1: Builup-error

2: Graph proof

(a) Prove by contradiction(infinite descending): Define

$$d_{-1}(v) := \#\{(w, v) \in E : w \in V\}$$

to be the number of in-neighbourhoods of city v. We claim that the city with the most number of in-neighbourhoods satisfies the conditon.

Let $c \in V$ be this vertex. Consider

$$I := \{v \in V : (v, c) \in E\}$$

$$O := \{v \in V : (c, v) \in E\}$$

Cities in I can reach c at one road distance. And $d_{-1}(c) = |I|$ is maximized over V. Pick $o \in O$.

Claim: there is $i \in I$ **such that** $(o, i) \in E$

If this claim is true, then $o \to i \to c$ are two roads to c. We can easily see that $V = I \cup O$. Thus, a city in either I or O can reach c through at most 2 roads.

Now we prove this claim. Assume the contrary, that this claim is false.

$$\forall o \in O \forall i \in I, (o, i) \notin E$$

This implies that $(i, o) \in E$ instead.

The counting of in-neighbourhoods of o shows that

$$\begin{aligned} d_{-1}(o) &= \#\{(c, o)\} + \sum_{(i,o)\in E} 1 \\ &= \#\{(c, o)\} + \sum_{i\in I} 1 \\ &= 1 + \sum_{i\in I} 1 = 1 + |I| > |I| \\ &= d_{-1}(c) \end{aligned}$$

contradicts to the maximality of $d_{-1}(c)$.

$\square$

(b)

By induction, if $m = 1$, according to Euler theorem, there is a euler tour from A to B. Now assume $m \geq 2$.

Let $(A_i, B_i), 1 \leq i \leq m$ be m pairs of vertices with odd degrees. According to theorem.XXX, there is a walk from A_m to B_m denoted by W_m. Remove W_m along with (A_m, B_m), we obtain a graph with $m - 1$ pairs of vertices with odd degress.

(c)

(sufficiency) Base case: there is neither odd cycle or even cycle(no cycle).

Suppose we have a cycle $T_1, T_2, T_3, \cdots, T_n, T_1$, Its length is even, by condition. We have $n = 2m$ Let

$$L = \{T_1, T_3, \cdots, T_{n-1}\}$$
$$R = \{T_2, T_4, \cdots, T_n\}$$

6: The bipartite graph

(a)

$$\begin{aligned}\sum_{v \in L} d(v) &= \sum_{v \in L} \sum_{(v,w) \in E} 1 \\ &= \sum_{v \in L} \sum_{w \in R} 1 \\ &= \sum_{w \in R} \sum_{v \in L} 1 \\ &= \sum_{w \in R} \sum_{(w,v) \in E} 1 \\ &= \sum_{w \in R} d(w)\end{aligned}$$

(b) By definition,

$$s = \frac{1}{|L|} \sum_{v \in L} d(v), t = \frac{1}{|R|} \sum_{w \in R} d(w)$$

Hence,

$$\begin{aligned}\frac{s}{t} &= \frac{\frac{1}{|L|} \cancel{\sum_{v \in L} d(v)}}{\frac{1}{|R|} \cancel{\sum_{w \in R} d(w)}} \\ &= \frac{\frac{1}{|L|}}{\frac{1}{|R|}} = \frac{|R|}{|L|}\end{aligned}$$

(c)

($\Longleftarrow$) Let G be a graph which can be two colored. Let's say, that $W \subseteq V$ is the set with every vertices colored white, and $B \subseteq V$ is the set with every vertices colored black.

Then we have

$$V = W \cup B, W \cap B = \{\}$$

. Since G is two-colored, we can't have $\{w_1, w_2\} \in E, \{b_1, b_2\} \in E$ where $w_1, w_2 \in W$ and $b_1, b_2 \in B$. Thus,

$$E = W \times B$$

By the definition, we conclude that G is bipartite.

$(\implies)$

Let $V = L \times R$. Color L with white, and R with black respectively. The proof that show that G is two-colored is trival.