Math 153, Fall 2023
Homework 11

1. Let $A \in \mathbb{R}^{2\times 2}$. Let $z(t)$ be a complex-valued function of $t \in \mathbb{R}$. Let $z(t) = x(t) + iy(t)$, where x and y are real-valued functions, so x is the real part of z, and y the complex part of z. Show that

$$\frac{dz}{dt} = Az \quad \text{if and only if} \quad \frac{dx}{dt} = Ax \quad \text{and} \quad \frac{dy}{dt} = Ay.$$

2. Let $A \in \mathbb{R}^{2\times 2}$. Prove that x is a real-valued solution of $\frac{dx}{dt} = Ax$ if and only if there exists a complex-valued solution z with $x = \mathrm{Re}(z)$.

3. Consider

$$\frac{dz_1}{dt} = z_1 - z_2, \quad \frac{dz_2}{dt} = z_1 + z_2.$$

   (a) Write down the matrix A and and show that its eigenvalues are $1+i$ and $1-i$, with associated eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

   (b) The general solution is now

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = c_1 e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \tag{1}$$

   for constants c_1 and c_2. These are all the *complex* solutions. But what are all the *real* solutions? From Problem 2, you know that you can take the real part of a solution of the form (1), and that will be a real solution. To make things simple, let's assume c_1 and c_2 are both real. Write the real part of (1) in terms of sines, cosines, and the real exponential function, under the assumption that c_1 and c_2 are real.

   (c) Sketch the solution from (b) in the (x_1, x_2)-plane.

   (d) In part (c), you should have seen that the orientation of the rotation is counter-clockwise. How could the sign of one particular entry in the matrix A have told you that?