Our Main Connectives

- The formal language of Sentential includes five connectives, which roughly correspond in meaning to certain English sentence connectives:

symbol	what it is called	rough meaning
$\neg$	negation	'It is not the case that ...'
$\wedge$	conjunction	'Both ... and ...'
$\vee$	disjunction	'Either ... or ...'
$\rightarrow$	conditional	'If ... then ...'
$\leftrightarrow$	biconditional	'... if and only if ...'

- They 'correspond' in meaning because they seem to behave like the Englush connectives in their consequences.

Truth Functions and Truth Tables

- Because a truth-function associates a unique truth value which any truth values you offer it, we can conveniently represent a truth-function by a TRUTH TABLE listing its 'inputs' and 'outputs'.
- The negation truth-function we introduced above, which flips the truth value to its opposite, can be represented in this SCHEMATIC truth table – schematic because it concerns any Sentential sentence, rather than some specific ones:

$\mathcal{A}$	$\neg\mathcal{A}$
T	F
F	T

  - For convenience, we have represented True by 'T' and False by 'F'.
  - This table says that, whatever sentence you choose, if its truth value is T, then its negation has the truth value F, and if it is F, then its negation is T.

Conjunction and Disjunction

- The schematic truth tables for conjunction and disjunction are as follows:

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \wedge \mathcal{B}$
T	T	T
T	F	F
F	T	F
F	F	F

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \vee \mathcal{B}$
T	T	T
T	F	T
F	T	T
F	F	F

- Note that both of these are COMMUTATIVE: $\mathcal{A} \wedge \mathcal{B}$ is always the same as $\mathcal{B} \wedge \mathcal{A}$.
  - This is because the truth-table assigns the same value to both of the rows on which we assign $\mathcal{A}$ and $\mathcal{B}$ different truth values. So their order cannot matter.

Conditional

- The conditional, as a truth-function, is unproblematic:

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \to \mathcal{B}$
T	T	T
T	F	F
F	T	T
F	F	T

- It is, correctly, non-commutative – 'if a coin landed heads 1000 times, a surprising thing has happened' is quite different in meaning and truth value from 'if a surprising thing has happened, then a coin landed heads 1000 times'.
- And it is correctly false when the antencedent condition holds, but the consequent isn't true. A case where I work hard and don't get a promotion is enough to make 'if I work hard, I'll get promoted' false.
- We'll return to the conditional later on.

Biconditional

- The biconditional has this associated truth-function:

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \leftrightarrow \mathcal{B}$
T	T	T
T	F	F
F	T	F
F	F	T

- It is obviously commutative.
- In a sense, you can think of the biconditional as saying that the two immediate constituents have the same truth value – it's true if they do, false otherwise.

Another Example

What about a truth table for '$(((P \to Q) \to (Q \to R)) \leftrightarrow (R \to P))$'?

			$\mathcal{A} \to \mathcal{B}$				
					$\mathcal{C} \to \mathcal{D}$		
			$\mathcal{E}$	$\to$	$\mathcal{H}$		
							$\mathcal{I} \to \mathcal{J}$
				$\mathcal{K}$		$\leftrightarrow$	$\mathcal{L}$
P	Q	R	$(((P \to Q)$	$\to$	$(Q \to R))$	$\leftrightarrow$	$(R \to P))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	F	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

Connective-Insensitive Rules: Reiteration

- The rule of Reiteration says, if you've proved something earlier in the range of some assumptions, you can repeat what you've already proved.

Reiteration

$$\begin{array}{r|ll} & \vdots & \\ m & \mathcal{A} & \\ & \vdots & \\ & \mathcal{A} & \quad \text{R}, m \end{array}$$

- CAREFUL: when you reiterate, keep the prior assumptions – i.e., be sure the reiterated sentence is in range of all the assumptions on which the earlier occurrence depended.
- These rules are useful in complex proofs, but ineffectual by themselves.

Conjunction Introduction, Officially

Conjunction Introduction

$$
\begin{array}{r|ll}
m & \mathcal{A} & \\
 & \vdots & \\
n & \mathcal{B} & \\
 & \vdots & \\
 & (\mathcal{A} \wedge \mathcal{B}) & \wedge\mathrm{I},\ m,\ n
\end{array}
\qquad\qquad
\begin{array}{r|ll}
m & \mathcal{A} & \\
 & \vdots & \\
n & \mathcal{B} & \\
 & \vdots & \\
 & (\mathcal{B} \wedge \mathcal{A}) & \wedge\mathrm{I},\ n,\ m
\end{array}
$$

- Note: m needn't be less than n. The rule just says: the first conjunct should be the first line mentioned in the commentary.
- So we could have used this rule to derive $(\mathcal{B} \wedge \mathcal{A})$.
- Note this is a schematic description of our proof rules – many actual examples fit this general template.

Conjunction Elimination, Officially

Conjunction Elimination

m	$(\mathcal{A} \wedge \mathcal{B})$		m	$(\mathcal{A} \wedge \mathcal{B})$	
	$\vdots$			$\vdots$	
	$\mathcal{A}$	$\wedge$E, m		$\mathcal{B}$	$\wedge$E, m

- Note that we do not call them '$\wedge$E-left' and '$\wedge$E-right' or anything – that would be too pedantic, even for this course.
- Worth emphasising: this rule only applies to a sentence with '$\wedge$' as its main connective. All of our connective-specific elimination rules are like this.

Conditional Elimination, Offically

Conditional Elimination

$$
\begin{array}{r|ll}
m & (\mathcal{A} \to \mathcal{B}) & \\
 & \vdots & \\
n & \mathcal{A} & \\
 & \vdots & \\
 & \mathcal{B} & \to\text{E}, m, n
\end{array}
$$

- Again, m needn't be less than n, as long as one is in the range of the assumptions of the other (shared vertical line).
- We cite the line with the conditional on it first in the commentary.

Biconditional Elimination

The biconditional rules behave just as if we had two conditionals, one going 'each way'.

Biconditional Elimination

m	$\mathcal{A} \leftrightarrow \mathcal{B}$	
	$\vdots$	
n	$\mathcal{A}$	
	$\vdots$	
	$\mathcal{B}$	$\leftrightarrow$E, m, n

m	$\mathcal{A} \leftrightarrow \mathcal{B}$	
	$\vdots$	
n	$\mathcal{B}$	
	$\vdots$	
	$\mathcal{A}$	$\leftrightarrow$E, m, n

Disjunction Introduction, Officially

- Given some assumptions, if a disjunct is true, then the disjunction is also true given those same assumptions.
- Accordingly, we have these two versions of our disjunction introduction rule:

Disjunction Introduction

$$\begin{array}{r|ll} m & \mathcal{A} & \\ & \vdots & \\ & (\mathcal{A} \vee \mathcal{B}) & \vee\mathrm{I}, m \end{array} \qquad\qquad \begin{array}{r|ll} m & \mathcal{A} & \\ & \vdots & \\ & (\mathcal{B} \vee \mathcal{A}) & \vee\mathrm{I}, m \end{array}$$

Conditional Introduction, Officially

- We are going to model the informal reasoning by treating the supposition as an additional assumption, and ceasing to make a supposition as discharging it.

Conditional Introduction

-

i	$\quad\mathcal{A}$	
	$\quad\vdots$	
j	$\quad\mathcal{B}$	
	$\vdots$	
	$(\mathcal{A} \to \mathcal{B})$	$\to$I, i–j

Biconditional Introduction

Just as ↔E works like two conditionals, so ↔I requires us to establish two conditionals, one going each way:

Biconditional Introduction

i	$\quad\mid\ \mathcal{A}$	
	$\quad\mid\ \vdots$	
j	$\quad\mid\ \mathcal{B}$	
k	$\quad\mid\ \mathcal{B}$	
	$\quad\mid\ \vdots$	
l	$\quad\mid\ \mathcal{A}$	
	$\mathcal{A} \leftrightarrow \mathcal{B}$	↔I, i–j, k–l

Disjunction Elimination, Officially

Disjunction Elimination

m	$(\mathcal{A} \vee \mathcal{B})$	
i	$\quad \mid \underline{\mathcal{A}}$	
	$\quad \mid \vdots$	
j	$\quad \mid \mathcal{C}$	
k	$\quad \mid \underline{\mathcal{B}}$	
	$\quad \mid \vdots$	
l	$\quad \mid \mathcal{C}$	
	$\mathcal{C}$	$\vee$E, m, i–j, k–l

Negation Introduction, Officially

Negation Introduction

$$
\begin{array}{l|l|ll}
i & & \mathcal{A} & \\
\cline{3-3}
 & & \vdots & \\
j & & \mathcal{B} & \\
 & & \vdots & \\
k & & \neg\mathcal{B} & \\
 & \vdots & & \\
 & \neg\mathcal{A} & & \neg\text{I}, i\text{–}j, i\text{–}k
\end{array}
$$

- There is no need for the line with $\mathcal{B}$ on it to occur before the line with $\neg\mathcal{B}$ on it.

Negation Elimination, Officially

Negation Elimination

$$
\begin{array}{l|l|ll}
i & & \neg\mathcal{A} & \\
\hline
 & & \vdots & \\
j & & \mathcal{B} & \\
 & & \vdots & \\
k & & \neg\mathcal{B} & \\
 & \vdots & & \\
 & \mathcal{A} & & \neg\text{E}, i\text{–}j, i\text{–}k
\end{array}
$$

Talking About Provability

Provability

If there is a proof conforming to our natural deduction rules which ends on a line containing $\mathcal{C}$, such that the assumptions still in effect on that last line are all among $\mathcal{A}_1, \ldots, \mathcal{A}_n$, then we say that $\mathcal{C}$ is PROVABLE FROM $\mathcal{A}_1, \ldots, \mathcal{A}_n$, which we write like this:

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \vdash \mathcal{C}.$$

Suppose we have the schematic proof at right.

$$\begin{array}{r|l}
1 & \mathcal{A}_1 \\ \cline{2-2}
2 & \begin{array}{|l} \mathcal{A}_2 \\ \hline \vdots \\ \\ \mathcal{C} \end{array} \\
 & \\
n &
\end{array}$$

This shows that

$$\mathcal{A}_1, \mathcal{A}_2 \vdash \mathcal{C}.$$

Fact (Deduction)

If $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B} \vdash \mathcal{C}$ then $\mathcal{A}_1, \ldots, \mathcal{A}_n \vdash \mathcal{B} \to \mathcal{C}$.

Proof *If $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B} \vdash \mathcal{C}$, the proof can be converted into a proof of $\mathcal{A}_1, \ldots, \mathcal{A}_n \vdash \mathcal{B} \to \mathcal{C}$, as follows:*

$$\begin{array}{r|l}
1 & \mathcal{A}_1 \\
 & \vdots \\
n & \mathcal{A}_n \\ \cline{2-2}
n+1 & \begin{array}{|l} \mathcal{B} \\ \hline \vdots \\ \\ \end{array} \\
k & \begin{array}{|l} \mathcal{C} \end{array}
\end{array}
\quad \rightrightarrows \quad
\begin{array}{r|ll}
1 & \mathcal{A}_1 & \\
 & \vdots & \\
n & \mathcal{A}_n & \\ \cline{2-2}
n+1 & \begin{array}{|l} \mathcal{B} \\ \hline \vdots \\ \\ \end{array} & \\
k & \begin{array}{|l} \mathcal{C} \end{array} & \\
k+1 & \mathcal{B} \to \mathcal{C} & \to I,\ n+1\text{–}k
\end{array}$$

Formalizing Universal Elimination

- The rule of Universal Elimination closely follows this model.
- Suppose $\mathcal{A}$ is a formula of Quantifier, with no variable occuring free other than x. (It could be something like '$(Fx \vee Gx)$').
- Then $\forall x\mathcal{A}$ is a sentence – e.g., '$\forall x(Fx \vee Gx)$'.
- And for any name c in the language, $\mathcal{A}|_{c\frown x}$ is a sentence.
  - Remember, this notation means $\mathcal{A}$ with c substituted for each free occurence of x. So the example could be something like '$(Fa \vee Ga)$', with 'a' as the chosen name.

Universal Elimination

m	$\forall x\mathcal{A}$	
	$\vdots$	
n	$\mathcal{A}\|_{c\frown x}$	$\forall$E, m

Here, c can be any name – there are no restrictions on whether it already occurs in $\mathcal{A}$ or elsewhere in the proof.

Universal Introduction

$$
\begin{array}{r|ll}
m & \mathcal{A}|_{c \curvearrowright x} & \\
 & \vdots & \\
 & \forall x \mathcal{A} & \forall\text{I}, m
\end{array}
$$

Here, c must not occur in any undischarged assumption when the rule is applied,
nor occur elsewhere in $\mathcal{A}$.

- The requirement that c be a newly introduced name is one way to ensure that its referent is an arbitrary object, since we can have assumed nothing about c already in that case.

Informal Existential Reasoning

- This seems like intuitively good reasoning:

  Suppose Alice is happy.
  Then surely someone is happy.

- This gives us the rule: if for some name 'c', $\mathcal{A}|_{c\curvearrowright x}$ is true, then there is something that is $\mathcal{A}$.

Existential Introduction

$$
\begin{array}{l|ll}
m & \mathcal{A}|_{c\curvearrowright x} & \\
 & \vdots & \\
 & \exists x\mathcal{A} & \exists\mathrm{I},\ m
\end{array}
$$

Here, there is no restriction on the name c.

Our Rule, formally pres

Existential Elimination

$$
\begin{array}{l|l}
m & \exists x \mathcal{A} \\
i & \quad \begin{array}{|l} \mathcal{A}|_{c \curvearrowright x} \\ \hline \vdots \\ \mathcal{B} \end{array} \\
j & \\
 & \vdots \\
 & \mathcal{B} \qquad\qquad\qquad \exists\mathrm{E},\ m,\ i\text{–}j
\end{array}
$$

c must not occur in any assumption still in effect in the subproof from lines i–j

c must not occur anywhere in $\mathcal{A}$

c must not occur anywhere in $\mathcal{B}$