MATH 465 - INTRODUCTION TO COMBINATORICS
LECTURE 10

1. Triangulations of an $(n+2)$-gon

Let $\mathcal{P}_{n+2}$ be a convex $(n+2)$-gon. We consider the edge as a convex 2-gon which has $C_0 = 1$ triangulation (the empty one). For example, the pentagon has $C_3 = 5$ triangulations (Figure 1).

Theorem 1.1. *The number of triangulations of $\mathcal{P}_{n+2}$ is C_n.*

Proof. Let e be an edge of $\mathcal{P}_{n+2}$ and let T be a triangulation of $\mathcal{P}_{n+2}$. If we remove e, we obtain two triangulated polygons $\mathcal{Q}_1$ and $\mathcal{Q}_2$ (in clockwise order starting from e) with a common vertex (see Figure 2). If $\mathcal{Q}_1$ is a $(k+2)$-gon and $\mathcal{Q}_2$ is an $(l+2)$-gon, we must have $k+l=n-1$. This procedure is clearly invertible, so if h_n denotes the number of triangulations of an $(n+2)$-gon, we obtain

$$h_n = \sum_{k+l=n-1} h_k h_l, \quad h_0 = 1,$$

the recurrence for Catalan numbers. □

2. Plane trees

A *plane tree* or *ordered rooted tree* P on $n+1$ vertices is a set with $n+1$ elements, called *vertices*, defined recursively as follows:

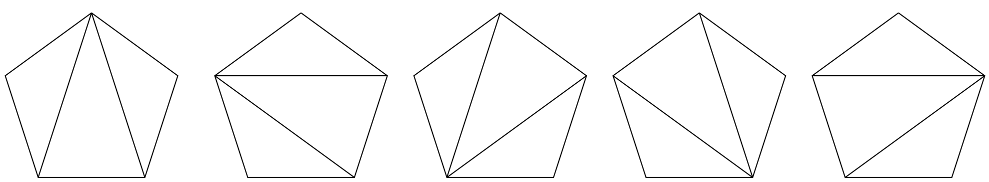


Figure 1. Triangulations of a pentagon.

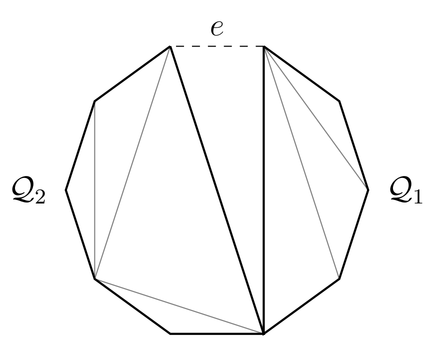


Figure 2. Removing an edge from a triangulation of $\mathcal{P}_{n+2}$.

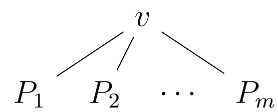


FIGURE 3. Drawing a plane tree.

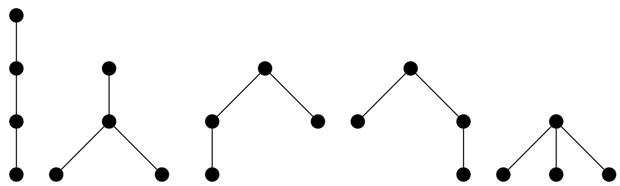


FIGURE 4. Plane trees with 4 vertices.

(1) P has a distinguished vertex v called the *root*. In particular, $P \neq \varnothing$.
(2) If $n = 0$, then $P = \{v\}$. If $n > 0$, P has a sequence $(P_1, \ldots, P_m)$ of sub-plane trees, where each $P_i, 1 \leq i \leq m$ is a plane tree with $\sum_{i=1}^{m} \#P_i = n$.

The roots of P_i are called the *children* of v, and P is drawn by placing the root v at the top, writing $P_1, \ldots, P_2$ from left to right below v and drawing edges from v to each of its children (Figure 3).

Lemma 2.1. *A plane tree with $n+1$ vertices has n edges.*

Proof. We prove this by strong induction. If $n = 0$, then $P = \{v\}$ and so P has 0 edges. Suppose the statement is true for all plane trees with fewer than $n+1$ vertices. Suppose P is a plane tree with $n+1$ vertices. Let $P_1, \ldots, P_m$ denote the set of subtrees of the root of P. Each P_i has fewer than $n+1$ vertices, so by the induction hypothesis, P_i has $\#P_i - 1$ edges. Therefore, P has $\sum_{i=1}^{m}(\#P_i - 1) + m = \#P - 1$ edges. $\square$

Theorem 2.2. *The number of plane trees with $n+1$ vertices is C_n.*

Proof. We will define a bijection between plane trees and ballot sequences. We first define an order called *preorder* on the vertices of a plane tree P. The definition is recursive. Let v be the root with subtrees $P_1, \ldots, P_m$. Define

$$\mathrm{ord}(P) = v, \mathrm{ord}(P_1), \ldots, \mathrm{ord}(P_m).$$

Traverse P in preorder, and each time we take a down step (i.e., away from the root), record a 1 and each time we take an up step (i.e., towards the root), record a -1. For the five plane trees in Figure 4, the respective sequences are

$$111 - --, \quad 11 - 1 - -, \quad 11 - -1-, \quad 1 - 11 - -, \quad 1 - 1 - 1 - .$$

By Lemma 2.1, there are n edges. Since each edge is traversed twice, the length of the sequence is $2n$. At any instant, the difference between the number of 1s and -1s is the depth below the root, so the ballot sequence condition is satisfied. This is a bijection between plane trees with $n+1$ vertices and ballot sequences of length $2n$. $\square$

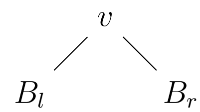


FIGURE 5. Drawing a binary tree.

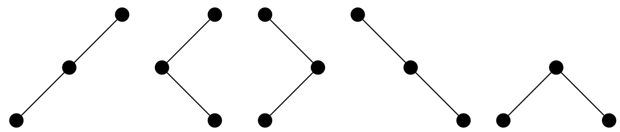


FIGURE 6. Binary trees with 3 vertices.

3. Binary trees

A *binary tree* on n vertices is a set B with $\#B = n$ defined recursively by the following rule:

(1) If $n = 0$, then $B = \varnothing$.
(2) If $n \neq 0$, then it has a root v, a left subtree B_l and a right subtree B_r with $\#B_l + \#B_r = n - 1$.

The root of B_l (resp., of B_r) is called the left (resp., right) child of v. We draw B as shown in Figure 5.

Theorem 3.1. *There are C_n binary trees with n vertices.*

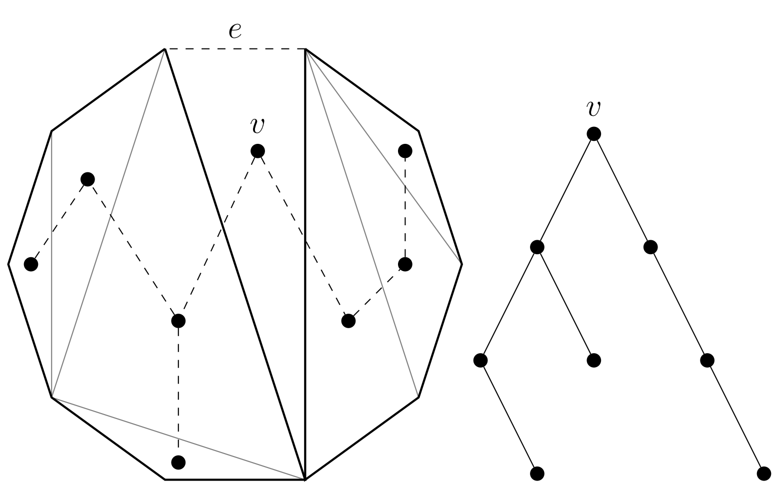
Proof. There is a bijection between binary trees with $n \geq 1$ vertices and pairs (B_l, B_r) of left and right subtrees with k, l vertices respectively such that $k + l = n - 1$. Therefore, if h_n is the number of binary trees with n vertices, we have

$$h_n = \sum_{k+l=n-1} h_k h_l, \quad h_0 = 1,$$

the Catalan recurrence. $\square$

Theorem 3.2. *Binary trees with n vertices are in bijection with triangulations of $(n+2)$-gons.*

Proof.



$\square$

4. Bracketings

A bracketing w of $n+1$ $*$s ($n \geq 0$) is defined recursively as follows:

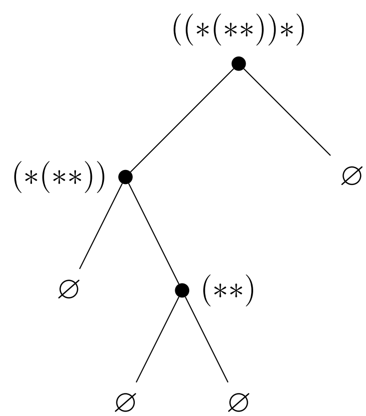
(1) If $n = 0$, then $w = *$.
(2) If $n > 0$, then $w = (st)$ where s and t are bracketings of $k+1$ and $l+1$ $*$s respectively for some k, l such that $k + l = n - 1$.

Theorem 4.1. *The number bracketings of a sequence of $n+1$ $*$s is C_n.*

$$(*(*(**))), \quad (*((**)*)), \quad ((**)(**)), \quad ((*(**))*), \quad (((**)*)*).$$

Proof. The parse tree construction provides a bijection between bracketings of $n+1$ $*$s and binary trees with n vertices. Given a binary tree B, we define the bracketing w_B recursively as follows:

(1) If $B = \varnothing$, let $w_B = *$.
(2) If v is the root with left and right subtrees B_l and B_r respectively, let $w_B = (w_{B_l} w_{B_r})$.



This procedure is invertible. Given a bracketing w, define the binary tree B_w recursively as follows: If $n = 0$, let $B_w = \varnothing$. Otherwise w is a product (st). Define the right and left subtrees of B_w to be B_s and B_t respectively. For example, if $w = (**)$, we have $s = *$ and $t = *$. Therefore, $B_s = B_t = \varnothing$, so B is the tree with one vertex.

$\square$

5. Bijections between Catalan objects

We constructed bijections between the following families of combinatorial objects counted by the Catalan numbers C_n:

$$\begin{array}{ccc}
\boxed{\begin{array}{c}\text{bracketings}\\ \text{of } n+1 \text{ letters}\end{array}} & & \boxed{\begin{array}{c}\text{Dyck paths}\\ \text{of length } 2n\end{array}} \\
\updownarrow & & \updownarrow \\
\boxed{\begin{array}{c}\text{binary trees}\\ \text{on } n \text{ vertices}\end{array}} & & \boxed{\begin{array}{c}\text{ballot sequences}\\ \text{of length } 2n\end{array}} \\
\updownarrow & & \updownarrow \\
\boxed{\begin{array}{c}\text{triangulations}\\ \text{of a convex } (n+2)\text{-gon}\end{array}} & & \boxed{\begin{array}{c}\text{plane trees}\\ \text{on } n+1 \text{ vertices}\end{array}}
\end{array}$$

Is there a bijection linking the families on the left and on the right?

Theorem 5.1. *There is a bijection between bracketings of $n+1$ letters and ballot sequences of length $2n$.*

Exercise. Prove that the following construction is a bijection. Remove all)'s and replace all ('s with a 1 and each letter with -1 except for the last letter. For example, for $n = 3$, we have

$$\begin{array}{ccccc}
(*(*(**))) & (*((**)*)) & ((**)(**)) & ((*(**))*) & (((**)*)*) \\
1-1-1- & 1-11-- & 11--1- & 11-1-- & 111---\,.
\end{array}$$

$\square$