Geometry of surfaces - Solutions

48. Let v and w be a pair of orthogonal directions and let $\mathbf{t}_1$ be the principal direction relative to κ_1. Let θ be the angle that v makes with $\mathbf{t}_1$ and denote by $\kappa_n(v)$ the normal curvature given by v. Then, by Euler's formula,

$$\kappa_n(v) = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta)$$

Since v and w are perpendicular, the angle that w makes with $\mathbf{t}_1$ is $\theta \pm \frac{\pi}{2}$. Euler's formula gives

$$\kappa_n(w) = \kappa_1 \cos^2\left(\theta \pm \frac{\pi}{2}\right) + \kappa_2 \sin^2\left(\theta \pm \frac{\pi}{2}\right) = \kappa_1 \sin^2(\theta) + \kappa_2 \cos^2(\theta).$$

Adding both equations leads to $\kappa_n(v) + \kappa_n(w) = \kappa_1 + \kappa_2$.

49. We have $\sigma_u(u,v) = (\dot f(u), \dot g(u), 0)$ and $\sigma_v(u,v) = (0,0,1)$, which determines the unit normal $\mathbf{N}(u,v) = \frac{(\sigma_u \times \sigma_v)(u,v)}{\|(\sigma_u \times \sigma_v)(u,v)\|} = (\dot g(u), -\dot f(u), 0)$. This gives $E(u,v) = \|\sigma_u(u,v)\|^2 = \dot f(u)^2 + \dot g(u)^2 = 1$, $F(u,v) = (\sigma_u \cdot \sigma_v)(u,v) = 0$ and $G(u,v) = \|\sigma_v(u,v)\|^2 = 1$. Hence the first fundamental form is $du^2 + dv^2$.

Next, we have $\sigma_{uu}(u,v) = (\ddot f(u), \ddot g(u), 0)$, $\sigma_{uv}(u,v) = (0,0,0)$ and $\sigma_{vv}(u,v) = (0,0,0)$, which implies $L(u,v) = (\sigma_{uu} \cdot \mathbf{N})(u,v) = (\ddot f \dot g - \ddot g \dot f)(u)$, $M(u,v) = (\sigma_{uv} \cdot \mathbf{N})(u,v) = 0$ and $N(u,v) = (\sigma_{vv} \cdot \mathbf{N})(u,v) = 0$. Hence the second fundamental form is $(\ddot f \dot g - \ddot g \dot f)(u)du^2$.

To find the principal curvatures we have to solve the equation

$$0 = \det(\mathcal{F}_{II} - \kappa \mathcal{F}_I) = \det \begin{pmatrix} (\ddot f \dot g - \ddot g \dot f)(u) - \kappa & 0 \\ 0 & -\kappa \end{pmatrix} = \kappa(\kappa - (\ddot f \dot g - \ddot g \dot f)(u)).$$

Obviously, the solutions are

$$\kappa_1(u,v) = 0 \ , \ \ \kappa_2(u,v) = (\ddot f \dot g - \ddot g \dot f)(u).$$

To find the principal directions, we need to solve the equations

$$\begin{pmatrix} (\ddot f \dot g - \ddot g \dot f)(u) - \kappa_i & 0 \\ 0 & -\kappa_i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For κ_1 we have the obvious solution $(u,v) = (0,1)$, which leads to the principal direction $\sigma_v(u,v) = (0,0,1)$ for κ_1. For κ_2 we have the obvious solution $(u,v) = (1,0)$, which leads to the principal direction $\sigma_u(u,v) = (\dot f(u), \dot g(u), 0)$ for κ_2.

50. In Exercise 47 we saw that the principal curvatures of $\tilde\sigma$ are $\tilde\kappa_1 = \frac{\kappa_1}{\lambda}$ and $\tilde\kappa_2 = \frac{\kappa_2}{\lambda}$. It follows that

$$\tilde K = \tilde\kappa_1 \tilde\kappa_2 = \frac{\kappa_1}{\lambda}\frac{\kappa_2}{\lambda} = \frac{1}{\lambda^2}K \text{ and } \tilde H = \frac{1}{2}(\tilde\kappa_1 + \tilde\kappa_2) = \frac{1}{2}\left(\frac{\kappa_1}{\lambda} + \frac{\kappa_2}{\lambda}\right) = \frac{1}{\lambda}H.$$

51. From Exercise 45 we know that the normal curvature of $\mathcal{S}$ along γ is zero. Assume $\kappa_1 \le \kappa_2$. Since $\kappa_1(p)$ (resp. $\kappa_2(p)$) is the minimum (resp. maximum) value of the normal curvature of all unit speed curves on the surface passing through a point $p \in \mathcal{S}$ we get $\kappa_1 \le 0 \le \kappa_2$ along γ. This implies $K = \kappa_1\kappa_2 \le 0$ along γ.

52. We have

$$K = \frac{LN - M^2}{EG - F^2} = 0 \ , \qquad H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{3}{4} \ ,$$

$$\kappa_1 = H - \sqrt{H^2 - K} = 0 \, , \qquad \kappa_2 = H + \sqrt{H^2 - K} = \frac{3}{2} \, .$$

Since $K = 0$ and $\kappa_2 \neq 0$, O is a parabolic point.

53. No, because $4H^2 = (\kappa_1 + \kappa_2)^2 = \kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2 \geq 2\kappa_1\kappa_2 = 2K$ and therefore $2H^2 - K \geq 0$, which does not hold if $H = 1$ and $K = 100$.

54. Since the Gaussian curvature $K = \kappa_1\kappa_2 > 0$, the principal curvatures κ_1 and κ_2 have the same sign. We can assume that $\kappa_1 \leq \kappa_2$.

Case 1: $\kappa_1, \kappa_2 > 0$. Then $\min\{|\kappa_1|, |\kappa_2|\} = \kappa_1$ and we need to prove that $\kappa \geq \kappa_1$. By Euler's Theorem, κ_1 is the smallest normal curvature and $\kappa_n \geq \kappa_1 > 0$, which implies $\kappa^2 = \kappa_g^2 + \kappa_n^2 \geq \kappa_n^2 \geq \kappa_1^2$ and thus $\kappa \geq \kappa_1$.

Case 2: $\kappa_1, \kappa_2 < 0$. Then $\min\{|\kappa_1|, |\kappa_2|\} = -\kappa_2$ and we need to prove that $\kappa \geq -\kappa_2$. By Euler's Theorem, κ_2 is the largest normal curvature and $\kappa_n \leq \kappa_2 < 0$, which implies $\kappa^2 = \kappa_g^2 + \kappa_n^2 \geq \kappa_n^2 \geq \kappa_2^2$ and thus $\kappa \geq -\kappa_2$.

For a sphere of radius R, the principal curvatures are $\kappa_1 = \kappa_2 = \frac{1}{R}$, and therefore $\kappa \geq \frac{1}{R}$ for every unit speed curve γ on the sphere.

55. Let $\kappa_1 \leq \kappa_2$ be the principal curvatures at p. Since p is a non-planar point, at least one of the two principal curvatures is nonzero. Since the mean curvature at p is zero, we therefore have $\kappa_1 = -\kappa_2 \neq 0$. Then Euler's Formula becomes $\kappa_n = \kappa_1(\cos^2(\theta) - \sin^2(\theta))$. Thus $\kappa_n = 0$ has two solutions (modulo π), namely $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. These two angles determine the two orthogonal asymptotic directions.