Geometry of Surfaces - Exercises

1. True or False? Justify your answer!

(a) Every regular curve is unit speed.
(b) Every unit speed curve is regular.
(c) There is exactly one unit speed parametrization of a given regular curve.
(d) Every curve whose image is a circle is regular.

2. Let $\gamma : (a,b) \to \mathbb{R}^n$ be a unit speed curve. Compute the length of γ.

3. Show that the length of a curve $\gamma : (a,b) \to \mathbb{R}^n$ does not depend on its parametrization.

4. Let $\gamma : (a,b) \to \mathbb{R}^n$ be a curve and $a < c < d < b$. Show that

$$\|\gamma(c) - \gamma(d)\| \leq \text{Length}(\gamma : (c,d) \to \mathbb{R}^n).$$

In other words, straight lines are the shortest curves joining two given points.

5. Let

$$\gamma : (0,1) \to \mathbb{R}^3 \ , \ t \mapsto \left(t\sin(t), t\cos(t), \frac{\sqrt{8}}{3} t^{\frac{3}{2}} \right).$$

Compute the length of γ and the arc-length starting at 0. Then, find a unit speed reparametrization of γ.

6. Let $\gamma : (a,b) \to \mathbb{R}^2$ be a regular curve and let ℓ be a straight line which does not intersect γ. Assume that $t_0 \in (a,b)$ so that $\gamma(t_0)$ is a point on γ that is closest to ℓ. Show that $\dot{\gamma}(t_0)$ is parallel to ℓ. [Recall that the distance from a point p to a line ℓ is $\frac{|v\cdot(p-p_0)|}{\|v\|}$ where v is any vector perpendicular to ℓ and p_0 is any point on ℓ.]

7. Consider the circular helix

$$\gamma : \mathbb{R} \to \mathbb{R}^3 \ , \ t \mapsto (\cos(t), \sin(t), 2t).$$

Calculate the tangent vectors of γ and show that they trace out a circle as t varies. What is the radius of such circle? Find a unit-speed reparametrization of γ.

8. Let $\gamma : (a,b) \to \mathbb{R}^3$ be a curve that does not pass through 0 and suppose that $\gamma(t_0)$, $t_0 \in (a,b)$, is its closest point to 0. Show that $\gamma(t_0) \cdot \dot{\gamma}(t_0) = 0$.