MATH 465 - INTRODUCTION TO COMBINATORICS
LECTURE 6

1. Stirling numbers of the first kind

For a permutation $w \in S_n$, recall that $c(w)$ denotes the number of cycles in w. Define the *signless Stirling number of the first kind*

$$c(n,k) := \#\{w \in S_n : c(w) = k\}, \quad 1 \le k \le n,$$

i.e., $c(n,k)$ is the number of permutations of $[n]$ with exactly k cycles.

Corollary 1.1. *We have*

$$\sum_{k=1}^{n} c(n,k)x^k = \sum_{w \in S_n} x^{c(w)} = x(x+1)(x+2)\cdots(x+n-1).$$

Proposition 1.2.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k).$$

Proof. The formula

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1)$$

implies the identity

$$\sum_{k=1}^{n} c(n,k)x^k = (x+n-1)\sum_{j=1}^{n-1} c(n-1,j)x^j.$$

Now compare the coefficients of x^k on both sides. □

Exercise 1.3. Give a combinatorial proof of the recurrence.

The recurrence

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

can be used to recursively compute the Stirling numbers $c(n,k)$.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 2 & & 3 & & 1 & & & & \\
 & & & 6 & & 11 & & 6 & & 1 & & & \\
 & & 24 & & 50 & & 35 & & 10 & & 1 & & \\
 & 120 & & 274 & & 225 & & 85 & & 15 & & 1 & \\
720 & & 1764 & & 1624 & & 735 & & 175 & & 21 & & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

The numbers $s(n,k) := (-1)^{n-k}c(n,k)$ are called *Stirling numbers of the first kind.*

Theorem 1.4. *We have*

$$\sum_{k=1}^{n} s(n,k)x^k = x(x-1)(x-2)\cdots(x-n+1) = (x)_n = n!\binom{x}{n}.$$

Proof. We have

$$\begin{aligned}\sum_{k=1}^{n} s(n,k)x^k &= \sum_{k=1}^{n}(-1)^{n-k}c(n,k)x^k \\ &= (-1)^n\sum_{k=1}^{n} c(n,k)(-x)^k \\ &= (-1)^n(-x)(-x+1)(-x+2)\cdots(-x+n-1) \\ &= x(x-1)(x-2)\cdots(x-n+1).\end{aligned}$$

□

Therefore, the Stirling numbers of the first kind appear when we express the falling powers (or binomial coefficients) in terms of ordinary powers.

2. Stirling numbers of the second kind

A *partition* of a finite set A is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of subsets of A such that $B_i \neq \varnothing$ for each i, $B_i \cap B_j = \varnothing$ if $i \neq j$ and $A = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k$. Each B_i is called a *block* of π. Contrast this with an ordered set partition $(B_1, B_2, \ldots, B_k)$ in which the blocks are linearly ordered. The *Stirling number of the second kind* $S(n,k)$ is defined to be the number of partitions of an n-element set A into k blocks.

Example 2.1. Let us find $S(4,2)$. Let $A = \{1,2,3,4\}$. The possible partitions are:

$$\begin{gathered}\{\{1,2,3\},\{4\}\}, \quad \{\{1,2,4\},\{3\}\}, \quad \{\{1,3,4\},\{2\}\}, \quad \{\{2,3,4\},\{1\}\}, \\ \{\{1,2\},\{3,4\}\}, \quad \{\{1,3\},\{2,4\}\}, \quad \{\{1,4\},\{2,3\}\}.\end{gathered}$$

Therefore, $S(4,2) = 7$.

Theorem 2.2.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Proof. To partition $[n]$ into k blocks, we can either:

(1) Partition $[n-1]$ into k blocks and add n to one of the k blocks in $kS(n-1,k)$ ways, or
(2) Partition $[n-1]$ into $k-1$ blocks and create a new block $\{n\}$ in $S(n-1,k-1)$ ways.

□

The recurrence

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

can be used to compute the Stirling numbers of the second kind:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 1 & & & & \\
 & & & 1 & & 7 & & 6 & & 1 & & & \\
 & & 1 & & 15 & & 25 & & 10 & & 1 & & \\
 & 1 & & 31 & & 90 & & 65 & & 15 & & 1 & \\
1 & & 63 & & 301 & & 350 & & 140 & & 21 & & 1 \\
.. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & ..
\end{array}
$$

Stirling numbers of the second kind $S(n,k)$ for $1 \leq k \leq n \leq 7$

Theorem 2.3.

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k = \sum_{k=1}^{n} S(n,k)k!\binom{x}{k}.$$

Therefore, the Stirling numbers of the second kind appear when we try to express ordinary powers in terms of falling powers or binomial coefficients.

Proof. It suffices to prove the identity when x is a positive integer [why?]. Then x^n is the number of ways to color $\{1,2,\ldots,n\}$ in x colors. $\sum_{k=1}^{n} S(n,k)(x)_k$ is the number of ways to split $\{1,2,\ldots,n\}$ into blocks and then color these blocks in different colors. $\square$

3. Stirling numbers as transition matrices

Consider the n-dimensional vector space

$$\{\text{polynomials in } x \text{ of degree} \leq n \text{ with constant term } 0\},$$

i.e., a vector is of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$. This vector space has two bases:

(1) The basis of monomials $B_1 = (x, x^2, x^3, \ldots, x^n)$.
(2) The basis of falling powers $B_2 = ((x)_1, (x)_2, \ldots, (x)_n)$.

Then Theorem 2.3 asserts that the $n \times n$ matrix $\mathbf{S} = [S(m,k)]_{1\leq k\leq n, 1\leq m\leq n}$ is the transition matrix between the basis B_2 and the basis B_1. Similarly Theorem 1.4 says that the matrix $\mathbf{s} = [s(m,k)]_{1\leq k\leq n, 1\leq m\leq n}$ is the transition matrix from B_1 to B_2. In particular, the matrices $\mathbf{S}$ and $\mathbf{s}$ are inverses of each other.

Example 3.1.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}^{-1}.$$

4. Summation formulas

Proposition 4.1.

$$\sum_{j=0}^{n} \binom{j}{k} = \sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}.$$

Proof. The LHS is the coefficient of x^k in $\sum_{j=0}^n (1+x)^j$. We have

$$\sum_{j=0}^n (1+x)^j = \frac{(1+x)^{n+1}-1}{x}.$$

The RHS is the coefficient of x^k in $\frac{(1+x)^{n+1}-1}{x}$. □

We can use this to find the sum of squares:

Proposition 4.2.

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof.

$$\begin{aligned}
\sum_{j=1}^n j^2 &= \sum_{j=1}^n \left(\binom{j}{1} + 2\binom{j}{2}\right) \\
&= \binom{n+1}{2} + 2\binom{n+1}{3} \\
&= \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{3} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

□

More generally, expressing a polynomial $f(j)$ in terms of binomial coeffients $\binom{j}{k}$ leads to a summation formula for f:

$$\begin{aligned}
f(j) &= \sum_k a_k \binom{j}{k} \\
\implies \sum_{j=0}^n f(j) &= \sum_{j=0}^n \sum_k a_k \binom{j}{k} \\
&= \sum_k a_k \binom{n+1}{k+1}.
\end{aligned}$$