Geometry of Surfaces

5CCM223A/6CCM223B

Video 15
Length of curves on surfaces

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Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and

$$\gamma : (\alpha, \beta) \to \mathbb{R}^3 \ , \ t \mapsto \sigma(u(t), v(t))$$

be a curve on the surface $\mathcal{S} = \sigma(U)$. The Chain Rule implies

$$\dot{\gamma}(t) = \sigma_u(u(t), v(t))\dot{u}(t) + \sigma_v(u(t), v(t))\dot{v}(t) \qquad \dot{} = \frac{d}{dt}$$

or briefly

$$\dot{\gamma} = \sigma_u \dot{u} + \sigma_v \dot{v}$$

Then

$$\|\dot{\gamma}\|^2 = (\sigma_u \cdot \sigma_u)\dot{u}^2 + 2(\sigma_u \cdot \sigma_v)\dot{u}\dot{v} + (\sigma_v \cdot \sigma_v)\dot{v}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

with

$$E = \sigma_u \cdot \sigma_u \ , \ F = \sigma_u \cdot \sigma_v \ , \ G = \sigma_v \cdot \sigma_v$$

Let $t_0 \in (\alpha, \beta)$ and consider the arc length function

$$s(t) = \int_{t_0}^{t} \left\| \frac{d\gamma}{dx} \right\| dx = \int_{t_0}^{t} \left(E\frac{du^2}{dx^2} + 2F\frac{du}{dx}\frac{dv}{dx} + G\frac{dv^2}{dx^2} \right)^{\frac{1}{2}} dx$$
$$= \int_{t_0}^{t} \left(Edu^2 + 2Fdudv + Gdv^2 \right)^{\frac{1}{2}}$$

The expression

$$Edu^2 + 2Fdudv + Gdv^2$$

is called the first fundamental form of σ. Since $s(t) = \int_{t_0}^{t} \left(ds^2\right)^{\frac{1}{2}}$, we also write

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

The first fundamental form depends on the parametrization but not on the curve γ

First fundamental form of plane

$$\sigma(u, v) = a + up + vq$$

with $a, p, q \in \mathbb{R}^3$ and p, q linearly independent. Then $\sigma_u = p$ and $\sigma_v = q$. Thus

$$E = \sigma_u \cdot \sigma_u = \|p\|^2 \ , \ F = \sigma_u \cdot \sigma_v = p \cdot q \ , \ G = \sigma_v \cdot \sigma_v = \|q\|^2$$

We can always choose $p, q \in \mathbb{R}^3$ with $\|p\| = 1 = \|q\|$ and $p \cdot q = 0$. Then the first fundamental form of the plane is of the form

$$ds^2 = du^2 + dv^2$$

First fundamental form of sphere

$$\sigma(\theta,\varphi) = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$$
$$\sigma_\theta(\theta,\varphi) = (-\sin(\theta)\cos(\varphi), -\sin(\theta)\sin(\varphi), \cos(\theta))$$
$$\sigma_\varphi(\theta,\varphi) = (-\cos(\theta)\sin(\varphi), \cos(\theta)\cos(\varphi), 0)$$

The coefficients of the first fundamental form are

$$E = \sigma_\theta \cdot \sigma_\theta = 1 \ , \ F = \sigma_\theta \cdot \sigma_\varphi = 0 \ , \ G = \sigma_\varphi \cdot \sigma_\varphi = \cos(\theta)^2$$

The first fundamental form of the sphere is

$$ds^2 = d\theta^2 + \cos(\theta)^2 d\varphi^2$$

First fundamental form of generalized cylinder

$$
\begin{aligned}
\sigma(u,v) &= (f(u), g(u), v)\\
\sigma_u(u,v) &= (\dot{f}(u), \dot{g}(u), 0)\\
\sigma_v(u,v) &= (0,0,1)
\end{aligned}
$$

The coefficients of the first fundamental form are

$$E = \sigma_u \cdot \sigma_u = \dot{f}^2 + \dot{g}^2 \ , \ \ F = \sigma_u \cdot \sigma_v = 0 \ , \ \ G = \sigma_v \cdot \sigma_v = 1$$

Since σ is regular, also γ is regular. If we parametrize γ by arc length, then $E = 1$ and the first fundamental form is

$$ds^2 = du^2 + dv^2$$

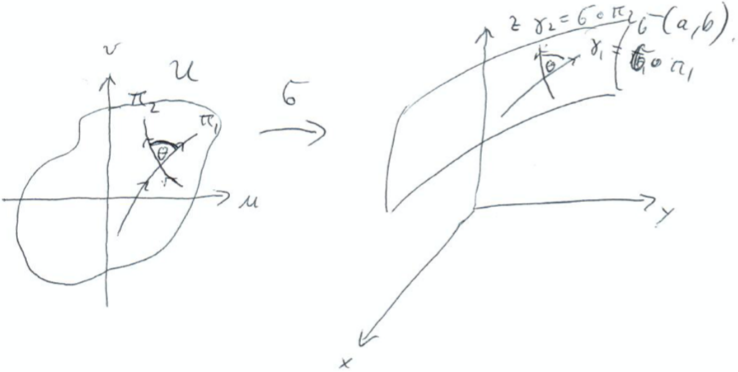
This is the same expression as for the plane!

Reparametrizations change E, F, G but not ds^2 (Exercise!)

Every surface has a conformal parametrization, that is, a parametrization for which $E = G$ and $F = 0$.

Proposition. *A surface parametrization $\sigma : U \to \mathbb{R}^3$ is conformal if and only if for any two curves $\pi_1 = (u_1, v_1)$, $\pi_2 = (u_2, v_2)$ in U with $\pi_1(t_0) = (a, b) = \pi_2(t_0)$ the angle of intersection of π_1 and π_2 at (a, b) is equal to the angle of intersection of $\gamma_1 = \sigma \circ \pi_1$ and $\gamma_2 = \sigma \circ \pi_2$ at $\sigma(a, b)$*

Briefly: Conformal parametrizations are angle-preserving parametrizations


v
U
π_2
π_1
θ
u
σ
z
$\gamma_2 = \sigma \circ \pi_2$
$\sigma(a,b)$
θ
$\gamma_1 = \sigma \circ \pi_1$
y
x

Proof. Assume σ is conformal. Let θ be the angle between $\dot{\gamma}_1(t_0)$ and $\dot{\gamma}_2(t_0)$. We have $\dot{\gamma}_i = \sigma_u \dot{u}_i + \sigma_v \dot{v}_i$ and

$$\begin{aligned}\cos(\theta) &= \frac{\dot{\gamma}_1 \cdot \dot{\gamma}_2}{\|\dot{\gamma}_1\|\|\dot{\gamma}_2\|}(t_0) \\ &= \frac{E\dot{u}_1\dot{u}_2 + F(\dot{u}_1\dot{v}_2 + \dot{v}_1\dot{u}_2) + G\dot{v}_1\dot{v}_2}{(E\dot{u}_1^2 + 2F\dot{u}_1\dot{v}_1 + G\dot{v}_1^2)^{\frac{1}{2}}(E\dot{u}_2^2 + 2F\dot{u}_2\dot{v}_2 + G\dot{v}_2^2)^{\frac{1}{2}}}(t_0) \qquad (*)\end{aligned}$$

Since σ is conformal, we have $E = G$ and $F = 0$ and thus

$$\cos(\theta) = \frac{\dot{u}_1\dot{u}_2 + \dot{v}_1\dot{v}_2}{(\dot{u}_1^2 + \dot{v}_1^2)^{\frac{1}{2}}(\dot{u}_2^2 + \dot{v}_2^2)^{\frac{1}{2}}} \qquad (**)$$

which is the angle between $\dot{\pi}_1(t_0)$ and $\dot{\pi}_2(t_0)$. Thus σ is angle-preserving

Conversely, assume σ is angle-preserving, which means $(*) = (**)$ for all curves π_1, π_2 with $\pi_1(t_0) = (a, b) = \pi_2(t_0)$. Take

$$\pi_1(t) = (a + t, b)\ ,\ \pi_2(t) = (a, b + t)\ ,\ t_0 = 0$$

Then $(*) = (**)$ gives $0 = F$. Take

$$\pi_1(t) = (a + t, b + t)\ ,\ \pi_2(t) = (a + t, b - t)\ ,\ t_0 = 0$$

Then $(*) = (**)$ gives $0 = E - G$.

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Isometries of surfaces

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Plane and round cylinder have same first fundamental form.

$$\text{plane} \sim \text{round cylinder}$$

Length of curve remains unchanged when deforming plane to round cylinder

$$\text{plane} \not\sim \text{round sphere}$$

expect different first fundamental form

$\sigma_1 : U_1 \to \mathbb{R}^3$, $\sigma_2 : U_2 \to \mathbb{R}^3$ regular surface patches,
$\mathcal{S}_1 = \sigma_1(U_1)$, $\mathcal{S}_2 = \sigma_2(U_2)$

A map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is smooth if there exists a smooth map $F : U_1 \to U_2$ with $f \circ \sigma_1 = \sigma_2 \circ F$

$$\begin{tikzcd} U_1 \arrow[r, "F"] \arrow[d, "\sigma_1"'] & U_2 \arrow[d, "\sigma_2"] \\ \mathcal{S}_1 \arrow[r, "f"'] & \mathcal{S}_2 \end{tikzcd}$$

A map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a diffeomorphism if f is bijective and f, f^{-1} are smooth maps ($\Longleftrightarrow F : U_1 \to U_2$ bijective and F, F^{-1} smooth)

Note: If $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$, $f = \mathrm{id}_{\mathcal{S}}$, then F is a reparametrization map

A diffeomorphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ preserving the length of curves is called an isometry

$\mathcal{S}_1$ and $\mathcal{S}_2$ are isometric (or σ_1 and σ_2 are isometric) if there exists an isometry $f : \mathcal{S}_1 \to \mathcal{S}_2$. We write $\mathcal{S}_1 \cong \mathcal{S}_2$ if $\mathcal{S}_1$ and $\mathcal{S}_2$ are isometric

The concept of isometric surfaces is independent of reparametrizations

Theorem. *Two surfaces are isometric if and only if they have reparametrizations $\sigma_1 : U_1 \to \mathbb{R}^3$ and $\sigma_2 : U_2 \to \mathbb{R}^3$ with the same first fundamental form*

Proof. Let $\tilde{\sigma}_1 : U_1 \to \mathbb{R}^3$, $\tilde{\sigma}_2 : U_2 \to \mathbb{R}^3$ be regular surface patches, $\mathcal{S}_1 = \tilde{\sigma}_1(U_1)$, $\mathcal{S}_2 = \tilde{\sigma}_2(U_2)$. Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be an isometry. There exists a bijective smooth map $F : U_1 \to U_2$ so that

$$\begin{tikzcd} U_1 \arrow[r, "F"] \arrow[d, "\tilde{\sigma}_1"'] & U_2 \arrow[d, "\tilde{\sigma}_2"] \\ \mathcal{S}_1 \arrow[r, "f"'] & \mathcal{S}_2 \end{tikzcd} \qquad \begin{tikzcd} U = U_1 \arrow[r, "F"] \arrow[d, "\sigma_1=\tilde{\sigma}_1"'] \arrow[dr, "\sigma_2"] & U_2 \arrow[d, "\tilde{\sigma}_2"] \\ \mathcal{S}_1 \arrow[r, "f"'] & \mathcal{S}_2 \end{tikzcd}$$

Define $U = U_1$, $\sigma_1 = \tilde{\sigma}_1 : U \to \mathcal{S}_1$, $\sigma_2 = \tilde{\sigma}_2 \circ F : U \to \mathcal{S}_2$. σ_2 is a reparametrization of $\tilde{\sigma}_2$. Consider curves

$$\gamma_1(t) = \sigma_1(u(t), v(t))\,,\ \gamma_2(t) = \sigma_2(u(t), v(t)) = (f \circ \sigma_1)(u(t), v(t))$$

Since f is an isometry, γ_1, γ_2 have the same length

$$\int_{t_0}^{t_1} \left(E_1 \dot{u}^2 + 2F_1 \dot{u}\dot{v} + G_1 \dot{v}^2\right)^{\frac{1}{2}} dt = \int_{t_0}^{t_1} \left(E_2 \dot{u}^2 + 2F_2 \dot{u}\dot{v} + G_2 \dot{v}^2\right)^{\frac{1}{2}} dt$$

Put $u_0 = u(t_0)$, $v_0 = v(t_0)$

$$\begin{aligned}
(u(t), v(t)) = (u_0 + t - t_0, v_0) &\Longrightarrow E_1 = E_2 \\
(u(t), v(t)) = (u_0, v_0 + t - t_0) &\Longrightarrow G_1 = G_2 \\
(u(t), v(t)) = (u_0 + t - t_0, v_0 + t - t_0) &\Longrightarrow F_1 = F_2
\end{aligned}$$

Thus σ_1 and σ_2 have the same first fundamental form

Conversely, assume that σ_1 and σ_2 have the same first fundamental form. Define $f : \mathcal{S}_1 \to \mathcal{S}_2$ by $f(\sigma_1(u,v)) = \sigma_2(u,v)$. Let

$$\gamma_1(t) = \sigma_1(u(t), v(t))\,,\ \gamma_2(t) = \sigma_2(u(t), v(t)) = (f \circ \sigma_1)(u(t), v(t))$$

Length of γ_1, γ_2 is obtained by integrating $\left(E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2\right)^{\frac{1}{2}}$, which is the same for both surfaces by assumption. It follows that f preserves the length of curves and hence is an isometry.

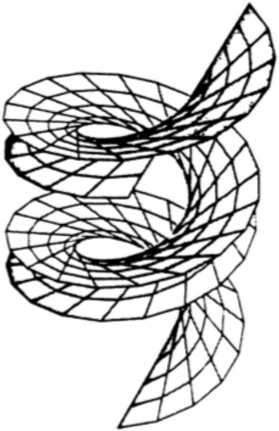
plane $\cong$ generalized cylinder $\cong$ generalized cone

A tangent developable is the union of tangent lines to a regular curve γ in $\mathbb{R}^3$ (can assume $\|\dot{\gamma}\| = 1$):

$$\begin{aligned}
\sigma(u,v) &= \gamma(u) + v\dot{\gamma}(u) \\
\sigma_u(u,v) &= \dot{\gamma}(u) + v\ddot{\gamma}(u) \\
\sigma_v(u,v) &= \dot{\gamma}(u) \\
(\sigma_u \times \sigma_v)(u,v) &= v\ddot{\gamma}(u) \times \dot{\gamma}(u) = v(\dot{\mathbf{t}} \times \mathbf{t})(u) \\
&= v\kappa(u)(\mathbf{n} \times \mathbf{t})(u) = -v\kappa(u)\mathbf{b}(u)
\end{aligned}$$

Thus σ regular if and only if $\kappa > 0$ and $v \neq 0$

The tangent developable of a circular helix



Theorem. *Any tangent developable is isometric to (part of) a plane.*

Proof.

$$\begin{aligned}
\sigma(u,v) &= \gamma(u) + v\dot{\gamma}(u)\\
\sigma_u(u,v) &= \dot{\gamma}(u) + v\ddot{\gamma}(u)\\
\sigma_v(u,v) &= \dot{\gamma}(u)\\
E = (\sigma_u\cdot\sigma_u)(u,v) &= \|\dot{\gamma}(u) + v\ddot{\gamma}(u)\|^2 = 1 + v^2\kappa(u)^2\\
F = (\sigma_u\cdot\sigma_v)(u,v) &= (\dot{\gamma}(u) + v\ddot{\gamma}(u))\cdot\dot{\gamma}(u) = 1\\
G = (\sigma_v\cdot\sigma_v)(u,v) &= \|\dot{\gamma}(u)\|^2 = 1\\
ds^2 &= (1 + v^2\kappa(u)^2)du^2 + 2dudv + dv^2
\end{aligned}$$

Tangent developable of a plane curve $\tilde{\gamma}$ with curvature κ has the same first fundamental form (tangents fill out parts of a plane). Thus both surfaces are isometric

One can prove that a surface that is isometric to (part of) a plane is (part of)

- plane
- generalized cylinder
- generalized cone
- tangent developable

Geometry of Surfaces

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Surface area

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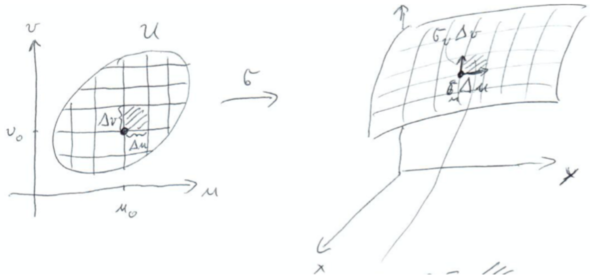
Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch. Let R be a rectangle in U with side lengths Δu and Δv. Then, for small Δu and Δv, the area of $\sigma(R)$ is approximately the area of the parallelogram with sides $\sigma_u \Delta u$ and $\sigma_v \Delta v$:

$$\|\sigma_u \Delta u \times \sigma_v \Delta v\| = \|\sigma_u \times \sigma_v\| \Delta u \Delta v$$



Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $R \subseteq U$. The area of $\sigma(R)$ is

$$\mathcal{A}_\sigma(R) = \iint_R \|\sigma_u \times \sigma_v\| dudv$$

Let θ be the angle between σ_u and σ_v. Then

$$\begin{aligned} \|\sigma_u \times \sigma_v\|^2 &= \|\sigma_u\|^2 \|\sigma_v\|^2 \underbrace{\sin(\theta)^2}_{1-\cos(\theta)^2} \\ &= \|\sigma_u\|^2 \|\sigma_v\|^2 - \underbrace{\|\sigma_u\|^2 \|\sigma_v\|^2 \cos(\theta)^2}_{(\sigma_u \cdot \sigma_v)^2} = EG - F^2 \end{aligned}$$

Therefore

$$\mathcal{A}_\sigma(R) = \iint_R \left(EG - F^2\right)^{\frac{1}{2}} dudv$$

Proposition. *Area is unchanged by reparametrizations.*

Proof. Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ be a reparametrization of σ with reparametrization map $\phi : U \to \tilde{U}$:

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \tilde{\sigma}(\phi(u, v)) = \sigma(u, v)$$

The Chain Rule gives

$$\sigma_u = \tilde{\sigma}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial u} + \tilde{\sigma}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u} \ , \ \sigma_v = \tilde{\sigma}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial v} + \tilde{\sigma}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}$$

$$\sigma_u \times \sigma_v = \underbrace{\left(\frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v} \frac{\partial \tilde{v}}{\partial u} \right)}_{\det(J\phi)} \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}$$

Thus

$$\sigma_u \times \sigma_v = \det(J\phi)\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}$$

Altogether,

$$\begin{aligned}
\mathcal{A}_\sigma(R) &= \iint_R \|\sigma_u \times \sigma_v\| du dv \\
&= \iint_R |\det(J\phi)| \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| du dv \\
&= \iint_R \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| d\tilde{u} d\tilde{v} = \mathcal{A}_{\tilde{\sigma}}(R)
\end{aligned}$$

by Change of Variables Formula