Geometry of surfaces - Solutions

9. For each question we give a counterexample or a proof.

(a) True. By definition $\kappa(s) = \|\ddot{\gamma}(s)\| \geq 0$, $\gamma(s)$ unit speed.
(b) False. For example, a circular helix can have negative torsion.
(c) True. $\kappa = 0 \implies \ddot{\gamma}(s) = 0 \implies \gamma(s) = as + b$ for some $a, b \in \mathbb{R}^3$.
(d) False. For example, a circular helix with nonzero pitch has constant curvature but is not (part of) a circle.

10. Put $x = 1 + \cos(2t), y = \sin(2t), z = 2\sin(t)$ and check that $x^2 + y^2 + z^2 = 4$ and $(x-1)^2 + y^2 = 1$. Taking $t = \frac{\pi}{4}$ gives $(1, 1, \sqrt{2}) = \gamma(\frac{\pi}{4}) \in \mathcal{C}$. Next, we calculate

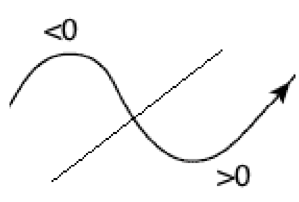
$$\gamma'(t) = (-2\sin(2t), 2\cos(2t), 2\cos(t)),$$
$$\gamma''(t) = (-4\cos(2t), -4\sin(2t), -2\sin(t)),$$

and thus $\gamma'(\frac{\pi}{4}) = (-2, 0, \sqrt{2})$ and $\gamma''(\frac{\pi}{4}) = (0, -4, -\sqrt{2})$. This gives $(\gamma' \times \gamma'')(\frac{\pi}{4}) = (2\sqrt{2}, -\sqrt{2}, 8)$. Using the formula in Proposition 2.1.2, we obtain

$$\kappa((1, 1, \sqrt{2})) = \kappa(\gamma(\pi/4)) = \frac{\|(\gamma' \times \gamma'')(\frac{\pi}{4})\|}{\|\gamma'(\frac{\pi}{4})\|^3} = \frac{1}{3}\sqrt{\frac{13}{3}}.$$

11. If $\|\gamma(t)\|$ has a maximum at t_0, so does $\|\gamma(t)\|^2$. Thus $0 = (\gamma \cdot \gamma)'(t_0) = 2\gamma'(t_0) \cdot \gamma(t_0)$ and $0 \geq (\gamma \cdot \gamma)''(t_0) = 2(\gamma'' \cdot \gamma)(t_0) + 2(\gamma' \cdot \gamma')(t_0) = 2(\gamma'' \cdot \gamma)(t_0) + 2$. Using the Cauchy-Schwarz inequality we obtain $\kappa(t_0)\|\gamma(t_0)\| = \|\gamma''(t_0)\|\|\gamma(t_0)\| \geq |\gamma''(t_0) \cdot \gamma(t_0)| \geq 1$, which proves the claim.

12. The signed curvature is the derivative of the counter-clockwise angle that a fixed vector makes with $\mathbf{t}$. Thus, the signed curvature is > 0 if $\mathbf{t}$ rotates counter-clockwise and < 0 if $\mathbf{t}$ rotates clockwise.



13. We have $\dot{\gamma}(s) = (\cos(\frac{s^5}{5}), \sin(\frac{s^5}{5}))$, which implies that γ is unit speed. Thus the signed curvature κ_s is the derivative of the angle $\varphi(s)$ that $\dot{\gamma}(s)$ makes with a fixed unit vector, say $(1, 0)$. Since, $\dot{\gamma}(s) = (\cos(\frac{s^5}{5}), \sin(\frac{s^5}{5}))$, we get $\varphi(s) = \frac{s^5}{5}$ and hence $\kappa_s(s) = s^4$. By Theorem 2.2.2, $\tilde{\gamma}$ differs from γ by a rigid motion of $\mathbb{R}^2$. Thus, there exists a rotation R_α and a translation by a vector $a \in \mathbb{R}^2$ such that $\tilde{\gamma} = R_\alpha\gamma + a$. Inserting the initial values $\tilde{\gamma}(0) = (1, 2)$ and $\tilde{\mathbf{t}}(0) = (0, 1)$ implies $(1, 2) = \tilde{\gamma}(0) = R_\alpha\gamma(0) + a = a$ and $(0, 1) = \tilde{\mathbf{t}}(0) = R_\alpha\dot{\gamma}(0) = R_\alpha(1, 0)$. Thus $\alpha = \frac{\pi}{2}$ and $a = (1, 2)$.

14. Let $\phi(s) = \int_0^s e^t dt = e^s - 1$. Then the curve

$$\tilde{\gamma}(s) = \left(\int_0^s \cos(e^t - 1)dt, \int_0^s \sin(e^t - 1)dt \right)$$

has signed curvature $\frac{d}{ds}\phi(s) = e^s$. Any other unit speed curve γ with signed curvature e^s is given by $\gamma = R_\alpha \tilde{\gamma} + a$ with some rotation R_α and $a \in \mathbb{R}^2$. From $(0,1) = \mathbf{t}(0) = \dot{\gamma}(0) = R_\alpha \dot{\tilde{\gamma}}(0) = R_\alpha(1,0)$ we get $\alpha = \frac{\pi}{2}$. Therefore $\gamma(s) = R_{\frac{\pi}{2}} \tilde{\gamma}(s) + a$.

15. We compute

$$\begin{aligned}
\mathbf{t}(s) = \dot{\gamma}(s) &= \left(-\frac{1}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right), \frac{1}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right), \frac{2}{\sqrt{5}} \right), \\
\ddot{\gamma}(s) &= \left(-\frac{1}{5} \cos\left(\frac{s}{\sqrt{5}}\right), -\frac{1}{5} \sin\left(\frac{s}{\sqrt{5}}\right), 0 \right), \\
\mathbf{n}(s) = \frac{\ddot{\gamma}(s)}{\|\ddot{\gamma}(s)\|} &= -\left(\cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), 0 \right), \\
\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s) &= \left(\frac{2}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right), -\frac{2}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right), \frac{1}{\sqrt{5}} \right).
\end{aligned}$$

16. Let $\gamma(s)$ be a unit speed curve in $\mathbb{R}^3$, M a rigid motion of $\mathbb{R}^3$ and $\gamma_1 = M\gamma$. Then $\dot{\gamma}_1 = \frac{d}{ds}(M\gamma(s)) = R\dot{\gamma}$. Thus, γ_1 is also unit speed and $\mathbf{t}_1 = R\mathbf{t}$. For the curvature we get $\kappa_1 = \|\dot{\mathbf{t}}_1\| = \|R\dot{\mathbf{t}}\| = \|\dot{\mathbf{t}}\| = \kappa$, which shows that the curvature is invariant under rigid motions. Next, we have $\mathbf{n}_1 = \frac{1}{\kappa_1}\dot{\mathbf{t}}_1 = \frac{1}{\kappa}R\dot{\mathbf{t}} = R(\frac{1}{\kappa}\dot{\mathbf{t}}) = R\mathbf{n}$. Thus $\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = R\mathbf{t} \times R\mathbf{n} = R(\mathbf{t} \times \mathbf{n}) = R\mathbf{b}$ and hence $\dot{\mathbf{b}}_1 = R\dot{\mathbf{b}}$. It follows that $\tau_1 = -\dot{\mathbf{b}}_1 \cdot \mathbf{n}_1 = -R\dot{\mathbf{b}} \cdot R\mathbf{n} = -\dot{\mathbf{b}} \cdot \mathbf{n} = \tau$, which shows that the torsion is invariant under rigid motions.