Geometry of Surfaces

5CCM223A/6CCM223B

Video 4

Curvature of curves

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What is curvature of curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$?

Intuition tells us:

1. lines should have curvature 0
2. curvature of large circle should be smaller than curvature of small circle
3. curvature should not depend on parametrization

Recall: $\ddot{\gamma} = 0 \implies \gamma \subset$ line

First attempt: define curvature as $\|\ddot{\gamma}\|$
Problem: $\|\ddot{\gamma}\|$ depends on parametrization of γ

Second attempt: assume $\|\dot{\gamma}\| = 1$. The curvature of γ at $\gamma(s)$ is

$$\kappa(s) = \|\ddot{\gamma}(s)\|$$

lines should have curvature 0

$$\begin{aligned}\kappa = 0 &\iff \|\ddot{\gamma}\| = 0 \\ &\iff \ddot{\gamma} = 0 \\ &\iff \dot{\gamma} \text{ constant} \\ &\iff \gamma \text{ (part of) straight line}\end{aligned}$$

curvature of large circle should be smaller than curvature of small circle

Consider circle centered at $(x_0, y_0) \in \mathbb{R}^2$ with radius $r > 0$

$$\begin{aligned}
\gamma(s) &= \left(x_0 + r\cos\left(\frac{s}{r}\right), y_0 + r\sin\left(\frac{s}{r}\right)\right) \\
\dot{\gamma}(s) &= \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right) \qquad \text{(thus } \|\dot{\gamma}\| = 1\text{)} \\
\ddot{\gamma}(s) &= \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right) \\
\kappa(s) &= \|\ddot{\gamma}(s)\| = \frac{1}{r}
\end{aligned}$$

curvature should not depend on parametrization

Consider unit speed reparametrization $\bar{\gamma}$ of γ, thus $\bar{\gamma} \circ \phi = \gamma$ with $\phi(t) = \pm s(t) + c$ ($s(t)$ arc length, $c \in \mathbb{R}$). Then

$$\phi'(t) = \pm s'(t) = \pm\|\dot{\gamma}(t)\| = \pm 1$$

Chain Rule implies

$$\begin{aligned} \dot{\gamma}(t) &= \dot{\bar{\gamma}}(\phi(t))\phi'(t) = \pm\dot{\bar{\gamma}}(\phi(t)) \\ \ddot{\gamma}(t) &= \pm\ddot{\bar{\gamma}}(\phi(t))\phi'(t) = \ddot{\bar{\gamma}}(\phi(t)) \\ \|\ddot{\gamma}(t)\| &= \|\ddot{\bar{\gamma}}(\phi(t))\| \end{aligned}$$

Problem. For given regular curve γ, it is often not possible to find an explicit unit speed reparametrization. How to calculate the curvature of such a curve?

Proposition *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n,\ t \mapsto \gamma(t)$ be a regular curve. Then*

$$\frac{\|\gamma''(\gamma' \cdot \gamma') - \gamma'(\gamma' \cdot \gamma'')\|}{\|\gamma'\|^4}(t) \qquad ' = \frac{d}{dt}$$

is the curvature of γ at $\gamma(t)$. For $n = 3$ this simplifies to

$$\frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}(t),$$

where $\times$ is the cross product on $\mathbb{R}^3$.

Proof. Let $\bar{\gamma} : (\bar{\alpha}, \bar{\beta}) \to \mathbb{R}^n,\ s \mapsto \bar{\gamma}(s)$ be a unit speed reparametrization of γ with reparametrization map $\phi : (\alpha, \beta) \to (\bar{\alpha}, \bar{\beta})$. Thus $s = \phi(t)$ and $t = \phi^{-1}(s)$, $\gamma(t) = \bar{\gamma}(\phi(t))$ and $\bar{\gamma}(s) = \gamma(\phi^{-1}(s))$.
We use the conventions $\dot{} = \frac{d}{ds}$ and $' = \frac{d}{dt}$.

$$\dot{\bar{\gamma}}(s) = \gamma'(\phi^{-1}(s))\frac{1}{\phi'(\phi^{-1}(s))} = \gamma'(t)\frac{1}{\phi'(t)}$$

Taking $\| \cdot \|$ implies $|\phi'(t)| = \|\gamma'(t)\|$ and hence

$$\phi'^2 = \|\gamma'\|^2 = \gamma' \cdot \gamma' \qquad (*)$$

Differentiating this equation leads to

$$2\phi'\phi'' = 2\gamma' \cdot \gamma'' \qquad (**)$$

Differentiating $\dot{\tilde{\gamma}}(s) = \gamma'(\phi^{-1}(s))\frac{1}{\phi'(\phi^{-1}(s))}$ gives

$$\begin{aligned}
\ddot{\tilde{\gamma}}(s) &= \gamma''(\phi^{-1}(s))\frac{1}{\phi'(\phi^{-1}(s))^2} + \gamma'(\phi^{-1}(s))\frac{-\phi''(\phi^{-1}(s))\frac{1}{\phi'(\phi^{-1}(s))}}{\phi'(\phi^{-1}(s))^2} \\
&= \gamma''(t)\frac{1}{\phi'(t)^2} - \gamma'(t)\frac{\phi''(t)}{\phi'(t)^3} = \left(\frac{\gamma''}{\phi'^2} - \frac{\gamma'\phi''}{\phi'^3}\right)(t) \\
&= \frac{\gamma''\phi' - \gamma'\phi''}{\phi'^3}(t) = \frac{\gamma''\phi'^2 - \gamma'\phi'\phi''}{\phi'^4}(t) \\
&= \frac{\gamma''(\gamma'\cdot\gamma') - \gamma'(\gamma'\cdot\gamma'')}{\|\gamma'\|^4}(t) \quad \text{(by } (*), (**))
\end{aligned}$$

Thus

$$\kappa(s) = \|\ddot{\tilde{\gamma}}(s)\| = \frac{\|\gamma''(\gamma'\cdot\gamma') - \gamma'(\gamma'\cdot\gamma'')\|}{\|\gamma'\|^4}(t)$$

$n = 3$: For the cross product on $\mathbb{R}^3$ we have

$$\gamma' \times (\gamma'' \times \gamma') = \gamma''(\gamma' \cdot \gamma') - \gamma'(\gamma' \cdot \gamma'')$$

Since $\gamma' \cdot (\gamma'' \times \gamma') = 0$, we have

$$\|\gamma' \times (\gamma'' \times \gamma')\| = \|\gamma'\| \|\gamma'' \times \gamma'\|$$

Altogether,

$$\begin{aligned}\frac{\|\gamma''(\gamma' \cdot \gamma') - \gamma'(\gamma' \cdot \gamma'')\|}{\|\gamma'\|^4} &= \frac{\|\gamma' \times (\gamma'' \times \gamma')\|}{\|\gamma'\|^4} \\ &= \frac{\|\gamma'\| \|\gamma'' \times \gamma'\|}{\|\gamma'\|^4} = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}\end{aligned}$$

Curvature is well-defined at regular points $(\gamma'(t)) \neq 0)$ of curves !!

Example. The circular helix

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \ \theta \mapsto (a\cos(\theta), a\sin(\theta), b\theta)$$

of radius $a > 0$ and pitch $2\pi|b| \neq 0$ has constant curvature

$$\kappa = \frac{|a|}{a^2 + b^2}$$

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Video 5
Signed curvature of plane curves

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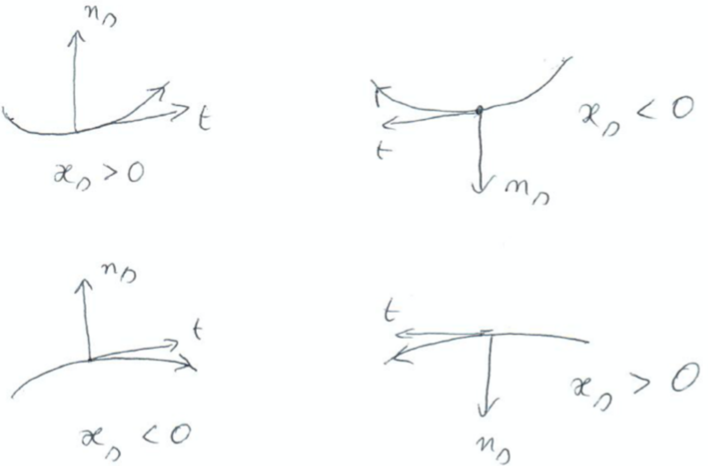
Let γ be a unit speed curve in $\mathbb{R}^2$. We define $\mathbf{t} = \dot{\gamma}$ and denote by $\mathbf{n}_s$ the unit vector obtained by rotating $\mathbf{t}$ anti-clockwise by a right angle.

Since $\|\dot{\gamma}\| = 1$, we have $\dot{\gamma} \cdot \ddot{\gamma} = 0$. Thus $\ddot{\gamma}$ is perpendicular to $\dot{\gamma}$ and hence proportional to $\mathbf{n}_s$. Therefore,

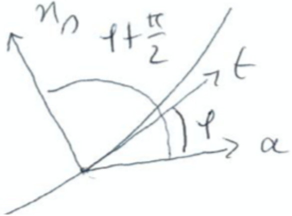
$$\exists \kappa_s : \ddot{\gamma} = \kappa_s \mathbf{n}_s$$

κ_s is called the signed curvature of γ

$$\kappa = \|\ddot{\gamma}\| = \|\kappa_s \mathbf{n}_s\| = |\kappa_s| \|\mathbf{n}_s\| = |\kappa_s|$$


n_n
t
$\varkappa_n > 0$
t
n_n
$\varkappa_n < 0$
n_n
t
$\varkappa_n < 0$
t
n_n
$\varkappa_n > 0$

Proposition Let γ be a unit speed curve in $\mathbb{R}^2$ and $a \in \mathbb{R}^2$ with $\|a\| = 1$. Denote by $\varphi(s)$ the angle through which a must be rotated anti-clockwise to bring it into coincidence with $\mathbf{t}(s)$.



Then

$$\kappa_s = \varphi'$$

Thus, κ_s measures rotation of $\mathbf{t}$ along γ.

Proof. By definition of φ we have

$$\cos(\varphi) = \mathbf{t} \cdot a$$

Differentiating this equation gives

$$-\sin(\varphi)\varphi' = \dot{\mathbf{t}} \cdot a = \ddot{\gamma} \cdot a = \kappa_s \mathbf{n}_s \cdot a = \kappa_s \cos(\varphi + \tfrac{\pi}{2}) = -\kappa_s \sin(\varphi)$$

Rigid motions of $\mathbb{R}^2$. Every rigid motion of $\mathbb{R}^2$ is of the form

$$M = M_{a,\alpha} = T_a \circ R_\alpha$$

where

$$\boxed{T_a} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \ , \ \ v \longmapsto v + a$$

is the translation by $a \in \mathbb{R}^2$ and

$$\boxed{R_\alpha} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \ , \ \ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

is the rotation by angle α

Theorem. *Let* $k : (a, b) \to \mathbb{R}$ *be smooth. Then there exists a unit speed curve* $\gamma : (a, b) \to \mathbb{R}^2$ *with signed curvature* k*. Moreover, if* $\gamma_1 : (a, b) \to \mathbb{R}^2$ *is a unit speed curve with signed curvature* k*, then there exists a rigid motion* M *of* $\mathbb{R}^2$ *so that* $\gamma_1 = M \circ \gamma$*.*

Proof. Existence. Let $s_0 \in (a, b)$ and define

$$\varphi(s) = \int_{s_0}^{s} k(u)du$$

$$\gamma(s) = \left(\int_{s_0}^{s} \cos(\varphi(t))dt, \int_{s_0}^{s} \sin(\varphi(t))dt\right)$$

Then

$$\dot{\gamma}(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$$

and hence $\|\dot{\gamma}\| = 1$. Moreover, by previous proposition,

$$\kappa_s = \varphi'(s) = k(s)$$

Uniqueness. Define $\varphi_1(s)$ by $\dot{\gamma}_1(s) = (\cos(\varphi_1(s)), \sin(\varphi_1(s)))$. Then

$$\gamma_1(s) = \left(\int_{s_0}^{s} \cos(\varphi_1(t))dt, \int_{s_0}^{s} \sin(\varphi_1(t))dt\right) + \gamma_1(s_0)$$

Since $(\varphi_1)' = k$, we have

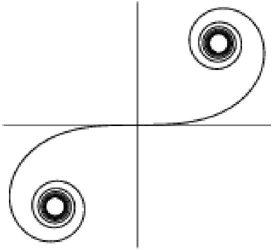
$$\varphi_1(s) = \int_{s_0}^{s} k(u)du + \varphi_1(s_0) = \varphi(s) + \varphi_1(s_0)$$

Define $a = \gamma_1(s_0)$ and $\alpha = \varphi_1(s_0)$. Then

$$\begin{aligned}\gamma_1(s) &= \left(\int_{s_0}^{s} \cos(\varphi(t) + \alpha)dt, \int_{s_0}^{s} \sin(\varphi(t) + \alpha)dt\right) + a \\ &= R_\alpha \underbrace{\left(\int_{s_0}^{s} \cos(\varphi(t))dt, \int_{s_0}^{s} \sin(\varphi(t))dt\right)}_{=\gamma(s)} + a\end{aligned}$$

Example. Every regular plane curve with constant curvature $\kappa > 0$ is part of a circle with radius $\frac{1}{\kappa}$

Example. The curve whose signed curvature is $k(s) = s$ is Cornu's spiral



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Video 6

Torsion of space curves

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Consider

1. circle with radius 1 in $\mathbb{R}^2 \subset \mathbb{R}^3$
2. helix in $\mathbb{R}^3$ with radius $\frac{1}{2}$ and pitch π

Both are curves in $\mathbb{R}^3$ with curvature 1 but cannot be transformed into each other by a rigid motion of $\mathbb{R}^3$

Conclusion: Curvature does not suffice to characterize space curves

New concept: torsion

Torsion measures in how far a curve is not contained in a plane

Let $\gamma(s)$ be a unit speed curve in $\mathbb{R}^3$, $\mathbf{t} = \dot{\gamma}$ tangent vector

If $\kappa(s) \neq 0$, define the principal normal of γ at $\gamma(s)$ by

$$\boxed{\mathbf{n}(s) = \frac{1}{\kappa(s)} \dot{\mathbf{t}}(s)}$$

We have

$$\|\mathbf{n}(s)\| = \left\| \frac{1}{\kappa(s)} \dot{\mathbf{t}}(s) \right\| = \frac{1}{\kappa(s)} \|\dot{\mathbf{t}}(s)\| = \frac{1}{\kappa(s)} \underbrace{\|\ddot{\gamma}(s)\|}_{=\kappa(s)} = 1$$

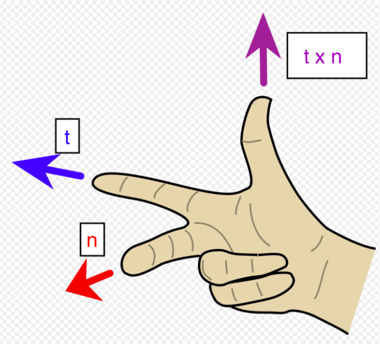
and

$$\mathbf{t} \cdot \mathbf{n} = \frac{1}{\kappa} \underbrace{\mathbf{t} \cdot \dot{\mathbf{t}}}_{=0} = 0 \text{ since } \|\mathbf{t}\| = 1$$

Define binormal of γ at $\gamma(s)$ by

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$$

$\mathbf{t}, \mathbf{n}, \mathbf{b}$ is a right-handed orthonormal basis of $\mathbb{R}^3$



$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \ , \ \mathbf{n} = \mathbf{b} \times \mathbf{t} \ , \ \mathbf{t} = \mathbf{n} \times \mathbf{b}$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \Longrightarrow \dot{\mathbf{b}} = \dot{\mathbf{t}} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}} = \kappa \underbrace{\mathbf{n} \times \mathbf{n}}_{=0} + \mathbf{t} \times \dot{\mathbf{n}} = \mathbf{t} \times \dot{\mathbf{n}}$$

$$\Longrightarrow \dot{\mathbf{b}} \cdot \mathbf{t} = (\mathbf{t} \times \dot{\mathbf{n}}) \cdot \mathbf{t} = 0$$

$$1 = \mathbf{b} \cdot \mathbf{b} \Longrightarrow 0 = \mathbf{b} \cdot \dot{\mathbf{b}}$$

Altogether: $\dot{\mathbf{b}}$ is perpendicular to $\mathbf{t}$ and $\mathbf{b}$. Thus there exists function τ so that

$$\boxed{\dot{\mathbf{b}} = -\tau \mathbf{n}}$$

τ is called the torsion of γ.

Note: torsion $\tau(s)$ is defined when curvature $\kappa(s)$ is non-zero

If $\gamma(t)$ is a regular curve, we define its torsion by using a unit speed reparametrization $\bar{\gamma}$. Is it well-defined? Suppose $\bar{\gamma}$ and $\tilde{\gamma}$ are two unit-speed reparametrizations of γ. Then $\tilde{\gamma}(\phi(t)) = \bar{\gamma}(t)$ with $\phi(t) = \pm s(t) + c$, $s(t)$ arc length of $\bar{\gamma}$. Using $\phi'(t) = \pm s'(t) = \pm\|\dot{\bar{\gamma}}(t)\| = \pm 1$ we obtain

$$\begin{aligned}
\bar{\mathbf{t}}(t) &= \tilde{\mathbf{t}}(\phi(t))\phi'(t) = \pm\tilde{\mathbf{t}}(\phi(t)) \\
\dot{\bar{\mathbf{t}}}(t) &= \pm\dot{\tilde{\mathbf{t}}}(\phi(t))\phi'(t) = \dot{\tilde{\mathbf{t}}}(\phi(t)) \\
\bar{\mathbf{n}}(t) &= \frac{1}{\bar{\kappa}(t)}\dot{\bar{\mathbf{t}}}(t) = \frac{1}{\tilde{\kappa}(\phi(t))}\dot{\tilde{\mathbf{t}}}(\phi(t)) = \tilde{\mathbf{n}}(\phi(t)) \\
\bar{\mathbf{b}}(t) &= \bar{\mathbf{t}}(t) \times \bar{\mathbf{n}}(t) = \pm\tilde{\mathbf{t}}(\phi(t)) \times \tilde{\mathbf{n}}(\phi(t)) = \pm\tilde{\mathbf{b}}(\phi(t)) \\
\dot{\bar{\mathbf{b}}}(t) &= \pm\dot{\tilde{\mathbf{b}}}(\phi(t))\phi'(t) = \dot{\tilde{\mathbf{b}}}(\phi(t))
\end{aligned}$$

Thus

$$\bar{\tau}(t)\bar{\mathbf{n}}(t) = -\dot{\bar{\mathbf{b}}}(t) = -\dot{\tilde{\mathbf{b}}}(\phi(t)) = \tilde{\tau}(\phi(t))\tilde{\mathbf{n}}(\phi(t)) = \tilde{\tau}(\phi(t))\bar{\mathbf{n}}(t)$$

It follows that

$$\bar{\tau}(t) = \tilde{\tau}(\phi(t))$$

and hence the definition of torsion is independent of the choice of unit speed reparametrization

How to compute the torsion of curves in $\mathbb{R}^3$?

Proposition. *Let* γ *be a regular curve in* $\mathbb{R}^3$ *with* $\kappa \neq 0$ *everywhere. Then*

$$\frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}(t) \qquad ' = \frac{d}{dt}$$

is the torsion of γ *at* $\gamma(t)$*.*

Proof. See [Pressley, Proposition 2.3.1]

Example. The circular helix

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \ \theta \mapsto (a\cos(\theta), a\sin(\theta), b\theta)$$

of radius $a > 0$ and pitch $2\pi|b| \neq 0$ has torsion

$$\tau = \frac{b}{a^2 + b^2}$$