CONDITIONAL PROBABILITY

DISCRETE STRUCTURES II

DARRYL HILL

BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Let's Make a Deal!

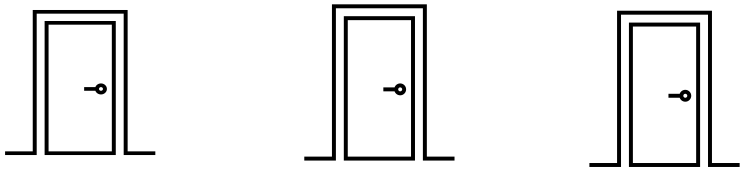
One door – Tesla Roadster
Two doors – Dogecoin

D_1 D_2 D_3

Three doors, and you do not know what is behind any of them.

The game is as follows:

1. Choose uniformly random door (but don't open it, ex. D_1)



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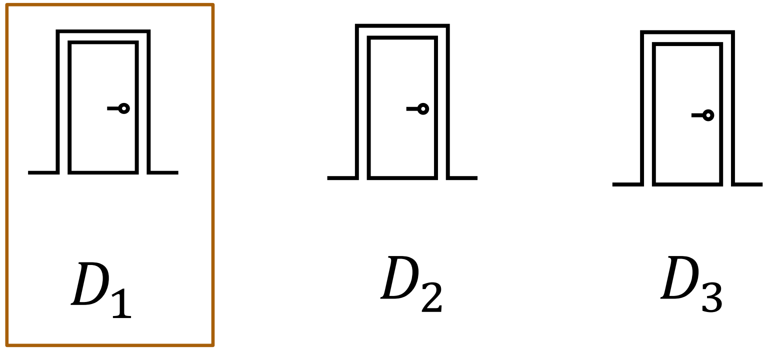
1. Choose uniformly random door (but don't open it, ex. D_1)
2. Out of the unselected doors (D_2 and D_3) Monty Hall opens one door with Dogecoin (ex. D_3).



D_1 D_2 D_3

You choose D_1

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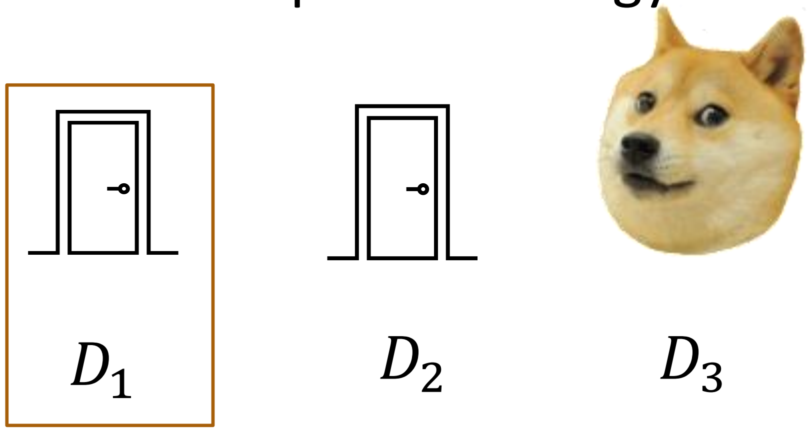
1. Choose uniformly random door (but don't open it, ex. D_1)
2. Out of the unselected doors (D_2 and D_3) Monty Hall opens one door with Dogecoin (ex. D_3).
3. Make decision – keep your door D_1 or open other door D_2?

What do you do?
Is there a superior strategy?



D_1 D_2 D_3

Do you keep D_1 or choose D_2?

You choose D_1

Monty shows you D_3

Let's Make a Deal!

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1. Choose uniformly random door (but don't open it, ex. D_1)
2. Monty Hall opens one door with Dogecoin (ex. D_3).
3. Make decision – keep your door D_1 or open other door D_2?

What sort of strategy should we use?

One thought - Monty shows you the Doge, then probability of car being behind each remaining door is ½

This is wrong – why?

What do we know?

Monty Hall knows where the Tesla is.

Monty Hall will never show you the door hiding the Tesla.

He will always show you a door hiding Doge.

Let's Make a Deal!

One door – Tesla Roadster
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1. Choose uniformly random door (but don't open it, ex. D_1)
2. Monty Hall opens one door with Dogecoin (ex. D_3).
3. Make decision – keep your door D_1 or open other door D_2?

Monty is actually giving you information because his selection is not random.

Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Let's work through this strategy.

The first door we pick has something random behind it.

What happens if we select a door with Dogecoin?

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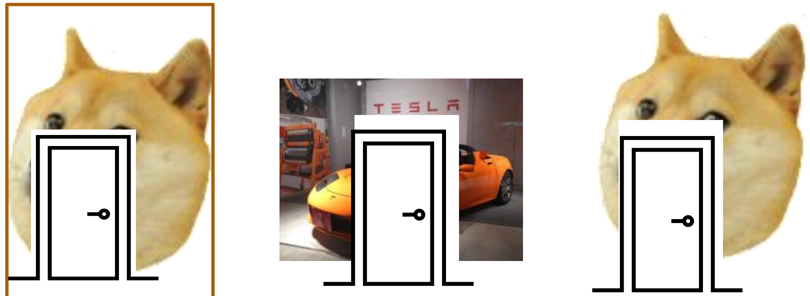
Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Dogecoin.

Monty reveals D_2 or D_3. But of course he must show D_3.



D_1 D_2 D_3

You choose D_1 Monty shows you D_3

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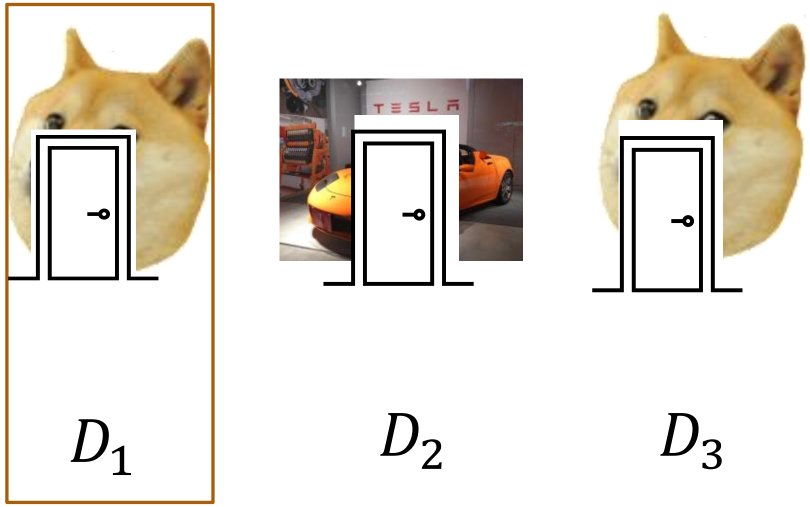
Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Dogecoin.

Monty reveals D_2 or D_3. But of course he must show D_3.

Now we switch:
We always win the Tesla if we first select Dogecoin



You choose D_1 Monty shows you D_3

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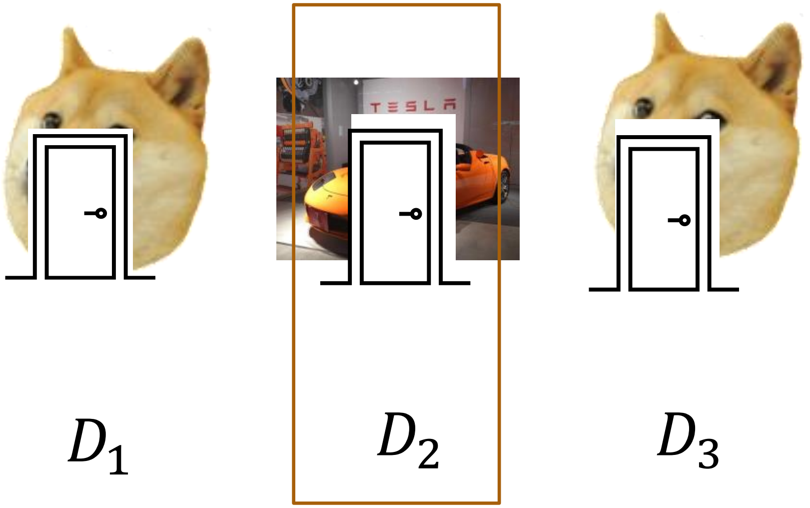
Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Tesla.

Monty reveals D_1 or D_3. In this case he can choose either.



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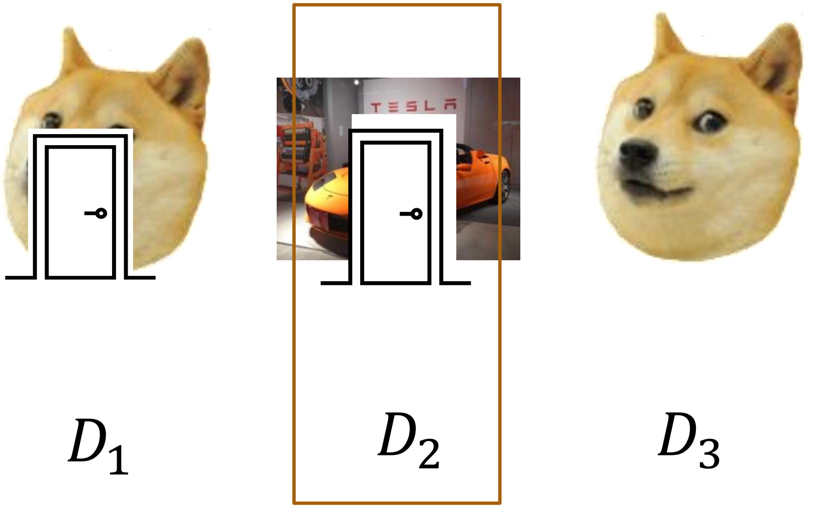
Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Tesla.

Monty reveals D_1 or D_3. In this case he can choose either.

We switch.

We always find Dogecoin.



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If we always switch doors:

Win Tesla $\leftrightarrow$ door chosen in step 1 has Dogecoin.

$$\Pr(\text{first door has Doge}) = \frac{2}{3}$$

Win Dogecoin $\leftrightarrow$ door chosen in step 1 has Tesla.

$$\Pr(\text{first door has Tesla}) = \frac{1}{3}$$

What is behind the door selected in step 1 is random. So what are the probabilities?
Always switching gives us probability $\frac{2}{3}$ of winning

Let's Make a Deal!

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1. Choose uniformly random door (but don't open it, ex. D_1)
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3. Make decision – keep your door D_1 or open other door D_2?

By knowing that Monty always reveals Doge, we arrived at different probabilities than we would suspect.

$$\Pr(\text{first door has Doge}) = \frac{2}{3}$$

$$\Pr(\text{first door has Tesla}) = \frac{1}{3}$$

This is conditional probability, which we will formalize (after the next example).

Anil's Kids

Anil Maheshwari has 2 kids.

We are told that at least 1 of his kids is a boy.

When each child was born:

$$\Pr(\text{child is a boy}) = \frac{1}{2}$$

$$\Pr(\text{child is a girl}) = \frac{1}{2}$$

Given that at least 1 kid is a boy, what is the $\Pr(\text{both are boys}) = ?$

We know 1 is a boy, so guess might be that the probability other is a boy is ½.

But again, we are given some (incomplete) information, and we should account for it.

Anil's Kids

Anil Maheshwari has 2 kids, at least 1 is a boy.

$\Pr(\text{child is a boy}) = ½$
$\Pr(\text{child is a girl}) = ½$

$\Pr(\text{both are boys}) = ?$

We know one is a boy, but we don't know which one.

Let's look at the sample space S: Anil has 2 kids.

All the possible combinations of 2 kids is

$$S = \{bb, bg, gb, gg\}$$

The first character represents the older child.

The second character represents the younger child.

Each of these outcomes has equal probability.

Our extra information – at least 1 is a boy – shrinks the sample space

Anil's Kids

Anil Maheshwari has 2 kids, at least 1 is a boy.

$\Pr(\text{child is a boy}) = ½$
$\Pr(\text{child is a girl}) = ½$

$\Pr(\text{both are boys}) = ?$

If we said the *oldest* is a boy, then $S = \{bb, bg\}$

$\Pr(\text{both are boys}) =$ ½.

$S = \{bb, bg, gb, gg\}$

We know that the outcome cannot be gg.

What are the outcomes of S that have at least 1 boy?

$$S' = \{bb, bg, gb\}$$

Now we have a sample space S' and an event $BB = \{bb\}$. What is the probability of BB?

$$\Pr(BB) = \frac{|BB|}{|S'|}$$

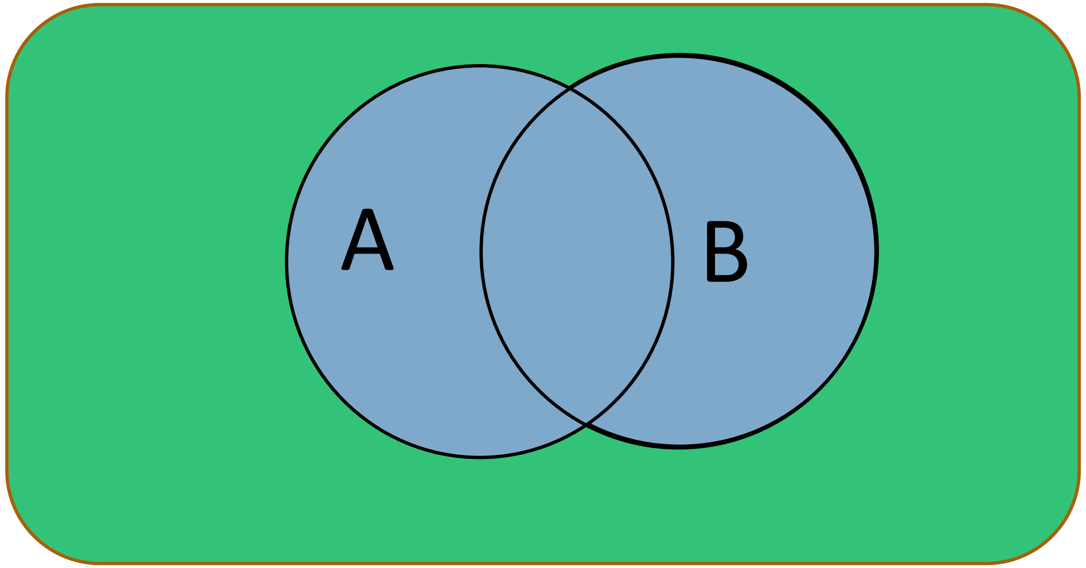
$$= \frac{1}{3}$$

Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) =$ probability of A given B

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$



One way to think of it is since we are told B is true, the set B becomes the new sample space.

Then $\Pr(A|B)$ is the probability of selecting an element of A from the sample space B.

Note that if we have uniform probability, then this is the probability of event $A \cap B$ occurring in the sample space B.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{|A \cap B|/|S|}{|B|/|S|} = \frac{|A \cap B|}{|B|}$$

This does NOT generalize (but can be useful).

Conditional Probability

Events $A, B, \Pr(B) > 0$

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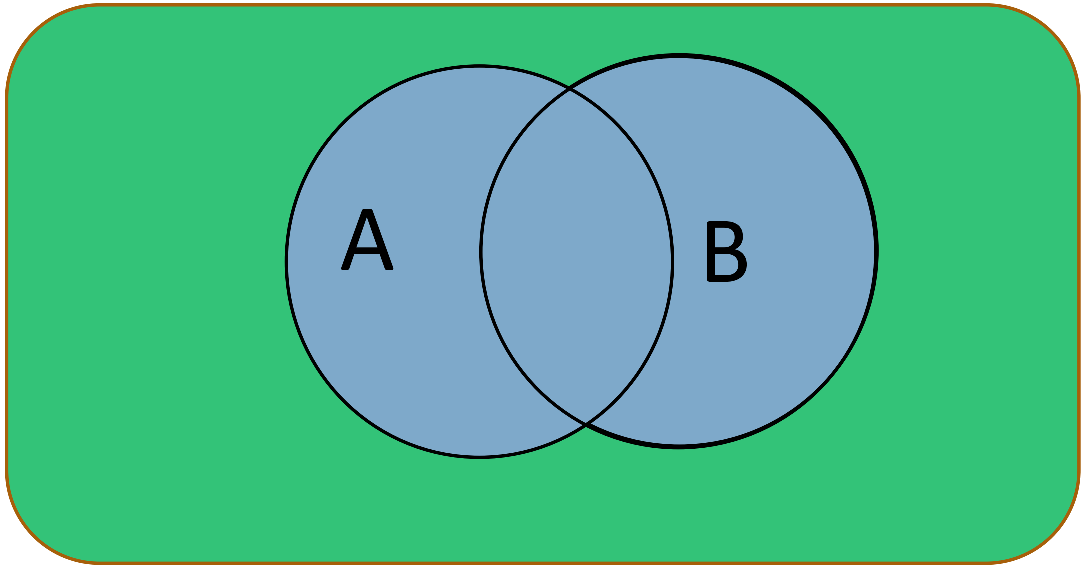


$$\Pr(A|A) = \frac{\Pr(A \cap A)}{\Pr(A)} = \frac{\Pr(A)}{\Pr(A)} = 1$$

Anil's kids:

Sample space S = two kids = $\{gg, gb, bg, bb\}$

Event B = at least one boy = $\{gb, bg, bb\}$

Event A = both are boys = $\{bb\}$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}$$

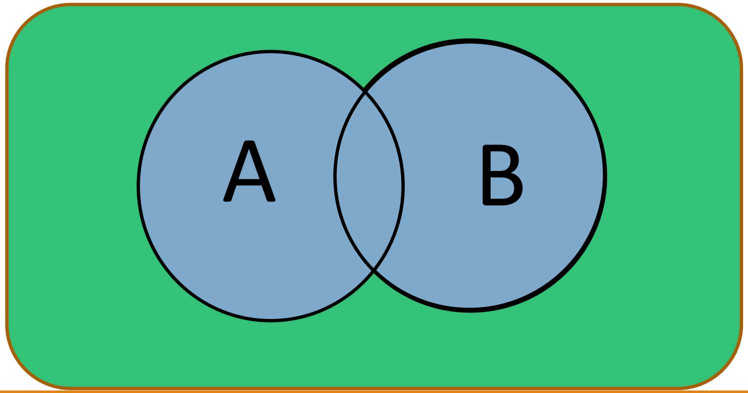
Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B)$ = probability of A given B

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

Is there a relationship between $\Pr(A|B)$ and $\Pr(B|A)$?



Roll fair die: $S = \{1, 2, 3, 4, 5, 6\}$

$A =$ "result is 3" $= \{3\}$
$B =$ "result is odd" $= \{1, 3, 5\}$
$C =$ "result is prime" $= \{2, 3, 5\}$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/6}{1/6} = 1$$

In general $\Pr(A|B) \neq \Pr(B|A)$

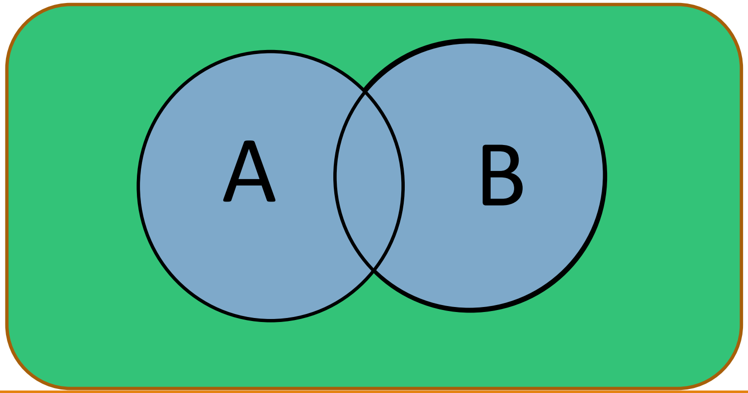
Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) =$ probability of A given B

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Is there a relationship between $\Pr(C|B)$ and $\Pr(C|\bar{B})$?



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$C =$ "result is prime" $= \{2, 3, 5\}$
$\bar{B} =$ "result is even" $= \{2, 4, 6\}$

$$\Pr(C|\bar{B}) = \frac{\Pr(C \cap \bar{B})}{\Pr(\bar{B})} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$\Pr(C|B) = \frac{\Pr(C \cap B)}{\Pr(B)} = \frac{2/6}{3/6} = \frac{2}{3}$$

$$\Pr(C|\bar{B}) + \Pr(C|B) = \frac{1}{3} + \frac{2}{3} = 1$$

Is this always true?

Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B)$ = probability of A given B

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

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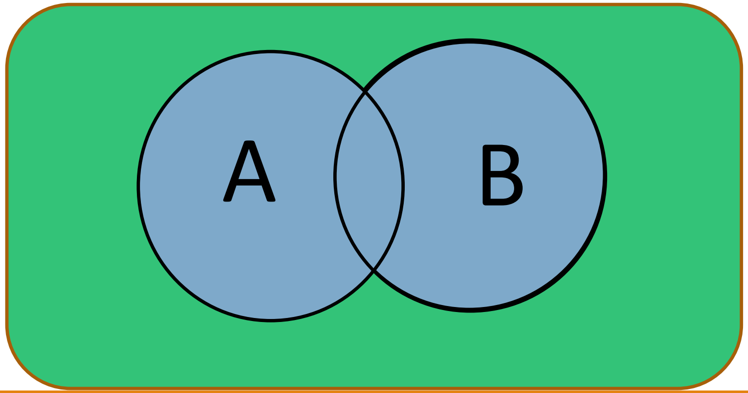

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$\bar{B} =$ "result is even" $= \{2, 4, 6\}$
$\bar{A} = \{1,2,4,5,6\}$

$$\Pr(C|A) + \Pr(C|\bar{A}) = \frac{\Pr(C \cap A)}{\Pr(A)} + \frac{\Pr(C \cap \bar{A})}{\Pr(\bar{A})}$$
$$= \frac{1/6}{1/6} + \frac{2/6}{5/6}$$
$$= \frac{1}{1} + \frac{2}{5} > 1$$

Not true in general.

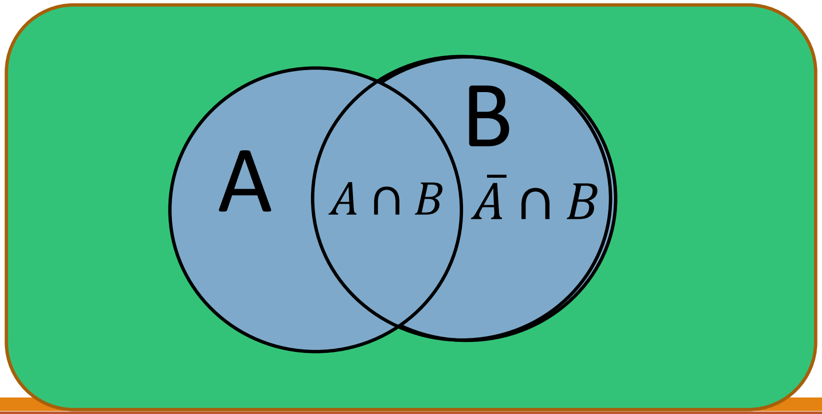
Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B)$ = probability of A given B

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

Is there a relationship between $\Pr(A|B)$ and $\Pr(\bar{A}|B)$?



Roll fair die: $S = \{1, 2, 3, 4, 5, 6\}$

$A =$ "result is 3" $= \{3\}$
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$C =$ "result is prime" $= \{2, 3, 5\}$
$\bar{B} =$ "result is even" $= \{2, 4, 6\}$
$\bar{A} = \{1,2,4,5,6\}$

$$\Pr(A|B) + \Pr(\bar{A}|B)$$
$$= \frac{\Pr(A \cap B)}{\Pr(B)} + \frac{\Pr(\bar{A} \cap B)}{\Pr(B)}$$
$$= \frac{1/6}{3/6} + \frac{2/6}{3/6}$$

$= \frac{1}{3} + \frac{2}{3} = 1$ always

Conditional Probability

Anil has 2 kids.

1 kid is a boy who was born on Sunday.

What is the probability that Anil has 2 boys?

The first thing we should do is determine the sample space.

The sample space is all combinations of 2 children. But now the children have 2 "stats".

Each child has a gender and they were born on some day of the week.

We can count S using the Product Rule, by building each individual element.

$S = \{(g_1, d_1, g_2, d_2)|$

for $i \in \{1,2\}, g_i =$ gender of child i
$d_i =$ day of the week child i was born on

where

$g_i \in \{\text{girl}, \text{boy}\}$
$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$

Thus $(\text{girl}, \text{Fri}) \in S$, $(\text{boy}, \text{Sun}) \in S$, etc.

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

2 ways to choose g_1
7 ways to choose d_1
2 ways to choose g_2
7 ways to choose d_2

Thus there are $2 \cdot 7 \cdot 2 \cdot 7 = 196$ elements in S, or $|S| = 196$.

Let A be the event that Anil has 2 boys.

Let B be the event that Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(B) = \frac{|\text{B}|}{|\text{S}|}$$

$B = B_1 \cup B_2$ where
$B_1 = \text{1st kid is a boy born on Sunday}$
$B_2 = \text{2nd kid is a boy born on Sunday}$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\}, \\ d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\} \}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$B_1 = 1^{\text{st}} \text{ kid is a boy born on Sunday}$$

$$B_1 = \{ (\text{boy}, \text{Sun}, g_2, d_2) | \\ g_2 \in \{\text{girl}, \text{boy}\}, \\ d_2 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in B_1?

1 way to choose g_1
1 way to choose d_1
2 ways to choose g_2
7 ways to choose d_2

$$|B_1| = 1 \cdot 1 \cdot 2 \cdot 7 = 14$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$B_2 =$ 2nd kid is a boy born on Sunday

$$B_2 = \{\ (g_1, d_1, \text{boy}, \text{Sun}) |$$
$$g_1 \in \{\text{girl}, \text{boy}\},$$
$$d_1 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in B_2?

2 ways to choose g_1
7 ways to choose d_1
1 ways to choose g_2
1 ways to choose d_2

$$|B_2| = 2 \cdot 7 \cdot 1 \cdot 1 = 14$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\}, d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$B =$ Anil has ≥ 1 boy born on Sunday

$B_1 =$ 1st kid is a boy born on Sunday
$B_2 =$ 2nd kid is a boy born on Sunday

$$B = B_1 \cup B_2$$

Thus $|B| = |B_1| + |B_2| - |B_1 \cap B_2|$

$$B_1 \cap B_2 = \{(\text{boy}, \text{Sun}, \text{boy}, \text{Sun})\}$$

How many elements in $B_1 \cap B_2$?

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\}, \\ d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\} \}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$B_1 =$ 1st $\text{kid is a boy born on Sunday}$
$B_2 =$ 2nd $\text{kid is a boy born on Sunday}$

How many elements in $\mathrm{B}_1 \cap B_2$?

$$B_1 \cap B_2 = \{ (\text{boy}, \text{Sun}, \text{boy}, \text{Sun})\}$$

$$|B_1 \cap B_2| = 1$$

$$B = B_1 \cup B_2$$

$$\begin{aligned} |B| &= |B_1| + |B_2| - |B_1 \cap B_2| \\ &= 14 + 14 - 1 \\ &= 27 \end{aligned}$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$
Let $A =$ Anil has 2 boys.
Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$A \cap B = \text{has 2 boys and}$$
$$\geq 1 \text{ boy was born on Sunday}$$

$$A \cap B = AB_1 \cup AB_2$$

Where:

$$AB_1 = \text{2 boys, first born Sun}$$
$$AB_2 = \text{2 boys, second born Sun}$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\}, d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\} \}$$

$|S| = 196$
Let A = Anil has 2 boys.
Let B = Anil has $\geq$ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$AB_1 = 2 \text{ boys, first born Sun}$$

$$AB_1 = \{ (\text{boy}, \text{Sun}, \text{boy}, d_2) | d_2 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in AB_1?

1 way to choose g_1
1 way to choose d_1
1 way to choose g_2
7 ways to choose d_2

$$|AB_1| = 1 \cdot 1 \cdot 1 \cdot 7 = 7$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$
Let $A =$ Anil has 2 boys.
Let $B =$ Anil has $\geq$ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$AB_2 = 2 \text{ boys, second born Sun}$$

$$AB_2 = \{\ (\text{boy}, d_1, \text{boy}, \text{Sun}) |$$
$$d_1 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in AB_2?

1 way to choose g_1
7 ways to choose d_1
1 way to choose g_2
1 way to choose d_2

$$|AB_2| = 1 \cdot 7 \cdot 1 \cdot 1 = 7$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$
Let A = Anil has 2 boys.
Let B = Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$AB_1 = 2 \text{ boys, first born Sun}$$
$$AB_2 = 2 \text{ boys, second born Sun}$$

$$AB_1 \cap AB_2 = \{\ (\text{boy}, \text{Sun}, \text{boy}, \text{Sun})\}$$

How many elements in $AB_1 \cap AB_2$?

$$|AB_1 \cap AB_2| = 1$$

$$A \cup B = AB_1 \cup AB_2$$

$$\begin{aligned} |AB_1 \cup AB_2| &= |AB_1| + |AB_2| - |AB_1 \cap AB_2| \\ &= 7 + 7 - 1 \\ &= 13 \end{aligned}$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{|\mathrm{A} \cap \mathrm{B}|/|\mathrm{S}|}{|\mathrm{B}|/|\mathrm{S}|}$$

$$= \frac{13/196}{27/196}$$

$$= \frac{13}{27} \approx 0.48$$

Flip and Flip or Roll

Given fair red coin and fair blue coin and fair die. Perform the experiment:

```
flip red coin
if heads:
    flip blue coin
    return (H, blue coin)
else:
    roll die
    return (T, die roll)
```

What is the probability that a 5 is rolled?

$$S = \{(H,H),(H,T),(T,1),(T,2),(T,3),\\ (T,4),(T,5),(T,6)\}.$$

Then the even A="a 5 is rolled" is

$$A = \{(T,5)\}.$$

We want $\Pr(A)$.

Is S a uniform probability space?

Flip and Flip or Roll

Given fair red coin and fair blue coin and fair die. Perform the experiment:

flip red coin
if heads:
    flip blue coin
    return (H, blue coin)
else:
    roll die
    return (T, die roll)

$$S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

$$A = \{(T,5)\}.$$

Let $R =$ "the red coin is tails"

$$R = \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$

We can cheat a bit, because we know $\Pr(R) = \frac{1}{2}$.

Also, $A = A \cap R$, thus $\Pr(A) = \Pr(A \cap R)$.

Now we can use the definition of conditional probability.

Flip and Flip or Roll

Given fair red coin and fair blue coin and fair die. Perform the experiment:

```
flip red coin
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else:
    roll die
    return (T, die roll)
```

$$S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

$$A = \{(T,5)\}.$$

$$R = \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$

$$\Pr(A|R) = \frac{\Pr(A \cap R)}{\Pr(R)}$$

$$\Pr(A|R) = \frac{\Pr(A)}{\Pr(R)}$$

$$\Pr(A) = \Pr(A|R) \cdot \Pr(R)$$

Flip and Flip or Roll

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$$A = \{(T,5)\}.$$

$$R = \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$

$$\Pr(A) = \Pr(A|R) \cdot \Pr(R)$$
$$\Pr(A) = \Pr(A|R) \cdot \frac{1}{2}$$

What is $\Pr(A|R)$? It is the probability that we roll a 5 given that we flipped heads on the red coin.

$$\Pr(A|R) = \frac{1}{6}, \Pr(A) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$