# Calculus: Homework #2

Due on February 12, 2014 at 3:10pm

*Professor Isaac Newton Section A*

**Josh Davis** **Davis Josh**

# Problem 1

We choose a number from the set $\{1, 2, \cdots, 100\}$ uniformly at random and denote this number by $X$. For each of the following choices decide whether the two events in question are independent or not.

1. $A = \{X \text{ is even}\}$, $B = \{X \text{ is divisible by } 5\}$
2. $C = \{x \text{ has two digits}\}$, $D = \{X \text{ is divisible by } 3\}$
3. $E = \{X \text{ is prime}\}$, $F = \{X \text{ has a digit } 5\}$

## Solution

Recall the definition of independency: if $X$ and $Y$ are two independent variables, then $P(X \cap Y) = P(X) \cdot P(Y)$.

From there, we have $P(A) = 0.5, P(B) = 0.2, P(A \cap B) = \frac{\#\{10,20,\cdots,100\}}{100} = \frac{10}{100} = \frac{1}{10} = 0.1$ We have

$$P(A) \cdot P(B) = 0.5 \times 0.2 = 0.1 = P(A \cap B)$$

Thus $A$ and $B$ are independent.

Likewise, we compute

$$P(C) = \frac{9}{10}$$

and

$$P(D) = \frac{\left[\frac{100}{3}\right]}{100} = \frac{34}{100}$$

and

$$P(C \cap D) = \frac{\#\{12, 15, 18, \cdots, 99\}}{100} = \frac{31}{100}$$

Since

$$P(C) \cdot P(D) = \frac{297}{1000} \neq \frac{31}{100} = P(C \cap D)$$

$C$ is not independent from $D$.

There are 25 primes less than 100, and we know that

$$F = \{5, 15, 25, 35, 45, 50, \cdots 59, 65, 75, 85, 95\}$$

Thus,

$$\#F = 5 + 10 + 4 = 19$$

We have

$$P(E) = \frac{1}{4}, P(F) = \frac{19}{100}$$

# Problem 2

Suppose there are two student assistants working as typists in the main office of the Statistics & Applied Probability Department at UCSB. The number of typos per page made by student assistant $A$ is a Poisson random variable with parameter $\lambda_A = 1$. The number of typos per page made by student assistant $B$ is also a Poisson random variable with an average of 10 typos per page. One of the professors in the department asks one of the students to type up a letter. From experience, this work will be done with 1/3 probability by student A and with 2/3 probability by student B.

(a) What is the probability that the typewritten letter will contain exactly one typo?
(b) It turns out that the typewritten letter does not contain any typos. Given this informa tion, what is the probability that student B typewrote this letter?

## Solution

(a) Let $T_A, T_B$ be the number of typos made by each assistant in each page respectively. First off we determine the Possion variable of student $B$, which is

$$P(T_B = k) = \frac{\lambda_B^k \exp(-\lambda_B)}{k!}, P(T_A = k) = \frac{1^k \exp(-1)}{k!}$$

where $\lambda_B = 10$. Thus by **total probability**,

$$\begin{aligned}
P(\text{exactly one typo}) &= P(\text{exactly one typo}|\text{student A selected})P(\text{student A selected}) \\
&+ P(\text{exactly one typo}|\text{student B selected})P(\text{student B selected}) \\
&= \frac{1}{3}P(T_A = 1|\text{student A selected}) + \frac{2}{3}P(T_B = 1|\text{ student B selected}) \\
&= \frac{1}{3}P\left(T_A = 1\right) + \frac{2}{3}P(T_B = 1) \\
&= \frac{1}{3}e^{-1} + \frac{2}{3}10e^{-10} \approx 12.3\%
\end{aligned}$$

(b) By the Bayes formula,

$$\begin{aligned}
P(\text{student B} \mid \text{no typo}) &= P(\text{no typo} \mid \text{student B}) \cdot \frac{P(\text{student B})}{P(\text{no typo})} \\
&= \frac{2}{3}\frac{e^{-10}}{P(T_A = 0|\text{student A})P(\text{student A}) + P(T_B = 0|\text{student B})P(\text{student B})} \\
&= \frac{2}{3}\frac{e^{-10}}{1/3 \cdot e^{-1} + 2/3 \cdot e^{-10}} \\
&\approx 0.2\%
\end{aligned}$$

# Problem 3

Suppose you are rolling a fair die 600 times independently. Let $X$ count the number of sixes that appear.

(a) What type of random variable is $X$? Specify all parameters needed to characterize $X$ as well as the state space $S_X$ of $X$.
(b) Find the probability that you observe the number 6 at most 100 times.
(c) Use a famous limit theorem (which one?) to show why the probability in (b) can be approximated by the value $\frac{1}{2}$

## Solution

(a) We observe that $X$ is Binomial distribution whose PMF is

$$P(X = k) = \binom{600}{k} \left(\frac{5}{6}\right)^{600-k} \left(\frac{1}{6}\right)^{k}$$

where $S_X = \{0, 1, 2, \cdots, 600\}$

(b) We need to compute

$$\begin{aligned} P(X \leq 100) &= \sum_{k=0}^{100} P(X = k) \\ &= \sum_{k=0}^{100} \binom{600}{k} \left(\frac{5}{6}\right)^{600-k} \left(\frac{1}{6}\right)^{k} \end{aligned}$$

From Wolfram Alpha, the summation above approximately equals to the numerical value 0.53
(c)