MATH 465 - INTRODUCTION TO COMBINATORICS
HOMEWORK 3

(1) Compute the Stirling numbers of the second kind $S(7, k)$, for $k = 1, ..., 7$. Show your computations.
(2) Find and prove a closed formula for $\sum_{j=1}^{n} j^4$.
(3) Let $a(n, k)$ denote the number of permutations of length n with 1 and 2 in the same cycle. Prove that for $n \geq 2$,
$$\sum_{k=1}^{n} a(n, k)x^k = x(x + 2)(x + 3) \cdots (x + n - 1).$$
(4) Let $a(n, k)$ be as in Problem 3. Let $t(n, k) = c(n, k) - a(n, k)$ be the number of permutations of length n with k cycles in which the entries 1 and 2 are not in the same cycle. Prove that $a(n, k) = t(n, k + 1)$ for all $k \leq n - 1$.
(5) Prove that
$$S(n + 1, k + 1) = \sum_{m=k}^{n} S(m, k)\binom{n}{m}.$$
(6) Let $I(n, k)$ denote the number of permutations of $[n]$ with k inversions. Prove that $I(n, k) = I(n, \binom{n}{2} - k)$.
(7) Let $I(n, k)$ be as in Problem 6. Find an explicit formula for $I(n, 3), n \geq 3$.
(8) For $n \geq 0$, prove that
$$\sum_{k=1}^{n} c(n, k)x^{n-k} = \prod_{k=1}^{n-1}(1 + kx),$$
where $c(n, k)$ is the signless Stirling number of the first kind.