TUTORIAL 11: Surface Integrals, The 3D Divergence Theorem, Stokes' Theorem

1. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{A}$, where $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j} + z^4\mathbf{k}$ and S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

2. Evaluate $\iint_S z^2\, dA$, where S is the hemispherical shell given by $x^2+y^2+z^2 = a^2$ with $z \geq 0$.

3. Let S be the spherical shell given by $x^2 + y^2 + z^2 = 1$ with $x \geq 0$ (not $z \geq 0$). Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{A}$, where $\mathbf{F} = (1, -z, y)$.

4.

(a) Let S be the cylindrical surface given parametrically by $x = a\cos(\theta)$ and $y = a\sin(\theta)$ with $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq b$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{A}$, where $\mathbf{F} = (x^2, y, 0)$.

(b) Now let E be the solid region bounded by S together with the planes $z = 0$ and $z = b$. Evaluate $\iiint_E div(\mathbf{F})dV$.

5. Use the 3D Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{A}$, where $\mathbf{F} = (y^2x, z^2y, x^2z)$ and S is the surface of the sphere of radius a centred at the origin.

6. Let E be the solid cylinder with equations $0 \leq x^2 + y^2 \leq a^2$ and $0 \leq z \leq b$. Let S be the surface of E, and let $\mathbf{F}$ be the vector field (x^3, y^3, z^3). Use the 3D Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{A}$.

7. Let E be the cone whose top is a flat disc of radius a centred on the z-axis at height b, and whose point is at the origin. Let S_1 be the flat top of E, and let S_2 be the lower curved surface (so S_1 and S_2 together form the whole boundary of E).

(a) Give the equations for S_1, S_2 and E in cylindrical polar coordinates.

(b) Put $\mathbf{F} = grad(f)$, where $f = x^2 + y^2 + z^2$. Show that $\int_{S_2} \mathbf{F} \cdot d\mathbf{A} = 0$, and calculate $\int_{S_1} \mathbf{F} \cdot d\mathbf{A}$.

(c) Use the 3D Divergence Theorem to deduce the volume of E.

8. Let C be the vertical circle given by $y = a\sin(t)$ and $z = a\cos(t)$ with $x = 0$. Use Stokes's Theorem to evaluate $\int_C (x^2y, z, 0) \cdot d\mathbf{r}$. Check your answer by calculating the integral directly.

9. Consider the points

$$P = (0, 0, c) \qquad Q = (a, 0, c) \qquad R = (a, b, c).$$

Let C be the triangular path that goes from P to Q to R and back to P. Use Stokes's Theorem to evaluate $\int_C (yz^2, x^3, xy^2).d\mathbf{r}$.

10. Let E be the solid region where $-1 \leq x, y \leq 1$ and $0 \leq z \leq (1-x^2)(1-y^2)$. Let S be the surface of E, and let $\mathbf{u}$ be the vector field $(x, y, 0)$. Verify the divergence theorem.

Useful identities

$$\sin(\alpha)\cos^2(\alpha) = \frac{1}{4}(\sin(3\alpha) + \sin(\alpha))$$

$$\cos^3(\alpha) = \frac{1}{4}\cos(3\alpha) + \frac{3}{4}\cos(\alpha).$$

The spherical position vector

$$\mathbf{r} = (r\cos\theta\cos\phi, r\sin\theta\sin\phi, r\cos\phi)$$

The spherical area and volume element

$$dA = r^2 \sin\theta d\theta d\phi, \quad dV = r^2 \sin\theta dr d\theta d\phi$$

Answers

1.

$$\int_S \mathbf{F} \cdot d\mathbf{A} = 3$$

2. $2\pi a^4/3$;
3. π;
4 $a^2 b\pi$;
5. $4\pi a^5/5$
6.

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \pi a^2 b \left(\frac{3a^2}{2} + b^2\right)$$

7b. $2\pi a^2 b$; 7c. $\pi a^2 b/3$
8. πa^2;
9. $3a^3 b/4 - abc^2/2$;
10. $32/9$