N_{v_i} - v_0$, then $|N'_{v_i}| = 1$.

Thus —V— $\geq \left|\{v_0\} \bigcup N_{v_0} \bigcup \left(N'_{v_1} \cup \cdots \cup N'_{v_k}\right)\right|$

$= 1 + k + |\left(N'_{v_1} \cup \cdots \cup N'_{v_k}\right)|$

$\geq 1 + k + |N'_{v_i}|$

$= 1 + k + k - 1$
$= 2k$
Solution for (b)
Pick $v_0, w_0 \in V$, such that $\{v_0, w_0\} \in E$, and set $w_k = v_0, v_k = w_0$. Because there is no triangle in G, $N_{v_0} \cap N_{w_0} = \{\}$. Let $N_{v_0} = \{v_1, v_2, \cdots, v_k\}, N_{w_0} = \{w_1, \cdots, w_k\}$ Let $N_v := \bigcup_{i=1}^{k-1}(N_{v_i} - \{v_0\}), N_w := \bigcup_{i=1}^{k-1}(N_{w_i} - \{w_0\})$ We have $N_v \cap N_w = \{\}$ otherwise there will be a quadrilateral $\{v_0, v_i, w_j, w_0\}$ for some $1 \leq i, j \leq k$. For the same reason, $N_{v_i} \cap N_{v_j} = \{\}$, and $N_{w_i} \cap N_{w_j} = \{\}, \forall 1 \leq i, j \leq k$. Hence

$$|N_v| = |N_w| = \sum_{s=1}^{k-1}(k-1) = (k-1)^2$$

Thus —V— $\geq |\{v_0, w_0\} \bigcup (N_{v_0} - \{w_0\}) \cup (N_{w_0} - \{v_0\}) \bigcup (N_v \cup N_w)|$
$= 2 + (k-1) + (k-1) + |N_v \cup N_w|$
$\geq 2 + (k-1) + (k-1) + |N_v|$
$= 2 + 2(k-1) + (k-1)^2$
$= 1 + (k-1+1)^2$
$= k^2 + 1$
Problem 4 (continued)Problem 4 continued on next page. . . Problem 5

Problem 5

Problem 5 continued on next page. . . Problem 5 (continued)Problem 5 continued on next page. . .
Please use Gale-Shapley's algorithm to find one stable matching for thepreference lists below.
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Solution.
We first consider boy A, Girl X is on his top of list, then match (A, X). Next Step consider Boy B. Y is on his top of list and not ye purposed, then match (B, Y). Now consider boy C. X is on the top of his list, yet X is matched with A. So we can't change anything here. Next we consider Y, Y is already matched with B and Y rank B over C. Thus no change here as well. Therefore, the only choice of C is Z Problem 5 (continued)Problem 5 continued on next page. . . Problem 6

Problem 6

Problem 6 continued on next page. . . Problem 6 (continued)Problem 6 continued on next page. . . Let M be a matching for a graph G. (Recall that a matching in a graph is a set of edges that do not have common vertices.) An alternating path is a path that begins with an unmatched vertex and whose edges belong alternately to the matching M and not to the matching. An M-augmenting path is an alternating path that starts from and ends on unmatched vertices. Show that M is a maximum matching if and only if G has no M-augmenting path.
Proof of Necessity
Assume the contrary that G has a M-augmenting path

$$P := v_0 m_0 v_1 m_1 v_2 m_2 \cdots m_k v_{k+1}$$

where $m_{2i+1} \in M$ On the other hand, let $M \bigoplus P := (M \cup P) - (M \cap P)$ denote the symmetric difference. We'll show that $M \bigoplus P$ is also a match and $|M \bigoplus P| = |M| + 1$. After action of the symmetric difference, edges in M not in P stay invariant, while all m_i with even subscription remains, which are not in M, and the number of which exceeds the number of m_j who are in M by exactly one(since $k + 2$ is odd). Thus $|M \bigoplus P| = |M| + 1$ which contradicts to the maximality of M.
Proof of Sufficiency
Assume that the largest matching is not M but M'. Let $S := M \bigoplus M'$. We'll show that P contains a M'-augmenting path. First off, we compare the number of element of M in S, and the number of M in S, which are $m := |M| - |M \cap M'|$ and $m' := |M'| - |M \cap M'|$ respectively. clearly $m' > m$. Let P be a path in S. Its edges must be alternative between M and M' and in this case the number of edge in M' should be equal to the number of edge in M'. Otherwise it is a M'-augmenting path or M-augmenting path; it would lead to a contradiction. Since $m' > m$, some paths in S must be M-augmenting which is also a contradiction.
Problem 6 (continued)Problem 6 continued on next page. . . Problem 7