COMP 2804B — Assignment 4

Due: Sunday December 3rd, 11:59 pm.

Assignment Policy:

- Your assignment must be submitted as a single .pdf file through Brightspace. Typesetting (using Latex, Word, Google docs, etc) is recommended but not required. Marks will be deducted for illegible or messy solutions. This includes but is not limited to excessive scribbling, shadows, blurry photos, messy handwriting, etc.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: You and your friend decide to play a dice game. You each know a little about probability - for instance, you know that if you roll 2 dice the most likely number to come up is 7. You suggest the following game. You keep rolling the dice until one of the following two events happens:

- If the roll is 3 or 11, your friend wins.
- If a 7 comes up *twice* before a 3 or 11 is rolled, you win.

Each roll is mutually independent.

a) What is the probability that you win and what is the probability that your friend wins?

b) You decide to play for money. Every time a 3 or 11 comes up, you pay your friend \$3. Every time a 7 comes up, he pays you \$2. What is your expected winnings per die roll?

c) How many dice rolls would you expect there to be before a 3, 7, or 11 is rolled?

d) How many dice roll would you expect before someone wins the original game? That is, how many dice rolls would you expect before a 7 is rolled twice or a 3 or 11 is rolled once?

The next question uses a standard deck of cards. A standard deck is a deck of 52 cards. If $R = \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$ is the set of possible ranks and $S = \{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$ are the set of suits, then a deck of 52 cards $D = S \times R = \{(\heartsuit, A), (\diamondsuit, A), (\spadesuit, A), (\clubsuit, A), (\heartsuit, 2), (\diamondsuit, 2), (\spadesuit, 2), (\clubsuit, 2), ..., (\spadesuit, K), (\clubsuit, K)\}$. The ranks J, Q, and K are known as "face cards".

Question 3:
Let D be a standard deck of cards. For a card $c \in D$, let the value of $X(c)$ be equal to its rank if the rank is an integer, 11 if the rank is A and 10 if the rank is in $\{J, Q, K\}$. For example, if $X(c)$ random variable mapping the card to its value, then $X(\heartsuit, A) = 11$, $X(\spadesuit, A) = 11$, $X(\spadesuit, 7) = 7$, $X(\spadesuit, 8) = 8$, $X(\spadesuit, K) = 10$, etc. Assume you are playing Blackjack. You are dealt 2 cards, and the dealer deals themselves 1 card up and 1 card down so you cannot see it (the card that is face down is known as the *hole card*). One strategy in Blackjack is to always play as if the dealer's hole card is worth 10. Let

$$\begin{aligned} V &= \text{The sum of the values of the two cards in your hand.} \\ F &= \text{The value of the dealer's card + 10} \\ Z &= \text{F - V} \end{aligned}$$

a) What is $E(Z)$?

b) Are F and Z independent random variables?

Question 4: You are in a class of 200 people. Let X be the number of different birthdays among these 200 people (assuming no one was born on February 29, i.e., on a leap year). Determine the expected value $E(X)$ of X.

Hint: Use indicator random variables.

Question 5: Consider a fair 6-sided die.

a) Roll the die twice. What is the expected value of the highest number?

b) Roll the die once. If the number is > 3 keep that number. Otherwise roll the die again and keep the highest of the two rolls. What is the expected value?

c) Roll the die once. If the number is > 4 keep that number. Otherwise roll the die again and keep the highest of the two rolls. What is the expected value? What strategy gives us the highest value on average?

Question 6: In this question we will consider bitstrings. A bit is called *lonely* if it is a 1 and every adjacent bit is a 0. A bit is *not lonely* if it is a 1 and it is adjacent to at least one other 1.

a) Consider a random bitstring of length 10. What is the expected number of lonely bits?

b) We choose a bitstring uniformly at random from all bitstrings of length 5 with exactly three 1's. What is the expected number of lonely bits?

c) What is the expected number of bits that are not lonely?

d) We choose a bitstring uniformly at random from all bitstrings of length 10 with exactly four 1's. What is the expected number of lonely and not lonely bits?

Question 7: We have a fair, 6-sided die. We roll this die until the sum of all rolls is ≥ 2. Let X be the number of rolls, and let Y be the sum of all the rolls.

a) What is $E(X)$? Use the formula $E(X) = \sum_{\forall k} k \cdot \Pr(X = k)$.

b) What is $E(Y)$? Use the formula $E(Y) = \sum_{\forall k} k \cdot \Pr(Y = k)$.

c) Let D be the value of a single die roll. We have seen in class that $E(D) = 3.5$. What is $E(X) \cdot E(D)$?

d) This is an example of Wald's Identity. Wald's Identity tells us that if X is the number of die rolls, and the value of X depends on a *stopping condition* (which it does in this case), then the expected sum $E(Y) = E(X) \cdot E(D)$. Find $E(X)$ if X is the number of rolls until the sum is ≥ 3. Then find the corresponding value $E(Y)$ using Wald's Identity.

Question 8: If X is a random variable that can take any value n where n is an integer and $n \geq 1$, and if A is an event, then the *conditional expected value* $E(X|A)$ is defined as

$$E(X|A) = \sum_{k=1}^{\infty} k \cdot \Pr(X = k|A).$$

You roll a fair six-side die repeatedly until you see the number 6. Define the random variable X to be the number of die rolls (including the last roll where you see 6). We have seen in class that $E(X) = 6$. Let A be the event

$A =$ "You do **not** roll 6 on the first two rolls".

Determine the conditional expected value $E(X|A)$.