10.3 GLOBAL SURFACES

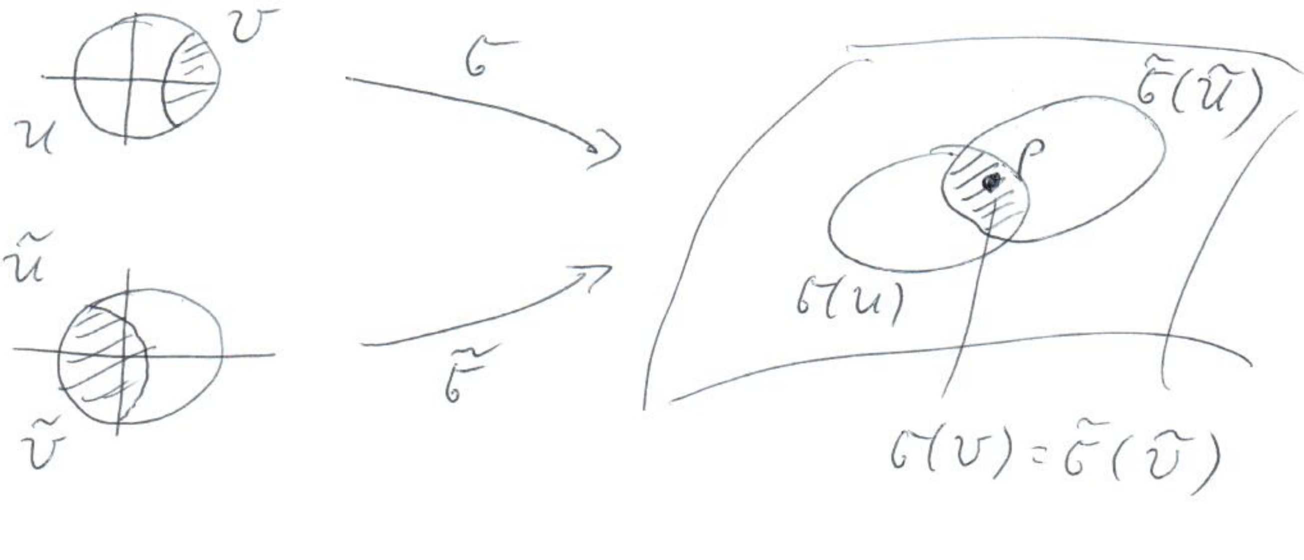
SURFACE ~ UNION OF SURFACE PATCHES

SPHERE ~ UNION OF SIX HEMISPHERES
(CUT WITH PLANES $x=0, y=0, z=0$)

HOW TO GLUE TOGETHER DIFFERENT SURFACES PATCHES ?

<u>DEFINITION 10.3.1</u> $\sigma : \mathcal{U} \to \mathbb{R}^3$, $\tilde{\sigma} : \tilde{\mathcal{U}} \to \mathbb{R}^3$ SURFACE PATCHES. $\sigma, \tilde{\sigma}$ ARE SAID TO BE COMPATIBLE IF $\sigma(\mathcal{U}) \cap \tilde{\sigma}(\tilde{\mathcal{U}}) = \emptyset$ OR $\forall p \in \sigma(\mathcal{U}) \cap \tilde{\sigma}(\tilde{\mathcal{U}})$ $\exists \mathcal{V} \subset \mathcal{U}, \tilde{\mathcal{V}} \subset \tilde{\mathcal{U}}$ OPEN :

(i) $p \in \sigma(\mathcal{V})$;
(ii) $\sigma(\mathcal{V}) = \tilde{\sigma}(\tilde{\mathcal{V}})$;
(iii) $\tilde{\sigma}^{-1} \circ \sigma : \mathcal{V} \to \tilde{\mathcal{V}}$, $\sigma^{-1} \circ \tilde{\sigma} : \tilde{\mathcal{V}} \to \mathcal{V}$ SMOOTH



DEFINITION 10.3.2 $S \subseteq \mathbb{R}^3$. AN ATLAS FOR S IS A COLLECTION OF SURFACE PATCHES $\sigma_i : \mathcal{U}_i \to \mathbb{R}^3$ SUCH THAT

(i) $S = \bigcup_{i \in I} \sigma_i(\mathcal{U}_i)$

(ii) $\forall i, j \in I$: σ_i, σ_j COMPATIBLE

(iii) $\forall p \in S$: $p \in \sigma_i(\mathcal{U}_i) \Rightarrow \exists V_i \subseteq \mathcal{U}_i$ OPEN : $p \in \sigma_i(V_i)$ AND $\sigma_i(V_i) = S \cap \mathcal{W}_i$ WITH SOME OPEN SET $\mathcal{W}_i \subseteq \mathbb{R}^3$.

A GLOBAL SURFACE IS A SUBSET $S \subset \mathbb{R}^3$ TOGETHER WITH AN ATLAS FOR S.

NOTE: (iii) IS INCLUDED TO RULE OUT CERTAIN EXAMPLES SATISFYING (i), (ii) BUT WE DO NOT WANT TO CONSIDER AS SURFACES.

EXAMPLE 10.3.3: $S^2 \subset \mathbb{R}^3$ WITH SIX HEMISPHERE PATCHES.

$$\sigma_1^+(u,v) = \left(\sqrt{1-u^2-v^2},\, u,\, v\right)$$
$$\sigma_1^-(u,v) = \left(-\sqrt{1-u^2-v^2},\, u,\, v\right) \qquad u^2+v^2<1$$
$$\sigma_2^+(u,v) = \left(u,\, \sqrt{1-u^2-v^2},\, v\right)$$
$$\vdots$$

$$\sigma_1^+(u,v) = \sigma_2^+(\tilde{u},\tilde{v})$$
$$\Longleftrightarrow \quad \tilde{u} = \sqrt{1-u^2-v^2}\,,\quad \tilde{v} = v$$
$$(u,v) \longmapsto \left(\sqrt{1-u^2-v^2},\, v\right) \quad \text{SMOOTH} \ldots$$

REMARKS: (1) THE UNION OF 2 ATLASES IS AGAIN AN ATLAS. GIVES EQUIVALENCE RELATION AMONG ATLASES.

(2) HOW MANY NON-EQUIVALENT ATLASES ARE THERE FOR GIVEN S? DIFFICULT QUESTION! COULD BE NONE.

SPECIAL CASE:

THEOREM 10.3.4: S COMPACT GLOBAL SURFACE IN $\mathbb{R}^3$. THEN ANY TWO ATLASES FOR S ARE EQUIVALENT.

REMARK: $S \subset \mathbb{R}^3$

S COMPACT $\Leftrightarrow$ S CLOSED AND BOUNDED

$\uparrow$

HEINE-BOREL THEOREM.

S CLOSED $\Leftrightarrow$ $\mathbb{R}^3 \setminus S$ OPEN

S BOUNDED $\Leftrightarrow$ $\exists R>0$: $S \subset$ BALL OF RADIUS R.

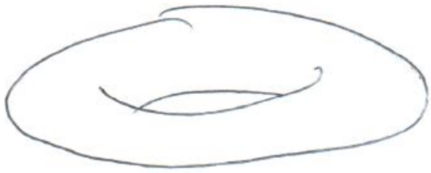
PROOF IS DIFFICULT.

THM 10.3.4 DOES NOT CLARIFY EXISTENCE OF ATLAS ON COMPACT SUBSET OF $\mathbb{R}^3$.

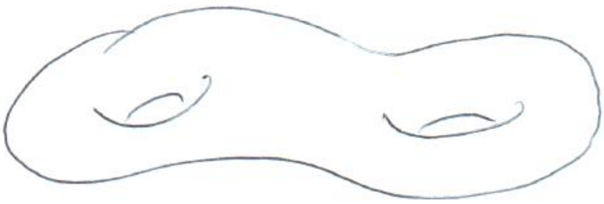
(IN FACT, DOES NOT EXIST IN GENERAL)

EXAMPLES OF COMPACT SURFACES IN $\mathbb{R}^3$:

S^2 SPHERE $\quad T_0$



T^2 TORUS $\quad T_1$



T_2

T_3

$\vdots$

T_g

g HOLES

<u>THEOREM 10.3.5.</u> T_g, $g \geq 0$, CAN BE GIVEN AN ATLAS MAKING IT A GLOBAL SURFACE. EVERY COMPACT GLOBAL SURFACE IS ONE OF T_g, $g \geq 0$.

10.4 GAUSS-BONNET: GLOBAL VERSION

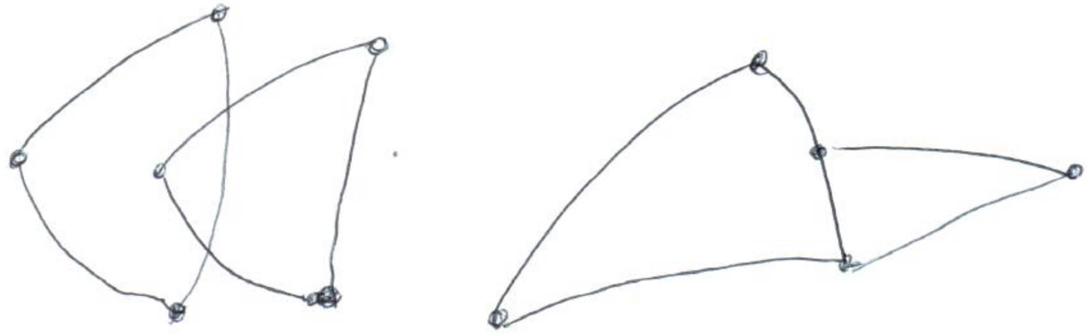
IDEA: COVER COMPACT GLOBAL SURFACE WITH CURVILINEAR POLYGONS, APPLY ABOVE RESULT TO EACH OF THEM AND ADD UP.

DEFINITION 10.4.1: S GLOBAL SURFACE WITH ATLAS $\{\sigma_i : U_i \to \mathbb{R}^3\}$. A TRIANGULATION OF S IS A COLLECTION OF CURVILINEAR POLYGONS, (THE INTERIOR OF) ~~WHICH~~ EACH OF WHICH IS CONTAINED IN ONE OF THE $\sigma_i(U_i)$ SUCH THAT

(i) EVERY POINT OF S IS IN AT LEAST ONE OF THE CURVILINEAR POLYGONS;

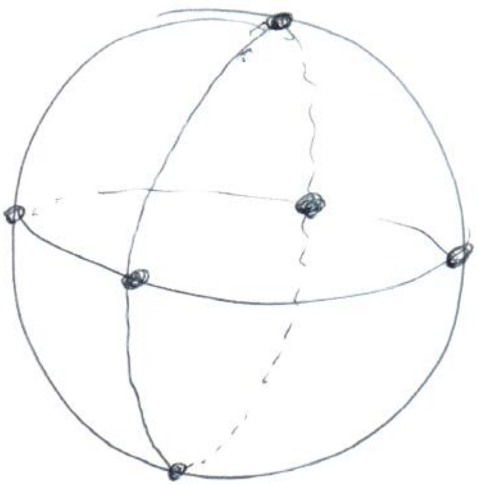
(ii) TWO CURVILINEAR POLYGONS ARE EITHER DISJOINT, OR THEIR INTERSECTION IS A COMMON EDGE OR COMMON VERTEX

(iii) EACH EDGE IS AN EDGE OF EXACTLY TWO POLYGONS.



NOT ALLOWED

A TRIANGULATION OF SPHERE :



8 POLYGONS.
(TRIANGLES)

THEOREM 10.4.2 EVERY COMPACT GLOBAL SURFACE HAS A TRIANGULATION WITH FINITELY MANY POLYGONS.

DEFINITION 10.4.3

THE EULER NUMBER χ OF A TRIANGULATION OF A COMPACT SURFACES IS

$$\chi = V - E + F$$

WHERE

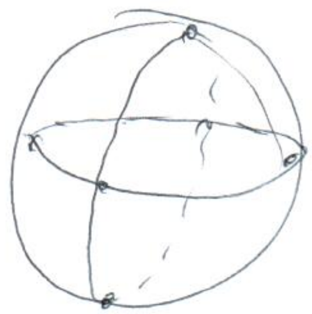
V = TOTAL NUMBER OF VERTICES

E = ——————— EDGES

F = ——————— POLYGONS (FACES)

OF THE TRIANGULATION.

EXAMPLE:



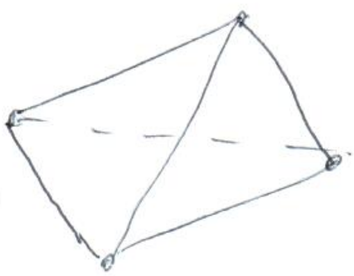
$V = 6$

$E = 12$

$F = 8$

$$\chi(S^2) = 6 - 12 + 8 = 2$$

INFLATE TETRAHEDRON TO GET OTHER TRIANGULATION OF S^2



$V = 4$

$E = 6$

$F = 4$

$$\chi = 4 - 6 + 4 = 2.$$

THIS IS A GENERAL FACT:

<u>THEOREM 10.4.4</u> S COMPACT GLOBAL SURFACE IN $\mathbb{R}^3$. THEN, FOR ANY TRIANGULATION OF S:

$$\iint_S K \, d\mathcal{A} = 2\pi\chi,$$

WHERE χ IS THE EULER NUMBER OF THE TRIANGULATION.

EXPLANATION OF $\iint_S K \, d\mathcal{A}$.

FIX TRIANGULATION OF S WITH POLYGONS P_i

$\forall P_i \ \exists \sigma_i : \mathcal{U}_i \to \mathbb{R}^3$ SURFACE PATCH IN ATLAS OF S ; $P_i = \sigma_i(\mathbf{R}_i)$

FOR SOME $R_i \subseteq \mathcal{U}_i$.

THEN

$$\iint_S K \, d\mathcal{A} = \sum_i \iint_{R_i} K \, d\mathcal{A}_\sigma$$

K GAUSSIAN CURVATURE OF σ_i

NEED TO SHOW THAT THIS IS INDEPENDENT OF

- CHOICE OF SURFACE PATCHES (OR ATLAS)
- CHOICE OF TRIANGULATION

ASSUME $\tilde{\sigma}_i : \tilde{U}_i \to \mathbb{R}^3$ IS COMPATIBLE WITH $\sigma_i : U_i \to \mathbb{R}^3$ AND $P_i = \tilde{\sigma}_i(\tilde{R}_i)$, $\tilde{R}_i \subset \tilde{U}_i$.

THEN

$$\iint_{R_i} K \, dA_{\sigma_i} = \iint_{\tilde{R}_i} K \, dA_{\tilde{\sigma}_i}$$

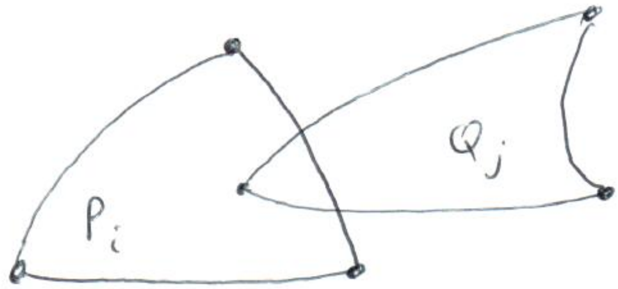
BECAUSE:

AREA UNCHANGED BY REPARAMETRIZATION (5.3.3)

K ——— " ———

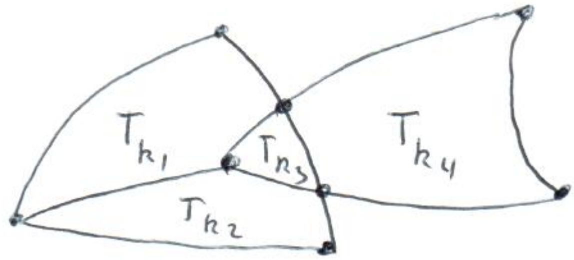
BY THEOREMA EGREGIUM.

NEXT, CONSIDER 2 TRIANGULATIONS $\{P_i\}$, $\{Q_j\}$. CAN FIND TRIANGULATION $\{T_k\}$ SUCH THAT EACH P_i AND EACH Q_j IS THE UNION OF CERTAIN T_k's :



$$P_i = T_{k_1} \cup T_{k_2} \cup T_{k_3}$$

$$Q_j = T_{k_3} \cup T_{k_4}$$



THEN

$$\sum_i \iint_{R_i} K \, dA_{\sigma_i} = \sum_k \iint_{R_k} K \, dA_{\sigma_k}$$

$$= \sum_j \iint_{R_j} K \, dA_{\sigma_j}$$

SINCE

$$\underset{\text{UNION OF POLYGONS}}{\iint \cdots} = \sum \underset{\text{POLYGONS}}{\iint \cdots}$$

DISJOINT OR
INTERSECTING
IN A COMMON
EDGE OR VERTEX

COROLLARY 10.4.5 THE EULER NUMBER χ OF A TRIANGULATION OF A COMPACT GLOBAL SURFACE S DEPENDS ONLY ON S AND NOT ON THE CHOICE OF TRIANGULATION.

PROOF OF THEOREM 10.4.4.:

FIX TRIANGULATION $\{P_i\}$ WITH $\sigma_i : U_i \to \mathbb{R}^3$, $P_i = \sigma_i(R_i)$, $R_i \subset U_i$.

BY 10.2.1:

$$\iint_{R_i} K \, dA_{\sigma_i} = \angle_i - (n_i - 2)\pi - \int_{\gamma_i} \kappa_g \, ds$$

$\angle_i$ = SUM OF INTERIOR ANGLES OF POLYGON

n_i = NUMBER OF VERTICES OF POLYGON P_i

γ_0 = BOUNDARY CURVE OF POLYGON P_i.

NEED TO UNDERSTAND $\sum_i$ OF THESE TERMS.

1) $\sum_i \angle_i$ = SUM OF ALL INTERNAL ANGLES OF POLYGONS.

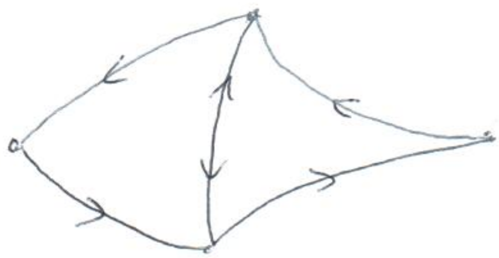
AT ONE VERTEX

SUM OF INTERNAL ANGLES IS 2π

$$\Rightarrow \sum_i \angle_i = 2\pi V$$

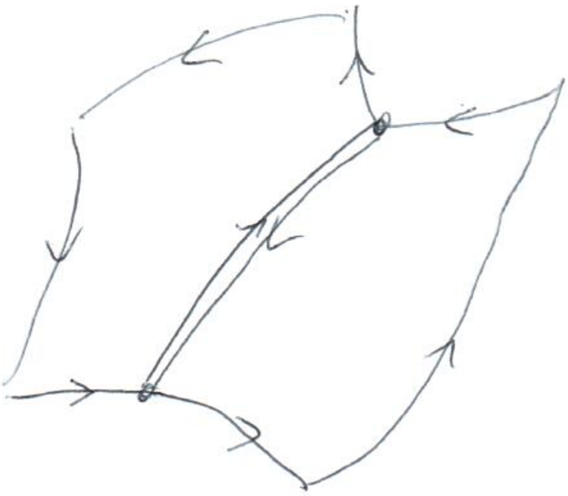
2) $$\sum_i (n_i - 2)\pi = \underbrace{\left(\sum_i n_i\right)}_{=2E} \pi - 2\pi F = 2\pi E - 2\pi F$$

EACH EDGE IS COUNTED TWICE (AS EACH EDGE IS AN EDGE OF EXACTLY 2 POLYGONS).



3) $\sum_i \int_{\gamma_i} \kappa_g \, ds = 0$

INTEGRATE TWICE ALONG EACH EDGE:



κ_g CHANGES SIGN WHEN REVERSING DIRECTION OF CURVE.

THUS CORRESPONDING PAIRS IN $\sum_i \int_{\gamma_i} \kappa_g \, ds$ CANCEL OUT EACH OTHER.

ALTOGETHER:

$$\iint_S K \, dA = \sum_i \iint_{R_i} K \, dA_\sigma$$

$$= \sum_i c_i - \sum_i (n_i - 2)\pi - \sum_i \int_{\gamma_i} \kappa_g \, ds$$

$$= 2\pi V - (2\pi E - 2\pi F) - 0$$

$$= 2\pi \chi .$$

□.

EXAMPLE: UNIT SPHERE S^2.

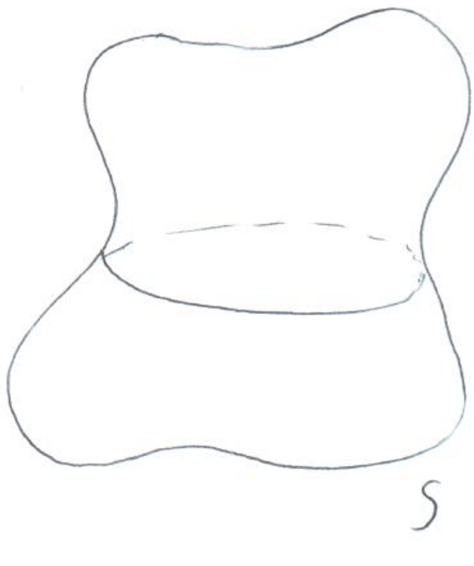
$\Rightarrow \chi = 2$:

THUS $\iint_{S^2} K \, dA = 4\pi$

$K = 1$ (STANDARD METRIC) :

4π = AREA OF SPHERE.

NOW DEFORM SPHERE (WITHOUT TEARING)



K NONCONSTANT

$\iint_S K \, dA = ?$ LOOKS DIFFICULT

TRIANGULATION OF S^2

$\rightsquigarrow$ TRIANGULATION OF DEFORMED SPHERE WITH SAME NUMBER OF VERTICES, EDGES AND POLYGONS.

$\Rightarrow \iint_S K \, dA = 4\pi$ REMARKABLE!

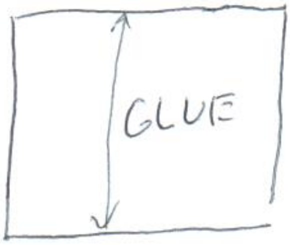
10.4.4.

THEOREM 10.4.6.

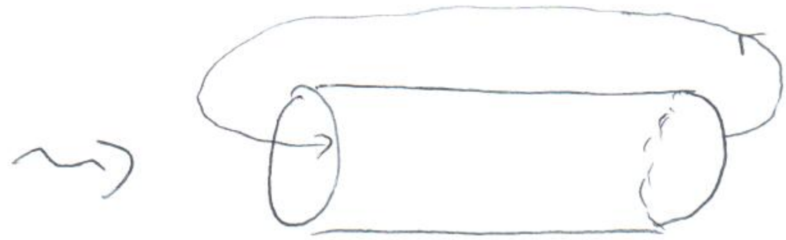
$$\chi(T_g) = 2 - 2g$$

PROOF: $g = 0$ T_0 = SPHERE ✓

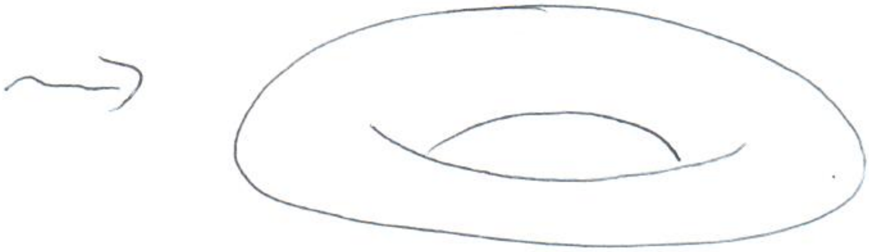
$g = 1$: TORUS



SQUARE



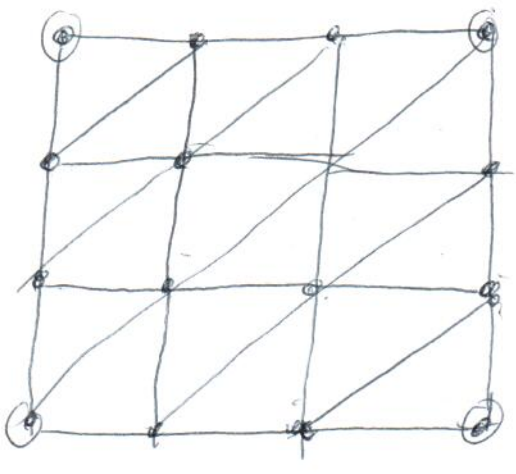
GLUE CYLINDER



TORUS

CONSTRUCT TRIANGULATION:

⊙ IS ONE VERTEX ON TORUS



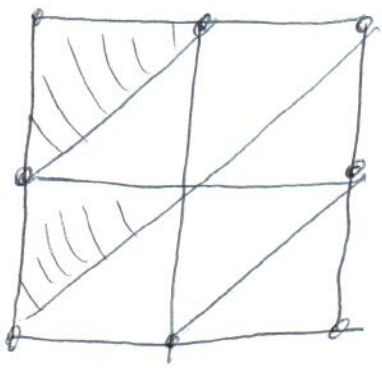
$V = 9$

$E = 27$

$F = 18$

NOTE:

NEED TO BE CAREFUL WITH CHOICE OF TRIANGULATION. FOR EXAMPLE:

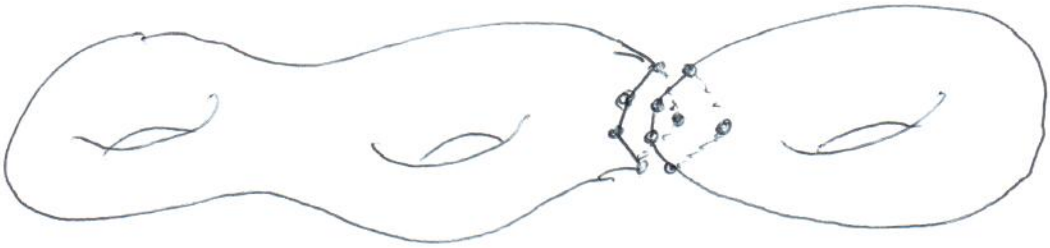


DOES NOT WORK. AFTER GLUING, TWO SHADED TRIANGLES ~~HAVE~~ INTERSECT IN TWO VERTICES!

USING ABOVE TRIANGULATION WE CALCULATE

$$\chi(T_1) = 9 - 27 + 18 = 0 = 2 - 2\cdot 1.$$

COMPLETE PROOF BY INDUCTION ON g: T_{g+1} OBTAINED FROM T_g BY GLUING ON ONE COPY OF T_1



REMOVE CURVILINEAR n-GON FROM T_g AND T_1 AND GLUE CORRESPONDING EDGES.

(AFTER HAVING FIXED SUITABLE TRIANGULATIONS OF T_g AND T_1)

V', E', F' = NUMBER OF VERTICES, EDGES, POLYGONS OF T_g

V'', E'', F'' = ⊢――――――――――――――――――――――― T_1

THEN

$$V = V' - n + V'' - n + n = V' + V'' - n$$

$$E = E' - n + E'' - n + n = E' + E'' - n$$

$$F = F' - 1 + F'' - 1 = F' + F'' - 2.$$

$$\Rightarrow \chi(T_{g+1}) = V - E + F$$
$$= (V' + V'' - n) - (E' + E'' - n) + (F' + F'' - 2)$$
$$= (V' - E' + F') + (V'' - E'' + F'') - 2$$
$$= \chi(T_g) + \chi(T_1) - 2$$
$$= 2 - 2g + 0 - 2 \quad \text{BY INDUCTION HYPOTHESIS}$$
$$= 2 - 2(g+1) \qquad \square$$

COROLLARY 10.4.7

$$\iint_{T_g} K\,dA = 4\pi(1-g)$$