Geometry of Surfaces

5CCM223A/6CCM223B

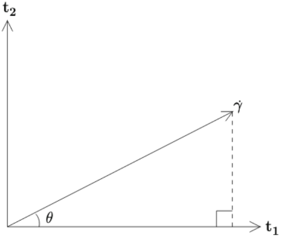
Video 22
Euler's Theorem

Jürgen Berndt
King's College London

Euler's Theorem. *Let γ be a unit speed curve on a surface σ. Let κ_1, κ_2 be principal curvatures of σ with unit principal vectors $\mathbf{t}_1, \mathbf{t}_2$. [We can assume that $\mathbf{t}_1, \mathbf{t}_2$ are perpendicular.] Then the normal curvature κ_n of γ satisfies*

$$\kappa_n = \kappa_1 \cos(\theta)^2 + \kappa_2 \sin(\theta)^2$$

with $\theta = \angle(\dot{\gamma}, \mathbf{t}_1)$.



Proof. Write

$$\mathbf{t}_i = \xi_i \sigma_u + \eta_i \sigma_v \ , \ \mathbf{T}_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$$

$$\dot{\gamma} = \xi \sigma_u + \eta \sigma_v \ , \ \mathbf{T} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

By definition of θ, we also have

$$\dot{\gamma} = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$$

Thus

$$\xi \sigma_u + \eta \sigma_v = \cos(\theta)(\xi_1 \sigma_u + \eta_1 \sigma_v) + \sin(\theta)(\xi_2 \sigma_u + \eta_2 \sigma_v)$$

$$\xi\sigma_u + \eta\sigma_v = \cos(\theta)(\xi_1\sigma_u + \eta_1\sigma_v) + \sin(\theta)(\xi_2\sigma_u + \eta_2\sigma_v)$$

Equivalently,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \cos(\theta) \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} + \sin(\theta) \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

Therefore,

$$\mathbf{T} = \cos(\theta)\mathbf{T}_1 + \sin(\theta)\mathbf{T}_2$$

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Then

$$\begin{aligned}
\kappa_n &= \mathbf{T}^\top \mathcal{F}_{II} \mathbf{T} \\
&= (\cos(\theta)\mathbf{T}_1 + \sin(\theta)\mathbf{T}_2)^\top \mathcal{F}_{II} (\cos(\theta)\mathbf{T}_1 + \sin(\theta)\mathbf{T}_2) \\
&= \cos(\theta)^2 \underbrace{\mathbf{T}_1^\top \mathcal{F}_{II} \mathbf{T}_1}_{=\kappa_1} + \sin(\theta)^2 \underbrace{\mathbf{T}_2^\top \mathcal{F}_{II} \mathbf{T}_2}_{=\kappa_2} \\
&\quad + \sin(\theta)\cos(\theta)(\underbrace{\mathbf{T}_1^\top \mathcal{F}_{II} \mathbf{T}_2}_{=0} + \underbrace{\mathbf{T}_2^\top \mathcal{F}_{II} \mathbf{T}_1}_{=0}) \\
&= \kappa_1 \cos(\theta)^2 + \kappa_2 \sin(\theta)^2
\end{aligned}$$

Corollary. *The principal curvatures are the maximum and minimum values of the normal curvatures of all unit speed curves on the surface passing through the point. The principal vectors (directions) are the tangent vectors of the curves giving these maximum resp. minimum values.*

Proof. Case 1: $\kappa_1 \neq \kappa_2$. Assume $\kappa_1 > \kappa_2$. From Euler's Theorem we get

$$\kappa_n = \kappa_1 \underbrace{\cos(\theta)^2}_{=1-\sin(\theta)^2} + \kappa_2 \sin(\theta)^2 = \kappa_1 - \underbrace{(\kappa_1 - \kappa_2)}_{>0} \sin(\theta)^2 \leqslant \kappa_1$$

with equality if and only if $\theta \in \{0, \pi\}$, that is, if $\dot{\gamma} \| \mathbf{t}_1$.
Analogously, $\kappa_n \geqslant \kappa_2$ with equality if and only if $\dot{\gamma} \| \mathbf{t}_2$.

Case 2: $\kappa_1 = \kappa_2$. From Euler's Theorem we get

$$\kappa_n = \kappa_1 \cos(\theta)^2 + \kappa_2 \sin(\theta)^2 = \kappa_i \cos(\theta)^2 + \kappa_i \sin(\theta)^2 = \kappa_i$$

and every non-zero tangent vector is a principal vector.

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Video 23

Principal curvatures and the shape of surfaces

Jürgen Berndt

King's College London

Principal curvatures at a point determine the shape of the surface near that point. Consider quadric surface

$$z = \kappa' x^2 + \kappa'' y^2 \qquad (\kappa', \kappa'' \in \mathbb{R})$$

with parametrization

$$\sigma(u, v) = (u, v, \kappa' u^2 + \kappa'' v^2)$$

We calculate

$$\sigma_u = (1, 0, 2\kappa' u) \ , \ \sigma_v = (0, 1, 2\kappa'' v)$$
$$\sigma_{uu} = (0, 0, 2\kappa') \ , \ \sigma_{uv} = (0, 0, 0) \ , \ \sigma_{vv} = (0, 0, 2\kappa'')$$

$$\sigma_u = (1, 0, 2\kappa' u) \ , \ \sigma_v = (0, 1, 2\kappa'' v)$$
$$\sigma_{uu} = (0, 0, 2\kappa') \ , \ \sigma_{uv} = (0, 0, 0) \ , \ \sigma_{vv} = (0, 0, 2\kappa'')$$

At $p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ this gives

$$\sigma_u = (1, 0, 0) \ , \ \sigma_v = (0, 1, 0)$$
$$E = \sigma_u \cdot \sigma_u = 1 \ , \ F = \sigma_u \cdot \sigma_v = 0 \ , \ G = \sigma_v \cdot \sigma_v = 1$$
$$\mathbf{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (0, 0, 1)$$
$$L = \sigma_{uu} \cdot \mathbf{N} = 2\kappa' \ , \ M = \sigma_{uv} \cdot \mathbf{N} = 0 \ , \ N = \sigma_{vv} \cdot \mathbf{N} = 2\kappa''$$

$$E = 1\ ,\ F = 0\ ,\ G = 1\ ,\ L = 2\kappa'\ ,\ M = 0\ ,\ N = 2\kappa''$$

Principal curvatures are roots of

$$\begin{aligned} 0 = \det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) &= \det\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} \\ &= \det\begin{pmatrix} 2\kappa' - \kappa & 0 \\ 0 & 2\kappa'' - \kappa \end{pmatrix} = (2\kappa' - \kappa)(2\kappa'' - \kappa) \end{aligned}$$

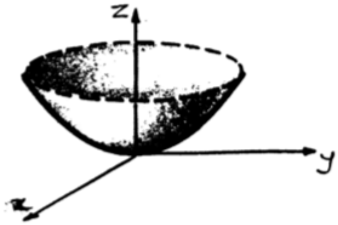
Hence $2\kappa'$ and $2\kappa''$ are the principal curvatures of the quadric surface. Conclusion: Near a point p of a surface at which the principal curvatures are κ_1, κ_2, the surface looks like the quadric

$$z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$$

Discuss four cases

Case 1: $\kappa_1 \kappa_2 > 0$

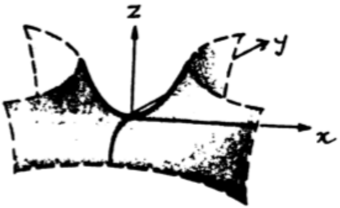
Surface looks like an elliptic paraboloid near p



p is called an elliptic point of the surface

Case 2: $\kappa_1 \kappa_2 < 0$

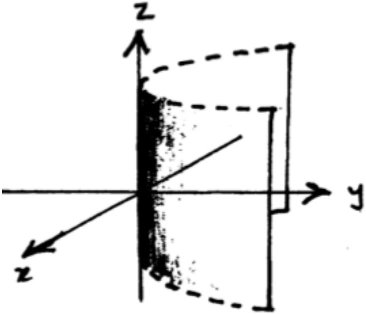
Surface looks like a hyperbolic paraboloid near p



p is called a hyperbolic point of the surface

Case 3: $\kappa_1\kappa_2 = 0$ and $\kappa_1 + \kappa_2 \neq 0$

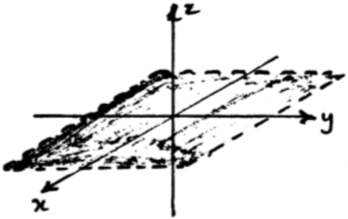
Surface looks like a parabolic cylinder near p



p is called a parabolic point of the surface

Case 4: $\kappa_1 = 0 = \kappa_2$

Surface looks like a plane near p

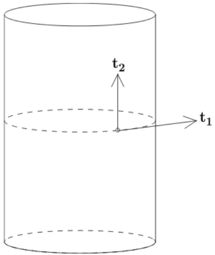


p is called a planar point of the surface

Need to examiner higher order derivatives to determine shape of surface more precisely

Example 1. All points on the sphere are elliptic and all non-zero tangent vectors are principal vectors.

Example 2. All points on the circular cylinder of radius 1 are parabolic points. More precisely, the principal curvatures are 1 and 0 and the corresponding principal vectors are perpendicular to the axis and tangent to the axis, respectively.



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Video 24

Gaussian curvature and mean curvature

Jürgen Berndt

King's College London

Let κ_1, κ_2 be the principal curvatures of a surface

$$K = \kappa_1 \kappa_2$$

is called the Gaussian curvature of the surface

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

is called the mean curvature of the surface

Consider a surface with
first fundamental form $Edu^2 + 2Fdudv + Gdv^2$
and second fundamental form $Ldu^2 + 2Mdudv + Ndv^2$

Proposition.

$$
\begin{aligned}
K &= \frac{LN - M^2}{EG - F^2} \\
H &= \frac{LG - 2MF + NE}{2(EG - F^2)} \\
\kappa_1, \kappa_2 &= H \pm \sqrt{H^2 - K}
\end{aligned}
$$

Proof. The principal curvatures κ_1, κ_2 are the roots of

$$
\begin{aligned}
0 &= \det\left(\mathcal{F}_{II} - \kappa \mathcal{F}_I\right) = \det \begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} \\
&= (L - \kappa E)(N - \kappa G) - (M - \kappa F)^2 \\
&= (EG - F^2)\kappa^2 - (LG - 2MF + NE)\kappa + (LN - M^2)
\end{aligned}
$$

Well-known fact: If $a\kappa^2 + b\kappa + c = 0$, then $-\frac{b}{a}$ is the sum of the roots and $\frac{c}{a}$ is the product of the roots of the quadratic equation. It follows that

$$
K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}\ , \ H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{LG - 2MF + NE}{2(EG - F^2)}
$$

and κ_1, κ_2 are the roots of $\kappa^2 - 2H\kappa + K = 0$ and hence $\kappa_1, \kappa_2 = H \pm \sqrt{H^2 - K}$

Example 1. The unit sphere S^2 has principal curvatures $\kappa_1 = \kappa_2 = 1$. Thus $K = 1$ and $H = 1$

Example 2. The circular cylinder of radius 1 has principal curvatures $\kappa_1 = 1$ and $\kappa_2 = 0$. Thus $K = 0$ and $H = \frac{1}{2}$

A sphere of radius $r > 0$ has constant Gaussian curvature $K = \frac{1}{r^2} > 0$. A circular cylinder (or a plane) has constant Gaussian curvature $K = 0$.

Problem. Is there a surface with constant Gaussian curvature $K < 0$?

Consider surface of revolution

$$\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

with profile curve

$$(f(u), 0, g(u)) \qquad f > 0\ ,\ \dot{f}^2 + \dot{g}^2 = 1$$

Compute E, F, G, L, M, N and use $\dot{f}^2 + \dot{g}^2 = 1$ to get

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{\ddot{f}}{f}$$

Then

$$K = -1 \iff \ddot{f} = f$$

Can choose for example

$$f(u) = e^u$$

Using $\dot{f}^2 + \dot{g}^2 = 1$ we get

$$g(u) = \int \sqrt{1 - e^{2u}} du = \ldots = \sqrt{1 - e^{2u}} - \cosh^{-1}(e^{-u})$$

It follows that the surface of revolution

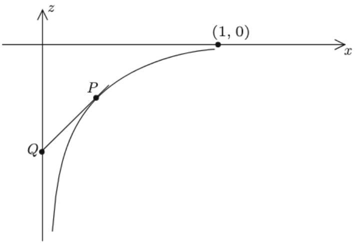
$$\sigma(u, v) = (e^u \cos(v), e^u \sin(v), \sqrt{1 - e^{2u}} - \cosh^{-1}(e^{-u}))$$

has constant Gaussian curvature $K = -1$. The profile curve

$$(e^u, 0, \sqrt{1 - e^{2u}} - \cosh^{-1}(e^{-u}))$$

is a tractrix.

Problem posed by Leibniz: What is the path of an object dragged along a horizontal plane by a string of constant length (here: length one) when the end of the string not joined to the object moves along a straight line in the plane?
Solution: the path is a tractrix



The surface we just constructed is called a pseudosphere
Beltrami (1868): Realization of hyperbolic geometry (locally)