Model Theory

COMP SCI 2LC3

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Motivation

- We will use a little bit different notation here!
- What is a theorem?
- What is a proof?
- What is truth?

Foundations

- As usual, everything starts from a language.
- Alphabet: $\{\wedge, \vee, \neg, (,), \forall, \exists, x, \ldots, R_1, \ldots, R_k\}$, where
  $\wedge, \vee, \neg$ - Boolean operations
  $(,)$ - parenthesis
  $\forall, \exists$ - quantifiers
  $x, \ldots$ - variables
  R_i - relations (n-ary)
- Formula - a well formed expression over the alphabet defined above
- $R_i(x_1, \ldots, x_n)$ - atomic formula
  n is and arity of the relational symbol R_i.
  All appearances of R_i must have the same arity.

- Φ is a **formula** iff:
  1. Φ is atomic
  2. $\Phi = \Phi_1 \wedge \Phi_2$, $\Phi = \Phi_1 \vee \Phi_2$, $\Phi = \neg\Phi$,
     where Φ_1, Phi_2 are formulas
  3. $\Phi = \exists[\Phi]$, $\Phi = \forall[\Phi]$
     (we do not need brackets but they improve readability)
- <u>Prenex Normal Form</u>: All quantifiers appera in the front of the formula.
  We assume all our formulas are in prenex normal form.
  It can be proved that every formula has its equivalent prenex normal form.
- <u>Free variable</u>: not bound by any quantifier
- <u>Sentence</u>, <u>statement</u>: no free variables

$R_1(x_1) \wedge R_2(x_1, x_2, x_3)$ x_1, x_2, x_3 - free variables
$\forall x_1 [R_1(x_1) \wedge R_2(x_1, x_2, x_3)]$ x_2, x_3 - free variables
$\forall x_1 \exists x_2 \exists x_3 [R_1(x_1) \wedge R_2(x_1, x_2, x_3)]$ - sentence

Models

- A **model** (**interpretation**, **structure**) is a tuple

$$M = (U, P_1, \dots, P_k),$$

  where U is a **universe** over which the variables may take values,
  P_i is a **relation** assigned to the symbol R_i
- A **language of a model** is the set of all formulas of the model
- If the formula ϕ is **true** in a model M, we say that M is a **model of** ϕ.

Example

$$\phi = \forall x.\forall y.\ R_1(x, y) \vee R_1(y, x)$$

- Model M_1: U - natural numbers, P_1 is $\leq$ (we write $a \leq b$ instead of $\leq(a, b)$ or $P_1(a, b)$)

$$\phi_{M_1} = \forall x.\forall y.\ x \leq y \vee y \leq x$$

  - the formula ϕ_{M_1} is **true** so M_1 is a model of ϕ.

- Model M_2: U - natural numbers, P_1 is $<$ (we write $a < b$ instead of $<(a, b)$ or $P_1(a, b)$)

$$\phi_{M_2} = \forall x.\forall y.\ x < y \vee y < x$$

  - the formula ϕ_{M_2} is **false** so M_2 is **not** a model of ϕ.

Example

$$\phi = \forall y.\exists x.\ R_1(x, x, y)$$

- Let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{I}$ denote the set of all integers, and let $PLUS \subseteq \mathbb{R}^3$ be a **relation** (written as an atomic predicate) defined as:
$$PLUS(a, b, c) = true \iff a + b = c.$$
- Model $M_3 = (\mathbb{R}, PLUS)$

$$\psi_{M_3} = \forall y.\exists x.\ x + x = y$$

  - the formula ϕ_{M_3} is **true** so M_3 is a model of ϕ.
- Model $M_4 = (\mathbb{I}, PLUS)$,

$$\psi_{M_4} = \forall y.\exists x.\ x + x = y$$

  - the formula ϕ_{M_4} is **false** so M_4 is **not** a model of ϕ.