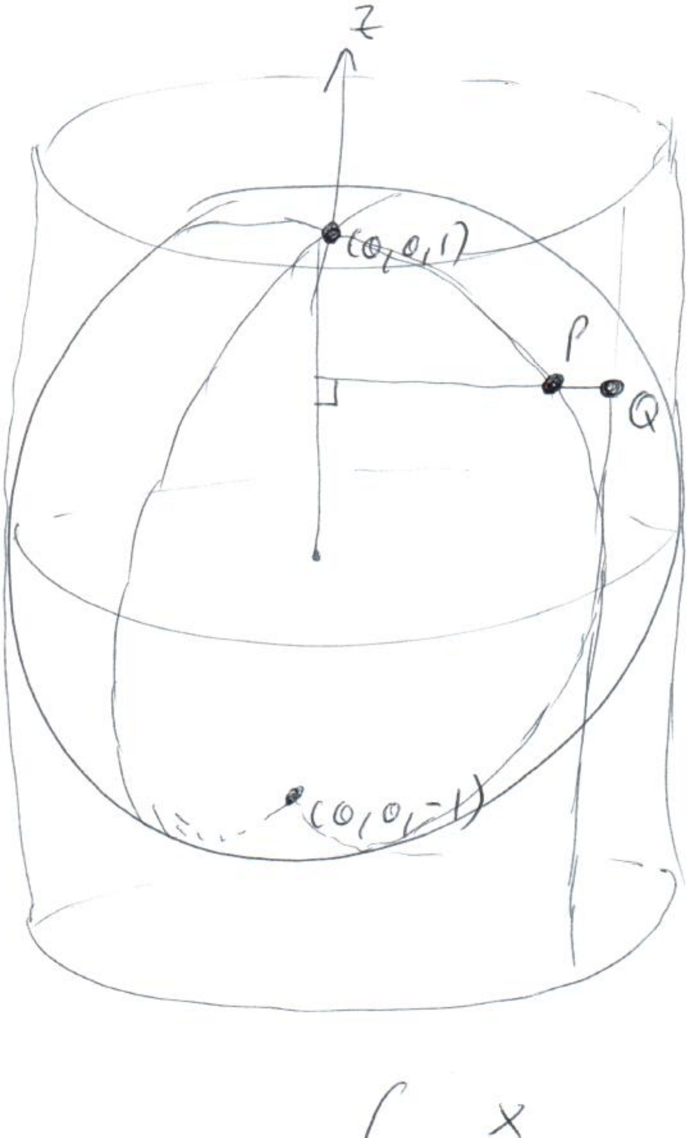
5.4 A THEOREM OF ARCHIMEDES

$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ UNIT SPHERE

$\mathcal{Z} = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$ CYLINDER

S^2 INSIDE $\mathcal{Z}$, TOUCHING AT CIRCLE $\{(x,y,0) : x^2+y^2=1\}$.



$$f : S^2 \setminus \{(0,0,\pm 1)\} \to \mathcal{Z}$$
$$P \mapsto Q$$

$$f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z \right)$$

ARCHIMEDES THM 5.4.1

f IS AREA-PRESERVING.

PROOF PARAMETRIZE S^2 BY

$$\sigma(\theta,\varphi) = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$$

$$\cancel{\theta} \; -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad 0 < \varphi < 2\pi$$

DEFINE

$$\tilde{\sigma}(\theta,\varphi) = f(\sigma(\theta,\varphi)) = (\cos(\varphi), \sin(\varphi), \sin(\theta))$$

PARAMETRIZATION OF Z (CHECK!)

$R \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$ $\quad$ f SMOOTH BY DEF!

CLAIM: $\mathcal{A}_\sigma(R) = \mathcal{A}_{\tilde{\sigma}}(R)$

$$\mathcal{A}_\sigma(R) = \iint_R (EG-F^2)^{\frac{1}{2}}\, d\theta\, d\varphi \qquad \mathcal{A}_{\tilde{\sigma}}(R) = \iint_R (\tilde{E}\tilde{G}-\tilde{F}^2)^{\frac{1}{2}}\, d\theta\, d\varphi \quad (*)$$

EX. 5.1.2: $E = 1$, $F = 0$, $G = \cos^2(\theta)$

$\tilde{\sigma}_\theta = (0, 0, \cos(\theta))$, $\tilde{\sigma}_\varphi = (-\sin(\varphi), \cos(\varphi), 0)$

$\Rightarrow \tilde{E} = \cos^2(\theta)$, $\tilde{F} = 0$, $\tilde{G} = 1$

IMPLIES $(*)$ $\qquad \square$.

NOTE:

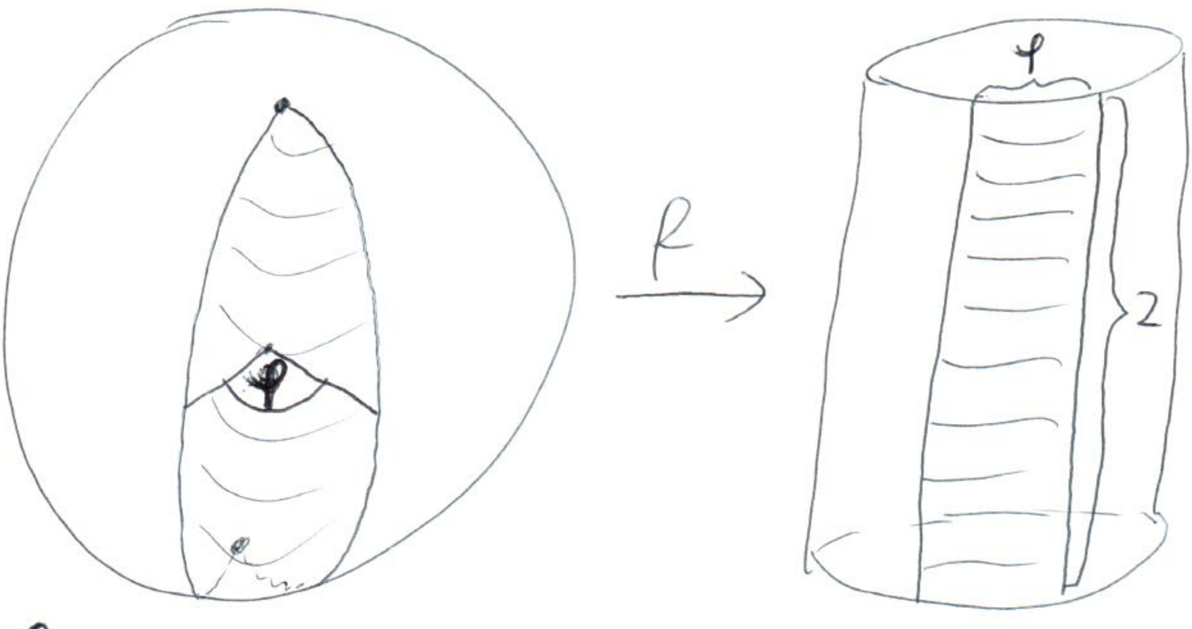
$f: S_1 \to S_2$ ISOMETRY $\Rightarrow$ f AREA-PRESERVING

$\not\Leftarrow$

$\uparrow$

NOT TRUE, ABOVE $f: S^2 \to Z$ IS COUNTEREXAMPLE.

EXAMPLE 5.4.2 (AREA OF LUNE)



$0 < \varphi < 2\pi$

Area of lune $\underset{\uparrow}{=}$ 2φ

ARCHIMEDES THM.

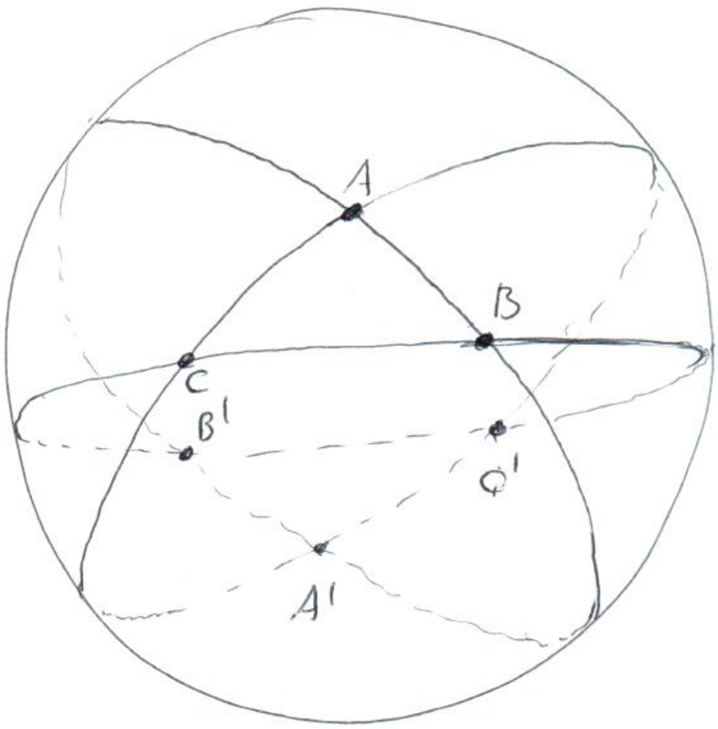
NOTE $\varphi \to 2\pi \Rightarrow$ Area $\to 4\pi = A(S^2)$

THM 5.4.3 LET ΔABC BE TRIANGLE ON S^2 (SIDES ARE ARCS OF GREAT CIRCLES) THEN

$$\mathcal{A}(\Delta ABC) = \angle A + \angle B + \angle C - \pi$$

PROOF

8 TRIANGLES



$$\mathcal{A}(ABC) + \mathcal{A}(A'BC) = 2\angle A$$
$$\mathcal{A}(ABC) + \mathcal{A}(AB'C) = 2\angle B$$
$$\mathcal{A}(ABC) + \mathcal{A}(ABC') = 2\angle C$$

$$2\mathcal{A}(ABC) + \Big(\underbrace{\mathcal{A}(ABC) + \overbrace{\mathcal{A}(A'BC)}^{=\mathcal{A}(AB'C')\ \text{(ANTIPODAL MAP IS ISOMETRY)}} + \mathcal{A}(AB'C) + \mathcal{A}(ABC')}_{=2\pi \text{ AS TRIANGLES FORM A HEMISPHERE}}\Big) = \begin{array}{l} 2\angle A \\ +2\angle B \\ +2\angle C \end{array}$$

$\square$

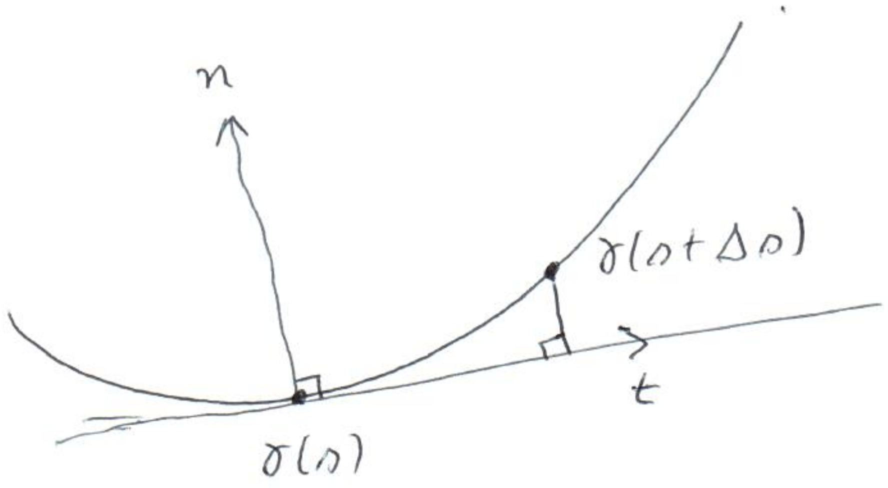
CHAPTER 10: GENERALISE TO

$S^2 \rightsquigarrow$ SURFACE, GREAT CIRCLES $\rightsquigarrow$ CURVES

6 CURVATURE OF SURFACES

6.1 THE SECOND FUNDAMENTAL FORM

MOTIVATION VIA CURVES:



TAYLOR'S THM:

$$\gamma(s+\Delta s) = \gamma(s) + \underset{\substack{\parallel \\ t}}{\dot\gamma(s)}\,\Delta s + \frac{1}{2}\,\underset{\substack{\parallel \\ \dot t = \varkappa n}}{\ddot\gamma(s)}\,(\Delta s)^2 + \cdots$$

$$\underbrace{\big(\gamma(s+\Delta s) - \gamma(s)\big)\cdot n}_{\text{DEVIATION FROM TANGENT LINE}} = \frac{1}{2}\,\varkappa\,(\Delta s)^2 + \cdots$$

GENERALIZE THIS TO SURFACES ...

σ SURFACE IN $\mathbb{R}^3$

$\vec{N}$ UNIT NORMAL

TAYLOR'S THM :

$$\sigma(u+\Delta u, v+\Delta v)$$

$$= \sigma(u,v) + \sigma_u \Delta u + \sigma_v \Delta v$$

$$+ \frac{1}{2}\left(\sigma_{uu}(\Delta u)^2 + 2\sigma_{uv}\Delta u \Delta v + \sigma_{vv}(\Delta v)^2\right) + \cdots$$

$$\Rightarrow \underbrace{\left(\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)\right)\cdot\vec{N}}_{\text{DEVIATION FROM TANGENT PLANE}}$$

$$= \frac{1}{2}\Big(\underbrace{L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2}_{=\text{ SECOND FUNDAMENTAL FORM OF } \sigma}\Big) + \cdots$$

WITH

$$L = \sigma_{uu}\cdot\vec{N}, \quad M = \sigma_{uv}\cdot\vec{N}, \quad N = \sigma_{vv}\cdot\vec{N}$$

$$2^{nd}\text{ FF OF } \sigma \approx \varkappa(\Delta s)^2 \text{ OF } \gamma$$

$$\| $$

$$L\,du^2 + 2M\,du\,dv + N\,dv^2$$

EXAMPLE 6.1.1 (PLANE)

$$\sigma(u,v) = a + up + vq$$

$\sigma_u = p$, $\sigma_v = q$

$\sigma_{uu} = 0$, $\sigma_{uv} = 0$, $\sigma_{vv} = 0$

$\Rightarrow 2^{nd}$ FF $= 0$

EXAMPLE 6.1.2 (SURFACE OF REVOLUTION)

$$\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

$f > 0$, $\dot{f}^2 + \dot{g}^2 = 1$, $(f(u), 0, g(u))$ REGULAR..

$\sigma_u(u,v) = (\dot{f}(u)\cos(v), \dot{f}(u)\sin(v), \dot{g}(u))$

$\sigma_v(u,v) = (-f(u)\sin(v), f(u)\cos(v), 0)$

$(\sigma_u \times \sigma_v)(u,v) =$

$= (-f(u)\dot{g}(u)\cos(v), -f(u)\dot{g}(u)\sin(v), f(u)\dot{f}(u))$

$\|(\sigma_u \times \sigma_v)(u,v)\|^2 = f^2(u)$

$\vec{N}(u,v) = (-\dot{g}(u)\cos(v), -\dot{g}(u)\sin(v), \dot{f}(u))$

$$\sigma_{uu}(u,v) = (\ddot{f}(u)\cos(v), \ddot{f}(u)\sin(v), \ddot{g}(u))$$

$$\sigma_{uv}(u,v) = (-\dot{f}(u)\sin(v), \dot{f}(u)\cos(v), 0)$$

$$\sigma_{vv}(u,v) = (-f(u)\cos(v), -f(u)\sin(v), 0)$$

$$L = \sigma_{uu} \cdot \vec{N} = \dot{f}\ddot{g} - \ddot{f}\dot{g}$$

$$M = \sigma_{uv} \cdot \vec{N} = 0$$

$$N = \sigma_{vv} \cdot \vec{N} = f\dot{g}$$

$\Rightarrow$ 2^{nd} FF IS

$$(\dot{f}\ddot{g} - \ddot{f}\dot{g})(u)\, du^2 + (f\dot{g})(u)\, dv^2$$

SPECIAL CASES: (1) UNIT SPHERE

$$f(u) = \cos(u), \quad g(u) = \sin(u)$$

2^{nd} FF IS $du^2 + \cos^2(u)\, dv^2$

(2) CIRCULAR CYLINDER OF RADIUS 1

$$f(u) = 1, \quad g(u) = u$$

2^{nd} FF IS dv^2

6.2 CURVATURE OF CURVES ON A SURFACE

$\sigma(u,v)$ SURFACE IN $\mathbb{R}^3$

$\gamma(s) = \sigma(u(s), v(s))$ CURVE ON σ WITH $\|\dot\gamma\| = 1$

$$\dot\gamma = \sigma_u \dot u + \sigma_v \dot v$$

$\dot\gamma(s), \vec N(\gamma(s)), \vec N(\gamma(s)) \times \dot\gamma(s)$ ORTHONORMAL

$$\|\dot\gamma\| = 1 \Rightarrow \dot\gamma \cdot \ddot\gamma = 0$$

$$\Rightarrow \ddot\gamma = \varkappa_n \vec N + \varkappa_g (\vec N \times \dot\gamma)$$

$\varkappa_n$ NORMAL CURVATURE OF γ

$\varkappa_g$ GEODESIC CURVATURE OF γ

NOTE: IF γ NOT UNIT SPEED, DEFINE $\varkappa_n, \varkappa_g$ VIA UNIT SPEED REPARAMETRIZATION OF γ.

BY CONSTRUCTION:

$$\varkappa_n = \ddot{\gamma} \cdot \vec{N} \quad , \quad \varkappa_g = \ddot{\gamma} \cdot (\vec{N} \times \dot{\gamma})$$

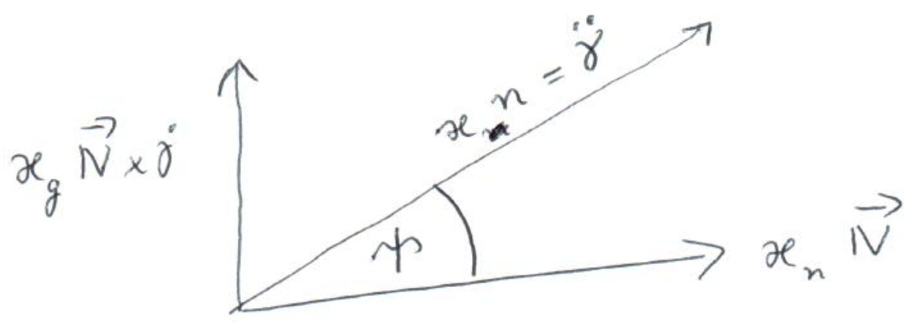
$$\underbrace{\|\ddot{\gamma}\|^2}_{} = \varkappa_n^2 + \varkappa_g^2 \qquad (*)$$

$= \varkappa^2$ CURVATURE OF γ

SINCE $\ddot{\gamma} = \varkappa n$ (n = PRINCIPAL NORMAL OF γ)

$$\varkappa_n = \varkappa n \cdot \vec{N} = \varkappa \cos(\varphi)$$

WITH $\varphi = \measuredangle(n, \vec{N})$



$$\underset{(*)}{\Longrightarrow} \quad \varkappa_g = \pm \varkappa \sin(\varphi)$$

SPECIAL CASE:

γ NORMAL SECTION AT $P \in \sigma$

$\Updownarrow$

γ = INTERSECTION OF σ WITH PLANE π ($P \in \pi$) PERPENDICULAR TO TANGENT PLANE AT P.

$\Rightarrow$ $n \parallel \pi$ (SINCE $\gamma \subset \pi$)

$\vec{N} \parallel \pi$ (BY CONSTRUCTION)

$\underset{n, \vec{N} \perp \dot{\gamma}}{\Longrightarrow}$ $n \parallel \vec{N}$ $\Longrightarrow$ $\varphi \in \{0, \pi\}$

$\Longrightarrow$ $\varkappa_n = \pm \varkappa$, $\varkappa_g = 0$.

STUDY $\varkappa_g$ IN CHAPTER 8.

NOW STUDY $\varkappa_n$.

6.3 THE NORMAL AND PRINCIPAL CURVATURES

PROP 6.3.1 $\gamma(s) = \sigma(u(s), v(s))$, $\|\dot{\gamma}\| = 1$. THEN

$$\varkappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

PROOF

$$\varkappa_n = \vec{N} \cdot \ddot{\gamma} = \vec{N} \cdot \frac{d\dot{\gamma}}{ds}$$

$$= \vec{N} \cdot \frac{d}{ds}\left(\sigma_u \dot{u} + \sigma_v \dot{v}\right)$$

$$= \vec{N} \cdot \Big(\sigma_u \ddot{u} + \sigma_v \ddot{v} + \left(\sigma_{uu}\dot{u} + \sigma_{uv}\dot{v}\right)\dot{u} + \left(\sigma_{uv}\dot{u} + \sigma_{vv}\dot{v}\right)\dot{v}\Big)$$

$$\underset{\substack{\uparrow \\ \sigma_u, \sigma_v \perp \vec{N}}}{=} L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 . \qquad \square$$

$$L = \sigma_{uu} \cdot \vec{N}, \quad M = \sigma_{uv} \cdot \vec{N}, \quad N = \sigma_{vv} \cdot \vec{N}$$

INTRODUCE MATRIX NOTATION:

$$\underset{SS}{\mathcal{F}_I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \underset{SS}{\mathcal{F}_{II}} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$E\,du^2 + 2F\,du\,dv + G\,dv^2 \quad L\,du^2 + 2M\,du\,dv + N\,dv^2$

LET

$$\left.\begin{aligned} t_1 &= \xi_1 \sigma_u + \eta_1 \sigma_v \\ t_2 &= \xi_2 \sigma_u + \eta_2 \sigma_v \end{aligned}\right\} \text{TANGENT VECTORS AT } P$$

THEN

$$t_1 \cdot t_2 =$$

$$= \xi_1 \xi_2 \underbrace{\sigma_u \cdot \sigma_u}_{E} + (\xi_1 \eta_2 + \xi_2 \eta_1) \underbrace{\sigma_u \cdot \sigma_v}_{F} + \eta_1 \eta_2 \underbrace{\sigma_v \cdot \sigma_v}_{G}$$

$$= (\xi_1 \ \eta_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

$$= T_1^T \mathcal{F}_I T_2 \qquad T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \approx t_i$$

FOR CURVE $\gamma(s) = \sigma(u(s), v(s))$,

$$\dot{\gamma} = \xi \sigma_u + \eta \sigma_v$$

$$\kappa_n = T^T \mathcal{F}_{II} T \qquad T = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

BY SIMILAR CALCULATION & PROP 6.3.1

DEF 6.3.2 THE PRINCIPAL CURVATURES OF A SURFACE ARE THE ROOTS OF

$$\det(\mathcal{F}_{II} - \varkappa \mathcal{F}_{I}) = 0$$

$$\parallel$$

$$\det \begin{pmatrix} L - \varkappa E & M - \varkappa F \\ M - \varkappa F & N - \varkappa G \end{pmatrix}$$

QUADRATIC EQUATION

$\rightsquigarrow$ 2 PRINCIPAL CURVATURES $\varkappa_1, \varkappa_2$ ($\varkappa_1 = \varkappa_2$ POSSIBLE)

DEF 6.3.3. IF $0 \neq T = \binom{\xi}{\eta}$ SATISFIES

$$(\mathcal{F}_{II} - \varkappa \mathcal{F}_{I}) T = 0$$

THEN

$$\mathbf{t} = \xi \sigma_u + \eta \sigma_v$$

IS CALLED A PRINCIPAL VECTOR OF σ CORRESPONDING TO THE PRINCIPAL CURVATURE $\varkappa$.

PROP 6.3.4 LET $\varkappa_1, \varkappa_2$ BE THE PRINCIPAL CURVATURES OF σ AT $P \in \sigma$.

(a) IF $\varkappa_1 \neq \varkappa_2$, THEN PRINCIPAL VECTORS t_1, t_2 CORRESPONDING TO $\varkappa_1, \varkappa_2$ ARE PERPENDICULAR.

(b) IF $\varkappa_1 = \varkappa_2$, THEN EVERY (NONZERO) TANGENT VECTOR OF σ AT P IS A PRINCIPAL VECTOR.

PROOF: (a) WRITE

$$t_i = \xi_i \sigma_u + \eta_i \sigma_v \quad , \quad T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$$

$$\Rightarrow t_1 \cdot t_2 = T_1^T \mathcal{F}_I T_2$$

BY ASSUMPTION

$$\mathcal{F}_{II} T_i = \varkappa_i \mathcal{F}_I T_i$$

$$\Rightarrow T_2^T \mathcal{F}_{II} T_1 = \varkappa_1 T_2^T \mathcal{F}_I T_1 = \varkappa_1 (t_1 \cdot t_2)$$

$$T_1^T \mathcal{F}_{II} T_2 = \varkappa_2 T_1^T \mathcal{F}_I T_2 = \varkappa_2 (t_1 \cdot t_2)$$

$$\parallel$$

$$(T_1^T \mathcal{F}_{II} T_2)^T = T_2^T \mathcal{F}_{II}^T (T_1^T)^T = T_2^T \mathcal{F}_{II} T_1 = \varkappa_1 (t_1 \cdot t_2)$$

$$\underset{\varkappa_1 \neq \varkappa_2}{\Longrightarrow} \quad t_1 \cdot t_2 = 0 .$$

(b) $t_i = \xi_i \sigma_u + \eta_i \sigma_v$ ANY TWO UNIT TANGENT VECTORS. WITH $t_1 \cdot t_2 = 0$

$T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$.

PUT $A = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} T_1 & T_2 \end{pmatrix}$

$$A^T \mathcal{F}_I A = \begin{pmatrix} T_1^T \mathcal{F}_I T_1 & T_1^T \mathcal{F}_I T_2 \\ T_2^T \mathcal{F}_I T_1 & T_2^T \mathcal{F}_I T_2 \end{pmatrix}$$

$$= \begin{pmatrix} t_1 \cdot t_1 & t_1 \cdot t_2 \\ t_2 \cdot t_1 & t_2 \cdot t_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

DEFINE $\mathcal{G}_{II} = A^T \mathcal{F}_{II} A$

$$\Rightarrow \mathcal{G}_{II}^T = (A^T \mathcal{F}_{II} A)^T = A^T \mathcal{F}_{II} A = \mathcal{G}_{II}$$

$$\underset{\text{LIN ALG}}{\Longrightarrow} \exists B \in O_2 : B^T \mathcal{G}_{II} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

PUT $C = AB$

$$\Rightarrow C^T \mathcal{F}_I C = B^T \underbrace{A^T \mathcal{F}_I A}_{= I_2} B = B^T B \underset{\substack{\uparrow \\ B \in O_2}}{=} I_2$$

$$C^T \mathcal{F}_{II} C = B^T \underbrace{A^T \mathcal{F}_{II} A}_{= \mathcal{G}_{II}} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

C INVERTIBLE, SINCE A, B INVERTIBLE.
THUS

$$\det(\mathcal{F}_{II} - \varkappa \mathcal{F}_{I}) = 0 \iff \det\left(C^T(\mathcal{F}_{II} - \varkappa \mathcal{F}_{I})C\right) = 0$$

$$\iff \det\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \varkappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

THUS PRINCIPAL CURVATURES ARE
λ_1, λ_2.

IF $\varkappa = \lambda_1 = \lambda_2$, THEN

$$C^T \mathcal{F}_{I} C = I_2, \quad C^T \mathcal{F}_{II} C = \varkappa I_2$$

$$\Rightarrow C^T(\mathcal{F}_{II} - \varkappa \mathcal{F}_{I}) C = 0$$

$$\Longrightarrow \mathcal{F}_{II} - \varkappa \mathcal{F}_{I} = 0$$

$$\Longrightarrow \forall T : (\mathcal{F}_{II} - \varkappa \mathcal{F}_{I}) T = 0$$

$\Rightarrow$ ALL NONZERO TANGENT VECTORS ARE PRINCIPAL. $\square$

REMARK PROOF REMINDS OF "EIGENVECTORS OF SYMMETRIC MATRIX CORRESPONDING TO DISTINCT EIGENVALUES ARE PERPENDICULAR" (LINEAR ALGEBRA)

SPECIAL CASE: $\mathcal{F}_I = I$: THEN PRINCIPAL CURVATURES ARE THE EIGENVALUES OF $\mathcal{F}_{II}$.

IN GENERAL: PRINCIPAL CURVATURES ARE EIGENVALUES OF $\mathcal{F}_I^{-1}\mathcal{F}_{II}$:

$$\det(\mathcal{F}_{II} - \varkappa \mathcal{F}_I) = 0 \Leftrightarrow$$

$$\Leftrightarrow \det(\mathcal{F}_I^{-1}\mathcal{F}_{II} - \varkappa I) = 0.$$

PRINCIPAL VECTORS ARE EIGENVECTORS OF $\mathcal{F}_I^{-1}\mathcal{F}_{II}$.

NOTE: $\mathcal{F}_I^{-1}\mathcal{F}_{II}$ IS <u>NOT</u> SYMMETRIC IN GENERAL. (SO PROP 6.3.4 DOES NOT FOLLOW FROM STANDARD LIN ALG.)