GEODESIC EQUATIONS ARE OF THE FORM

$$(*)\quad \begin{cases} \ddot{u} = f(u, v, \dot{u}, \dot{v}) \\ \ddot{v} = g(u, v, \dot{u}, \dot{v}) \end{cases} \qquad f, g \text{ SMOOTH}$$

EXISTENCE & UNIQUENESS RESULTS ABOUT ODE'S TELL US:

$\forall a, b, c, d \in \mathbb{R}; \forall t_0 \in I\ \exists!$ SOLUTION $(u(t), v(t))$ OF $(*)$ WITH $u(t_0) = a$, $v(t_0) = b$, $\dot{u}(t_0) = c$, $\dot{v}(t_0) = d$ AND $|t - t_0| < \varepsilon$ FOR $\varepsilon > 0$ SUFFICIENTLY SMALL.

APPLY THIS TO GEODESIC EQUATIONS:

<u>PROPOSITION 8.1.10</u> LET $p \in \sigma(\mathcal{U}) = S$, $X \in T_pS$, $\|X\| = 1$. THEN THERE EXISTS A UNIQUE GEODESIC $\gamma(t) = \sigma(u(t), v(t))$ ON σ WITH $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = X$.

<u>PROOF</u>: WRITE

$$p = \sigma(a, b), \quad X = c\,\sigma_u(a, b) + d\,\sigma_v(a, b)$$

$$\gamma(t) = \sigma(u(t), v(t))$$

$$\gamma(t_0) = p \iff u(t_0) = a, \ v(t_0) = b$$

$$\dot{\gamma}(t_0) = \dot{u}(t_0)\,\sigma_u(a,b) + \dot{v}(t_0)\,\sigma_v(a,b)$$
$$\shortparallel$$
$$X = c\,\sigma_u(a,b) + d\,\sigma_v(a,b)$$

$$\Leftrightarrow \quad \dot{u}(t_0) = c \quad , \dot{v}(t_0) = d .$$

ODE RESULT IMPLIES ASSERTION. $\square$.

$$\boxed{\text{THERE IS A UNIQUE GEODESIC THROUGH ANY GIVEN POINT OF A SURFACE IN ANY GIVEN DIRECTION.}}$$

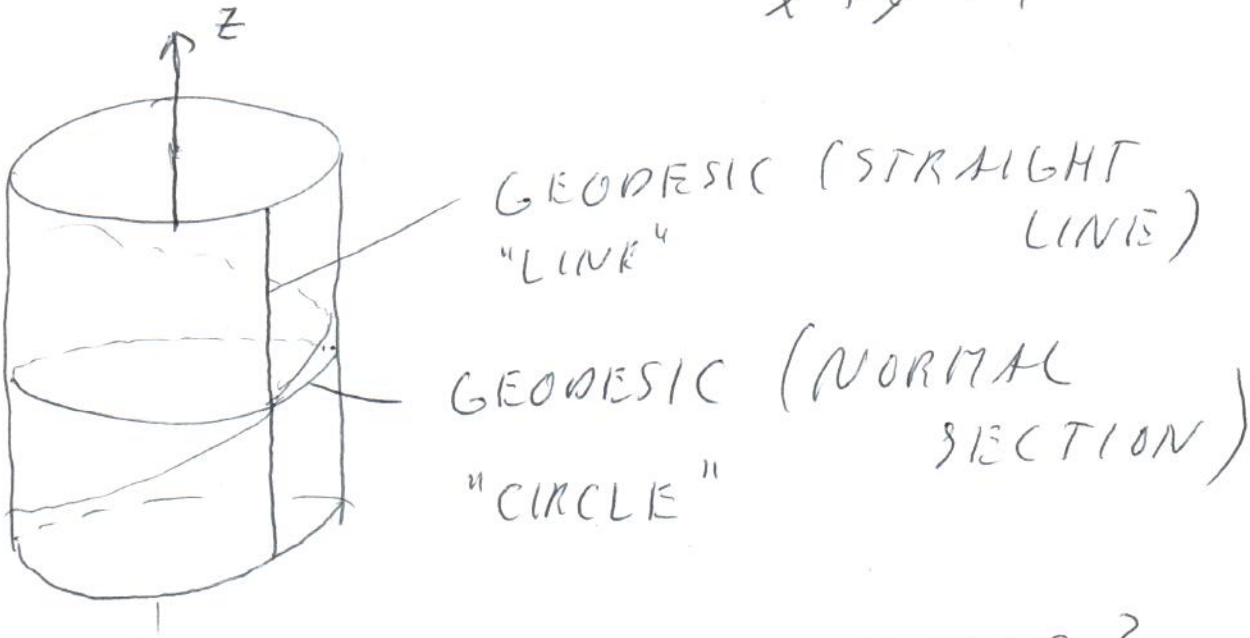
APPLICATION: GEODESICS ON PLANES, SPHERES, CYLINDERS & CONES.

<u>EXAMPLE 8.1.11.</u> THE GEODESICS IN THE PLANE ARE THE STRAIGHT LINES.

<u>EXAMPLE 8.1.12</u> THE GEODESICS IN THE SPHERE ARE THE GREAT CIRCLES.

EXAMPLE 8.1.13 CIRCULAR CYLINDER

$x^2+y^2=1$



WHAT ARE THE OTHER GEODESICS?

PLANE $\rightarrow$ CYLINDER ISOMETRY

$(0, u, v) \mapsto (\cos(u), \sin(u), v)$

MAPS GEODESICS ON PLANE TO GEODESICS ON CYLINDER

$v = mu + c \mapsto \gamma(u) = (\cos(u), \sin(u), mu+c)$

CIRCULAR HELIX

"LINE" AND "CIRCLE" ARE LIMIT CASES

"$m \to \infty$" $\quad m = 0$

EXAMPLE 8.1.14 SURFACE OF REVOLUTION

$$\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

WITH $f > 0$, $\left(\frac{df}{du}(u)\right)^2 + \left(\frac{dg}{du}(u)\right)^2 = 1$

[NOTE: WE USE $\dot{}$ FOR $\frac{d}{dt}$]

$$\sigma_u(u,v) = \left(\frac{df}{du}(u)\cos(v), \frac{df}{du}(u)\sin(v), \frac{dg}{du}(u)\right)$$

$$\sigma_v(u,v) = (-f(u)\sin(v), f(u)\cos(v), 0)$$

$$E = (\sigma_u \cdot \sigma_u)(u,v) = \left(\frac{df}{du}(v)\right)^2 + \left(\frac{dg}{du}(u)\right)^2 = 1$$

$$F = (\sigma_u \cdot \sigma_v)(u,v) = 0$$

$$G = (\sigma_v \cdot \sigma_v)(u,v) = f(u)^2$$

GEODESIC EQUATIONS ARE

$$\ddot{u} = f(u)\frac{df}{du}(u)\dot{v}^2 \qquad (1)$$

$$\frac{d}{dt}\left(f(u)^2\dot{v}\right) = 0 \qquad (2)$$

NOTE THAT

$$1 = \|\dot{\gamma}\|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \qquad (3)$$
$$= \dot{u}^2 + f(u)^2\dot{v}^2$$

CONCLUSIONS:

(a) EVERY MERIDIAN $v = v_0$ IS A GEODESIC

$$v = v_0 \Rightarrow \dot v = 0 \underset{(1)}{\Rightarrow} \ddot u = 0 \Rightarrow \dot u \text{ CONST}$$

SATISFIES GEODESIC EQUATION

(b) A PARALLEL $u = u_0$ IS A GEODESIC

$$\Leftrightarrow \frac{df}{du}(u_0) = 0.$$

"$\Rightarrow$" $u = u_0 \underset{(3)}{\Rightarrow} 1 = f(u_0)^2 \dot v^2 \Rightarrow \dot v = \pm \frac{1}{f(u_0)}$

$$\underset{(1)}{\Rightarrow} \frac{df}{du}(u_0) = 0$$

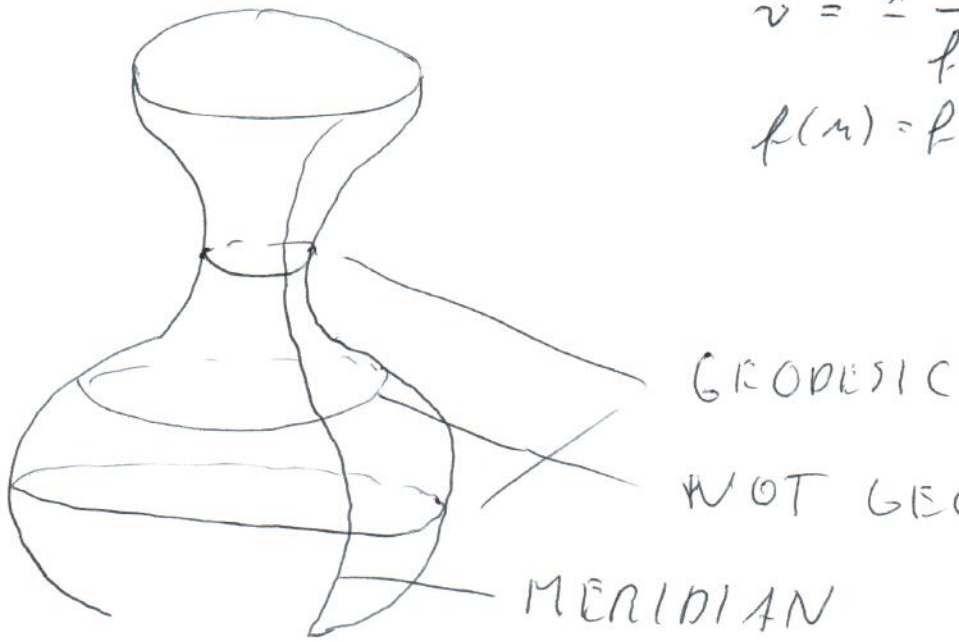
"$\Leftarrow$" $\frac{df}{du}(u_0) = 0 \Rightarrow$ (1) HOLDS OBVIOUSLY

(2) HOLDS BECAUSE

$\dot v = \pm \frac{1}{f(u_0)}$ BY (3)

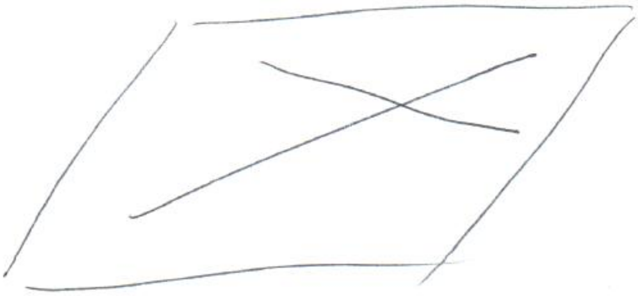
$f(u) = f(u_0)$ CONST

BY ASSUMPTION

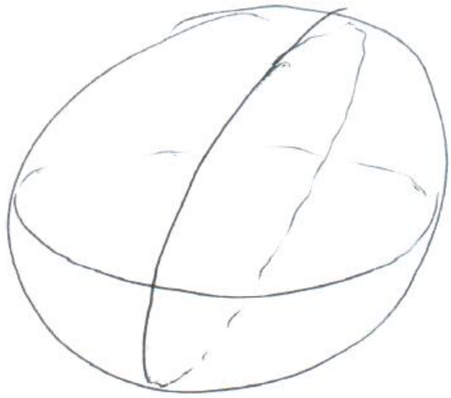


OTHER GEODESICS: EXERCISE.

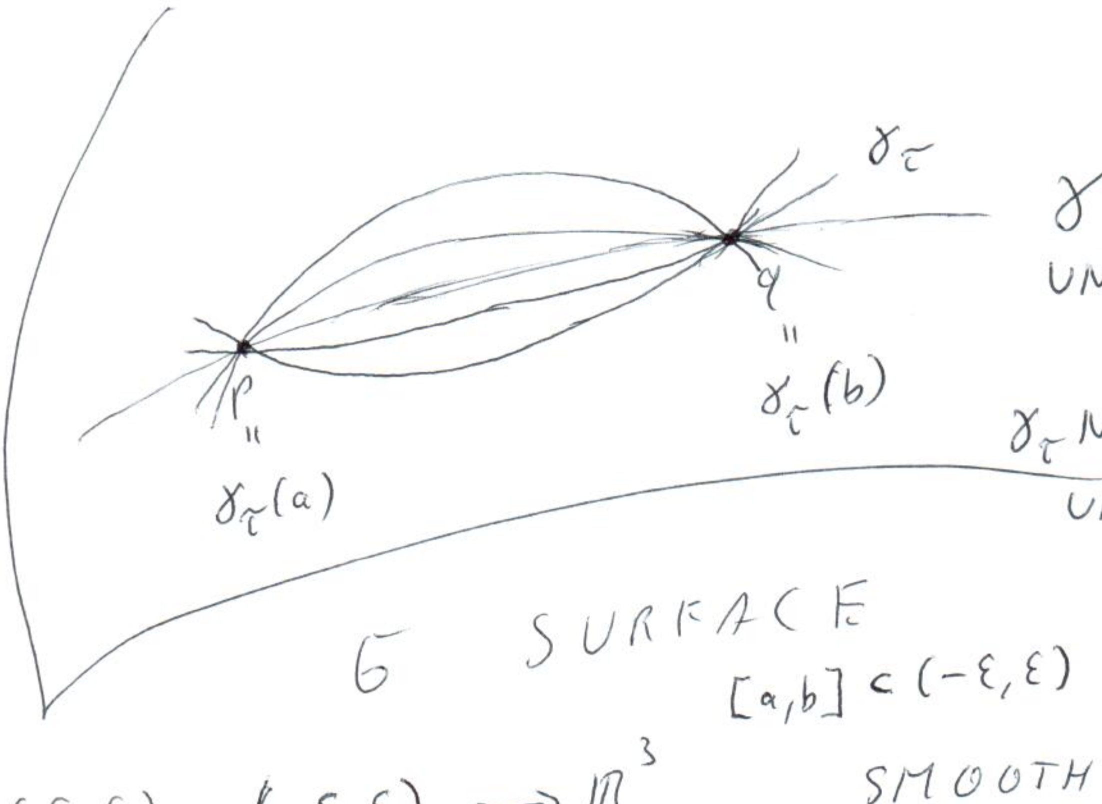
8.2 GEODESICS AS SHORTEST PATHS



SHORTEST PATH GLOBALLY



SHORTEST PATH LOCALLY



$\gamma = \gamma_0$ UNIT SPEED CURVE

γ_τ NOT NECESSARILY UNIT SPEED

$\mathcal{G}$ SURFACE

$[a,b] \subset (-\varepsilon, \varepsilon)$

$$\begin{aligned} (-\delta, \delta) \times (-\varepsilon, \varepsilon) &\longrightarrow \mathbb{R}^3 \\ (\tau, t) &\longmapsto \gamma_\tau(t) \end{aligned}$$

SMOOTH MAP "VARIATION OF CURVES"

$$\text{LENGTH}\left(\gamma_\tau\big|_{[a,b]}\right) = \int_a^b \|\dot{\gamma}_\tau(t)\| \, dt = \mathcal{L}(\gamma_\tau)$$

THEOREM 8.2.1

$$\gamma \text{ GEODESIC} \iff \left.\frac{d}{dt}\right|_{\tau=0} \mathcal{L}(\gamma_\tau) = 0$$

FOR ALL SMOOTH VARIATIONS OF γ

PROOF: "$\Rightarrow$"

$$\frac{d}{d\tau}\mathcal{L}(\gamma_\tau) = \frac{d}{d\tau}\int_a^b \|\dot\gamma_\tau(t)\|\,dt \qquad \dot{} = \frac{d}{dt}$$

$$= \frac{d}{d\tau}\int_a^b (\underbrace{E\dot u^2 + 2F\dot u\dot v + G\dot v^2}_{=:\,g(\tau,t)})^{\frac12}\,dt$$

$$= \int_a^b \frac{d}{d\tau} g(\tau,t)^{\frac12}\,dt = \frac12\int_a^b g(\tau,t)^{-\frac12}\frac{\partial g}{\partial\tau}\,dt$$

$$\frac{\partial g}{\partial\tau} = \frac{\partial E}{\partial\tau}\dot u^2 + 2\frac{\partial F}{\partial\tau}\dot u\dot v + \frac{\partial G}{\partial\tau}\dot v^2$$

$$+ 2E\dot u\frac{\partial\dot u}{\partial\tau} + 2F\left(\frac{\partial\dot u}{\partial\tau}\dot v + \dot u\frac{\partial\dot v}{\partial\tau}\right) + 2G\dot v\frac{\partial\dot v}{\partial t}$$

$$= \left(E_u\frac{\partial u}{\partial\tau} + E_v\frac{\partial v}{\partial\tau}\right)\dot u^2 + 2\left(F_u\frac{\partial u}{\partial\tau} + F_v\frac{\partial v}{\partial\tau}\right)\dot u\dot v$$

$$+ \left(G_u\frac{\partial u}{\partial\tau} + G_v\frac{\partial v}{\partial\tau}\right)\dot v^2 + 2E\dot u\frac{\partial^2 u}{\partial\tau\partial t}$$

$$+ 2F\left(\frac{\partial^2 u}{\partial\tau\partial t}\dot v + \dot u\frac{\partial^2 v}{\partial\tau\partial t}\right) + 2G\dot v\frac{\partial^2 v}{\partial\tau\partial t}$$

$$= (E_u\dot u^2 + 2F_u\dot u\dot v + G_u\dot v^2)\frac{\partial u}{\partial\tau}$$

$$+ (E_v\dot u^2 + 2F_v\dot u\dot v + G_v\dot v^2)\frac{\partial v}{\partial\tau}$$

$$+ 2(E\dot u + F\dot v)\frac{\partial^2 u}{\partial\tau\partial t} + 2(F\dot u + G\dot v)\frac{\partial^2 v}{\partial\tau\partial t}.$$

$$\int_a^b g^{-\frac{1}{2}}\left((E\dot{u}+F\dot{v})\frac{\partial^2 u}{\partial\tau\partial t}+(F\dot{u}+G\dot{v})\frac{\partial^2 v}{\partial\tau\partial t}\right)dt$$

$$\underset{\text{INTEGR. BY PARTS}}{=} g^{-\frac{1}{2}}\left((E\dot{u}+F\dot{v})\underbrace{\frac{\partial u}{\partial\tau}}_{=0}+(F\dot{u}+G\dot{v})\underbrace{\frac{\partial v}{\partial\tau}}_{=0}\right)\Bigg|_{t=a}^{t=b} \quad \text{FOR } t=a,\, t=b$$

$$-\int_a^b\left(\frac{d}{dt}\left\{g^{-\frac{1}{2}}(E\dot{u}+F\dot{v})\right\}\frac{\partial u}{\partial\tau}+\frac{d}{dt}\left\{g^{-\frac{1}{2}}(F\dot{u}+G\dot{v})\right\}\frac{\partial v}{\partial\tau}\right)dt$$

$$0 \underset{\uparrow}{=} \frac{\partial\gamma_\tau}{\partial\tau}=\frac{\partial u}{\partial\tau}\sigma_u+\frac{\partial v}{\partial\tau}\sigma_v \quad \text{FOR } t\in\{a,b\}$$

SINCE $\gamma_\tau(a)$, $\gamma_\tau(b)$ INDEPENDENT OF τ

THUS

$$\frac{d}{d\tau}\mathcal{L}(\gamma_\tau) = \int_a^b\left(\mathcal{U}\frac{\partial u}{\partial\tau}+V\frac{\partial v}{\partial\tau}\right)dt$$

WITH

$$\mathcal{U}(\tau,t)=\frac{1}{2}g(\tau,t)^{-\frac{1}{2}}\left(E_u\dot{u}^2+2F_u\dot{u}\dot{v}+G_u\dot{v}^2\right)$$
$$-\frac{d}{dt}\left\{g(\tau,t)^{-\frac{1}{2}}(E\dot{u}+F\dot{v})\right\}$$

$$V(\tau,t)=\frac{1}{2}g(\tau,t)^{-\frac{1}{2}}\left(E_v\dot{u}^2+2F_v\dot{u}\dot{v}+G_v\dot{v}^2\right)$$
$$-\frac{d}{dt}\left\{g(\tau,t)^{-\frac{1}{2}}(F\dot{u}+G\dot{v})\right\}$$

NOTE THAT

$$u(0,t) = 0 \quad , \quad v(0,t) = 0$$

ARE THE GEODESIC EQUATIONS

[HERE WE USE THAT $g(0,t) = \|\dot{\gamma}_0\| = 1$].

THUS $\left.\frac{d}{d\tau}\right|_{\tau=0} \ell(\gamma_\tau) = 0$.

~~CONVERS~~ " $\Leftarrow$ "

ASSUME

$$\int_a^b \left(u \frac{\partial u}{\partial \tau} + v \frac{\partial v}{\partial \tau} \right) dt = 0 \quad \text{WHEN } \tau = 0$$

FOR ALL ~~SMOO~~ SMOOTH VARIATIONS OF γ

~~[illegible]~~ WE HAVE TO PROVE THAT

$$u(0,t) = 0 \quad , \quad v(0,t) = 0 \text{ FOR ALL } t \in [a,b]$$

ASSUME $u(0,t) \neq 0$.

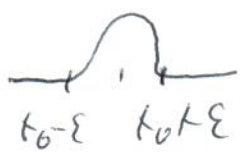
$\Rightarrow \exists t_0 \in (a,b) : u(0,t_0) \neq 0$, SAY > 0.

$\underset{u \text{ CONT.}}{\Longrightarrow} \exists \eta > 0 \ \forall t \in (t_0 - \eta, t_0 + \eta) : u(0,t) > 0$

CHOOSE SMOOTH FUNCTION ϕ WITH

$$\phi(t) > 0 \quad \text{IF } t \in (t_0 - \eta, t_0 + \eta)$$

$$\phi(t) = 0 \quad \text{IF } t \notin (t_0 - \eta, t_0 + \eta)$$



WRITE $\gamma(t) = \sigma(u(t), v(t))$

AND CONSIDER VARIATION

$$\gamma_\tau(t) = \sigma\big(\underbrace{u(t) + \tau\phi(t)}_{=: u(\tau,t)}, \underbrace{v(t)}_{=: v(\tau,t)}\big)$$

$$\Rightarrow \frac{\partial u}{\partial \tau} = \phi, \quad \frac{\partial v}{\partial \tau} = 0$$

THUS

$$0 = \int_a^b \left(U\frac{\partial u}{\partial \tau} + V\frac{\partial v}{\partial \tau}\right)\Big|_{\tau=0} dt = \int_{t_0-\eta}^{t_0+\eta} \underbrace{U(0,t)}_{>0}\underbrace{\phi(t)}_{>0} dt$$

CONTRADICTION.

THUS $U(0,t) = 0$.

SIMILARILY: $V(0,t) = 0$. $\square$

NOTE: FOR ϕ CAN TAKE

$$\phi(t) = \varphi\left(\frac{t-t_0}{\eta}\right),$$

$$\varphi(t) = \theta(1+t)\,\theta(1-t)$$

$$\theta(t) = \begin{cases} e^{-\frac{1}{t^2}} & t > 0 \\ 0 & t \leq 0 \end{cases} \qquad \theta^{(n)}(0) = 0 \quad \forall n$$

COMMENTS

(a) ASSUME γ IS SHORTEST PATH FROM p TO q.

$\Rightarrow \mathcal{L}(\gamma_\tau)$ HAS ABSOLUTE MINIMUM WHEN $\tau = 0$

$$\Rightarrow \frac{d}{d\tau} \mathcal{L}(\gamma_\tau)\Big|_{\tau=0} = 0$$

$\Rightarrow \gamma$ GEODESIC

THM 8.2.1

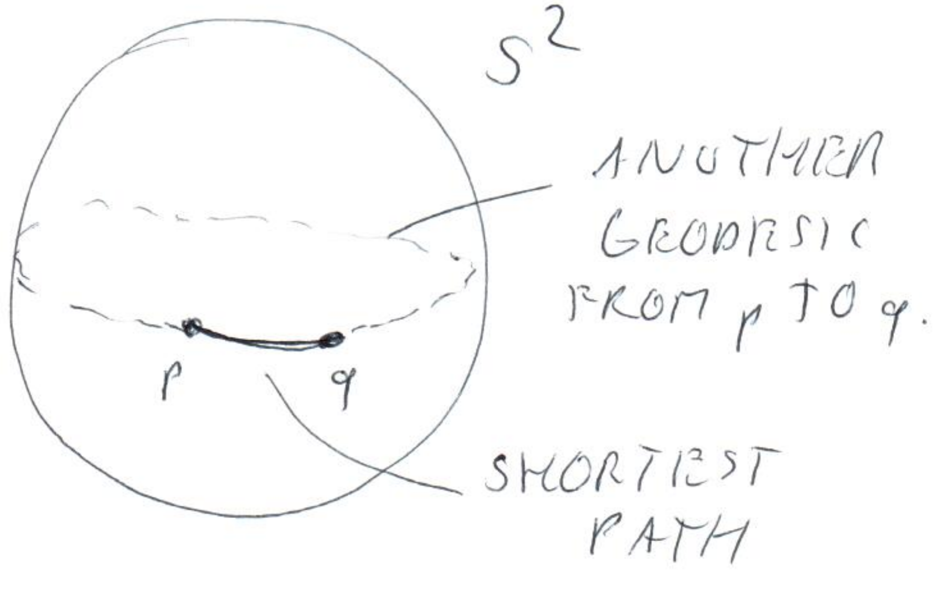
(b) ASSUME γ GEODESIC THROUGH p, q

$$\Rightarrow \frac{d}{d\tau} \mathcal{L}(\gamma_\tau)\Big|_{\tau=0} = 0 \quad , \text{ SO}$$

$\mathcal{L}(\gamma_\tau)$ HAS EXTREMUM, WHEN $\tau = 0$

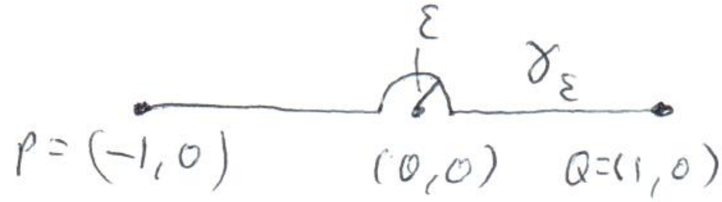
DOES NOT NEED TO BE ABSOLUTE MINIMUM.

EXAMPLE:



(c) SHORTEST PATHS MAY NOT EXIST! 150

$\mathcal{S} = \mathbb{R}^2 \setminus \{(0,0)\}$



THERE IS NO SHORTEST PATH FROM P TO Q.

$$L(\gamma_\varepsilon) = 2(1-\varepsilon) + \pi\varepsilon = 2 + (\pi - 2)\varepsilon$$

$$\lim_{\varepsilon \to 0} L(\gamma_\varepsilon) = 2$$

$$\Rightarrow \inf\{L(\gamma) : \gamma \text{ PATH } \overbrace{}^{\text{in } \mathcal{S}} \text{ FROM } p \text{ TO } q\} = 2$$

BUT THERE IS NO CURVE IN $\mathcal{S}$ FROM p TO q OF LENGTH $= 2$.

(d) IF SURFACE S IS CLOSED SUBSET OF $\mathbb{R}^3$, THEN THERE ALWAYS EXISTS A SHORTEST PATH BETWEEN 2 POINTS IN S.