THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 2 Assessment, 2019

School of Mathematics and Statistics

MAST10006 Calculus 2

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

Common content with: MAST10009 Accelerated Mathematics 2

This paper consists of 6 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- No written or printed materials may be brought into the examination.
- No calculators of any kind may be brought into the examination.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- All questions may be attempted.
- Marks may be awarded for
  - Correct use of appropriate mathematical techniques
  - Accuracy and validity of any calculations or algebraic manipulations
  - Clear justification or explanation of techniques and rules used
  - Clear communication of mathematical ideas through diagrams
  - Use of correct mathematical notation and terminology
- Write your answers in the script book provided. Additional script books are available from the invigilators if required. Any answers you write on this exam paper will not be assessed.
- Start each question on a new page of the script book. Clearly label each page with the number of the question that you are attempting.
- There is a separate 1 page formula sheet provided, which you may use in the examination.
- There are 10 questions with marks as shown. The total number of marks available is 110.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.
- Initially students are to receive the exam paper, the 1 page formula sheet, and two 10 page script books.

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Question 1 (13 marks)

In this question you must state if you use any standard limits, limit laws, continuity, l'Hôpital's rule or the sandwich theorem.

(a) (i) Explain, with reference to continuity theorems, why the function $f(x) = e^{x^2-1}$ is continuous for all $x \in \mathbb{R}$.

(ii) Find the value of $a \in \mathbb{R}$, if any, such that the function

$$g(x) = \begin{cases} e^{x^2-1} & x \leq 1 \\ -x^3 + ax^2 + x - 1 & x > 1 \end{cases}$$

is continuous at $x = 1$. Justify your answer with reference to the definition of continuity.

(b) Calculate the following limit, or explain why it does not exist:

$$\lim_{x\to\infty} e^{\cos(x)} \sin\left(\frac{1}{x}\right)$$

Question 2 (14 marks)

In this question you must state if you use any standard limits, limit laws, continuity, l'Hôpital's rule, the sandwich theorem or convergence tests for series.

(a) Determine if each of the following series converges:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} + 2019^{-n}}{5n + 3\sqrt{n} + 8}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$

(b) Consider the series

$$\sum_{n=1}^{\infty} \sinh\left(\left(\frac{n+a}{n}\right)^n\right)$$

where $a \in \mathbb{R}$. For which value(s) of a does the series converge? Justify your answer mathematically.

Question 3 (6 marks)

(a) Sketch the graphs of $y = \tanh(x)$ and $y = \coth(x)$.

(b) Prove that

$$\operatorname{arccoth}(x) = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$$

(c) For which value(s) of x is the equation in part (b) valid?

Question 4 (12 marks)

(a) Evaluate the integral $\displaystyle\int \arctan(x)\,dx$

(b) Evaluate the integral $\displaystyle\int \sqrt{4x^2-1}\,dx$

(c) Use the complex exponential to evaluate the integral $\displaystyle\int e^{3x}\sin(10x)\,dx$

Question 5 (5 marks) Consider the differential equation

$$\frac{dy}{dx} = \frac{2x\sinh^2(x+y)}{\cosh(x+y)} - 1$$

(a) Make the substitution $u = \sinh(x+y)$ and show that the differential equation above reduces to

$$\frac{du}{dx} = 2u^2x$$

(b) Find the general solution $y(x)$.

Question 6 (13 marks) A lake of volume 6 GL (gigalitres) contains fresh water. A river flows into the lake at a rate of 24 GL per year, mixes uniformly with the lake water, and flows out the other side of the lake at the same rate of 24 GL per year.

An industrial accident upstream of the lake releases a pollutant into the river. Let $x(t)$ be the amount (in tonnes) of pollutant in the lake at time t years after the accident.

(a) Suppose that the river water flowing into the lake contains pollutant at a constant concentration of 0.5 tonnes per GL. Assume that the lake initially contains no pollutant.

(i) Show that $x(t)$ is described by the ordinary differential equation (ODE)

$$\frac{dx}{dt} = 12 - 4x \qquad (t \geq 0)$$

(ii) Find the equilibrium solution(s) of this ODE, and use a graphical technique to determine their stability.

(iii) According to this model, what happens to the amount of pollutant in the lake in the long term?

(b) Suppose that the pollutant concentration of the water flowing into the lake decreases exponentially over time, due to pollution clean-up efforts. Assume that the concentration of pollutant in the river water flowing into the lake at time t years is $0.5e^{-t}$ tonnes per GL. In this case, the ODE describing $x(t)$ is

$$\frac{dx}{dt} = 12e^{-t} - 4x \qquad (t \geq 0)$$

(you do not need to prove this). Assume that the lake initially contains no pollutant.

(i) Solve the ODE to find the amount of pollutant $x(t)$ in the lake at any time t.

(ii) According to this model, what happens to the amount of pollutant in the lake in the long term?

Question 7 (14 marks)

(a) Find the solution of the differential equation

$$y'' - 2y' + y = t^2 - 2t + 1$$

subject to the boundary conditions $y(0) = 0,\ y(1) = 6$.

(b) Find the general solution of the differential equation

$$y'' - 2y' + y = -4\cos^2\left(\frac{t}{2}\right).$$

Question 8 (12 marks) Consider a mass of 10 kilograms attached to a spring hanging vertically from a fixed support. The spring has spring constant $k = 30$ Newtons per metre. When the system is at equilibrium, the spring is stretched a distance of $\frac{10}{3}$ metres. Air resistance acts on the mass with magnitude (in Newtons) proportional to the instantaneous velocity (in metres per second), where the constant of proportionality is $\beta = 20$. Assume that the gravitational constant on Earth is $g = 10$ metres per second per second.

Let $y(t)$ be the displacement (in metres) of the mass below its equilibrium position at time t seconds.

Initially, the mass is released from rest at a distance of 1 metre above its equilibrium position.

(a) Draw a diagram of the system when the mass is below the equilibrium position and moving up. Show all forces acting on the mass and label them with their magnitudes.

(b) Use Newton's 2nd Law to show that the system satisfies the equation of motion:

$$10\ddot{y} + 20\dot{y} + 30y = 0$$

(c) Find the general solution of the equation of motion.

(d) State the initial conditions for the system.

(e) Sketch y versus t. *Hint: it is not necessary to solve for the arbitrary constants in the general solution.*

(f) Suppose that an identical system is set up on the moon, where we assume the gravitational constant is $g = 2$ metres per second per second. Assume that air resistance in the artificial atmosphere is the same as on Earth, and the mass of the object, the spring constant and the initial conditions are unchanged. Describe how the motion of the mass on the moon would compare to its motion on Earth. Explain your answer. Additional calculations are not required.

Question 9 (15 marks) Let f be the function

$$f(x, y) = \log\left(x^2 + y^2\right)$$

and let S be the surface given by $z = f(x, y)$.

(a) State the maximal domain and range of f.

(b) Find an expression for the level curve of this surface when $z = c$.

(c) Sketch the level curve for $c = 0$ and $c = 2$. Label each axis intercept with its value.

(d) Find an expression for the cross section of the surface in the y-z plane, and sketch the cross-section.

(e) Sketch the surface S.

(f) Find the equation of the tangent plane to the surface S at the point where $(x, y) = (2, 2)$.

(g) Find the value(s) of y for which the direction of steepest increase of f at the point $(2, y)$ is parallel to the vector $\mathbf{i} + 2\mathbf{j}$.

Question 10 (6 marks) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^3 + 2x^2 + y^3 - 4y^2 - 4x + 12.$$

Find all stationary points of f, and classify them as either local minima, local maxima, or saddle points.

End of Exam—Total Available Marks = 110

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