Math 115B   Final Exam   Spring 2023

Due Tuesday, June 13, 2023

Name ____.

Perm Number ____________________________________.

100 Points Total

Your Total......................

1 ...... **2** ...... **3** ...... **4** ...... **5** ...... **6** ...... **7** ...... **8** ......

1. (10 points) Prove that 2 is a primitive root $(\bmod 11)$.

2. (14 points) Suppose p and q are primes, $p = 4q + 1$. Prove that q is *not* a primitive root $(\bmod p)$.

3. (14 points) Suppose p and q are primes,

$$p = 2q + 1, \qquad p \equiv 2 (\bmod 5).$$

Prove that 5 is a primitive root $(\bmod p)$.

4. (14 points) Suppose

$$m_1 > 2, \quad m_2 > 2, \quad (m_1, m_2) = 1.$$

Prove that there is *no* primitive root $(\bmod m_1 m_2)$.

5. (14 points) Let $\Lambda(n)$ be given by

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^k,\ k > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where p denotes a prime. It is known that

$$\sum_{n \le x} \Lambda(n) \left[\frac{x}{n}\right] = \sum_{n \le x} \log n. \tag{1}$$

Let

$$\Psi(y) = \sum_{n \le y} \Lambda(n).$$

(i). Prove that

$$\sum_{n \le x} \log n = x \log x - x + O(\log x) \quad \text{if} \quad x \ge 2. \tag{2}$$

(ii). Prove that

$$\sum_{n \le x} \Psi\left(\frac{x}{n}\right) = \sum_{n \le x} \Lambda(n) \left[\frac{x}{n}\right] \tag{3}$$

Hint: The left side of (3) is equal to the double sum

$$\sum_{n \le x} \sum_{m \le x/n} \Lambda(m).$$

Changing the order of summation to obtain (3).

By (1) and (2) we have

$$\sum_{n \le x} \Psi\left(\frac{x}{n}\right) = x \log x - x + O(\log x) \quad \text{if} \quad x \ge 2. \tag{4}$$

6. (14 points) Prove that

$$\sum_{n=1}^{\infty} \Psi\left(\frac{x}{n}\right) - 2\sum_{n=1}^{\infty} \Psi\left(\frac{x}{2n}\right) = x \log 2 + O(\log x) \quad \text{if} \quad x \ge 4. \tag{5}$$

Hint: Note that $\Psi(y) = 0$ if $0 < y \le 1$. The left side of (5) is equal to

$$\sum_{n \le x} \Psi\left(\frac{x}{n}\right) - 2 \sum_{n \le x/2} \Psi\left(\frac{x}{2n}\right).$$

Apply (4) to the first sum and the second sum with $x/2$ in place of x.

7. (14 points) Prove that

$$\sum_{n=1}^{\infty} \Psi\left(\frac{x}{n}\right) - 2\sum_{n=1}^{\infty} \Psi\left(\frac{x}{2n}\right) \le \Psi(x) \tag{6}$$

and

$$\sum_{n=1}^{\infty} \Psi\left(\frac{x}{n}\right) - 2\sum_{n=1}^{\infty} \Psi\left(\frac{x}{2n}\right) \ge \Psi(x) - \Psi\left(\frac{x}{2}\right). \tag{7}$$

Hint: The left sides of (6) and (7) are equal to

$$\begin{aligned}
&\Psi(x) + \Psi\left(\frac{x}{2}\right) + \Psi\left(\frac{x}{3}\right) + \cdots - 2\left\{\Psi\left(\frac{x}{2}\right) + \Psi\left(\frac{x}{4}\right) + \cdots\right\} \\
&= \left\{\Psi(x) - \Psi\left(\frac{x}{2}\right)\right\} + \left\{\Psi\left(\frac{x}{3}\right) - \Psi\left(\frac{x}{4}\right)\right\} + \cdots \\
&= \Psi(x) - \left\{\Psi\left(\frac{x}{2}\right) - \Psi\left(\frac{x}{3}\right)\right\} - \left\{\Psi\left(\frac{x}{4}\right) - \Psi\left(\frac{x}{5}\right)\right\} - \cdots
\end{aligned}$$

Use the fact that the function $\Psi(y)$ is increasing in y.

8. (6 points) Conclude that

$$\Psi(x) \geq x \log 2 + O(\log x)$$

and

$$\Psi(x) - \Psi\left(\frac{x}{2}\right) \leq x \log 2 + O(\log x)$$

if $x \geq 4$.