MATH 465 - INTRODUCTION TO COMBINATORICS
LECTURE 12

Theorem 0.1. *Let S be a finite set and let $A_1, \dots, A_n$ be subsets of S. Then,*

$$|S - (A_1 \cup \cdots \cup A_n)| = |S| + \sum_{j=1}^{n} (-1)^j \sum_{\{i_1,\dots,i_j\} \subseteq [n]} |A_{i_1} \cap \cdots \cap A_{i_j}|,$$

or equivalently,

$$|S - (\cup_{i \in [n]} A_i)| = \sum_{j=0}^{n} (-1)^j \sum_{I \in \binom{[n]}{j}} |\cap_{i \in I} A_i|.$$

1. Euler's totient function

Euler's totient function $\phi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is defined by

$$\phi(n) = |\{k \in \mathbb{Z} \mid 1 \leq k \leq n, \quad \gcd(k,n) = 1\}|.$$

For example,

$$\phi(100) = |\{1, 3, 7, 9, 11, 13, 17, 19, \dots, 91, 93, 97, 99\}| = 40.$$

Theorem 1.1 (Euler)**.** *Let $\{p_1, \dots, p_r\}$ be the set of distinct prime divisors of n. Then,*

$$\phi(n) = n \prod_{j=1}^{r} (1 - \tfrac{1}{p_j}).$$

For example,

$$\phi(100) = 100 \cdot \tfrac{1}{2} \cdot \tfrac{4}{5} = 40.$$

Let $S = \{1, \dots, n\}$ and $A_j = \{p_j, 2p_j, 3p_j, \dots, n\}$ for $j = 1, \dots, r$. Then

$$\begin{aligned}
\phi(n) = |S - (A_1 \cup \cdots \cup A_r)| &= |S| + \sum_{j=1}^{r} (-1)^j \sum_{\{i_1,\dots,i_j\} \subseteq [r]} |A_{i_1} \cap \cdots \cap A_{i_j}| \\
&= n + \sum_{j=1}^{r} (-1)^j \sum_{\{i_1,\dots,i_j\} \subseteq [r]} \frac{n}{p_{i_1} \cdots p_{i_j}} \\
&= n \prod_{j=1}^{r} (1 - \tfrac{1}{p_j}).
\end{aligned}$$

2. Compositions with restrictions

Problem 2.1. Count weak compositions of 7 with 7 parts none of which equals 2.

Solution.

$$\begin{aligned}
S &= \{x_1 + \cdots + x_7 = 7,\ x_1, \ldots, x_7 \geq 0\} \\
A_i &= \{(x_1, \ldots, x_7) \in S \mid x_i = 2\} \\
|S| &= |\{x_1 + \cdots + x_7 = 7\}| = \tbinom{13}{6} \\
|A_i| &= |\{x_1 + \cdots + x_6 = 5\}| = \tbinom{10}{5} \\
|A_i \cap A_j| &= |\{x_1 + \cdots + x_5 = 3\}| = \tbinom{7}{4} \\
|A_i \cap A_j \cap A_k| &= |\{x_1 + \cdots + x_4 = 1\}| = \tbinom{4}{3} \\
|S - (A_1 \cup \cdots \cup A_7)| &= \tbinom{13}{6} - \tbinom{7}{1}\tbinom{10}{5} + \tbinom{7}{2}\tbinom{7}{4} - \tbinom{7}{3}\tbinom{4}{3} = 547.
\end{aligned}$$

□

Problem 2.2. Count weak compositions of 10 with 10 parts all of which are ≤ 2.

Solution.

$$\begin{aligned}
S &= \{x_1 + \cdots + x_{10} = 10,\ x_1, \ldots, x_{10} \geq 0\} \\
A_i &= \{(x_1, \ldots, x_{10}) \in S \mid x_i \geq 3\} \\
|S| &= |\{x_1 + \cdots + x_{10} = 10\}| = \tbinom{19}{9} \\
|A_i| &= |\{y_1 + \cdots + y_{10} = 7\}| = \tbinom{16}{9} \\
|A_i \cap A_j| &= |\{z_1 + \cdots + z_{10} = 4\}| = \tbinom{13}{9} \\
|A_i \cap A_j \cap A_k| &= |\{u_1 + \cdots + u_{10} = 1\}| = \tbinom{10}{9} \\
|S - (A_1 \cup \cdots \cup A_{10})| &= \tbinom{19}{9} - 10 \cdot \tbinom{16}{9} + \tbinom{10}{2}\tbinom{13}{9} - \tbinom{10}{3}\tbinom{10}{9} = 8953.
\end{aligned}$$

□

3. Derangements

A *derangement* is a permutation w such that $w(i) \neq i$ for all i. Let d_n denote the number of derangements in S_n.

Example 3.1.

$n = 1$		$d_1 = 0$
$n = 2$	21	$d_2 = 1$
$n = 3$	231 312	$d_3 = 2$
$n = 4$	2143 2341 2413 3142 3412 3421 4123 4312 4321	$d_4 = 9$
$n = 5$	21453 21534 23154 23451 23514 24153 24513 24531 25134 25413 25431	$d_5 = 44$

Theorem 3.2. *The number d_n of derangements in S_n is given by*

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Proof. Set $S = S_n$ and $A_i = \{w \in S_n \mid w(i) = i\}$ for $i = 1, \dots, n$. Then

$$\begin{aligned} d_n &= |S - (A_1 \cup \cdots \cup A_n)| \\ &= n! - n(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots \\ &= \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \end{aligned}$$

□

Example 3.3.

$$\begin{aligned} d_3 &= 6\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6}\right) = 2 \\ d_4 &= 24\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24}\right) = 9. \end{aligned}$$

$$\tfrac{1}{e} = e^{-1} = \tfrac{1}{0!} - \tfrac{1}{1!} + \tfrac{1}{2!} - \tfrac{1}{3!} + \tfrac{1}{4!} - \tfrac{1}{5!} + \cdots$$

Corollary 3.4. *The number of derangements d_n is the closest integer to $\frac{n!}{e}$.*

Thus the probability that a uniformly random permutation is a derangement is approximately equal to $\frac{1}{e}$.

Theorem 3.5. *The derangement numbers d_n satisfy the recurrence*

$$d_n = n d_{n-1} + (-1)^n.$$

$$\begin{aligned} d_3 &= 6\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6}\right) = 2 \\ d_4 &= 24\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24}\right) = 9 \\ d_5 &= 120\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120}\right) = 44 \end{aligned}$$

4. Counting permutations with prescribed ascents and descents

Problem 4.1. How many permutations $(a, b, c, d, e, f, g) \in S_7$ satisfy

$$a < b < c > d > e < f < g\,?$$

Solution.

$$\begin{aligned}
S &= \{(a,\dots,g) \in S_7 \mid a<b<c, \qquad\quad e<f<g\} \\
A_1 &= \{(a,\dots,g) \in S_7 \mid a<b<c<d, \quad e<f<g\} \\
A_2 &= \{(a,\dots,g) \in S_7 \mid a<b<c, \quad d<e<f<g\} \\
A_1 \cap A_2 &= \{(a,\dots,g) \in S_7 \mid a<b<c<d<e<f<g\} \\
|S-(A_1\cup A_2)| &= \binom{7}{3}\binom{4}{3} - \binom{7}{4} - \binom{7}{3} + 1 = 71.
\end{aligned}$$

□