Problem 1: (a) Keywords: linear span, matrix multiplication, addition of vector space

Let A be an $m \times k$-matrix, and let B be a $k \times n$-matrix. (a) Prove that the column space of AB is contained in the column space of A.

Subset proof format: Prove the implication

$$\vec{v} \in Col(AB) \implies \vec{v} \in Col(A)$$

.

Proof.

Definition of $Col(A)$:

$$Col(A) := \{A\mathbf{v} | \mathbf{v} \in \mathbb{R}^n\}$$
$$Row(A) := \{A^T\mathbf{v} | \mathbf{v} \in \mathbb{R}^n\}$$

Let $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k]$, $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n]$ Then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, A\mathbf{b}_3, \cdots, A\mathbf{b}_n]$$

Let $\vec{v} \in Col(AB)$, then

$$\vec{v} = \sum_{i=1}^{n} c_i A\mathbf{b}_i$$

By **conservation of addition of vector space**.

Problem 1: (b) Keywords: Definition of Nullspace.

Assume that $k = m$ and that A is invertible. Prove that the null space of AB is equal to the null space of B.

Proof.($\implies$)

Let $x \in \text{Null}(B)$, then we can easily see that $ABx = A(Bx) = A \cdot \vec{0} = \vec{0}$ Thus, $x \in \text{Null}(AB)$

$(\Longleftarrow)$

Let $x \in \text{Null}(AB)$, then $ABx = 0$. Since A is invertible, $\text{Null}(A) = \{\mathbf{0}\}$. This forces $Bx = 0$. Therefore, $x \in \text{Null}(B)$.

Problem 1: (c) keywords: Rank-nullity theorem, Dimension and rank difference

We know that B and AB is $m \times n$. Therefore $\dim AB = \dim B = n$. From part (b) we know that $\text{nullity}(AB) = \text{nullity}(B)$. Thus, by rank-nullity theorem,

$$\begin{aligned}
\text{rank}(AB) &= \dim(AB) - \text{nullity}(AB) \\
&= \dim(B) - \text{nullity}(B) \\
&= \text{rank}(B)
\end{aligned}$$

Comment: Rank of a matrix is the actual dimension of this matrix

Problem 2: 2