Discrete Mathematics PS4

1. Let $G = (V,E)$ be a simple graph where V is the set of vertices and E is the set of edges. We denote the number of vertices by $n := |V|$ and the number of edges by $m := |E|$. For a vertex $a \in V$, we denote the degree of vertex a by $d(a)$. Furthermore, denote the minimum degree of all vertex degrees by δ and the maximum degree by Δ. Please prove the statements below.

   (a) The number of vertices that have odd degrees must be even;

   (b) $m \leq \binom{n}{2}$;

   (c) $\delta \leq \frac{2m}{n} \leq \Delta$;

   (d) The length of a path is defined to be the number of edges in the path. Assume $n > 0$. Let $k \geq 0$ be an integer. If $\delta \geq k$, then G has a path with length k.

2. Let G be a graph with $n - 1$ edges where n is the number of vertices of G. Show the following three statements are equivalent:

   (a) G is connected; (b) G has no cycles; (c) G is a tree.

   (**Remark:** You could prove the equivalence by proving (a) → (b), (b) → (c) and (c) → (a).)

3. A *walk* in a graph G is a sequence $W := v_0e_1v_1 \ldots v_{l-1}e_lv_l$, whose terms are alternately vertices and edges of G (not necessarily distinct), such that v_{i-1} and v_i are the ends of e_i, $1 \leq i \leq l$. A *closed walk* is a walk that starts from and ends on the same vertex. So an Eulerian cycle is a closed walk that traversed each edge exactly once and we also call it *Eulerian tour*. (Be cautious that an Eulerian cycle is not a cycle. Recall that, by definition, a cycle should be a connected (sub)graph whose vertices are all of degree 2.)

   We say that a graph is *Eulerian* if it contains an Eulerian tour. So we know an Eulerian graph has no vertices of odd degree. Let G be Eulerian.

   (a) Prove that G contains a cycle;

   (b) For two cycles with no edges in common, we call they are edgedisjoint. Please show that the edge set of G can be partitioned into edge sets corresponding to edge-disjoint cycles in the graph.

4. A k-regular graph is a graph in which every vertex is of degree k. The girth of a graph is the length of the shortest cycle of the graph. Prove that

   (a) A k-regular graph of girth four has at least $2k$ vertices;

   (b) A k-regular graph of girth five has at least $k^2 + 1$ vertices.

5. Please use Gale-Shapley's algorithm to find one stable matching for thepreference lists below.

Boys	1_{st}
A	X
B	Y
C	X

Girls	1_{st}	2_{nd}	3_{rd}
X	B	A	C
Y	A	B	C
Z	A	B	C

6. Let M be a matching for a graph G. (Recall that a matching in a graph is a set of edges that do not have common vertices.) An alternating path is a path that begins with an unmatched vertex and whose edges belong alternately to the matching M and not to the matching. An Maugmenting path is an alternating path that starts from and ends on unmatched vertices. Show that M is a maximum matching if and only if G has no M-augmenting path.

   (**Remark:** This theorem is the principle of an algorithm for finding a maximum matching.)

   (**Hint:** For necessity, consider the symmetric difference of the matching and an M-augmenting path, and for sufficiency, prove by contradiction: assume there is another maximum matching and analyse the subgraph induced by the symmetric difference of this maximum matching and the orginal matching.)

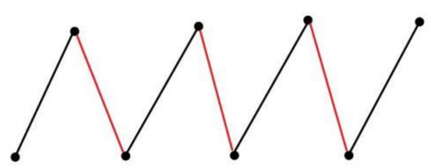

Figure 1: An illustration of M-augmenting path, the red edges are in the matching M.