# MATH 465 - INTRODUCTION TO COMBINATORICS
# LECTURE 13

## 1. Partitions

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ of $n$ is a sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ of integers $\lambda_i$ such that $\lambda_1 + \lambda_2 + \cdots = n$. The nonzero $\lambda_i$ are called the *parts* of $\lambda$ and we write $|\lambda| = n$.

**Example 1.1.** $\lambda = (4, 2, 2, 1)$ is a partition of $n = 9$ with 4 parts. When the parts are small, we typically write $\lambda = 4221$ instead of $(4, 2, 2, 1)$ etc.

Equivalently, a partition is a way of writing $n$ as a sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k$, where we disregard the order of the summands. Compare this with a weak composition of $n$ where the summands are ordered.

We denote by $p(n)$ the number of partitions of $n$.

**Example 1.2.**

| $n$ | Partitions of $n$ | $p(n)$ |
|---|---|---|
| 0 | $\varnothing$ (empty partition) | 1 |
| 1 | $1$ | 1 |
| 2 | $2, 11$ | 2 |
| 3 | $3, 21, 111$ | 3 |
| 4 | $4, 31, 22, 211, 1111$ | 5 |
| 5 | $5, 41, 32, 311, 221, 2111, 11111$ | 7 |
| 6 | $6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111$ | 11 |

There is no closed formula for $p(n)$, but there is a nice generating function.

**Theorem 1.3** (Euler)**.**

$$\sum_{n=0}^{\infty} p(n)x^n = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots$$
$$= \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

**Remark 1.4** (Technical remark-the $(x)$-adic topology)**.** There is an infinite product in the statement of the theorem, so we should say what it means. If $F_1(x), F_2(x), \dots$ are formal power series, we say that $F_k(x)$ converges to the formal power series $F(x) = \sum_{n \geq 0} a_n x^n$ (written $\lim_{k \to \infty} F_k(x) = F(x)$) if for every $n$, there is a number $\delta(n)$ such that the coefficient of $x^n$ in $F_k(x)$ is $a_n$ if $k \geq \delta(n)$. In other words, the coefficient of $x^n$ in $F_k(x)$ eventually becomes constant and equal to $a_n$. Let us look at some examples.

(1) Let $F_k(x) := 1 + x + x^2 + \cdots + x^k$. We have

$$\lim_{k \to \infty} F_k(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x},$$

because every coefficient stabilizes to 1. More generally, if $F(x) = \sum_{n\geq 0} a_n x^n$ and $F_k(x) := \sum_{n=0}^{k} a_n x^n$, then $\lim_{k\to\infty} F_k(x) = F(x)$.
(2) Let $F_k(x) := x^k$. Then, $\lim_{k\to\infty} F_k(x) = 0$.
(3) Let $F_k(x) := \frac{1}{k}x^k$. Then, $\lim_{k\to\infty} F_k(x)$ does not exist since the coefficient of $x$ does not stabilize.
(4) Let $F_k(x) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$. Then, we define

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} := \lim_{k\to\infty} F_k(x).$$

The sequence $F_1(x), F_2(x), \ldots$ converges because for any $k$, the coefficient of $x^k$ stabilizes after $F_k$, i.e, $[x^k]F_k(x) = [x^k]F_{k+1}(x) = [x^k]F_{k+2}(x) = \cdots$, since

$$F_{k+1}(x) = F_k(x)(1 + x^{k+1} + x^{2(k+1)} + \cdots) = F_k(x) + \text{terms divisible by } x^{k+1}.$$

Notice that computing any coefficient is still a finite process and does not involve taking any limits of numbers.

*Proof.* Let $A_k$ denote the set of partitions all of whose parts are $k$, so $k, kk, kkk$ etc. Since any partition splits uniquely into a partition $\mu_1$ with all parts equal to 1, a partition $\mu_2$ with all parts equal to 2 etc, we have a bijection

$$\{\text{partitions}\} \xrightarrow{\sim} A_1 \times A_2 \times A_3 \times \cdots$$

Consider the weight function $\alpha(\lambda) = |\lambda|$ on $\{\text{partitions}\}$ and the weight function $\beta_k(\mu_k) = |\mu_k|$ on $A_k$. Then, we have the additivity property $\alpha(\lambda) = \beta_1(\mu_1) + \beta_2(\mu_2) + \cdots$. Using the multiplication principle for generating functions, we get

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)x^n &= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots \\ &= \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.\end{aligned}$$

□

**Theorem 1.5** (Euler)**.** *The generating function for partitions with parts $\leq k$ is*

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

**Theorem 1.6** (Euler)**.** *The generating function for partitions with distinct parts is*

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots.$$

**Theorem 1.7** (Euler)**.** *The generating function for partitions with distinct parts all of which are $\leq k$ is*

$$(1+x)(1+x^2)\cdots(1+x^k).$$

## 2. Odd parts vs. distinct parts

**Theorem 2.1** (Euler)**.** *The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into distinct parts.*

**Example 2.2.**

| | |
|---|---|
| $11$ | $2$ |
| $3, 11$ | $3, 21$ |
| $31, 1111$ | $4, 31$ |
| $5, 311, 11111$ | $5, 41, 32$ |
| $51, 33, 3111, 111111$ | $6, 51, 42, 321$ |
| $7, 511, 331, 31111, 1111111$ | $7, 61, 52, 43, 421$ |

*Proof 1.*

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^n} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

□

*Proof 2.* Let $S$ be the set of partitions of $n$. The first number is $|S-(A_1\cup A_2\cup\cdots)|$ where

$$A_i = \{\text{partitions of } n \text{ containing a part equal to } 2i\}.$$

The second number is $|S-(B_1\cup B_2\cup\cdots)|$ where

$$B_i = \{\text{partitions of } n \text{ containing at least two parts equal to } i\}.$$

We next convince ourselves that, for distinct $i, j, k, \ldots$, we have

$$\begin{array}{rcl} |A_i| = & p(n-2i) & = |B_i| \\ |A_i\cap A_j| = & p(n-2i-2j) & = |B_i\cap B_j| \\ |A_i\cap A_j\cap A_k| = & p(n-2i-2j-2k) & = |B_i\cap B_j\cap B_k| \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

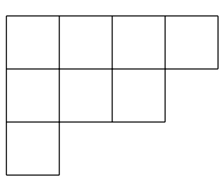
By the inclusion-exclusion formula, the theorem follows. □

## 3. Ferrers shapes/Young diagrams

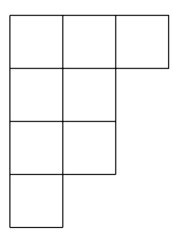
The *Ferrers shape* (also called a *Young diagram*) associated with a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a collection of unit boxes on the square grid which is made up of contiguous rows of lengths $\lambda_1, \lambda_2, \ldots$ located one under another so that their left ends are aligned. The column lengths of this shape form the *conjugate partition* $\lambda'$. In other words, the number of parts of $\lambda'$ that equal $i$ is $\lambda_i - \lambda_{i+1}$.

**Example 3.1.** Consider the partition $\lambda = 431$. The conjugate partition must have

$$4-3 = 1 \text{ part equal to } 1,$$
$$3-1 = 2 \text{ parts equal to } 2,$$
$$1-0 = 1 \text{ parts equal to } 3.$$



$\lambda = 431$



$\lambda' = 3221$

**Proposition 3.2.** *The number of partitions of $n$ with the largest part equal to $k$ is equal to the number of partitions of $n$ with exactly $k$ parts.*

*The number of partitions of $n$ with the largest part at most $k$ is equal to the number of partitions of $n$ with at most $k$ parts.*

*Proof.* The largest part of $\lambda$ is the number of parts of $\lambda'$. □

**Corollary 3.3.** *The generating function for partitions with at most $k$ parts is given by*

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

## 4. Partitions in a box

Let $g(m, n, a)$ denote the number of partitions of $a$ whose shape fits into the $m \times n$ rectangle. Equivalently, the number of parts is $\leq m$, and the largest part is $\leq n$.

**Example 4.1.** $g(3, 4, 6) = |\{42, 411, 33, 321, 22\}| = 5$.

**Proposition 4.2.**

$$\sum_a g(m, n, a) = \binom{m+n}{m}.$$

*Proof.* The partitions counted on the left-hand side are in bijection with lattice paths connecting two opposite corners of an $m \times n$ rectangle. □

**Proposition 4.3.** $g(m, n, a) = g(m-1, n, a-n) + g(m, n-1, a)$.

*Proof.* Each lattice path $(0, 0) \to (n, m)$ passes either through $(n, m-1)$ or through $(n-1, m)$, but not both. This gives a bijection between

- lattice paths $(0, 0) \to (n, m)$ carving out a shape of size $a$, and
- the disjoint union of the following two categories of lattice paths:
  - lattice paths $(0, 0) \to (n, m-1)$ carving out a shape of size $a-n$;
  - lattice paths $(0, 0) \to (n-1, m)$ carving out a shape of size $a$.

□