## Geometry of surfaces - Solutions

**56.** The unit normal to $\sigma$ is

$$\mathbf{N}(x,y) = \frac{1}{\sqrt{4x^2+4y^2+1}}(-2x,-2y,1).$$

The region covered by the Gauss map $\mathcal{G} : \mathbb{R}^2 \to S^2$ , $(x,y) \mapsto \mathbf{N}(x,y)$ is the upper hemisphere.

**57.** Parametrize the equator by the unit speed curve $\gamma(t) = (\cos(t), \sin(t), 0)$. We have $\ddot{\gamma}(t) = (-\cos(t), -\sin(t), 0) = -\gamma(t) = -\mathbf{N}(\gamma(t))$, where $\mathbf{N}$ is the usual unit normal vector of $S^2$. Thus $\ddot{\gamma}$ is parallel to $\mathbf{N}$ along $\gamma$, which means by definition that $\gamma$ is a geodesic in $S^2$.

**58.** Since $\gamma$ is an asymptotic curve, its normal curvature $\kappa_n$ is zero (see Exercise 55). Since $\gamma$ is a geodesic, its geodesic curvature $\kappa_g$ is zero (see Proposition 8.1.2). Since the curvature $\kappa$ of $\gamma$ satisfies $\kappa^2 = \kappa_n^2 + \kappa_g^2$, it follows that $\kappa = 0$ and hence $\gamma$ is (part of) a straight line (see Proposition 1.1.4).

**59.** Note that $\gamma(u) = \sigma(u,0)$. We $\sigma_u(u,v) = \dot{\gamma}(u) + v\dot{\mathbf{b}}(u)$ and $\sigma_v(u,v) = \mathbf{b}(u)$. Thus $\mathbf{N}(u,0) = \sigma_u(u,0)\times\sigma_v(u,0) = \dot{\gamma}(u)\times\mathbf{b}(u) = \mathbf{t}(u)\times\mathbf{b}(u) = -\mathbf{n}(u)$. Since $\ddot{\gamma}(u) = \kappa(u)\mathbf{n}(u)$, this shows that $\ddot{\gamma}(u)$ is parallel to $\mathbf{N}(u,0)$ for all $u$, which by definition means that $\gamma$ is a geodesic on the surface.

**60.** The surface $\sigma$ is isometric to (part of) the plane (see Theorem 5.2.3). Therefore the geodesics are the images under $\sigma$ of (part of) lines and of the form $\sigma(at+b, ct+d)$ with $a,b,c,d \in \mathbb{R}$ and at least one of $a,b$ non-zero.

**61.** Put $u(t) = c$ and $v(t) = e^t$. Then $\dot{u}(t) = 0$, $\dot{v}(t) = e^t$ and

$$\|\dot{\gamma}(t)\|^2 = E(u(t),v(t))\dot{u}(t)^2 + 2F(u(t),v(t))\dot{u}(t)\dot{v}(t) + G(u(t),v(t))\dot{v}(t)^2 = \frac{e^{2t}}{e^{2t}} = 1,$$

which shows that $\gamma$ is a unit speed curve. We now check that $\gamma$ satisfies the geodesic equations (see Theorem 8.1.8). Since $\dot{u} = 0$ and $F = 0$ we need to verify that

$$0 = \frac{1}{2}G_u(u(t),v(t))\dot{v}(t)^2 \text{ and } \frac{d}{dt}(G(u(t),v(t))\dot{v}(t)) = \frac{1}{2}G_v(u(t),v(t))\dot{v}(t)^2.$$

The first equation holds since $G_u = 0$. We have $G(u(t),v(t))\dot{v}(t) = e^{-2t}e^t = e^{-t}$ and thus $\frac{d}{dt}(G(u(t),v(t))\dot{v}(t)) = -e^{-t}$. On the other hand, $\frac{1}{2}G_v(u(t),v(t))\dot{v}(t)^2 = \frac{1}{2}(-2)e^{-3t}e^{2t} = -e^{-t}$. Altogether this shows that the geodesic equations are satisfied and therefore $\gamma$ is a geodesic.