Geometry of surfaces - Solutions

24. We have $\sigma_u(u,v) = (1,0,2u)$ and $\sigma_v(u,v) = (0,1,-2v)$, hence $(\sigma_u \times \sigma_v)(u,v) = (-2u, 2v, 1) \neq 0$ for all u, v. Thus σ is regular and hence the tangent plane is well-defined at each point $\sigma(u,v)$ and equal to the span of $\sigma_u(u,v)$ and $\sigma_v(u,v)$.

For $(u,v) = (0,0)$ we have $\sigma_u(0,0) = (1,0,0)$ and $\sigma_v(0,0) = (0,1,0)$. Thus the tangent plane of the surface at $\sigma(0,0) = (0,0,0)$ is $\{(x,y,0) \mid x, y \in \mathbb{R}\}$. Next,

For $(u,v) = (1,1)$ we have $\sigma_u(1,1) = (1,0,2)$ and $\sigma_v(1,1) = (0,1,-2)$. Thus the tangent plane of the surface at $\sigma(1,1) = (1,1,0)$ is $\{(x,y,2(x-y)) \mid x, y \in \mathbb{R}\}$.

25. Using the Frenet-Serret equations we get

$$\begin{aligned}
\sigma_s(s,\theta) &= \dot\gamma(s) + a(\cos(\theta)\dot{\mathbf{n}}(s) + \sin(\theta)\dot{\mathbf{b}}(s)) \\
&= \mathbf{t}(s) + a(\cos(\theta)(-\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s)) + \sin(\theta)(-\tau(s)\mathbf{n}(s)) \\
&= (1 - a\kappa(s)\cos(\theta))\mathbf{t}(s) - a\tau(s)\sin(\theta)\mathbf{n}(s) + a\tau(s)\cos(\theta)\mathbf{b}(s), \\
\sigma_\theta(s,\theta) &= -a\sin(\theta)\mathbf{n}(s) + a\cos(\theta)\mathbf{b}(s).
\end{aligned}$$

Using the relations $\mathbf{n} \times \mathbf{b} = \mathbf{t}$, $\mathbf{b} \times \mathbf{t} = \mathbf{n}$ and $\mathbf{t} \times \mathbf{n} = \mathbf{b}$, we obtain

$$\begin{aligned}
(\sigma_s \times \sigma_\theta)(s,\theta) &= a\cos(\theta)(a\kappa(s)\cos(\theta) - 1)\mathbf{n}(s) + a\sin(\theta)(a\kappa(s)\cos(\theta) - 1)\mathbf{b}(s) \\
&= a(a\kappa(s)\cos(\theta) - 1)(\cos(\theta)\mathbf{n}(s) + \sin(\theta)\mathbf{b}(s)).
\end{aligned}$$

By assumption we have $a\kappa(s) < 1$ for all s, which implies $a\cos(\theta)\kappa(s) - 1 \neq 0$ for all s, θ. Since $\cos(\theta)$ and $\sin(\theta)$ cannot be equal to 0 simultaneously, and $a \neq 0$ by assumption, we see that $(\sigma_s \times \sigma_\theta)(s,\theta) \neq 0$ for all s, θ, and hence the surface is regular everywhere.

26. We have $\sigma_s(s,v) = \dot\gamma(s) + v\dot{\mathbf{t}}(s) = \mathbf{t}(s) + v\kappa(s)\mathbf{n}(s)$ and $\sigma_v(s,v) = \mathbf{t}(s)$. Since $\kappa(s) \neq 0$ for all s, $\sigma_s(s,v)$ and $\sigma_v(s,v)$ are linearly independent if and only if $v \neq 0$. Thus $\sigma(s,v)$ is a regular point of the surface if and only if $v \neq 0$. If $v \neq 0$, the tangent plane at $\sigma(s,v)$ is spanned by $\sigma_s(s,v) = \mathbf{t}(s) + v\kappa(s)\mathbf{n}(s)$ and $\sigma_v(s,v) = \mathbf{t}(s)$. This plane coincides with the span of $\mathbf{t}(s)$ and $\mathbf{n}(s)$, which by definition is the osculating plane of γ at $\gamma(s)$.

27. The unit normal N satisfies $N \cdot N = 1$. Differentiating this equation with respect to u (resp. v) gives $N_u \cdot N = 0$ (resp. $N_v \cdot N = 0$). Thus N_u and N_v are perpendicular to N.

Define the map $f : U \to \mathbb{R}^3$, $(u,v) \mapsto \sigma(u,v) + \alpha(u,v)N(u,v)$. By assumption, f is constant and $f(U) = \{p\}$. Thus $0 = f_u = \sigma_u + \alpha_u N + \alpha N_u$ and $0 = f_v = \sigma_v + \alpha_v N + \alpha N_v$. Since N is perpendicular to $\sigma_u, \sigma_v, N_u, N_v$, we get $\alpha_u = 0 = \alpha_v$ and thus α is constant. As $\sigma = p - \alpha N$, this shows that $\sigma(U)$ lies on the sphere with centre p and radius $|\alpha|$.

28. We have $\sigma_u(u,v) = \alpha'(u)$ and $\sigma_v(u,v) = \beta'(v)$. Thus $\sigma(u,v)$ is a regular point of the surface if and only if $\alpha'(u)$ and $\beta'(v)$ are linearly independent. In this situation the tangent plane at $\sigma(u,v)$ is the span of $\alpha'(u)$ and $\beta'(v)$. In particular, the tangent planes at all regular points of the form $\sigma(u,0)$ contain the line spanned by $\beta'(0)$.