King's College London

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BSc and MSci Examination

6CCM223B Geometry of surfaces

Summer 2020

Time Allowed: Two Hours

This paper consists of two sections, Section A and Section B.
Section A contributes 45% of the total marks for the paper.
Answer all questions in Section A.
All questions in Section B carry equal marks, but if more than TWO questions are attempted, then only the best two will count.

You may consult lecture notes and use a calculator.

Section A

All ten questions in Section A carry equal marks.
Answer all questions for full marks.

A 1. Calculate the arc length of $\gamma : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\cos(e^t), \sin(e^t))$ starting at $\gamma(0)$.

A 2. Calculate the curvature of $\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (t, t^2, t^3)$ at $\gamma(0)$.

A 3. Calculate the torsion of $\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \left(1 + \cos(t), \sin(t), 2\sin\left(\frac{t}{2}\right)\right)$ at $\gamma(0)$.

A 4. Calculate the tangent plane of the surface $\sigma(u, v) = (u, v, u^3 - v^2)$ at $\sigma(1, 1)$.

A 5. Calculate the geodesic curvature of the curve $\gamma(s) = (\cos(s), \sin(s), 1)$ on the surface $\sigma(u, v) = (u, v, u^2 + v^2)$.

A 6. Calculate the first fundamental form of the surface $\sigma(u, v) = (u + v, u, u^3 v)$.

A 7. Calculate the second fundamental form of the surface $\sigma(u, v) = (v^2, u - v, uv)$ at $\sigma(1, -1)$.

A 8. Calculate the principal curvatures of the surface $\mathcal{S}$ at $p \in \mathcal{S}$ whose coefficients of the first and second fundamental form at p are given by $E = 1$, $F = 2$, $G = 3$, $L = 1$, $M = 0$, $N = 1$.

A 9. Calculate the Gaussian curvature of the surface $\sigma(u, v) = (u - v, 2u, u^2 + v^2)$ at $\sigma(0, 0)$.

A 10. Calculate the mean curvature of the surface $\sigma(u, v) = (u\cos(v), u\sin(v), v)$ at $\sigma(0, 0)$.

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Section B

All three questions in Section B carry equal marks.
Answer TWO questions for full marks.

B 11. (i) Find a unit speed reparametrization of the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (\cosh(t), \sinh(t), t).$$

(ii) Let γ be a unit speed curve in $\mathbb{R}^3$ with curvature $\kappa > 0$ everywhere. Let τ be the torsion of γ and assume that $\frac{\tau}{\kappa}$ is constant. Prove that there exists a unit vector $a \in \mathbb{R}^3$ such that $\dot{\gamma} \cdot a$ is constant.

(iii) Let γ be a unit speed curve in $\mathbb{R}^3$ with curvature $\kappa > 0$ everywhere. Let τ be the torsion of γ. Prove that $\rho = \dot{\gamma}$ is a regular curve in $\mathbb{R}^3$ and the curvature κ_ρ of ρ is given by

$$\kappa_\rho = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}.$$

B 12. (i) For each of the following maps $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, decide whether the map defines a surface patch. Justify your answers.

(a) $\sigma(u, v) = (u, uv, v)$

(b) $\sigma(u, v) = (u^2, u^3, v)$

(c) $\sigma(u, v) = (u, u^2, v + v^3)$

(d) $\sigma(u, v) = (\cos(2\pi u), \sin(2\pi u), v)$

(ii) Calculate the image of the Gauss map of the surface

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u, v) \mapsto (u, v, uv).$$

(iii) Consider the surface

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u, v) \mapsto (u, v, u^3 - 3uv^2)$$

Prove that $\sigma(0, 0)$ is a planar point of the surface. Find two lines in the surface passing through $\sigma(0, 0)$. What is the normal curvature and the geodesic curvature of these lines?

(iv) Does there exist a surface with constant mean curvature $H = -1$ and constant Gaussian curvature $K = +1$? Justify your answer!

B 13. (i) Consider the surface given by

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u,v,u^2+v^3).$$

Find all points on the surface at which the tangent plane is perpendicular to $(2,3,-1)$.

(ii) Let $\sigma : U \to \mathbb{R}^3$ be a surface with $R = (0,1) \times (0,1) \subset U \subset \mathbb{R}^2$. Assume that the first fundamental form of σ satisfies

$$E = \frac{1}{u+v} + \frac{1}{(1-u)(1-v)}\ ,\ F = \frac{1}{u+v}\ ,\ G = \frac{1}{u+v} - \frac{1}{(1+u)(1+v)}.$$

Calculate the area of $\sigma(R)$. [Note that $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x)$.]

(iii) Let f be an isometry between two surfaces. Do the two surfaces always have the same mean curvature at corresponding points? Justify your answer.

(iv) Does there exist a local isometry between the unit sphere in $\mathbb{R}^3$ and the cylinder in $\mathbb{R}^3$ with radius 1 around the z-axis? Justify your answer.

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Solutions

For each question I state one possible solution that is based on the material taught in the course. For some questions, in particular proofs, there are of course other solutions for which a student can get full marks.

A 1. To get the arc length function $s(t)$ starting at $\gamma(0)$, we calculate

$$\begin{aligned}
\gamma(t) &= (\cos(e^t), \sin(e^t)) \\
\gamma'(t) &= (-\sin(e^t)e^t, \cos(e^t)e^t) = e^t(-\sin(e^t), \cos(e^t)) \\
\|\gamma'(t)\|^2 &= e^{2t}(\sin^2(e^t) + \cos^2(e^t)) = e^{2t} \\
s(t) &= \int_0^t \|\gamma'(u)\| du = \int_0^t \sqrt{e^{2u}} du = \int_0^t e^u du = e^t - 1
\end{aligned}$$

A 2. To get the curvature $\kappa(0)$ of γ at $\gamma(0)$, we calculate

$$\begin{aligned}
\gamma(t) &= (t, t^2, t^3) \\
\gamma'(t) &= (1, 2t, 3t^2) \\
\gamma''(t) &= (0, 2, 6t) \\
\gamma'(0) \times \gamma''(0) &= (1, 0, 0) \times (0, 2, 0) = (0, 0, 2) \\
\kappa(0) &= \frac{\|\gamma'(0) \times \gamma''(0)\|}{\|\gamma'(0)\|^3} = \frac{\|(0, 0, 2)\|}{\|(1, 0, 0)\|^3} = 2
\end{aligned}$$

A 3. To get the torsion $\tau(0)$ of γ at $\gamma(0)$, we calculate

$$\begin{aligned}
\gamma(t) &= (1 + \cos(t), \sin(t), 2\sin(\tfrac{t}{2})) \\
\gamma'(t) &= (-\sin(t), \cos(t), \cos(\tfrac{t}{2})) \\
\gamma''(t) &= (-\cos(t), -\sin(t), -\tfrac{1}{2}\sin(\tfrac{t}{2})) \\
\gamma'''(t) &= (\sin(t), -\cos(t), -\tfrac{1}{4}\cos(\tfrac{t}{2})) \\
\gamma'(0) \times \gamma''(0) &= (0, 1, 1) \times (-1, 0, 0) = (0, -1, 1) \\
\gamma'''(0) &= (0, -1, -\tfrac{1}{4}) \\
\tau(0) &= \frac{(\gamma'(0) \times \gamma''(0)) \cdot \gamma'''(0)}{\|\gamma'(0) \times \gamma''(0)\|^2} = \frac{\frac{3}{4}}{2} = \frac{3}{8}
\end{aligned}$$

A 4. To get the tangent plane $T_{\sigma(1,1)}\mathcal{S}$ of the surface $\mathcal{S}$ given by $\sigma(u,v) = (u, v, u^3 - v^2)$ at $\sigma(1,1)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (1, 0, 3u^2)\\
\sigma_v(u,v) &= (0, 1, -2v)\\
\sigma_u(1,1) &= (1, 0, 3)\\
\sigma_v(1,1) &= (0, 1, -2)\\
T_{\sigma(1,1)}\mathcal{S} &= \{(x, y, 3x - 2y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}
\end{aligned}$$

A 5. To get the geodesic curvature $\kappa_g(s)$ of $\gamma(s) = (\cos(s), \sin(s), 1)$ on the surface $\sigma(u,v) = (u, v, u^2 + v^2)$, we observe that $\gamma(s) = \sigma(\cos(s), \sin(s))$ and calculate

$$\begin{aligned}
\dot\gamma(s) &= (-\sin(s), \cos(s), 0) \text{ (thus } \gamma \text{ is unit speed)}\\
\ddot\gamma(s) &= (-\cos(s), -\sin(s), 0)\\
\sigma_u(\cos(s), \sin(s)) &= (1, 0, 2\cos(s))\\
\sigma_v(\cos(s), \sin(s)) &= (0, 1, 2\sin(s))\\
(\sigma_u \times \sigma_v)(\cos(s), \sin(s)) &= (-2\cos(s), -2\sin(s), 1)\\
\mathbf{N}(\cos(s), \sin(s)) &= \tfrac{1}{\sqrt{5}}(-2\cos(s), -2\sin(s), 1)\\
\mathbf{N}(\cos(s), \sin(s)) \times \dot\gamma(s) &= \tfrac{1}{\sqrt{5}}(-\cos(s), -\sin(s), -2)\\
\kappa_g(s) &= \ddot\gamma(s) \cdot (\mathbf{N}(\cos(s), \sin(s)) \times \dot\gamma(s)) = \tfrac{1}{\sqrt{5}}
\end{aligned}$$

A 6. To get the first fundamental form ds^2 of the surface $\sigma(u,v) = (u + v, u, u^3 v)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (1, 1, 3u^2 v)\\
\sigma_v(u,v) &= (1, 0, u^3)\\
E(u,v) &= \|\sigma_u(u,v)\|^2 = 1 + 1 + 9u^4v^2 = 2 + 9u^4v^2\\
F(u,v) &= \sigma_u(u,v) \cdot \sigma_v(u,v) = 1 + 3u^5 v\\
G(u,v) &= \|\sigma_v(u,v)\|^2 = 1 + u^6\\
ds^2 &= (2 + 9u^4v^2)du^2 + 2(1 + 3u^5v)dudv + (1 + u^6)dv^2
\end{aligned}$$

A 7. To get the second fundamental form $II(1,-1)$ of $\sigma(u,v) = (v^2, u-v, uv)$ at $\sigma(1,-1)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (0,1,v) \\
\sigma_v(u,v) &= (2v,-1,u) \\
\sigma_u(1,-1)\times\sigma_v(1,-1) &= (0,1,-1)\times(-2,-1,1) = (0,2,2) \\
\mathbf{N}(1,-1) &= \tfrac{1}{\sqrt{8}}(0,2,2) = \tfrac{1}{\sqrt{2}}(0,1,1) \\
\sigma_{uu}(u,v) &= (0,0,0) \\
\sigma_{uv}(u,v) &= (0,0,1) \\
\sigma_{vv}(u,v) &= (2,0,0) \\
L(1,-1) &= \sigma_{uu}(1,-1)\cdot\mathbf{N}(1,-1) = 0 \\
M(1,-1) &= \sigma_{uv}(1,-1)\cdot\mathbf{N}(1,-1) = \tfrac{1}{\sqrt{2}} \\
N(1,-1) &= \sigma_{vv}(1,-1)\cdot\mathbf{N}(1,-1) = 0 \\
II(1,-1) &= 2M(1,-1)dudv = \sqrt{2}dudv
\end{aligned}$$

A 8. To get the principal curvatures of $\mathcal{S}$ at p, we calculate

$$\begin{aligned}
0 &= \det\begin{pmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G\end{pmatrix} = \det\begin{pmatrix} 1-\kappa & -2\kappa \\ -2\kappa & 1-3\kappa\end{pmatrix} \\
&= (1-\kappa)(1-3\kappa)-4\kappa^2 = -\kappa^2-4\kappa+1,
\end{aligned}$$

which has solutions $-2\pm\sqrt{5}$. Thus the principal curvatures of $\mathcal{S}$ at p are $-2-\sqrt{5}$ and $-2+\sqrt{5}$.

A 9. To get the Gaussian curvature $K(0,0)$ of the surface $\sigma(u,v) = (u-v, 2u, u^2+v^2)$ at $\sigma(0,0)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (1,2,2u)\\
\sigma_v(u,v) &= (-1,0,2v)\\
(\sigma_u \times \sigma_v)(0,0) &= (1,2,0) \times (-1,0,0) = (0,0,2)\\
\mathbf{N}(0,0) &= (0,0,1)\\
\sigma_{uu}(u,v) &= (0,0,2)\\
\sigma_{uv}(u,v) &= (0,0,0)\\
\sigma_{vv}(u,v) &= (0,0,2)\\
E(0,0) &= \sigma_u(0,0) \cdot \sigma_u(0,0) = 5\\
F(0,0) &= \sigma_u(0,0) \cdot \sigma_v(0,0) = -1\\
G(0,0) &= \sigma_v(0,0) \cdot \sigma_v(0,0) = 1\\
L(0,0) &= \sigma_{uu}(0,0) \cdot \mathbf{N}(0,0) = 2\\
M(0,0) &= \sigma_{uv}(0,0) \cdot \mathbf{N}(0,0) = 0\\
N(0,0) &= \sigma_{vv}(0,0) \cdot \mathbf{N}(0,0) = 2\\
K(0,0) &= \tfrac{LN-M^2}{EG-F^2}(0,0) = \tfrac{4}{4} = 1
\end{aligned}$$

A 10. To get the mean curvature $H(0,0)$ of the surface $\sigma(u,v) = (u\cos(v), u\sin(v), v)$ at $\sigma(0,0)$, we calculate

$$\begin{aligned}
\sigma_u(u,v) &= (\cos(v), \sin(v), 0)\\
\sigma_v(u,v) &= (-u\sin(v), u\cos(v), 1)\\
(\sigma_u \times \sigma_v)(0,0) &= (1,0,0) \times (0,0,1) = (0,-1,0)\\
\mathbf{N}(0,0) &= (0,-1,0)\\
\sigma_{uu}(u,v) &= (0,0,0)\\
\sigma_{uv}(u,v) &= (-\sin(v), \cos(v), 0)\\
\sigma_{vv}(u,v) &= (-u\cos(v), -u\sin(v), 0)\\
E(0,0) &= \sigma_u(0,0) \cdot \sigma_u(0,0) = 1\\
F(0,0) &= \sigma_u(0,0) \cdot \sigma_v(0,0) = 0\\
G(0,0) &= \sigma_v(0,0) \cdot \sigma_v(0,0) = 1\\
L(0,0) &= \sigma_{uu}(0,0) \cdot \mathbf{N}(0,0) = 0\\
M(0,0) &= \sigma_{uv}(0,0) \cdot \mathbf{N}(0,0) = -1\\
N(0,0) &= \sigma_{vv}(0,0) \cdot \mathbf{N}(0,0) = 0\\
H(0,0) &= \tfrac{LG-2MF+NE}{2(EG-F^2)}(0,0) = \tfrac{0}{2} = 0
\end{aligned}$$

B 11. (i) [7 marks] We have $\gamma(t) = (\cosh(t), \sinh(t), t)$ and $\gamma'(t) = (\sinh(t), \cosh(t), 1)$. Thus

$$\|\gamma'(t)\|^2 = \sinh^2(t) + \cosh^2(t) + 1 = 2\cosh^2(t).$$

The arc length function is

$$s(t) = \int_0^t \|\gamma'(u)\|du = \sqrt{2}\int_0^t \cosh(u)du = \sqrt{2}(\sinh(t) - \sinh(0)) = \sqrt{2}\sinh(t).$$

This implies $t = \sinh^{-1}(\frac{s}{\sqrt{2}})$ and hence

$$\mathbb{R} \to \mathbb{R}^3,\ s \mapsto \Big(\cosh(\sinh^{-1}(\tfrac{s}{\sqrt{2}})), \tfrac{s}{\sqrt{2}}, \sinh^{-1}(\tfrac{s}{\sqrt{2}})\Big)$$

is a unit speed parametrization of γ.

(ii) [9 marks] By assumption, there exists $\theta \in \mathbb{R}$ such that $\frac{\tau}{\kappa} = \cot(\theta)$. Consider the function

$$f(s) = \cos(\theta)\mathbf{t}(s) + \sin(\theta)\mathbf{b}(s).$$

Using the Frenet-Serret equations, we obtain

$$\begin{aligned}\dot{f}(s) &= \cos(\theta)\dot{\mathbf{t}}(s) + \sin(\theta)\dot{\mathbf{b}}(s)\\ &= \cos(\theta)\kappa(s)\mathbf{n}(s) - \sin(\theta)\tau(s)\mathbf{n}(s)\\ &= (\cos(\theta)\kappa(s) - \sin(\theta)\tau(s))\mathbf{n}(s) = 0.\end{aligned}$$

Thus f is constant and hence there exists $a \in \mathbb{R}^3$ so that $f(s) = a$ for all s. Since $\|f(s)\|^2 = \cos^2(\theta) + \sin^2(\theta) = 1$, we have $\|a\| = 1$. Then

$$\begin{aligned}\dot{\gamma}(s)\cdot a = \dot{\gamma}(s)\cdot f(s) &= \dot{\gamma}(s)\cdot(\cos(\theta)\mathbf{t}(s) + \sin(\theta)\mathbf{b}(s))\\ &= \mathbf{t}(s)\cdot(\cos(\theta)\mathbf{t}(s) + \sin(\theta)\mathbf{b}(s)) = \cos(\theta)\end{aligned}$$

is constant.

(iii) [9 marks] We have

$$\rho' = \ddot{\gamma} = \dot{\mathbf{t}} = \kappa\mathbf{n}.$$

Since $\kappa > 0$ everywhere, we see that ρ is a regular curve. Its curvature is therefore given by $\kappa_\rho = \frac{\|\rho'\times\rho''\|}{\|\rho'\|^3}$. We have

$$\rho'' = \kappa'\mathbf{n} + \kappa\dot{\mathbf{n}} = \kappa'\mathbf{n} + \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) = -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}.$$

This gives

$$\rho'\times\rho'' = -\kappa^3\mathbf{n}\times\mathbf{t} + \kappa^2\tau\mathbf{n}\times\mathbf{b} = \kappa^3\mathbf{b} + \kappa^2\tau\mathbf{t} = \kappa^2(\kappa\mathbf{b} + \tau\mathbf{t}).$$

Altogether this gives

$$\kappa_\rho = \frac{\|\rho'\times\rho''\|}{\|\rho'\|^3} = \frac{\kappa^2(\sqrt{\kappa^2+\tau^2})}{\kappa^3} = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}$$

B 12. (i) [4 marks] All four maps are smooth, so we just need to check injectivity. The map σ is injective for (a), (b), (c), but not for (d). Thus (a),(b),(c) define surface patches, but not (d).

(ii) [5 marks] For $\sigma(u,v) = (u, v, uv)$ we have $\sigma_u = (1, 0, v)$ and $\sigma_v = (0, 1, u)$. Then $\sigma_u \times \sigma_v = (-v, -u, 1)$ and hence $\mathbf{N} = \frac{1}{\sqrt{1+u^2+v^2}}(-v, -u, 1)$. The image of the Gauss map of σ is the image of $\mathbf{N}$, which is the upper hemisphere $\{(x, y, z) \in S^2 : z > 0\}$.

(iii) [7 marks] For $\sigma(u,v) = (u, v, u^3 - 3uv^2)$ we have $\sigma_u(u,v) = (1, 0, 3u^2 - 3v^2)$ and $\sigma_v(u,v) = (0, 1, -6uv)$. Then $\sigma_{uu}(u,v) = (0, 0, 6u)$, $\sigma_{uv}(u,v) = (0, 0, -6v)$ and $\sigma_{vv}(u,v) = (0, 0, -6u)$. Thus the second fundamental form is equal to zero for $(u,v) = (0,0)$, which implies that both principal curvatures at $\sigma(0,0)$ are equal to 0. Thus $\sigma(0,0)$ is a planar point.

[2 marks] We have $u^3 - 3uv^2 = u(u^2 - 3v^2) = u(u - \sqrt{3}v)(u + \sqrt{3}v)$. Thus $\sigma(\sqrt{3}t, t) = (\sqrt{3}t, t, 0)$ and $\sigma(-\sqrt{3}t, t) = (-\sqrt{3}t, t, 0)$ are two lines in the surface passing through $\sigma(0,0)$.

[2 marks] A line has curvature $\kappa = 0$. The normal curvature κ_n and geodesic curvature κ_g satisfy $0 = \kappa^2 = \kappa_n^2 + \kappa_g^2$. It follows that $\kappa_n = 0 = \kappa_g$.

(iv) [5 marks] Yes! The unit sphere S^2 (or an open part of it) has constant Gaussian curvature $+1$. The two principal curvatures of S^2 with respect to the standard parametrization using spherical coordinates are both equal to $+1$. If we change the parametrization so that the unit normal changes sign, then the principal curvatures change sign and hence are both equal to -1. Then the mean curvature is equal to -1.

B 13. (i) [5 marks] For $\sigma(u,v) = (u, v, u^2 + v^3)$ we have $\sigma_u(u,v) = (1, 0, 2u)$ and $\sigma_v(u,v) = (0, 1, 3v^2)$. These two vectors span the tangent plane at $\sigma(u,v)$. We have $0 = (1,0,2u) \cdot (2,3,-1) = 2 - 2u$ if and only if $u = 1$ and $0 = (0,1,3v^2) \cdot (2,3,-1) = 3 - 3v^2$ if and only if $v = \pm 1$. Thus $(2,3,-1)$ is perpendicular to the tangent plane at $\sigma(u,v)$ if and only if $(u,v) = (1, \pm 1)$.

(ii) [10 marks] The area of $\sigma(R)$ is $\mathcal{A}_\sigma(R) = \iint_R \sqrt{EG - F^2}\,dudv$. With the given E, F, G we calculate

$$\begin{aligned}
EG &= \left(\frac{1}{u+v} + \frac{1}{(1-u)(1-v)}\right)\left(\frac{1}{u+v} - \frac{1}{(1+u)(1+v)}\right) \\
&= \frac{1}{(u+v)^2} + \frac{(1+u)(1+v) - (1-u)(1-v) - (u+v)}{(u+v)(1-u)(1-v)(1+u)(1+v)} \\
&= \frac{1}{(u+v)^2} + \frac{u+v}{(u+v)(1-u)(1-v)(1+u)(1+v)} \\
&= \frac{1}{(u+v)^2} + \frac{1}{(1-u^2)(1-v^2)}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{A}_\sigma(R) &= \iint_R \sqrt{EG - F^2}\,dudv = \iint_R \sqrt{\frac{1}{(1-u^2)(1-v^2)}}\,dudv \\
&= \left(\int_0^1 \frac{1}{\sqrt{1-u^2}}du\right)\left(\int_0^1 \frac{1}{\sqrt{1-v^2}}dv\right) \\
&= (\arcsin(1) - \arcsin(0))^2 = \frac{\pi^2}{4}.
\end{aligned}$$

(iii) [5 marks] No! Consider for example a plane and a round cylinder, which we know are (locally) isometric to each other. The plane has zero mean curvature, whereas the cylinder has nonzero mean curvature.

(iv) [5 marks] No! The unit sphere has constant Gaussian curvature 1 and the cylinder has constant Gaussian curvature 0. An isometry preserves Gaussian curvature by the Theorem Egregium. It follows that there cannot be a local isometry from the sphere to the cylinder.