## 1:

Let $A \in \mathbb{R}^{2\times 2}$. Let $z(t)$ be a complex-valued function of $t \in \mathbb{R}$. Let $z(t) = x(t) + iy(t)$, where $x$ and $y$ are real-valued functions, so $x$ is the real part of $z$, and $y$ the complex part of $z$. Show that

$$\frac{dz}{dt} = Az \text{ if and only if } \frac{dx}{dt} = Ax \text{ and } \frac{dy}{dt} = Ay$$

**Solution:**

Substitute $z(t) = x(t) + iy(t)$ into $\frac{dz}{dt} = Az$, we have:

$$\begin{aligned}\frac{dz}{dt} &= \frac{d}{dt}A(x(t) + iy(t)) \\ &= A\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)\end{aligned}$$

Also, we have:

$$\begin{aligned}Az &= A(x(t) + iy(t)) \\ &= Ax(t) + iAy(t)\end{aligned}$$

By comparing the real and imaginary parts of $\frac{dz}{dt}$ and $Az$, we have:

$$\begin{aligned}&\frac{dz}{dt} = Az \\ \iff &\frac{dx}{dt} + i\frac{dy}{dt} = Ax(t) + iAy(t) \\ \iff &\frac{dx}{dt} = Ax(t) \text{ and } \frac{dy}{dt} = Ay(t)\end{aligned}$$

$\square$

## 2:

Let $A \in \mathbb{R}^{2\times 2}$. Prove that $x$ is a real-valued solution of $\frac{dx}{dt} = Ax$ if and only if there is a complex-valued solution $z$, with $x(t) = \mathrm{Re}(z)$.

**Solution:**

$$z(t) = x(t) + ib(t)$$

,

$$Ax + ibA = A(x(t) + ib(t)) = \frac{d(x(t) + ib(t))}{dt} = \frac{dx}{dt} + i\frac{db}{dt}$$

$$Ax = \frac{dx}{dt}$$

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**3:**

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Now the solution looks like

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = c_1 e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Determine the values of $c_1$ and $c_2$ such that the solution is real-valued

$$\begin{aligned} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} &= c_1 e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= e^t \left( c_1(\cos t + i \sin t) \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2(\cos(-t) + i \sin(-t)) \begin{bmatrix} -i \\ 1 \end{bmatrix} \right) \\ &= e^t \left( \begin{bmatrix} ic_1 \cos t + i^2 c_1 \sin t) \\ c_1 \cos t + c_1 i \sin t \end{bmatrix} + \begin{bmatrix} -ic_2 \cos t + c_2 i^2 \sin t \\ c_2 \cos t - ic_2 \sin t \end{bmatrix} \right) \\ &= e^t \left( \begin{bmatrix} -c_1 \sin t - c_2 \sin t \\ c_1 \cos t + c_2 \cos t \end{bmatrix} + i \begin{bmatrix} (c_1 - c_2) \cos t \\ (c_1 - c_2) \sin t \end{bmatrix} \right) \end{aligned}$$

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**4:**

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What are the solutions of $\frac{dx}{dt} = ix$?

**Solution:**

The solution are

$$x = ce^{it} = (a + bi)(\cos t + i \sin t) = (a \cos t - b \sin t) + i(a \sin t + b \cos t)$$

(b) Show that a function is the real part of a solution of $dy/dx = ix$ if and only if it it is linear combination of $\sin t$ and $\cos t$.

**Solution:**

(c)

Find all real solutions.

$x = 0$ is the only possible solution by observing the diff-eq.