Geometry of Surfaces - Exercises

Exercises marked with $*$ are to be answered (partially) in the online quiz for this week on the Keats page for this module.

48. Use Euler's formula to show that the sum of the normal curvatures for any pair of orthogonal directions is equal to the sum $\kappa_1 + \kappa_2$ of the principal curvatures.

49. Compute the principal curvatures and principal directions of a generalized cylinder given by $\sigma(u, v) = (f(u), g(u), v)$ with $\dot{f}^2 + \dot{g}^2 = 1$.

50.* Let K be the Gaussian curvature and H be the mean curvature of a surface σ. What are the Gaussian curvature $\tilde{K}$ and the mean curvature $\tilde{H}$ of the surface $\tilde{\sigma} = \lambda\sigma$ with $0 \neq \lambda \in \mathbb{R}$?

51. Let $\mathcal{S}$ be a surface and suppose there is a plane that is tangent to $\mathcal{S}$ along a unit speed curve γ. Prove that the Gaussian curvature of $\mathcal{S}$ along γ is ≤ 0.

52.* Let $\mathcal{S}$ be a surface with $O \in \mathcal{S}$ and assume that the coefficients of the first and second fundamental form at O are $E = 1$, $F = 0$, $G = 2$, $L = 1$, $M = 1$ and $N = 1$. Compute the Gaussian curvature, the mean curvature and the principal curvatures of $\mathcal{S}$ at O. Is O an elliptic, hyperbolic or parabolic point?

53.* Can the Gaussian curvature be 100 and the mean curvature be 1 at the same point?

54. Let γ be a unit speed curve on a surface $\mathcal{S}$ with positive Gaussian curvature. Show that the curvature κ of γ satisfies

$$\kappa \geq \min\{|\kappa_1|, |\kappa_2|\}$$

at each point. Deduce that the curvature of a unit speed curve on a sphere of radius R is always $\geq \frac{1}{R}$.

55. A nonzero tangent vector to a surface $\mathcal{S}$ is called asymptotic (or an asymptotic direction) if the normal curvature in that direction is zero. Show that if the mean curvature is zero at a nonplanar point $p \in \mathcal{S}$, then there are two orthogonal asymptotic directions at p.