Relations and Functions

CS 2LC3

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Tuples

- For expressions b and c, the 2-tuple (b, c) (or $\langle b, c\rangle$) is called an *ordered pair*, or simply a *pair*.
- Defining ordered pairs with sets: for any expressions b and c:

  Ordered pair: $(b, c) = \{\{b\}, \{b, c\}\}$.
- $(c, b) = \{\{c\}, \{b, c\}\} \neq \{\{b\}, \{b, c\}\} = (b, c)$
  $(a, a) = \{\{a\}\}$
- Alternative (logical) approach:

  Axiom, Pair equality: $(b, c) = (b', c') \equiv b = b' \wedge c = c'$

Cartesian (Cross) Product

- *Cartesian product* $S \times T$ of two sets S and T is the set of pairs (b, c) such that b is in S and c is in T.

  Axiom, Cartesian product:
  $S \times T = \{b, c \mid b \in S \wedge c \in T : (b, c)\}$
- Alternative, more popular notation

  $S \times T = \{(b, c) \mid b \in S \wedge c \in T\}$
- For example:
  $\{2, 5\} \times \{1, 2, 3\} = \{(2, 1), (2, 2), (2, 3), (5, 1), (5, 2), (5, 3)\}$.

Theorems for Cartesian Product (Numbers from Textbook)

(14.4) **Membership:** $\langle x, y\rangle \in S \times T \;\equiv\; x \in S \;\wedge\; y \in T$

(14.5) $\langle x, y\rangle \in S \times T \;\equiv\; \langle y, x\rangle \in T \times S$

(14.6) $S = \emptyset \;\Rightarrow\; S \times T = T \times S = \emptyset$

(14.7) $S \times T = T \times S \;\equiv\; S = \emptyset \vee T = \emptyset \vee S = T$

(14.8) **Distributivity of $\times$ over $\cup$:**

$$S \times (T \cup U) = (S \times T) \cup (S \times U)$$
$$(S \cup T) \times U = (S \times U) \cup (T \times U)$$

(14.9) **Distributivity of $\times$ over $\cap$:**

$$S \times (T \cap U) = (S \times T) \cap (S \times U)$$
$$(S \cap T) \times U = (S \times U) \cap (T \times U)$$

(14.10) **Distributivity of $\times$ over $-$:**

$$S \times (T - U) = (S \times T) - (S \times U)$$

Theorems for Cartesian Product and Extension from 2 to n

(14.11) **Monotonicity:** $T \subseteq U \;\Rightarrow\; S \times T \subseteq S \times U$

(14.12) $S \subseteq U \wedge T \subseteq V \;\Rightarrow\; S \times T \subseteq U \times V$

(14.13) $S \times T \subseteq S \times U \wedge S \neq \emptyset \;\Rightarrow\; T \subseteq U$

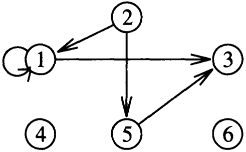
(14.14) $(S \cap T) \times (U \cap V) \;=\; (S \times U) \cap (T \times V)$

(14.15) For finite S and T, $\#(S \times T) = \#S \cdot \#T$

- We can extend the notion of a Cartesian product from two sets to n sets.
- For example, $\mathbb{Z} \times \mathbb{N} \times \{3, 4, 5\}$ is the set of triples (x, y, z) where x is an integer, y is a natural number, and $z \in \{3, 4, 5\}$.
- The theorems shown for the Cartesian product of two sets extend to theorems for the Cartesian product of n sets in the expected way.

Relations

- A *relation* on a Cartesian product $B_1 \times \ldots \times B_n$ is simply a subset of $B_1 \times \ldots \times B_n$.
- Thus, a relation is a set of n-tuples (for some fixed n).
- A *binary relation* over $B \times C$ is a subset of $B \times C$.
- If B and C are the same, so that the relation is on $B \times B$, we call it simply a (binary) relation on B .
- Any binary relation can be described by a *directed graph*. The graph has one *vertex* for each element of the set, and there is a directed edge from vertex b (say) to vertex c iff (b, c) is in the binary relation.



Examples of (binary) relations

(a) The *empty relation* on $B \times C$ is the empty set, $\emptyset$.

(b) The *identity relation* $\imath_B$ on B is $\{x \mid x \in B : \langle x, x \rangle\}$.

(c) Relation *parent* on the set of people is the set of pairs $\langle b, c \rangle$ such that b is a parent of c . Relation *child* on the set of people is the set of pairs $\langle b, c \rangle$ such that b is a child of c . Relation *sister* on the set of people is the set of pairs $\langle b, c \rangle$ such that b is a sister of c .

(d) Relation *pred* (for predecessor) on $\mathbb{Z}$ is the set of pairs $\langle b - 1, b \rangle$ for integers b , $pred = \{b{:}\mathbb{Z} \mid \langle b - 1, b \rangle\}$. Relation *succ* (for successor) is defined by $succ = \{b{:}\mathbb{Z} \mid \langle b + 1, b \rangle\}$.

(e) Relation *sqrt* on $\mathbb{R}$ is the set $\{b, c{:}\mathbb{R} \mid b^2 = c : \langle b, c \rangle\}$.

(f) An algorithm P can be viewed as a relation on states. A pair $\langle b, c \rangle$ is in the relation iff some execution of P begun in state b terminates in state c . $\square$

Notations, Domain and Range of Relations

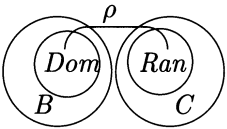
- **Notations:** Let $R \subseteq B \times C$ be a binary relation. We may write

$$bRc \text{ or } (b, c) \in R.$$

- For example: $x < y$ or $(x, y) \in <$ (or sometimes $(x, y) \in R_<$).
- We have $bR_1cR_2d \equiv bR_1c \wedge cR_2d$, etc.
- The *domain* $Dom.R$ (or $Dom(R)$) and *range* $Ran.R$ (or $Ran(R)$) of the relation R on $B \times C$ are defined by:

$$Dom.R = \{b : B \mid (\exists c \mid: bRc\} = \{b \mid \exists c \in C\ (b, c) \in R\}$$
$$Ran.R = \{c : C \mid (\exists b \mid: bRc\} = \{c \mid \exists b \in B\ (b, c) \in R\}$$

- B and $Dom.R$ need not to be the same!



Operations on Relations: $\cup, \cap, \setminus, \sim, ^{-1}$

- Suppose R and Q are relations on $B \times C$.
- Since a relation is a set, $R \cup Q$, $R \cap Q$, $R \setminus Q$, and $\sim R$ (where $\sim R = (B \times C) \setminus R$) are also relations on $B \times C$.
- The *inverse* R^{-1} of a relation R on $B \times C$ is the relation defined by:

  $$(b, c) \in R^{-1} \equiv (c, b) \in R \text{ (for all } b \in B, c \in C).$$
- Properties of inverse
  (a) $Dom(R^{-1}) = Ran(R), \quad Ran(R^{-1}) = Dom(R)$
  (b) $R \subseteq B \times C \implies R^{-1} \subseteq C \times B$
  (c) $(R^{-1})^{-1} = R$
  (d) $R \subseteq Q \equiv R^{-1} \subseteq Q^{-1}$

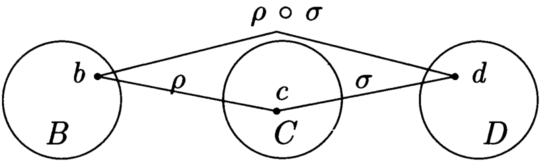
Operations on Relations: composition/product

- Let R be a relation on $B \times C$ and Q be a relation on $C \times D$.
- The *composition* or *product* of R and Q, denoted by $R \circ Q$, is the relation defined by

  $$(b, d) \in R \circ Q \equiv (\exists c \mid c \in C : (b, c) \in R \wedge (c, d) \in Q),$$

  or, using the alternative notation, by

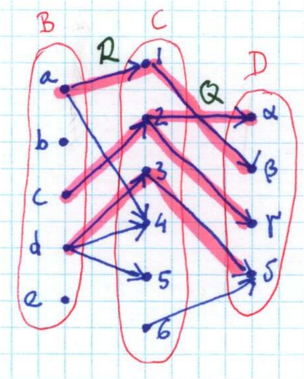
  $$b(R \circ Q)d \equiv (\exists c \mid : bRcQd).$$



$b\ (\rho \circ \sigma)\ d$ holds iff $b\ \rho\ c\ \sigma\ d$ holds for some c.

Example of Composition

$B = \{a, b, c, d, e\}, \; C = \{1, 2, 3, 4, 5, 6\}, \; D = \{\alpha, \beta, \gamma, \delta\},$
$R = \{(a, 1), (a, 4), (c, 2), (d, 3), (d, 4), (d, 5)\} \subseteq B \times C,$
$Q = \{(1, \beta), (2, \alpha), (2, \gamma), (3, \delta), (6, \delta)\} \subseteq C \times D,$
$R \circ Q = \{(a, \beta), (c, \alpha), (c, \gamma), (d, \delta)\} \subseteq B \times D.$



- The figure above represents another useful representation of compositin of finite relations.

Properties of Composition/Product of Relations

- **Associativity of $\circ$:** $R \circ (Q \circ S) = (R \circ Q) \circ S$
- **Distributivity of $\circ$ over $\cup$:** $R \circ (Q \cup S) = R \circ Q \cup R \circ S$
  $(Q \cup S) \circ R = Q \circ R \cup S \circ R$
- **Distributivity of $\circ$ over $\cap$:** $R \circ (Q \cap S) \subseteq R \circ Q \cap R \circ S$
  $(Q \cap S) \circ R \subseteq Q \circ R \cap S \circ R$

Power of Relations

- Let $R \subseteq B \times B$ be a relation. The relation $R \circ R$ is often written as R^2, $R \circ R \circ R$ as R^3, etc.
- Formally, R^n, $n \geq 0$ can be defined as follows:

  $R^0 = id_B = \{(b, b) \mid : b \in B\}$ (identity on B, diagonal of B)
  $R^{n+1} = R^n \circ R$ (for $n \geq 0$)
- Properties of powers:
  (a) $R^m \circ R^n = R^{m+n}$ (for $m \geq 0, n \geq 0$)
  (b) $(R^m)^n = R^{m \cdot n}$ (for $m \geq 0, n \geq 0$)

Functions as Relations

- **Definition.** A binary relation f on $B \times C$ is called a **function** iff it is *determinate*:

  Determinate: $(\forall b, c, c' \mid bfc \wedge bfc' : c = c')$
  or, using more traditional notation:
  $\forall b, c, c'(bfc \wedge bfc' \implies c = c')$.
- **Definition.** A function f on $B \times C$ is *total* if

  Total: $B = Dom(f)$;

  otherwise it *partial*.
- We write $f : B \rightarrow C$ for the type of f if f is *total*.
- We write $f : B \rightsquigarrow C$ (or $f : B \rightharpoonup C$, $f : B \hookrightarrow C$, $f : B \nrightarrow C$, or just $f : B \rightarrow C$ and a comment "partial") if f is *partial*.
- Composition of partial functions might be *undefined*!
- For each partial function $f : B \rightsquigarrow C$, the function $f : Dom(B) \rightarrow C$ is total.

Examples of Functions as Relations

(a) Binary relation $<$ is not a function, because it is not determinate —both $1 < 2$ and $1 < 3$ hold.

(b) Identity relation $\imath_B$ over B is a total function $\imath_B : B \to B$; $\imath.b = b$ for all b in B.

(c) Total function $f{:}\mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$ is the relation $\{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \ldots\}$.

(d) Partial function $f{:}\mathbb{N} \rightsquigarrow \mathbb{Q}$ defined by $f(n) = 1/n$ is the relation $\{\langle 1, 1/1\rangle, \langle 2, 1/2\rangle, \langle 3, 1/3\rangle, \ldots\}$. It is partial because $f.0$ is not defined.

(e) Function $f{:}\mathbb{Z}^+ \to \mathbb{Q}$ defined by $f(b) = 1/b$ is total, since $f.b$ is defined for all elements of $\mathbb{Z}^+$, the positive integers. However, $g{:}\mathbb{N} \rightsquigarrow \mathbb{Q}$ defined by $g.b = 1/b$ is partial because $g.0$ is not defined.

(f) The partial function f that takes each lower-case character to the next character can be defined by a finite number of pairs: $\{\langle \text{'}a\text{'}, \text{'}b\text{'}\rangle, \langle \text{'}b\text{'}, \text{'}c\text{'}\rangle, \ldots, \langle \text{'}y\text{'}, \text{'}z\text{'}\rangle\}$. It is partial because there is no pair whose first component is 'z'. $\square$

Composition of Total Functions

- **Theorem.** Let $f : B \to C$ and $g : C \to D$. We have:

  $(f \circ g)(b) = g(f(c))$, for all $b \in B$.

  The proof is simple:

  $$
  \begin{aligned}
  & (f \circ g)(b) \\
  = & \langle\ f \circ g \text{ as a relation } \rangle \\
  & b(f \circ g)d \\
  = & \langle\ \text{Definition of composition for relations} \rangle \\
  & (\exists c \mid: bfc \wedge cgd) \\
  = & \langle\ f, g \text{ are total functions } \rangle \\
  & (\exists c \mid: f(b) = c \wedge g(c) = d) \\
  = & \langle\ \text{Trading } \rangle \\
  & (\exists c \mid: c = f(b) \wedge d = g(c)) \\
  = & \langle\ \text{One-point rule for quantifiers } \rangle \\
  & g(f(b))
  \end{aligned}
  $$

- **Definition**. For functions f and g, $f \bullet g = g \circ f$.

Powers of Functions

- The theory of binary relations tells us that function composition is associative: $(f \bullet g) \bullet h = f \bullet (g \bullet h)$.
- *Powers* of a function $f : B \to B$ are defined as follows:

  f^0 is the *identity function*: $f^0(b) = b$,
  $f^{n+1}(b) = f(f^n(b))$, fro all for$n \geq 0$.

Functions as Sets

- Let $f : B \to C$ and $g : B \to C$ be total functions.
- The relation $f \cap g$ is a *partial* function $(f \cup g) : C \rightsquigarrow B$, unless $f = g$. Moreover

$$(f \cap g)(x) = \begin{cases} f(x) & \text{if } f(x) = g(x) \\ \text{undefined} & \text{otherwise} \end{cases}$$

- If $f \neq g$ then the relation $f \cup g$ is *never a function*, even partial. If $f \neq g$ then there is at least one x such that $f(x) \neq g(x)$, so we have $(x, f(x)) \in f \cup g$ and $(x, g(x)) \in f \cup g$, so $f \cup g$ is not determinate.
- Example: $f(x) = 5 \cdot x$, $g(x) = x^2$. then $\{(2,10),(2,4)\} \subseteq f \cup g$.

Basic Classes of Relations

- A few classes of relations that enjoy certain properties are used frequently.
- The most popular properties are given below, in two equivalent forms.

	Name	Property	Alternative
(a)	reflexive	$(\forall b \mid : b \; \rho \; b)$	$\imath_B \subseteq \rho$
(b)	irreflexive	$(\forall b \mid : \neg(b \; \rho \; b))$	$\imath_B \cap \rho \; = \; \emptyset$
(c)	symmetric	$(\forall b, c \mid : b \; \rho \; c \; \equiv \; c \; \rho \; b)$	$\rho^{-1} \; = \; \rho$
(d)	antisymmetric	$(\forall b, c \mid : b \; \rho \; c \; \wedge \; c \; \rho \; b \; \Rightarrow \; b = c)$	$\rho \cap \rho^{-1} \; \subseteq \; \imath_B$
(e)	asymmetric	$(\forall b, c \mid : b \; \rho \; c \; \Rightarrow \; \neg(c \; \rho \; b))$	$\rho \cap \rho^{-1} \; = \; \emptyset$
(f)	transitive	$(\forall b, c, d \mid : b \; \rho \; c \; \wedge \; c \; \rho \; d \; \Rightarrow \; b \; \rho \; d)$	$\rho = (\cup i \mid i > 0 : \rho^{\imath})$

Examples of Classes Of Relations

(a) Relation $\leq$ on $\mathbb{Z}$ is reflexive, since $b \leq b$ holds for all integers b. It is not irreflexive. Relation $<$ on $\mathbb{Z}$ is not reflexive, since $2 < 2$ is false. It is irreflexive.

(b) Consider relation *square* on $\mathbb{Z}$ that is defined by $b \; square \; c$ iff $b = c \cdot c$. It is not reflexive because it does not contain the pair $\langle 2, 2 \rangle$. It is not irreflexive because it does contain the pair $\langle 1, 1 \rangle$. Thus, a relation that is not reflexive need not be irreflexive.

(c) Relation $=$ on the integers is symmetric, since $b = c \;\equiv\; c = b$. Relation $<$ is not symmetric.

(d) Relation $\leq$ is antisymmetric since $b \leq c \wedge c \leq b \;\Rightarrow\; b = c$. Relation $<$ is antisymmetric: since $b < c \;\wedge\; c < b$ is always *false*, we have $b < c \wedge c < b \Rightarrow b = c$ for all b, c. Relation $\neq$ is not antisymmetric.

(e) Relation $<$ is asymmetric, since $b < c$ implies $\neg(c < b)$. Relation $\leq$ is not asymmetric.

(f) Relation $<$ is transitive, since if $b < c$ and $c < d$ then $b < d$. Relation *parent* is not transitive. However, relation *ancestor* is transitive, where $b \; ancestor \; c$ holds if b is an ancestor of c. $\square$

Closures

- The *closure* of a relation R with respect to some property (e.g. reflexivity) is the *smallest* relation that both has that property and contains R.
- To construct a closure, add pairs to R, but not too many, until it has the property.
- For example, the reflexive closure of $<$ over the integers is the relation constructed by adding to relation $<$ all pairs (b, b) for b an integer.
- Therefore, $\leq$ is the *reflexive closure* of $<$.
- The construction of a closure does not always make sense.
- For example, the irreflexive closure of a relation containing $(1, 1)$ doesn't exist, since it is precisely the *presence* of this pair that makes the relation not irreflexive.

Definition (14.30 in the textbook)

Let R be a relation, $R \subseteq B \times B$. The *reflexive* (*symmetric*, *transitive*, etc.,) closure of R is the relation $R' \subseteq B \times B$ that satisfies:

(a) *reflexive* (*symmetric*, *transitive*, etc.,),

(b) $R \subseteq R'$,

(c) If R'' is *reflexive* (*symmetric*, *transitive*, etc.,) and $R'' \subseteq R$, then $R' \subseteq R''$.

Notation:

- $r(R)$ - the reflexive closure of R;
- $s(R)$ - the symmetric closure;
- R^+ - the transitive closure;
- R^* - the reflexive transitive closure.

Examples and Simple Properties of Closures

(a) The reflexive closure $r(<)$ of $<$ on the integers is $\leq$.

(b) The symmetric closure $s(parent)$ of $parent$ is $parent \cup child$, since if $\langle b, c\rangle$ is in the symmetric closure, then so is $\langle c, b\rangle$.

(c) The transitive closure $parent^+$ of $parent$ is $ancestor$, since whenever $\langle b, c\rangle$ and $\langle c, d\rangle$ are in the transitive closure, then so is $\langle b, d\rangle$.

(d) The reflexive transitive closure $parent^*$ of $parent$ is the relation $ancestor\text{-}or\text{-}self$. $\square$

Theorem

A reflexive relation is its own reflexive closure;
a symmetric relation is its own symmetric closure;
and a transitive relation is its own transitive closure.

Alternative Explicit Definitions of Closures

Theorem

Let $R \subseteq B \times b$. *Then*

(a) $r(R) = R \cup id_B$, *where* id_B *is the identity relation on* B.

(b) $s(R) = R \cup R^{-1}$.

(c) $R^{+} = \bigcup(i \mid 0 < i : R^i)$, *or equivalently* $R^{+} = \bigcup_{i=1}^{\infty} R^i$.

(d) $R^{*} = R^{+} \cup id_B = \bigcup(i \mid 0 \leq i : R^i)$, *or equivalently*
$R^{*} = \bigcup_{i=0}^{\infty} R^i$.

Proof.

We will prove only (c) as this is the only one that is non trivial. Our proof will be a little bit of high level, a very detailed low level proof in in the textbook. ■

Proof.

We will prove: $R^+ = \bigcup(i \mid 0 < i : R^i)$, or $R^+ = \bigcup_{i=1}^{\infty} R^i$, is a transitive closure according to Definition from page 22.

(a) We need to show that R^+ is transitive. Assume bR^+c and cR^+d. This means there are $i_1 > 0$ and $i_2 > 0$ such that $bR^{i_1}c$ and $cR^{i_2}d$. Hence $bR^{i_1}c \circ R^{i_2}d = bR^{i_1+i_2}d \Rightarrow bR^+d$. So R^+ is transitive.

(b) Clearly $R \subseteq R^+ = R \cup R^2 \cup R^3 \cup \ldots$

(c) Assume R' is transitive and $R \subseteq R'$. We will show that $R^+ \subseteq R'$. Since $R^+ = \bigcup_{i=1}^{\infty} R^i$, it suffices to show that $R^i \subseteq R'$ for all $i \geq 1$. First note that if a relation Q is transitive than $Q^i \subseteq Q$. This just follows from the fact that

$b_0 Q b_1 Q b_2 \ldots b_{i-1} Q b_i \Rightarrow b_0 Q b_i$ for any transitive relation Q.

Since $R \subseteq R'$, then $R^i \subseteq (R')^i \subseteq R'$, so we are done. ∎