Geometry of surfaces - Solutions

29. We have $\sigma_u = (1, 0, -\sin(u))$ and $\sigma_v = (0, 1, \cos(v))$. Thus $E(0,0) = \|\sigma_u(0,0)\|^2 = \|(1,0,0)\|^2 = 1$, $F(0,0) = \sigma_u(0,0)\cdot\sigma_v(0,0) = (1,0,0)\cdot(0,1,1) = 0$ and $G(0,0) = \|\sigma_v(0,0)\|^2 = \|(0,1,1)\|^2 = 2$.

30. We have $\sigma_u(u,v) = (1,1,2u)$ and $\sigma_v(u,v) = (-1,1,2v)$. Thus $E(u,v) = \|\sigma_u(u,v)\|^2 = 2+4u^2$, $F(u,v) = \sigma_u(u,v)\cdot\sigma_v(u,v) = 4uv$ and $G(u,v) = \|\sigma_v(u,v)\|^2 = 2+4v^2$. The first fundamental form of σ therefore is $ds^2 = (2+4u^2)du^2 + 8uvdudv + (2+4v^2)dv^2$.

31. We have $\tilde{\sigma}_u = \lambda\sigma_u$ and $\tilde{\sigma}_v = \lambda\sigma_v$. Thus $\tilde{E} = \|\tilde{\sigma}_u\|^2 = \|\lambda\sigma_u\|^2 = \lambda^2\|\sigma_u\|^2 = \lambda^2 E$, $\tilde{G} = \|\tilde{\sigma}_v\|^2 = \|\lambda\sigma_v\|^2 = \lambda^2\|\sigma_v\|^2 = \lambda^2 G$ and $\tilde{F} = \tilde{\sigma}_u\cdot\tilde{\sigma}_v = (\lambda\sigma_u)\cdot(\lambda\sigma_v) = \lambda^2(\sigma_u\cdot\sigma_v) = \lambda^2 F$. Thus the first fundamental form of $\tilde{\sigma}$ is $\lambda^2(Edu^2 + 2Fdudv + Gdv^2)$.

32. Since $\sigma(u,v)^2 = 1$, the image of σ lies in the unit sphere S^2. It is also clear that σ is injective and smooth. We have $\sigma_u = \frac{2}{(1+u^2+v^2)^2}(1-u^2+v^2, -2uv, 2u)$ and $\sigma_v = \frac{2}{(1+u^2+v^2)^2}(-2uv, 1+u^2-v^2, 2v)$. Then $E = \|\sigma_u\|^2 = \frac{4}{(1+u^2+v^2)^2} = \|\sigma_v\|^2 = G$ and $F = \sigma_u\cdot\sigma_v = 0$, which means that σ is a conformal parametrization.

33. We write $\gamma(t) = \sigma(u(t), v(t))$ with $u(t) = t$ and $v(t) = t^2$. The coefficients of the first fundamental form of σ are $E(u,v) = 1$, $F(u,v) = \frac{1}{2}(1-u)$ and $G(u,v) = \frac{3u^2}{4v}$. The length of γ is equal to

$$\begin{aligned}
\text{Length}(\gamma) &= \int_0^1 \sqrt{E(u(t),v(t))\dot{u}(t)^2 + 2F(u(t),v(t))\dot{u}(t)\dot{v}(t) + G(u(t),v(t))\dot{v}(t)^2}dt \\
&= \int_0^1 \sqrt{1 + 2\frac{1}{2}(1-t)2t + \frac{3t^2}{4t^2}4t^2}dt = \int_0^1 \sqrt{1+2t+t^2}dt \\
&= \int_0^1 \sqrt{(1+t)^2}dt = \int_0^1 (1+t)dt = \left(t + \frac{t^2}{2}\right)\Bigg|_{t=0}^{t=1} = \frac{3}{2}\,.
\end{aligned}$$

34. We have to prove three items:

(a) The identity map $A \to A$, $p \mapsto p$ is an isometry.

(b) If A is isometric to B, then there exists an isometry $f : A \to B$. Then $f^{-1} : B \to A$ is a diffeomorphism and it remains to prove that it preserves distances. Let γ_B be a curve in B and let $\gamma_A = f^{-1}(\gamma_B)$. Since f is an isometry, $\text{Length}(\gamma_A) = \text{Length}(f(\gamma_A))$. Since $\text{Length}(f(\gamma_A)) = \text{Length}(f(f^{-1}(\gamma_B))) = \text{Length}(\gamma_B)$, then $\text{Length}(\gamma_A) = \text{Length}(\gamma_B)$ and f^{-1} is an isometry.

(c) Let $f : A \to B$ and $g : B \to C$ be isometries. Since the composition of two diffeomorphisms is a diffeomorphism, $h = g\circ f : A \to C$ is a diffeomorphism. Let γ be a curve in A. Since f is an isometry, we have $\text{Length}(f(\gamma)) = \text{Length}(\gamma)$. Since g is an isometry, we have $\text{Length}(g(f(\gamma))) = \text{Length}(f(\gamma))$. Altogether this implies $\text{Length}(h(\gamma)) = \text{Length}(\gamma)$. Thus h preserves the lengths of curves and hence is an isometry.

35. We have $\sigma_u(u,v) = (\dot{f}(u), \dot{g}(u), 0)$ and $\sigma_v(u,v) = (0,0,1)$. This implies $E(u,v) = \|\sigma_u(u,v)\|^2 = \dot{f}(u)^2 + \dot{g}(u)^2 = 1$, $F(u,v) = \sigma_u(u,v) \cdot \sigma_v(u,v) = 0$ and $G(u,v) = \|\sigma_v(u,v)\|^2 = 1$. Hence the first fundamental form of σ is $du^2 + dv^2$, which is the same as the first fundamental form for the plane in standard coordinates (u,v). The assertion then follows from Theorem 5.2.3.

36. We have $\sigma_u(u,v) = (-\sin(u)v, \cos(u)v, 0)$ and $\sigma_v(u,v) = (\cos(u), \sin(u), 1)$. This implies $E(u,v) = \|\sigma_u(u,v)\|^2 = v^2$, $F(u,v) = \sigma_u(u,v) \cdot \sigma_v(u,v) = 0$ and $G(u,v) = \|\sigma_v(u,v)\|^2 = 2$. Hence the first fundamental form of σ is $v^2du^2 + 2dv^2$.

Now consider the parametrization $\tilde{\sigma}(u,v) = \left(\sqrt{2}\cos\left(\frac{u}{\sqrt{2}}\right)v, \sqrt{2}\sin\left(\frac{u}{\sqrt{2}}\right)v, 0\right)$ of (part of) the plane. Note that this is a slight modification of polar coordinates. Then we have $\tilde{\sigma}_u(u,v) = \left(-\sin\left(\frac{u}{\sqrt{2}}\right)v, \cos\left(\frac{u}{\sqrt{2}}\right)v, 0\right)$ and $\tilde{\sigma}_v(u,v) = \left(\sqrt{2}\cos\left(\frac{u}{\sqrt{2}}\right), \sqrt{2}\sin\left(\frac{u}{\sqrt{2}}\right), 0\right)$. This implies $\tilde{E}(u,v) = \|\tilde{\sigma}_u(u,v)\|^2 = v^2$, $\tilde{F}(u,v) = \tilde{\sigma}_u(u,v) \cdot \tilde{\sigma}_v(u,v) = 0$ and $\tilde{G}(u,v) = \|\tilde{\sigma}_v(u,v)\|^2 = 2$. Hence the first fundamental form of $\tilde{\sigma}$ is $v^2du^2 + 2dv^2$.

Since both surfaces have the same first fundamental form, they are isometric.

37. We have $E(u,v) = u^2v^3 + v^3$, $F(u,v) = v$ and $G(u,v) = \frac{1}{v}$. For the area $\mathcal{A}_\sigma(\mathcal{S})$ we then get

$$\begin{aligned}\mathcal{A}_\sigma(\mathcal{S}) &= \int_0^1 \int_0^1 \sqrt{E(u,v)G(u,v) - F(u,v)^2}dudv = \int_0^1 \int_0^1 \sqrt{(u^2v^3 + v^3)\frac{1}{v} - v^2}dudv \\ &= \int_0^1 \int_0^1 \sqrt{u^2v^2}dudv = \int_0^1 \int_0^1 uvdudv = \frac{1}{2}\int_0^1 vdv = \frac{1}{4}\end{aligned}$$

38. We can parametrize the paraboloid by

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3 \ , \ (u,v) \mapsto (u, v, u^2 + v^2).$$

Put $R = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$. We have $\sigma_u(u,v) = (1,0,2u)$ and $\sigma_v(u,v) = (0,1,2v)$. This implies $E(u,v) = \|\sigma_u(u,v)\|^2 = 1 + 4u^2$, $F(u,v) = \sigma_u(u,v) \cdot \sigma_v(u,v) = 4uv$ and $G(u,v) = \|\sigma_v(u,v)\|^2 = 1 + 4v^2$. For the area we then get

$$\mathcal{A}_\sigma(R) = \iint_R \sqrt{E(u,v)G(u,v) - F(u,v)^2}dudv = \iint_R \sqrt{1 + 4u^2 + 4v^2}dudv.$$