MATH 465 - INTRODUCTION TO COMBINATORICS
HOMEWORK 5

(1) Let $C(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating function for the Catalan numbers. Use the recurrence to show that

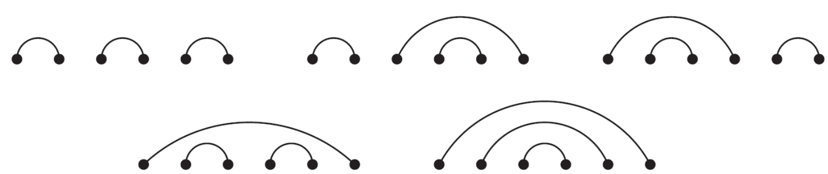
$$1 - C(x) + xC(x)^2 = 0.$$

(2) Prove that the number of sequences $a_1, \ldots, a_{2n}$ such that:
  (a) every $a_i \in \{\pm 1\}$;
  (b) $a_1 + a_2 + \cdots + a_{2n} = 0$;
  (c) every partial sum satisfies $a_1 + \cdots + a_i > -2$

is a Catalan number. For example, when $n = 2$, there are 5 such sequences:

$$11--, \quad 1-1-, \quad 1--1, \quad -11-, \quad -1-1.$$

(3) Show that the number of non-crossing (complete) matchings on $2n$ vertices, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n non-intersecting arcs, each arc connecting two of the points and lying above the points, is C_n. For example when $n = 3$, there are 5 non-crossing matchings.



(4) Prove that the number of $2 \times n$ matrices whose entries are $1, 2, \ldots, 2n$, whose rows increase left-to-right, and whose columns increase top-down, is the Catalan number C_n. For example, for $n = 3$, the matrices are:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

(5) Prove that the number of pairs (α, β) of compositions of n with the same number of parts, such that $\alpha \geq \beta$ in the dominance order (that is, $\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$ for all i) is C_n. For $n = 3$, the pairs are:

$$(111, 111) \quad (12, 12) \quad (3, 3) \quad (21, 12) \quad (21, 21).$$

(6) Show that the number of sequences $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ such that $a_i \leq i$ for all $1 \leq i \leq n$ is C_n. For $n = 3$, there are 5 such sequences:

$$111 \quad 112 \quad 113 \quad 122 \quad 123.$$

(7) Prove that the number of permutations $w_1 \cdots w_n \in S_n$ satisfying the condition

$$(*) \quad \text{there are no indices } i < j < k \text{ for which } w_k < w_i < w_j$$

is a Catalan number. For example, when $n = 3$, all permutations in S_3 satisfy $(*)$ except 231.