Geometry of Surfaces - Exercises

Exercises marked with $*$ are to be answered (partially) in the online quiz for this week on the Keats page for this module.

29.* Let $\mathcal{S}$ be the surface given by the surface patch

$$\sigma : (-1,1) \times (-1,1) \to \mathbb{R}^3 \ , \ \ (u,v) \mapsto (u, v, \cos(u) + \sin(v)).$$

Calculate the coefficients of the first fundamental form of $\mathcal{S}$ at $\sigma(0,0) = (0,0,1)$.

30.* Compute the first fundamental form of the surface $\sigma(u,v) = (u-v, u+v, u^2+v^2)$.

31. Let σ be a surface and $Edu^2 + 2Fdudv + Gdv^2$ its first fundamental form. Compute the first fundamental form of the surface $\tilde{\sigma} = \lambda\sigma$ with $0 \neq \lambda \in \mathbb{R}$.

32. Show that

$$\sigma(u,v) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1)$$

is a conformal parametrization of the unit sphere S^2 minus $N = (0,0,1)$. [Note that σ is the inverse map of the stereographic projection $S^2 \setminus \{N\} \to \mathbb{R}^2$.]

33.* Let $\sigma : (0,1) \times (0,1) \to \mathbb{R}^3$ be a regular surface patch whose first fundamental form is given by

$$ds^2 = du^2 + (1-u)dudv + \frac{3u^2}{4v}dv^2.$$

Compute the length of the curve

$$\gamma : (0,1) \to \mathbb{R}^3 \ , t \mapsto \sigma(t, t^2).$$

34. Prove that the concept of isometric surfaces is an equivalence relation. More precisely, prove that:

(a) A surface A is always isometric to itself.
(b) If a surface A is isometric to a surface B, then B is isometric to A.
(c) If a surface A is isometric to a surface B and B is isometric to a surface C, then A is isometric to C.

35. Prove that the generalized cylinder given by $\sigma(u,v) = (f(u), g(u), v)$ with $\dot{f}^2 + \dot{g}^2 = 1$ is (locally) isometric to a plane.

36. Prove that the cone given by $\sigma(u,v) = (\cos(u)v, \sin(u)v, v)$ with $0 < u < 2\pi$ and $0 < v < \infty$ is isometric to (part of) the plane.

37.* Let $\mathcal{S}$ be a surface with surface patch $\sigma : (0,1) \times (0,1) \to \mathbb{R}^3$ and first fundamental form

$$ds^2 = (u^2v^3 + v^3)du^2 + 2vdudv + \frac{1}{v}dv^2.$$

Compute the area of $\mathcal{S}$.

38. Write down an integral formula for the area of the paraboloid $z = x^2 + y^2$ with $z \leq 1$.