EULER'S THM 6.3.5 LET γ BE UNIT SPEED CURVE ON SURFACE σ;
κ_1, κ_2 PRINCIPAL CURVATURES OF σ
WITH UNIT PRINCIPAL VECTORS t_1, t_2.
THEN

$$\kappa_n = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta)$$

WITH $\theta = \sphericalangle(\dot{\gamma}, t_1)$

PROOF: WRITE

$$t_i = \xi_i \sigma_u + \eta_i \sigma_v \quad , \quad T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$$

$$\dot{\gamma} = \xi \sigma_u + \eta \sigma_v \quad , \quad T = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

WE HAVE

$$\dot{\gamma} = \cos(\theta)\, t_1 + \sin(\theta)\, t_2 .$$

THUS

$$\cos(\theta)(\xi_1 \sigma_u + \eta_1 \sigma_v) + \sin(\theta)(\xi_2 \sigma_u + \eta_2 \sigma_v)$$

$$= \xi \sigma_u + \eta \sigma_v$$

$$\Rightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \cos(\theta) \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} + \sin(\theta) \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

$$\Rightarrow T = \cos(\theta)\, T_1 + \sin(\theta)\, T_2 .$$

PROVED EARLIER

$$
\begin{aligned}
\varkappa_n &\overset{\downarrow}{=} T^t \mathcal{F}_{II} T \\
&= \left(\cos(\theta) T_1^T + \sin(\theta) T_2^T\right) \mathcal{F}_{II} \left(\cos(\theta) T_1 + \sin(\theta) T_2\right) \\
&= \cos^2(\theta) \underbrace{T_1^T \mathcal{F}_{II} T_1}_{=\varkappa_1} + \sin^2(\theta) \underbrace{T_2^T \mathcal{F}_{II} T_2}_{=\varkappa_2} \\
&\quad + \sin(\theta)\cos(\theta) \Big(\underbrace{T_1^T \mathcal{F}_{II} T_2}_{=0} + \underbrace{T_2^T \mathcal{F}_{II} T_1}_{=0} \Big) \\
&\overset{\#}{=} \varkappa_1 \cos^2(\theta) + \varkappa_2 \sin^2(\theta). \qquad \square .
\end{aligned}
$$

<u>COROLLARY 6.3.6</u> THE PRINCIPAL CURVATURES ARE THE MAXIMUM AND MINIMUM VALUES OF THE NORMAL CURVATURE OF ALL UNIT SPEED CURVES ON THE SURFACE PASSING THROUGH THE POINT. THE PRINCIPAL VECTORS (DIRECTIONS) ARE THE TANGENT VECTORS OF THE CURVES GIVING THESE MAXIMUM/MINIMUM VALUES.

<u>PROOF</u>: (a) $\varkappa_1 \neq \varkappa_2$. ASSUME $\varkappa_1 > \varkappa_2$

$$
\begin{aligned}
\varkappa_n &= \varkappa_1 \underbrace{\cos^2(\theta)}_{=1-\sin^2(\theta)} + \varkappa_2 \sin^2(\theta) \\
&= \varkappa_1 - \underbrace{(\varkappa_1 - \varkappa_2)}_{>0} \sin^2(\theta)
\end{aligned}
$$

$\Rightarrow\ \varkappa_n \le \varkappa_1$ AND "=" $\Leftrightarrow\ \theta \in \{0, \pi\}$.
$\Leftrightarrow\ \dot{\gamma} \parallel t_1$

SIMILARLY:

$\varkappa_n \ge \varkappa_2$ AND "=" $\Leftrightarrow\ \dot{\gamma} \parallel t_2$

(b) $\varkappa_1 = \varkappa_2$. THEN $\varkappa_n = \varkappa_1 = \varkappa_2$ AND EVERY NONZERO TANGENT VECTOR IS PRINCIPAL VECTOR. $\square$

PRINCIPAL CURVATURES TELL US SHAPE OF SURFACE NEAR POINT.

CONSIDER QUADRIC SURFACE

$$z = \varkappa' x^2 + \varkappa'' y^2 \qquad \varkappa', \varkappa'' \in \mathbb{R}.$$

PARAMETRIZATION:

$$\sigma(u,v) = (u, v, \varkappa' u^2 + \varkappa'' v^2)$$

$$\sigma_u = (1, 0, 2\varkappa' u)$$
$$\sigma_v = (0, 1, 2\varkappa'' v)$$
$$\sigma_{uu} = (0, 0, 2\varkappa'), \quad \sigma_{vv} = (0, 0, 2\varkappa'')$$
$$\sigma_{uv} = 0.$$

AT $p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\sigma_u = (1,0,0) \quad , \quad \sigma_v = (0,1,0)$$

$$E = \sigma_u \cdot \sigma_u = 1 \, , \; F = \sigma_u \cdot \sigma_v = 0 \, , \; G = \sigma_v \cdot \sigma_v = 1$$

$$\vec{N} = \frac{\sigma_u \times \sigma_v}{\| \sigma_u \times \sigma_v \|} = (0,0,1)$$

$$L = \sigma_{uu} \cdot \vec{N} = 2\varkappa' \, , \; M = \sigma_{uv} \cdot \vec{N} = 0$$

$$N = \sigma_{vv} \cdot \vec{N} = 2\varkappa''.$$

PRINCIPAL CURVATURES OF ~~S~~ σ ARE ROOTS OF

$$\det \left(\begin{pmatrix} 2\varkappa' & 0 \\ 0 & 2\varkappa'' \end{pmatrix} - \varkappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0 :$$

$$\boxed{2\varkappa' \, , \; 2\varkappa''}$$

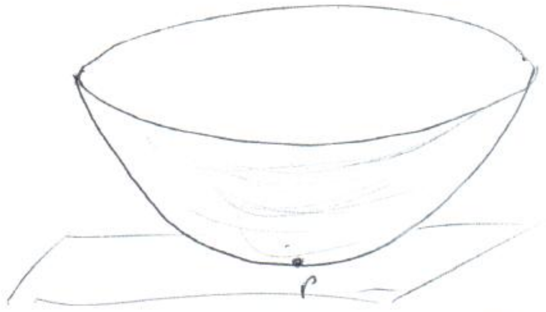
CONCLUSION: NEAR A POINT p OF A SURFACE AT WHICH THE PRINCIPAL CURVATURES ARE $\varkappa_1, \varkappa_2$, THE SURFACE LOOKS LIKE THE QUADRIC

$$z = \frac{1}{2} \left(\varkappa_1 x^2 + \varkappa_2 y^2 \right) .$$

DISCUSS 4 CASES:

(i) $\kappa_1, \kappa_2 > 0$ OR $\kappa_1, \kappa_2 < 0$. $\kappa_1 \cdot \kappa_2 > 0$

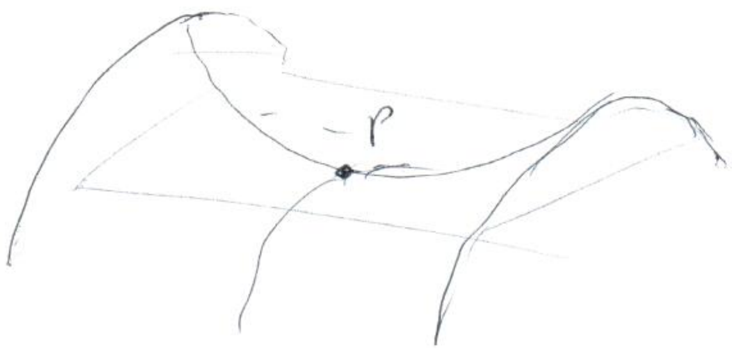
ELLIPTIC PARABOLOID:



P "ELLIPTIC" POINT

(ii) $\kappa_1 \cdot \kappa_2 < 0$

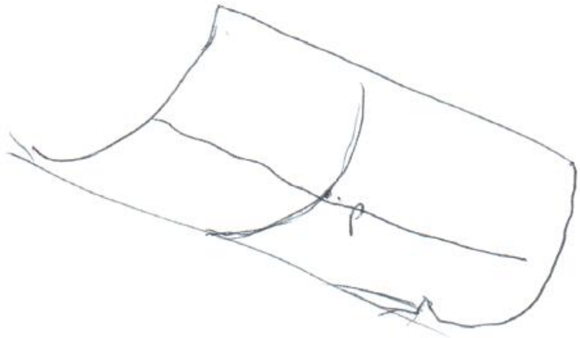
HYPERBOLIC PARABOLOID



P "HYPERBOLIC" POINT +

(iii) $x_1 x_2 = 0$, $x_2 \neq 0$

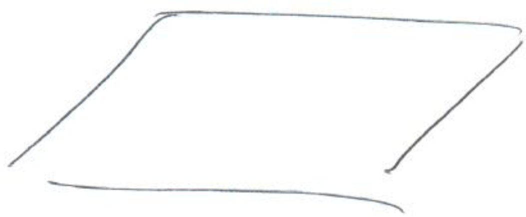
PARABOLIC CYLINDER



P "PARABOLIC" POINT

(iv) $x_1 = x_2 = 0$

PLANE



P "PLANAR" POINT.

IN THIS CASE NEED TO ~~DETERMINE~~ EXAMINE DERIVATIVES OF HIGHER ORDER TO DETERMINE SHAPE.

EXAMPLE 6.3.7 (SPHERE)

INTUITION: SAME CURVATURE EVERYWHERE.

5.1.2: $E = 1$, $F = 0$, $G = \cos^2(\theta)$

6.1.2: $L = 1$, $M = 0$, $N = \cos^2(\theta)$

PRINCIPAL CURVATURES ARE ROOTS OF

$$\det\left(\begin{pmatrix} 1 & 0 \\ 0 & \cos^2(\theta) \end{pmatrix} - \varkappa \begin{pmatrix} 1 & 0 \\ 0 & \cos^2(\theta) \end{pmatrix}\right) = 0$$

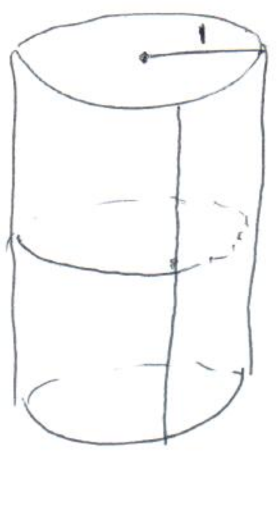
$$= (1-\varkappa)^2 \cos^2(\theta)$$

$$\Longrightarrow \quad \varkappa_1 = \varkappa_2 = 1$$

ALL POINTS ARE ELLIPTIC

EVERY NONZERO TANGENT VECTOR IS A PRINCIPAL VECTOR.

EXAMPLE 6.3.8. (CIRCULAR CYLINDER)



INTUITION:

$\varkappa_1 = 1$

$\varkappa_2 = 0$.

$$\sigma(u,v) = (\cos(v), \sin(v), u)$$

5.1.3: $E = 1$, $F = 0$, $G = 1$

6.1.2.: $L = 0$, $M = 0$, $N = 1$

$$0 = \det\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \kappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= -\kappa(1-\kappa)$$

$\Rightarrow$ PRINCIPAL CURVATURES ARE 0, 1

ALL POINTS ARE PARABOLIC.

FIND PRINCIPAL VECTORS:

$$t_i = \xi_i \sigma_u + \eta_i \sigma_v, \quad T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$$

SOLVE

$$(\mathcal{F}_{II} - \kappa_i \mathcal{F}_I) T_i = 0$$

$$\shortparallel$$

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \kappa_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) T_i$$

$$\shortparallel$$

$$\begin{pmatrix} -\kappa_i & 0 \\ 0 & 1-\kappa_i \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$$

$$x_1 = 1 : \quad \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \eta_1\end{pmatrix} = 0$$

$$\parallel$$

$$\begin{pmatrix}-\xi_1\\ 0\end{pmatrix}$$

$\Rightarrow$ t_1 MULTIPLE OF σ_v

~~t_1 MULT~~

$$t_1 \in \mathbb{R}\left(-\sin(v), \cos(v), 0\right)$$

$$x_2 = 0 : \quad 0 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi_2\\ \eta_2\end{pmatrix} = \begin{pmatrix}0\\ \eta_2\end{pmatrix}$$

$\Rightarrow$ t_2 MULTIPLE OF σ_u

$$t_2 \in \mathbb{R}(0, 0, 1)$$

COINCIDES WITH INTUITION!

7 GAUSSIAN CURVATURE AND THE GAUSS MAP

7.1 GAUSSIAN AND MEAN CURVATURES

DEF 7.1.1. LET $\varkappa_1, \varkappa_2$ BE THE PRINCIPAL CURVATURES OF A SURFACE. DEFINE

GAUSSIAN CURVATURE $K = \varkappa_1 \varkappa_2$

MEAN CURVATURE $H = \frac{1}{2}(\varkappa_1 + \varkappa_2)$

PROP 7.1.2

(i) $$K = \frac{LN - M^2}{EG - F^2}$$

(ii) $$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

(iii) $$\varkappa_{1/2} = H \pm \sqrt{H^2 - K}$$

PROOF x_1, x_2 ARE THE ROOTS OF

$$0 = \det \begin{pmatrix} L - xE & M - xF \\ M - xF & N - xG \end{pmatrix}$$

$$= (L - xE)(N - xG) - (M - xF)^2$$

$$= (EG - F^2)x^2 - (LG - 2MF + NE)x + (LN - M^2)$$

RECALL: IF $ax^2 + bx + c = 0$, THEN

$-\frac{b}{a}$ = SUM OF ROOTS

$\frac{c}{a}$ = PRODUCT OF ROOTS

THUS

$$K = x_1 x_2 = \frac{LN - M^2}{EG - F^2}$$

$$H = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}$$

AND x_1, x_2 ARE ROOTS OF

$$x^2 - 2Hx + K = 0$$

$$\Rightarrow x_1, x_2 = H \pm \sqrt{H^2 - K}. \quad \square$$

EXAMPLES 7.1.3. (UNIT SPHERE)

6.3.7 : $\varkappa_1 = \varkappa_2 = 1$

$$\Rightarrow \quad K = H = 1$$

(CIRCULAR CYLINDER OF RADIUS 1)

6.3.8 : $\varkappa_1 = 1$, $\varkappa_2 = 0$

$$\Rightarrow \quad K = 0, \quad H = \frac{1}{2}.$$

K CONSTANT > 0 : SPHERE

K CONSTANT $= 0$: CYLINDER

K CONSTANT < 0 : ?

EXAMPLE 7.1.4 (PSEUDOSPHERE)

$$\sigma(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

SURFACE OF REVOLUTION

ASSUME $f > 0$, $\dot{f}^2 + \dot{g}^2 = 1$ (SEE 6.1.2)

6.1.2:

$$E = 1 \ , \ F = 0 \ , \ G = f^2$$

$$L = \dot f \ddot g - \ddot f \dot g \ , \ M = 0 \ , \ N = f \dot g$$

THUS

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(\dot f \ddot g - \ddot f \dot g) f \dot g}{f^2}$$

FROM $1 = \dot f^2 + \dot g^2$ WE GET

$$0 = \dot f \ddot f + \dot g \ddot g$$

$$\Rightarrow (\dot f \ddot g - \ddot f \dot g) \dot g = - \dot f^2 \ddot f - \ddot f \dot g^2$$

$$= - \ddot f (\dot f^2 + \dot g^2) = - \ddot f$$

$$\Rightarrow \quad K = - \frac{\ddot f}{f}$$

THUS

$$K \equiv -1 \iff \ddot f = f$$

$$\iff f(u) = a e^{u} + b e^{-u} \quad , a, b \in \mathbb{R}.$$

CHOOSE $f(u) = e^{u}$.

$\Rightarrow g = ??$

$$\dot{g}^2 = 1 - \dot{f}^2 = 1 - e^{2u}$$

$$\Rightarrow g(u) = \int \sqrt{1 - e^{2u}}\, du$$

PUT $v = e^{u}$. THEN $\dfrac{dv}{du} = e^{u} = v$

$$\Rightarrow du = \frac{1}{v}\, dv$$

$$\Rightarrow g(u) = \int \sqrt{1-e^{2u}}\, du = \int \frac{\sqrt{1-v^2}}{v}\, dv$$

$$= \int \frac{1-v^2}{v} \frac{1}{\sqrt{1-v^2}}\, dv = \int \left(\frac{1}{v} - v\right) \frac{1}{\sqrt{1-v^2}}\, dv$$

$$= \underbrace{\int \frac{1}{v\sqrt{1-v^2}}\, dv}_{} + \sqrt{1-v^2}$$

$\{ w = \dfrac{1}{v} \Rightarrow \dfrac{dw}{dv} = -\dfrac{1}{v^2} = -w^2$

$$= -\int \frac{1}{\frac{1}{w} \cdot \sqrt{1 - \frac{1}{w^2}}} \frac{1}{w^2}\, dw = -\int \frac{dw}{\sqrt{w^2-1}} = -\cosh^{-1}(w)$$

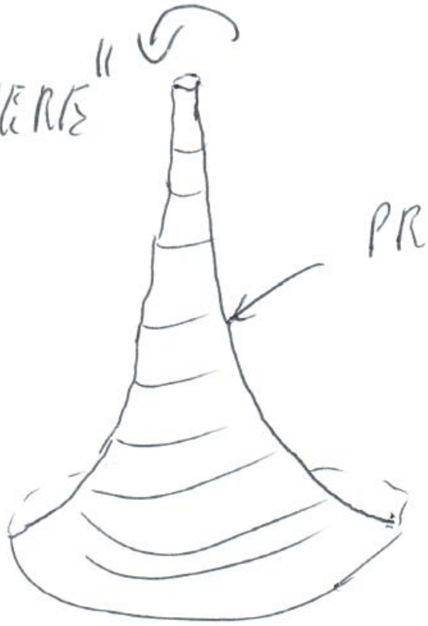
$$= -\cosh^{-1}\left(\frac{1}{v}\right) + \sqrt{1-v^2}$$

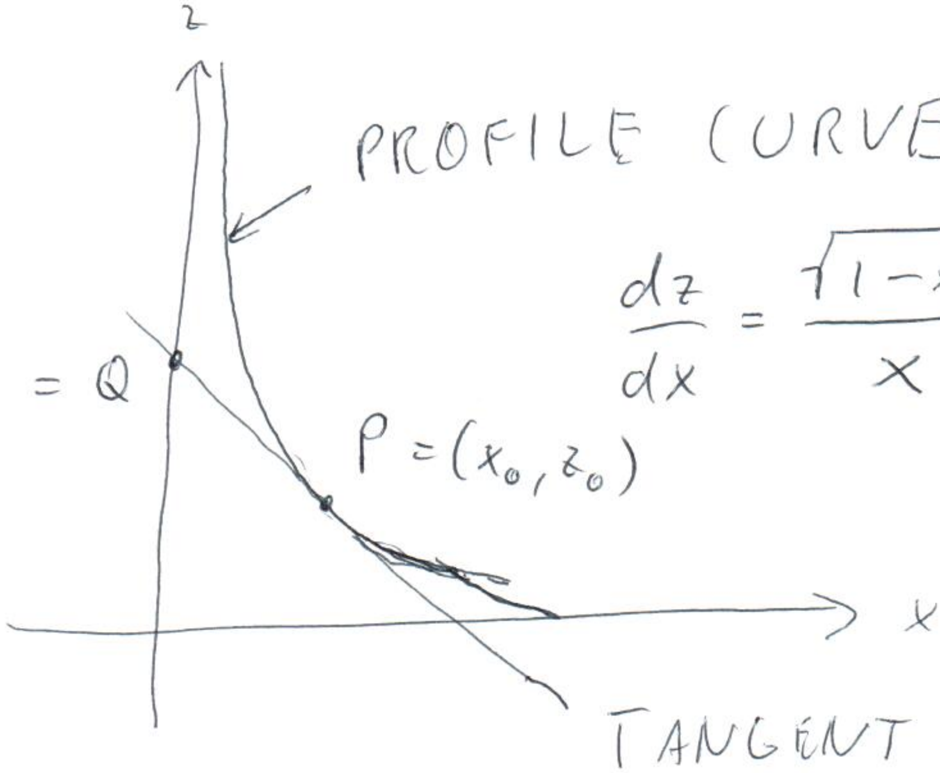
$$= -\cosh^{-1}(e^{-u}) + \sqrt{1-e^{2u}} \quad \Big(+ \underset{\substack{\| \text{ WLOG} \\ 0}}{\text{const}} \Big)$$

PUT $x = f(u) = e^{u}$

$\Rightarrow z = g(u) = \sqrt{1-x^2} - \cosh^{-1}\left(\frac{1}{x}\right)$

IS EQUATION OF PROFILE CURVE IN xz-PLANE





$$\frac{dz}{dx} = \frac{\sqrt{1-x^2}}{x}$$

EQUATION OF TANGENT LINE:

$$z - z_0 = \frac{\sqrt{1-x_0^2}}{x_0}(x - x_0)$$

$\Rightarrow Q = (0, z_1)$ WITH

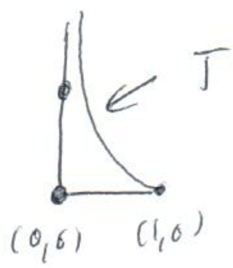
$$z_1 - z_0 = \frac{\sqrt{1-x_0^2}}{x_0}(0 - x_0) = -\sqrt{1-x_0^2}$$

$$\Rightarrow z_1 = z_0 - \sqrt{1-x_0^2}$$

$$\Rightarrow |PQ|^2 = (\cancel{0} - x_0)^2 + (z_1 - z_0)^2$$

$$= x_0^2 + 1 - x_0^2 = 1$$

STORY: DONKEY PULLS A BOX OF STONES BY A ROPE OF LENGTH ONE.



$\leftarrow$ TRACTRIX.