Problem 1:

or the single two-state spin $s = \pm 1$, with energy described by Eq. 8.1, find the entropy as a function of the temperature T and applied field H.

Solution: For a single two-state spin, the partition function is

$$Z = \sum_{s=\pm 1} e^{-\beta E_s} = e^{\beta H} + e^{-\beta H} \tag{1}$$

The entropy is given by

$$\begin{aligned} S &= -k_B \sum_{s=\pm 1} p_s \ln p_s \\ &= -k_B \sum_{s=\pm 1} \frac{e^{-\beta E_s}}{Z} \ln \frac{e^{-\beta E_s}}{Z} \\ &= -k_B \sum_{s=\pm 1} \frac{e^{-\beta E_s}}{Z} \left(-\beta E_s - \ln Z\right) \\ &= k_B \left(\beta H e^{\beta H} + \beta H e^{-\beta H} - \ln\left(e^{\beta H} + e^{-\beta H}\right)\right) \end{aligned}$$

where $\beta = \frac{1}{k_B T}$.

Problem 2:

Calculate expressions for the free energy and mean magnetization of a system of three coupled two-state spins s_1, s_2 and $s_3 = \pm 1$, with Hamiltonian (total energy)
$E = -J(s_1 s_2 + s_2 s_3) - H(s_1 + s_2 + s_3)$. Show that the mean magnetization of this system vanishes in the absence of an applied field H.

Solution:

Using the transfer matrix, the partition function is

$$\begin{aligned} Z &= \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \sum_{s_3=\pm 1} e^{-\beta E} \\ &= \mathbf{Tr}(\mathbf{T}^3) \end{aligned}$$

where

$$\mathbf{T} = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix} \tag{2}$$

Evaluating the trace of $\mathbf{T}^3$, we have

$$\begin{aligned} \mathbf{T}^3 &= \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix}^3 \\ &= \begin{pmatrix} e^{3\beta(J+H)} + e^{3\beta(J-H)} + 2e^{\beta J} e^{-\beta H} & e^{3\beta(J-H)} + e^{3\beta(J+H)} - 2e^{\beta J} e^{-\beta H} \\ e^{3\beta(J-H)} + e^{3\beta(J+H)} - 2e^{\beta J} e^{-\beta H} & e^{3\beta(J+H)} + e^{3\beta(J-H)} + 2e^{\beta J} e^{-\beta H} \end{pmatrix} \end{aligned}$$

The partition function is then

$$\begin{aligned} Z &= \mathbf{Tr}(\mathbf{T}^3) \\ &= e^{3\beta(J+H)} + e^{3\beta(J-H)} + 2e^{\beta J}e^{-\beta H} + e^{3\beta(J-H)} + e^{3\beta(J+H)} - 2e^{\beta J}e^{-\beta H} \\ &= 2e^{3\beta(J+H)} + 2e^{3\beta(J-H)} \end{aligned}$$

The free energy is given by

$$\begin{aligned} F &= -k_B T \ln Z \\ &= -k_B T \ln\left(2e^{3\beta(J+H)} + 2e^{3\beta(J-H)}\right) \end{aligned}$$

When $H = 0$, the mean magnetization is given by

$$\begin{aligned} \langle s_1 + s_2 + s_3 \rangle &= \frac{\partial(-\beta F)}{\partial(\beta H)} \\ &= \left.\frac{6e^{3\beta(J+H)} - 6e^{3\beta(J-H)}}{2e^{3\beta(J+H)} + 2e^{3\beta(J-H)}}\right|_{H=0} \\ &= 0 \end{aligned}$$

Problem 3:

Let

$$\mathbf{T} = \begin{pmatrix} e^{\beta(J'+H')} & e^{-\beta J'} \\ e^{-\beta J'} & e^{\beta(J'-H')} \end{pmatrix} \tag{3}$$

(b) If $H' = 0$, show that $\mathbf{U}^{-1}\mathbf{T}\mathbf{U}$ is diagonal for the unitary rotation matrix

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{4}$$

Solution::

Since $H' = 0$,

$$\mathbf{T} = \begin{pmatrix} e^{\beta J'} & e^{-\beta J'} \\ e^{-\beta J'} & e^{\beta J'} \end{pmatrix} \tag{5}$$

The eigenvalues of $\mathbf{T}$ are

$$\lambda_1 = e^{\beta J'} + e^{-\beta J'}, \quad \lambda_2 = e^{\beta J'} - e^{-\beta J'} \tag{6}$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7}$$

Thus the diagonal matrix is

$$\mathbf{D} = \begin{pmatrix} e^{\beta J'} + e^{-\beta J'} & 0 \\ 0 & e^{\beta J'} - e^{-\beta J'} \end{pmatrix} \tag{8}$$

Therefore $\mathbf{U}^{-1}\mathbf{T}\mathbf{U}$ is diagonal. And

$$\mathbf{U}^{-1}\mathbf{T}\mathbf{U} = \begin{pmatrix} e^{\beta J'} + e^{-\beta J'} & 0 \\ 0 & e^{\beta J'} - e^{-\beta J'} \end{pmatrix} \tag{9}$$

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Solution:

$$\begin{aligned} \frac{\lambda_1^{N-r}\lambda_2^r + \lambda_2^{N-r}\lambda_1^r}{\lambda_1^N + \lambda_2^N} &= \frac{\lambda_1^N \frac{\lambda_2}{\lambda_1} + \lambda_2^N \frac{\lambda_1}{\lambda^2}}{\lambda_1^N + \lambda_2^N} \\ &= \frac{\left(\frac{\lambda_1}{\lambda_2}\right)^N}{\left(\frac{\lambda_1}{\lambda_2}\right)^N + 1} \end{aligned}$$

Problem 5:

In the chiral clock model, a one-dimensional chain of sites each carries an integer variable ni that can take values $0, 1$ or 2, representing three equally-spaced directions around a clock-face. Nearest neighbours interact so that their lowest-energy states occur when each site is one position further round the clock-face than its neighbour to the left, so that the energy is given by

$$E = -J \sum_{i=1}^{N} \cos\left(\frac{2\pi(n_i - n_{i-1} + 1)}{3}\right) \tag{10}$$

with a coupling constant J and periodic boundary conditions $n_{N+1} = n_1$. Show that the transfer matrix for this model has only one eigenvalue, equal to $2\exp(-\beta J/2) + \exp(\beta J)$, and hence write down the free energy per site.

Solution:

The partition function is

$$Z =$$

Problem 6:

Problem 7: (Page 90-91)

A mean field theory of a particular thermodynamic system yields an expression for the free energy density f as a function of volume fraction ϕ of conserved particles as

$$f = \frac{1}{\phi^2(1-\phi)^2} + 250\phi(1-\phi) \tag{11}$$

Use the common-tangent construction to discover the binodal values, for which this system exhibits coexistence. [Hint: You may find it helpful to re-write this function in a more symmetrical way by making a substitution for ϕ.]

Solution: Here we make a substitution for ϕ:

$$\phi = \varphi + \frac{1}{2} \tag{12}$$

where α is a new variable. Then

$$\begin{aligned} f &= \frac{1}{\phi^2(1-\phi)^2} + 250\phi(1-\phi) \\ &= \frac{1}{\left(\alpha+\frac{1}{2}\right)^2\left(1-\alpha-\frac{1}{2}\right)^2} + 250\left(\alpha+\frac{1}{2}\right)\left(1-\alpha-\frac{1}{2}\right) \\ &= \frac{1}{(\alpha^2-1/4)^2} + 250\left(\alpha^2-\frac{1}{4}\right) \end{aligned}$$

We set up two equations:

$$\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=\alpha_1} = \left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=\alpha_2} \tag{13}$$

which is equivalent to

$$-\frac{4\alpha_1}{(\alpha_1^2-1/4)^3} + 500\alpha_1 = -\frac{4\alpha_2}{(\alpha_2^2-1/4)^3} + 500\alpha_2 \tag{14}$$

and

$$\alpha_1\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=\alpha_1} - f(\alpha_1) = \alpha_2\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=\alpha_2} - f(\alpha_2) \tag{15}$$

which is equivalent to

$$-\frac{4\alpha_1^2}{(\alpha_1^2-1/4)^3}+500\alpha_1^2-\frac{1}{(\alpha_1^2-1/4)^2}-250\alpha_1^2+125 = -\frac{4\alpha_2^2}{(\alpha_2^2-1/4)^3}+500\alpha_2^2-\frac{1}{(\alpha_2^2-1/4)^2}-250\alpha_2^2+125 \tag{16}$$

Solving the equations, we have

Assume that $f(x)$ is integrable over any interval (a, b), and $g(x)$ is the probability density function of X within the interval (a, b), with $\int_a^b g(x) = 1$. We need to prove that after repeating the sampling process n times (where n is sufficiently large), the mean of $\frac{f(x_i)}{g(x_i)}$ converges to $\int_a^b f(x)dx$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{f(x_i)}{g(x_i)} = \int_a^b f(x)dx \tag{17}$$

According to the Law of Large Numbers, if $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with finite mean μ and variance σ^2, then for any given positive number ϵ, when n is sufficiently large, we have

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| < \epsilon\right) \approx 1. \tag{18}$$

Now let's prove that the mean of $\frac{f(x_i)}{g(x_i)}$ converges to $\int_a^b f(x)dx$ after repeating the sampling process n times.

Assume that we perform n independent samples, resulting in random variables $X_1, X_2, \ldots, X_n$, where each X_i has the probability density function $g(x)$.

Define the random variable $Y_i = \frac{f(X_i)}{g(X_i)}$, and the expectation of Y_i is

$$E[Y_i] = \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} g(x)dx = \int_{-\infty}^{\infty} f(x)dx. \tag{19}$$

According to the Law of Large Numbers, for any given positive number ϵ, when n is sufficiently large, we have

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - \int_{-\infty}^{\infty} f(x)dx\right| < \epsilon\right) \approx 1. \tag{20}$$

Since $f(x)$ is integrable over any interval (a, b), we can transform the integral $\int_{-\infty}^{\infty} f(x)dx$ into $\int_a^b f(x)dx$, as the values of $f(x)$ outside (a, b) do not affect the integral result.

Therefore, when n is sufficiently large, we have

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - \int_a^b f(x)dx\right| < \epsilon\right) \approx 1. \tag{21}$$

In other words, when n is sufficiently large, $\frac{1}{n}\sum_{i=1}^{n} Y_i$ is highly likely to fall within an ϵ