Notes for Differential Equations

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Chapter 1

Introduction

Differential equations (abbreviated DEs) are equations involving functions and their derivatives. For example,

$$\frac{dy}{dx} = y \tag{1.1}$$

is a differential equation: y is a function of the independent variable x, and the rate of change of y with respect to x (ie $\frac{dy}{dx}$) is equal to the value of y. We would like to determine y.

The solution to this equation is

$$y = Ce^x,$$

with C an arbitrary constant. We can check that it is a solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}Ce^x \\ &= Ce^x \\ &= y.\end{aligned}$$

That is, if we plug $y = Ce^x$ into equation (1.1), both sides match. If we modify this a little bit, say letting $y = Ce^x + x$, then the left hand side doesn't match the right:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(Ce^x + x) \\ &= Ce^x + 1 \\ &= y + 1 \\ &\neq y,\end{aligned}$$

so it isn't a solution to equation (1.1).

An initial value problem (abbreviated as IVP) is a differential equation combined with one or more initial conditions. For example,

$$\frac{dy}{dx} = y, \qquad y(0) = 1$$

is an initial value problem. The differential equation is solved by $y = Ce^x$, for any value of C. However, the only way to satisfy

$$y(0) = 1$$

with the solution $y = Ce^x$ is to set $C = 1$. That is, the solution to the initial value problem is

$$y = e^x,$$

since it satisfies both the differential equation and the initial conditions.

In these brief notes we will describe techniques that can solve a small variety of different types of differential equations. In practice, some DEs can be solved analytically, and some can't; when we can't solve them analytically, we can sometimes find a numerical solution that gives us an answer that we can use. But the first step is always to try and find an exact, analytic solution, and it is these techniques which we will demonstrate in these notes.

Chapter 2

Separable and Exact DEs

2.1 Separable equations

Separable equations are differential equations of the form

$$\boxed{h(y)\frac{dy}{dx} = g(x)}. \tag{2.1}$$

These are simple to solve: from equation (2.1), we can just split the derivative and integrate: $h(y)\,dy = g(x)\,dx$ becomes

$$\boxed{\int h(y)\,dy = \int g(x)\,dx},$$

and then all we need to do is find the integral and solve for y.

Example: Consider the equation

$$y\frac{dy}{dx} = \sin(x) \tag{2.2}$$

with initial condition $y(0) = 1$. Separating and integrating,

$$\int y\,dy = \int \sin(x)\,dx,$$

yields

$$\frac{y^2}{2} = -\cos(x) + C$$

where C is a constant that will be determined by the initial conditions. Let us now solve for y:

$$y(x) = \pm\sqrt{2C - 2\cos(x)}.$$

Notice that we have two different solutions a positive one and a negative one. We can use the initial condition $y(0) = 1$ to eliminate one. Since

$$\begin{aligned} y(0) &= \pm\sqrt{2C - 2\cos(0)} \\ &= \pm\sqrt{2C - 2} \\ &= 1, \end{aligned}$$

and 1 is positive, we must choose the positive root. Thus,

$$2C - 2 = 1$$

so

$$C = 3/2,$$

and the solution is

$$y(x) = \sqrt{3 - 2\cos(x)}.$$

It's a good idea to check that the solution that you got is correct. This is pretty easy; just plug the solution into the original equation. From the above example, $y(x) = \sqrt{3 - 2\cos(x)}$, so

$$\frac{dy}{dx} = \frac{\sin(x)}{\sqrt{3 - 2\cos(x)}}.$$

Putting this into the left-hand side of equation (2.2) yields

$$\begin{aligned} y\frac{dy}{dx} &= \left(\sqrt{3 - 2\cos(x)}\right) \times \left(\frac{\sin(x)}{\sqrt{3 - 2\cos(x)}}\right) \\ &= \sin(x), \end{aligned}$$

which matches the right-hand side of equation (2.2), and we can be sure that we got the correct answer. □

2.2 Exact Equations

Definition: *Exact equations.* Homogeneous first-order differential equations can be written in the form $M(x, y)dx + N(x, y)dy = 0$. If

$$\boxed{\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}}$$

then the equation is called *exact*.

The nice thing with exact differentials is that Poincare's lemma implies the existence of a function F such that $dF = M(x, y)dx + N(x, y)dy$, and dF, called the differential of F, can be thought of as "how much F changes". Since $dF = 0$, F does not change, i.e. it's constant. Solutions to the exact differential equation are given implicitly by $F(x, y) =$ constant.

Exact equations are straightforward to solve: after a little bit of trickery, we simply integrate. Let

$$\boxed{F(x, y) = \int M(x, y)\, dx + g(y)}. \tag{2.3}$$

We require that $\frac{\partial}{\partial y} F(x,y) = N(x,y)$, which will allow us to determine $g(y)$. That is,

$$\begin{aligned} \frac{\partial F(x,y)}{\partial y} &= \frac{\partial}{\partial y}\left(\int M(x,y)\,dx + g(y)\right) \\ &= \int \frac{\partial M(x,y)}{\partial y}\,dx + \frac{\partial g(y)}{\partial y} \\ &= N(x,y) + g'(y). \end{aligned}$$

This tells us what $g'(y)$ is, so we now know $g(y)$ up to some constant C. We can put this into equation (2.3), which gives us the implicit solution

$$\boxed{F(x,y) = C}. \tag{2.4}$$

In many cases, we can solve for y, thus getting an explicit solution $y = f(x)$ so that $F(x, f(x)) = C$, but sometimes all we have for a solution is something in the form of equation (2.4).

Example: Solve

$$y\,dx + \left(y^2 + x\right)dy = 0$$

Using the notation above, we have $M(x,y) = y$, and $N(x,y) = y^2 + x$. This equation is exact, since

$$\frac{d}{dy} y = 1 = \frac{d\left(y^2 + x\right)}{dx}.$$

So write

$$\begin{aligned} F(x,y) &= \int y\,dx + g(y) \\ &= xy + g(y). \end{aligned}$$

We now need to set $\frac{\partial}{\partial y} F = N$, so

$$\frac{\partial}{\partial y}[xy + g(y)] = y^2 + x$$

which implies that $g'(y) = y^2$. We integrate this to get

$$g(y) = \frac{y^3}{3} + C. \tag{2.5}$$

The solution is then given by

$$F(x,y) = xy + \frac{y^3}{3} = C. \quad \square \tag{2.6}$$

(Note that in the above equation we have played fast and loose with the undetermined constant C; in fact, C changed sign between equation (2.5) and equation (2.6). We perform this abuse of notation because C is *undetermined*, so there's not much point nailing it down to a value until the very last moment.)

2.3 Problems

1. Does $y = \sqrt{3 - 2\sin x}$ solve the differential equation

$$y\frac{dy}{dx} = \sin(x)?$$

2. Solve the equation

$$(2xy + 3)dx + (x^2 - 1)dy = 0$$

3. Solve the logistic map $\frac{dy}{dx} = y - y^2$ as a separable equation.

4. Give the general solution to the differential equation

$$\frac{dy}{dt} = 1 + \frac{1}{y^2}$$

Chapter 3

Linear Equations & Transformations

3.1 Linear Equations

Linear equations have the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}. \tag{3.1}$$

To solve these equations, we use an *integrating factor*. That is, if we define the integrating factor μ as

$$\boxed{\mu(x) \doteq \exp\left[\int P(x)\,dx\right]},$$

then notice that

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \frac{d\mu}{dx}y + \mu\frac{dy}{dx} \\ &= \mu P(x)y + \mu\frac{dy}{dx}.\end{aligned}$$

So if we multiply equation (3.1) by μ, we get

$$\frac{d}{dx}(\mu y) = \mu Q,$$

which we can now solve by taking the integral of both sides with respect to x. Doing this, we get

$$d(\mu y) = \mu Q\,dx$$

which implies that

$$\mu y = \int \mu Q\,dx + C.$$

This provides us with the general formula for the integrating factor,

$$\boxed{y = \frac{\int \mu Q\,dx + C}{\mu}.} \tag{3.2}$$

Example: Consider the equation

$$\frac{dy}{dx} + \frac{y}{x} = 2e^x.$$

We identify $P(x) = 1/x$ and $Q(x) = 2e^x$. The integrating factor is

$$\begin{aligned}\mu(x) &= \exp\left[\int \frac{1}{x}\,dx\right] \\ &= e^{\ln x} \\ &= x.\end{aligned}$$

Then,

$$\frac{d}{dx}(xy) = 2xe^x, \tag{3.3}$$

integrating this gives

$$xy = \int 2xe^x \, dx. \tag{3.4}$$

From the formula (equation (3.2)), we have

$$\begin{aligned} y &= \frac{\int \mu Q \, dx + C}{\mu} \\ &= \frac{\int 2xe^x \, dx + C}{x}. \end{aligned} \tag{3.5}$$

To solve this, we use integration by parts. That is, $\int u \, dv = uv - \int v \, du$. We choose $u = x$ and $dv = e^x \, dx$, so $du = dx$, and $v = e^x$. Thus,

$$\begin{aligned} \int xe^x \, dx &= xe^x - \int e^x \, dx \\ &= xe^x - e^x. \end{aligned}$$

Putting this into equation (3.5) yields

$$y = 2e^x \left(1 - \frac{1}{x}\right) + \frac{C}{x}. \quad \square$$

If we had been provided with initial conditions, we would now use them to determine the value of C.

3.2 Equations that can be solved via substitution

Sometimes it is possible to make a substitution to transform a differential equation into something that we already know how to solve. For example, the integrating factor technique transforms linear equations into separable equations. This can be a very powerful technique. Unfortunately, each type of equation needs its own particular substitution and this makes substitution not as straight forward as other techniques.

The most important examples of substitution are linear, homogeneous, and Bernoulli equations. We also consider two other types of equations that can be solved by substitution, which are shown in the flow chart in Figure 3.1.

Homogeneous Equations

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}.$$

We solve this by substituting $v = y/x$, so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \quad \Rightarrow \quad v + x\frac{dv}{dx} = f(v),$$

which can be written as the separable equation

$$\boxed{\frac{dv}{dx} = \frac{f(v) - v}{x}}.$$

Example:

$$\frac{dy}{dx} = \frac{x}{y}, \text{ with } y(1) = 2.$$

This is homogeneous, so let $v = y/x$, and then $f(v) = \frac{1}{v}$, so

$$v + x\frac{dv}{dx} = \frac{1}{v}.$$

The equation is now separable:

$$\frac{1}{\frac{1}{v} - v}\, dv = \frac{1}{x}\, dx.$$

Integrating both sides gives

$$\int \frac{v}{1 - v^2} dv = \int \frac{1}{x}\, dx.$$

Let $u = 1 - v^2$ to solve the integral:

$$-\frac{1}{2} \int \frac{1}{u} du = \ln x + \ln C$$

which implies that

$$\begin{aligned} \ln(u) &= \ln\left(1 - v^2\right) \\ &= \ln \frac{C}{x^2}. \end{aligned}$$

In terms of v, we have which implies that

$$v = \pm\sqrt{1 - \frac{C}{x^2}}.$$

Expressing this in terms of the original variable y gives

$$y = \pm x\sqrt{1 - \frac{C}{x^2}}.$$

The initial condition is $y(1) = 2$, so we choose the positive root, and find C by setting $x = 1$, $y = 2$, i.e.

$$2 = 1 \times \sqrt{1 - C}$$

so $C = -3$. The solution of the initial value problem is

$$y = x\sqrt{1 + \frac{3}{x^2}}.$$

This would be a good one to check. Note that

$$\frac{dy}{dx} = \sqrt{1 + \frac{3}{x^2}} - \frac{3/x^2}{\sqrt{1 + \frac{3}{x^2}}} = \frac{1}{\sqrt{1 + \frac{3}{x^2}}} = \frac{x}{y},$$

so the solution is correct. $\square$

Bernoulli Equations

Bernoulli equations are differential equations of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$$

where n can be an integer or a rational number. Note that if $n = 0$ or $n = 1$, then this is just a linear equation.

We solve this by substituting $v = y^{1-n}$, so that

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

and rearranging gives

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x).$$

We can write this as a linear equation:

$$\boxed{\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).}$$

Example:

$$\frac{dy}{dx} - \frac{1}{2}\frac{y}{x} = -e^x y^3$$

Since $n = 3$, choose $v = y^{-2}$. Then

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dv}{dx} = -2y^{-3}\left(\frac{1}{2}\frac{y}{x} - e^x y^3\right) \quad \Rightarrow \quad \frac{dv}{dx} + \frac{v}{x} = 2e^x,$$

which is the linear equation given as the example from section 3.1. □

3.2.1 Flow chart for solving first-order DEs

As we have seen, it is sometimes necessary to use several transformations in order to solve a given DE. Each type of first order DE that we have seen so far is ultimately separable, as outlined in the flow chart below, or exact.

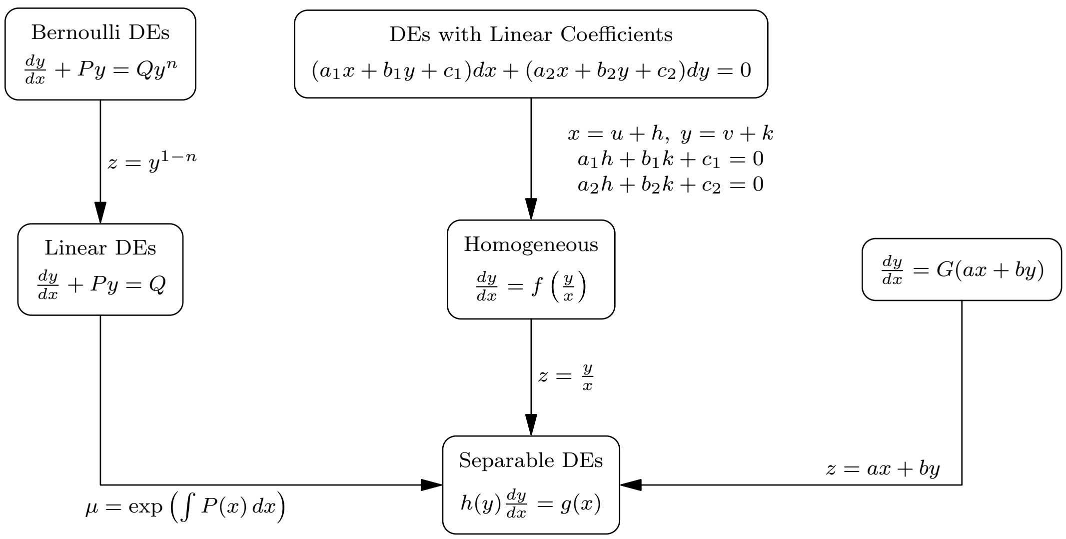

Figure 3.1: Flow chart for solving first-order DEs

3.3 Problems

1. Solve the initial value problem

$$(x^2 + 1)\frac{dy}{dx} + 4xy = 4x,$$

with initial condition

$$y(0) = \quad 0 \quad \text{or} \quad 1$$

(choose one, and circle your choice).

2. Joe headed out to the bar in his new Thinsulate jacket, but drank too much and passed out on the way home. Ignoring the heat that his body produces, his temperature is determined by Newton's law of cooling. Determine Joe's temperature T at time t by solving Newton's law of cooling, both as a separation problem and a linear problem

$$\frac{dT}{dt} = -r(T - T_{\text{env}}),$$

where $T(0) = 37$, $T_{\text{env}} = -40$, and $r = 5$ (Thinsulate's r-value).

3. Solve the logistic map,

$$\frac{dy}{dx} = y - y^2.$$

as a Bernoulli equation

4. Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + x^2 y^2.$$

5. Equations of the form $\frac{dy}{dx} = f(ax + by)$ may be transformed into a separable equation via the substitution $v = ax + by$. Using this technique, solve the differential equation

$$\frac{dy}{dx} = -(4x - y)^2$$

6. Equations of the form

$$(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0$$

can be transformed into homogeneous equations by using the substitution $x = u + h$ and $y = v + k$, with h and k constants which obey the relationship

$$\begin{aligned} a_1 h + b_1 k + c_1 &= 0 \\ a_2 h + b_2 k + c_2 &= 0. \end{aligned}$$

This reduces the problem to

$$\frac{dv}{du} = -\frac{a_1 + b_1(v/u)}{a_2 + b_2(v/u)},$$

which is homogeneous in u and v. Solve the differential equation

$$(2y + 2)dx + (x + y + 2)dy = 0$$

using this technique.

Chapter 4

Second-order Linear Equations

Let a, b, c be real numbers. Second order equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \tag{4.1}$$

are linear in y. If both y_1 and y_2 are solutions to equation (4.1), and α and β are constants, then

$$\alpha y_1 + \beta y_2$$

is a linear combination of y_1 and y_2. If we put this linear combination into the original differential equation, we get:

$$\begin{aligned}
&a\frac{d^2(\alpha y_1 + \beta y_2)}{dt^2} + b\frac{d(\alpha y_1 + \beta y_2)}{dt} + c(\alpha y_1 + \beta y_2) \\
&= \alpha\left(a\frac{d^2y_1}{dt^2} + b\frac{dy_1}{dt} + cy_1\right) + \beta\left(a\frac{d^2y_2}{dt^2} + b\frac{dy_2}{dt} + cy_2\right) \\
&= \alpha \times 0 + \beta \times 0 \\
&= 0.
\end{aligned}$$

In other words, $\alpha y_1 + \beta y_2$, which is a linear combination of y_1 and y_2, is also a solution to equation (4.1). Being able to use this linearity is a powerful tool that we can use to solve this very important type of differential equation. We will, in general, have two solutions to these second-order differential equations, and we will need two initial values to fully determine the solution. This is expressed in the following theorem:

Theorem 4.1: *Let b, c, Y_0, and $Y_1 \in \mathbb{R}$. Then, the initial value problem*

$$y'' + by' + cy = 0, \qquad y(0) = Y_0, \quad y'(0) = Y_1$$

has a unique solution.

4.1 Homogeneous Linear Equations

The behaviour of these systems is basically exponential. To see this, set $y = e^{rt}$. Then, putting this into equation (4.1), we get

$$\begin{aligned} ar^2e^{rt} + bre^{rt} + ce^{rt} &= e^{rt}\left(ar^2 + br + c\right) \\ &= 0. \end{aligned}$$

Since e^{rt} is never zero, there is no harm in dividing by it. This leaves us with the *characteristic equation*,

$$\boxed{ar^2 + br + c = 0},$$

which allows us to determine r using the quadratic formula. In this way we get two solutions,

$$\boxed{y_1 = e^{r_1 t} \text{ and } y_2 = e^{r_2 t}},$$

from the two solutions r_1 and r_2 of the characteristic equation.

Example: Solve the second-order homogeneous equation

$$\frac{d^2y}{dt^2} - y = 0,$$

with initial conditions $y(0) = 1$, $y'(0) = 0$.
Solution: Setting $y = e^{rt}$, this becomes

$$\begin{aligned} e^{rt}\left(r^2 - 1\right) &= e^{rt}(r-1)(r+1) \\ &= 0, \end{aligned}$$

so $r_1 = -1$, and $r_2 = 1$. Thus,

$$y_1(t) = e^{-t}, \qquad y_2(t) = e^{t}.$$

The solution y is therefore a linear combination of y_1 and y_2. That is,

$$\begin{aligned} y(t) &= \alpha y_1(t) + \beta y_2(t) \\ &= \alpha e^{-t} + \beta e^{t}, \end{aligned}$$

for some constants α and β that are determined by the initial conditions. Since $y(0) = 1$,

$$\begin{aligned} y(0) &= \alpha + \beta \\ &= 1. \end{aligned} \tag{4.2}$$

Since $y'(0) = 0$, and $y'(t) = -\alpha e^{-t} + \beta e^{t}$,

$$\begin{aligned} y'(0) &= -\alpha + \beta \\ &= 0. \end{aligned} \tag{4.3}$$

Combining equations (4.2) and (4.3), it is easy to see that $\alpha = \beta = \frac{1}{2}$. The solution is therefore

$$y = \frac{e^{-t} + e^{t}}{2}. \quad \square$$

4.2 Dealing with complex roots

So far, we've seen only problems where the roots of the characteristic equation are real. Of course, this isn't always the case, but we can deal with this using *Euler's Formula*,

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}.$$

For example, if we get $r_{1,2} = 4 \pm 2i$, then solutions are a linear combination of

$$\begin{aligned} e^{(4+2i)t} &= e^{4t}e^{i2t} \\ &= e^{4t}(\cos(2t) + i\sin(2t)) \end{aligned}$$

and

$$\begin{aligned} e^{(4-2i)t} &= e^{4t}e^{-i2t} \\ &= e^{4t}(\cos(2t) - i\sin(2t)). \end{aligned}$$

An easy way to get all linear combinations is to just set

$$y_1(t) = e^{4t}\cos(2t), \qquad y_2(t) = e^{4t}\sin(2t).$$

Example: Give the general solution to

$$\frac{d^2y}{dt^2} + 1 = 0$$

with initial conditions $y(0) = 1$, $y'(0) = 0$.
Solution: The characteristic equation is

$$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i.$$

The solution is then a linear combination of

$$y_1(t) = e^{0t}\cos(t) = \cos(t), \qquad y_2(t) = e^{0t}\sin(t) = \sin(t)$$

Setting $y(t) = \alpha\cos(t) + \beta\sin(t)$, we can input the initial conditions to get

$$\begin{aligned} y(0) &= \alpha\cos(0) + \beta\sin(0) = \alpha = 1, \\ y'(0) &= -\alpha\sin(0) + \beta\cos(0) = \beta = 0. \end{aligned}$$

The solution to the IVP is then $y(t) = \cos(t)$. □

4.3 Dealing with repeated roots

In the above section, we were lucky, since we had two independent roots. Two independent roots gave two independent solutions y_1 and y_2, which we used to solve the two initial values for the problem. When we have repeated roots, we still need to make sure that we have two independent solutions. How do we get this? Well, just multiply one solution by t[1].

[1]Note that if $y = e^{rt}$ is a solution, with r a double-root. To simplify matters, let $a = 1$. Then,

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \left(\frac{d}{dt} - r\right)\left(\frac{d}{dt} - r\right)y.$$

For example, suppose that we solved the characteristic equation and got $r_1 = r_2 = r$. We still get one solution out of this, namely

$$y_1(t) = e^{rt}.$$

To get the second solution, just take

$$\begin{aligned} y_2(t) &= ty_1(t) \\ &= te^{rt}. \end{aligned}$$

This works since $\frac{d^2t}{dt^2} = 0$, so the extra t in y_2 is eventually killed, and everything cancels out nicely.

Example: Give the general solution to

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 1 = 0$$

with initial conditions $y(0) = 1$, $y'(0) = 0$.
Solution: The characteristic equation for this problem is

$$\begin{aligned} r^2 - 2r + 1 &= (r-1)(r-1) \\ &= 0, \end{aligned}$$

which has the double root $r_1 = r_2 = 1$. Thus, set $y_1(t) = e^t$ and $y_2 = te^t$, so

$$y = \alpha e^t + \beta te^t$$

Now,

$$y(0) = \alpha = 1.$$

For the second condition, calculate that $y'(t) = \alpha e^t + \beta e^t + \beta te^t$. Thus

$$y'(0) = \alpha + \beta = 0.$$

Thus, $\alpha = 1$ and $\beta = -1$, so the solution is

$$y(t) = e^t - te^t. \quad \square$$

Now, $\left(\frac{d}{dt} - r\right)e^{rt} = 0$, so it's easy to see that $\left(\frac{d}{dt} - r\right)\left[te^{rt}\right] = e^{rt}$. Then,

$$\frac{d^2\left(te^{rt}\right)}{dt^2} + b\frac{d\left(te^{rt}\right)}{dt} + c\left(te^{rt}\right) = \left(\frac{d}{dt} - r\right)\left(\frac{d}{dt} - r\right)\left(te^{rt}\right) = \left(\frac{d}{dt} - r\right)e^{rt} = 0,$$

so te^{rt} is indeed another solution. Moreover, it is clear that e^{rt} and te^{rt} are linearly independent.

4.3.1 Flow chart for 2^{nd}-order linear homogeneous DEs

Second-order linear differential equations with real-valued coefficients will produce two (possibly equal) real roots or a pair of complex roots. In this case, you will never encounter repeated complex roots, and you can use the following flow-chart to solve these types of equations:

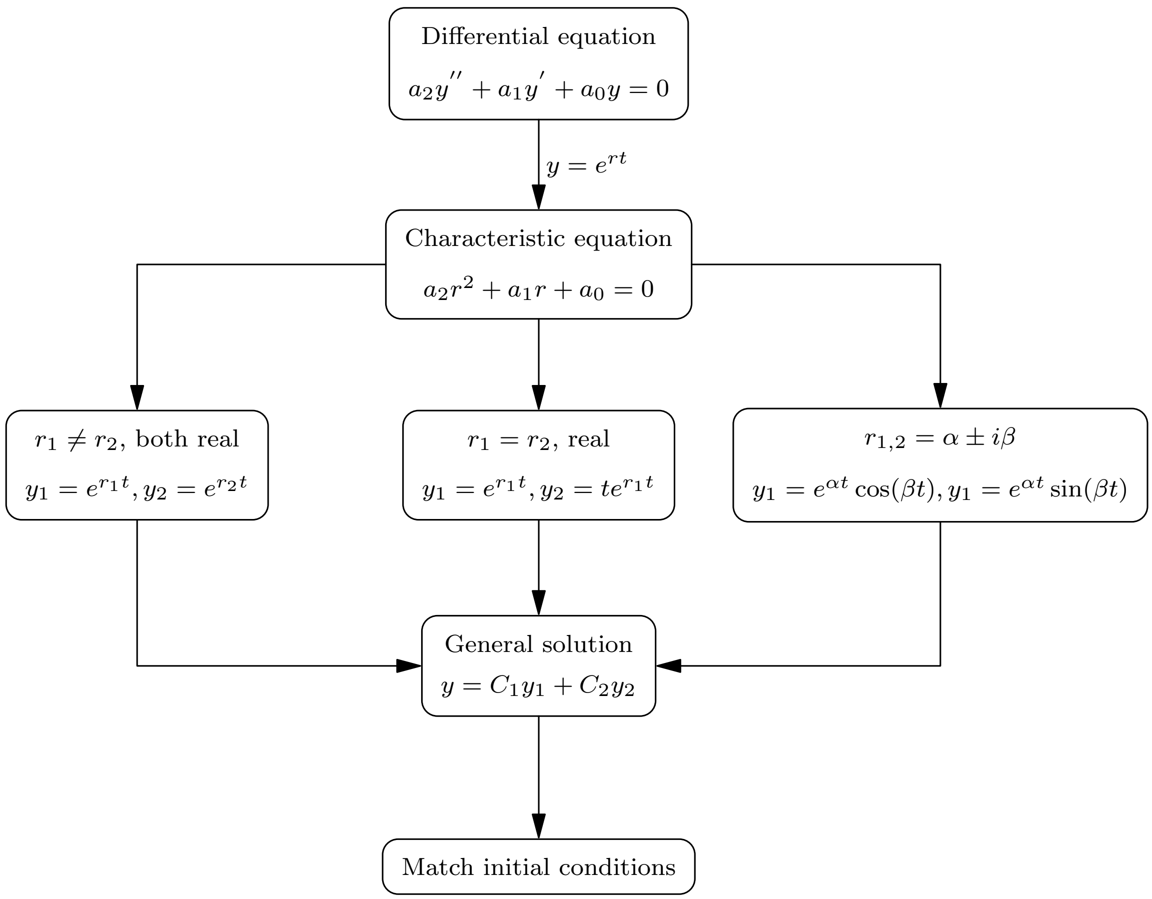

Figure 4.1: Flow chart for solving second-order homogeneous DEs

4.4 Problems

1. Solve the IVP
$$2y'' + 5y' + 2y = 0$$
with initial conditions $y(0) = 0$ and $y'(0) = 3/2$.

2. Solve the IVP
$$2y'' + 8y = 0$$
with initial conditions $y(0) = 1$ and $y'(0) = 2$.

3. Solve the IVP
$$y'' + 2y' + 4y = 0$$
with initial conditions $y(0) = 0$ and $y'(0) = 2$.

4. Prove Euler's formula using differential equations.
Consider the IVP
$$y'' + y = 0, \qquad y(0) = 1, \quad y'(0) = i.$$

   - Step 1: Show that $\{y_1 = \cos(t), y_2 = \sin(t)\}$ are solutions to the differential equation. Find a solution $y = \alpha y_1 + \beta y_2$ to the IVP.
   - Step 2: Using the characteristic equation, find a different pair of solutions $\{y_1, y_2\}$ made up of complex exponentials. Find a solution $y = \alpha y_1 + \beta y_2$ to the IVP.
   - Step 3: Use the uniqueness theorem to show that these two solutions must be identical, thereby proving Euler's formula.

5. Find the general solution of the differential equation
$$10{,}000\, y'' - 100{,}000\, y' + 250{,}000\, y = 0.$$

6. Solve the IVP
$$y'' + 4y' + 5y = 0$$
with $y(\pi) = e^{-\pi}$, $y'(0) = \sqrt{\pi} + 2e^{\pi}$.

Chapter 5

Non-homogeneous 2^{nd} Order Linear DEs

5.1 Method of Undetermined Coefficients

The next step is to add a function to the right-hand side of equation (4.1), so that we get

$$ay'' + by' + cy = f(t).$$

Then what we have is a *non-homogeneous equation* and we say that equation $ay'' + by' + cy = 0$ is the corresponding homogeneous equation.

In this section, we are going to restrict ourselves to a couple of choices for $f(t)$ so that we can use the method of *undetermined coefficients*, also known as *we can probably guess what the answer is, so let's do that.*

The idea behind this method is trying to find the particular solution y_p, which has the property

$$ay_p'' + by_p' + cy_p = f(t), \tag{5.1}$$

though it need not match the initial conditions. For instance, if $f(t) = \sin(t)$, guessing that $y_p = A\sin(t) + B\cos(t)$ would probably do the trick. We can just substitute $y_p = A\sin(t) + B\cos(t)$ into equation (5.1) in order to find the values of A and B that work.

Here is a table that will guide you, young Jedi:

$f(t)$	y_p
ke^{at}	Ce^{at}
kt^n	$C_nt^n + C_{n-1}t^{n-1} + \cdots + C_1t + C_0$
$k\cos(at)$ or $k\sin(at)$	$K\cos(at) + M\sin(at)$
kt^ne^{at}	$e^{at}\big(C_nt^n + C_{n-1}t^{n-1} + \cdots + C_1t + C_0\big)$
$kt^n\cos(at)$ or $kt^n\sin(at)$	$(C_nt^n + \cdots + C_0)\cos(at) + (D_nt^n + \cdots + D_0)\sin(at)$
$ke^{at}\cos(bt)$ or $ke^{at}\sin(bt)$	$e^{at}(K\cos(at) + M\sin(at))$
$kt^ne^{at}\cos(bt)$ or $kt^ne^{at}\sin(bt)$	$(C_nt^n + \cdots + C_0)e^{at}\cos(bt) + (D_nt^n + \cdots + D_0)e^{bt}\sin(bt)$

Finally, if your guess ends up being a polynomial times a linear combination of the solutions to the corresponding homogeneous equation (known as y_1 and y_2 in previous lab), then multiply your guess by t until it isn't. For example, if

$$y_1 = e^t, \quad y_2 = e^{2t}, \quad \text{and} \quad f(t) = t^2 e^t,$$

then you would choose

$$y_p = t\big(C_2 t^2 + C_1 t + C_0\big)e^t.$$

If $y_1 = e^t, \quad y_2 = te^t \quad \text{and} \quad f(t) = t^2 e^t$, then you would choose

$$y_p = t^2\big(C_2 t^2 + C_1 t + C_0\big)e^t.$$

Once you have determined y_p, combine it with the homogeneous solutions to give the general solution

$$\boxed{y(t) = \alpha y_1(t) + \beta y_2(t) + y_p(t)}$$

to match the initial conditions.

Example: Solve the IVP

$$y'' - 3y' + 2y = t \tag{5.2}$$

with $y(0) = 3/4$ and $y'(0) = 3/2$.
Solution: This has characteristic equation

$$r^2 - 3r + 2 = (r-1)(r-2) \quad \Rightarrow \quad r_1 = 1,\, r_2 = 2,$$

so the homogeneous solutions are $y_1(t) = e^t$, $y_2 = e^{2t}$. Since $f(t) = t$ is not a linear combination of y_1 and y_2, we can just choose $y_p = C_1 t + C_0$. Putting this into equation (5.2), we get

$$\frac{d^2(C_1 t + C_0)}{dt^2} - 3\frac{d(C_1 t + C_0)}{dt} + 2(C_1 t + C_0) = t$$

which implies that

$$-3C_1 + 2C_1 t + 2C_0 = t,$$

so $C_1 = 1/2$ and $C_0 = 3/4$. Thus, $y_p = \frac{t}{2} + \frac{3}{4}$, and the general solution is

$$y = \alpha e^t + \beta e^{2t} + \frac{t}{2} + \frac{3}{4}.$$

Now, we satisfy the initial conditions:

$$y(0) = \alpha + \beta + \frac{3}{4} = \frac{3}{4} \quad \Rightarrow \quad \alpha + \beta = 0,$$

and

$$y'(0) = \alpha + 2\beta + \frac{1}{2} = \frac{3}{2} \quad \Rightarrow \quad \alpha + 2\beta = 1$$

so $\alpha = -1$ and $\beta = 1$. The solution is therefore

$$y(t) = -e^t + e^{2t} + \frac{t}{2} + \frac{3}{4}.$$

Let's check this solution. We have

$$\begin{aligned}
y(t) &= -e^t + e^{2t} + \frac{t}{2} + \frac{3}{4} \\
y'(t) &= -e^t + 2e^{2t} + \frac{1}{2} \\
y''(t) &= -e^t + 4e^{2t}.
\end{aligned}$$

Putting these into equation (5.2) yields

$$\begin{aligned}
y'' - 3y' + 2y &= \left(-e^t + 4e^{2t}\right) - 3\left(-e^t + 2e^{2t} + \frac{1}{2}\right) + 2\left(-e^t + e^{2t} + \frac{t}{2} + \frac{3}{4}\right) \\
&= t,
\end{aligned}$$

as required. $\square$

5.2 Superposition Principle

Suppose that we know the solution $y_{1,p}$ to

$$ay'' + by' + cy = f(t)$$

and the solution $y_{2,p}$ to

$$ay'' + by' + cy = g(t).$$

We can use these to determine the solution to the more difficult problem

$$ay'' + by' + cy = Af(t) + Bg(t) \tag{5.3}$$

by using the fact that the differential operator

$$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c$$

is linear. That is, for constants α and β

$$\begin{aligned}
L(\alpha y_{1,p} + \beta y_{2,p}) &= a(\alpha y_{1,p} + \beta y_{2,p})'' + b(\alpha y_{1,p} + \beta y_{2,p})' + c(\alpha y_{1,p} + \beta y_{2,p}) \\
&= \alpha\left(ay_{1,p}'' + by_{1,p}' + cy_{1,p}\right) + \beta\left(ay_{2,p}'' + by_{2,p}' + cy_{2,p}\right) \\
&= \alpha L(y_{1,p}) + \beta L(y_{2,p}),
\end{aligned}$$

just like "linearity" is defined in linear algebra. Then since $L(y_{1,p}) = f(t)$ and $L(y_{2,p}) = g(t)$,

$$L(Ay_{1,p} + By_{2,p}) = Af(t) + Bg(t),$$

which gives us the particular solution for equation (5.3) without having to do a lot of extra work.

Example: Find the general solution to

$$y'' - 3y' - 4y = t + 10e^{-t}.$$

Solution: The characteristic equation for the homogeneous part is $0 = r^2 - 3r - 4 = (r+1)(r-4)$ which gives $r_1 = -1$ and $r_2 = 4$. The solution space is then spanned by $\{y_1 = e^{-t}, y_2 = e^{4t}\}$. We can use the superposition principle to break this problem up into two smaller problems:

1.
$$y'' - 3y' - 4y = t \tag{5.4}$$

   Notice that t is not a multiple of y_1 or y_2, so, looking at the table, we use $y_{1,p} = at + b$. Plugging this into equation (5.4), we get

$$-3(a) - 4(at + b) = t$$

   Grouping the terms that have a factor of t, we get $-4at = t \Rightarrow a = -1/4$. Similarly, the constant terms give us $-3a - 4b = 0$, which, with $a = -1/4$, gives $b = 3/16$. Thus

$$y_{1,p} = \frac{-t}{4} + \frac{3}{16}.$$

2.
$$y'' - 3y' - 4y = e^{-t} \tag{5.5}$$

   The situation here is slightly different, since e^{-t} is both part of the homogeneous solution, and on the right-hand side. Thus, we need to choose $y_{2,p} = cte^{-t}$. Plugging this into equation (5.5), we get

$$(ct - 2c)e^{-t} - 3(-ct + c)e^{-t} - 4cte^{-t} = e^{-t}.$$

   Dividing by e^{-t} and gathering powers of t, we get the following system:

$$ct + 3ct - 4ct = 0 \quad \Rightarrow \quad 0 = 0$$

   which isn't useful at all. But,

$$-2c - 3c = 1 \quad \Rightarrow \quad c = -\frac{1}{5}$$

   Notice that the first equation didn't tell us anything, but the second equation gave us everything that we needed to solve the system. That is

$$y_{2,p} = -\frac{te^{-t}}{5}.$$

We combine these two solutions to give us the particular solution for the problem: we want $L(y_p) = t + 10e^{-t}$. We found that $y_{1,p}$ gives us t, and $y_{2,p}$ gives us e^{-t}, so we choose

$$\begin{aligned} y_p &= y_{1,p} + 10y_{2,p} \\ &= \frac{-t}{4} + \frac{3}{16} - 2te^{-t}. \end{aligned}$$

The general solution is then

$$\begin{aligned} y &= \alpha y_1 + \beta y_2 + y_p \\ &= \alpha e^{-t} + \beta e^{4t} - 2te^{-t} - \frac{t}{4} + \frac{3}{16}. \quad \square \end{aligned}$$

5.3 Problems

1. Solve the IVP
$$y'' - y = \cos(t)$$
with initial conditions $y(0) = 0$ and $y'(0) = 1$.

2. Solve the IVP
$$y'' + y = \cos(t)$$
with initial conditions $y(0) = 0$ and $y'(0) = 1$.

3. Solve the IVP
$$y'' + y = \cos(t) + t$$
with initial conditions $y(0) = 0$ and $y'(0) = 1$.
Hint: first solve $y'' + y = \cos(t)$, and then $y'' + y = t$. Combine the two to solve $y'' + y = \cos(t) + t$, to which you can apply the initial conditions.

4. Use the method of undetermined coefficients to find a particular solution of the differential equation
$$x''(t) - 3x'(t) = 27t^2e^{3t}$$

Chapter 6

Variation of Parameters

The method of undetermined coefficients is pretty easy, but it only works when we have certain functions on the right-hand side. The method of variation of parameters gives us a more general way to determine y_p. The technique is actually an application of Cramer's rule from linear algebra, and can be generalized to linear differential equations of any order.

Given a differential equation

$$y'' + by' + cy = f(t)$$

with homogeneous solutions $\{y_1(t), y_2(t)\}$, we are going to look for a particular solution of the form $y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$. This means that $y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$, which we can simplify by setting[1]

$$u_1'y_1 + u_2'y_2 = 0. \tag{6.1}$$

Substituting this into the original differential equation produces the linear system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f. \end{aligned} \tag{6.2}$$

Using Cramer's rule to solve the linear system, the solution is

$$\boxed{y_p = y_1 \int \frac{-fy_2}{w}\, dt + y_2 \int \frac{fy_1}{w}\, dt}$$

where

$$\boxed{w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

is the Wronskian. The general solution is, as always,

$$y = \alpha y_1 + \beta y_2 + y_p.$$

[1] If you work out the details, it's easy to see that equation (6.1) makes the system much easier to solve because it reduces the order of the system for u_1 and u_2. The addition of equation (6.1) gives us two equations to solve for two unknowns.

Example: Give the general solution to

$$y'' - 2y' + 2y = 2e^t \tag{6.3}$$

using variation of parameters.
Solution: The characteristic equation is

$$r^2 - 2r + 2 = 0$$

which has roots $r = 1 \pm i$. Thus, $y_1 = e^t \cos t$, and $y_2 = e^t \sin t$. The Wronskian is

$$\begin{aligned} w &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= e^{2t}[\cos t(\cos t + \sin t) - \sin t(\cos t - \sin t)] \\ &= e^{2t}. \end{aligned}$$

The particular solution is

$$\begin{aligned} y_p &= e^t \cos t \int \frac{-2e^t e^t \sin t}{e^{2t}}\, dt\ + e^t \sin t \int \frac{2e^t e^t \cos t}{e^{2t}}\, dt \\ &= 2e^t\left(\cos^2 t + \sin^2 t\right) \\ &= 2e^t. \end{aligned}$$

Check: When we put y_p into equation (6.3), we get

$$2e^t - 4e^t + 4e^t = 2e^t,$$

so we got the correct y_p. □

6.0.1 Flow chart for 2^{nd}-order linear nonhomogeneous DEs

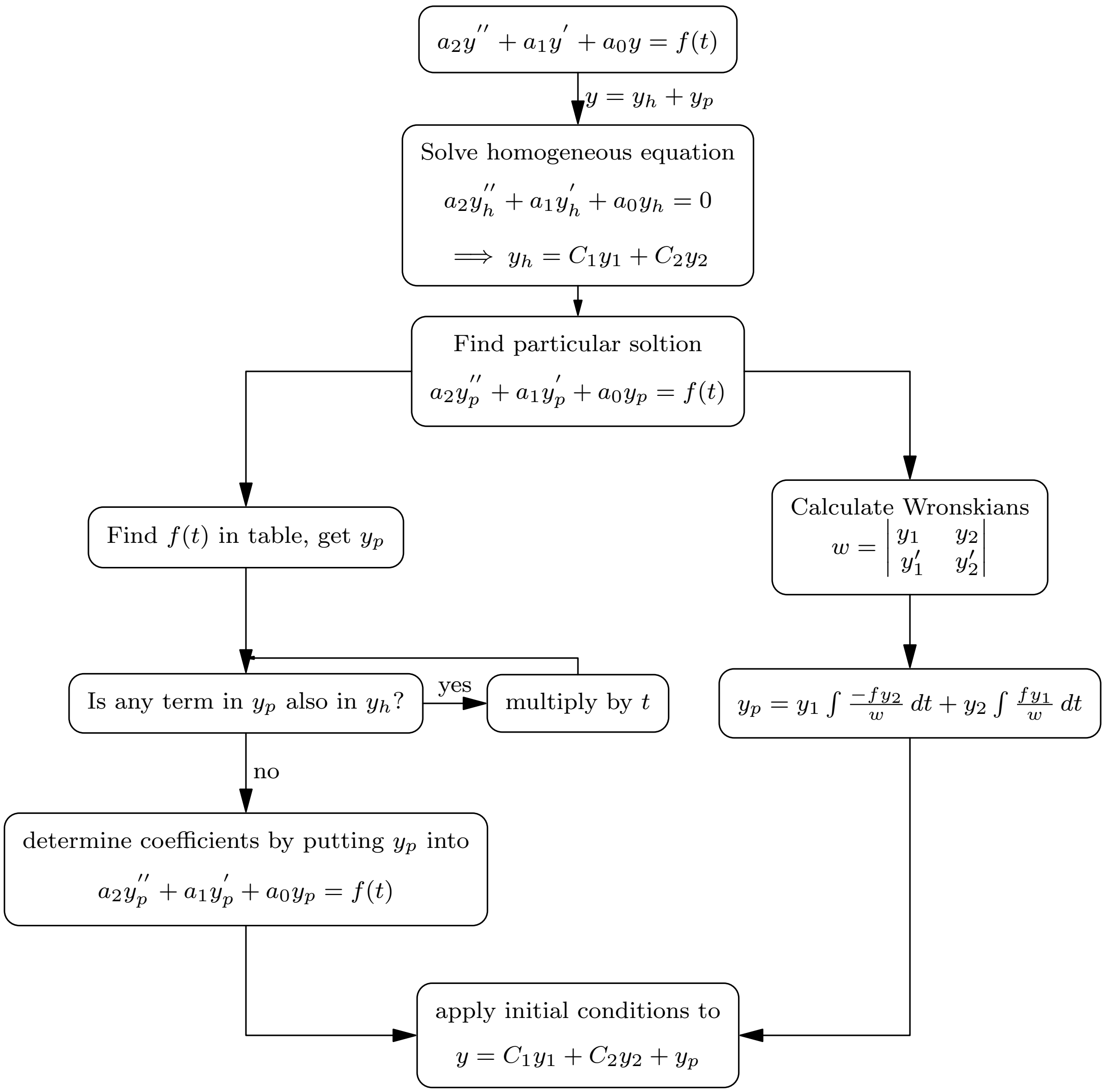

Figure 6.1: Flow chart for solving second-order non-homogeneous DEs

6.1 Problems

1. Use the method of variation of parameters to find the general solution of

$$y'' - 2y' + y = t^{-1}e^{t}.$$

2. Use the method of variation of parameters to find the general solution of

$$y'' + 4y = \sin(2t).$$

3. Find the general solution to

$$y'' + y = \tan t + t.$$

4. Derive the method of variation of parameters for first-order linear equations, i.e. equations of the form

$$y' + by = f(t).$$

   Hint: consider the method in terms of the determinants of matrices.

Chapter 7

Laplace Transforms

Transformations are a very interesting part of mathematics because they give us another perspective from which to look at a problem. If we're lucky with our choice of transformation, then the answer just pops out at us.

The Laplace transform of a function $f(t)$, which we denote by either $\mathcal{L}(f(t))$ or $F(s)$, is defined as

$$\boxed{\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.}$$

This is itself a function, though we have changed the independent variable from t to s. So long as f is sufficiently smooth and doesn't grow too fast as t goes to infinity, we can always find its Laplace transform.

Example: Find the Laplace transform of

$$f(t) = t$$

Solution: The Laplace transform of t is given by

$$F(s) = \int_0^\infty te^{-st}\, dt$$

Using integration by parts with $u = t$ and $dv = e^{-st}$,

$$\begin{aligned} F(s) &= t\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt \\ &= \frac{1}{s^2}, \end{aligned}$$

that is, $\mathcal{L}(t) = 1/s^2$.

Well, that was an excellent example, and we all had a lot of fun. The question remains, however, "why is this useful?" Consider the Laplace transform of the first derivative of $y(t)$:

$$\begin{aligned}\mathcal{L}(y') &= \int_0^\infty y'(t)e^{-st}\,dt \\ &= y(t)e^{-st}\Big|_0^\infty - \int_0^\infty y\big(-se^{-st}\big)\,dt,\end{aligned}$$

so

$$\boxed{\mathcal{L}(y') = s\mathcal{L}(y) - y(0)}.$$

That is, the Laplace transform changes derivatives into polynomials in s, which can be much easier to deal with! The Laplace transform of the nth derivative of y is

$$\boxed{\mathcal{L}(y^{(n)}) = s^n\mathcal{L}(y) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)},$$

which we will apply to higher-order differential equations.

Example: Compute the Laplace transform of the function

$$f(t) = e^{2t} + e^{3t}.$$

Solution 1: By the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \int_0^\infty e^{-st}f(t)dt \\ &= \int_0^\infty \big(e^{2t}+e^{3t}\big)e^{-st}dt \\ &= \int_0^\infty e^{2t}e^{-st}dt + \int_0^\infty e^{3t}e^{-st}dt \\ &= \frac{e^{(2-s)t}}{2-s}\bigg|_0^\infty + \frac{e^{(3-s)t}}{3-s}\bigg|_0^\infty \\ &= \frac{1}{s-2} + \frac{1}{s-3}. \quad \square\end{aligned}$$

Alternatively, we can solve this using the table of Laplace transforms:
Solution 2: The Laplace transform is linear, so

$$\begin{aligned}\mathcal{L}\{e^{2t}+e^{3t}\} &= \mathcal{L}\{e^{2t}\} + \mathcal{L}\{e^{3t}\} \\ &= \frac{1}{s-2} + \frac{1}{s-3}. \quad \square\end{aligned}$$

Of course, you should know how to use both techniques.

7.1 Problems

1. Find the Laplace transform of $f(t) = t^n$ where $n \in \mathbb{N}$, $n > 0$.
2. Find the $\mathcal{L}(e^{\alpha t} \cos t)$ and $\mathcal{L}(e^{\alpha t} \sin t)$ by finding the Laplace transform of $e^{(\alpha+i\beta)t}$.
3. Prove that the Laplace transform is linear.
4. Prove that
$$\mathcal{L}(y^{(n)}) = s^n \mathcal{L}(y) - s^{n-1} y(0) - \cdots - y^{(n-1)}(0).$$

Chapter 8

Solving DEs with Laplace Transforms

Consider a first-order IVP of the form

$$y' + by = f(t), \qquad y(0) = y_0.$$

Taking the Laplace transform of the DE and applying the properties of Laplace transforms yields

$$s\mathcal{L}(y) - y(0) + b\mathcal{L}(y) = \mathcal{L}(f).$$

Rearranging for $\mathcal{L}(y)$, we get

$$\mathcal{L}(y) = \frac{\mathcal{L}(f) + y_0}{s + b}.$$

We need to invert the Laplace transform in order to determine y. Since the inverse Laplace transform is kind of complicated, the easiest way to deal with this is to use the table of Laplace transforms found in section A.2, page 65.

It is often necessary to use the technique of partial fractions to disentangle the Laplace transform of the solution so we can take the inverse transform. Here's an example of how it works in an initial value problem:

Example: Solve the following differential equation using Laplace transforms

$$y'' + 4y = 4t^2 - 4t + 10, \qquad y(0) = 0, \qquad y'(0) = 3$$

Solution: Since

$$\begin{aligned} \mathcal{L}(y'') &= s^2\mathcal{L}(y) - sy(0) - y'(0) \\ &= s^2\mathcal{L}(y) - 3, \end{aligned}$$

the Laplace transform of the left hand side is

$$s^2\mathcal{L}(y) - 3 + 4\mathcal{L}(y)$$

which is equal to the Laplace transform of the right hand side,

$$\frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s}.$$

Rearranging for $\mathcal{L}(y)$ yields

$$\mathcal{L}(y) = \frac{8 - 4s + 10s^2 + 3s^3}{s^3(s^2+4)}$$

and we need to use partial fractions to deal with this:

$$\frac{8 - 4s + 10s^2 + 3s^3}{s^3(s^2+4)} = \frac{As^2 + Bs + C}{s^3} + \frac{Ds + E}{s^2+4}$$

implies that

$$8 - 4s + 10s^2 + 3s^3 = (A + D)s^4 + (B + E)s^3 + (C + 4A)s^2 + 4Bs + 4C.$$

We can match the coefficients of each power of s to get

$$\begin{aligned} 4C = 8 &\Rightarrow C = 2 \\ 4B = -4 &\Rightarrow B = -1 \\ 2 + 4A = 10 &\Rightarrow A = 2 \\ -1 + E = 3 &\Rightarrow E = 4 \\ A + D = 0 &\Rightarrow D = -2. \end{aligned}$$

So we have

$$\mathcal{L}(y) = \frac{2s^2 - s + 2}{s^3} + \frac{-2s + 4}{s^2 + 4}.$$

Now use the table of Laplace transforms to get

$$y(t) = t^2 - t + 2 + 2\sin 2t - 2\cos 2t. \quad \square$$

8.1 Laplace Transforms of Periodic Functions

Periodic functions behave particularly nicely under Laplace transforms. Suppose the function $f(t)$ is periodic with period T. That is, for all t,

$$f(t + T) = f(t).$$

The Laplace transform of f is

$$\begin{aligned} \mathcal{L}\{f\} &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots + \int_{nT}^{(n+1)T} f(t)e^{-st}dt + \ldots \end{aligned}$$

Then we perform the change of variables $\tau = t - nT$ to see that this is just

$$\begin{aligned} &\int_0^T f(\tau)e^{-s\tau}d\tau + e^{-sT}\int_0^T f(\tau)e^{-s\tau}d\tau + \cdots + e^{-nsT}\int_0^T f(\tau)e^{-s\tau}d\tau + \ldots \\ &= \left(1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT} + \ldots\right) \times \int_0^T f(\tau)e^{-s\tau}d\tau. \end{aligned}$$

Now, the first factor is just the geometric series $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ with $r = e^{-sT}$, so

$$\boxed{\mathcal{L}\{f\} = \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}}. \tag{8.1}$$

Example:

Find the Laplace transform of a saw wave with period $T = 1$,

$$f(t) = \begin{cases} t & \text{if } t \in (0,1); \\ f(t-N) & \text{if } t \in (N, N+1) \text{ for } N \in \mathbb{N}. \end{cases}$$

Solution: From equation (8.1), we have

$$\begin{aligned} \mathcal{L}\{f\} &= \frac{\int_0^1 te^{-st}dt}{1-e^{-sT}} \\ &= \frac{\left[t\frac{e^{-st}}{-s}\right]_0^1 - \int_0^1 \frac{e^{-st}}{-s}dt}{1-e^{-s}} \\ &= \frac{se^{-s} - e^{-s} + 1}{s^2(1-e^{-s})}. \quad \square \end{aligned}$$

8.2 Problems

1. Find $\mathcal{L}^{-1}F(s)$ if

$$F(s) = \frac{5s^2 + s - 3}{(s+3)(s^2 - 2s - 3)}$$

2. Solve the following IVP:

$$y'' + y = e^t, \qquad y(0) = 1, \qquad y'(0) = 0.$$

3. Find the Laplace transform of the square wave with period $T = 2$,

$$f(t) = \begin{cases} -1 \text{ if } t \in (0,1); \\ \;\;\,1 \text{ if } t \in (1,2). \end{cases}$$

4. Solve the following IVP:

$$y' + 2y = e^{-2t}, \qquad y(0) = 0.$$

Chapter 9

Laplace Transforms: Convolutions, Impulses and Delta functions

9.1 Convolutions

The Laplace transform is linear, so it deals with linear DEs very well. But not all DEs are linear. In fact, the most aren't.

The Laplace transform is a good tool when the nonlinear term is a convolution. The convolution of two functions $f(t)$ and $g(t)$ from $(0, \infty)$ to $\mathbb{R}$ is written $f * g$ and is defined as

$$\boxed{(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau.} \tag{9.1}$$

It has a particularly nice Laplace transform. If we write $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$, then

$$\boxed{\mathcal{L}(f * g) = F(s)G(s).} \tag{9.2}$$

That is, the Laplace transform of a convolution is the product of their Laplace transforms.
Example:

Solve $y'' = g(t)$, $y(0) = c_0$, $y'(0) = c_1$ for $y(t)$ using Laplace transforms.

Solution: Taking the Laplace transform of both sides, we have

$$s^2 Y - sc_0 - c_1 = G(s).$$

Solving for Y yields

$$Y = \frac{G + sc_0 + c_1}{s^2} = \frac{G}{s^2} + \frac{c_0}{s} + \frac{c_1}{s^2}.$$

The solution is then the inverse Laplace transform, i.e.

$$\begin{aligned} y &= \mathcal{L}^{-1}\left(G(s)\frac{1}{s^2} + c_0 + c_1 t\right) \\ &= g(t) * t + c_0 + c_1 t. \quad \square \end{aligned}$$

This technique can be extended to the full harmonic oscillator,

$$y'' + by' + cy = g, \qquad y(0) = c_0, \qquad y'(0) = c_1,$$

which then has solution

$$y = g(t) * \mathcal{L}^{-1}\left(\frac{1}{s^2 + bs + c}\right) + \mathcal{L}^{-1}\left(\frac{sc_0 + bc_0 + c_1}{s^2 + bs + c}\right).$$

9.2 Laplace Transforms of Discontinuous Functions

The Laplace transform, like the method of variation of parameters, allows us to solve differential equations by integrating. This can be tremendously useful, since we can deal with differential equations with functions that are not smooth; they need only be integrable.

The step function, also known as the Heaviside function, is defined as

$$\boxed{u_c(t) = \begin{cases} 0 \text{ if } t < c; \\ 1 \text{ if } t > c. \end{cases}} \tag{9.3}$$

and is useful for describing processes that start at a certain time.

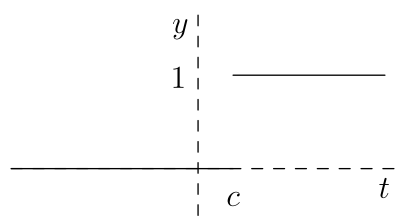

Figure 9.1: Graph of the step function $u_c(t)$.

The Laplace transform of the step function is straightforward to calculate. If we take $c > 0$, then

$$\mathcal{L}(u_c(t)) = \int_0^\infty u_c(t)e^{-st}dt = \int_c^\infty e^{-st}dt = \frac{e^{-cs}}{s}.$$

The step function often shows up multiplied by other functions, but it's easy to spot. Whenever you have an exponential in the frequency domain, the inverse Laplace transform involves a step function: just use the formula

$$\boxed{\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s).}$$

Example:

Solve the differential equation

$$y' = tu_c(t), \qquad y(0) = 0$$

using Laplace transforms.
Solution:

Take the Laplace transform of both sides of the differential equation, yielding

$$\begin{aligned} sY &= \mathcal{L}[tu_c(t)] \\ &= \mathcal{L}[u_c(t)(t-c)] + c\mathcal{L}[u_c(t)] \\ &= e^{-cs}\frac{1}{s^2} + c\frac{e^{-cs}}{s} \end{aligned}$$

Thus,

$$Y = \frac{ce^{-sc}}{s^2} + \frac{e^{-sc}}{s^3}.$$

From the table of Laplace transforms, we know that $\mathcal{L}(f(t-c)u_c(t)) = e^{-cs}F(s)$, so

$$y = c(t-c)u_c(t) + \frac{1}{2}(t-c)^2u_c(t).$$

□

9.3 The Impulse Function

While the step function describes functions which start suddenly (like turning on a switch), the delta function, $\delta(t)$, is slightly more complicated. It describes processes that happen in an instant, like the impact from a hammer. It is (loosely) defined as having the following properties:

$$\boxed{\delta(t) = \begin{cases} \infty & \text{if } t = 0; \\ 0 & \text{if } t \neq 0. \end{cases}} \tag{9.4}$$

and, for any function $f(t)$,

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)}. \tag{9.5}$$

Example: Solve the following differential equation using Laplace transforms.

$$2y'' + y' + 2y = \delta(t-5), \qquad y'(0) = y(0) = 0$$

Solution: Taking the Laplace transform of both sides, we have

$$\left(2s^2 + s + 2\right)Y = e^{-5s}. \tag{9.6}$$

To see that the Laplace transform of $\delta(t-5)$ is e^{-5s}, consider the definition of the Laplace transform:

$$\mathcal{L}(\delta(t-5)) = \int_0^{\infty} e^{-st}\delta(t-5)\,dt. \tag{9.7}$$

If we let $\tau = t-5$, then this is the same as

$$\int_{-5}^{\infty} e^{-s(\tau+5)}\delta(\tau)\,d\tau.$$

Then we apply equation (9.5) to see that this is just e^{-5s}.

Going back to equation (9.6), solving for Y and completing the square yields

$$Y = \frac{e^{-5s}}{2}\left(\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right).$$

Finding the inverse transform is the hardest part of this process, but we can break it up into smaller steps. We can just pull the 1/2 out because the Laplace transform is linear and then if we rearrange this, we get

$$\begin{aligned} Y &= \frac{2}{\sqrt{15}}\, e^{-5s}\, \frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \\ &= \frac{2}{\sqrt{15}}\, e^{-5s}\, \mathcal{L}\left[e^{\frac{-t}{4}} \sin\left(\frac{\sqrt{15}}{4}t\right)\right] \end{aligned}$$

Now we have an exponential term, e^{-5s}, times a term that is the Laplace transform of $e^{\alpha t}\sin(\beta t)$ with $\alpha = 1/4$ and $\beta = \sqrt{15/16}$. So we will use the Laplace transform table to see that Y is the Laplace transform of

$$y = \frac{2}{\sqrt{15}} u_5(t) e^{\frac{-(t-5)}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right). \quad \square$$

9.4 Problems

1. Solve the integro-differential equation

$$y(t) + \int_0^t y(v)(t-v)dv = 1.$$

2. Find the Laplace transform of

$$f = \begin{cases} e^t \text{ if } t < c; \\ t^2 \text{ if } t \geq c. \end{cases}$$

3. Prove that $\mathcal{L}(f * g) = F(s)G(s)$.

4. Solve $y'' = t \times u_c(t)$, $y(0) = y'(0) = 0$, for $y(t)$ using Laplace transforms.

5. Solve the IVP

$$y'' + y = M\delta(t-1), \qquad y'(0) = y(0) = 0.$$

Chapter 10

Solving Systems of Differential Equations

Up to now, we could have solved every problem with the method of variation of parameters. You've probably noticed that some questions are easier to solve with certain techniques than with others, of course, but VoP will handle any nonhomogeneous part, so long as you can calculate the integral. Here's something that it won't handle:

Example: Solve the system of initial value problems for both x and y:

$$\begin{aligned} x' &= -y \qquad & x(0) &= 0 \\ y' &= x \qquad & y(0) &= 1 \end{aligned}$$

Well, you can actually solve this by doing tricky things like taking the derivative of one of the equations, but let's use Laplace transforms:

Solution: The Laplace transform of the original system is

$$\begin{aligned} sX &= -Y \\ sY - 1 &= X. \end{aligned}$$

Solving for Y yields

$$sY - 1 = \frac{-Y}{s} \quad \Rightarrow \quad Y = \frac{s}{s^2+1}. \tag{10.1}$$

Taking the inverse Laplace transform of equation (10.1) gives us

$$y(t) = \cos t.$$

We can solve for X in a similar way, or just notice that $x = y'$, i.e.

$$x(t) = -\sin t.$$

Systems of differential equations model situations where there are two or more quantities are changing in time, for example heat and reaction rate, or predator and prey populations. While they are obviously of great use, they can get quite complicated as the number of variables increases (and are greatly simplified by writing them in terms of matrices).

Example: Solve the following system of differential equations:

$$\begin{aligned} x' - 3x + 2y &= \sin t \\ 4x - y' - y &= \cos t \end{aligned}$$

with initial conditions $x(0) = y(0) = 0$.

Solution:
We take the Laplace transform of each equation and put in the initial conditions, which yields

$$\begin{aligned} sX - 3X + 2Y &= \frac{1}{s^2+1} \\ 4X - sY - Y &= \frac{s}{s^2+1}. \end{aligned}$$

We can solve the second equation for X:

$$X = \frac{1}{4}\left(\frac{s}{s^2+1} + (s+1)Y\right)$$

Put this into the first equation, so

$$(s-3)\left(\frac{1}{4}\frac{s}{s^2+1} + \frac{s+1}{4}Y\right) + 2Y = \frac{1}{s^2+1}$$

which gives

$$\begin{aligned} Y &= -\frac{(s-4)(s+1)}{(s^2+1)(s^2-2s+5)} \\ &= \frac{11s+7}{10(s^2+1)} + \frac{-11s+5}{10(s^2-2s+5)} \\ &= \frac{11s}{10(s^2+1)} + \frac{7}{10(s^2+1)} + \frac{-11}{10}\frac{s-1}{(s-1)^2+2^2} - \frac{3}{10}\frac{2}{(s-1)^2+2^2} \end{aligned} \tag{10.2}$$

by partial fractions and completing the square in the last two terms. We are now in a position to use an inverse Laplace transform to get $y(t)$:

$$y(t) = \frac{7}{10}\sin t + \frac{11}{10}\cos t - \frac{11}{10}e^t\cos 2t - \frac{3}{10}e^t\sin 2t$$

It now remains to solve for $x(t)$. We use the solution for X and the Laplace transform of y. This is to say

$$
\begin{aligned}
X &= \frac{1}{4}\left(\frac{s}{s^2+1} + (s+1)Y\right) \\
&= \frac{-1+7s}{10(s^2+1)} + \frac{-7s+15}{10(s^2-2s+5)} \\
&= \frac{7s}{10(s^2+1)} + \frac{-1}{10(s^2+1)} + \frac{-7}{10}\frac{s-1}{(s-1)^2+2^2} + \frac{2}{5}\frac{2}{(s-1)^2+2^2}
\end{aligned} \tag{10.3}
$$

So we can now use an inverse Laplace transform to get $x(t)$.

$$x(t) = \frac{-1}{10}\sin t + \frac{7}{10}\cos t - \frac{7}{10}e^t\cos 2t + \frac{2}{5}e^t\sin 2t$$

Alternatively, we could have solved this as a linear system, since

$$\begin{bmatrix} (s-3) & 2 \\ 4 & -(s+1) \end{bmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1}{s^2+1} \\ \frac{s}{s^2+1} \end{pmatrix}$$

which is easily solved for X and Y to give

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{s^2+1}\cdot\frac{1}{s^2-2s+5}\begin{pmatrix} 3s+1 \\ s^2-3s-4 \end{pmatrix},$$

which brings us to equations (10.2) and (10.3) with a lot less work. □

10.1 Problems

1. Solve the following system of differential equations:

$$
\begin{aligned}
x' &= z, \qquad & x(0) &= 0 \\
y' &= x, \qquad & y(0) &= 0 \\
z' &= y, \qquad & z(0) &= 1.
\end{aligned}
$$

2. Solve the following system of differential equations

$$
\begin{aligned}
x' - y &= e^t, \qquad & x(0) &= 1. \\
y' + x &= e^t, \qquad & y(0) &= 0.
\end{aligned}
$$

3. Solve the system of differential equations

$$
\begin{aligned}
x' + y &= 0, \qquad & x(0) &= 1 \\
y' &= 2\cosh(2t), \qquad & y(0) &= 1
\end{aligned}
$$

Chapter 11

Series Solutions to DEs

An appropriately smooth function $f(t)$ may be represented as a *Taylor series*,

$$\boxed{f(t) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(t-a)^i}.$$

The Taylor series breaks $f(t)$ up into powers of t; this can be very useful in solving some differential equations. It's important to when a series solution is valid, which is to say, when the Taylor series converges to the value of the function. To answer this question, we have a variety of tests:

1. *The comparison test*: if $\lim_{n\to\infty} |a_n/b_n| \in (0,\infty)$, then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ converges.
2. *The ratio test*: if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = c < 1$, then $\sum_{n=0}^{\infty} a_n$ converges.
3. *The root test*: if $\lim_{n\to\infty} |a_n|^{1/n} = c < 1$, then $\sum_{n=0}^{\infty} a_n$ converges.
4. *The integral test*: if there is a function f with $f(n) = a_n$, then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\int_0^{\infty} f(t)dt$ is finite.
5. *The alternating series test*: if $a_n = (-1)^n c_n$, $c_n \geq 0$, then $\sum_{n=0}^{\infty} a_n$ converges so long as $\lim_{n\to\infty} c_n = 0$ and $c_{n+1} < c_n$ for large enough n.

If we know the Taylor series, we can use these tests to determine the range in which the series equals the function. If you try and get a solution outside of this range, you might get an answer, but it's not going to be a correct one.

Example:

Find the Taylor series for $f = e^x$ about $x = 0$ and determine its interval of convergence.

Solution: First, we need to find the derivatives of e^x. This is easy, since

$$f^{(n)}(x) = \frac{d^n}{dx^n} e^x = e^x.$$

Thus, $f^{(n)}(0) = e^0 = 1$, so the Taylor series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

For what values of x will this series converge? To determine this, we'll have to use a convergence test. Our test are: Let's try the ratio test. That is, fix x and look at

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1.$$

That is, for any $x \in \mathbb{R}$, $n+1$ will eventually be greater than x. Thus, the limit is zero, and the series converges for all $x \in (-\infty, \infty) = \mathbb{R}$. $\square$

In the above example, the series converged everywhere. This isn't always the case: often, a series solution to a differential equation is only valid in some neighbourhood of the initial conditions, and the series becomes divergent when $|t - a| > r$, and we call r the *radius of convergence*.

Now that we can calculate Taylor series and know when they are valid, we can use them to calculate solutions to differential equations. To simplify matters, we will restrict ourselves to initial value problems which begin at $t = 0$, so that $a = 0$ in all the above formulae. Now, we can express the solution $y(t)$ as a power series,

$$y(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{11.1}$$

which is really just a Taylor series:

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n. \tag{11.2}$$

Since Taylor series are unique, equations (11.1) and (11.2) must agree term-by-term. That is, for $n = 0, 1, 2, \ldots,$

$$a_n = \frac{y^{(n)}(0)}{n!}.$$

That is, initial conditions specify coefficients: if you are given $y(0) = c_0$, then you know that

$$\begin{aligned} a_0 &= y(0)/0! \\ &= c_0. \end{aligned}$$

Similarly, $y'(0) = c_1$ specifies a_1, and so on.

We assume that this solution is analytic, so we can take the derivative of the series term-by-term. That is,

$$\frac{dy}{dt} = \frac{d}{dt} \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} \frac{d}{dt}(a_n t^n) = \sum_{n=0}^{\infty} a_n n t^{n-1} = \sum_{n=1}^{\infty} a_n n t^{n-1}.$$

The last equality is because when $n = 0$ we have that $a_0 \times 0 \times t^{0-1} = 0$, so we can ignore this term.

To solve differential equations using power series, we put the power series representation for $y(t)$, $y'(t)$, and so on, into the differential equation. Matching like powers of t gives us a a recurrence relation, which expresses a_n in terms of a_m with $m < n$, which we can solve to get a_n

in terms of n and the initial conditions.

Example:

Solve the IVP

$$y'' = -y, \qquad y(0) = 1, \qquad y'(0) = 0.$$

using a Taylor series for y.

Solution:

Again, let $y = \sum_{n=0}^{\infty} a_n t^n$. Then

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)t^{n-2}.$$

The differential equation is then

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} = -\sum_{n=0}^{\infty} a_n t^n \tag{11.3}$$

If we set $m = n - 2$, then we can shift the summation index on the LHS to start at zero. That is,

$$\begin{aligned} y'' &= \sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} \\ &= \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)t^m. \end{aligned}$$

This makes it much easier to solve equation (11.3), since we now have

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)t^m = -\sum_{n=0}^{\infty} a_n t^n$$

and we can put it all together to get

$$\sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) + a_n]t^n = 0$$

In order for this to hold for all t in a neighbourhood of $t = 0$, it must be that

$$a_{n+2}(n+2)(n+1) + a_n = 0$$

which gives us the recurrence relation

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}. \tag{11.4}$$

From the initial conditions, we have $a_0 = 1$ and $a_1 = 0$. We can solve for a_2 by setting $n = 0$ in equation (11.4), which yields $a_2 = -\frac{1}{2}$.

If we have n even, then $n = 2p$, and

$$\begin{aligned}
a_{2p} &= -\frac{a_{2p-2}}{2p(2p-1)} \\
&= \frac{a_{2p-4}}{2p(2p-1)(2p-3)(2p-4)} \\
&= \frac{(-1)^p a_0}{(2p)!} \\
&= \frac{(-1)^p}{(2p)!}
\end{aligned}$$

It's easy to see that $a_3 = 0$, and, in fact, $a_n = 0$ if n is odd. Since we now know the values for all the a_n, we can write the solution:

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} t^n \begin{cases} \frac{(-1)^{n/2}}{n!} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p}}{(2p)!}.
\end{aligned}$$

You may recall from previous classes that this is just the Taylor series for $\cos t$. □

Sometimes we just don't know how to solve a differential equation, and the best that we can do is to give an approximation of the solution which is valid over some range. One way to do this is to use the *Taylor Polynomial Approximation* (or simply a Taylor polynomial). It's easy to deal with both analytically and numerically, and it can be used to get at least a basic understanding of just about any initial value problem. The nth degree Taylor polynomial is

$$\begin{aligned}
f_n(t) &= f(a) + f'(a)(t-a) + \cdots + \frac{f^{(n)}(a)}{n!}(t-a)^n \\
&= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i,
\end{aligned}$$

and it exists for any function f whose first n derivatives are smooth for $t \in [a, t]$. Let $R_n(t) = f(t) - f_n(t)$ denote the error associated with the nth Taylor polynomial. Then, Taylor's theorem states that there exists a number $\xi \in [a, t]$ where

$$\boxed{R_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(t-a)^{n+1}}.$$

So long as $\lim_{n\to\infty} R_n(t) = 0$, f_n will be a better and better approximation as n increases.

11.1 Problems

1. Determine the convergence set of the series

$$\sum_{n=0}^{\infty} \frac{3^n}{n}(x-2)^n.$$

2. Determine the first four non-zero terms of the Taylor series of the solution to the Chebyshev equation,
$$(1-x^2)y'' - xy' + p^2 y = 0$$
with initial conditions $y(0) = a_0$, $y'(0) = a_1$.

3. Solve
$$y' = \frac{1}{1+x^2}, \qquad y(0) = 0$$
using a Taylor series. Do not leave your solution as an infinite series, but equate it to a known function.

4. Give a recursion formula for a_n if
$$a_{3n+1} = \frac{a_1}{(3n+1)\times(3n)\times(3n-2)\ldots 6\times 4\times\times 3}.$$

5. Given the recursion relationship
$$a_n = \frac{-2}{n^2}a_{n-1},$$
find a_n explicitly.

6. Given the recursion relationship
$$a_{n+4} = \frac{-k^2}{(n+4)(n+3)}a_{n-1},$$
with k a constant, find a_n explicitly.

7. Solve the differential equation
$$y'' - 2xy' + \lambda y = 0,$$
with λ a constant, using a Taylor series about $x = 0$. For what values of λ is the solution a polynomial?

8. Find the general solution to $y''' + \lambda y = 0$.

9. Find the general solution to $y' = 2xy$.

Chapter 12

Series Solutions to DEs at Regular Singular Points

Differential equations of the form

$$y'' + p(t)y' + q(t)y = 0 \tag{12.1}$$

can behave poorly at $t = 0$, and may not admit solutions of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. However, we can still solve these problem by modifying the series expansion. The easiest case is when the singular point is a regular singular point, for which we use *the method of Froebenius.*

Definition: *Singular points.* The point $t = 0$ is a *singular point* if either $\lim_{t\to 0} p(t) = \infty$ or $\lim_{t\to 0} q(t) = \infty$.

Definition: *Regular singular points.* The point $t = 0$ is a *regular singular point* of equation (12.1) if the two limits $\lim_{t\to 0} tp(t) = p_0$ and $\lim_{t\to 0} t^2 q(t) = q_0$ exist and are finite.

Series solutions at regular singular points have one solution of the form

$$\boxed{y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r}},$$

for some $r \in \mathbb{C}$ such that $a_0 \neq 0$.

Since we now have one more variable for which to solve (i.e. r in addition to $a_0, a_1, \dots$), we require one more equation in order to determine the solution uniquely. Then if we take the first and second derivative of $y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$ and put them into the left-hand side of equation (12.1), we can see that the leading-order (which is to say the t^{r-2}) coefficient is

$$a_0 r(r-1)t^{r-2} + a_0 \frac{p_0}{t} r t^{r-1} + a_0 \frac{q_0}{t^2} t^r$$

In order to satisfy equation (12.1), this equation must equal 0. We can multiply by t^2 and since $a_0 \neq 0$, we see that

$$\boxed{r(r-1) + p_0 r + q_0 = 0.}$$

This is called the *indicial equation*. It's a quadratic so it has two solutions, $r = r_1, r_2$.

Second-order differential equations always have two independent solutions. For regular singular points, there are three cases which determine the form of y_2:

1. If $r_1 \neq r_2$, and $r_1 - r_2$ is *not* an integer:

   then $y_2 = \sum_{n=0}^{\infty} b_n t^{n+r_2}$.

2. If $r_1 = r_2$:

   then $y_2 = \ln(t) y_1(t) + \sum_{n=0}^{\infty} b_n t^{n+r_1}$.

3. If $r_1 \neq r_2$, and $r_1 - r_2$ *is* an integer:

   then $y_2 = C \ln(t) y_1(t) + \sum_{n=0}^{\infty} b_n t^{n+r_2}$, for some $C \in \mathbb{C}$.

Once one has determined r_1 and r_2, the constants a_n and b_n can be determined by matching powers as per the previous section. We're often only interested in determining y_1.

Example:

Consider Bessel's equation,

$$t^2 y'' + t y' + (t^2 - \alpha^2) y = 0. \tag{12.2}$$

Solution:

We can divide Bessel's equation by t^2 to get it in the form of equation (12.1), that is

$$y'' + \frac{1}{t} y' + \left(1 - \frac{\alpha^2}{t^2}\right) y = 0,$$

which is clearly singular. We can identify $p = 1/t$, and $q = 1 - \frac{\alpha^2}{t^2}$. Then,

$$p_0 = \lim_{t \to 0} t p(t) = 1, \quad \text{and} \quad q_0 = \lim_{t \to 0} t^2 q(t) = -\alpha^2.$$

The indicial equation is then $r(r-1) + r - \alpha^2 = 0 \Rightarrow r_{1,2} = \pm\alpha$. Thus,

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n t^{n+\alpha} \\ y_1' &= \sum_{n=0}^{\infty} a_n (n+\alpha) t^{n+\alpha-1}, \\ y_1'' &= \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) t^{n+\alpha-2}. \end{aligned}$$

Putting this into equation (12.2), we have

$$\sum_{n=0}^{\infty} a_n \left[(n+\alpha)(n+\alpha-1) + (n+\alpha) - \alpha^2\right] t^{n+\alpha} + \sum_{n=0}^{\infty} a_n t^{n+2+\alpha} = 0.$$

Changing the index of the second sum, we have

$$\sum_{n=0}^{\infty} a_n \left[(n+\alpha)(n+\alpha-1)+(n+\alpha)-\alpha^2\right] t^{n+\alpha} + \sum_{n=2}^{\infty} a_{n-2} t^{n+\alpha} = 0.$$

The a_0 and a_1 terms are determined by initial conditions. For $n \geq 2$, the terms must cancel, so we have

$$a_n \left[(n+\alpha)^2 - \alpha^2\right] t^{n+\alpha} + a_{n-2} t^{n+\alpha} = 0$$

which implies that

$$\begin{aligned} a_n &= \frac{-a_{n-2}}{(n+\alpha)^2 - \alpha^2} \\ &= \frac{-a_{n-2}}{n(n+2\alpha)}. \end{aligned}$$

Associating y_1 with $a_0 = 1$ and $a_1 = 0$, we can set $n = 2m$, so the recursion relationship is

$$\begin{aligned} a_{2m} &= \frac{-a_{2(m-1)}}{2m(2m+2\alpha)} \\ &= \frac{-a_{2(m-1)}}{2^2 m(m+\alpha)}. \end{aligned}$$

Solving the recursion relationship then yields

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(m+\alpha)(m+\alpha-1)\ldots(1+\alpha)}.$$

The solution is then

$$y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m!\,(m+\alpha)(m+\alpha-1)\ldots(1+\alpha)(\alpha)} t^{m+\alpha}. \quad \square \tag{12.3}$$

We can simplify this a little by setting $a_0 = 1/(2^\alpha \Gamma(\alpha+1))$. Then we would get the standard form of Bessel's function of the first kind of order α,

$$J_\alpha(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\alpha+n)} \left(\frac{t}{2}\right)^{2n-\alpha}.$$

Here $\Gamma(x)$ is called the Gamma function and it is an extension of the factorial function. So in particular, when $\alpha = 0$, we get

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} (n!)^2}.$$

12.1 Problems

1. Find a general solution to the Cauchy-Euler (equidimensional) equation,

$$(\pi x)^2 y''(x) + \pi(\pi - 2)xy'(x) + y(x) = 0$$

for $x > 0$.

2. Find the solution to Bessel's equation corresponding to $r = -\alpha$, assuming that 2α is not an integer.

3. Find the first solution (i.e. y_1 in the above notation) to the differential equation

$$x^2 y'' - xy' + (1 - x)y = 0.$$

Chapter 13

Partial Differential Equations

Let $u(x,t)$ be the temperature of a one-dimensional rod with thermal conductivity k. The equation that describes the evolution of y is called the heat equation:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.}$$

This is an important example of a partial differential equation (PDE), in which the derivatives with respect to one variable are related to derivatives with respect to another variable, in this case $\partial/\partial t$ and $\partial/\partial x$.

The basic technique for solving partial differential equations is to transform it into many ordinary differential equations, which we can then solve using any of the techniques that we have discussed so far. One very powerful technique to do this is to transform the function y into a *Fourier series.*

13.1 Separation of Variables

As in the previous section, we use a series solution for y and expand around $x = 0$. However, instead of the coefficients being constant, we allow them to be functions of time, and, instead of using x^n, we let $X_n(x)$ be a more general function of x. That is, we let

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} T_n(t)X_n(x).}$$

As you will see in the next lab, we can choose the functions X_n in a way that makes the PDE solvable. For now, let's consider the case where the X_n are either sine or cosine, i.e. when u is a Fourier sine or cosine series.

13.2 Fourier Cosine and Sine Series

Consider functions that are periodic with period $2T$. If we set $X_n = \cos(n\pi x/T)$, we have the *Fourier cosine series* for $f(x)$,

$$\boxed{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right)}$$

(note the annoying 1/2 on the first term). Setting $X_n = \sin(n\pi x/T)$, we get the *Fourier sine series* for $f(x)$,

$$\boxed{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right).}$$

The coefficients a_n and b_n are determined by integration, with

$$\boxed{a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(\frac{n\pi x}{T}\right)dx}, \tag{13.1}$$

and

$$\boxed{b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\left(\frac{n\pi x}{T}\right)dx}.$$

More generally, the *Fourier series* for $f(x)$ is the sum of these, i.e.

$$\boxed{\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right)\right]}.$$

Often, we just look at functions that have period 2π, which makes for the simpler formulae,

$$\begin{aligned}
&\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right],\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx,\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx,
\end{aligned} \tag{13.2}$$

which we will refer to as “the” Fourier series, unless otherwise stated.

Well, now we have yet another series that we can derive from a given function, but we have to show that the series converges to the function if we want to do anything useful. For this we have the following lemma:

Theorem 13.1 (Riemann-Lebesque lemma): *If* $\int_{-\infty}^{\infty} |f(x)|\,dx$ *exists, then*

$$\lim_{z\to\pm\infty}\int_{-\infty}^{\infty} f(x)e^{izx}dx = 0$$

which implies that $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = 0$, *and*

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].}$$

The Fourier series is the expression of a function as the sum an infinite series of waves with different amplitudes. That is, we transform from "x-space" to "frequency-space", which can be incredibly useful. For example, the `Ogg Vorbis` audio format is just a modified cosine-series. Fourier series (and Fourier transforms, which are not covered in this course) lie at the heart of signal analysis. In terms of solving PDEs, we make use of the fact that

$$\frac{d^2}{dx^2}\sin(\lambda x) = -\lambda^2 \sin(\lambda x),$$

(i.e. that sine is an Eigenfunction of d^2/dx^2) to turn differential equations into algebraic equations. But more on that later. First, let's just find the Fourier series for a normal function.
Example:

Find the Fourier series of $f(x) = x$ for $x \in (-\pi, \pi)$.
Solution:

To find the Fourier series, we just need to determine a_n and b_n using equation (13.2). The coefficients for the cosine series are

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx \\
&= \frac{1}{\pi}\left[x\frac{\sin(nx)}{n}\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n}\,dx \right] \\
&= \frac{1}{\pi}\left[0 - \frac{1}{n^2}\cos(nx)|_{-\pi}^{\pi} \right] \\
&= 0.
\end{aligned}$$

This could have been expected, since $f(x) = x$ is an odd function, and $\cos(nx)$ is even, and the integration domain is symmetric. Notice, then, that the cosine series of a function is equal to the even part of the function. Now, for the sine series, we have

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx \\
&= \frac{1}{\pi}\left[\frac{-x\cos(nx)}{n}\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-\cos(nx)}{n}\,dx \right] \\
&= \frac{1}{\pi}\left[\frac{-\pi\cos(\pi n)}{n} - \frac{-\pi\cos(-n\pi)}{n} - 0 \right] \\
&= \frac{1}{\pi}\left[\frac{-2\pi}{n}\cos(n\pi) \right] \\
&= \frac{-2(-1)^n}{n} \\
&= (-1)^{n+1}\frac{2}{n}.
\end{aligned}$$

The Fourier series for $f(x) = x$ is then

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx). \quad \square \tag{13.3}$$

This does indeed converge to $f(x) = x$ around $x = 0$, as can be seen in figure 13.1.

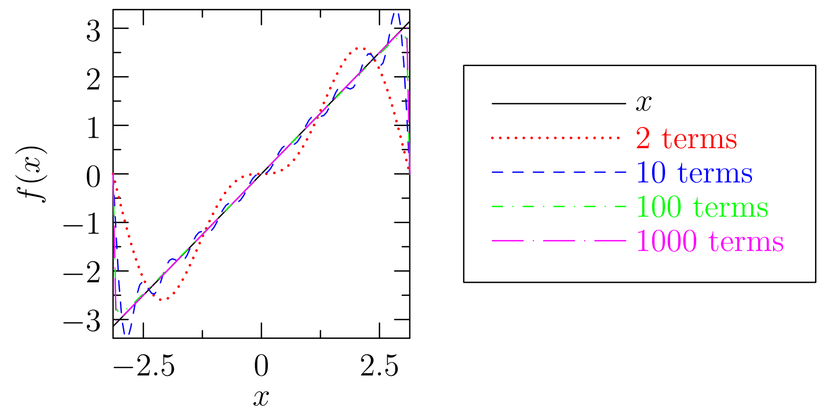

Figure 13.1: Fourier series approximations to $f(x) = x$.

13.3 Problems

1. What are the Fourier sine and cosine series for $y = \sin x$, $x \in (-\pi, \pi)$?
2. What is the Fourier series for $f(x) = \delta(x - 1)$, $x \in (-\pi, \pi)$?
3. What is the Fourier series for $f(x) = 2u_0(x) - 1$, $x \in (-\pi, \pi)$?
4. What is the Fourier series in x for $f(x, t)$, $x \in (-\pi, \pi)$, if

$$f(x,t) = \begin{cases} -t \text{ if } x < 0; \\ t \text{ if } x \geq 0 \end{cases} ?$$

Chapter 14

Partial Differential Equations: Actually Solving Them

Let us now return to the heat equation,

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \tag{14.1}$$

and add some *initial conditions*

$$u(x,0) = f(x),$$

and *boundary conditions*

$$u(0,t) = 0, \qquad u(\pi,t) = 0. \tag{14.2}$$

Physically, this corresponds to modelling the temperature on a rod of length π in contact at both ends with a heat sink with temperature 0 (note that this is not necessarily absolute zero: if we take $u = u + C$, the equation remains the same, so our base temperature is arbitrary.) The rod starts with the temperature at position x given by $f(x)$.

We'll solve this using separation of variables:

$$u = \sum_{n=0}^{\infty} T_n(t) X_n(x), \tag{14.3}$$

where we have taken $X_n(x)$ to be orthogonal[1]. Putting equation (14.3) into equation (14.1), we get

$$\sum_{n=0}^{\infty} \frac{dT_n(t)}{dt} X_n(x) = \beta \sum_{n=0}^{\infty} T_n(t) \frac{d^2 X_n(x)}{dx^2} \tag{14.4}$$

Now, since the X_n's are orthogonal, this actually holds term-by-term. That is, for each n, we have

$$T_n'(t) X_n(x) = \beta T_n(t) X_n''(x)$$

[1] That is, if $i \neq j$, then $\int_0^\pi X_i(x) X_j(x) dx = 0$. This is the case with elements of $\{1, \cos(nx), \sin(nx), n = 1, 2, \dots\}$.

which we can rearrange to get

$$\frac{1}{\beta}\frac{T_n'(t)}{T_n(t)} = \frac{X_n''(x)}{X_n(x)}.$$

Now, the LHS is independent of x, so the RHS must also be independent of x. Since the RHS is also clearly independent of t, it must be constant. That is,

$$\boxed{\frac{1}{\beta}\frac{T_n'(t)}{T_n(t)} = \frac{X_n''(x)}{X_n(x)} = K_n,}$$

where K_n can only depend on n.

This gives us an ODE for X_n,

$$\boxed{X_n'' = K_n X_n.} \tag{14.5}$$

Now, depending on the sign of K_n, we have three possibilities, which we will deal with by enforcing the boundary conditions. Since the X_n are orthogonal, the boundary conditions must be satisfied for each n. The cases are:

1. $K_n = k_n^2 > 0$. In this case, $X_n(x) = C_1 \cosh(k_n x) + C_2 \sinh(k_n x)$. But then, $X_n(0) = 0$ implies that $C_1 = 0$, and $X_n(\pi) = 0$ implies that $C_2 = 0$, so this Eigenfunction is zero.

2. $K_n = 0$. In this case, we get $X_n(x) = C_1 + C_2 x$. Again, the boundary conditions imply that $C_1 = C_2 = 0$, so this Eigenfunction is again zero.

3. $K_n = -k_n^2 < 0$. This gives us periodic behaviour,

$$X_n(x) = C_1 \cos(k_n x) + C_2 \sin(k_n x).$$

We require that $X_n(0) = C_1 = 0$, so we can remove all the cosines. The other boundary condition gives us

$$X_n(\pi) = C_2 \sin(k_n \pi) = 0,$$

which implies that either $C_2 = 0$ or k_n is an integer. Since this is our last chance to have X_n not be zero everywhere, we can't have $C_2 = 0$, so we set k_n to be an integer. In particular, set $k_n = n$.

Thus, $K_n = -n^2$, and

$$\boxed{X_n = \sin(nx).}$$

Much simpler!

We now have enough information to start determining T_n. We know that $K_n = -n^2$, so T_n obeys the equation

$$T_n' = \beta K_n T_n = -\beta n^2 T_n$$

which is solved by

$$\boxed{T_n = T_n(0)e^{-\beta n^2 t}.}$$

Putting this together, our solution (so far) is

$$u(x,t) = \sum_{n=0}^{\infty} T_n(0) e^{-\beta n^2 t} \sin(nx).$$

To get this, we have used the original equation and the boundary conditions. We still have to determine the values of $T_n(0)$, for which we will use the initial conditions, $u(x,0) = f(x)$. That is,

$$\begin{aligned} u(x,0) &= f(x) \\ &= \sum_{n=0}^{\infty} T_n(0) e^{-\beta n^2 t} \sin(nx) \\ &= \sum_{n=0}^{\infty} T_n(0) \sin(nx). \end{aligned}$$

In other words, this is just a sine-series for $f(x)$! However, instead of integrating over $(-\pi, \pi)$, we only know $f(x)$ for $x \in (0, \pi)$. We can solve this problem by extending $f(x)$ as an odd function by setting $f(-x) = -f(x)$, so the coefficients are given by

$$\begin{aligned} T_n(0) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) dx, \end{aligned}$$

since the integrand is even.

The solution to the heat equation, for the type of initial and boundary conditions given above, is

$$\boxed{u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\int_0^{\pi} f(x) \sin(nx)\, dx \right] e^{-\beta n^2 t} \sin(nx).}$$

Example:

The initial temperature in a rod of length π is given by

$$u(x,0) = 2\sin(x) + \sin(5x), \tag{14.6}$$

and the temperature at the ends of the rod is kept at zero. Assuming that

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2},$$

find $u(x,t)$ for $x \in (0, \pi)$, $t > 0$. *Solution*:
The boundary conditions match those given in equation (14.2), so the above analysis shows that we can express u as a linear combination of $\{\sin(nx), n = 1, 2, \dots\}$. That is,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(0) e^{-\beta n^2 t} \sin(nx).$$

The initial conditions are that $u(x,0) = 2\sin(x) + \sin(5x)$, so $T_1(0) = 2$, $T_5(0) = 1$, and all others are zero. We can then write the solution as

$$u(x,t) = 2e^{-\beta t}\sin(x) + e^{-25\beta t}\sin(5x).$$

This result is shown for various times in figure 14.1 for $\beta = 1$.

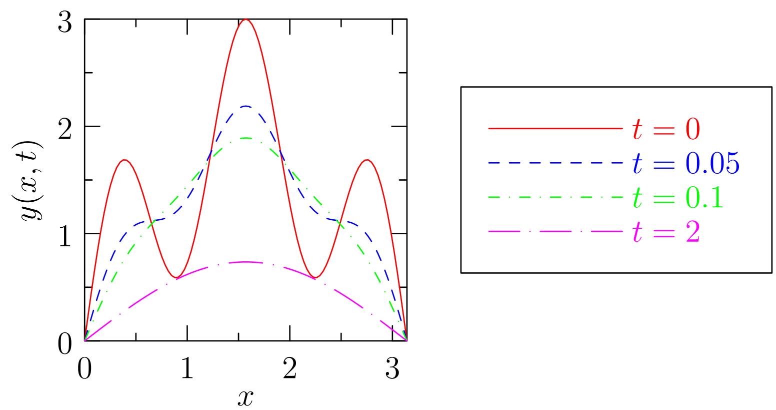

Figure 14.1: The temperature at different times with initial conditions given in equation (14.6). Notice that the $\sin(5x)$ term, which decays like $e^{-25\beta t}$, is relevant only for small t.

14.1 Non-homogeneous constant boundary conditions

Suppose that the boundary conditions were instead

$$u(0,t) = A, \qquad u(\pi,t) = B.$$

Notice that these conditions match the case $K_n = 0$ with

$$u(x,t) = \frac{B-A}{\pi}x + A.$$

This obeys the heat equation, since

$$\frac{\partial^2}{\partial x^2}(mx+b) = 0 = \frac{\partial}{\partial t}(mx+b),$$

and is independent of t, i.e. it is a steady state solution. Then, setting

$$g(x) = f(x) - \left(A + \frac{B-A}{\pi}x\right),$$

$$u(x,t) = A + \frac{B-A}{\pi}x + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^{\pi} g(x)\sin(nx)dx\right)e^{-\beta n^2 t}\sin(nx) \tag{14.7}$$

matches both the initial and boundary conditions.

Example:

Solve the initial boundary problem

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \tag{14.8}$$

$$u(0) = 1, \quad u(\pi) = 1 + \pi,$$

$$u(x, 0) = f(x) = x^2 + x + 1$$

for $u(x, t)$.

Solution: The steady-state solution is $x + 1$. Let

$$\begin{aligned} g(x) &= f(x) - (x + 1) \\ &= x^2 \end{aligned}$$

Then,

$$u(x, t) = 1 + x + \sum_{n=1}^{\infty} T_n(0) e^{-\beta n^2 t} \sin(nx).$$

And in order to satisfy $u(x, 0) = 1 + x + x^2$, we have

$$x^2 = \sum_{n=1}^{\infty} T_n(0) \sin(nx).$$

The coefficients are given by the formula

$$\begin{aligned} T_n(0) &= \frac{2}{\pi} \int_0^\pi x^2 \sin(nx)\, dx \\ &= \frac{2}{\pi} \left(x^2 \frac{-\cos(nx)}{n} \bigg|_0^\pi + \frac{2}{n} \int_0^\pi x \cos(nx) dx \right) \\ &= \frac{2}{n\pi} \left[-\pi^2 \cos(n\pi) + \frac{2}{n} \left(x \sin(nx)|_0^\pi - \int_0^\pi \sin(nx) dx \right) \right] \\ &= \frac{-2\pi}{n} \cos(n\pi) - \frac{4}{n^3 \pi} \left[-\cos(nx) \right]_0^\pi \\ &= \frac{-2\pi}{n} (-1)^n + 4 \frac{(-1)^n - 1}{n^3 \pi} \end{aligned}$$

The solution is therefore

$$u(x, t) = 1 + x + \sum_{n=1}^{\infty} \left[\frac{-2\pi}{n} (-1)^n + 4 \frac{(-1)^n - 1}{n^3 \pi} \right] e^{-\beta n^2 t} \sin(nx).$$

14.1.1 Flow chart for solving the heat equation

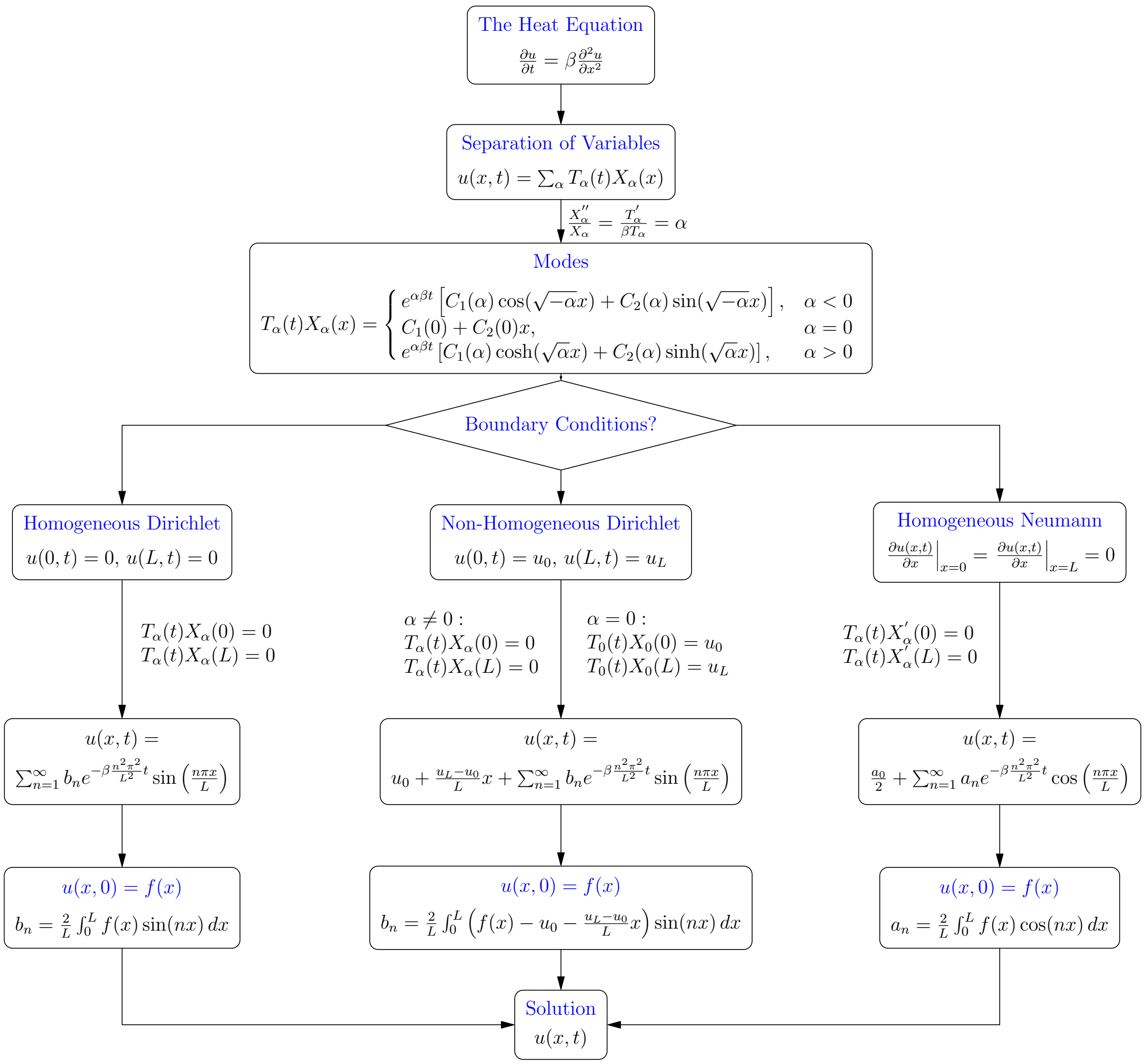

Figure 14.2: Flow chart for solving the heat equation.

14.2 Problems

1. Show that equation (14.7) does indeed solve the heat equation with the given boundary and initial conditions.

2. Solve the heat equation
$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$
with boundary conditions $u(0,t) = u(\pi,t) = 0$ and initial condition $u(x,0) = 2\sin(x) - \sin(3x)$.

3. An inanimate carbon rod of length π is hit with a "laser", transferring an amount of heat H to a point at its centre, where H is a constant. That is, the initial temperature distribution along its length is given by
$$u(x,0) = H\delta\Big(x - \frac{\pi}{2}\Big).$$
If the ends of the rod are kept at constant temperature 0, and the temperature in the rod obeys the relationship
$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$
find $u(x,t)$ for all $t \geq 0$.

4. If the boundary conditions were instead *homogeneous Neumann boundary conditions* at $x = 0$ and $x = \pi$,
$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=\pi} = 0,$$
and the initial condition $u(x,0) = \cos(x)$, what is the solution to the heat equation, $\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$?

Appendix A

Tables

A.1 Table of Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots$$

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \dots$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

A.2 Table of Laplace Transforms

$f(t)$	$\mathcal{L}(f) = F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
t^n	$\dfrac{n!}{s^{n+1}}$
$e^{\alpha t}$	$\dfrac{1}{s-\alpha}$
$e^{\alpha t}f(t)$	$F(s-\alpha)$
$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
$\cos(\beta t)$	$\dfrac{s}{s^2+\beta^2}$
$\sin(\beta t)$	$\dfrac{\beta}{s^2+\beta^2}$
$\cosh(\beta t)$	$\dfrac{s}{s^2-\beta^2}$
$\sinh(\beta t)$	$\dfrac{\beta}{s^2-\beta^2}$
$e^{\alpha t}\cos(\beta t)$	$\dfrac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$e^{\alpha t}\sin(\beta t)$	$\dfrac{\beta}{(s-\alpha)^2+\beta^2}$
$u_c(t),\ c>0$	e^{-cs}/s
$u_c(t)f(t-c),\ c>0$	$e^{-cs}F(s)$
$\delta(t-c),\ c>0$	e^{-cs}
$\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau \doteq f*g$	$F(s)G(s)$
$f(t)$ with $f(t+T)=f(t)$	$\dfrac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$
$t^nf(t)$	$(-1)^n\dfrac{d^n}{ds^n}F(s)$

A.3 Table of Integrals

$$\int u dv = uv - \int v du$$

$$\int \cos x dx = -\sin x$$

$$\int \sin x dx = \cos x$$

$$\int \tan x dx = -\ln|\cos x|$$

$$\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$$

$$\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$\int \tan^2 x dx = \tan x - x$$

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x dx$$

$$\int \sin ax \sin bx dx = -\frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)}, \quad a^2 \neq b^2$$

$$\int \cos ax \cos bx dx = \frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)}, \quad a^2 \neq b^2$$

$$\int \sin ax \cos bx dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}, \quad a^2 \neq b^2$$

$$\int \sec^2 x dx = \tan x$$

$$\int \csc^2 x dx = -\cot x$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \arccos \frac{a}{x}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a},$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right|,$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left| x + \sqrt{x^2 + a^2} \right|$$

$$\int \sinh x dx = \cosh x$$

$$\int \cosh x dx = \sinh x$$

$$\int e^{ax} \sin nx dx = \frac{e^{ax}(a \sin nx - n \cos nx)}{a^2 + n^2}$$

$$\int e^{ax} \cos nx dx = \frac{e^{ax}(a \cos nx + n \sin nx)}{a^2 + n^2}$$