MATH 111 & 112 document

A document for the students of PCC.

FUNCTIONS 1

Contents

1.1 The Basics of Function Vocabulary

Section Themes, Concepts, Issues, Competencies, and Skills:

- *You will have an understanding of the definition of a function.*
- *You will be able to use standard notation concerning functions correctly, and recognize when notation has been used incorrectly.*
- *You will recognize some real examples of functions in your life.*

Most of us are familiar with the $\sqrt{}$ symbol. This symbols is used to turn numbers into their square roots. Sometimes it's simple to do this on paper or in our heads, and sometimes it helps a lot to have a calculator. We can see some calculations in Table 1.1.

Table 1.1: Values of $\sqrt{x}$

$\sqrt{9} = 3$
$\sqrt{1/4} = 1/2$
$\sqrt{2} \approx 1.41$

The $\sqrt{}$ symbol signifies a *process*; it's a way for us to turn numbers into other numbers. This idea of having a process for turning numbers into other numbers is fundamental to the science and mathematics that uses college-level algebra.

Definition 1 (Function)
A function is a process for turning numbers into (potentially) different numbers.

This definition is so broad that you probably use functions all the time.

Example 1 Think about each of these examples, where some process is used for turning one number into another.

- If you use a person's birth year to determine how old they are, you are using a function.
- If you look up the Kelly Blue Book value of a Mazda Protegé based on how old it is, you are using a function.
- If you use the the amount of money that you have available to determine how much you wish to spend on a birthday gift for your friend, you are using a function. ■

The process of using $\sqrt{}$ to change numbers might feel more "mathematical" than these examples. Let's continue thinking about $\sqrt{}$ for now, since it's a formula-like symbol that we are familiar with. One concern with $\sqrt{}$ is that although we live in the modern age of computers, this symbol is not found on most keyboards. And yet computers still tend to be capable of producing square roots. Computer technicians write sqrt( ) when they want to compute a square root, as we see in Table 1.2.

Table 1.2: Values of $\sqrt{x}$

$\mathrm{sqrt}(9) = 3$
$\mathrm{sqrt}(1/4) = 1/2$
$\mathrm{sqrt}(2) \approx 1.41$

The parentheses in sqrt( ) are very important. To see why, try to put yourself in the "mind" of a computer, and look closely at

$$\mathrm{sqrt}\,16$$

The computer will recognize sqrt and know that it needs to compute a square root. But sometimes computers have myopic vision and they might not see the entire number 16. A computer might think that it needs to compute sqrt 1 and then append a "6" to the end, which would produce a final result of 16. This is probably not what was intended. And so the purpose of the parentheses in sqrt(16) is to denote exactly what number needs to be operated on.

This use of sqrt( ) serves as a model for the standard notation that is used worldwide to write down most functions. By having a standard notation for communicating about functions, people in China, Venezuela, Senegal, and the United States can all communicate mathematics with each other more easily.

Functions have their own names. We've seen a function named sqrt, but any name you can imagine is allowable. In the sciences, it is common to name functions with whole words, like weight or health_index. In mathematics, we often abbreviate such function names to w or h. And of course, since the word "function" itself starts with "f", we will often name a function f.

It's crucial to continue reminding ourselves that functions are *processes* for changing numbers; they are not numbers themselves. And that means that we have a potential for confusion that we need to stay aware of. In some contexts, the symbol t might represent a variable - a number that is represented by a letter. But in other contexts, t might represent a function - a process for changing numbers into other numbers. By staying conscious of the context of an investigation, we avoid confusion.

Next we need to discuss how we go about using a function's name.

Function notation
The standard notation for referring to functions involves giving the function itself a name, and then writing

$$\begin{array}{c}\text{name}\\ \text{of}\\ \text{function}\end{array}\left(\ \text{input}\ \right)$$

Example 2 $f(13)$ is pronounced "f of 13". The word "of" is very important, because it reminds us that f is a process and we are about to apply that process to the input value 13. So f is the function, 13 is the input, and $f(13)$ is the output we'd get from using 13 as input.

$f(x)$ is pronounced "f of x". This is just like the previous example, except that the input is not any specific number. The value of x could be 13 or any other number. Whatever x's value, $f(x)$ means the corresponding output from the function f.

BudgetDeficit(2009) is pronounced "BudgetDeficit of 2009". This is probably about a function that takes a year as input, and gives that year's federal budget deficit as output. The process here of changing a year into a dollar amount might not involve any mathematical formula, but rather looking up information from the Congressional Budget Office's website.

Celsius(F) is pronounced "Celsius of F". This is probably about a function that takes a Fahrenheit temperature as input and gives the corresponding Celsius temperature as output. Maybe a formula is used to do this; maybe a chart or some other tool is used to do this. Here, Celsius is the function, F is the input variable, and Celsius(F) is the output from the function. ■

Function Notation (continued)
While a function has a name like f, and the input to that function often has a variable name like x, the expression $f(x)$ represents the output of the function. To be clear, $f(x)$ is *not* a function. Rather, f is a function, and $f(x)$ its output when the number x was used as input.

As mentioned earlier, we need to remain conscious of the context of any symbol we are using. It's possible for f to represent a function (a process), but it's also possible for f to represent a variable (a number). Similarly, parentheses might indicate the input of a function, or they might indicate that two numbers need to be multiplied. It's up to our judgment to interpret mathematical expressions in the right context. Consider the expression $a(b)$. This could easily mean the output of a function a with input b. It could also mean that two numbers a and b need to be multiplied. It all depends on the context in which these symbols are being used.

★ **try it yourself** ★

Problem 1
Describe your own example of a function that has real context to it. You will need some kind of input variable, like "number of years since 2000" or "weight of the passengers in my car". You will need a process for using that number to bring about a different kind of number. The process does not need to involve a formula; a verbal description would be great, as would a formula.

Give your function a name. Write the symbol(s) that you would use to represent input. Write the symbol(s) that you would use to represent output.

make sure you try it!

FIX

Sometimes it's helpful to think of a function as a machine. This illustrates how complicated functions are. A number is just that - a number. But a function has the capacity to take in all kinds of different numbers into it's hopper (feeding tray) and do different things to each of them.

1.1.1 Tables and Graphs

Since functions are potentially so complicated, we seek out new ways to understand them better. Two basic tools for understanding a function better are tables and graphs.

Example 3 Consider the function BudgetDeficit, that takes in a year as its input and outputs the US federal budget deficit for that year. For example, the Congressional Budget Office's website tells us that BudgetDeficit(2009) is \$1.41 trillion. If we'd like to understand this function better, we might make a table of all the inputs and outputs we can find. Using the CBO's website[1], we put together Table 1.3.

Table 1.3

input x (year)	output BudgetDeficit(x) (\$trillion)
2007	0.16
2008	0.46
2009	1.41
2010	1.29
2011	1.30

How is this table helpful? There are things about the function that we can see now by looking at the numbers in this table.

- We can see that the budget deficit has grown by quite a bit over the entire five-year period.
- We can see that there was a particularly large jump in 2008.
- We can see that the deficit reduced by a little bit between 2009 and 2010, and then remained stable.

These observations serve to help us understand the function BudgetDeficit a little better. ■

Example 4 Let's return to our example of the function sqrt. Tabulating some inputs and outputs reveals Table 1.4.

Table 1.4

input x	output sqrt(x)
0	0
1	1
2	≈ 1.41
3	≈ 1.73
4	2

How is this table helpful? Here are some observations that we can make now.

- We can see that when input numbers increase, so do output numbers.
- We can see even though outputs are increasing, they increase by less and less with each step forward in x.

These observations help us understand sqrt a little better. For instance, based on these observations which do you think is larger: the difference between sqrt(23) and sqrt(24), or the difference between sqrt(85) and sqrt(86)? ■

Another powerful tool for understanding functions better is a graph. Given a function f, one way to make its graph is to take a table of input and output values, and read each row as the coordinates of a point in the xy-plane.

[1] Congressional Budget Office

Example 5 Returning to the function BudgetDeficit that we studied in Example 3, in order to make a graph of this function we view Table 1.3 as a list of points with x and y coordinates, as in Table 1.5. We then plot these points on a set of coordinate axes, as in Figure 1.1. The points have been connected with a curve so that we can see the overall pattern given by the progression of points. Since there was not any actual data for inputs in between any two years, the curve is dashed. That is, this curve is dashed because it just represents someone's best guess as to how to connect the plotted points. Only the plotted points themselves are precise.

Table 1.5

(input, output)
$(x, \text{BudgetDeficit}(x))$
$(2007, 0.16)$
$(2008, 0.46)$
$(2009, 1.41)$
$(2010, 1.30)$
$(2011, 1.29)$

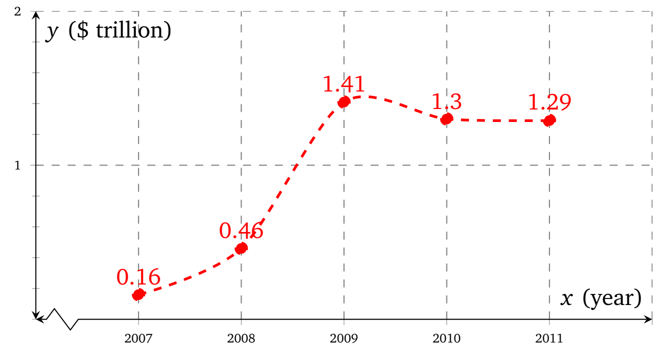


Figure 1.1: $y = \text{BudgetDeficit}(x)$

How has this graph helped us to understand the function better? All of the observations that we made in Example 3 are perhaps even more clear now. For instance, the spike in the deficit between 2008 and 2009 is now visually apparent. Seeking an explanation for this spike, we recall that there was a financial crisis in late 2008. Revenue from income taxes dropped at the same time that federal money was spent to prevent further losses. ■

Example 6 Let's now construct a graph for sqrt. Tabulating inputs and outputs gives the points in Table 1.6, which in turn gives us the graph in Figure 1.2. Just as in the previous example,

Table 1.6

(input, output)
$(x, \text{sqrt}(x))$
$(0, 0)$
$(1, 1)$
$\approx (2, 1.41)$
$\approx (3, 1.73)$
$(4, 2)$

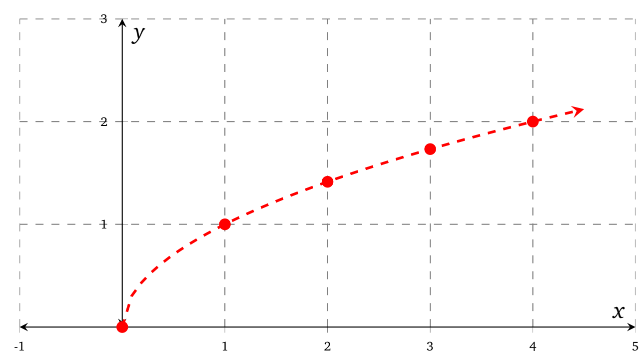


Figure 1.2: $y = \text{BudgetDeficit}(x)$

we've plotted points where we have concrete coordinates, and then we have made our best attempt to connect those points with a curve. Unlike the previous example, here we believe that points could continue to be computed and plotted indefinitely to the right, and so we have added an arrowhead to the graph.

What has this graph done to improve our understanding of sqrt? As inputs (x-values) increase, the outputs (y-values) increase too, although not at the same rate. In fact we can see that our graph is steep on its left, and less steep as we move to the right. This confirms our earlier observation in Example 4 that outputs increase by smaller and smaller amounts as the input increases. ■

The graph of a function
Given a function f, when we refer to a *graph of* f we are *not* referring to an entire picture, like Figure 1.2. A graph of f is only *part* of that picture - the curve and the points that it connects. Everything else: axes, tick marks, the grid, labels, and the

surrounding white space is just useful decoration, so that we can read the graph more easily.

It is also common to refer to the graph of f as the graph of the *equation* $y = f(x)$. However we should never refer to "the graph of $f(x)$". That would indicate a fundamental misunderstanding of our notation. We have decided that $f(x)$ is the output for a certain input x. That means that $f(x)$ is just a number; a relatively uninteresting thing compared to f the function, and not worthy of any two-dimensional picture.

While it is important to be able to make a graph of a function f, we also need to be capable of looking at a graph and reading it well. A graph of f provides us with helpful specific information about f; it tells us what f does to its input values. When we were making graphs, we plotted points of the form

$$(\text{input}, \text{output})$$

Now given a graph of f, we interpret coordinates in the same way.

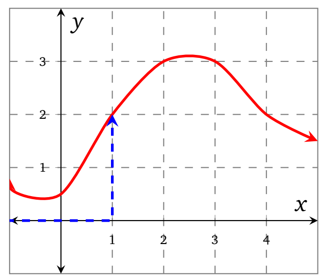


Figure 1.3: $y = f(x)$

In Figure 1.3 we have a graph of a function f. If we wish to find $f(1)$, we recognize that 1 is being used as an input. So we would want to find a point of the form $(1, \ )$. Seeking out x-coordinate 1 in Figure 1.3, we find that the only such point is $(1,2)$. Therefore the output for 1 is 2; in other words $f(1) = 2$.

★ try it yourself ★

Problem 2
Use the graph of f in Figure 1.3 to find $f(0)$, $f(3)$, and $f(4)$.

make sure you try it!

Example 7 Suppose that u is the unemployment function of time. That is, $u(t)$ is the unemployment rate in the United States in year t. The graph of the equation $y = u(t)$ is given in Figure 1.4[2].

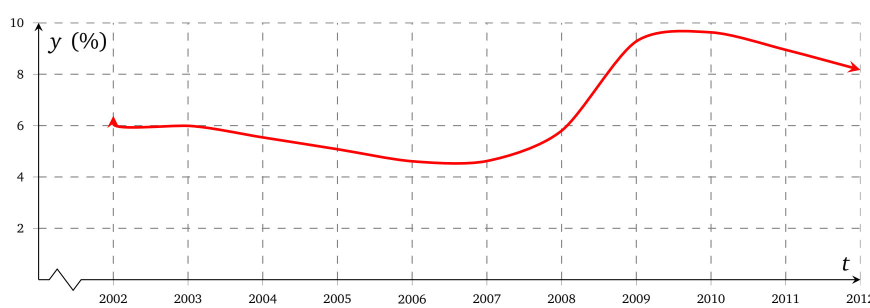


Figure 1.4: Unemployment in the United States

What was the unemployment in 2002? It is a straightforward matter to use Figure 1.4 to find that unemployment was about 6 % in 2002. Asking this question is exactly the same thing as asking to find $u(2002)$. That is, we have one question that can either be asked in a everyday-English way or which can be asked in a terse, mathematical notation-heavy way:

"What was unemployment in 2002?" "Find $u(2002)$."

[2] US Bureau of Labor Statistics

If we use the table to establish that $u(2009) \approx 9.25$, then we should be prepared to translate that into everyday-English using the context of the function: In 2009, unemployment in the U.S. was about 9.25 %.

If we ask the question "when was unemployment at 5 %", we can read the graph and see that there were two such times: early 2005 and mid-2007. But there is again a more mathematical notation-heavy way to ask this question. Namely, since we are being told that the output of u is 5, we are being asked to solve the equation $u(t) = 5$. So the following communicate the same thing:

"When was unemployment at 5 %?" "Solve the equation $u(t) = 5$."

And our answer to this question is:

"Unemployment was at 5 % in early 2005 and mid-2007" "$t \approx 2005$ or $t \approx 2007.5$" ■

★ try it yourself ★

Use the graph of u in Figure 1.4 to respond to the following.

Problem 3
Find $u(2011)$ and interpret it.

Problem 4
Solve the equation $u(t) = 6$ and interpret your solution(s).

make sure you try it!

1.1.2 Translating Between Four Descriptions of the Same Function

We have noted that functions are complicated, and that we are seeking out ways to make them easier to understand. It's common to encounter a problem involving a function and not know how to proceed toward a solution to that problem. As it happens, most functions have at least four standard ways to think about them. If we become capable of translating between these four perspectives, perhaps we will find that one of them makes a given problem easy to solve.

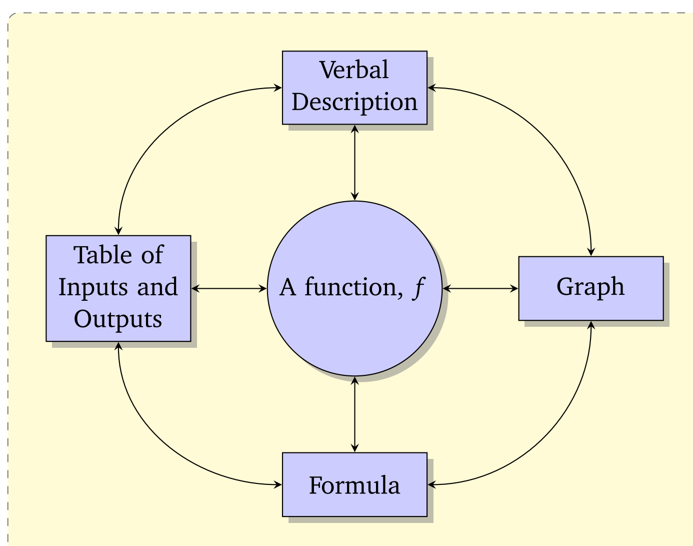


Figure 1.5: Translating Between Perspectives

The four modes for working with a given function are:

- a verbal description
- a table of inputs and outputs
- a graph of the function
- a formula for the function

This has been visualized in Figure 1.5.

Example 8 Consider a function f that squares its input and then adds 1. Translate this verbal description of f into a table, a graph, and a formula.

Solution To make a table for f, we'll have to select some input x-values. These choices are left entirely up to us, so we might as well choose small, easy-to-work-with values. However we shouldn't shy away from negative input values. Given the verbal description, we should be able to compute a column of output values. Table 1.7 is one possible table that we might end up with.

Once we have a table for f, it's a simple matter to make a graph for f as in Figure 1.6, using the table to plot points.

Table 1.7

x	$f(x)$
-2	$(-2)^2+1=5$
-1	$(-1)^2+1=2$
0	$0^2+1=1$
1	$1^2+1=2$
2	5
3	10
4	17

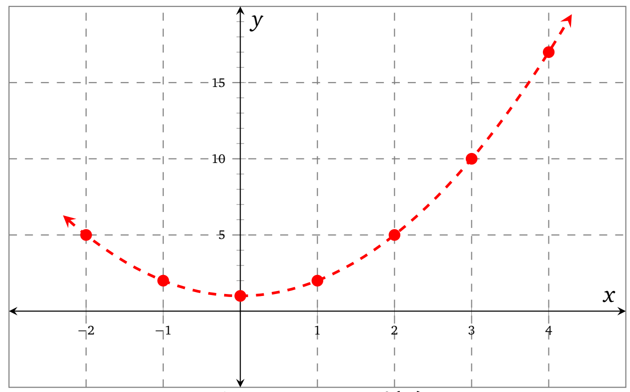


Figure 1.6: $y=f(x)$

Lastly, we must find a formula for f. This means we need to write a mathematical expression that says the same thing about f as the verbal description, the table, and the graph. For this example, we can focus on the verbal description. Since f takes its input, squares it, and adds 1, then we have that

$$f(x)=x^2+1$$

■

Example 9 Let F be the function that takes a Celsius temperature as input and outputs the corresponding Fahrenheit temperature. Translate this verbal description of F into a table, a graph, and a formula.

Solution To make a table for F, we will need to rely on what we know about Celsius and Fahrenheit temperatures. It is a fact that the freezing temperature of water at sea level is 0 °C, which equals 32 °F. Also, boiling temperature of water at sea level is 100 °C, which is the same as 212 °F. One more piece of information we might have is that standard human body temperature is 37 °C, or 98.6 °F. All of this is compiled in Table 1.8. Note that we tabulated inputs and outputs by working with the context of the function, not with any computations.

Once a table is established, making a graph by plotting points is a simple matter, as in Figure 1.7. The three plotted points seem to be in a straight line, so we think it is reasonable to connect them in that way.

To find a formula for F, the verbal definition is not of much direct help. But F's graph does seem to be a straight line. And linear equations are familiar to us. This line has a y-intercept at $(0,32)$ and a slope we can calculate: $\frac{212-32}{100-0}=\frac{180}{100}=\frac{9}{5}$. So the equation of this line is $y=\frac{9}{5}C+32$. On the other hand, the equation of this graph is $y=F(C)$, since it is a graph of the function F. So evidently,

$$F(C)=\frac{9}{5}C+32$$

■

Example 10 – Referencing a function: Label each of the following snippets as correct use of vocabulary (✓) or incorrect use (×); if the usage is incorrect, give a brief reason why.

Table 1.8

C	$F(C)$
0	32
37	98.6
100	212

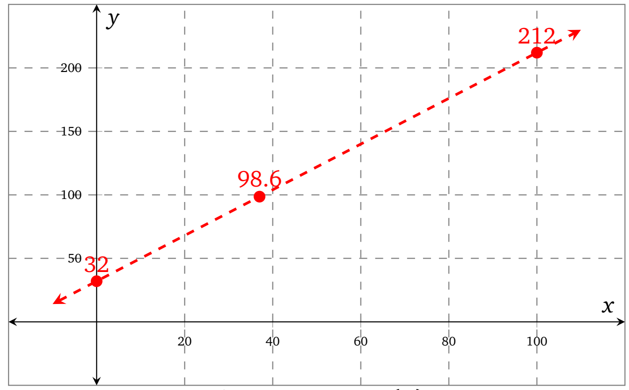


Figure 1.7: $y = F(x)$

(a) The function f is shown in Figure ...

(b) The function $f(x)$ is shown in Figure ...

(c) Consider the function g that has formula $g(x) = x^2 + 3$.

(d) Consider the function $g(x) = x^2 + 1$ ...

Solution Items (a) and (c) are fine usage of vocabulary (✓). Item (b) is not (×), since $f(x)$ is the value of f at x; it is not a function. Also item (d) is bad usage (×) because g is the function; $g(x) = x^2 + 1$ is the *formula* for the function g. ■

We will be using the correct language throughout this document; pay close attention to it and try your best to use it in all of your mathematical work, both verbal and written. ■

Exercises

Problem 5 (Slope and vertical intercept)
In each of the following problems you are given a formula for a linear function. State the slope of each function, and state the vertical intercept of each function as an ordered pair.

5.1 $f(x) = 3x - 1$ **5.2** $g(x) = 5 - \frac{x}{2}$ **5.3** $h(x) = \pi - 10x$ **5.4** $k(x) = mx + b$

Problem 6 (Linear or quadratic)
Decide which of the following formulas correspond to a linear function, and which correspond to a quadratic function.

6.1 $f(x) = 2x + 1$

6.2 $g(t) = t^2 + 2$

6.3 $h(s) = 9 - 5s$

6.4 $j(u) = 4^2 - \frac{u}{5}$

6.5 $\alpha(c) = c^2 + 2c + 4$

6.6 $\beta(m) = 4$

6.7 $\gamma(h) = \frac{4}{3} - h^2$

6.8 $\delta(z) = z$

Problem 7 (Vertex of a quadratic function)
Each of the following formulas corresponds to a quadratic function. State the vertex and range of each function (the domain of each function is $(-\infty, \infty)$).

7.1 $f(x) = (x - 3)^2 + 4$

7.2 $g(x) = 4 - (x - 3)^2$

7.3 $h(x) = 2(x - 5)^2$

7.4 $j(x) = 5(3x - 4)^2 + 7$

7.5 $F(x) = x^2 + 1$

7.6 $G(x) = 3x^2 + 5x$

7.7 $H(x) = 4 - \frac{1}{2}x^2$

7.8 $J(x) = \frac{1}{3} - \frac{x}{2} - \frac{x^2}{5}$

1.2 Domain and Range

A function is a process for turning input values into output values. Occasionally a function f will have input values for which the process breaks down.

Example 1 Let P be the population of Portland as a function of the year. According to Google[3] we can say that:

$$P(2011) = 593\,820 \qquad P(1990) = 487\,849$$

But what if we asked to find $P(1600)$? The question doesn't really make sense anymore. While there were indigenous peoples living in the area then, the city of Portland was not incorporated until 1851. We say that $P(1600)$ is *undefined*. ■

Example 2 If m is a person's mass in kg, let $w(m)$ be their weight in lb. There is an approximate formula for w:

$$w(m) \approx 2.2m$$

From this formula we can find:

$$w(50) \approx 110 \qquad w(80) \approx 176$$

which tells us that a 50-kg person weighs 110 lb, and an 80-kg person weighs 176 lb.

What if we asked for $w(-100)$? In the context of this example, we would be asking for the weight of a person whose mass is -100 kg. This is clearly nonsense. That means that $w(-100)$ is *undefined*. Note that the *context* of the example is telling us that $w(-100)$ is undefined even though the formula alone might suggest that $w(-100) = -220$. ■

Example 3 Let g have the formula

$$g(x) = \frac{x}{x-7}$$

For most x-values, $g(x)$ is perfectly computable:

$$g(2) = -\frac{2}{5} \qquad g(14) = 2$$

But if we try to compute $g(7)$, we run into an issue of arithmetic.

$$\begin{aligned} g(7) &= \frac{7}{7-7} \\ &= \frac{7}{0} \end{aligned}$$

The expression $\frac{7}{0}$ is *undefined*. There is no number that this could equal. ■

★ try it yourself ★

Problem 1

Find an input for the function f that would cause an undefined output, where $f(x) = \dfrac{x+2}{x+8}$.

make sure you try it!

Definition 2 (Domain)
The *domain* of a function f is the collection of all of its valid input values.

Example 4 Referring to the functions from Examples 1–3,

- The domain of P is all years starting from 1851 and later. It would be reasonable to say the the domain is actually all years from 1851 up to the current year, since we cannot guarantee that Portland will still exist into the future.

[3] http://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&idim=place:4159000&dl=en&hl=en&q=population+of+portland

- The domain of w is all positive real numbers. It is nonsensical to have a person with negative mass or even one with zero mass. While there is some lower bound for the smallest mass a person could have, and also an upper bound for the largest mass a person could have, these boundaries are gray. We can say for sure that nonpositive numbers should never be used as input for w.
- The domain of g is all real numbers except 7. This is the only number that causes a breakdown in g's formula. ■

★ **try it yourself** ★

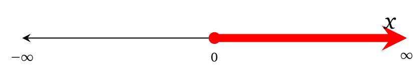
Problem 2
What is the domain of the function sqrt?

make sure you try it!

1.2.1 Interval, Set, and Set-Builder Notation

Communicating the domain of a function can be wordy. In mathematics, we can communicate the same information using concise notation that is accepted for use almost worldwide. Table 1.9 contains example functions from this section, their domains, and illustrates *interval notation* for these domains.

Table 1.9: Domains from Example 2, Example 3, and Problem 2

function	verbal domain	illustration of domain	interval notation for domain
w from Example 2, computing mass from weight	all real numbers greater than 0		$(0,\infty)$
g from Example 3, with formula $g(x)=\frac{x}{x-7}$	all real numbers except 7		$(-\infty,7)\cup(7,\infty)$
sqrt from Problem 2	all numbers greater than or equal to 0		$[0,\infty)$

Interval notation comes in many forms. Each of the expressions (a,b), $(a,b]$, $[a,b)$, and $[a,b]$ are examples of simple intervals. The notation is communicating that we wish to consider all real numbers between a and b. If a round parenthesis is used, then that number itself should be excluded from consideration. If a square bracket is used, then that number itself should be included under consideration. The 'a' might be the symbol $-\infty$, and the 'b' might be the symbol ∞. If these symbols are used, then there is no lowermost or uppermost bound to the interval. Lastly, two or more simple intervals can be joined together with the union symbol $\cup$. Table 1.10 gives more examples of interval notation in use.

Sometimes we will consider collections of only a small, finite number of numbers. In those cases, we use *set notation*. With set notation, we have a list of numbers in mind, and we simply list all of those numbers. Curly braces are standard for encasing the list. Table 1.11 illustrates set notation in use.

While most collections of numbers that we will encounter can be described using a combination of interval notation and set notation, there is another commonly used notation that all students of college algebra should be exposed to: *set-builder notation*. Set-builder notation also uses curly braces. Set-builder notation provides a template for what a number that is under consideration might look like, and then it gives you restrictions on how to use that template. A very basic example of set-builder notation is

$$\{x \mid x \geq 3\}$$

Table 1.10: Interval Notation

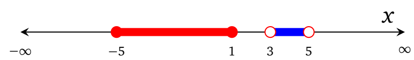
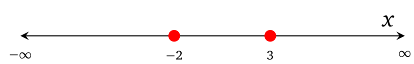
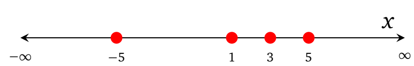
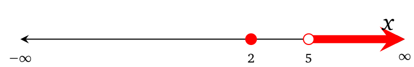
picture of interval	interval notation
	$(-2,3]$
	$[-5,1]\cup(3,5)$
	$(-\infty,2)\cup(2,5)\cup(5,\infty)$
	$(-\infty,\infty)$

Table 1.11: Set Notation

picture of set	set notation
	$\{-2,3\}$
	$\{-5,1,3,5\}$
	$\{2\}\cup(5,\infty)$

Verbally, this is 'the set of all x such that x is greater than or equal to 3'. Table 1.12 gives more examples of set-builder notation in use.

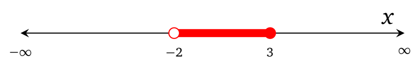
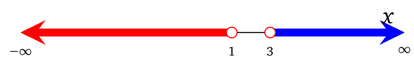
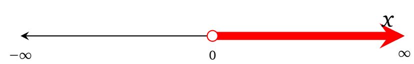
The domain of a function is the collection of its possible inputs; there is a similar notion for *output*.

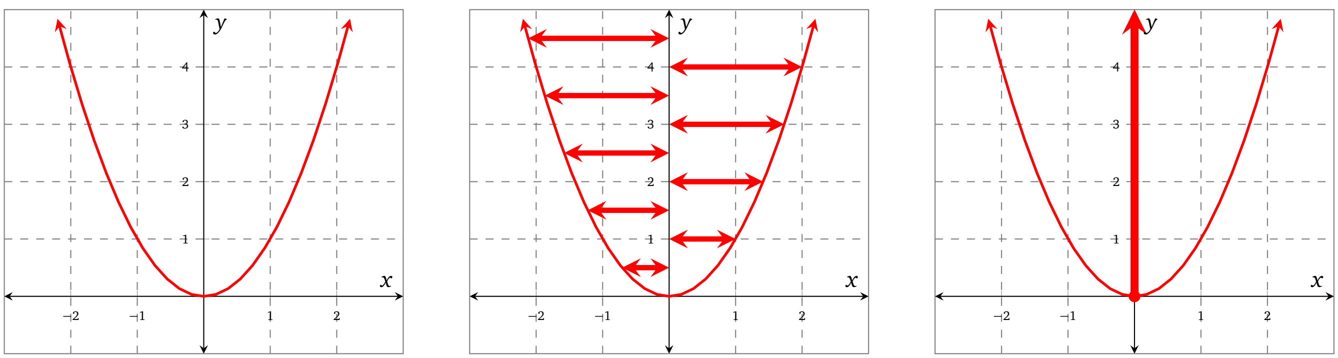
Definition 3 (Range)
The *range* of a function f is the collection of all of its possible output values.

Example 5 Let f be the function defined by the formula $f(x)=x^2$. Finding f's domain is particularly basic. Any number anywhere can be squared to produce an output, so f has domain $(-\infty,\infty)$. What is the *range* of f?

Solution We would like to describe the collection of possible numbers that f can give as outputs. First we will use a graphical approach. Figure 1.8 displays a graph of f, and the visualization that reveals f's range.

Table 1.12: Set-Builder Notation

picture of set	set-builder notation
	$\{x \mid -2 < x \text{ and } x \leq 3\}$
	$\{x \mid x < 1 \text{ or } x > 3\}$
	$\left\{x \,\middle|\, x^2 \leq 16\right\}$
	$\left\{x^2 \,\middle|\, x \text{ is a real number}\right\}$



(a) (b) (c)

Figure 1.8: $y = f(x)$, where $f(x) = x^2$

Output values are the y-coordinates in a graph. If we 'slide the ink' across to the y-axis (Figure 1.8b) to emphasize what the y-values in the graph are, we have y-values that start from 0 and continue upward forever. Therefore the range is $[0, \infty)$ (see Figure 1.8c). ■

Solution Here is an alternative solution. Occasionally it is possible to find the range directly , without the help of a graph. In the case of this function, we understand that the outputs must be nonnegative, since any real number squared is not negative. We also understand that any nonnegative output y that you could imagine (0, 0.01, 244, ...) is a possible output if we feed f the right input, namely $\sqrt{y}$. So the domain of f is $[0, \infty)$. ■

Finding range from a formula
Example 5 shows us that it is sometimes possible to compute a range without the aid of a graph. However until students learn some topics that will be covered later in this text and in a calculus course, it will often be difficult to do so. Therefore when you are asked to find the range of a function based on its formula, your first approach should be a graphical one.

Example 6 Given the function g graphed in Figure 1.9, find the domain and range of g.

Solution To find the domain, we can visualize all of the x-values that are valid inputs for this function,

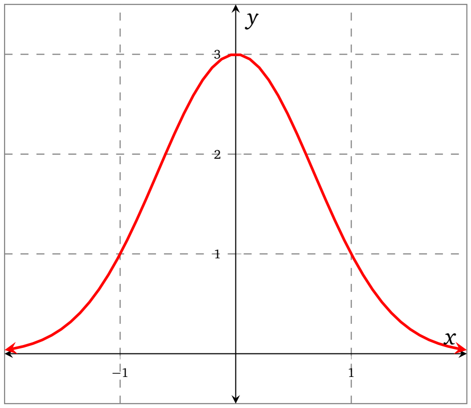


Figure 1.9: $y = g(x)$

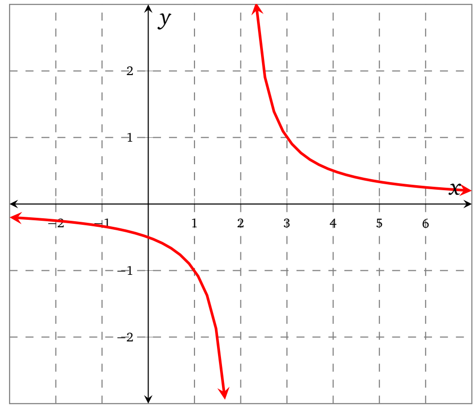


Figure 1.10: $y = h(x)$

by 'sliding the ink down onto the x-axis. The arrows indicate that whatever pattern we see in the graph continues off to the left and right. Here, we see that the arms of the graph are tapering down to the x-axis and extending left and right forever. Every x value is covered, so the domain is (∞, ∞).

If we visualize the possible outputs by 'sliding the ink' sideways onto the y-axis, we find that outputs as high as 3 are possible (including 3 itself). The outputs appear to be very close to 0 when x is large, but they aren't quite equal to 0. So the range is $(0, 3]$. ■

★ **try it yourself** ★

Problem 3
Find the domain and range of the function h graphed in Figure 1.10.

make sure you try it!

The examples of finding domain and range so far have all involved either a verbal description of a function, a formula for that function, or a graph of that function. Recall that there is a fourth persepctive on functions: the table. In the case of a table, we have very limited information about the function's inputs and outputs. If the table is all that we have, then there are a handful of input values listed in the table for which we know outputs. For any other input, the output is undefined.

Table 1.13: The function k

x	$k(x)$
3	4
8	5
10	5

Example 7 Consider the function k given in Table 1.13. What is the domain and range of k?

Solution All that we know about k is that $k(3) = 4$, $k(8) = 5$, and $k(10) = 5$. Without any other information such as a formula for k or a context for k that tells us its verbal description, we must assume that its domain is $\{3, 8, 10\}$; these are the only valid input for k. Similarly, k's range is $\{4, 5\}$. ■

Note that we have used set notation, not interval notation, since the answers here were lists of x-values and not intervals. Also note that we could graph the information that we have regarding k, as in Figure 1.11, and the visualization of 'sliding ink' to determine domain and range still works. ■

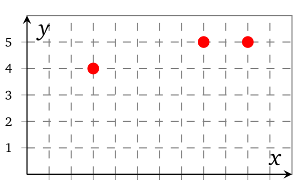


Figure 1.11: $y = k(x)$

Exercises

Problem 4 (Domain of radical functions)
Find the domain of each of the functions associated with the following formulas.

4.1 $f(x) = \sqrt{x}$

4.2 $g(x) = \sqrt{x + 10}$

4.3 $h(x) = \sqrt[3]{5x}$

4.4 $j(x) = \sqrt[4]{5x + 2}$

4.5 $k(x) = \sqrt[7]{3 - x}$

4.6 $l(x) = \sqrt[6]{2 - x}$

4.7 $m(x) = 4 - \sqrt[9]{x^2}$

4.8 $n(x) = 2 - \sqrt[8]{x^2 + 1}$

1.3 Increasing, decreasing, concave up/down

Many functions that are worth studying use time as the input variable. This is quite convenient when you have a graph of those functions. As we read the graph and follow the curve left-to-right, we can imagine time moving forward. From this perspective, in Figure 1.12 we see a function whose outputs grow as time passes, while in Figure 1.13 the outputs decrease. We would like to take this understanding and declare that some functions are *increasing* functions, while others are *decreasing* functions.

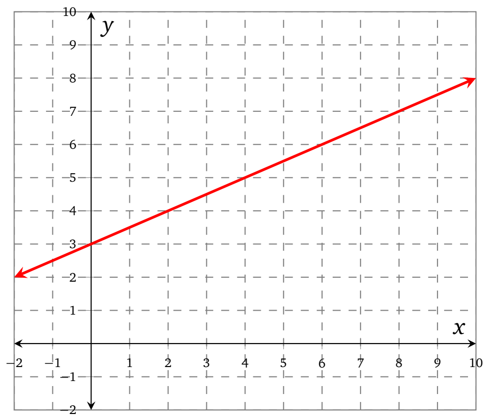


Figure 1.12: $y = f(x)$

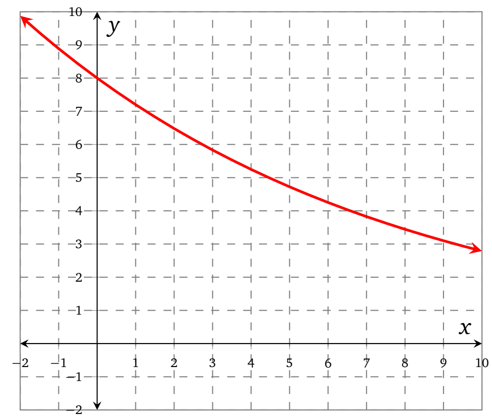


Figure 1.13: $y = g(x)$

Given a graph of some function f, it is usually apparent whether or not we would want to call the function increasing, decreasing, or neither. But if we left it that, then these new vocabulary terms wouldn't be very helpful. We would not be able to use them to prove anything of consequence, because their definitions would be vague. For this reason more formal definitions have been developed.

Definition 4 (Increasing and Decreasing)
A function f is an *increasing* function if whenever $b > a$ with both b and a in f's domain, the nature of f implies that $f(b) > f(a)$.

Similarly, f is a *decreasing* function if whenever $b > a$ with both b and a in f's domain, the nature of f implies that $f(b) < f(a)$.

This definition is consistent with our graphical intuition for what "increasing" and "decreasing" should mean. For instance in Figure 1.13, you can choose any two numbers that you like on the input-axis and label the larger number b and the smaller one a. Once you do this, see that $g(b) < g(a)$. This confirmation of cause ($b > a$) and effect ($g(b) < g(a)$) on your part makes g meet the definition of "decreasing" in Definition 4. But the defintion can be used in other nongraphical ways as the Examples 1 and 2 show.

Example 1 Suppose we have a function h given by $h(x) = 3x+1$. Is h an increasing function, a decreasing function, or neither? It is important to learn to answer a question like this according to the formal definition that we have introduced.

Solution Suppose that b and a are two numbers with $b > a$. Both of these numbers are in h's domain, since h's domain is $(-\infty, \infty)$. We must decide if the nature of h guarantees that $f(b) > f(a)$, that $f(b) < f(a)$, or guarantess no such thing.

Well,

$$\begin{aligned} & b > a \\ \implies \quad & 3b > 3a \\ \implies \quad & 3b+1 > 3a+1 \\ \implies \quad & h(b) > h(a) \end{aligned}$$

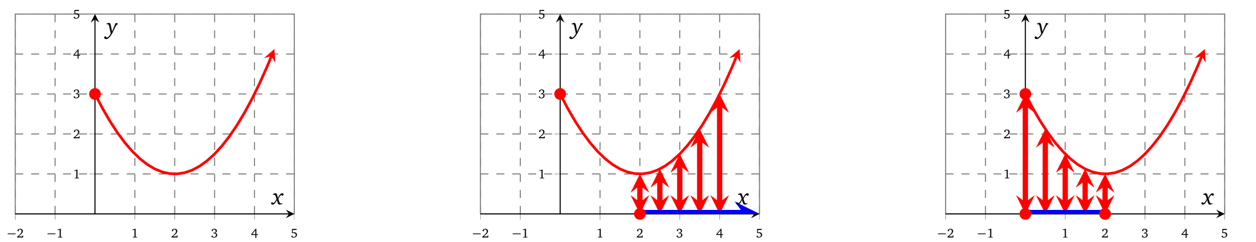
So we have confirmed that h is increasing. ■

Table 1.14: k

x	$k(x)$
2	3
3	8
4	7

Example 2 Let k be the function given in Table 1.14. We only know outputs of k for inputs in $\{2,3,4\}$. We can see inputs $3 > 2$ with outputs $k(3) > k(2)$. This is evidence that k might be increasing. But then se see inputs $4 > 3$ and $k(4) < k(3)$, which is evidence that k might be decreasing. So the only conclusion we can make is that k is neither increasing nor decreasing. ■

It's not satisfying to look at the graph of function like f in Figure 1.14a and simply state that it is neither increasing nor decreasing. Part of that graph shows increasing behavior, and part of it shows decreasing behavior. We'd like to be able to specify this. Since f seems to be decreasing above the interval $[0,2]$ and increasing above the interval $[2,\infty)$, we are motivated to introduce more definitions.



(a) $y = f(x)$ (b) f is increasing on... (c) f is decreasing on...

Figure 1.14: Incresing and decreasing behavior

Definition 5 (Increasing and Decreasing on an Interval)
Given a function f, we say that f is *increasing on the interval* I if whenever $b > a$ with both b and a in I, the nature of f implies that $f(b) > f(a)$. Here, I could be an interval of any form: (p,q), $(p,q]$, etc.

Similarly, f is *decreasing on the interval* I if whenever $b > a$ with both b and a in I, the nature of f implies that $f(b) < f(a)$.

★ **try it yourself** ★

Problem 1
Find all intervals on which the function f in Figure 1.15 is increasing.

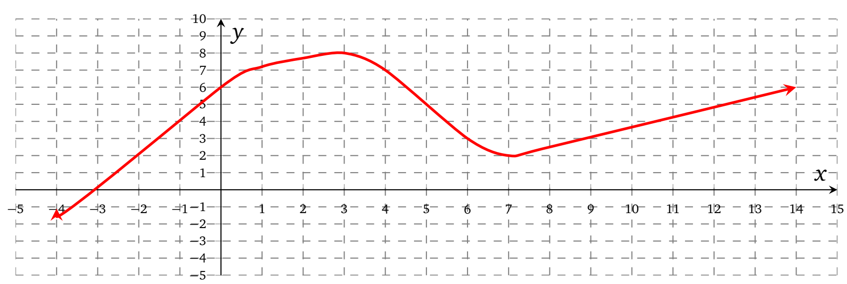


Figure 1.15: $y = f(x)$

make sure you try it!

1.3.1 Concavity

Table 1.15: $f(x) = 3x + 2$

x	$f(x)$
1	5
2	8
3	11
4	14
5	17

In earlier math classes you spent a lot of time talking about linear functions. One defining trait of a linear function is that its rate of change is constant; we call this constant rate of change the slope of the function. For example, the slope of the function f, where $f(x) = 3x + 2$, is three. This tells us (among other things) that every time the value of x increases by 1, the value of $f(x)$ increases by 3. This is reflected in the values shown in in Table 1.15.

On the other hand, the function g defined by $g(x) = x^2$ does not change at a constant rate. If we look at how the function behaves over the positive integers (see Table 1.16), we clearly see that as the value of x continually increases by 1, the value of the function increases at a faster and faster rate; another way to express this is to say that g is *concave up*. This vocabulary is motivated by the graph of g, which has a concavity above it.

Table 1.16: $g(x) = x^2$

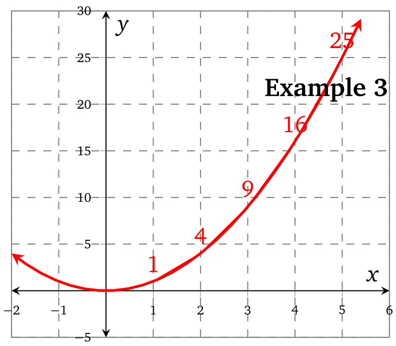
x	$g(x)$
1	1
2	4
3	9
4	16
5	25

Definition 6 (Concavity)
A function f is *concave up* on an interval I if every way of taking two numbers a and b from I, locating $(a, f(a))$ and $(b, f(b))$ on the graph of f, and connecting them with a straight line segment yields a line segment that is above the graph of f, touching the graph of f only at the segment's endpoints.

Here, I may be an interval of any type: (p, q), $(p, q]$, etc.

A function f is *concave down* on an interval I if such line segments are below the graph of f.

Figure 1.16: $y = x^2$



Example 3 Figures 1.17 and 1.18 demonstrate some functions and their concavities.

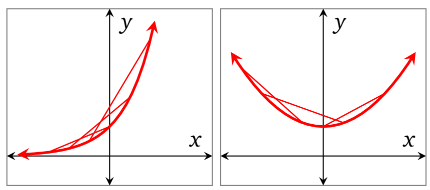


Figure 1.17: Concave up on (∞, ∞)

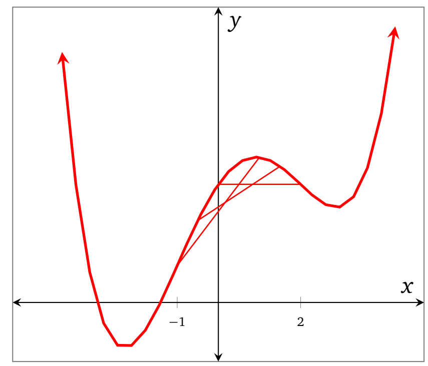


Figure 1.18: Concave down on $[-1, 2]$

Example 4 Graph each of the functions defined by the following formulas on the interval $(-5, 5)$, using either a table of values or technology; state if each function is concave up or concave down on $(-\infty, \infty)$.

$$f(x) = \frac{1}{4}(x-2)^4 \qquad g(x) = 3 + 2x - x^2 \qquad k(x) = 5 - 2^x$$

Solution We graph the functions f, g, and h in Figures 1.19–1.21. We observe that

- f is concave up on $(-\infty, \infty)$;
- g is concave down on $(-\infty, \infty)$;
- k is concave down on $(-\infty, \infty)$. ■

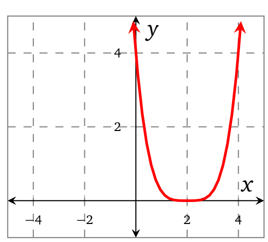


Figure 1.19: f

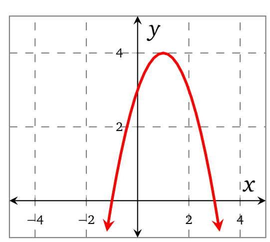


Figure 1.20: g

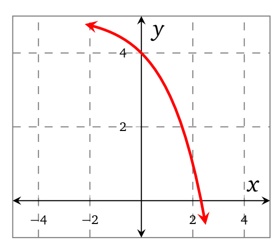


Figure 1.21: k

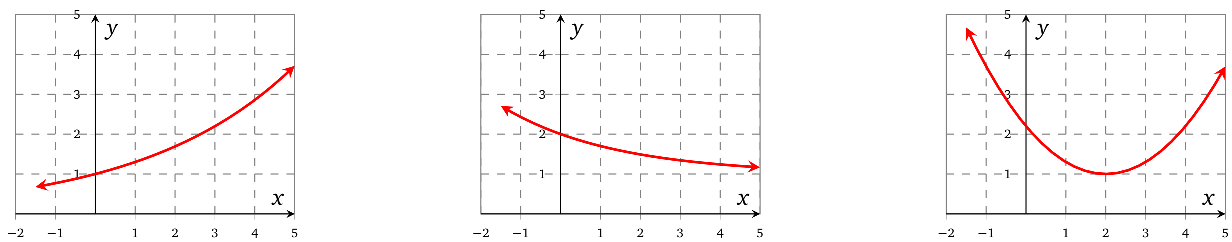
★ try it yourself ★

Problem 2
Graph each of the functions defined by the following formulas on the interval $(-5, 5)$, using either a table of values or technology; state if each function is concave up or concave down on $(-\infty, \infty)$.

$$f(x) = -x^2 \qquad g(x) = 0.1x^4 + 0.5x^2 - 3x + 2 \qquad h(x) = x^3$$

make sure you try it!

We introduced this section on concavity by discussing the rate of change of the functions in Tables 1.15 and 1.16, but then defined concavity as a geometric property of a function's graph. There is a connection. Let's take a look at the functions in Figures 1.22a–1.22c which are concave up on $(-\infty, \infty)$, reading each graph left-to-right.



(a) Concave up and increasing (b) Concave up and decreasing (c) Concave up

Figure 1.22: Concave up functions

In Figure 1.22a, the graph is increasing everywhere. At the beginning it is increasing slowly, and as we move to the right, it increases with a higher and higher rate of change.

In Figure 1.22b, the graph is decreasing everywhere. At the beginning it is decreasing quickly, and as we move to the right, it decreases with a smaller and smaller rate of change.

In Figure 1.22c, the graph is decreasing at first, and then increases. At the beginning it is decreasing quickly, and as we move toward the low point, it decreases with a smaller and smaller rate of change. Beyond the low point, it is increasing slowly at first, and then increases with a larger and larger rate of change.

These three situations capture what it could mean for a funciton to be concave up.

Concavity and rates of change
A function f is concave up on an interval I if any of the following statements is true (see Figures 1.22a–1.22c):

- f increases at a faster and faster rate;
- f decreases at a slower and slower rate;
- f transitions from decreasing at a slower and slower rate to increasing at a faster and faster rate.

Here, I may be an interval of any type: (p, q), $(p, q]$, etc.

A function f is concave down on an interval I if any of the following statements is true:

- f decreases at a faster and faster rate;
- f increases at a slower and slower rate;
- f transitions from increasing at a slower and slower rate to decreasing at a faster and faster rate.

If you are comfortable with negative numbers, then there is an even simpler way to summarize this. In **??**, the rate of change begins small and positive and gradually becomes larger. In

??, the rate of change begins large and negative and gradually becomes a smaller negative number; that is the rate of change becomes larger on a number line. In **??**, the rate of change begins negative and gradually moves higher on a number line until it is positive.

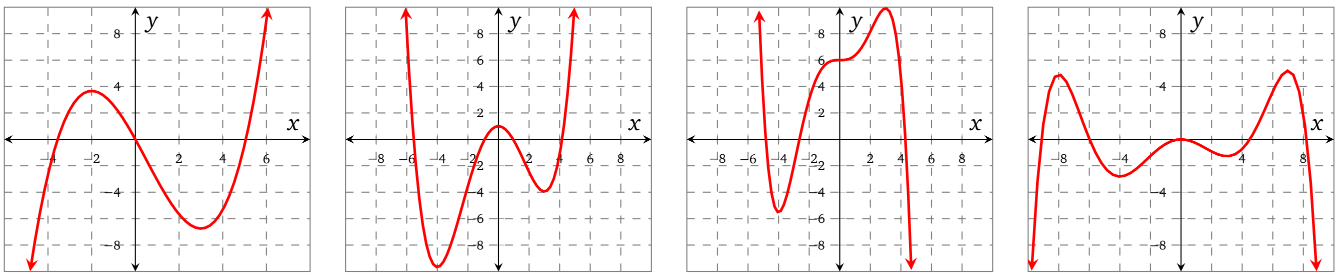
Concavity and rates of change again
A function f is concave up on an interval I if the rate of change becomes larger and larger in a number-line sense.

Here, I may be an interval of any type: (p, q), $(p, q]$, etc.

A function f is concave up on an interval I if the rate of change becomes smaller and smaller in a number-line sense.

Exercises

Problem 3 (Intervals of increase, decrease and concavity)
Figure 1.23 shows the graphs of four functions p, q, r, and s.



(a) $y = p(x)$ (b) $y = q(x)$ (c) $y = r(x)$ (d) $y = s(x)$

Figure 1.23: Graphs for Problem 3.

3.1 Approximate the zeros of each function.

3.2 Approximate the local maximums and minimums of each of the functions.

3.3 Approximate the global maximums and minimums of each of the functions.

3.4 Approximate the intervals on which each function is increasing and decreasing.

3.5 Approximate the intervals on which each function is concave up and concave down.

Problem 4 (Given properties, sketch a function)
In each of the following problems, sketch a function that has the given properties.

4.1 Increasing and concave up.

4.2 Decreasing and concave up.

4.3 Decreasing and concave down.

4.4 Increasing and concave down.

Problem 5 (Increasing or decreasing from a table)
Tables 1.17a–1.17d show values of functions f, g, h, and j. Decide if each function is increasing or decreasing.

Problem 6 (Determine concavity from a table)
Decide if each of the functions depicted in Tables 1.17a–1.17d are concave up or concave down. You might like to plot the values from each table to help you visualize the concavity.

Problem 7 (Counter examples)
The following statements are *all false*. Provide the formula for a function that contradicts each statement– note that there are many different functions that will work and as we progress through the later chapters, you will be able to provide more interesting examples.

7.1 All increasing functions are concave up.

7.2 All increasing functions are concave down.

Table 1.17: Tables for Problem 5

(a) $y = f(x)$

x	y
0	0
1	-1
2	-4
3	-9
4	-16
5	-25
6	-36
7	-49
8	-64

(b) $y = g(x)$

x	y
-10	π
-6	π
-2	π
2	π
6	π
10	π
14	π
18	π
22	π

(c) $y = h(x)$

x	y
-4	-7
-3	-5
-2	-3
-1	-1
0	1
1	3
2	5
3	7
4	9

(d) $y = j(x)$

x	y
-4	$1/16$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8
4	16

7.3 All decreasing functions are concave up.

7.4 All decreasing functions are concave down.

7.5 Quadratic functions always increase.

7.6 Quadratic functions always decrease.

1.4 Simplification Issues

Throughout your algebra course (and beyond), you are going to encounter the algebra simplification steps that we are about to discuss. Algebra simplification is a skill - like cooking noodles or painting a wall. It's not always the most exciting thing in the world, but it does serve a greater purpose. Also like cooking noodles or painting a wall, it's not usually difficult. And yet there are common avoidable mistakes that people make. With practice from this section, you'll have experience to prevent yourself from overcooking the noodles or ruining your paintbrush.

Let's start by reminding ourselves one more time what the meaning of our function notation is. When we write $f(x)$, we have a process f that is working its business on an input value x. Whatever is inside those parentheses is the input to the function.

Example 1 If $f(x) = x^2 + 3x - 4$, find and simplify a formula for $f(-x)$.

Solution Those parentheses encase "$-x$", so we are meant to treat "$-x$" as the input. The rule that we have been given for f is

$$f(x) = x^2 + 3x - 4$$

But the x's that are in this formula are just place holders. What f does to a number can just as well be communicated with

$$f(\quad) = (\quad)^2 + 3(\quad) - 4$$

So now that we are meant to treat "$-x$" as the input, we will insert "$-x$" into those slots, after which we can do more familiar algebraic simplification:

$$\begin{aligned} f(-x) &= (-x)^2 + 3(-x) - 4 \\ &= x^2 - 3x - 4 \end{aligned}$$

■

Example 2 If $f(x) = 2x^2 + 8$, find and simplify a formula for $f(3x)$.

Solution Those parentheses encase "$3x$", so we are meant to treat "$3x$" as the input.

$$\begin{aligned} f(\quad) &= 2(\quad)^2 + 8 \\ f(3x) &= 2(3x)^2 + 8 \\ &= 2(9x^2) + 8 \\ &= 18x^2 + 8 \end{aligned}$$

■

Example 3 If $f(x) = x^2 - 3x$, find and simplify a formula for $f(x - 4)$.

Solution This kind of example is often challenging for college algebra students. But let's focus on those parentheses one more time. They encase "$x - 4$", so we are meant to treat "$x - 4$" as the input.

$$\begin{aligned} f(\quad) &= (\quad)^2 - 3(\quad) \\ f(x-4) &= (x-4)^2 - 3(x-4) \\ &= x^2 - 8x + 16 - 3x + 12 \\ &= x^2 - 11x + 28 \end{aligned}$$

■

The tasks that are shown in Examples 1–3 are the kind of task that will make it easier to understand interesting and useful material in later chapters and sections, particularly in Section 1.6. This skill is also essential for getting off the ground in a calculus course, which might be in your future.

Example 4 Consider the function f given by $f(x) = \frac{1}{4}x^2 + x + 2$, graphed in Figure 1.24. Let's introduce a value on the x-axis and call that value a. Figure 1.25 illustrates a possible location for

a, but we do not wish to specify any particular number for a. Directly above a, we have the point $(a, f(a))$ on the graph of f. Let's imagine stepping 3 units forward on the x-axis[4]. What x-value would we now be at?

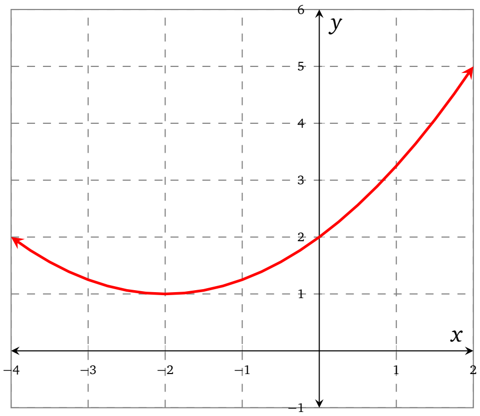


Figure 1.24: figure
$y = f(x)$

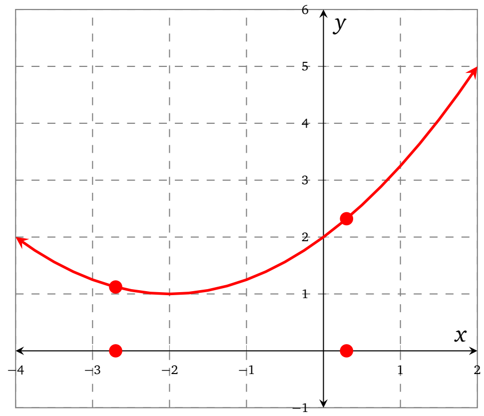


Figure 1.25: figure
$y = f(x)$

FIX

After stepping 3 units forward, the new x-value would be $a+3$. Since points on the graph are all of the form (input, output), then above this on the graph is the point $(a+3, f(a+3))$. This is also marked in Figure 1.25.

We are interested in the slope of the line that connects these two points on f's graph. Our task is to find and simplify an expression for that slope.

Solution As we recall, the slope of a line can be computed by measuring the rise and run between two points on that line, and taking their ratio. We see a run of 3 between these two points. What is the rise between them? The right point has y-value $f(a+3)$ and the left point has y-value $f(a)$. So the rise is $f(a+3) - f(a)$. And that means the slope of the line is given by

$$\frac{f(a+3) - f(a)}{3}$$

To simplify this, we will again pay careful attention to the meaning of those parentheses. In general,

$$f(\ \ ) = \frac{1}{4}(\ \ )^2 + (\ \ ) + 2$$

So

$$\begin{aligned}
\frac{f(a+3) - f(a)}{3} &= \frac{\frac{1}{4}(a+3)^2 + (a+3) + 2 - \left(\frac{1}{4}(a)^2 + (a) + 2\right)}{3} \\
&= \frac{\frac{1}{4}(a^2 + 6a + 9) + a + 5 - \frac{1}{4}a^2 - a - 2}{3} \\
&= \frac{\frac{1}{4}a^2 + \frac{3}{2}a + \frac{9}{4} + a + 5 - \frac{1}{4}a^2 - a - 2}{3} \\
&= \frac{\frac{3}{2}a + \frac{21}{4}}{3} \\
&= \frac{1}{2}a + \frac{7}{4}
\end{aligned}$$

To clarify what we just computed: wherever we place the x-value a, the slope of the segment that connects the graph to a point 3 units further to the right will always be $\frac{1}{2}a + \frac{7}{4}$. ■

[4] There is nothing special about 3; we are just choosing a number for the example.

1.5 Composition

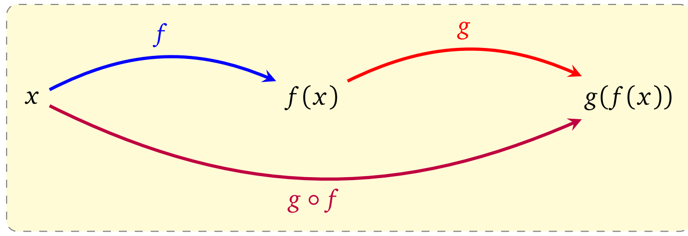


Figure 1.26

Example 1 – Coupons: Jon and Kevin are shopping for a pair of jeans; they have a coupon for \$5 off a pair of jeans. When they arrive at the store, they find that all jeans have an extra 25 % marked off the price. They find a pair of jeans for \$55; they would like to save as much money as possible, but can't agree on a good stratedgy:

(a) Jon wants to use the coupon first, and then apply the 25 % off.

(b) Kevin wants to apply the 25 % off first, and then use the coupon.

Help them resolve their dispute.

Solution (a) If they use the coupon first, then the total cost of the jeans is calculated using

$$(55 - 5) \cdot (0.75) = 37.5$$

The jeans cost \$37.50 using Jon's method.

(b) If they deduct 25 % first, then the cost of the jeans is calculated using

$$(0.75) \cdot 55 - 5 = 36.25$$

The jeans cost \$36.25 using Kevin's method.

If Jon and Kevin want to minimize cost, they should take 25 % off the price first, and then use the coupon. ■

Example 2 – Coupons continued: Jon and Kevin (from Example 1) are still thinking about jeans, and decide to try and generalize their findings to jeans that cost x dollars.

They let f be the function that represents the cost of the jeans after using the \$5 coupon, and g be the function that represents the cost of the jeas after applying the 25 % discount. They write the following formulas for $f(x)$ and $g(x)$

$$\begin{aligned} f(x) &= x - 5 \\ g(x) &= 0.75x \end{aligned}$$

Jon suggests using a function r to represent the cost of the jeans when using the coupon first and then applying the 25 % off; Kevin suggests using a function s to represent the cost of the jeans when applying the 25 % first, and then the coupon. They find the following formulas for $r(x)$ and $s(x)$

$$\begin{aligned} r(x) &= (g \circ f)(x) \\ &= g(x - 5) \\ &= 0.75(x - 5) \end{aligned} \qquad \begin{aligned} s(x) &= (f \circ g)(x) \\ &= f(0.75x) \\ &= 0.75x - 5 \end{aligned}$$

They decide to test their formulas by evaluating $r(55)$ and $s(55)$ as follows

$$\begin{aligned} r(55) &= 0.75(55 - 5) \\ &= 37.5 \end{aligned} \qquad \begin{aligned} s(55) &= 0.75(55) - 5 \\ &= 36.25 \end{aligned}$$

Both calculations agree with what they found in Example 1, as expected. ■

Example 3 – Composition: Let f and g be the functions that have formulas

$$f(x) = x^2, \qquad g(x) = 2x + 1$$

We can *compose* the functions to form new functions $f \circ g$ and $g \circ f$ as follows

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= (2x+1)^2\end{aligned} \qquad\qquad \begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= 2x^2 + 1\end{aligned}$$

■

Example 4 Isla is considering the functions f and g that have formulas

$$f(x) = \sqrt{x-3}, \qquad g(x) = \frac{3}{x}$$

Help Isla find the domain of the following functions:

(a) $g \circ f$

(b) $f \circ g$

Solution (a) The function $g \circ f$ has formula

$$(g \circ f)(x) = \frac{3}{\sqrt{x-3}}$$

The domain of $g \circ f$ is $(3, \infty)$.

(b) Isla composes f and g to form the function $f \circ g$ that has formula

$$(f \circ g)(x) = \sqrt{\frac{3}{x} - 3}$$

and says that the domain of $f \circ g$ is $(-\infty, 1)$ because

$$\begin{aligned}\frac{3}{x} - 3 \geq 0 \Rightarrow \frac{3}{x} &\geq 3 \\ \Rightarrow 1 &\geq x\end{aligned}$$

Oscar stops by and says that since 0 is supposedly in the domain of $f \circ g$, he should be able to compute $(f \circ g)(0)$, but immediately runs into trouble since $g(0)$ is undefined.

Oscar retraces Isla's steps and remembers that

$$\frac{3}{x} \geq 3 \Rightarrow 1 \geq x$$

but *only when* $x > 0$; when $x < 0$ it switches the inequality symbol and implies that $x \geq 1$ which is clearly a contradiction. Oscar visualizes this algebra in Figure 1.27; in particular, the shaded region highlights the interval on which $\frac{3}{x} \geq 3$.

Isla and Oscar therefore conclude that the domain of $f \circ g$ is actually $(0, 1]$. ■

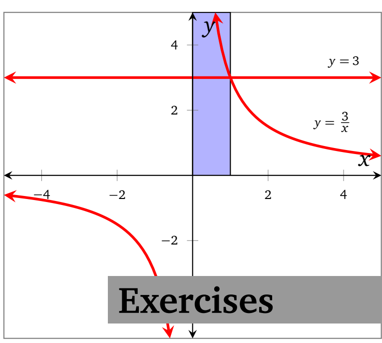


Figure 1.27

Exercises

Problem 1 (Composition using tables)
Tables 1.18a–1.18d show values of the functions F, G, H, and J. Use these tables to find the following values; if the values are defined, state so.

1.1 $(F \circ G)(8)$

1.2 $(G \circ H)(-3)$

1.3 $(H \circ J)(11)$

1.4 $(J \circ F)(7)$

1.5 $(G \circ F)(5)$

1.6 $(H \circ G)(7)$

1.7 $(J \circ H)(4)$

1.8 $(F \circ J)(17)$

Problem 2 (Composition using graphs)
The functions p, q, r, and s are shown in Figure 1.28. Use the graphs to evaluate each of the following– if the value is undefined, then state so.

Table 1.18: Tables for Problem 1

(a) $y = F(x)$

x	y
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8

(b) $y = G(x)$

x	y
0	8
1	7
2	6
3	5
4	4
5	3
6	2
7	1
8	0

(c) $y = H(x)$

x	y
-4	2
-3	3
-2	5
-1	7
0	11
1	13
2	17
3	19
4	23

(d) $y = J(x)$

x	y
2	0
3	1
5	2
7	3
11	4
13	5
17	6
19	7
23	8

2.1 $(q \circ p)(8)$

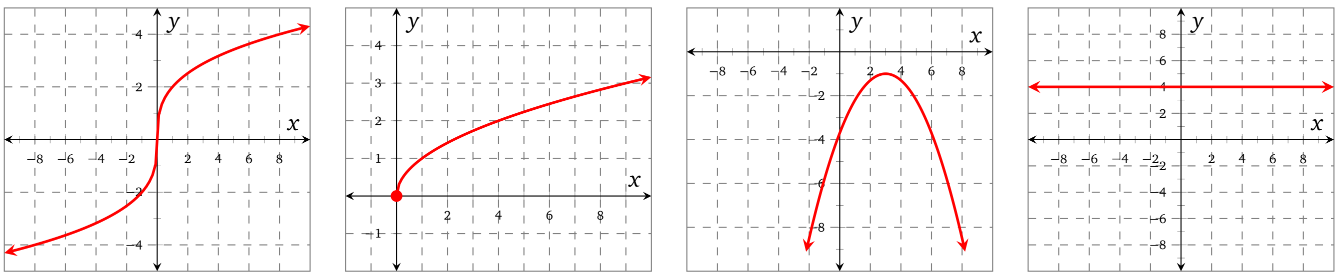
2.2 $(p \circ q)(0)$

2.3 $(q \circ r)(0)$

2.4 $(p \circ r)(-2)$

2.5 $(q \circ r)(6)$

2.6 $(q \circ s)(6)$



(a) $y = p(x)$ (b) $y = q(x)$ (c) $y = r(x)$ (d) $y = s(x)$

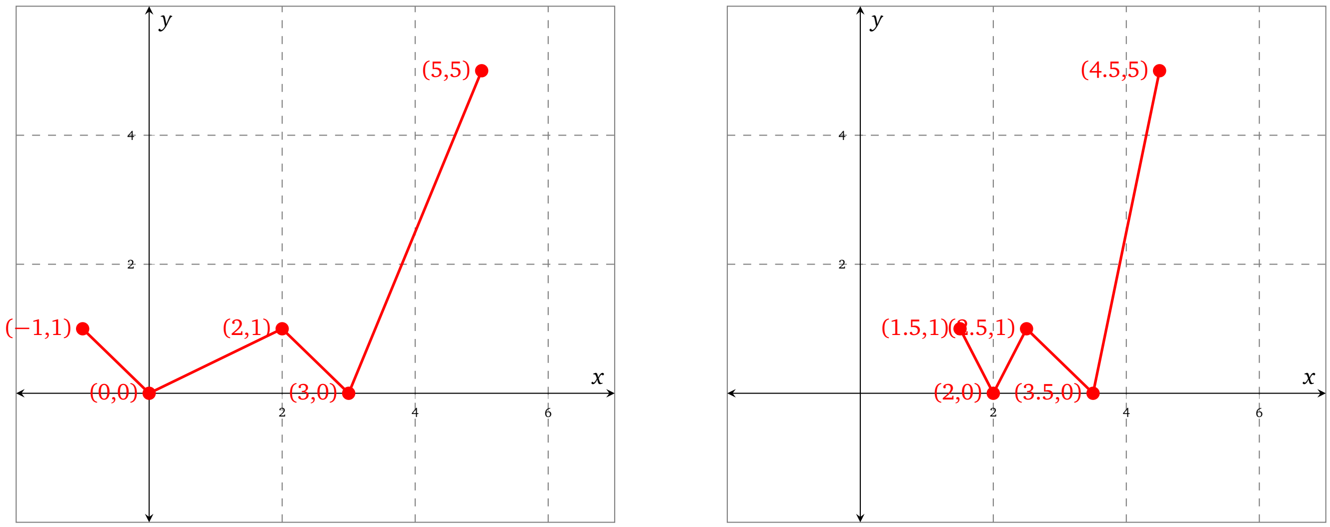
Figure 1.28

1.6 Transformations

Example 1 Figure 1.29a shows a function f; Figure 1.29b shows a function g that is a horizontal transformation of f. The formula for $g(x)$ is

$$g(x) = f(2x - 4) \tag{1.1}$$

Find the sequence of transformations that transforms $f(x)$ into $g(x)$.



(a) $y = f(x)$ (b) $y = g(x)$

Figure 1.29: A function f and a transformation of f

Solution Let's consider how we can get the new x-values, x_n, for $g(x)$ in terms of the old x-values, x_o, from $f(x)$.

We set the argument of $g(x)$ equal to an *old* x-value, and then solve for the *new* value

$$\begin{aligned} 2x_n - 4 &= x_0 && \text{original function} \\ 2x_n &= x_0 + 4 && \text{shift to the } \textit{right} \text{ by 4 units} \\ x_n &= \frac{x_0 + 4}{2} && \text{compress by a factor of } \frac{1}{2} \text{ towards the } y\text{-axis} \end{aligned}$$

You might prefer to look at this using ordered pairs; Table 1.19 shows how the steps above can be applied to the ordered pairs of the original function f depicted in Figure 1.29a.

Table 1.19: Transforming $f(x)$ into $g(x)$ numerically

(a) Shift *right* 4

$f(x)$	New
$(-1,1)$	$(-1+4,1) = (3,1)$
$(0,0)$	$(0+4,0) = (4,0)$
$(2,1)$	$(6,1)$
$(3,0)$	$(7,0)$
$(5,5)$	$(9,5)$

(b) Compress by factor of $\frac{1}{2}$

Result from Table 1.19a	$g(x)$
$(3,1)$	$(3 \div 2, 1) = (1.5, 2)$
$(4,0)$	$(4 \div 2, 0) = (2, 0)$
$(6,1)$	$(3,1)$
$(7,0)$	$(3.5,0)$
$(9,5)$	$(4.5,0)$

The original formula for $g(x)$ in Equation (1.1) can be expressed in a slightly different way

$$\begin{aligned} g(x) &= f(2x - 4) \\ &= f(2(x - 2)) \end{aligned}$$

It seems that this formula for $g(x)$ will lead us to make different transformations from $f(x)$ to $g(x)$; let's see if we can replicate our previous result.

We set the argument of $g(x)$ equal to an *old* x-value, and then solve for the *new* value

$$\begin{aligned}
2(x_n - 2) &= x_o && \text{the original function}\\
x_n - 2 &= \frac{x_0}{2} && \text{compress by a factor of } \frac{1}{2} \text{ towards the } y\text{-axis}\\
x_n &= \frac{x_0}{2} + 2 && \text{shift to the } \textit{right} \text{ by 2 units}
\end{aligned}$$

If you prefer an approach using ordered pairs, then study Table 1.20.

Table 1.20: Transforming $f(x)$ into $g(x)$ (alternative approach)

(a) Compress by a factor of $\frac{1}{2}$

$f(x)$	New
$(-1,1)$	$(-1\div 2,1)=(-0.5,1)$
$(0,0)$	$(0\div 0,0)=(0,0)$
$(2,1)$	$(1,1)$
$(3,0)$	$(1.5,0)$
$(5,5)$	$(2.5,5)$

(b) Shift right by 2 units

Result from Table 1.20a	$g(x)$
$(-0.5,1)$	$(-0.5+2,0)=(1.5,0)$
$(0,0)$	$(0+2,0)=(2,0)$
$(1,1)$	$(3,1)$
$(1.5,0)$	$(3.5,0)$
$(2.5,5)$	$(4.5,5)$

Notice that in the end, both sets of transformations yeild the same formula for $g(x)$. ■

EXPONENTIAL FUNCTIONS 2

Contents

2.1 Introduction

Section Themes, Concepts, Issues, Competencies, and Skills:

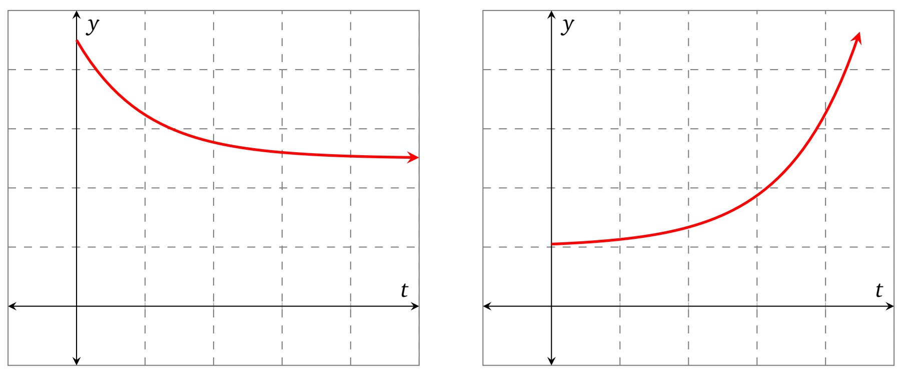
- *Explore increasing and decreasing functions, particularly in the context of concavity;*
- *Determine a function's concavity based on a table of values, a graph, or a description.*

Core problems in this section (★): 2.1, 3.1, 3.3

In our mathematical adventures so far we have studied linear, quadratic, and radical functions. The simplicity of these functions is useful when introducing new concepts such as transformations, composition, and inverse functions; but it is somewhat restrictive when we wish to consider interesting real-world application problems.

For example, let's say that we were interested in modeling the temperature of a hot cup of coffee since it was first poured. We cannot write a formula for such a model yet, but perhaps Figure 2.1a depicts a reasonable approximation of the graph of it. Or perhaps we would like to model the growth in population of the world; again, we can not write a formula for such a model at this stage, but you might agree that Figure 2.1b is a likely candidate for the graph of the model.

Clearly the functions depicted in Figure 2.1 belong to a different class than those that we have considered so far. In fact, they belong to the class known as *exponential* functions, the study of which is a fascinating topic that encompasses many applications, and a lot of interesting mathematical features. Prepare yourself for a colorful and exciting journey that will take us through the landscape of some of the most useful functions that we will every encounter.



(a) (b)

Figure 2.1

Example 1 Congratulations, you've been offered a job! Human Resources told you that you would start out making 2 ¢ on the first day, and every day you work thereafter your pay will double. Would you take this job?

Table 2.1

d (days worked)	p (cents)
1	2
2	4
3	8
4	16
5	32
10	1024
30	1 073 741 824

Table 2.2

d (days worked)	p (cents)
1	2
2	2^2
3	2^3
4	2^4
5	2^5
10	2^{10}
30	2^{30}

Table 2.1 shows how much money you would make per day, in cents, for the first 5 days, and how much you would make on the 10th and 30th days. The amount you make on day 30 is 1073741824 cents, which is

$$\$10{,}737{,}418.24$$

That's over 10 million dollars in a single day! How did this happen? Can we develop a formula to help us understand the mathematics behind this?

It seems that the dollar amount is multiplied by 2 each day. An alternative way of writing our daily income is shown in Table 2.2. Can we write a formula that calculates the pay, p, in cents, as a function of the number of days worked, d? According to Table 2.2, it appears that the day of the month is in the exponent, so let's write

$$p = 2^d$$

where d is a positive integer. This is our first example of an exponential function – exploring these types of functions is our primary goal in this chapter. ■

Definition 7 (Exponential functions)
An exponential function is a function f that can be described with the formula

$$f(x) = a\,b^x$$

where a is a non-zero real number ($a \in \mathbb{R}, a \neq 0$) and b is a positive number other than 1 ($b > 0$, $b \neq 1$). Notice that the variable is in the exponent and the base is the fixed constant b.

Note that in an exponential term the base is fixed and the variable is in the exponent (e.g. $5 \cdot 4^x$), whereas in a polynomial term the exponent is fixed the base is the variable (e.g $6x^3$).

Example 2 – Rice on a chessboard: Many years ago there lived a Queen who loved to play games; so much so, that she had a jester dedicated to devising interesting games for her. The Queen particularly enjoyed mathematical games.

One day the jester brought her a chessboard (see Figure 2.2) and a bucket filled with rice.

Table 2.3

square on board x	grains of rice $g(x)$
1	3
2	9
3	27
4	81
⋮	⋮
⋮	⋮
20	3 486 784 401
⋮	⋮
⋮	⋮
x	3^x

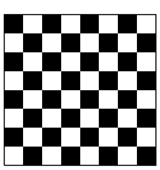


Figure 2.2

The jester asked the Queen to follow these instructions

- Place 3 grains of rice on the square in the lower left hand corner of the board.
- Place 9 grains of rice on the square immediately to the right of the square you were just working with.
- Place 27 grains of rice on the square immediately to the right of the square you were just working with.

The Queen starts to notice a pattern, and records her findings in Table 2.3. She also notes that as she progresses from square to square, the number of grains appears to be *tripling* each time.

The Queen, being mathematically inclined, decides to try to model the game using a formula. She decides to let x be the number corresponding to the square on the chessboard, and let

$g(x)$ represent the number of grains of rice on that square, and assumes that she works each row from left to right as she moves up the chessboard.

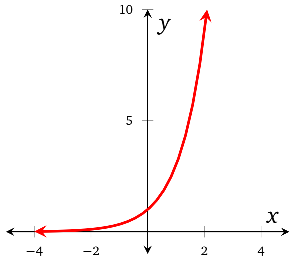


Figure 2.3: g

The Queen notices that each of the numbers she writes in Table 2.3 can be written as a power of 3, and concludes that the formula for the number of grains on square x of the chessboard is

$$g(x) = 3^x$$

The Queen decides to test her formula by calculating the number of grains on the 17^{th} square. She finds that

$$g(17) = 129\,140\,163$$

and says that there are (or would be, if they could fit) 129 140 163 grains of rice on the 17^{th} square – wow!

Of course, the Queen knows that this formula only works when x takes the integer values $\{1, 2, \ldots, 64\}$, but wonders what would happen if she graphed g on her calculator (which she always has with her for just such a situation). The Queen graphs g on her calculator assuming that g can take values outside of the contextual domain, and obtains the graph in Figure 2.3.

The Queen concludes that g is increasing at a faster and faster rate, and cannot imagine ever being able to fit the appropriate amount of rice on each square. ■

★ try it yourself ★

Problem 1

Repeat Example 2, but instead of *tripling* the number of grains of rice on each square, try *quadrupling* them.

make sure you try it!

We have so far seen two exponential functions, p and g, both of which increase at a faster and faster rate. You may wonder if all exponential functions behave in this way – the next example demonstrates that they do not.

Example 3 – Folding paper: Have you ever tried to fold a piece of paper in half more than 7 times? No matter the size of the paper, it becomes quite difficult – the MythBusters tried quite an elaborate experiment along these lines.[1]

We are going to experiment with paper folding and study the mathematics behind the results. The area of a 'letter' sheet of paper is $8.5\,\text{in} \times 11\,\text{in}$, or $93.5\,\text{in}^2$. We will use two decimal places in what follows.

If we fold a sheet of letter paper in half, the visible surface area is

$$\begin{aligned} \frac{93.52\,\text{in}^2}{2} &= \left(\frac{1}{2}\right) 93.5\,\text{in}^2 \\ &= 46.75\,\text{in}^2 \end{aligned}$$

Note that we say, 'visible', because the actual surface area of the paper has not changed.

If we fold the sheet in half again, the visible surface area is

$$\begin{aligned} \frac{46.75\,\text{in}^2}{2} &= \frac{93.52\,\text{in}^2}{4} \\ &= \left(\frac{1}{2}\right)^2 93.5\,\text{in}^2 \\ &\approx 23.38\,\text{in}^2 \end{aligned}$$

If we fold the sheet in half a third time, the visible surface area is

$$\begin{aligned} \frac{93.5\,\text{in}^2}{8} &= \left(\frac{1}{2}\right)^3 93.5\,\text{in}^2 \\ &\approx 11.69\,\text{in}^2 \end{aligned}$$

[1] http://www.youtube.com/watch?v=kRAEBbotuIE

Let's try and generalize our results by letting x be the number of paper folds; x will start at 0, and increase in integer values. Table 2.4 has two columns, one for the number of folds (up to 7), and one for the visible surface area of the (folded) paper.

Table 2.4

number of folds x	visible area (in^2)
0	93.50
1	46.75
2	23.38
3	11.69
4	5.84
5	2.92
6	1.46
7	0.73

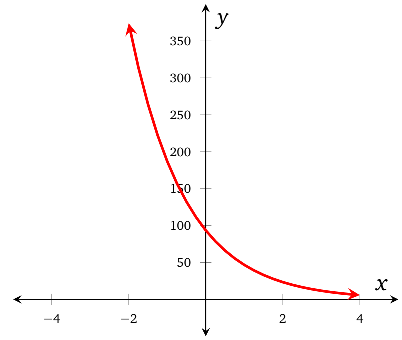


Figure 2.4: $y = A(x)$

If we let $A(x)$ represent the visible surface area of the paper after x folds, then a formula for A is

$$A(x) = 93.5\left(\frac{1}{2}\right)^x$$

where $x = 0, 1, 2, \ldots$. If we allow A to take values outside of its contextual domain, then we can graph $y = A(x)$ on a graphing calculator, and obtain Figure 2.4. Notice in particular that A decreases at a slower and slower rate.

The graphs of exponential functions have certain features that tell us a lot about the quantities they are modeling. Graphical features like increasing/decreasing and the position of any asymptotes translate to important information about population sizes, bank accounts, and more.

Example 4 Consider the function f in Figure 2.5. There are a number of features that we can note:

- f is increasing;
- f is concave up; in particular, f is increasing at a faster and faster rate;
- the line $y = 3$ is a horizontal asymptote of f as $x \to -\infty$;
- $f(x) \to \infty$ as $x \to \infty$;
- the range of f is $(3, \infty)$.

Note that f never touches its horizontal asymptote (see Figure 2.6).

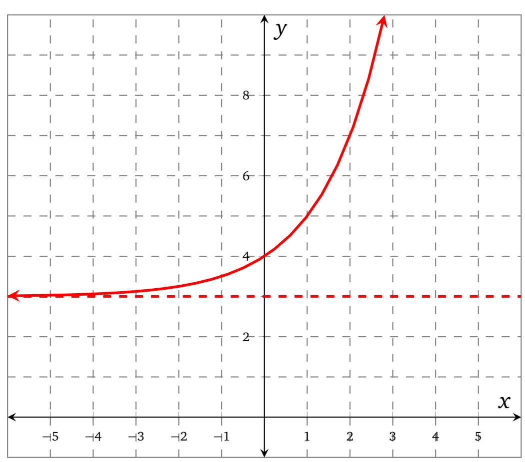


Figure 2.5: $f(x) = 2^x + 3$

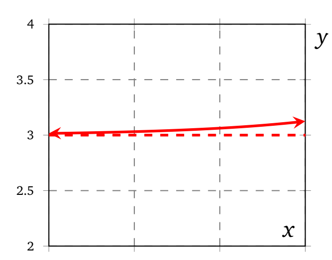


Figure 2.6: Close up!

FIX

★ try it yourself ★

Problem 2
Repeat Example 4 for each of the functions defined by the following formulas.

★ **2.1** $g(x) = 5 - 4^x$

2.2 $h(x) = \left(\frac{2}{3}\right)^x - 4$

make sure you try it!

Exponential modeling will require familiarity with percentages. This example aims to help you (re)acquaint yourself with them.

Example 5 Wild honeybee colonies tend to have around 15 % drones (males). If a colony has 4,280 bees, about how many of them are drones?

Solution The percentage 15 % should be converted to a decimal in order to do arithmetic with it: 0.15.

$$\begin{aligned} \text{drone count} &= 15\,\% \text{ of total bee count} \\ &= 0.15 \cdot 4280 \\ &= 642 \end{aligned}$$

So there are about 642 drones in the colony. ■

Definition 8 (Growth factor and growth rate)
An exponential function f can be written in (at least) two ways.

$$f(t) = a\,b^t \qquad\qquad f(t) = a(1+r)^t$$

- The constant b is called the *growth factor*. When t increases by 1, the value of $f(t)$ changes by a factor of b; that is, every unit of time the value of $f(t)$ is multiplied by b.
- The constant r is called the *growth rate*. We usually write r as a decimal and interpret it as a percent. When t increases by 1, the amount of change in $f(t)$ is r. For example, if $r = 0.10$, when t increases by 1 the value of $f(t)$ increases by 10 %. Whereas if $r = -0.05$, when t increases by 1 the value of $f(t)$ decreases by 5 %.

Example 6 A population is modeled by the formula $P(t) = P_0(1.15)^t$, where t is the amount of time that has passed (in years) since the population was P_0. Find each of the following:

(a) 1-year growth factor and 1-year growth rate
(b) 2-year growth factor and 2-year growth rate
(c) 10-year growth factor and 10-year growth rate

Solution

(a) The 1-year growth factor is what we multiply P_0 by after one year: $(1.15)^1 = 1.15$. So the 1-year growth factor is 1.15 and the 1-year growth rate is 15 %.

(b) The 2-year growth factor is what P_0 would be multiplied by after two years: $(1.15)^2 \approx 1.32$. So the 2-year growth factor is about 1.32 and the 2-year growth rate is about 32 %.

(c) The 10-year growth factor is $(1.15)^{10} \approx 4.05$, and the 10-year growth rate is approximately 305 %. ■

Example 7 The bacterium *Clostridium perfringens* can reproduce every 9 minutes. Suppose that there are initially 50 bacteria in a jar and that they have access to an adequate supply of nutrients. Write a model for this situation and find the following:

(a) 9-minute growth factor and 9-minute growth rate

(b) 1-minute growth factor and 1-minute growth rate

(c) 1-hour growth factor and 1-hour growth rate

Solution Let $P(t)$ be the number of bacteria where t is the amount of time that has passed (in minutes) since the population was 50 bacteria.

Table 2.5 shows values of $P(t)$ for the first 27 hours.

Table 2.5

t	$P(t)$	Exponential form
0	50	$50 \cdot 2^{0/9} = 50 \cdot 2$
9	100	$50 \cdot 2^{9/9} = 50 \cdot 2^1$
18	200	$50 \cdot 2^{18/9} = 50 \cdot 2^2$
27	400	$50 \cdot 2^{27/9} = 50 \cdot 2^3$

We can deduce from Table 2.5 that

$$P(t) = 50 \cdot 2^{t/9}$$

(a) The 9-minute growth factor is $2^{9/9} = 2.00$, and the 9-minute growth rate is 100 % (not surprising since we knew the population doubled in 9 minutes).

(b) The 1-minute growth factor is $2^{1/9} \approx 1.08$, and the 1-minute growth rate is approximately 8 %.

(c) The 1-hr growth factor is $2^{60/9} \approx 101.59$ (or 10 159 %), and the 1-hr growth rate is approximately 10 059 %. ■

Investigations

Problem 3 (The Legend of Payasam)
Example 2 is a version of the Legend of Payasam.

According to legend, Lord Krishna once appeared in the form of a sage in the court of a king who ruled a region of southern India. Lord Krishna challenged the king to a game of chess. The king, being a chess enthusiast, gladly accepted the challenge.

The players decided to put a wager on the game; the king let the sage choose the prize. The sage told the king that he was a man of few material needs, and thus all he wished for was a few grains of rice.

The sage suggested that the amount of rice should be determined using the chessboard in the following manner. Two grains of rice will be placed on the first square, four grains on the second square, eight on the third square, and so on. That is, every square will have double the number of grains as its predecessor.

Upon hearing the demand, the king was unhappy since the sage requested only a few grains of rice instead of other riches from the kingdom.

We are going to attach monetary value to our calculations. We will assume that

- there are approximately 7200 grains of rice in a cup;
- there are 3 cups of rice in a 1-lb bag;
- a 1-lb bag of rice is worth $2.

★ **3.1** Approximate the number of grains of rice that are in a 1-lb bag.

3.2 What is the first square on the chessboard that could be used to fill a 1-lb bag (without using rice from the previous squares)?

★ **3.3** Before we begin our money calculations, write down how much money you would like to get as a prize from the king.

3.4 If you were to exchange the rice on the 16th square for money, how much would you get?

3.5 Using the value you obtained in Problem 3.4, determine the value of the rice on the 17th square.

3.6 Using the value you obtained in Problem 3.4, determine the first square that would give you more than \$1,000,000 worth of rice.

3.7 Using the value you obtained in Problem 3.4, determine the value of the rice put on the last square.

3.8 How does the value of the rice on the last square compare to the amount you wrote down in Problem 3.3?

Problem 4 (Changing Rates of Change)

4.1 The graphs of several increasing functions are given in Figures 2.7–2.9. For each function, decide whether the function increases at a constant rate, increases at an increasing rate (concave up), or increases at a slower and slower rate (concave down).

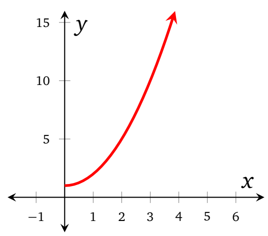


Figure 2.7: $y = m(x)$

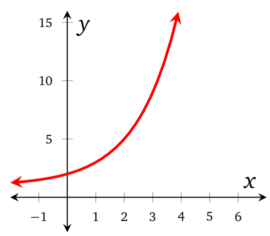


Figure 2.8: $y = n(x)$

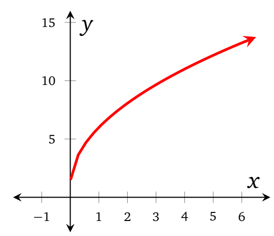


Figure 2.9: $y = o(x)$

4.2 Tables 2.6–2.8 show values for 3 increasing functions. For each function, decide whether the function increases at a constant rate, increases at an increasing rate (concave up), or increases at a slower and slower rate (concave down).

Table 2.6: $y = p(x)$

x	y
1	3
2	8
4	18
8	38
16	78

Table 2.7: $y = q(x)$

x	y
1	6
4	7
9	8
16	9
25	10

Table 2.8: $y = r(x)$

x	y
1	4
2	8
3	16
4	32
5	64

4.3 Several functions are described below. For each function, decide whether the function increases at a constant rate, increases at a faster and faster rate (concave up), or increases at a slower and slower rate (concave down).

(a) The amount in a bank account where \$5000 is initially invested and the money sits and earns interest at a rate of 6 % per year.

(b) The distance your car has traveled t seconds after you slammed on the brakes.

(c) The elevation of a typewriter that is falling, t seconds after it is dropped from a plane flying at an elevation of 30,000 feet.

4.4 For each function below, decide whether the function increases at a constant rate, increases at an increasing rate (concave up), or increases at a slower and slower rate (concave down).

(a) The function f, where $f(x) = 3 + 2\sqrt{x}$

(b) The function g, where $g(x) = 3 + 2x$

(c) The function h, where $h(x) = 3 + 2(4^x)$

Problem 5 (Medication)

A medication is injected into your body. The amount of medication in your body decays exponentially over time. The original dose you receive is 4 cc, and the amount in your body decays at a rate of 8.5 % per hour.

5.1 Let $Q(t)$ be the amount of medication in your body (in cc) at time t in hours since it was injected. Write a formula for $Q(t)$.

5.2 What are the growth rate and growth factor of Q?

5.3 According to your model, does the medication ever go away completely? Why or why not?

Problem 6 (Melting of Arctic Sea Ice)
Using satellite imagery, scientists now believe that the Arctic sea ice cover is being reduced in area by 8% every ten years. In September of 2005 the area of the ice cover was 5.35 million km^2, according to the National Snow and Ice Data Center[2].

6.1 Generate an exponential model for the melting of the Arctic sea ice cover.

6.2 Find the half-life of the ice. In what year will the Arctic sea ice reach half of its 2005 level?

6.3 According to your model, what was the area in 1995?

6.4 Use the two data points you determined in Problems 6.2 and 6.3 to generate a *linear* model for the melting of the Arctic sea ice cover. Comment on the differences in your two models' predictions for 2010, 2030, and 2050.

6.5 When do each of your models (from Problems 6.1 and 6.4) predict the Arctic sea ice cover will melt to less than $100\,\text{km}^2$?

6.6 Graph your exponential model twice. For the first graph, choose a scale which supports the view that these changes are minimal and nothing to worry about. For the second graph, choose a scale which supports the view that these changes are drastic and of great concern. When reading graphs produced by someone else which seem to support a particular opinion, what aspects of the graphs are important to consider?

6.7 If all the ice melts in the summer, does this mean that the Arctic sea ice cover has permanently disappeared? What effect does the disappearing Arctic sea ice cover have on the planet? Are there consequences beyond the Arctic region?

Exercises

Problem 7 (Exponential or not)
Decide if the following formulas correspond to exponential functions or not.

7.1 $f(x) = 5^x$

7.2 $g(x) = x^5$

7.3 $h(x) = 3 \cdot 2^x$

7.4 $k(x) = -3 \cdot 2^x$

7.5 $m(x) = 3x^2$

7.6 $n(x) = \left(\frac{1}{2}\right)^x$

7.7 $p(x) = 4$

7.8 $q(x) = 0$

7.9 $r(x) = \left(\frac{2}{3}\right)^x$

7.10 $s(x) = \left(-\frac{2}{3}\right)^x$

7.11 $t(x) = 5x$

7.12 $u(x) = \pi$

Problem 8 (Identify a and b)
Each of the following formulas define exponential functions, and have the form $f(x) = a\, b^x$. Identify a and b in each case.

8.1 $f(x) = 2 \cdot 3^x$

8.2 $g(x) = -4 \cdot 5^x$

8.3 $h(t) = \left(\frac{2}{3}\right)^t$

8.4 $k(y) = -\left(\frac{2}{3}\right)^y$

8.5 $F(s) = 3^{-s}$

8.6 $G(r) = \frac{2}{3^r}$

8.7 $H(w) = -\frac{4^w}{5}$

8.8 $K(z) = -10 \cdot 5^{-z}$

Problem 9 (Exponential function evaluation)
Evaluate each of the following formulas at -10, -5, 0, 5, and 10. Give the exact answer, and an approximation (where appropriate) using two figures after the decimal.

9.1 $f(x) = 2^x$

9.2 $g(x) = \left(\frac{1}{3}\right)^x$

9.3 $h(x) = -5^x$

9.4 $k(x) = -\left(\frac{2}{5}\right)^x$

[2] http://nsidc.org/

Problem 10 (Features of exponential functions)
Refer to the functions f, g, h, and k defined in Problem 9 throughout this problem.

10.1 Decide if each function is always increasing or always decreasing.

10.2 Decide if each function is concave up or concave down.

10.3 Determine the domain and range of each function.

Problem 11 (Prerequisite percentage skills)
Answer these questions concerning percentages.

FIX

11.1 Wild honeybee colonies tend to have around 15 % drones (males). If a colony has 4,280 bees, about how many of them are drones?

11.2 The human body is made up of approximately 66 % water. If a person weighs 180 lb, how much water do they contain?

11.3 The air we breathe is roughly 20 % oxygen. If you are in a room with dimensions 40 ft × 4 ft × 10 ft, approximately how much oxygen is there in the room?

11.4 You are at a restaurant and receive the check for \$26. You tip 15 %. How much is the total bill?

11.5 You are working in a sales job and manage to secure a client worth \$100,000. You get a 1 % commission, of which your supervisor gets 50 %. How much do you get?

Problem 12 (Growth factor)
For each of the following, identify the growth factor, the growth rate, and the initial value.

12.1 $y = 5 \cdot 3^x$

12.2 $y = 6 \cdot (0.5)^x$

12.3 $y = 2 \cdot (3/4)^x$

12.4 $y = 500 \cdot (2.5)^x$

Problem 13 (Growth factor applications)
Use the following descriptions to determine the growth rate and growth factor as decimals.

13.1 A store has 100 hats and is increasing their stock at a rate of 10 % per day.

13.2 A fire department is losing 6 % of its annual funding with each passing year, where they had \$10,500 in annual funding when the station opened.

Problem 14 (Increasing exponential functions)
Consider the functions f and g that have formulas $f(x) = 2^x$ and $g(x) = 3^x + 4$.

Table 2.9

x	$f(x)$	$g(x)$
-3		
-2		
-1		
0		
1		
2		

Table 2.10

x	$m(x)$	$n(x)$
-3		
-2		
-1		
0		
1		
2		

14.1 Graph the functions f and g over the interval $-3 \le x \le 2$ after first filling in Table 2.9. Label the functions f and g on your graph.

14.2 What are the horizontal asymptotes of f and g?

14.3 Notice that both f and g are increasing as x increases. Which function is increasing at the faster rate?

14.4 Can the value of $f(x)$ be zero? Can it be negative? Why or why not?

14.5 What are the domain and range of the functions f and g?

Problem 15 (Decreasing exponential functions)
Consider the functions m and n that have formulas $m(x) = (1/2)^x$ and $n(x) = (1/3)^x - 2$.

15.1 Graph these functions over the interval $-2 \le x \le 3$ after first filling in Table 2.9. Label the functions m and n.

15.2 What are the horizontal asymptotes of m and n?

15.3 What are the domain and range of the functions m and n?

15.4 Notice that both of these functions are decreasing as x increases. Why do these functions decrease when the functions in Problem 14 increase?

Problem 16 (Horizontal asymptotes when $b > 1$)
Consider the functions f, g, h, and j that have formulas

$$f(x) = 3^x - 1, \qquad g(x) = -4^x - 3, \qquad h(x) = 2^x + 1, \qquad j(x) = -5^x + 2$$

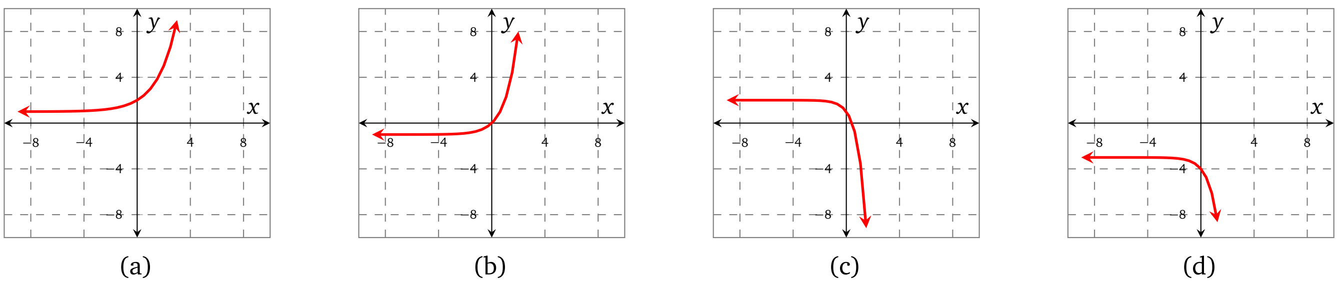


Figure 2.10: Graphs for Problem 16.

16.1 Match each of the functions f, g, h, and j with one of the graphs in Figure 2.10.

16.2 As $x \to -\infty$, $f(x) \to -1$. This tells us two things:

- the horizontal asymptote as $x \to -\infty$ is the line $y = -1$;
- since the function is increasing, the range of the function is $(-1, \infty)$.

For the remaining three functions g, h, and j, deduce the behavior as $x \to -\infty$, the horizontal asymptote as $x \to -\infty$, and the range.

16.3 Using the appropriate function in Figure 2.10, we can observe that $f(x) \to \infty$, as $x \to \infty$. Make analogous statements about the remaining functions in Figure 2.10.

16.4 Recall that an equivalent way of writing that $f(x) \to \infty$ as $x \to \infty$ is to use limit notation:

$$\lim_{x\to\infty} f(x) = \infty$$

Similarly, we can say

$$\lim_{x\to-\infty} f(x) = 1.$$

Use limit notation to re-write your answers from Problems 16.2 and 16.3.

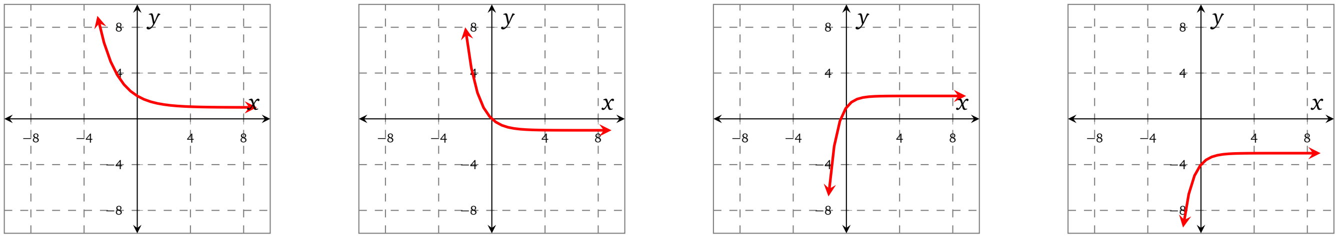
Problem 17 (Horizontal asymptotes when $0 < b < 1$)
Consider the functions F, G, H and J that have formulas

$$F(x) = \left(\frac{1}{3}\right)^x - 1, \qquad G(x) = -\left(\frac{1}{4}\right)^x - 3, \qquad H(x) = -\left(\frac{1}{5}\right)^x + 2, \qquad J(x) = \left(\frac{1}{2}\right)^x + 1$$

17.1 Match each of the functions F, G, H, and J with one of the graphs in Figure 2.11.

17.2 Using the appropriate function in Figure 2.11, we can say that $\left(\frac{1}{2}\right)^x + 1 \to 1$ as $x \to \infty$. Similarly, we can say that $\left(\frac{1}{2}\right)^x + 1 \to \infty$ as $x \to -\infty$. Make analogous statements about the remaining functions in Figure 2.11.

17.3 Use limit notation to re-write your answers from Problem 17.2.



(a) (b) (c) (d)

Figure 2.11: Graphs for Problem 17.

Problem 18 (Increasing and decreasing functions)
Let f be an exponential function that has formula

$$f(x) = a\, b^x$$

where $b > 0$. In each of the following cases, give values of a and b that satisfy the given requirements.

18.1 f is always increasing, and $b > 1$

18.2 f is always increasing, and $0 < b < 1$

18.3 f is always decreasing, and $0 < b < 1$

18.4 f is always decreasing, and $b > 1$

Problem 19 (Successive ratios)
In this activity, we are going to focus on functions that grow at faster and faster rates (concave up). In casual conversation, all such functions are sometimes said to be “growing exponentially”. While that’s just fine while you are muttering about the price of gas, in the sciences the phrase “exponential growth” has a much more precise meaning.

One thing implied by the definition of an exponential function is that if $f(x) = a\, b^x$ (where $b > 0$, $b \neq 1$), then for all values of x, $\dfrac{f(x+1)}{f(x)} = b$.

19.1 Verify that last assertion by simplifying the expression $\frac{ab^{x+1}}{ab^x}$.

19.2 For each of the functions graphed in Figures 2.14–2.13, determine whether or not the function is exponential. If it is, determine the base b of the function. Your first step might be to choose 3 ordered pairs that have successive x-values (e.g 1, 2, 3), and compare the ratio of the y-values.

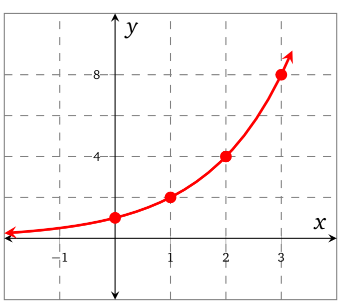


Figure 2.12: $y = n(x)$

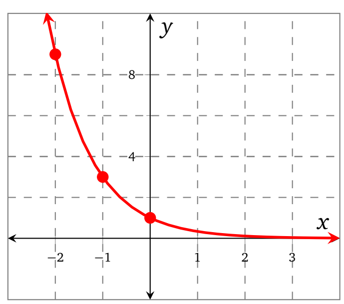


Figure 2.13: $y = r(x)$

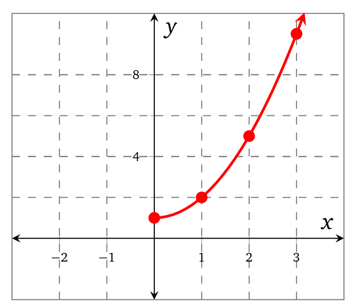


Figure 2.14: $y = m(x)$

19.3 For each of the functions implied by the data in Tables 2.11–2.13, determine whether or not the function is exponential. If it is, determine the base b of the function.

Problem 20 (True or false?)
Consider the function f that has formula

$$f(x) = a\, b^x + c$$

where a and c are any real numbers and b is a positive real number not equal to 1. Label each of the following statements as true (T) or false (F); if you believe a statement is false, provide an example that supports your answer.

20.1 If $b > 1$, then f is an increasing function for any value of a.

Table 2.11: $y = g(x)$

x	y
1	3
2	9
3	27
4	81
5	241

Table 2.12: $y = h(x)$

x	y
1	5
4	25
7	125
10	625
13	3125

Table 2.13: $y = k(x)$

x	y
1	4
2	16
4	64
8	256
16	1024

20.2 If $0 < b < 1$, then f is a decreasing function for any value of a.

20.3 If $b > 1$, then $f(x) \to c$ as $x \to -\infty$.

20.4 If $0 < b < 1$, then $f(x) \to c$ as $x \to \infty$.

20.5 If $b > 1$, then $f(x) \to \infty$ as $x \to \infty$.

20.6 If $0 < b < 1$, then $f(x) \to \infty$ as $x \to -\infty$.

20.7 If $0 < b < 1$, then $\lim\limits_{x \to \infty} f(x) = 0$.

20.8 If $b > 1$, then $\lim\limits_{x \to -\infty} f(x) = \infty$.

20.9 If $b > 1$, then $\lim\limits_{x \to \infty} f(x) = \infty$ or $\lim\limits_{x \to \infty} f(x) = -\infty$.

20.10 If $b > 1$, then the line defined by $y = 0$ is an asymptote of f.

20.11 If $0 < b < 1$, then the line defined by $y = 0$ is an asymptote of f.

20.12 If $0 < b < 1$, then the line defined by $x = 0$ is an asymptote of f.

2.2 Solving exponential equations

As we have seen in the past, we are often interested in finding what value or values of a variable will cause certain conditions to be met. We've solved for the unknown variable x in equations like $2x+4=5$ and $x^2+8x-6=0$. Now we will solve for x in equations where x is in the exponent. Before we begin though, we will refresh ourselves on the rules of exponents.

Properties of exponents

Recall some of the basic rules for exponents, assuming that a, b, x, and y are real numbers:

(E_1) $a^x b^x = (ab)^x$

(E_2) $a^x a^y = a^{x+y}$

(E_3) $a^{(xy)} = (a^x)^y$

We can often make use of these rules to simplify exponential expressions.

essential skills

The following problems contain prerequisite skills that are essential for success. Make sure that you can complete them before moving on!

Problem 1

Evaluate each of the following without the use of a calculator.

1.1 2^2

1.2 -2^3

1.3 -2^4

1.4 $(-3)^2$

1.5 $\left(\frac{2}{3}\right)^2$

1.6 $\frac{4}{9^2}$

1.7 $-\frac{5^2}{6}$

1.8 $-\left(\frac{7}{10}\right)^2$

1.9 $\left(-\frac{7}{10}\right)^2$

1.10 4^0

1.11 -58^0

1.12 $-3\left(-2^{768}\right)^0$

Example 1 Use properties of exponents to write the following formulas in the form $f(t)=a\,b^t$. Leave your answer in exact form.

(a) $f(t)=2\left(3^t\right)\left(3^{3t}\right)$ (b) $g(t)=7(1.08)^t(3.2)^t$ (c) $h(t)=8^t\left(2^{\frac{t+1}{2}}\right)$

Solution We use the properties of exponents, and demonstrate the simplification one step per-line

(a)
$$\begin{aligned} f(t) &= 2\left(3^t\right)\left(3^{3t}\right) \\ &= 2\left(3^{t+3t}\right) && \text{property } (E_2) \\ &= 2\left(3^{4t}\right) \\ &= 2(3^4)^t && \text{property } (E_3) \\ &= 2(81)^t \end{aligned}$$

(b)
$$\begin{aligned} g(t) &= 7(1.08)^t(3.2)^t \\ &= 7(1.08\cdot 3.2)^t && \text{property } (E_1) \\ &= 7(3.456)^t \end{aligned}$$

(c)
$$\begin{aligned}
h(t) &= 8^t \left(2^{\frac{t+1}{2}}\right) \\
&= (2^3)^t \left(2^{\frac{t+1}{2}}\right) \\
&= 2^{3t} \left(2^{\frac{t+1}{2}}\right) \\
&= 2^{3t+\frac{t+1}{2}} && \text{property } (E_2) \\
&= 2^{\frac{7t+1}{2}} \\
&= \left(2^{1/2}\right)^{7t+1} && \text{property } (E_1) \\
&= \left(\sqrt{2}\right)\left(\sqrt{2}\right)^{7t} && \text{property } (E_2) \\
&= \left(\sqrt{2}\right)\left(\left(\sqrt{2}\right)^{7}\right)^{t} && \text{property } (E_1) \\
&= \left(\sqrt{2}\right)\left(\sqrt{128}\right)^{t}
\end{aligned}$$

■

★ **try it yourself** ★

Problem 2
Use properties of exponents to write the following formulas in the form $f(t) = a\,b^t$. Leave your answer in exact form.

$$f(t) = 2^{t+1} \cdot 2^{3t}$$

make sure you try it!

Example 2 Use your knowledge of exponents to solve the following equations:

(a) $3^x = 27$ (b) $2^{x^2} = 16$

Solution (a) We know that $3^3 = 27$, so $x = 3$.

(b) We know that $2^4 = 16$, which means that

$$x^2 = 4$$

The two solutions are 2 and -2. It is left an exercise to check that both of these solutions satisfy the equation $2^{x^2} = 16$. ■

★ **try it yourself** ★

Problem 3
Solve the following exponential equation. Check your solution.

$$2^x = 4$$

make sure you try it!

Example 3 Solve the following equation by factoring. Hint: put $t = 3^x$ as your first step.

$$3^{2x} - 10 \cdot 3^x + 9 = 0 \tag{2.1}$$

Solution Following the hint, we substitute $t = 3^x$ into Equation (2.1), which gives

$$t^2 - 10t + 9 = 0$$

This equation can be factored and written as $(t-9)(t-1) = 0$. We therefore have to solve the equations

$$3^x = 9, \qquad 3^x = 1$$

Using our knowledge of exponents, we see that $x = 2$ or $x = 0$. It is left an exercise to check that both of these solutions satisfy Equation (2.1). ■

★ **try it yourself** ★

Problem 4
Solve the following equation. Try putting $t = 4^x$ as your first step. Check your solutions.

$$4^{2x} - 5 \cdot 4^x - 4 = 0$$

make sure you try it!

Not all solutions are integers; in fact in applied problems solutions are almost never integers. We don't yet have the tools to find exact solutions to most exponential equations, so we are going to explore techniques for approximating solutions to equations.

Example 4 Use a table of values or a graph to solve the following exponential equation

$$2^x = 13$$

Solution Table 2.14 shows the solution is on the interval $[3, 4]$; given that 13 is closer to 16 than it is to 8, we expect our solution to be closer to 4 than it is to 3.

Figure 2.15 shows $y = 2^x$ and $y = 13$. The solution to the equation $2^x = 13$ is the x-coordinate of the point of intersection; using a calculator, we find the x-coordinate is approximately 3.70. ■

Table 2.14

x	2^x
1	2
2	4
3	8
4	16

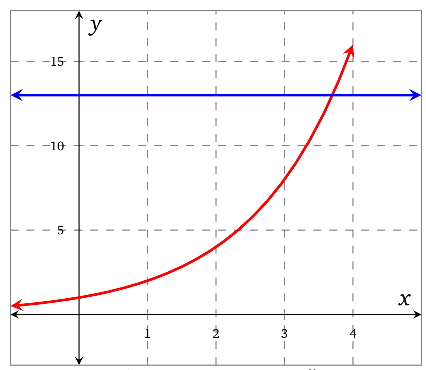


Figure 2.15: $y = 2^x$

★ **try it yourself** ★

Problem 5
Solve the following exponential equation using either a graph or a table of values; state your solution to 2 decimal places, and check your answer.

$$2^x = 10$$

make sure you try it!

Investigations

Problem 6 (To the Moon)
Let x represent the horizontal distance (in inches) that you are from your front door. Imagine that for every inch that you move away from your front door, your elevation above the ground y (in inches) magically increases according to the rule $y = 2^x$.

6.1 What is your elevation above the ground when you are at a horizontal distance of 1 ft from your front door? State your answers in ft, using the fact that there are 12 in in 1 ft.

6.2 What is your elevation above the ground when you are at a horizontal distance of 2 ft from your front door? State your answers in mi, using the fact that there are 63 360 in in 1 mi.

6.3 Given that the distance between the Earth and the Moon is approximately 1.595×10^{10} in, what is your horizontal distance from your front door when you reach the Moon? Use a graphing calculator to help you. [3]

[3] The distance from the Earth to the Moon (in miles) is 227 037 mi.

Exercises

Problem 7 (Algebraic manipulation)
Use properties of exponents to write each of the following formulas in the form $f(t) = a\,b^t$. Leave your answers in exact form.

7.1 $f(t) = 4^{t+1} \cdot 2^{3t}$

7.2 $f(t) = 4^{\frac{t+1}{2}} \cdot 3^{2t}$.

Problem 8 (Integer solutions)
Solve each of the following exponential equations. Check your solutions.

8.1 $2^{-x} = \frac{1}{8}$

8.2 $4^x = 64$

8.3 $3 \cdot 5^x = 75$

8.4 $2^{x^2} - 16 = 0$

8.5 $2^{x^2-5x} = 2^{-6}$

8.6 $7^{x^2} = 49^x$

Problem 9 (Non-integer solutions)
Solve each of the following exponential equations using either a graph or a table of values; state your solutions to 2 decimal places, and check your answers.

9.1 $3^x = 11$

9.2 $4^x = 29$

9.3 $5^x = 61$

9.4 $7^x = -1$

9.5 $-6^x = -31$

9.6 $3 \cdot 5^x = 7$

9.7 $2^x + 5 = 10$

9.8 $7^x = 50$

Problem 10 (Factoring)
Use the technique demonstrated in Example 3 to help you solve the following equations. For Problem 10.1 try putting $t = 4^x$ as your first step. Check your solutions.

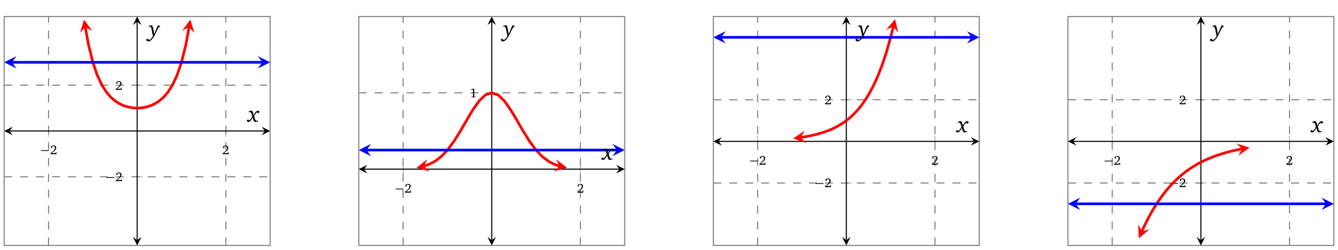
10.1 $4^{2x} + 2 \cdot 4^x = 3$

10.2 $5^{2x} - 1 = 0$

10.3 $125^x - 26 \cdot 25^x + 25 \cdot 5^x = 0$

Problem 11 (Matching graphs to equations)
Jake and Marisa are solving the following exponential equations. To guide them, they have graphed the functions involved in Figure 2.16. Match the equations with the appropriate graph.

$$4^{-x^2} = \frac{1}{4}, \qquad -\left(\frac{1}{3}\right)^x = -3, \qquad 5^x = 5, \qquad 3^{x^2} = 3$$



(a) (b) (c) (d)

Figure 2.16: Graphs for Problem 11

Problem 12 (Zeros)
Consider the functions f, g, h, and k that have formulas

$$f(x) = 4^{-x^2} - \frac{1}{4}, \qquad g(x) = 3 - \left(\frac{1}{3}\right)^x, \qquad h(x) = 5^x - 5, \qquad k(x) = 3^{x^2} - 3$$

Use your work from Problem 11 to help you decide how many zeros each function has.

Problem 13 (True or false?)
Myron and Win have studied all of the exponential equations so far, and are trying to generalize their results. They begin with the equation

$$2^x = c$$

where c can be any real number. Help them decide if the following statements are true of false, if they are false, provide an example that supports your answer.

13.1 If $c > 0$ then the equation has a solution.

13.2 If $c < 0$ then the equation has a solution.

13.3 If $c = 0$ then the equation has a solution.

Problem 14 (Beyond exponential equations)
FIX The function f that has formula $f(x) = \left(\frac{1}{2}\right)^x + \frac{1}{2}$ is shown in Figure 2.17, along with a mystery function g.

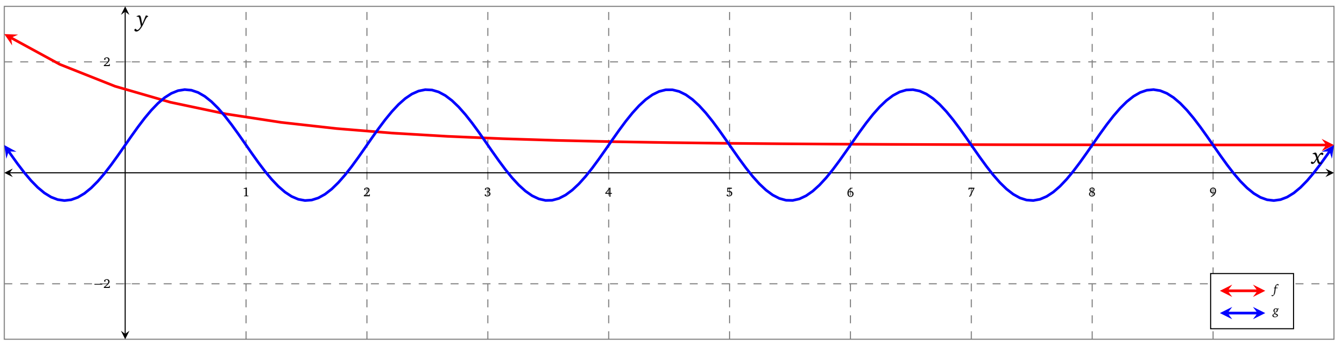


Figure 2.17

14.1 Use Figure 2.17 to decide how many solutions there are to the equation $f(x) = g(x)$ on the interval $[0, 9]$.

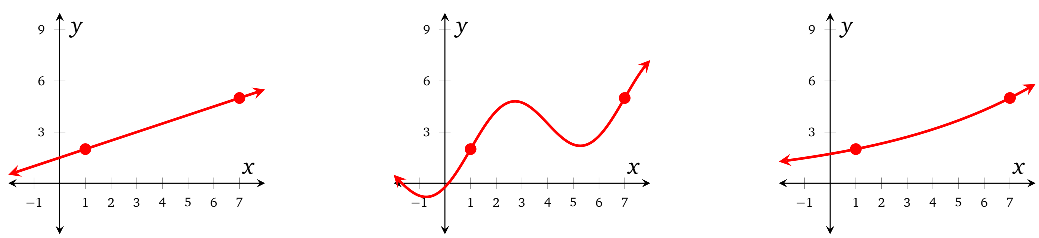
14.2 Hence determine how many zeros the function $h(x) = f(x) - g(x)$ has on the interval $[0, 9)$.

2.3 Finding an exponential function given two points

Section Themes, Concepts, Issues, Competencies, and Skills:

- *You will be able to find a formula for an exponential curve that connects two points in the plane.*
- *You will be able to predict the population of the United States and other countries in the year 2050.*
- *You will be able to model the amount of* CO_2 *emitted in the United States over time.*

Consider a puzzle where two points are given on the plane, and you are asked to connect them with a curve. There are infinitely many ways to do this. In the past, you've connected two points with a straight line (as in Figure 2.18a) and found the equation that represents that line. Of course, you could also connect the points with a random curve of your liking (as in Figure 2.18b). In this section we will try to connect the points with an *exponential* curve (as in Figure 2.18c) and simultaneously find that curve's equation.



(a) Points connected with a line

(b) Points connected with an arbitrary curve

(c) Points connected with an exponential curve

Figure 2.18: Connecting two points with various curves

essential skills

The following problems contain prerequisite skills that are essential for success. Make sure that you can complete them before moving on!

Problem 1
Simplify the given expressions.

1.1 $\dfrac{b}{b^{-2}}$ **1.2** $\dfrac{b^{-4}}{b^{-10}}$ **1.3** $\dfrac{b^{-3}}{b^{3}}$ **1.4** $\left(\dfrac{b^{-1}}{b^{-5}}\right)^{-2}$

Problem 2
Simplify the given expressions.

2.1 $8^{1/3}$ **2.2** $8^{-2/3}$ **2.3** $\left(8^5\right)^{1/3}$ **2.4** $\sqrt[3]{27^4}$

Problem 3
Solve the equations for x. Give both exact values and decimal approximations.

3.1 $x^4 = 81$ **3.2** $x^5 = -32$ **3.3** $x^6 = -64$ **3.4** $x^4 = 19$

In many application problems we will encounter situations where we wish to find an exponential function that goes through the two points (x_1, y_1) and (x_2, y_2). Let's practice this skill before moving on to the applied problems.

Example 1 Find an exponential function f given by $f(t) = a\,b^t$ whose graph goes through the points $\left(-2, \frac{3}{4}\right)$ and $(2, 12)$ as shown in Figure 2.19.

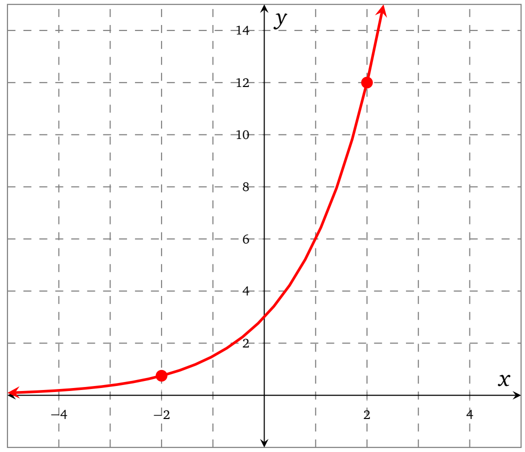


Figure 2.19

Solution We will need to find a and b in the equation $f(t) = a\,b^t$. We begin by using the given ordered pairs to write a system of equations

$$3/4 = a\,b^{-2}$$
$$12 = a\,b^2$$

We can eliminate a by equating the quotients formed by the two sides of the equations. One result is

$$\frac{12}{3/4} = \frac{a\,b^2}{a\,b^{-2}} \qquad \Longrightarrow \qquad 16 = b^4$$

This implies that either $b = 2$ or $b = -2$. Since the base of an exponential function must be positive ($b > 0$), we can conclude that $b = 2$. Substituting this into one of the original equations, we find that

$$12 = a\,(2)^2 \qquad \Longrightarrow \qquad a = 3$$

Note that we can substitute the value of b into either of the original equations.

We conclude that $f(t) = 3 \cdot 2^t$, which is graphed in Figure 2.19. The number 2 is called the growth factor; every time t increases by 1, the value of $f(t)$ grows by a factor of 2. ■

★ try it yourself ★

Problem 4
Find an exponential function f of the form $f(t) = a\,b^t$ whose graph goes through the ordered pairs $(1, 8)$ and $(3, 128)$. Identify the growth factor.

make sure you try it!

When dealing with application problems the values of a and b will rarely evaluate to integers. The method for finding an exponential model remains the same though.

Example 2 Find an exponential function $f(x) = a\,b^x$ that goes through the points $(1, 7)$ and $(10, 53)$.

Solution We will need to find a and b in the equation $f(x) = a\,b^x$. We begin by using the given ordered pairs to write a system of equations

$$7 = a\,b$$
$$53 = a\,b^{10}$$

We can eliminate a by equating the quotients formed by the two sides of the equations.

$$\frac{53}{7} = \frac{a\,b^{10}}{a\,b}$$

which simplifies to $b^9 = \frac{53}{7}$. This means that

$$b = \left(\frac{53}{7}\right)^{1/9}$$

If we substitute this value of b into the first equation, we obtain

$$\begin{aligned} a &= \frac{7}{b} \\ &= \frac{7}{\left(\frac{53}{7}\right)^{1/9}} \\ &= \frac{7^{10/9}}{53^{1/9}} \end{aligned}$$

We conclude that

$$\begin{aligned} f(x) &= \frac{7^{10/9}}{53^{1/9}} \left(\frac{53}{7}\right)^{x/9} \\ &= \frac{7^{10/9}}{53^{1/9}} \left(\left(\frac{53}{7}\right)^{1/9}\right)^{x} \\ &\approx 5.59(1.252237)^{x} \end{aligned}$$

■

★ try it yourself ★

Problem 5

Find an exponential function f of the form $f(x) = a\,b^x$ whose graph goes through the ordered pairs $(2, 10)$ and $(11, 23)$. Identify the growth factor.

make sure you try it!

Suppose that we have some reason to model a situation using an exponential function. In real-world applications, we might only have two points of data to work with. If we treat these data points as two points on a plane, we can use the skills we have been practicing to explicitly give the exponential model that we would like to use.

Example 3 An outbreak of avian flu occurs in a crowded city. Doctors immediately identify 25 patients who are infected. One week later, there are 2391 people infected. If we assume that the number of infected patients can be modeled with exponential growth, what is the rule for the exponential model?

Solution We choose to model this situation by treating the given data as points on a plane. Initially (or when $t = 0$) there are 25 patients, so $(0, 25)$ is one point. At time 7 (measured in days) there are 2391 patients, so $(7, 2391)$ is the other point. These points are sketched in Figure 2.20.

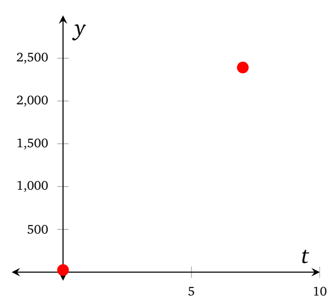


Figure 2.20: Flu patient data

Since we are trying to use an exponential model, we are searching for values of a and b such that the curve $y = a\,b^t$ will pass through these points. We need

$$\begin{aligned} 25 &= a\,b^0 \\ 2391 &= a\,b^7 \end{aligned}$$

The first equation immediately tells us that $a = 25$. Now the second equation reduces to

$$\begin{aligned} 2391 = 25\,b^7 \quad &\Longrightarrow \quad \frac{2391}{25} = b^7 \\ &\Longrightarrow \quad b = \left(\frac{2391}{25}\right)^{1/7} \end{aligned}$$

So we can model the number of infected people as a function of the number of days since the outbreak was first noticed using the model

$$\begin{aligned} f(t) &= 25\left(\left(\frac{2391}{25}\right)^{1/7}\right)^{t} \\ &= 25\left(\frac{2391}{25}\right)^{t/7} \end{aligned}$$

■

Once we have this model we can answer interesting questions with it. For example, how many people will be infected 3 days after the initial outbreak?

Solution Since

$$\begin{aligned} f(3) &= 25\left(\frac{2391}{25}\right)^{3/7} \\ &\approx 177 \end{aligned}$$

we can say that there will be about 177 infected people 3 days after the initial outbreak. ■

We can also look at a graph of the model (as in Figure 2.21) and answer other interesting questions. For example, when will the number of infected persons reach 20,000?

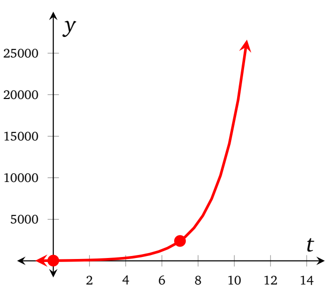


Figure 2.21: Flu patient model

Solution The graph suggests that around the tenth day after the initial outbreak, there will be 20000 infected persons. ■

Example 4 Many buildings in Detroit have been vacant and unattended for several years. Suppose that the amount of wood (in ft^3) that remains attached to a building t years after it has been abandoned is an exponential function of time. A particular building was abandoned on July 3, 1999. On July 3, 2002 there were 53 cubic yards of wood attached to the building and on July 3, 2009 there were 47 cubic yards of wood attached to the building.

Find a function w, given by $w(t) = a\,b^t$, that outputs the volume of remaining wood on the structure t years after July 3, 1999. Round the value of a to the nearest tenth and the value of b to the nearest thousandth.

Solution We are given that $w(3) = 53$ and $w(10) = 37$ so

$$\begin{aligned} a\,b^3 &= 53 \\ a\,b^{10} &= 47 \end{aligned} \tag{2.2}$$

Eliminating for b gives $b^7 = \frac{47}{53}$ and therefore

$$\begin{aligned} b &= \sqrt[7]{\frac{47}{53}} \\ &\approx 0.983 \end{aligned}$$

Using this value of b in Equation (2.2) gives

$$\begin{aligned} a &= \frac{53}{\left(\frac{47}{53}\right)^{3/7}} \\ &\approx 55.8 \end{aligned}$$

We conclude that

$$w(t) \approx 55.8(0.983)^t$$

■

Investigations

Problem 6 (US population)
The population of the United States has increased roughly exponentially from 76.2 million people in 1900 to 309 million people in 2010.[4]

6.1 Find a formula that approximates the number of people in the U.S.A., $p(t)$, in millions, t years after 1900 assuming that the population grows exponentially.

6.2 Use your model to approximate the population of the U.S.A. in the year 2000.

6.3 The actual population of the U.S.A. in 2000 was about 282 million people.[5] Did your approximation from Problem 6.2 underestimate or overestimate, and by how much?

6.4 Use the actual populations of the U.S.A. in the years 2000 and 2010 to find a formula that approximates the number of people in the U.S.A., $P(t)$, in millions, t years after 2000 assuming that the population grows exponentially.

6.5 The population of the U.S.A. was 151 million people in 1950[6]. How close of an approximation does your model from Problem 6.4 give? How close of an approximation does your model from Problem 6.1 give? Why are both models off? Which model is better for making a prediction about 1950?

Problem 7 (Greenhouse gases)
As our cars, homes, businesses and industry become more efficient, each person is responsible for the production of less carbon dioxide (CO_2) a greenhouse gas. The US Energy Information Administration provides the estimates for future quantities of CO_2 that each person will produce. They know that in 2010, on average each US citizen was responsible for the production of 18.1 metric tons of CO_2, and they predict that in 2031 the average US citizen will be responsible for 16.3 metric tons of CO_2.[7]

7.1 Find a formula for $c(t)$, an approximation for the average amount of CO_2 each US citizen will be responsible for emitting, in metric tons, t years after the year 2010 assuming that the amount of CO_2 produced decays exponentially.

7.2 Use your model to approximate the average amount of CO_2 each US citizen was responsible for in the year 2000.

7.3 If you have an answer to Problem 6.4, use it to find a formula for $C(t)$, an approximation for the *total* amount of CO_2 that all citizens of the U.S.A. will be responsible for emitting, in millions of metric tons, t years after the year 2000.

Problem 8 (Tapfish app)
Tapfish is a free mobile device app where you set up a digital aquarium. We can use the application to create approximations of exponential functions. We have simplified the nature of the application to make this problem easier. The app allows you to buy and sell fish once per day. The fish grow in size (and therefore value) after owning them for just one day. Our goal is to maximize the number of fish in our aquarium over a period of 9 days.

[4] http://www.wolframalpha.com/input/?i=us+population+1900
[5] http://www.wolframalpha.com/input/?i=us+population+2000
[6] http://www.wolframalpha.com/input/?i=us+population+1950
[7] http://www.eia.doe.gov/forecasts/aeo/

Each fish costs 10 coins and sells for 15 coins after 1 day (because of its larger size). Given that we start with 20 coins on day 0, the transactions for the first 3 days will be as follows:

- on day 0 we buy 2 fish;
- on day 1 we sell both fish for a total of 30 coins, and then buy 3 fish;
- on day 2 we sell our 3 fish for a total of 45 coins,and then buy 4 fish; we have 5 coins remaining as we cannot have fractions of fish.

Table 2.15

t	$F(t)$	$A(t)$
0	2	
1	3	
2	4	
3		
4		
5		
6		
7		
8		

8.1 Let $F(t)$ be the number of fish we have in our aquarium on day t. Complete the $F(t)$ column in Table 2.15. Assume that we always buy and sell the maximum number of fish, that we use whatever leftover coins we have from previous transactions, and that we cannot buy fractions of fish.

8.2 Make a graph of $y = F(t)$ using the points in your table.

8.3 The function F represents the exact number of fish that we have on a given day. We are going to find an approximate formula for F using an exponential function. To be clear, we will use A to represent the function that *approximates* F.

Pick any two of the points from Table 2.15 and find a function of the form, $A(t) = a\,b^t$, that approximates the number of fish in the tank t days after starting. (Answers will vary depending on which two points you use.)

8.4 Complete the $A(t)$ column in Table 2.15 using 6 decimal places, and compare the values of $A(t)$ to the values of $F(t)$.

Exercises

Problem 9 (Prerequisite simplification skills)
Simplify the given expressions.

9.1 $12^x 12^y$

9.2 $a^{5x}a^{x+3}$

9.3 $\left(3x^3y^2\right)^2$

9.4 $\left(2b^9c^4\right)\left(5bc^3\right)$

9.5 $\dfrac{14x^5}{12x^2}$

9.6 $\dfrac{6x^7}{27x^{-3}}$

9.7 $\left(\dfrac{3x^{-4}}{4x^2}\right)^{-3}$

9.8 $8^{x/3}$

9.9 $100^{x/2}$

9.10 $\left(3b^{4/3}\right)^{3x}$

9.11 $16^t(64)^{t/3}$

9.12 $\left(27t^3\right)^{4/3}$

Problem 10 (Prerequisite solving skills)
Solve each of the following equations. Give both the exact value and the decimal approximation where appropriate.

10.1 $2x^2 = 32$

10.2 $y^3 = \dfrac{27}{8}$

10.3 $\dfrac{1}{x^2} = \dfrac{1}{25}$

10.4 $\dfrac{1}{x^5+1} = 1$

10.5 $x^4 - 9 = 0$

10.6 $x^6 - 2x^3 - 35 = 0$

Problem 11 (Find a formula, a and b rational)
In each of the following problems, find an exponential function of the form $f(t) = a\,b^t$ that goes through the given ordered pairs. Identify the growth factor in each problem.

11.1 $(0,-3)$, $(4,-1875)$

11.2 $(2,36)$, $(5,7776)$

11.3 $\left(2,\frac{2}{9}\right)$, $\left(4,\frac{2}{81}\right)$

11.4 $\left(-3,\frac{2125}{8}\right)$, $\left(3,\frac{136}{125}\right)$

11.5 $(-1,279)$, $(1,124)$

11.6 $(2,64)$, $(-1,125)$

Problem 12 (Find a formula, a and b irrational)
In each of the following problems, find an exponential function of the form $f(x) = a\,b^x$ that goes through the given ordered pairs. Be sure to give both the exact form and the approximate form of f.

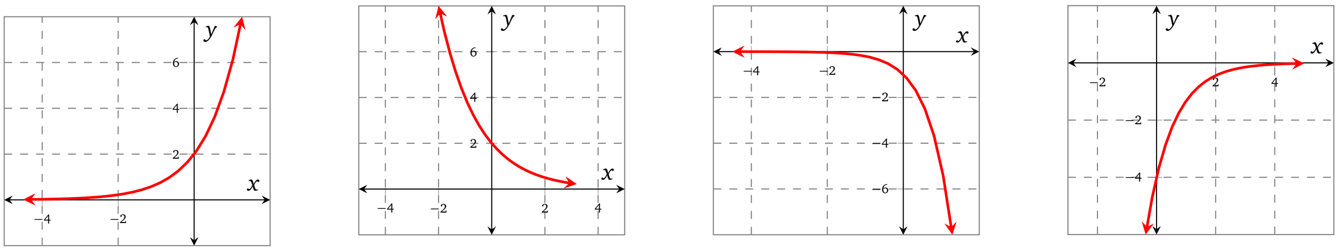
12.1 $(-5, 10)$, $(3, 4)$

12.2 $(4, 7)$, $(8, 2)$

12.3 $(3, 6)$, $(13, 29)$

12.4 $(-5, 6)$, $(7, 20)$

Problem 13 (Find a formula from a graph)
For each of the functions in Figure 2.22, assume that the vertical intercept is an integer value. Use this, together with the information in each caption, to find a formula for each function.



(a) $f(1) = 6$ (b) $g(2) = \frac{1}{2}$ (c) $h(-4) = -\frac{1}{625}$ (d) $k(3) = -\frac{4}{27}$

Figure 2.22

Problem 14 (Concavity)
For the functions in Figure 2.22, which graphs are concave up? Which are concave down?

Problem 15 (Find a formula from a table)
Find a formula for each of the functions implied by the data values in Tables 2.16–2.19. Assume that each function has the form $f(x) = a\, b^x$.

Table 2.16

x	y
1	10
2	100
3	1000
4	10 000
5	100 000

Table 2.17

x	y
−5	$9/16897$
−4	$9/2401$
−3	$9/343$
−2	$9/49$
−1	$9/7$

Table 2.18

x	y
2	$-2/3$
4	$-2/27$
6	$-2/243$
8	$-2/2187$
10	$-2/19683$

Table 2.19

x	y
−12	732 421 875
−9	5 859 375
−6	46 875
−3	375
0	3

Problem 16 (Increasing and decreasing functions)
Refer to the functions in Figure 2.22 and Tables 2.16–2.19 to help you decide if the following statements are true or false. If the answer is false, provide an example that supports your answer.

16.1 Exponential functions always increase.

16.2 Exponential functions always decrease.

16.3 Some exponential functions increase and some decrease.

16.4 If x increases by a constant amount, then y increases by a constant amount.

16.5 If x increases by a constant amount, then the successive ratios of y are the same.

16.6 If x increases by a constant amount, then the successive ratios of y increase by the same amount.

Problem 17 (Solar capacity)
The global solar photovoltaic power capacity (in MW) grew from 2000-2007[8] and is shown in Figure 2.23.

17.1 Let $S(t)$ represent the Solar PV Power Capacity (in MW) at time t in years since 2000. Use the first and last data points in Figure 2.23 to write down two ordered pairs that lie on the graph of S.

17.2 Use your ordered pairs to find a formula for S in the form $S(t) = a\, b^t$. State b to three decimal places.

17.3 What are the growth rate and growth factor for your function S?

[8] http://www.energyandcapital.com/articles/solar-stock-outlook/750

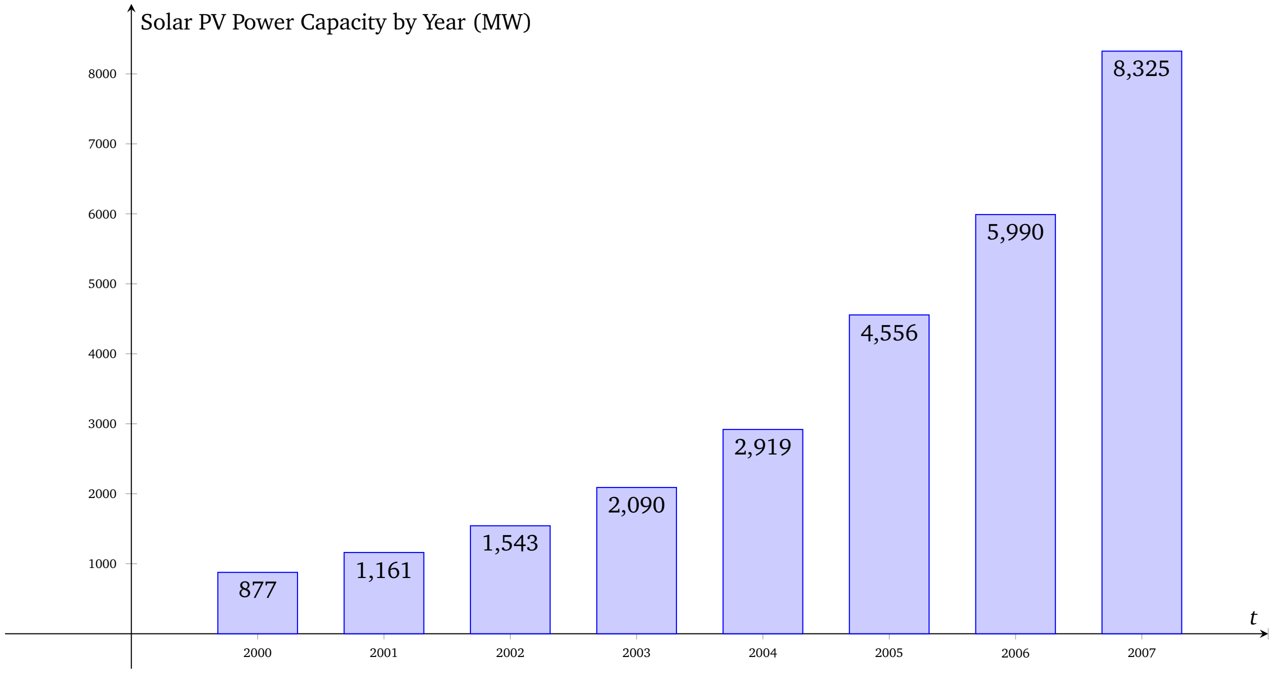


Figure 2.23: Solar capacity.

17.4 According to the article, the data implies 'a compounded annual growth rate (CAGR) of 37.9 %.' Is your model consistent with this statement?

17.5 According to your model, what is the Solar PV Power Capacity in 2010? Compare this with the actual value of 13 729 MW, and give reasons for the difference.

2.4 Exponential modeling

Section Themes, Concepts, Issues, Competencies, and Skills:

- *stuff*

In this section we explore some common applications of exponential functions. This will require a working knowledge of percentages.

essential skills

The following problems contain prerequisite skills that are essential for success. Make sure that you can complete them before moving on!

Problem 1

Perform the given percentage calculations. Give your answer correct to two decimal places when an approximation is appropriate.

1.1 Find 10 % of 100.

1.2 Find 20 % of 10.

1.3 Find 13 % of 28.

1.4 Find 81 % of 3.

1.5 Increase 17 by 28 %.

1.6 Increase 42 by 67 %.

1.7 Decrease 107 by 10 %.

1.8 Decrease 243 by 76 %.

Simple population models

Exponential functions are often used to model the size of a population as time passes. A population could be growing or declining at an exponential rate.

Definition 9 (Simple population model)

If an initial population, P_0, changes by the same percentage each year, then the population at time t in years, $P(t)$, is given by the formula

$$P(t) = P_0(1+r)^t,$$

where r is the decimal form of the percentage change.

When dealing with human population models, we generally restrict the domain of P to integer values of t. This means that we are only modeling population values on one specific day of the year.

Note that r can be positive (for a growing population) or negative (for a declining population). Also, note that we don't have to measure time in years. Using years makes sense for modeling human populations, but a smaller unit like hours might make sense for a bacteria population.

Example 1 Since 2001 the population of an inner city has been decreasing as people move to the suburbs. It is decreasing at a rate of 1 % per year; find a model for this situation.

Solution Let $P(t)$ be the population of the inner city at time t in years since 2001. Let P_0 be the population of the inner city in 2001. Then, since $r = -0.01$,

$$P(t) = P_0(0.99)^t$$

■

★ try it yourself ★

Problem 2

Repeat Example 1 for a population that decreases at a rate of 8 % per year.

make sure you try it!

Example 2 At the beginning of the year 2000, the population of One Horse was 10,000 people. By the beginning of 2005 the population had decreased to 9700. Assuming exponential decay, determine the population at the beginning of 2012. Round your answer to the nearest person.

Solution Over 5 years, the population decreased by 300 people. As $\frac{300}{10000} = 0.03$, the population decreased by 3 % over the five-year period. The growth factor for this five-year period is 0.97 and therefore the growth factor for one year is $(0.97)^{1/5}$. Using this, the population $P(t)$ at time t in years after 2000 can be represented by

$$P(t) = 10000\,(0.97)^{t/5}$$

To estimate the population in 2012, we evaluate $P(12)$:

$$\begin{aligned} P(12) &= 10000\,(0.97)^{12/5} \\ &\approx 9295 \end{aligned}$$

In 2012, the population of One Horse is expected to be approximately 9,295 people. ■

★ try it yourself ★

Problem 3 (Oregonians)
In 2000 there were 3.83 million people living in Oregon. By 2010, the population had increased by 12.0 %.[9] Assuming the population continues to grow at this rate, find a model representing the size of the population t years after 2000.

make sure you try it!

Radioactive decay

One common application of decaying exponential functions involves radioactivity. If you have a substance where some of the atoms are radioactive, the radioactive atoms will decay into other atoms in a very predictable way. In fact, each radioactive element has an associated half-life, which is the time it takes for 50 % of the radioactive atoms to decay into something else.

Example 3 Suppose that we have 8000 atoms of Carbon-12 (which is non-radioactive). Without an intervening act, 1,000,000 years from now that sample of carbon will still have 8000 atoms of Carbon-12. However, if we start with a sample of 8000 radioactive C-14 atoms, over time fewer and fewer of those atoms remain radioactive.

If we define $R(t)$ to be the number of radioactive atoms that remain in the sample t years from today, then we can model the function R using the template $R(t) = a\,b^t$ for unknown constants a and b. The number of remaining radioactive atoms t years from today is shown in Table 2.20.

Table 2.20: Decaying Radioactive Carbon-14

Number of years from today	Number of remaining radioactive atoms
0	8000
5730	4000
11 460	2000
17 190	1000
22 920	500

Using the data in Table 2.20 we can determine the formula for $R(t)$, the number of atoms that remain radioactive after t years; choosing two ordered pairs and assuming that $R(t) = a\ b^t$

$$\begin{aligned} 8000 &= a b^0 \\ 4000 &= a b^{5730} \end{aligned}$$

This clearly means that $a = 8000$, and

$$\frac{1}{2} = b^{5730} \Longrightarrow b = \left(\frac{1}{2}\right)^{\frac{1}{5730}}$$

[9] 2000 census

We can therefore say that

$$R(t) = 8000\left(\frac{1}{2}\right)^{\frac{1}{5730}t}$$

We can now determine how many atoms remain radioactive after 100 years

$$R(100) \approx 7904$$

Approximately 7904 atoms remain radioactive after 100 years.

How long it would take until there are only 6000 radioactive atoms remaining? We need to solve the equation

$$6000 = 8000\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

Using a graphing calculator, we obtain $t \approx 2378$. We conclude that 6000 radioactive atoms remain after approximately 2378 years. ■

Simple interest

Let's assume that we invest an amount P into an account that pays an interest rate r per year.

At the end of year 1 we have

$$\begin{aligned} A &= \overset{\substack{\text{beginning}\\ \text{balance}\\ \downarrow}}{P} + \overset{\substack{\text{interest}\\ \text{added}\\ \downarrow}}{P \cdot r} \\ &= P(1+r) \end{aligned}$$

At the end of year 1 we have

FIX

$$\begin{aligned} A &= \overbrace{P}^{\substack{\text{beginning}\\ \text{balance}}} + \overbrace{P \cdot r}^{\substack{interest\\ added}} \\ &= P(1+r) \end{aligned}$$

After 2 years we have

$$\begin{aligned} A &= \overset{\substack{\text{year 1}\\ \text{balance}\\ \downarrow}}{P(1+r)} + \overset{\substack{\text{interest}\\ \text{added}\\ \downarrow}}{P(1+r)\cdot r} \\ &= P(1+r)\cdot 1 + P(1+r)\cdot r \\ &= P(1+r)(1+r) \\ &= P(1+r)^2 \end{aligned}$$

After 3 years we have

$$\begin{aligned} A &= \overset{\substack{\text{year 2}\\ \text{balance}\\ \downarrow}}{P(1+r)^2} + \overset{\substack{\text{interest}\\ \text{added}\\ \downarrow}}{P(1+r)^2\cdot r} \\ &= P(1+r)^2(1+r) \\ &= P(1+r)^3 \end{aligned}$$

After t years we have

$$\begin{aligned} A &= \overset{\substack{\text{year } t-1\\ \text{balance}\\ \downarrow}}{P(1+r)^{t-1}} + \overset{\substack{\text{interest}\\ \text{added}\\ \downarrow}}{P(1+r)^{t-1}\cdot r} \\ &= P(1+r)^{t-1}(1+r) \\ &= P(1+r)^t \end{aligned}$$

Definition 10 (Simple interest)
If we invest an amount P in an account with interest rate r (per year), then the amount in the account is

$$A(t) = P(1+r)^t \tag{2.3}$$

where t is the amount of time that has passed (in years) since the initial investment.

Example 4 You invest \$5000 in an account that pays 3 % simple interest. What is the balance after four years? How long will it take your investment to double in value?

Solution We use Equation (2.3), with $P = 5000$, $r = 0.03$, and $t = 4$.

$$\begin{aligned} A(4) &= 5000(1+0.03)^4 \\ &\approx 5627.54 \end{aligned}$$

There is approximately \$5627.54 in the account after 4 years.

To find when the investment will double in value, our first thought might be to solve the equation

$$10000 = 5000(1.03)^t$$

for t. This is problematic in that the solution to this equation is almost certainly non-integer and the domain of our function is restricted to the integers. A more straight forward approach might be to simply find the first year the balance is at least \$10,000.

Table 2.21: Simple interest

t	$A(t)$
0	5000.00
1	5150.00
⋮	⋮
23	9867.93
24	10 163.97

Using Table 2.21, we conclude that the investment will double in about 24 years. ∎

Compound interest

We have just studied simple interest. Most banks, however, use a slightly more complicated system.

Example 5 – An offer you can't refuse: Two brothers, Michael and Fredo, each deposited \$100 into brand new bank accounts.

Michael's bank account gave him a 12 % annual interest rate. How much money did he have in his account after one year if he earned 12 % in simple annual interest?

We need to calculate $100(1.12) = 112$. Michael had \$112 after one year.

Fredo's bank account gave him a 12% annual rate compounded monthly, which means that each month he received one-twelfth of the 12 % (which is 1 % interest) on his account:

- after one month, he had $\$100(1.01) = \$101.00$ in his account;
- after two months he had $\$100(1.01)^2 = \$102.01$ in his account.

Table 2.22: Fredo's account

t	$A(t)$
0	100.00
1	101.00
2	102.01
3	103.03
4	104.06
5	105.10
6	106.15
7	107.21
8	108.29
9	109.37
10	110.46
11	111.57
12	112.68

We complete Table 2.22 using 2 decimal places, where t indicates the number of months since first investing and $A(t)$ is the amount of money that was in his account (in dollars) at time t.

If we look closely at what happened in Table 2.22, we notice that to find $A(2)$ we calculate

$$\begin{aligned} A(2) &= 100(1.01)(1.01) \\ &= 100(1.01)^2 \end{aligned}$$

Similarly, $A(12) = 100(1.01)^{12}$.

Fredo didn't actually earn 12 % interest after one year on his initial investment, so the 12 % is in name only, and is called the *nominal* interest rate.

After one year, what percent did Fredo actually earn? We calculate the *effective rate*:

$$\left(1 + \frac{0.12}{12}\right)^{12} - 1 \approx 0.126825$$

The effective rate is approximately 12.68 %.

Note that although both Michael and Fredo both had accounts that were nominally at 12 % annual interest, they ended up with different effective annual interest rates due to the different compounding periods.

A third brother, Sonny, invested \$100 into another account. His account earned 12 % nominal rate compounded daily. This means that he earned $\frac{12\,\%}{365}$, which is approximately 0.032 876 7 % each day. Find how much money Sonny had in his account after 1 year.

We calculate

$$100\left(1+\frac{0.12}{365}\right)^{365} \approx 112.75$$

and therefore Sonny had \$112.75 after one year.

We can calculate the effective interest rate on Sonny's account using

$$\left(1+\frac{0.12}{365}\right)^{365} - 1 \approx 0.127475$$

The effective rate was approximately 12.75 %. ■

Table 2.23

Words	n
yearly	1
annually	1
semi-annually	2
quarterly	4
monthly	12
biweekly	26
weekly	52
daily	365
non-stop	??

Definition 11 (Compound interest)
The compound interest formula is

$$A(t) = P_0\left(1+\frac{r}{n}\right)^{nt} \tag{2.4}$$

where:

- P_0 is the initial amount invested;
- r is the nominal interest rate;
- n is the compounding frequency (see Table 2.23);
- t is the amount of time that has passed since the initial investment (in years);
- $A(t)$ is the account balance (in dollars).

Example 6 We have \$7000 to invest into an account that earns interest at a 3 % nominal rate. Find the amount that we will have after four years assuming each of the following compounding frequencies:

(a) yearly (b) monthly (c) weekly (d) daily

Solution Apply the compound interest formula (Equation (2.4)) to each case.

(a) yearly:

$$\begin{aligned} A(4) &= 7000\left(1+\frac{0.03}{1}\right)^{1\cdot 4} \\ &\approx 7878.56 \end{aligned}$$

(b) monthly:

$$\begin{aligned} A(4) &= 7000\left(1+\frac{0.03}{12}\right)^{12\cdot 4} \\ &\approx 7891.30 \end{aligned}$$

(c) weekly:

$$\begin{aligned} A(4) &= 7000\left(1+\frac{0.03}{52}\right)^{52\cdot 4} \\ &\approx 7892.20 \end{aligned}$$

(d) daily:

$$\begin{aligned} A(4) &= 7000\left(1+\frac{0.03}{365}\right)^{365\cdot 4} \\ &\approx 7892.44 \end{aligned}$$

After 4 years, assuming compounding frequencies of yearly, monthly, weekly, daily, the balance will be, respectively, \$7878.56, \$7891.30, \$7892.20, and \$7892.44. ■

We have covered a lot of rates including growth rates and decay rates. In the context of finance we've talked about nominal interest rates and effective interest rates. Recall that the nominal interest rate is the stated rate before any compounding has been applied. Now we will formally define effective interest rate.

Definition 12 (Effective rate)
Given a nominal annual interest rate, r, and a compounding frequency, n, the effective rate is the interest rate that is *actually* earned over the course of one year. The effective rate is calculated using the expression

$$\left(1+\frac{r}{n}\right)^n - 1$$

The quantity $\left(1+\frac{r}{n}\right)^n$ represents the annual growth factor defined in Definition 8. This comes from the compound interest formula (Definition 11) and the equality

$$\left(1+\frac{r}{n}\right)^{nt} = \left(\left(1+\frac{r}{n}\right)^n\right)^t$$

Example 7 Find the effective rate of interest for each of the problems in Example 6.

Solution We will keep many decimal places in our solutions so that we can see the differences between the results.

(a) $\left(1+\frac{0.03}{1}\right) - 1 = 0.03$ (not surprising)

(b) $\left(1+\frac{0.03}{12}\right)^{12} - 1 \approx 0.030415957$

(c) $\left(1+\frac{0.03}{52}\right)^{52} - 1 \approx 0.030445620$

(d) $\left(1+\frac{0.03}{365}\right)^{365} - 1 \approx 0.030453264$

These have been tabulated in Table 2.24. ■

Table 2.24

n	nominal rate (%)	effective rate (%)
1	3	3
12	3	3.041 595 7
52	3	3.044 562 0
365	3	3.045 326 4

Example 7 demonstrates that the nominal interest rate is usually different from the effective interest rate. In practice, an effective interest rate is almost always a bit more than the nominal interest rate.

Investigations

Problem 4 (Atmospheric C-14)
During the Cold War, many above-ground thermonuclear tests were done in the South Pacific region, making many islands completely uninhabitable for decades (if not centuries) due to lingering radiation. In fact, the thermonuclear testing drastically increased the amount of atmospheric C-14 above normal levels worldwide. In this problem, we are going to investigate the long-term effects of worldwide nuclear testing half a world away from the South Pacific.

Figure 2.24 charts the percent of atmospheric C-14 above the natural pre-nuclear test level in the air above Austria[10]. The decay that you see has to do with C-14 being flushed out of the atmosphere and into the ground and ocean, not radioactive decay.

4.1 Write a sentence that contextually interprets the data point $(1963, 95)$.

4.2 Using the first and last data point determine a formula that gives an approximation of C-14 in the atmosphere in Austria as a function of the number of years since 1963. Round your

[10] http://en.wikipedia.org/wiki/Carbon-14

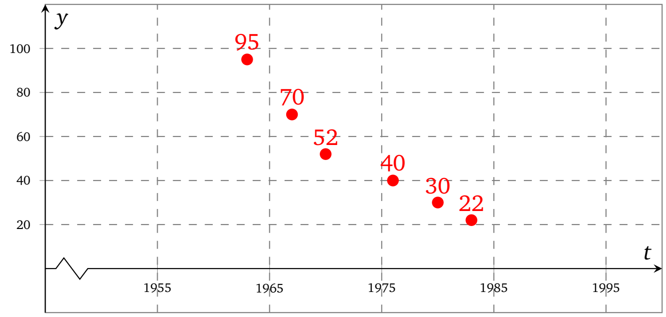


Figure 2.24: Atmospheric C-14 in Austria (% above pre-nuclear age levels)

growth factor to the second digit after the decimal point. Use your calculator to check the relative accuracy of your graph.

4.3 Repeat Problem 4.2 using the data points $(1970, 52)$ and $(1976, 40)$.

4.4 Which model is the better fit? [11]

Problem 5 (CSI)
Almost every compound in your body is replaced frequently by the food, water, and air that you take in. One of the few exceptions is the enamel in your teeth. Enamel is formed in early childhood, and is never replaced. Whatever C-14 level was in the atmosphere at that time becomes permanently locked in your teeth. Because the half-life of C-14 is so long, the level will remain steady throughout your life.

Forensic scientists have found a fascinating application for this information. After the devastating tsunami in Indonesia in 2004, there were many unidentified bodies. To help identify bodies with the names on lists of missing persons, researchers examined the C-14 level in the teeth of the tsunami victims. Through comparison with known C-14 levels, the forensic scientists were able to determine the birth year of victims to within 18 months.

Suppose that skeletal human remains are found in a forest in Austria. A first-molar from the remains is found to have 145 % of pre-nuclear age C-14. First-molars develop while a person is about 6 or 7 years of age. In what year was the person born? Use the model determined in Problem 4.2 while working this problem.

Problem 6 (Resources)
In 2005 a town has 1000 barrels of oil in a well. Some of the oil is easy to reach and some of it is hard to reach. As more of the easily accessible oil is drilled, the amount that they can drill in a given year is reduced. In fact, limitations of drilling equipment lead to a 20 % reduction in drilling capacity each year. In 2005, 200 barrels of oil were drilled. Let $Q(t)$ represent the amount of oil (in barrels) that is drilled where t is the number of years after 2005. Assume that in each year the town drills to its maximum capacity.

Table 2.25

t (years)	Q (barrels)
0	
1	
2	
3	
4	
5	

6.1 Complete Table 2.25 to two decimal places.

6.2 Find a formula for $Q(t)$.

6.3 What is the first year that the amount of oil drilled will be less than 100 barrels?

6.4 How long should it take until no more oil can be drilled? Make sure to justify your answer.

Problem 7 (Revisiting greenhouse gases)
Recall in Problem 6 on page 52 we modeled the US population and in Problem 7 on page 52 we modeled the average amount of CO_2 emissions for which each US citizen is responsible. The functions we came up with were, respectively,

$$P(t) = 282\left(\frac{309}{282}\right)^{t/10} \quad \text{and} \quad C(t) = 18.1\left(\frac{16.3}{18.1}\right)^{-10/21}\left(\frac{16.3}{18.1}\right)^{t/21}$$

[11] In fact in statistics you learn techniques called regression that allows you to take all of the data points into consideration and develop the most accurate model.

7.1 The total amount of CO_2 that the U.S.A. emits can be found by multiplying the amount that each person emits with the total U.S. population. Find and simplify a formula for $T(t)$, the total amount of CO_2 released by the U.S.A., in millions of metric tons, t years since 2000.

7.2 Based on your answer to Problem 7.1, is the total amount of CO_2 produced in the U.S.A. each year increasing or decreasing, and by what percent?

7.3 The US Energy Information Administration estimates that the U.S.A. will produce 5,679 million metric tons of CO_2 in the year 2015. Use your model from Problem 7.1 to estimate the average CO_2 emissions in the year 2015. How close is your approximation to the USEIA estimate?

Exercises

Problem 8 (Given description, write formula)
In each of the following, assume that the population of a town changes at the given rate. Write a formula for $P(t)$, the population at time t, measured in years since 2012.

8.1 $P_0 = 500$, increasing at 6 % per year.

8.2 $P_0 = 1500$, increasing at 12 % per year.

8.3 $P_0 = 2700$, increasing at 23 % per year.

8.4 $P_0 = 3600$, increasing at 52 % per year.

8.5 $P_0 = 700$, decreasing at 6 % per year.

8.6 $P_0 = 2405$, decreasing at 12 % per year.

8.7 $P_0 = 4302$, decreasing at 23 % per year.

8.8 $P_0 = 7300$, decreasing at 52 % per year.

Problem 9 (Given formula, write interpretaion)
Each of the following formulas model the population of a town at time t (in years) since 1998. Determine the initial population, P_0, and the percentage change, r, and give a sentence that describes the model.

9.1 $P(t) = 1000(1.1)^t$

9.2 $P(t) = 1800(1.07)^t$

9.3 $P(t) = 200(0.87)^t$

9.4 $P(t) = 907(0.76)^t$

Problem 10 (US population)
In the year 2000, the population of the U.S.A. was around 281 million people with an estimated percentage growth rate of 0.7 % per year[12].

10.1 Let $P(t)$ represent the number of people in the U.S.A., in millions of people, at time t in years since the year 2000. Write a formula for $P(t)$.

10.2 According to your model, how many people lived in the U.S.A. in 2010? How many will live in the U.S.A. in 2035? Round your answers to the nearest million.

10.3 Graph P using the grid in Figure 2.25. When graphing the population, we graph it continuously to see the trend, even though we are only modeling it for one specific day of the year.

10.4 What are the strengths and weaknesses of an exponential model like this? Are there any other factors or information you might want to know before making a more informed decision on whether this percentage growth rate changes or stays the same? What other information might help solidify or change your model?

Problem 11 (China population)
In the year 2009 China had a population around 1.3 billion people and its government had a goal of reducing the population to 700 million by 2050. It aimed to do this by limiting the number of children people had. Assume that with natural death rates and the restricted birth rate the population is decreasing by 0.5 % each year.

11.1 Let $P(t)$ represent the number of people in China, in billions of people, at time t in years since 2009. Write a formula for P.

11.2 According to your model, how many people will live in China in 2025? 2050?

11.3 Graph P using the grid in Figure 2.26. When graphing the population, we graph it continuously to see the trend, even though we are only modeling it for one specific day of the year.

[12] 2000 census

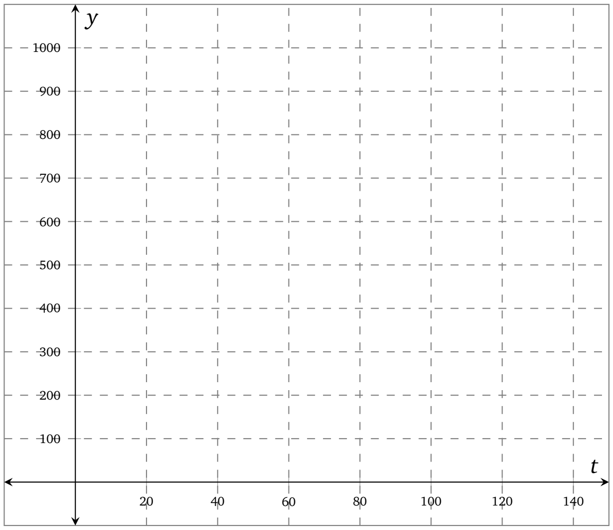


Figure 2.25: US population

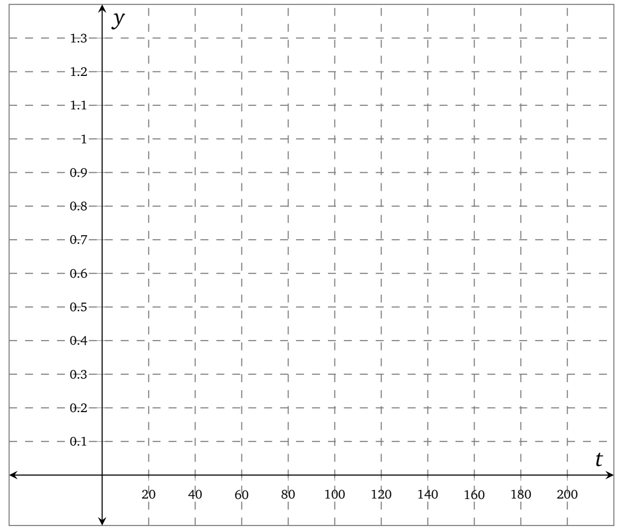


Figure 2.26: China population

11.4 According to your model, in what year will China reach its goal of having a population of 700 million people?

11.5 Do you think the decay rate is likely to stay at 0.5 % per year? If not, how do you think it might change?

11.6 What are the strengths and weaknesses of an exponential model like this? Are there any other factors or information you might want to know before making a more informed decision on whether this population rate changes or stays the same? What other information might help solidify or change your model?

Problem 12 (Textiles)
When textiles are made from a certain plant that grows alongside the Nile river, each square inch contains 142 billion atoms of radioactive C-14.

12.1 Using the half-life of 5730 years for C-14, determine a formula for the number of radioactive atoms (in billions), $N(t)$, that remain in a one-square-inch sample of this textile after t years.

12.2 How many atoms in the sample would remain radioactive after 3000 years?

12.3 Archaeologists found clothing preserved in a tomb made from this textile, and the clothing contained 93 billion atoms of radioactive C-14 per square inch. Roughly how old is the clothing?

Problem 13 (Simple interest)
Use Equation (2.3) to find the given unknowns in each of the following problems. Give your answers to 2 decimal places.

13.1 $P = 1000$, $r = 4\%$; find $A(3)$

13.2 $P = 2500$, $r = 7\%$; find $A(4)$

13.3 $A(4) = 2500$, $r = 7\%$; find P

13.4 $A(6) = 3600$, $r = 10\%$; find P

13.5 $A(6) = 3600$, $P = 1200$; find r

13.6 $A(20) = 5000$, $P = 600$; find r

Problem 14 (Investment)
You invest \$15,000 in the year 2010 into an investment account earning 5% simple interest annually.

14.1 Find a formula for the amount of money you have in total, $A(t)$, at time t in years since 2010.

14.2 How much will this investment be worth in 2020? 2030? 2040? Give your answers to two decimal places.

14.3 How long will it be until you have at least \$20,000 in the account?

Problem 15 (Compound interest)
Assume that you invest \$8000 into an account that earns interest at a 5 % nominal rate.

15.1 How much will you have after four years if the interest is compounded yearly?

15.2 How much will you have after four years if the interest is compounded weekly?

15.3 How much will you have after four years if the interest is compounded daily?

15.4 If the interest is compounded daily, how long will it take your investment to double? Triple? Use two decimal places in your answers.

15.5 The bank manager gives you the option to invest in a mystery account. You are told that the interest is compounded daily, and that if you invest your \$8000 for 3 years you will have \$10479.40. What is the interest rate in this mystery account?

Problem 16 (Exploring effective rate)
We have an amount P_0 to invest in an account that earns interest at a 6 % nominal rate. Answer each of the following through the eighth digit after the decimal point.

16.1 Find the effective rate if the interest is compounded annually.

16.2 Find the effective rate if the interest is compounded quarterly.

16.3 Find the effective rate if the interest is compounded daily.

16.4 Find the effective rate if the interest is compounded every second.

16.5 Do you think there would be much difference if the interest was compounded every half second compared to compounding it every second?

Problem 17 (Credit card)
A credit card company advertises an annual rate of 24%. What is the effective annual interest rate if your bill is compiled monthly?

Problem 18 (Payday loan)
A payday loan company makes this offer to a customer: they will receive \$375 today and when they get paid in two weeks, they owe the payday loan company \$450.

18.1 What percent interest is charged over the two-week period?

18.2 Using your answer in Problem 18.1, compute the effective annual rate.

Problem 19 (Find the rate)
The amount of money in an account grows by a total of 45 % over a period of 10 years. Find the nominal interest rate (to 5 decimal places) if the account was compounded:

19.1 annually

19.2 monthly

2.5 The number e

Section Themes, Concepts, Issues, Competencies, and Skills:

- *learn about e*

Continuously compounded interest and the number e

Congratulations, you've just won the grand prize of one dollar in the giga-millions Genuine Wheel of Fortune lottery game. One dollar doesn't sound like much of a grand prize, but here's the thing – you can deposit that dollar into a savings account that earns interest at a rate of 100 % per year! So after one year your prize will have grown to \$2, after two years it will have grown to \$4, wait a minute . . . even after 10 years it has only grown to \$1024. But you know how these increasing exponential functions work; once they get going, they really take off. If you can manage to hold off your withdrawal for twenty years you'll reap a little over a million dollars and wait just five more years after that and you'll take home over thirty-three million dollars!

All of those figures are based upon the assumption that the interest is compounded only once a year – that's where the wheel comes in. You spin the wheel to see the way in which the interest will be compounded. Table 2.26 shows the amount that will be in your account at the end of year one if the interest is compounded yearly, monthly, weekly, daily, every hour, every minute, and every second. Make sure that you see the connection between the expression $\left(1+\frac{1}{n}\right)^n$ and the compound interest formula as it applies to this application.

Table 2.26

n	$\left(1+\dfrac{1}{n}\right)^n$
1	2
12	2.613 035 29
52	2.692 596 95
365	2.714 567 48
8760	2.718 126 69
525 600	2.718 279 22
31 536 000	2.718 282 47

We can see that through the fifth digit after the decimal point the amount in your account will be the same whether the compounding is done by the minute or by the second. If we were to continue increasing the number of times we compound, we would find that the digits start to get fixed farther and farther to the right of the decimal point. However, no matter how many times we compound, if we look far enough to the right of the decimal point the digits will be different if the interest is compounded just one additional time. That is to say

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

is an irrational number (a number that in decimal form never terminates and never forms a forever-repeating pattern).

It turns out that this very same irrational number pops up in a wide variety of applications, and as such it is worthy of a name. The number is called Euler's number and is symbolized by the letter e. If we let the number of times the interest is compounded increase without bound, we say that interest is being compounded continuously. In our lottery game, if the interest is compounded continuously, then the amount that would be in your giga-millions account at the end of year one would be e dollars where $e \approx 2.71828$.

Definition 13 (The number e)
The number e is called Euler's number named after the famous Swiss mathematician Leonhard Euler (1707-1783). It is also called the natural base of an exponential function.

$$\begin{aligned} e &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\ &\approx 2.718281828 \end{aligned}$$

Another definition of e is hinted at in Problem 18.

Continuous growth and decay

We can use the natural base, e, to model real life situations that involve *continuous* growth and decay. The following definition will guide us in what follows.

Definition 14 (Continuous growth and decay)
If we are modeling a situation that involves continuous growth or decay, then the model will have the following form:

$$Q(t) = Q_0\, e^{kt}$$

- If $k > 0$, then the function Q is increasing, and k is called the *continuous growth rate*.
- If $k < 0$, then the function Q is decreasing, and $|k|$ is called the *continuous decay rate*.

In the context of a continuously compounded interest problems, k is the nominal interest rate and $e^k - 1$ is the effective interest rate.

Example 1 We have \$7000 to invest in an account that has a nominal interest rate of 3 % compounded *continuously*.

(a) Find a model for this situation.

(b) Find the amount in the account after 4 years.

(c) Compare the answer with the results of Example 6 on page 60.

(d) Find the effective annual rate of interest.

Solution (a) Using Definition 14,

$$Q(t) = 7000\, e^{0.03t}$$

(b) We evaluate the function Q when $t = 4$

$$\begin{aligned} Q(4) &= 7000\, e^{0.03(4)} \\ &\approx 7892.48 \end{aligned}$$

The amount in the account after four years is \$7892.48.

(c) This is larger than any of the values we found in Example 6.

(d) We calculate the effective annual rate of interest using Definition 14

$$e^{0.03} - 1 \approx 0.030454534$$

The effective annual rate of interest is approximately 3.045 453 4 %. Note that this is greater than any of the values we found in Example 7 on page 61. ■

★ **try it yourself** ★

Problem 1
Repeat Example 1 using an investment of \$4000, and a nominal rate of 2 %.

make sure you try it!

Example 2 You have \$2000 to invest in an interest-bearing account that has a nominal interest rate of 5 %.

(a) Calculate the effective annual growth rates if the interest is compounded daily and if the interest is compounded continuously.

(b) State the annual growth factor for each account.

(c) Calculate the account balance after 10 years if the interest is compounded annually, daily, and continuously.

Solution (a) If the interest is compounded daily, then we calculate

$$\left(1 + \frac{0.05}{365}\right)^{365\cdot 1} - 1 \approx 0.051267496$$

The effective annual rate is approximately 5.126 749 6 %.

If the interest is compounded *continuously* then we calculate

$$e^{0.05} - 1 \approx 0.05127110$$

The effective annual rate is approximately 5.127 110 %.

(b) If the interest is compounded daily, then the annual growth factor is calculated using

$$\left(1 + \frac{0.05}{365}\right)^{365 \cdot 1} \approx 1.051267496$$

The growth factor is approximately 1.051267496.

If the interest is compounded continuously, then the annual growth factor is calculated using

$$e^{0.05} \approx 1.05127110$$

The growth factor is approximately 1.05127110.

(c) The account balance after 10 years if the interest is compounded annually, daily, and continuously is calculated using (respectively)

$$\begin{aligned} 2000(1.05)^{10} &\approx 3257.79 \\ 2000\left(1 + \frac{0.05}{365}\right)^{10 \cdot 365} &\approx 3297.33 \\ 2000e^{0.05 \cdot 10} &\approx 3297.44 \end{aligned}$$

We conclude that

- when the interest is compounded annually the balance will be \$3257.79;
- when the interest is compounded daily the balance will be \$3297.33;
- when the interest is compounded continuously the balance will be \$3297.44. ■

★ **try it yourself** ★

Problem 2
Repeat Example 2 using an initial investment of \$15,000 at a nominal interest rate of 8 %.

make sure you try it!

Example 3 We have \$600 to invest in an account that compounds interest continuously.

(a) If the nominal interest rate is 8 %, what will the effective annual growth rate be?

(b) If the effective annual interest rate is 8 %, what will the nominal interest rate be?

Solution (a) The amount (in dollars) in the account will be $600\,e^{0.08t}$ t years after the money is invested. We need to find the value of r in the equation

$$A(t) = 600(1 + r)^t.$$

Well,

$$\begin{aligned} A(t) &= 600\,e^{0.08t} \\ &= 600\left(e^{0.08}\right)^t \\ &\approx 600\,(1.08328707)^t \end{aligned}$$

So the growth rate is $e^{0.08} - 1$, or about 8.328 707 %.

(b) This time the account will have $600(1.08)^t$, t years after the money is invested. We need to find the value of r in the equation

$$A(t) = 600\,e^{rt}.$$

Since the two expressions for $A(t)$ must be equal,

$$600\,e^{rt} = 600(1.08)^t \implies e^{rt} = (1.08)^t$$
$$\implies e^r = 1.08$$

Later we will learn how to solve for r exactly using logarithms. For now, we can find an approximate solution for r using a graphing calculator. The value of r is about 0.077, so the nominal interest rate is about 7.7 %. ■

Radioactive decay

You may remember that we studied radioactive decay in Example 3 on page 57. In fact it is often more appropriate to use the natural base, e, in such applications.

Example 4 The number of radioactive atoms in a sample of Carbon-14 decays according to the model

$$Q(t) = Q_0\,e^{-0.000120968t},$$

where Q_0 is the initial mass of the radioactive atoms and $Q(t)$ is the mass of radioactive atoms t years after the sample was established.

Assuming that the radioactive atoms have an initial mass of 10 mg ($Q_0 = 10$), what is the mass after 5730 years?

Solution We evaluate

$$Q(5730) \approx 5$$

The radioactive atoms have a mass of approximately 5 mg after 5730 years; in other words, the sample has decayed by half. ■

Another occurrence of *e*

By definition, a linear function is a function with constant slope. Amongst other things, this definition implies that non-linear functions do not have constant slope. That begs the question, just what do we mean when we talk about the slope of a non-linear function?

The slope of non-linear functions is dealt with in calculus, but it boils down to finding the slope of the line that best mimics the direction of motion along the function at the point of interest. These lines are called tangent lines, and the tangent lines at the point $(0, 1)$ are shown for three different exponential functions in Figures 2.27–2.29.

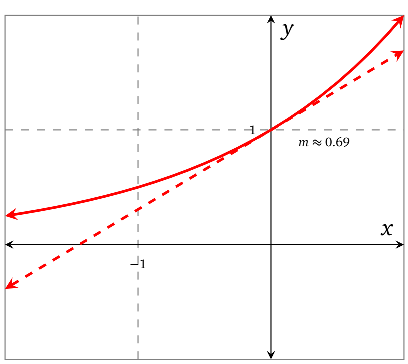


Figure 2.27: $f(x) = 2^x$

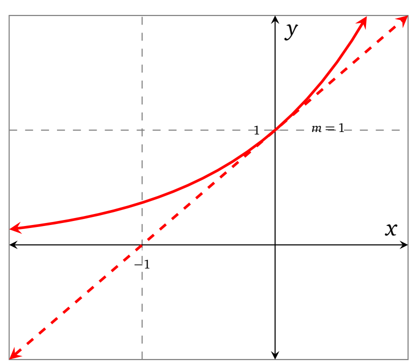


Figure 2.28: $g(x) = e^x$

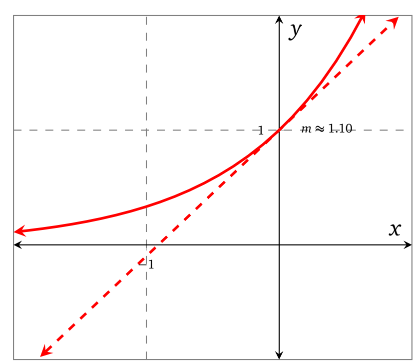


Figure 2.29: $h(x) = 3^x$

It can be proven (using calculus) that at any given point the slope of a function of the form $y = b^x$ is directly proportional to the y-coordinate of the point; the proportionality constant is whatever the slope is at the point $(0, 1)$. That is, for functions of the form $y = b^x$, at any given point (x, y) the slope of the function is ky where k is the slope of the curve at $(0, 1)$. For example, the function of the form $y = b^x$ that has a slope of 4 at the point $(0, 1)$ has a slope of 12 at the point where the y-coordinate is 3 and a slope of 80 at the point where y-coordinate is 20.

One implication of the proportionality between the slope of the function and the y-coordinate of the function is that the function of the form $y = b^x$ that has a slope of one at $(0,1)$ has a very unique property – at any given point the slope of the function is exactly equal to the y-coordinate of the point. As suggested in Figure 2.28, it turns out that the base that creates this unique situation is that same number that showed up in the continuously compounded interest application, the number e. So on the graph of $y = e^x$, at any given point the slope of the curve is exactly equal to the y-coordinate of the point.

Investigations

Problem 3 (Trusting the definition of e)
Definition 13 says that

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Let's see if we can verify this by doing some numerical calculations. Let f be the function that has formula

$$f(x) = \left(1+\frac{1}{x}\right)^x$$

3.1 Evaluate $f(10)$, $f(100)$, $f(1000)$, $f(10000)$, and $f(100000)$. Give each answer correct to 5 decimal places

3.2 Now use your calculator to evaluate e correct to 5 decimal places, and compare your answer to those you obtained in Problem 3.1.

3.3 Why can't we just put $f(\infty)$ into our calculator?

Problem 4 (Compounding continuously)
Imagine that you deposit \$100 into a bank account which accrues interest at a nominal rate of 5 % compounded continuously. The amount $Q(t)$ in the account t years after opening the account is given by $Q(t) = 100\,e^{0.05t}$.

4.1 Find $Q(1)$ correct to two decimal places and interpret the result.

4.2 What is the effective annual growth rate? State your answer to five decimal places.

4.3 Use your calculator to graph

$$Q(t) = 100\,e^{0.05t} \qquad \text{and} \qquad P(t) = 100(1.05127)^t$$

on the same axes. What do you notice?

Exercises

Problem 5 (Given description, write formula)
In each of the following, assume that the population of a town changes at the given rate. Write a formula for $P(t)$, the population at time t, measured in years since 2012.

5.1 $P_0 = 600$, increasing at 7 % per year.

5.2 $P_0 = 1500$, increasing at 7 % continuously per year.

5.3 $P_0 = 2300$, increasing at 27 % per year.

5.4 $P_0 = 3600$, increasing at 52 % continously per year.

5.5 $P_0 = 450$, decreasing at 6 % per year.

5.6 $P_0 = 2405$, decreasing at 12 % continuously per year.

5.7 $P_0 = 4402$, decreasing at 19 % per year.

5.8 $P_0 = 7203$, decreasing at 31 % continuously per year.

Problem 6 (Given formula, write interpretaion)
Each of the following formulas model the population of a town at time t (in years) since 1998. Determine the initial population, P_0, and the percentage change, r, and give a sentence that describes the model.

6.1 $P(t) = 1000(1.1)^t$

6.2 $P(t) = 1000e^{0.11t}$

6.3 $P(t) = 300(0.83)^t$

6.4 $P(t) = 907e^{-0.08t}$

Problem 7 (The number e)
Put the following numbers in ascending order:

$$e, \quad 1/3, \quad 9, \quad e^{-1}, \quad 1, \quad 1/4, \quad e, \quad 3, \quad 1/e^2, \quad 2, \quad e^2$$

Problem 8 (Investing in an account)
You have \$2000 to invest in an account that accrues interest at a nominal rate of 3.75 %. Write a formula for $A(t)$, the amount of money in the account t years after opening the account, assuming that the interest is compounded in each of the following ways. Calculate the effective annual rate in each case.

8.1 Annually.

8.2 Monthly

8.3 Daily

8.4 Continuously

Problem 9 (Solving an equation involving e)
You saved \$7,000 toward the purchase of a car costing \$9,000. How long would the \$7,000 have to be invested in an account that earns 8 % compounded continuously to grow to \$9,000?

Problem 10 (Annual effective rate vs continuous growth rate)
Imagine that you have a bank account with a principal of \$2500 earning an annual *effective* rate of 10 %.

10.1 Write a formula for $A(t)$, the amount of money in the account t years after opening the account.

10.2 If the interest is compounded continuously, use your calculator to find the continuous growth rate to 5 decimal places.

Problem 11 (Newton's Law of Cooling)
You may have noticed that when you leave a cup of hot coffee in a room, the coffee's temperature will decrease to room temperature. Provided that the difference between the temperature of the liquid and its surrounding is not too great, then *Newton's Law of Cooling* applies.

One day you buy a cup of coffee that starts at 90 °C, and you are stood outside where the temperature is 0 °C. Let $T(t)$ represent the temperature of your coffee at time t (in minutes) since you bought it.

11.1 What is $T(0)$?

11.2 Do you expect $T(t)$ to tend toward a value as $t \to \infty$?

11.3 The temperature of the coffee *decreases* at a continuous rate of 7 % per minute. Write a formula for $T(t)$.

11.4 According to your model, what is the temperature of the coffee 10 min after you bought it?

11.5 According to your model, how long after you bought your coffee is the temperature of the coffee 5 °C?

11.6 According to your model, is the temperature of the coffee ever 0 °C?

Problem 12 (The RC circuit)
A *capacitor* is a device that stores electrical energy in the form of charged particles. The *voltage* on the capacitor is a result of the electric field created by the particles and is proportional to the amount of charge stored. A *resistor* is a device that dissipates electrical energy. If a capacitor is charged up and then connected across a resistor, the capacitor discharges and the voltage drops.

The voltage (in V), on the capacitor as it is being discharged is modeled by the function V that has formula

$$V(t) = V_0 e^{-\frac{t}{RC}}$$

where V_0 is the initial capacitor voltage, R is the value of the resistor (in Ω), C is the value of the capacitor (in F) and t is time (in s).

12.1 Suppose that a 2×10^{-6}-farad capacitor, initially charged up to 10 V, is connected across a 50,000-ohm resistor. Write the formula for $V(t)$.

12.2 Construct a table of values of $V(t)$ using $t = 0, 0.1, \dots, 0.5$.

12.3 Based on your answer to Problem 12.2, do you think the graph of $y = V(t)$ will be concave up or concave down? Why?

12.4 Graph the function V.

12.5 By what percentage has the voltage decreased on the capacitor after 0.1 s? After 0.2 s?

Problem 13 (Charging a capacitor)
If a voltage source, such as a battery, is connected to a resistor and a uncharged capacitor, the capacitor will charge up. The voltage (in V), on the capacitor as it is being charged is modeled by the function V that has formula

$$V(t) = V_S\left(1 - e^{-\frac{t}{RC}}\right)$$

where V_S represents the voltage of the source (in V), R is the value of the resistor (in Ω), C is the value of the capacitor (in F) and t is time (in s).

13.1 Suppose that a 10-volt battery is connected to a 2×10^{-6}-farad capacitor and a 50,000-ohm resistor. Write the formula for $V(t)$.

13.2 Construct a table of values of $V(t)$ using $t = 0, 0.1, ..., 0.5$.

13.3 Based on your answer to Problem 13.2, do you think the graph of $y = V(t)$ will be concave up or concave down? Why?

13.4 Graph the function V.

13.5 After 0.1 s, the voltage on the capicitor is what percentage of the battery voltage? After 0.2 s?

13.6 Notice that even though the charge on the capacitor increasing, the formula modeling the voltage contains a decaying exponential. Explain why, mathematically, the function V is increasing even though it contains a decaying exponential.

Problem 14 (Half-life exploration)
In each of the following problems, assume that

$$Q(t) = Q_0\, e^{kt}$$

models the mass of radioactive atoms in a substance (in mg) t years after the sample was established. Use a graphing calculator to approximate the half-life for the given values of Q_0 and k. State your answers correctly to 2 decimal places.

14.1 $Q_0 = 4$, $k = -0.0001$

14.2 $Q_0 = 15$, $k = -0.05$

14.3 Does the half-life depend upon Q_0? Does it depend upon k?

Problem 15 (Continuous growth rate)
For each of the exponential functions defined by the formulas below, find the continuous growth rate and the growth rate.

15.1 $f(t) = e^t$

15.2 $g(t) = e^{0.2t}$

15.3 $h(t) = e^{-0.1t}$

Problem 16 (The function $f(x) = e^x$)
Let f be the function that has formula $f(x) = e^x$.

16.1 Use your calculator to help you construct a table of values for $f(x)$ when x takes all integer values between -3 and 3. Use 5 decimal places for each value of $f(x)$.

16.2 What is the domain of f? What is the range of f? Is f concave up or concave down?

16.3 Now graph the functions g and h that have formulas $g(x) = e^x + 4$ and $h(x) = -e^x$. What are the domain and range of g and h?

16.4 Figure 2.30 shows $y = 2^x$, $y = e^x$, and $y = 3^x$. Match each curve to the appropriate formula.

16.5 Figure 2.31 shows $y = \left(\frac{1}{2}\right)^x$, $y = e^{-x}$, and $y = \left(\frac{1}{3}\right)^x$. Match each curve to the appropriate formula.

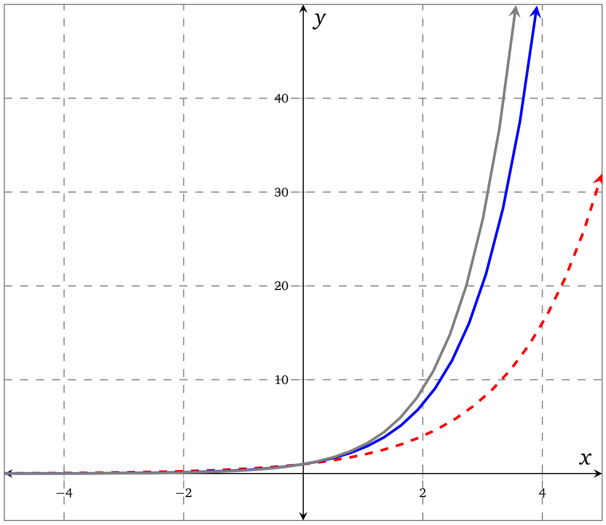


Figure 2.30: Graph for Problem 16.4

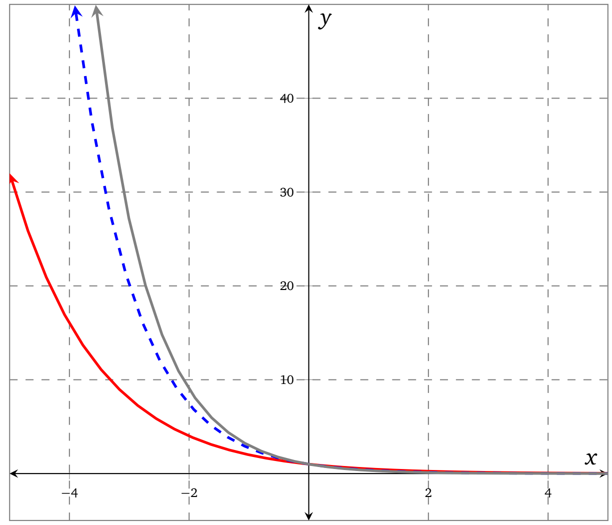


Figure 2.31: Graph for Problem 16.5

Problem 17 (Slopes)
In this problem you are going to explore the proportionality relationship between the slope and y-coordinate of an exponential function.

17.1 In Figure 2.27 we see that the graph of $y = 2^x$ has a slope of about 0.69 at the point $(0,1)$. That means the slope of the curve at the point $(1,2)$ is about 0.69(2) which is roughly 1.4. Lay out your ruler with a slope of 1.4 at the point $(1,2)$ and see that it follows the direction of the curve at that point.

17.2 In Figure 2.29 we see that the slope at $(0,1)$ is about 1.10. What is the slope at the point where the y-coordinate is 2? Verify this slope using your ruler.

Problem 18 (Factorials)
There is a function called the factorial function which is symbolized by an exclamation point. The domain of the function is limited to the non-negative integers. Table 2.27 shows enough values of the function for you to hopefully see the way the function works.

Table 2.27

k	$k!$
0	1
1	1
2	$2 \cdot 1$
3	$3 \cdot 2 \cdot 1$
4	$4 \cdot 3 \cdot 2 \cdot 1$
5	$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
6	$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Table 2.28

Exact value	Decimal value
$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}$	
$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}$	
$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}$	
$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}$	

Find, correct to 5 decimal places, the decimal values of each of the expressions in Table 2.28. What do you observe?

2.6 Comparing linear and exponential functions

Section Themes, Concepts, Issues, Competencies, and Skills:

- *Determine if real-world data establish a linear pattern or an exponential pattern (or neither).*
- *Find a reasonable formula for modeling social data over time.*
- *Understand the long-term differences between investing your money linearly versus exponentially.*

In professions where people work with a lot of data, it is frequently necessary to determine the formula that best fits the data at hand. In statistics, you learn techniques called regressions that allow you to determine the best model for a given set of data. You also learn how to quantify the validity of the model and where and when it is appropriate to apply the model.

essential skills

The following problems contain prerequisite skills that are essential for success. Make sure that you can complete them before moving on!

Problem 1 (Exponential or linear)
Decide if the functions defined by the following formulas are linear or exponential.

1.1 $f(x) = 2^x$

1.2 $g(x) = 2x - 10$

1.3 $k(x) = 10 - 5x$

1.4 $l(x) = -13 \cdot 6^x$

Problem 2 (Sketching linear functions)
Sketch a graph of each of the linear functions defined by the following formulas.

2.1 $f(x) = \frac{1}{2}x + 10$

2.2 $f(x) = 15 - 0.8x$

2.3 $f(x) = \frac{1}{3}(x - 7) + 4$

Problem 3 (Find the slope)
Find the slope between each pair of points.

3.1 $(12, 30)$, $(5, 28)$

3.2 $(2001, 18.4)$, $(2012, 15.3)$

3.3 $(t, 118)$, $(17, y)$

Problem 4
Determine if the data in Tables 2.29–2.31 could reasonably be modeled with a linear function.

Table 2.29

x	y
2005	40
2006	46
2007	52
2008	58
2009	64
2010	70

Table 2.30

x	y
42	1.7
43	1.92
44	2.11
45	2.31
46	2.53
47	2.72

Table 2.31

x	y
0	312
3	330
6	338
9	341
12	340
15	330

There are formal techniques for deciding the proper type of function to use to model data. It can be useful to first think about the type of function that might work best. For example, you might be presented with data where you need to decide whether the data is best modeled by a linear function, an exponential function, or something else. How might you make such a decision?

Example 1 Decide if the function that is described is more like an exponential function, a linear function, or neither.

Every hour, the number of yeast cells in a vat of fermenting wine increases. When there are more cells present, more cells are able to split and reproduce. The number of cells present in the vat is a function of the number of hours the wine has been fermenting.

Solution After the first hour, the cells will reproduce and the number of cells will be larger. After the second hour, there will be *even more* new cells added, since we will have more cells in the first place capable of reproducing. This pattern continues. The population is increasing at a faster and faster rate, so an exponential model may be appropriate. ■

Example 2 Decide if the function that is described is more like an exponential function, a linear function, or neither.

Your house is being repainted, and every hour the workers paint an additional 400 ft^2. The amount that has been painted is a function of the number of hours that have been worked.

Solution After the first hour, the workers paint 400 ft^2. After the second hour they have painted an additional 400 ft^2. The amount of space that has been painted is increasing at a constant rate, so a linear model is appropriate. ■

Suppose that you have some actual data and that you want to determine whether it would be best to model the data using a linear model or an exponential model. How might you make that decision? If the data is truly linear, then when the input values change at a steady pace, the output values also change at a steady pace. This is illustrated in Table 2.32 where the values of x increase by 4 from row to row and the values of y decrease by 9 from row to row.

Table 2.32

x	y
−7	30
−3	21
1	12
5	3
9	−6
13	−15

Table 2.33

x	y
22	3.5
28	7
34	14
40	28
46	56
52	112

Table 2.34

x	y
−8	72
−6	18
−4	4.5
−2	1.125
0	0.281 25
2	0.070 312 5

Table 2.35

x	y
−15	7
−10	2
−5	−3
0	−8
5	−13
10	−18

If the data is truly exponential, then when the input values change at a steady pace, the values of the output change at a constant ratio. This is illustrated in Table 2.33 where the value of x increases by 6 from row to row and the ratio of the successive values of y is always 2.

Example 3 For each of the data sets in Tables 2.34 and 2.35, determine if the data is linear, exponential, or neither. If the data is either linear or exponential, find a formula that models the data.

In Table 2.34, every time the value of x increases by 2, the ratio of the successive y-values is $\frac{1}{4}$. This is indicative of an exponential function.

If $f(x) = a\,b^x$, then from the data points $(-8, 72)$ and $(-6, 18)$ we get $\frac{a\,b^{-6}}{ab^{-8}} = \frac{18}{72}$. So $b^2 = \frac{1}{4}$ which means that the base of the function b must be $\frac{1}{2}$ (since the base cannot be negative). Using the data point $(0, 0.28125)$ along with our newly discovered base we find $a(0.5)^0 = 0.28125$, so $a = 0.28125$.

In conclusion, the data in Table 2.34 is modeled by the exponential function f where $f(x) = 0.28125\left(\frac{1}{2}\right)^x$.

In Table 2.35, every time the value of x increase by 5, the value of y decreases by 5. This is the behavior of a linear function with a slope of -1. Using either the slope-intercept form of a linear equation or the point-slope form of a linear equation, we can deduce that the data is modeled by the linear function g where $g(x) = -x - 8$. ■

In nature, data sets are almost never exactly linear nor exactly exponential. When dealing with data one might need to decide which model *best* fits the data, linear or exponential.

Example 4 Is the data in Table 2.36 better modeled with a linear function or an exponential function?

Table 2.36

x	$f(x)$
10	8
11	12.01
12	16.02
13	20
14	24

Solution Let's compute the successive differences and the successive ratios:

$$\begin{aligned} 12.01 - 8 &= 4.01 & 12.01/8 &\approx 1.50 \\ 16.02 - 12.01 &= 4.01 & 16.02/12.01 &\approx 1.33 \\ 20 - 16.01 &= 3.98 & 10/16.01 &\approx 1.24 \\ 24 - 20 &= 4 & 24/20 &= 1.2 \end{aligned}$$

The differences are not constant, but they are all fairly close to having the constant value of 4. The ratios however are not close to being constant. It might be appropriate to model this data with a linear function, but it would not be appropriate to model it with an exponential function. ■

★ try it yourself ★

Problem 5
Determine if each of the data sets in Tables 2.37–2.40 suggest an exponential relationship, a linear relationship, or neither.

Table 2.37

x	y
1	2
2	4
3	6
4	8
5	10
6	12

Table 2.38

x	y
2005	1
2006	3
2007	9
2008	27
2009	81
2010	243

Table 2.39

x	y
1950	17.3
1960	19.6
1970	21.9
1980	24.1
1990	26.4
2000	28.7

Table 2.40

x	y
0	546
4	502
8	462
12	425
16	391
20	360

make sure you try it!

Our examples in this section so far have not had any context; they have just been tables of numbers. As a student of the natural and social sciences, you will encounter data sets that come from interesting sources. Understanding how to model these data sets can help you understand that subject better.

Example 5 A study of reproductive health care found the data in Table 2.41 concerning the percentage of births in the U.S.A. that were delivered via a Cæsarian section.

Can we model this data with a linear or exponential function?

Solution It is always a good idea to plot data like this. Some patterns might be quickly evident from a graph that are not so quickly evident numerically.

Figure 2.32 shows us that there is a clear upward trend in Cæsarian section deliveries. Is the trend linear, exponential, or neither?

You might be able to see a slight upward bend to the plot, suggesting an exponential growth pattern. Then again, your eyes might see these points as lying in a relatively straight line, suggesting a linear growth pattern. Let's look at successive ratios and differences in Table 2.42.

In Table 2.42, we see successive differences that become larger and larger. This tells us that a linear model would not be appropriate for the Cæsarian section data. On the other hand, the successive ratios all bounce around close to 1.074. This suggests that an exponential model would be appropriate.

Table 2.41: US Cæsarian Section Data

Year	C-sections (%)
1996	21.0
1998	22.6
2000	24.2
2002	26.0
2004	27.9
2006	30.0
2008	32.2

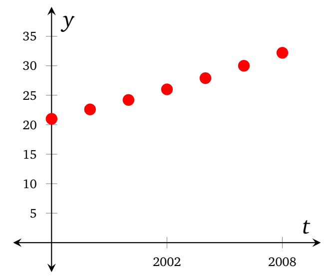


Figure 2.32: US Cæsarian section data

Table 2.42: US Cæsarian Section Data

Differences	Ratios
1.6	1.076...
1.6	1.071...
1.8	1.074...
1.9	1.073...
2.1	1.075...
2.2	1.073...

If we have decided that an exponential model is appropriate, can we explicitly write down a model?

Solution We have many data points, but we only require two in order to determine a formula for an exponential function. And it's likely that different choices of points will lead to slightly different models. In an intermediate statistics course, students learn how to deal with this issue. For now, we will use the following rule of thumb: use data points that are a little inward from the edges. In the current example, we will use the data from 1998 and 2006.

Since we typically wish to associate $t = 0$ to a year where the data was relevant, we will identify $t = 0$ with the year 2000. Now if $f(t) = a\,b^t$, then the data from 1998 and 2006 tell us that

$$\begin{cases} 22.6 = a\,b^{-2} \\ 30.0 = a\,b^{6} \end{cases}$$

We can eliminate a by equating the quotients formed by the two sides of the equations.

$$\frac{30.0}{22.6} = \frac{a\,b^6}{a\,b^{-2}} \qquad \Longrightarrow \qquad b = \left(\frac{30.0}{22.6}\right)^{1/8} \approx 1.036$$

The approximation of b by 1.036 is particularly valid in an application such as this, where we know that other choices of data points would have given different values of b anyway.

Solving for a in the equation for 2006:

$$30.0 = a\left(\frac{30.0}{22.6}\right)^{6/8} \qquad \Longrightarrow \qquad a = \frac{30}{\left(\frac{30.0}{22.6}\right)^{3/4}} \approx 24.26$$

Therefore a model for the Cæsarian section data is

$$\begin{aligned} f(t) &= 30^{1/4}22.6^{3/4}\left(\frac{30.0}{22.6}\right)^{t/8} \\ &= 30^{1/4}22.6^{3/4}\left(\left(\frac{30.0}{22.6}\right)^{1/8}\right)^{t} \\ &\approx 24.26(1.036)^t \end{aligned}$$

and it is acceptable to say that the percentage of C-section births is growing exponentially in the U.S.A. For completeness, we can examine a graph of this model overlaying the data in Figure 2.33. ■

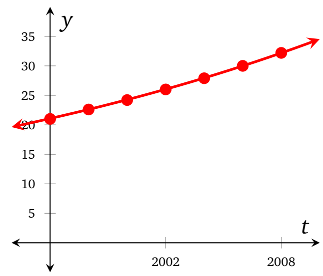


Figure 2.33: US Cæsarian section model

We have so far modeled data using exponential and linear functions; this has helped us determine how the two classes of function behave differently in the short term. You may wonder how the two classes of function behave differently in the long term.

Example 6 Consider the functions f and g that have formulas

$$f(x) = 4x + 1, \qquad g(x) = 4^x$$

Describe the behavior of both functions as $x \to \infty$ and as $x \to -\infty$.

Solution The functions f and g are graphed in Figure 2.34. Note that both functions grow without bound as $x \to \infty$. Another way to express this is to say

$$f(x) \to \infty \qquad \text{and} \qquad g(x) \to \infty$$

as $x \to \infty$.

However, even though both functions grow without bound, the exponential function g does so at a *much faster rate*.

The behavior of the functions as $x \to -\infty$ is quite different. Note that g has a horizontal asymptote ($y = 0$) and that f does not; in fact $f(x) \to -\infty$ as $x \to -\infty$. ■

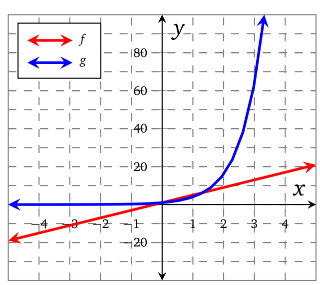


Figure 2.34

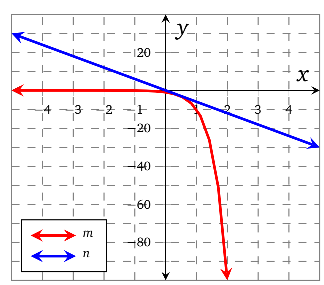


Figure 2.35

Example 7 Now consider the functions m and n that have formulas

$$m(x) = -10^x, \qquad n(x) = -6x$$

Describe the behavior of both functions as $x \to \infty$ and as $x \to -\infty$.

Solution The functions m and n are both graphed in Figure 2.35. Note that both functions decrease without bound as $x \to \infty$. However, m does so at *a much faster rate*.

As in Example 6, the behavior of m and n as $x \to -\infty$ is different. Notice that m has a horizontal asymptote ($y = 0$) and that n does not. In this case $n(x) \to \infty$ as $x \to -\infty$. ■

Investigations

Problem 6 (Cell phones)
A cell phone company is studying the number of cell phone subscriptions in the U.S.A. In 2003, there were 158 million cell phone users. In 2006, there were 233 million cell phone users.[13]

6.1 Write two ordered pairs suggested by the information above. Then find the slope between the two points and *interpret* its meaning including units. (Write a complete sentence that explains the meaning of the slope that you have calculated, without using the word 'slope'.)

6.2 Assuming that the relationship between years and cell phone users is linear, find a formula that determines the number of cell phone users as a function of the number of years since 2000.

6.3 Assuming that the relationship between years and cell phone users is exponential, find a formula that determines the number of cell phone users as a function of the number of years since 2000.

6.4 According to your models in Problems 6.2 and 6.3, how many cell phone users were there in 2008? In 2008 there were approximately 263 million cell phone users in the United States. Which of the two models does a better job of predicting this number?

Problem 7 (Wind power)
The wind energy production capacity (in MW) for the world[14] is shown in Figure 2.36.

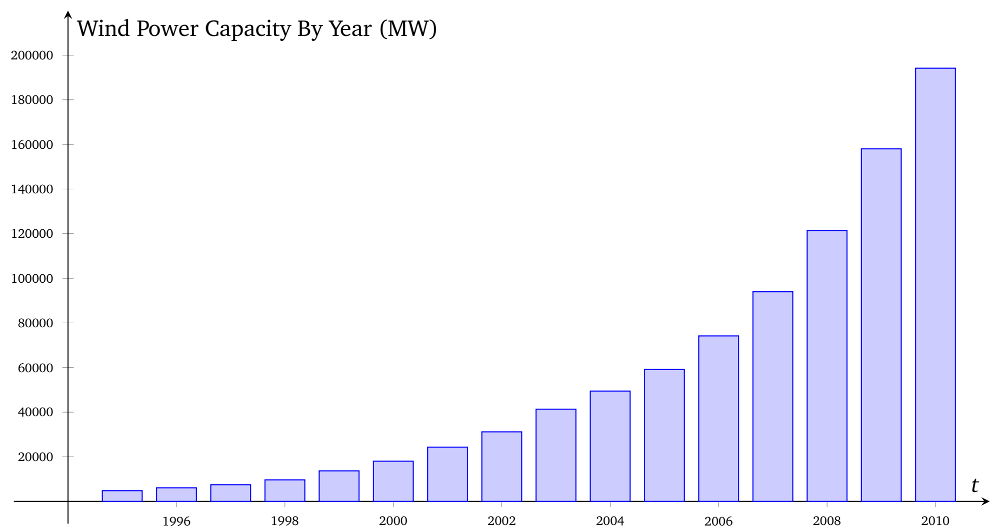


Figure 2.36: Wind power.

7.1 Which would be more appropriate to model this data, an exponential function or a linear function?

7.2 Let $W(t)$ represent the wind energy production (in MW) at time t in years since 1995. In 1995 the wind power capacity was 4800 MW and in 2010 the wind power capacity was 194 154 MW. Using two ordered pairs, $(0, 4800)$ and $(15, 194154)$, and assuming that an exponential model is appropriate, we can show that a formula that approximates W is

$$\begin{aligned} W(t) &= 4800\left(\frac{194154}{4800}\right)^{t/15} \\ &= 4800\left(\left(\frac{194154}{4800}\right)^{1/15}\right)^{t} \\ &\approx 4800(1.279756)^{t} \end{aligned}$$

Using a table of values or a graph, find when the world's wind power capacity will be 250 000 MW.

[13] CTIA - The Wireless Association
[14] http://www.thewindpower.net/statistics_world.php

Problem 8 (Population of Africa)
Table 2.43 gives the population of Africa for each year form 1999 to 2009.[15]

Table 2.43: Human Population of Africa

Year	Millions
1999	800.2
2000	819.5
2001	839.0
2002	858.9
2003	879.2
2004	899.9
2005	921.1
2006	942.7
2007	964.7
2008	987.1
2009	1009.9

8.1 Give a good reason why it would be appropriate to model this data with an exponential function.

8.2 Find an explicit exponential model for the population of Africa as a function of time. That is, find an exponential function P where $P(t) = a\,b^t$ such that $P(t)$ approximates the population of Africa at time t. For convenience, take $t = 0$ to mean the year 2000, with time measured in years.

8.3 Use the model that you found in Problem 8.2 to estimate the population of Africa in the year 2020.

8.4 Use a graph of your model to estimate when the population of Africa might reach 1.5 billion people. (1.5 billion is 1500 million.)

8.5 How good is your model? Use your model to add a third column to Table 2.43 that displays the model's predicted population. Discuss the accuracy of the model.

Problem 9 (Tortoise and the Hare)
Aesop's fable of the Tortoise and the Hare depicts an unlikely race between the two animals. The Hare, being known as a quick and lively animal, brims with confidence. The Tortoise, who is known as a slow and more methodical creature is a little nervous. We are going to model a version of this fable.

The Tortoise and the Hare are going to race over 1000 m. The Hare boasts that he can run at 20 m/s for as long as he likes. The Tortoise doesn't know how fast he can run, but he says that his distance from the their starting line obeys the rule $y = 2^t$, where t is the time (in seconds) since they begin the race; he does ask the Hare if he can have a 1 m head start. The Hare laughs at the Tortoise and says, 'Fine by me!'

9.1 Let $H(t)$ represent the Hare's distance from the starting line (in meters) t seconds after the race begins. Write a formula for H.

9.2 Let $T(t)$ represent the Tortoise's distance from the starting line (in meters) t seconds after the race begins. Write a formula for T.

9.3 Graph H and T on the same axis for t in $[0, 10]$. Who wins the race?

Unbeknown to both competitors, the Hare's older sister has been watching the race. She approaches the Tortoise and says that if she had been racing, she would easily have won because she can run at 40 m/s for as long as she likes.

9.4 Let $S(t)$ represent the Hare's sister's distance from the starting line (in meters) t seconds after the race begins. Write a formula for S.

9.5 Let's assume that all 3 animals race together. Graph H, T, and S on the same axis for t in $[0, 10]$. Who wins the race?

[15] http://www.earth-policy.org/books/wote

Exercises

Problem 10 (Linear or exponential?)
Decide if the functions defined by the following formulas are linear or exponential.

10.1 $B(y) = -10 \cdot 17^y$ **10.2** $A(t) = 4t + 10$ **10.3** $C(\alpha) = 5\alpha - \pi$ **10.4** $D(z) = \pi^z$

Problem 11 (Linear, exponential, or neither?)
Decide if the function that is described is more like an exponential function, a linear function, or neither.

11.1 The number of plants that germinate is a function of the number of seeds sown. For a particular crop of peas, exactly 70 % of the seeds sown germinate.

11.2 The number of tulip bulbs in a garden is a function of the number of years since first planting. Every year, each of the tulip bulbs divides into two.

11.3 The amount of money in Ross' bank account is a function of the number of years since the account was opened. Every year the amount of money in Ross' account increases by $5000.

11.4 The amount of money in Serena's account is a function of the number of years since she opened the account. Every year the amount of money Serena's account increases by 2 % of the amount that was in the account the year before.

11.5 The number of pages you still have to read in a book is a function of the page number you're on.

Problem 12 (Linear or exponential?)
Consider the data sets in Tables 2.44–2.50.

12.1 For each data set in Tables 2.44–2.47, state whether a linear or exponential function (or neither) would better model the data. If a linear or exponential model is appropriate, find an exact formula for the model.

Table 2.44

x	$f(x)$
-1	-4.5
0	-3.0
1	-1.5
2	0.0
3	1.5

Table 2.45

x	$g(x)$
-1	64
0	16
1	4
2	1
3	$1/4$

Table 2.46

x	h(x)
-1	2.000
0	3.000
1	4.500
2	6.750
3	10.125

Table 2.47

x	$k(x)$
-1	20
0	16
1	12
2	8
3	4

Table 2.48

x	y
4	7.32
5	8.05
6	8.86
7	9.74
8	10.72

Table 2.49

x	y
2	4.27
3	5.4
4	6.52
5	7.63
6	8.75

Table 2.50

x	y
5	20
6	30
7	42
8	56
9	72

12.2 For each data set in Tables 2.48–2.50, state whether a linear or exponential function (or neither) would better model the data. If a linear or exponential model is appropriate, find an approximate formula for the model.

Problem 13 (Which is greater?)
Let f and g be functions that have the formulas

$$f(x) = 1000x + 1 \times 10^6, \qquad g(x) = 2^x$$

13.1 Evaluate $f(1)$ and $g(1)$. Which is greater?

13.2 Evaluate $f(10)$ and $g(10)$. Which is greater?

13.3 Do you think that that $g(x)$ will ever be greater than $f(x)$?

13.4 Evaluate $f(20)$ and $g(20)$. Does this change your answer to Problem 13.3?

Problem 14
Consider the ordered pairs $(3,10)$ and $(7,15)$.

14.1 Find the formula for the linear function, f, that goes through the ordered pairs.

14.2 Find the formula for the exponential function, g, that goes through the ordered pairs.

14.3 What is the first integer value of x that makes $g(x) > f(x)$?

Problem 15 (True or false)
Answer the following questions as True or False; if you believe the answer to be False, provide justification that supports your answer.

15.1 Linear functions are concave down.

15.2 Linear functions are concave up.

15.3 It is possible to write a linear function that has a slope of 2.

15.4 It is possible to write an exponential function that has a slope of 2.

15.5 There is an exponential function that decreases at a constant rate of 5.

Problem 16 (Classify that function!)
Carlos and Anita are playing a game the call, 'Classify that function!' One of them describes how to plot the points or features of the graph, and the other has to say if it is linear or exponential. Help them decide if the following describe linear or exponential functions.

16.1 Over 2 up 3, over 2 up 3, over 2 up 3, ...

16.2 Start negative. Over 1, 5 times farther down, over 1, 5 times farther down, over 1, 5 times farther down, ...

16.3 Left 5 up 1, left 10 up 2, left 15 up 3, left 20 up 4, ...

16.4 A straight line that goes through the points $(0,0)$ and $(20,19)$.

16.5 A function that is concave up, and has a horizontal asymptote of $y = 0$ as $x \to -\infty$.

Problem 17 (Matching stories with formulas)
Match each of the following formulas with one of the given statements. Note that y and x have deliberately been used in each formula to avoid any extra hints; you will also notice that there are more choices than questions so you will not be able to use all choices.

(i) $y = 2\pi^x$
(ii) $y = 2\pi x$
(iii) $y = 2^x$
(iv) $y = 2x$
(v) $y = \frac{5}{9}(x - 32)$
(vi) $y = \frac{5}{9}x - 32$
(vii) $y = \frac{9}{5}x + 32$
(viii) $y = 100(0.9)^x$
(ix) $y = 100 - 10x$

17.1 To convert from Fahrenheit to Celsius, subtract 32 and then multiply by $5/9$.

17.2 A population starts with 100 people, and decreases by 10% per year.

17.3 The circumference of a circle is calculated by multiplying the radius by 2π.

17.4 What is the biggest city in the World? Dublin, because it keeps on doublin' and doublin', and...

17.5 To convert from Celsius to Fahrenheit, multiply by $9/5$ and then add 32.

17.6 A population starts with 100 people and decreases by 10 people per year.

Problem 18
Did you complete Problem 8 (from Section 2.3) about the Tapfish app? If you did, do the values of $F(t)$ suggest that an exponential model might be appropriate? Use successive ratios to decide.

Problem 19 (Long-run behavior of linear and exponential functions)
We are going to explore long-run behavior of exponential functions that have base less than one ($b < 1$).

19.1 Let f and n be the functions that have formulas

$$f(x) = \left(\frac{1}{4}\right)^x, \qquad n(x) = -3x$$

which are shown in Figure 2.37. Describe the behavior of f and n as $x \to \infty$ and $x \to -\infty$.

19.2 Let g and m be the functions that have formulas

$$g(x) = -\left(\frac{1}{5}\right)^x, \qquad m(x) = 6x$$

which are shown in Figure 2.38. Describe the behavior of g and m as $x \to \infty$ and $x \to -\infty$.

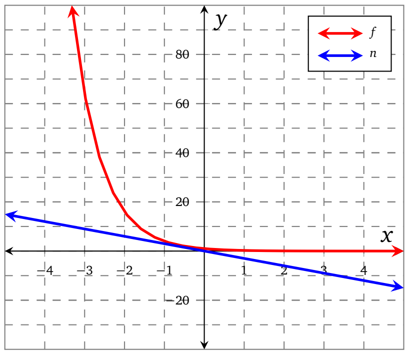


Figure 2.37: f and n

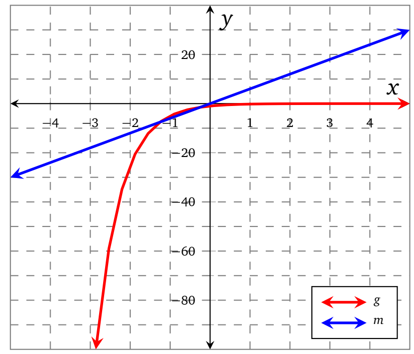
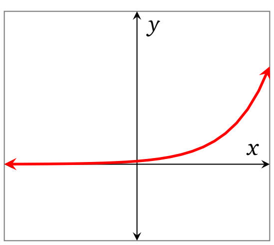


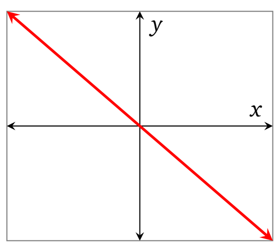
Figure 2.38: g and m

Problem 20 (Match formulas to graphs)
Match each of the following formulas with one of the graphs in Figure 2.39. Note that axis ticks and scaling have deliberately been omitted to encourage you to think about long-run behavior.

(i) $y = -4^x$	(iii) $y = -\pi x$	(v) $y = 10x + 2$	(vii) $y = x$
(ii) $y = \left(\frac{1}{2}\right)^x$	(iv) $y = 4$	(vi) $y = 10^x$	(viii) $y = -\left(\frac{1}{2}\right)^x$



(a)



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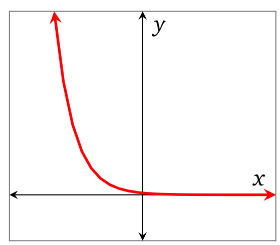

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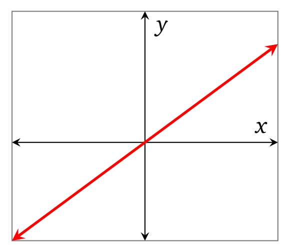

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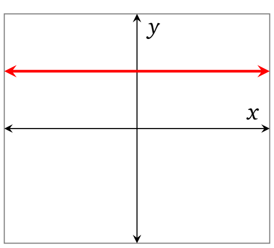

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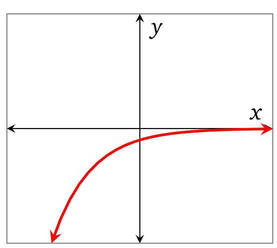

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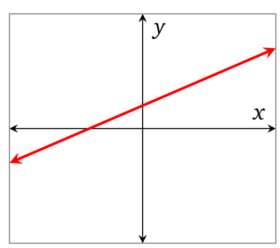
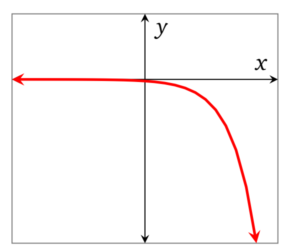


(g)



(h)

Figure 2.39

2.7 Extensions

Section Themes, Concepts, Issues, Competencies, and Skills:

- *Investigate the Logistic model for population growth and decay.*
- *Revisit composition and piecewise defined functions.*

Core problems in this section (★): 5.1, 5.2, 5.3

A more realistic population model

We considered population models in Section 2.4. Each model had the form

$$P(t) = a\,b^t$$

and therefore implied that as $t \to \infty$, the population either decays to zero, or grows without bound.

Intuitively, these models are unrealistic. A decreasing population does not necessarily decay to zero, and an increasing population encounters limitations on food and other resources that will prevent it from growing without bound. The Logistic Model takes resource limitations into account.

Definition 15 (The Logistic Model)
According to the Logistic Model, the population, $P(t)$, t years after the population started its logistic growth is given by the formula

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kt}}$$

where

- P_0 is the initial population: $P_0 = P(0)$;
- M is the *carrying capacity*, the maximum population that can be supported by the available resources;
- k approximates the relative growth rate when the population is small, relative to the carrying capacity, interpreted as a percent per year.

Example 1 Let $P(t)$ represent the population of a country at time t (in years) since 2000. Use Definition 15, with $k = 0.08$ and $M = 1000$, to study the effect of changing P_0 from 100 to 1400. What happens to P as $t \to \infty$?

Solution Using $k = 0.08$ and $M = 1000$ in Definition 15, we have

$$P(t) = \frac{1000P_0}{P_0 + (1000 - P_0)e^{-0.08t}}$$

If we put $P_0 = 100$ and then $P_0 = 1400$ then we have, respectively,

$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}, \qquad P(t) = \frac{7000}{7 - 2e^{-0.08t}}$$

graphs of which are shown in Figures 2.40 and 2.41.

Notice that in both models, $P(t) \to 1000$ as $t \to \infty$. Remember that we called M the carrying capacity in Definition 15, which represents the maximum population that the environment can support.

In contrast to the models presented in Section 2.4, the populations neither grow without bound nor decay toward zero. ■

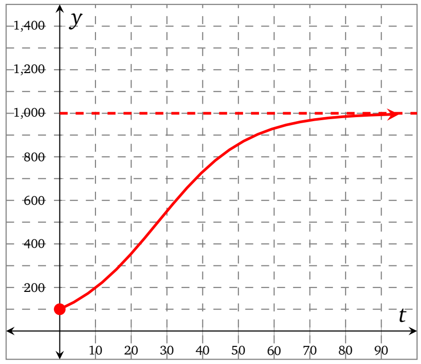


Figure 2.40: $P(t) = \dfrac{1000}{1+9e^{-0.08t}}$

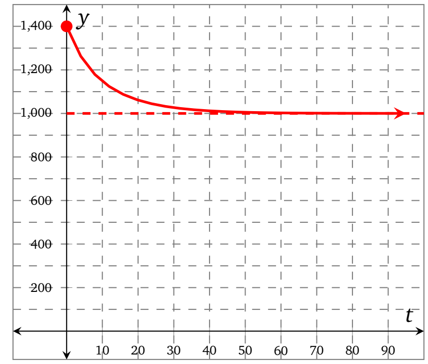


Figure 2.41: $P(t) = \dfrac{7000}{7-2e^{-0.08t}}$

Exercises

Problem 1 (Logistic model)
Use Definition 15 and Example 1 to guide you in this problem. Assume that $k = 0.05$, $P_0 = 100$ and $M = 800$.

1.1 Find a logistic population model, $P(t)$.

1.2 Find $P(1)$ and determine the relative growth over the first year. How does this compare to k?

1.3 Graph your function P on your calculator and describe its behavior as $t \to \infty$.

1.4 Is your function P increasing or decreasing?

Problem 2 (Rats!)
In the 1800s, a ship landed on a remote island and 35 black rats (*Rattus rattus*) escaped to colonize the island.

2.1 If the initial relative growth rate was about 90 % per year and the island has a carrying capacity of 20,000 rats, find a formula for $P(t)$, the number of rats on the island at time t, assuming logistic growth.

2.2 Use a graphing calculator to determine how long will it take for the population to reach 15,000 rats.

Problem 3 (The invasive blackberry)
Himalayan blackberry is an invasive species. Some seeds found their way into a remote valley and grew into 8 kg of blackberry biomass by the next year. Suppose that the initial relative growth rate of blackberry biomass was 350 % per year and that the valley has a carrying capacity of 50 000 kg.

3.1 If $t = 0$ corresponds to the time when there was 8 kg of biomass, find a formula for the amount of blackberry biomass in the valley after t years, assuming logistic growth.

3.2 Use a graphing calculator to determine how long will it take for the blackberry biomass to reach half of the valley's carrying capacity.

Problem 4

4.1 Suppose that P is a decreasing logistic function with $k = 0.05$ and $P(t) \to 800$ as $t \to \infty$. Find a formula for P and graph the function on your calculator.

4.2 What does your formula become if $P_0 = 800$? Is P increasing or decreasing in this case?

Problem 5 (Composition)
Let f and g be functions that have formulas $f(x) = 2^x$ and $g(x) = 3^x$. Find each of the following.

★ **5.1** $(f \circ g)(1)$

★ **5.2** $(g \circ f)(2)$

★ **5.3** $(f \circ g)(0)$

5.4 $(g \circ f)(0)$

5.5 $(f \circ g)(x)$

5.6 $(g \circ f)(x)$

Problem 6 (Decomposition)
In each of the following problems, you are given a formula for function h. Decompose h into two functions f and g such that $h = f \circ g$.

6.1 $h(x) = 2^{x^2}$ **6.2** $h(x) = -4^{x^3+2x}$ **6.3** $h(x) = 2^{x^2} + 3^{x^2}$ **6.4** $h(x) = e^{-x^2+2}$

Problem 7 (Inverse function evaluation)
The function f that has formula $f(x) = 2^x$ is invertible. Evaluate each of the following.

7.1 $f^{-1}(8)$ **7.2** $f^{-1}(16)$ **7.3** $f^{-1}\left(\frac{1}{4}\right)$ **7.4** $f^{-1}(1)$

Problem 8 (Piecewise functions)
Let k be the function that has formula

$$k(t) = \begin{cases} 2^t, & t < -5 \\ -10, & -5 \le t < 3 \\ 6t, & 3 < t < 7 \\ -4^t, & t > 7 \end{cases}$$

Evaluate each of the following, and leave your answers in exact form.

8.1 $k(-6)$ **8.3** $k(0)$ **8.5** $k(5)$ **8.7** $k(7)$

8.2 $k(-4)$ **8.4** $k(2.99)$ **8.6** $k(6)$ **8.8** $k(8)$

Problem 9 (Function algebra)
Let f and g be the exponential functions that have formulas

$$f(x) = 3^x, \qquad g(x) = \left(\frac{1}{4}\right)^x$$

Evaluate each of the following (if possible).

9.1 $(f+g)(0)$ **9.2** $(f-g)(2)$ **9.3** $(f \cdot g)(-2)$ **9.4** $\left(\frac{f}{g}\right)(1)$

Problem 10 (Transformations: given the transformation, find the formula)
Let f be the exponential function that has formula $f(x) = 7^x$. In each of the following problems apply the given transformation to the function f and write a formula for the transformed version of f.

10.1 Shift f to the right by 2 units. **10.4** Shift f down by 1 unit.

10.2 Shift f to the left by 5 units. **10.5** Reflect f over the horizontal axis.

10.3 Shift f up by 11 units. **10.6** Reflect f over the vertical axis.

Problem 11 (Transformations: given the formula, describe the transformation)
Describe each of the functions defined by the following formulas in terms of transformations of the exponential function f that has formula $f(x) = \left(\frac{2}{3}\right)^x$.

11.1 $g(x) = \left(\frac{2}{3}\right)^{x+7}$ **11.2** $h(x) = \left(\frac{2}{3}\right)^{x-13}$ **11.3** $j(x) = \left(\frac{2}{3}\right)^{2(x+9)}$ **11.4** $k(x) = 7\left(\frac{2}{3}\right)^{-x}$

FUNCTIONS 3

Contents

3.1 Function algebra

Definition 16 (Function algebra)
Given two functions f and g, we may combine the two functions to form new functions

$$f+g, \qquad f-g, \qquad f\cdot g, \qquad \frac{f}{g}$$

The formula for each function can be found using

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (f\cdot g)(x) &= f(x)\cdot g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\end{aligned}$$

The domain of each of the functions $f+g$, $f-g$, and $f\cdot g$ is

$$(\text{domain of } f)\cap(\text{domain of } g)$$

The domain of the function $\frac{f}{g}$ is

$$(\text{domain of } f)\cap(\text{domain of } g \text{ such that } g(x)\neq 0)$$

Example 1 – Function algebra domain: In each of the following cases you are given the formulas for two functions f and g. In each case, find the domain of $f+g$ and $\frac{f}{g}$.

(a) $f(x)=\sqrt{x}$, $g(x)=\sqrt{1-x}$

(b) $f(x)=\sqrt{x-1}$, $g(x)=\sqrt{1-x}$

(c) $f(x)=\dfrac{1}{x+3}$, $g(x)=\sqrt{x+5}$

Solution

(a) The domain of f is $[0,\infty)$ and the domain of g is $(-\infty,1]$. Therefore, the domain of the function $f+g$ is

$$[0,\infty)\cap(-\infty,1]=[0,1]$$

The domain of the function $\frac{f}{g}$ is found in a similar way, except we must have the additional condition that $g(x)\neq 0$; we therefore must exclude 1 from the domain. The domain of $\frac{f}{g}$ is therefore

$$[0,1)$$

(b) The domain of f is $[1,\infty)$ and the domain of g is $(-\infty,1]$. Therefore the domain of $f+g$ is

$$[1,\infty)\cap(-\infty,1]=\{1\}$$

The domain of the function $\frac{f}{g}$ is found in a simlar way, but we must exclude all values of x that make $g(x)=0$. Since $g(1)=0$ we must exclude 1 from the domain of $\frac{f}{g}$; we therefore conclude that the domain of $\frac{f}{g}$ is the empty set, $\emptyset$.

(c) The domain of f is $(-\infty,-3)\cup(-3,\infty)$ and the domain of g is $[-5,\infty]$. The domain of $f+g$ is therefore

$$((-\infty,-3)\cup(-3,\infty))\cap[-5,\infty)=[-5,-3)\cup(-3,\infty)$$

We must exclude -5 from the domain of $\frac{f}{g}$ since $g(-5)=0$; the domain of $\frac{f}{g}$ is

$$(-5,-3)\cup(-3,\infty)$$

■

Exercises

Problem 1 (Function algebra using formulas)
In each of the following problems you are given formulas for functions f and g. Find the domain of $f \cdot g$ and $\frac{f}{g}$ in each case.

1.1 $f(x) = x$, $g(x) = x^2 + 1$

1.2 $f(x) = 3x + 2$, $g(x) = \sqrt{x}$

1.3 $f(x) = \sqrt[4]{x-1}$, $g(x) = x^2 + 5x + 4$

1.4 $f(x) = \sqrt[5]{x}$, $g(x) = x^2 - 9x - 10$

Problem 2 (Function algebra numerically)
Values of the functions f, g, h, and j are shown in Tables 3.1a–3.1d

Table 3.1: Tables for Problem 2

(a) $y = f(x)$

x	y
−4	−56
−3	−18
−2	0
−1	4
0	0
1	−6
2	−8
3	0
4	24

(b) $y = g(x)$

x	y
−4	−16
−3	−3
−2	0
−1	−1
0	0
1	9
2	32
3	75
4	144

(c) $y = h(x)$

x	y
−4	2
−3	4
−2	6
−1	8
0	10
1	12
2	14
3	16
4	18

(d) $y = j(x)$

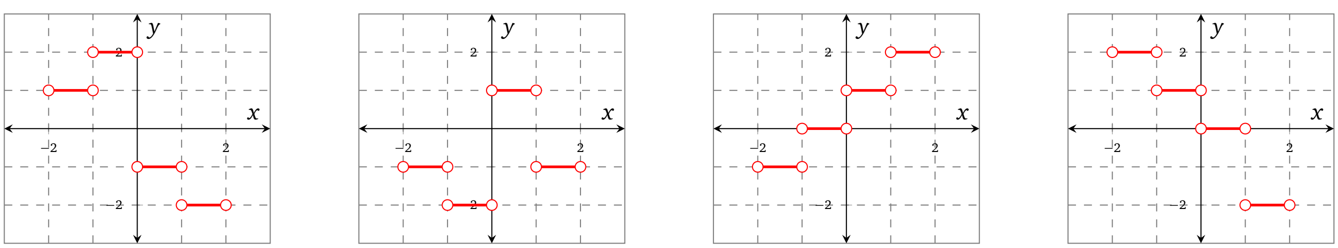
x	y
−4	30
−3	21
−2	12
−1	3
0	−6
1	−15
2	15
3	96
4	760

Construct a table of values for each of the following functions, marking with an X any that undefined.

2.1 $f + g$ **2.2** $f - g$ **2.3** $g \cdot h$ **2.4** $h + j$ **2.5** $\left(\frac{j}{h}\right)$ **2.6** $\left(\frac{j}{f}\right)$

Problem 3 (Function algebra graphically)
Consider the functions F, G, H, and J that have been graped in Figure 3.1. Use the graphs to plot each of the following functions.

3.1 $F + G$ **3.2** $G \cdot H$ **3.3** $\frac{H}{J}$ **3.4** $J - F$



(a) $y = F(x)$ (b) $y = G(x)$ (c) $y = H(x)$ (d) $y = J(x)$

Figure 3.1

Problem 4 (Function algebra numerically)
Table 3.2 shows some values of the functions f, g, and some functions obtained by using some function algebra on f and g. Use the given values to complete Table 3.2.

Problem 5 (Function algebra graphically)
Consider the functions α, β, γ, and δ which have been graphed in Figure 3.2. Evaluate each of the following.

Table 3.2

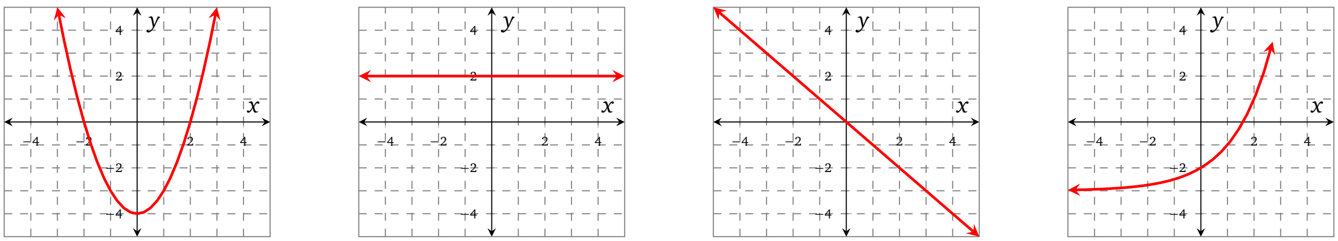
x	-6	-4	-2	0	2	4	6
$f(x)$	2		3			π	
$g(x)$	8	1		5		-1	
$(f+g)(x)$		2			1		
$(f-g)(x)$					3		10
$(f\cdot g)(x)$				0			
$\left(\frac{f}{g}\right)(x)$			1				6

5.1 $(\alpha+\beta)(0)$

5.2 $(\beta-\gamma)(3)$

5.3 $(\gamma\cdot\delta)(2)$

5.4 $\left(\frac{\delta}{\alpha}\right)(0)$



(a) $y=\alpha(x)$ (b) $y=\beta(x)$ (c) $y=\gamma(x)$ (d) $y=\delta(x)$

Figure 3.2

3.2 Piecewise-defined functions

The functions that we have considered so far have had just one formula throughout their domain; for example, the quadratic function q that has formula

$$q(x) = 5 - 3x^2$$

is defined for all real numbers.

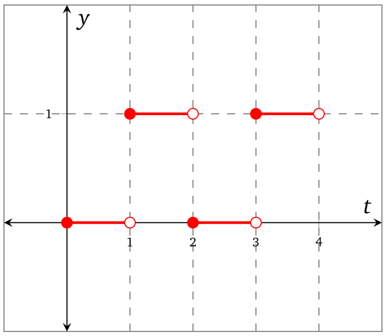


Figure 3.3: A switch function

There are many applications for which this is too restrictive; for example, electrical engineers often work with switches that are turned on (with a value of 1) and off (with a value of 0). An example of a function that might model such a switch over time, t, is shown in Figure 3.3. It is clear that this function takes the value 0 on some intervals, and 1 on other intervals. We can write a formula for such a function by first noting that is a *piecewise-defined* function.

Definition 17 (Piecewise-defined functions)
A piecewise-defined function has different formulas for different parts of its domain.

The formula for a piecewise-defined function is written using a *left brace* $\{$ and is read from top to bottom as we move from left to right through its domain.

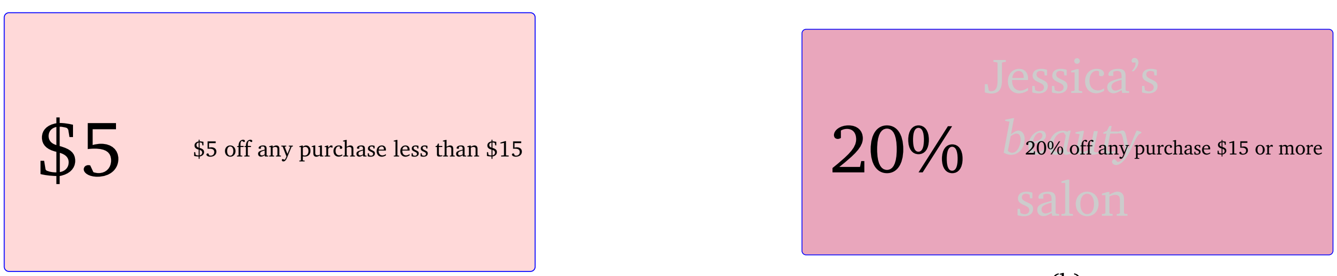
Example 1 Find a formula for the function that is graphed in Figure 3.3.

Solution Let's assume that the function shown in Figure 3.3 is called f. It seems that $f(t)$ takes the value 0 on the intervals $[0,1)$ and $[2,3)$; similarly, $f(t)$ takes the value 1 on the intervals $[1,2)$ and $[3,4)$. We can translate this into a formula for the function f as follows

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & 2 \le t < 3 \\ 1, & 3 \le t < 4 \end{cases}$$

Note that we use the left brace, $\{$, to link the formula together. Note also that the domain of f is $[0,4)$ and that as we read the formula from top to bottom, the values of t go from left to right. This will be true in every piecewise-defined function that we see. ■

Example 2 – Coupons: Wendy is going shopping at Jessica's beauty salon. Wendy has the coupons shown in Figure 3.4. Wendy is very interested in modeling the total amount of money that she will spend after applying the discounts from the coupons.



(a) (b)

Figure 3.4: Wendy's coupons

Wendy observes that the amount of money that she will save depends on the total cost of the items. She decides to let the function d represent the cost of the items after applying the discount to items that cost x dollars initially. Wendy realizes that she needs one formula for items that cost below \$15, and one for items that cost \$15 or more; she decides to write a formula for $d(x)$ using a piecewise-defined formula

$$d(x) = \begin{cases} x - 5, & 0 < x < 15 \\ 0.8x, & x \ge 15 \end{cases}$$

Wendy decides to test her formula by finding how much an item that costs \$13 initially will cost after using the coupon. She evaluates $d(13)$

$$\begin{aligned} d(13) &= 13 - 5 \\ &= 8 \end{aligned}$$

The item will cost her \$8.

Wendy also uses her formula to find her savings on a \$40 item by evaluating

$$\begin{aligned} d(40) &= 0.8 \cdot 40 \\ &= 32 \end{aligned}$$

and concludes that she will save \$8 using her coupon. ■

Example 3 – Function evaluation: Let g be the piecewise-defined function that has formula

$$g(x) = \begin{cases} -13, & x \leq -4 \\ 2, & -4 < x < 3 \\ 7, & x > 3 \end{cases}$$

Evaluate each of the following

(a) $g(-5)$ (b) $g(-4)$ (c) $g(0)$ (d) $g(3)$ (e) $g(53)$

Solution (a) To evaluate $g(-5)$ we first need to identify which part of the domain is appropriate. Since $-5 \leq -4$, we use the formula in the *first* row of $g(x)$, and therefore

$$g(-5) = -13$$

(b) Since $-4 \leq -4$, we use the *first* row in the formula for $g(x)$ again, so

$$g(-4) = -13$$

(c) Since $-4 < 0 < 3$ we need to use the *second* row in the formula for $g(x)$, so

$$g(0) = 2$$

(d) To evaluate $g(3)$ we need to find the appropriate interval in the formula for $g(x)$. Notice that 3 does not fall into any of the intervals! This means that $g(3)$ is undefined.

(e) We note that $53 > 3$, so we need to use the *third* row of the formula for $g(x)$, so

$$g(53) = 7$$

■

FIX

Example 4

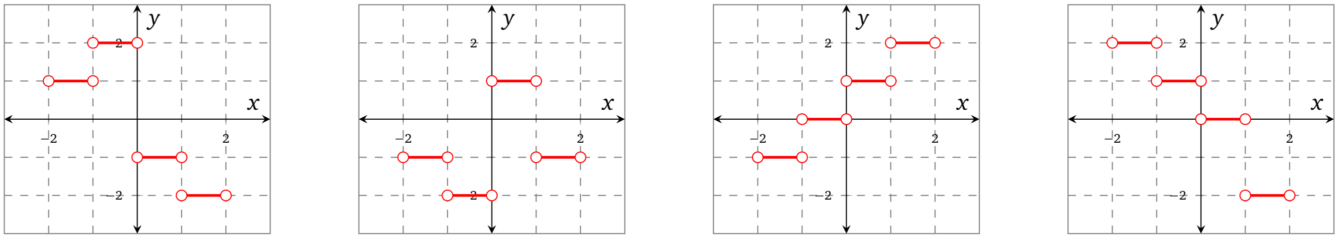
$$f(t) = \begin{cases} t^2, & t < -3 \\ 4 - 5t, & -3 \leq t < 6 \\ \sqrt{t} & t > 6 \end{cases}$$

■

Exercises

Problem 1 (Find a formula from a graph)
Consider the functions F, G, H, and J that have been graped in Figure 3.5. Find a formula for each function.

1.1 F **1.2** G **1.3** H **1.4** J



(a) $y = F(x)$ (b) $y = G(x)$ (c) $y = H(x)$ (d) $y = J(x)$

Figure 3.5

LOGARITHMS 4

Contents

4.1 Logarithmic functions

In our chapter on exponential functions we considered applications that lead to equations such as

$$10^x = 19$$

We can approximate solutions to such equations using graphical and numerical techniques. How can we solve these equations *algebraically* though? The answer is to use *logarithmic* functions.

Definition 18 (The logarithm function)
The logarithmic function, base b, where $b > 0$ and $b \neq 1$, is defined by

$$y = \log_b(x)$$

if, and only if,

$$b^y = x$$

The domain of the logarithmic function $y = \log_b(x)$ is $(0, \infty)$, and the range is $(-\infty, \infty)$.

Definition 18 says that if we are given a *logarithmic* expression then we can convert it into an equivalent *exponential* expression. This is useful when evaluating logarithmic expressions.

Example 1 Use a sentence to describe each of the following logarithmic expressions, and then evaluate each expression

(a) $\log_2(32)$ (b) $\log_3(81)$ (c) $\log_5(25)$ (d) $\log_{73}(1)$

Solution (a) The logarithm, base 2, of 32. In order to evaluate the expression, we need to answer the question: what power do we raise 2 to get 32? The answer is 5, so

$$\log_2(32) = 5$$

(b) The logarithm, base 3, of 81. What power do we raise 3 to get 81? The answer is 4, so

$$\log_3(81) = 4$$

(c) The logarithm, base 5, of 25. What power do we raise 5 to get 25? The answer is 2, so

$$\log_5(25) = 2$$

(d) The logarithm, base 73, of 1. We need to raise 73 to the power 0 to get 1, so

$$\log_{73}(1) = 0$$

■

Example 2 Convert each of the following exponential equations into their equivalent logarithm form

(a) $3^5 = 243$ (b) $7^0 = 1$ (c) $16^{1/2} = 4$ (d) $33^{-1} = \frac{1}{33}$

Solution (a) $3^5 = 243$ is equivalent to

$$\log_3(243) = 5$$

(b) $7^0 = 1$ is equivalent to

$$\log_7(1) = 0$$

(c) $16^{1/2} = 4$ is equivalent to

$$\log_{16}(4) = \frac{1}{2}$$

(d) $33^{-1} = \frac{1}{33}$ is equivalent to

$$\log_{33}\left(\frac{1}{33}\right) = -1$$

■

Example 3 Convert each of the following logarithmic equations into their equivalent exponential form

(a) $\log_4\left(\frac{1}{4}\right) = -1$ (b) $\log_6(36) = 2$ (c) $\log_{\frac{1}{2}}(4) = -2$ (d) $\log_e(e^8) = 8$

Solution (a) $\log_4\left(\frac{1}{4}\right) = -1$ is equivalent to

$$4^{-1} = \frac{1}{4}$$

(b) $\log_6(36) = 2$ is equivalent to

$$6^2 = 36$$

(c) $\log_{\frac{1}{2}}(4) = -2$ is equivalent to

$$\left(\frac{1}{2}\right)^{-2} = 4$$

(d) $\log_e(e^8) = 8$ is equivalent to

$$e^8 = e^8$$

In fact when evaluating a logarithm base e we use a special notation, as we'll soon see.

■

We have been able to perform all of our calculations so far using our knowledge of arithmetic and exponents. When faced with a logarithmic calculation that goes beyond this, we need to use a calculator to compute the value. Most modern calculators can work in any base, but of all the possible choices that we have available there are two bases that are particularly important.

Definition 19 (The common and natural logarithm functions)
When working with logarithmic functions that have base b and formula $y = \log_b(x)$,

- the *common* logarithmic function has base 10 and is written as

$$y = \log(x)$$

- the *natural* logarithmic function has base e and is written as

$$y = \ln(x)$$

It may help to recall from Definition 13 on page 66 that e is called the *natural* base.

Example 4 – Domain: Find the domain of each the functions implied by the following formulas

(a) $f(x) = \log(x)$ (c) $h(x) = \ln(4x - 5)$

(b) $g(x) = \log_3(2 + x)$ (d) $j(x) = \log_7(x^2)$

Solution (a) The domain of f is $(0, \infty)$. Note that the base of f is 10; f is the common logarithmic function.

(b) To find the domain of g we need to solve the inequality $2 + x > 0$. The domain of g is, therefore, $(-2, \infty)$.

(c) To find the domain of h we need to solve the inequality $4x - 5 > 0$. The domain of h is $\left(\frac{5}{4}, \infty\right)$. Note that the base of h is e; h is the natural logarithmic function.

(d) To find the domain of j we need to solve the inequality $x^2 > 0$. The domain of g is therefore $(-\infty, 0) \cup (0, \infty)$. ■

One of the implications of Definition 18 is that there is a relationship between logarithmic functions and exponential functions. Explicitly, if f is the exponential function that has formula

$$f(x) = b^x$$

then the inverse function, f^{-1}, has formula

$$f^{-1}(x) = \log_b(x)$$

FIX

We can use our knowledge of inverse functions (see) to help us graph logarithmic functions.

Example 5 – Graphing: Use your knowledge of the function f that has formula $f(x) = 2^x$ to help you graph its inverse function, f^{-1}, that has formula $f^{-1}(x) = \log_2(x)$.

Solution Let's start by constructing a table of values of the function f in Table 4.1. We can easily construct a table of values of $f^{-1}(x)$ by simply swapping the input and output values, which we have done in Table 4.2.

Table 4.1: f

x	$f(x)$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

Table 4.2: f^{-1}

x	$f^{-1}(x)$
$1/8$	-3
$1/4$	-2
$1/2$	-1
1	0
2	1
4	2
8	3

If we plot the values we obtained in Tables 4.1 and 4.2 and connect them using a smooth curve, then we obtain the curves given in Figure 4.1.

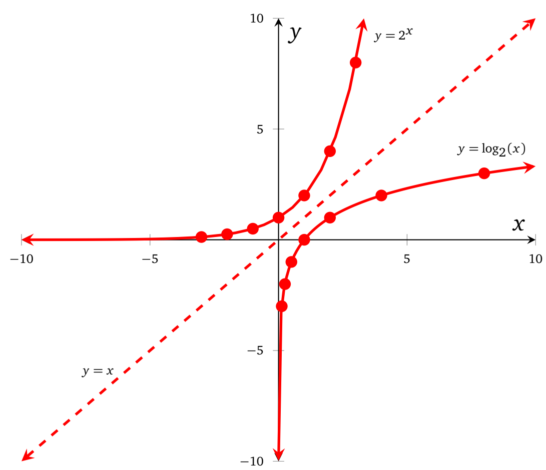


Figure 4.1

There are a few more observations that we can make about f and its inverse, using Figure 4.1 as a guide:

- the domain of f is $(-\infty, \infty)$, and the range of f is $(0, \infty)$; this means that the domain of f^{-1} is $(0, \infty)$, and the range of f^{-1} is $(-\infty, \infty)$;
- the function f has a *horizontal* asymptote with equation $y = 0$; this necessarily means that the function f^{-1} has a *vertical* asymptote with equation $x = 0$;

- the function f does not have a *vertical* asymptote| this therefore implies that the function f^{-1} does not have a *horizontal* asymptote;
- the curves of f and f^{-1} are symmetric about the line $y = x$. ■

★ **try it yourself** ★

Problem 1
Repeat Example 5 using the function f that has formula $f(x) = 3^x$.

make sure you try it!

There is a strong relationship between the logarithmic function f that has formula $f(x) = \log_b(x)$ and its inverse exponential function f^{-1} that has formula $f^{-1}(x) = b^x$. We can think of both functions as a type of *mapping* from their domains to their respective ranges. There are many possible ways to visualize the mapping| one such image is shown in Figure 4.2. Notice that the mapping lends itself well to highlighting properties (l_5) and (l_6), which detail the composition of logarithmic and exponential functions

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

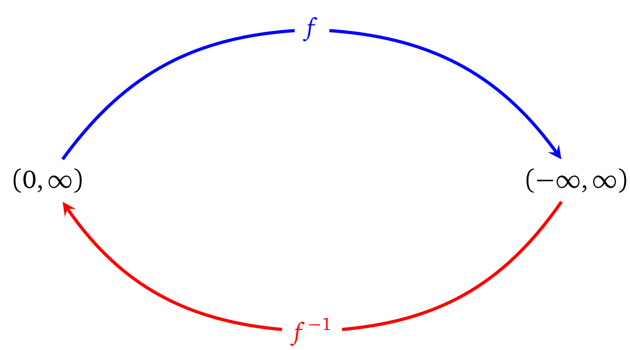


Figure 4.2: Visualizing the mappings of f and f^{-1}, where f has formula $f(x) = \log_b(x)$ and f^{-1} has formula $f^{-1}(x) = b^x$.

Our examples so far have concentrated on familiarizing ourselves with logarithmic functions but we have yet to see an application. The logarithmic functions have a myriad of applications| in particular, they can be used to help us study examples that otherwise could only be attempted from a graphical or numerical perpesctive.

Example 6 The number of radioactive atoms in a sample of Carbon-14 decays according to the model

$$Q(t) = Q_0 e^{-0.000120968t},$$

where Q_0 is the initial mass of the radioactive atoms and $Q(t)$ is the mass of radioactive atoms t years after the sample was established.

What is the half-life of the sample?

Solution We need to find the value of t that satisfies the equation $Q(t) = \frac{Q_0}{2}$. We proceed using the following steps

$$\begin{aligned}
\frac{Q_0}{2} = Q_0 e^{-0.000120968t} &\Rightarrow \frac{1}{2} = e^{-0.000120968t} \\
&\Rightarrow \ln\left(\frac{1}{2}\right) = -0.000120968t \\
&\Rightarrow t = -\frac{1}{-0.000120968} \ln\left(\frac{1}{2}\right) \\
&= 5370
\end{aligned}$$

We conclude that the half-life of the sample is 5370 years. ■

Example 7 – The RC circuit: A *capacitor* is a device that stores electrical energy in the form of charged particles. The *voltage* on the capacitor is a result of the electric field created by the particles and is proportional to the amount of charge stored. A *resistor* is a device that dissipates electrical energy. If a capacitor is charged up and then connected across a resistor, the capacitor discharges and the voltage drops.

The voltage (in V), on the capacitor as it is being discharged is modeled by the function V that has formula

$$V(t) = V_0 e^{-\frac{t}{RC}}$$

where V_0 is the initial capacitor voltage, R is the value of the resistor (in Ω), C is the value of the capacitor (in F) and t is time (in s).

(a) Suppose that a 1.0×10^{-6}-farad capacitor, initially charged to 12 V, is connected across a 10.000 Ω-resistor. How long will it take for the voltage on the capacitor to drop to half of its original value?

(b) Suppose the capacitor is initially charged to 20 V. How long will it take for the voltage to drop to one half of its original value?

(c) Suppose the capacitor is initially charged up to 100 V. How long will it take for the voltage to drop to one half of its original value?

(d) What effect will doubling the *resistance* have on the time it takes for the voltage to drop to one half of its initial value?

Solution (a) We need to solve the equation $\frac{1}{2}V_0 = V_0 e^{-\frac{t}{RC}}$:

$$\begin{aligned} 6 = 12e^{-100t} &\Rightarrow \frac{1}{2} = e^{-100t} \\ &\Rightarrow \ln\left(\frac{1}{2}\right) = -100t \\ &\Rightarrow t = -\frac{1}{100}\ln\left(\frac{1}{2}\right) \\ &\approx 0.007 \end{aligned}$$

It takes about 0.007 s for the voltage of the capacitor to reach one half of its initial value.

(b) We need to solve the equation $\frac{1}{2}V_0 = V_0 e^{-\frac{t}{RC}}$:

$$\begin{aligned} 10 = 20e^{-100t} &\Rightarrow \frac{1}{2} = e^{-100t} \\ &\Rightarrow \ln\left(\frac{1}{2}\right) = -100t \\ &\Rightarrow t = -\frac{1}{100}\ln\left(\frac{1}{2}\right) \\ &\approx 0.007 \end{aligned}$$

It takes about 0.007 s for the voltage of the capacitor to reach one half of its initial value. Does this sound familiar?

(c) We need to solve the equation $\frac{1}{2}V_0 = V_0 e^{-\frac{t}{RC}}$:

$$\begin{aligned} 50 = 100e^{-100t} &\Rightarrow \frac{1}{2} = e^{-100t} \\ &\Rightarrow \ln\left(\frac{1}{2}\right) = -100t \\ &\Rightarrow t = -\frac{1}{100}\ln\left(\frac{1}{2}\right) \\ &\approx 0.007 \end{aligned}$$

It takes about 0.007 s for the voltage of the capacitor to reach one half of its initial value. There seems to be a pattern here...

(d) If we double the resistance to 20.000 Ω then we need to solve the equation $\frac{1}{2}V_0 = V_0 e^{-50t}$; note that the value of V_0 does not affect our calculations

$$\begin{aligned}
\frac{1}{2}V_0 = V_0 e^{-50t} \Rightarrow \frac{1}{2} &= e^{-50t} \\
\Rightarrow \ln\left(\frac{1}{2}\right) &= -50t \\
\Rightarrow t &= -\frac{1}{50}\ln\left(\frac{1}{2}\right) \\
&\approx 0.014
\end{aligned}$$

We conclude that doubling the resistance doubles the time it takes (to about 0.014 s) for the voltage on the capacitor to reach half of its initial value. ■

Exercises

Problem 2 (Domain)
Find the domain of each of the functions implied by the following formulas.

2.1 $f(x) = \log_4(x+7)$ **2.3** $h(x) = 5\log(3x)$ **2.5** $k(x) = \log_6(x^2-9)$ **2.7** $m(x) = \ln(2^x)$

2.2 $g(x) = \log_9(x-2)$ **2.4** $j(x) = 8 - \log_2(4x+3)$ **2.6** $l(x) = 3\log_8(4-2x^2)$ **2.8** $n(x) = 2^{\log(x)}$

Problem 3 (Transformations: given the formula, describe the transformation)
Describe each of the functions g, h, j, and k in terms of transformations of the logarithmic function f that has formula $f(x) = \log(x)$. State the domain of each function.

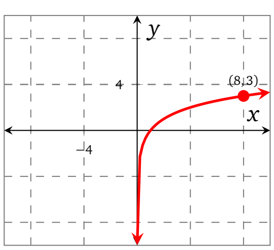
3.1 $g(x) = \log(x+3)$ **3.2** $h(x) = \log(x-5)$ **3.3** $j(x) = \log(2(x+7))$ **3.4** $k(x) = 5\log(-x)$

Problem 4 (Transformations: given the transformation, find the formula)
Let f be the function that has formula $f(x) = \log(x)$. In each of the following problems apply the given transformation to the function f and write a formula for the transformed version of f.

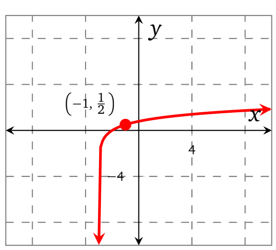
4.1 Shift f to the right by 2 units. **4.3** Shift f up by 11 units.

4.2 Shift f to the left by 5 units. **4.4** Shift f down by 1 unit.

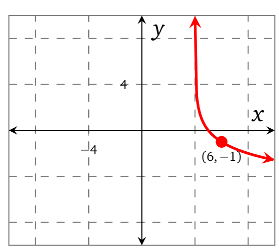
Problem 5 (Find the base from graphs)
Consider the functions graphed in Figure 4.3. Each function has a formula of the form $y = \log_b(x+a)$, where b is the base, and a is given for each function. Use the ordered pair given in each graph to find the base, b.



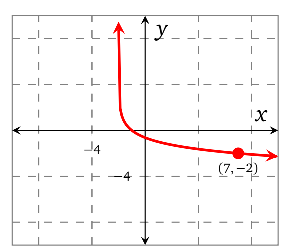
(a) $y = \log_b(x)$



(b) $y = \log_b(x+3)$



(c) $y = \log_b(x-4)$



(d) $y = \log_b(x+2)$

Figure 4.3: Graphs for Problem 5

Problem 6 (Solving exponential equations with base 10 and base e)
Use Definition 18 to solve each of the following equations. Give both the exact and an approximate solution.

6.1 $e^x = 7$

6.2 $e^x + 5 = 10$

6.3 $e^{x+5} = 10$

6.4 $e^{5x+7} - 4 = 2$

6.5 $10^x = 1$

6.6 $10^{x+1} = 11$

6.7 $10^{2x} = 4$

6.8 $10^{4-x} = 21$

6.9 $5e^{2x} + 1 = 10$

6.10 $8 - 7e^{-3x} = -10$

6.11 $9e^{5-x} - 1 = 0$

6.12 $e^{3x} - 4 = -5e^{3x}$

Problem 7 (Solving logarithmic equations with base 10 and base e)
Use Definition 18 to solve each of the following equations. Give both the exact and an approximate solution.

7.1 $\ln(x) = 7$

7.2 $2\ln(x) = -3$

7.3 $5 - 4\ln(2x) = 13$

7.4 $(\ln(x))^2 = 5$

7.5 $\log(x) = 7$

7.6 $3 - \log(x) = 0$

7.7 $\log(5x + 2) = -3$

7.8 $\log(5 - x) = \log_2(8)$

Problem 8 (Find the base from tables)
Tables 4.3a–4.3d show values of four different functions; each function has the form $y = \log_b(ax)$ where b is the base, and a is given for each function. Use any ordered pair you wish from each table to find the base, b, for each function.

Table 4.3: Tables for Problem 8

(a) $y = \log_b(3x)$

x	y
$1/16$	1
$1/3$	0
$2/3$	1
$4/3$	2
$8/3$	3

(b) $y = \log_b(5x)$

x	y
$1/135$	-3
$1/45$	-2
$1/15$	-1
$1/5$	0
$3/5$	1

(c) $y = \log_b(x)$

x	y
16	-2
4	-1
1	0
$1/4$	1
$1/16$	2

(d) $y = \log_b(-2x)$

x	y
$-9/8$	-2
$-3/4$	-1
-1	0
$-1/3$	1
$-2/9$	2

Problem 9 (Inverse functions)
Let f be the function that has formula $f(x) = 4^x$.

9.1 Construct a table of values of f, allowing x to take integer values on the interval $[-3, 3]$.

9.2 Use your answer to Problem 9.1 to construct a table of values of the function f^{-1}.

9.3 Use your answer to Problem 9.2 to evaluate each of the following

(a) $f^{-1}(4)$ (b) $f^{-1}(16)$ (c) $f^{-1}\left(\frac{1}{4}\right)$ (d) $f^{-1}\left(\frac{1}{16}\right)$

9.4 Give the formula for $f^{-1}(x)$.

Problem 10 (Inverse functions)
Each of the functions defined by the following formulas are invertable. For each function

(a) state its domain and range;

(b) find its inverse;

(c) state the domain and range of the inverse function.

10.1 $f(x) = 2^{5x}$

10.2 $g(t) = e^{3t+4}$

10.3 $h(s) = 5 - 4^{s-7}$

10.4 $j(u) = 3 \cdot 5^{-u}$

10.5 $k(v) = \ln(3v + 2)$

10.6 $l(w) = 5\log(2 - 7w)$

10.7 $m(\alpha) = \log_8(2\alpha - 1)$

10.8 $n(\beta) = \frac{2}{3}\log_3(4\beta - 7)$

Problem 11 (The RL Circuit)
An *inductor* is a device that stores electrical energy in a magnetic field. As long as *current* is flowing through the inductor, energy is stored. When connected to a voltage source, such as a battery, and a resistor, the current, (in A), through the inductor is modeled by the function I that has formula

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

where V is voltage of the source (in V), R is the value of the resistor (in Ω), L is the value of the inductor (in H), and t is time (in s) since the inductor was connected to the voltage source.

11.1 A switch is thrown in a circuit that connects a 5-henry inductor to a 200-ohm resistor and a 12-volt battery. Write the formula for $I(t)$.

11.2 What is the current in the circuit after 0.025 s? After 0.05 s?

11.3 What is the maximum value the current will reach?

11.4 Since we know that, mathematically, our model will never actually reach this value, how long will it take the current to reach 95 % of this value?

Problem 12 (Factoring)
Use your factoring skills to solve the following exponential equations (if possible). Give both the exact and an approximate solution.

12.1 $e^{2x} - 6e^x + 8 = 0$ **12.2** $e^{2x} - 4e^x - 3 = 0$ **12.3** $e^{2x} - 8e^x - 20 = 0$ **12.4** $e^{2x} + 11e^x + 30 = 0$

12.5 $10^{2x} - 6 \cdot 10^x + 8 = 0$ **12.6** $10^{2x} - 4 \cdot 10^x - 3 = 0$ **12.7** $10^{2x} - 8 \cdot 10^x - 20 = 0$ **12.8** $10^{2x} + 11 \cdot 10^x + 30 = 0$

Problem 13 (Factoring with logarithms)
Use your factoring skills to solve the following logarithmic equations (if possible). Give both the exact and an approximate solution.

13.1 $(\ln(x))^2 + 3\ln(x) + 2 = 0$ **13.2** $(\ln(x))^2 - 3\ln(x) + 2 = 0$ **13.3** $(\ln(x))^2 - 16 = 0$

13.4 $(\log(x))^2 + 6\log(x) - 7 = 0$ **13.5** $(\log(x))^2 + 7\log(x) + 12 = 0$ **13.6** $(\log(x))^2 - 1 = 0$

4.2 Properties of logarithms

Properties of logarithms

Assuming that x, y, and b are any positive real numbers (where b is the base) then the following properties of logarithms hold

(l_1) $\log_b(x) + \log_b(y) = \log_b(xy)$

(l_2) $\log_b(x) - \log_b(y) = \log_b\left(\frac{x}{y}\right)$

(l_3) $\log_b(x^t) = t\log_b(x)$ where t is any real number

(l_4) $x = y \iff \log_b(x) = \log_b(y)$

(l_5) $b^{\log_b(x)} = x$

(l_6) $\log_b(b^x) = x$

Note that, in particular, properties (l_5) and (l_6) say that if f and g are functions that have formulas

$$f(x) = b^x, \qquad g(x) = \log_b(x)$$

then f and g are inverse functions.

Example 1 Use the properties of logarithms to help solve the equation

$$\ln(x-3) + \ln(x+6) = \ln(10) \tag{4.1}$$

Solution When solving equations such as Equation (4.1), it is often helpful to use the symbol $\Rightarrow$ which means, 'implies that'; this also allows us to annotate each line (if necessary).

$$\begin{aligned}
\ln(x-3) + \ln(x+6) = \ln(10) &\Rightarrow \ln((x-3)(x+6)) = \ln(10) && \text{property } (l_1)\\
&\Rightarrow \ln(x^2+3x-18) = \ln(10) && \text{distribute}\\
&\Rightarrow x^2+3x-18 = 10 && \text{property } (l_4)\\
&\Rightarrow x^2+3x-28 = 0 && \\
&\Rightarrow (x+7)(x-4) = 0 && \text{factor}\\
&\Rightarrow x = -7, 4 &&
\end{aligned}$$

It seems that we have two solutions| we need to check both of them by substituting each into Equation (4.1):

$$\begin{aligned}
\ln(-7-3) + \ln(-7+6) &\stackrel{?}{=} \ln(10) \\
\ln(-10) + \ln(-1) &\stackrel{?}{=} \ln(10) \\
&\text{domain error!}
\end{aligned}
\qquad
\begin{aligned}
\ln(4-3) + \ln(4+6) &\stackrel{?}{=} \ln(10) \\
\ln(1) + \ln(10) &\stackrel{?}{=} \ln(10) \\
\ln(10) &\stackrel{?}{=} \ln(10) \\
&\text{true}
\end{aligned}$$

FIX

Since -7 gives a domain error when substituted into Equation (4.1) and 4 *does* satisfy Equation (4.1), we conclude that 4 is the only solution to the equation. ■

Example 2 another solving equation problem ■

The change of base formula

The change of base formula for logarithms is

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \tag{4.2}$$

where a, b, and x are real, positive numbers.

The change of base formula may seem like a little strange, but it is fairly simple to derive, as we show in the following steps

$$\begin{aligned} y = \log_a(x) &\Rightarrow a^y = x \\ &\Rightarrow \log_b(a^y) = \log_b(x) \\ &\Rightarrow y \log_b(a) = \log_b(x) \\ &\Rightarrow y = \frac{\log_b(x)}{\log_b(a)} \end{aligned}$$

The change of base formula is particularly useful when calculating logarithms that have a base other than e or 10. We can use it to help us explore graphical and numerical features of logarithmic functions with such bases; even though most modern calculators can evaluate logarithmic expressions of any base, the principle remains useful.

Example 3 Use the change of base formula, Equation (4.2), to help you graph the function f that has formula

$$f(x) = \log_{\frac{1}{4}}(x)$$

Compare the graph with that of the function g that has formula $g(x) = \left(\frac{1}{4}\right)^x$.

Solution We begin by using Equation (4.2) to rewrite the formula for f

$$\begin{aligned} f(x) &= \log_{\frac{1}{4}}(x) \\ &= \frac{\ln(x)}{\ln\left(\frac{1}{4}\right)} \end{aligned}$$

Note that the change of base formula allows us to use *any base we choose*; we have chosen to use the *natural* base simply because the function $\ln(x)$ is easily accessible on most calculators. Typically we will use either $\ln(x)$ or $\log(x)$ when changing base. We have plotted f and g (which has formula $g(x) = \left(\frac{1}{4}\right)^x$) in Figure 4.4.

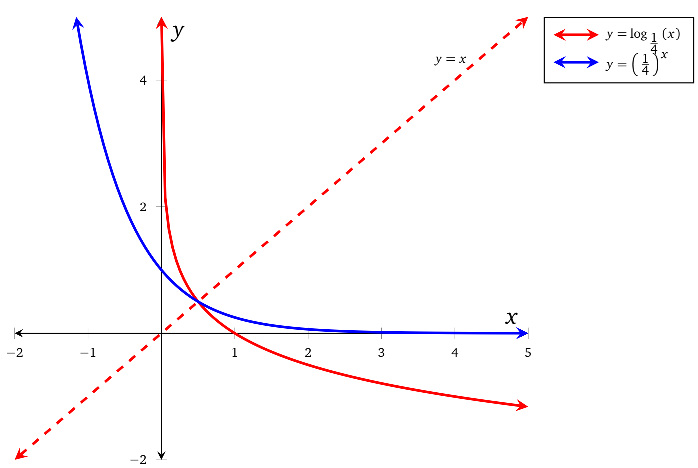


Figure 4.4

We can make observations about the graphs of f and $g = f^{-1}$ (which are similar to the observations we made in Example 5 on page 97)

- the domain of f is $(0, \infty)$, and the range of f is $(-\infty, \infty)$; this means that the domain of f^{-1} is $(-\infty, \infty)$, and the range of f^{-1} is $(0, \infty)$;
- the function f has a *vertical* asymptote with equation $x = 0$; this necessarily means that the function f^{-1} has a *horizontal* asymptote with equation $y = 0$;
- the function f does not have a *horizontal* asymptote| this therefore implies that the function f^{-1} does not have a *vertical* asymptote;

- the curves of f and f^{-1} are symmetric about the line $y = x$. ∎

Example 4 – Investing in an account: You have \$2000 to invest in an account that accrues interest at a nominal rate of 3.75% Assuming that $A(t)$ is the amount of money in the account t years after opening the account, calculate the amount of time it will take the money in each account to reach \$3000 when the interest is compounded in each of the following ways

(a) Annually. (b) Monthly (c) Daily (d) Continuously

You may like to refresh your knowledge about compound interest using Definition 11 on page 60.

Solution (a) $A(t) = 2000(1.0375)^t$; to calculate when $A(t) = 3000$ we need to solve the equation

$$\begin{aligned}3000 = 2000(1.0375)^t \Rightarrow \frac{3}{2} &= (1.0375)^t\\ \Rightarrow \ln\left(\frac{3}{2}\right) &= \ln(1.0375)^t\\ \Rightarrow \ln\left(\frac{3}{2}\right) &= t\ln(1.0375)\\ \Rightarrow t &= \frac{\ln\left(\frac{3}{2}\right)}{\ln(1.0375)}\\ &\approx 11.0139\end{aligned}$$

If the interest is compounded *annually*, it will take about 11 years for the initial investment to reach \$3000.

(b) $A(t) = 2000\left(1 + \frac{0.0375}{12}\right)^{12t}$; to calculate when $A(t) = 3000$ we need to solve the equation

$$\begin{aligned}3000 = 2000\left(1 + \frac{0.0375}{12}\right)^{12t} \Rightarrow \frac{3}{2} &= \left(1 + \frac{0.0375}{12}\right)^{12t}\\ \Rightarrow \ln\left(\frac{3}{2}\right) &= \ln\left(1 + \frac{0.0375}{12}\right)^{12t}\\ \Rightarrow \ln\left(\frac{3}{2}\right) &= 12t\ln\left(1 + \frac{0.0375}{12}\right)\\ \Rightarrow 12t &= \frac{\ln\left(\frac{3}{2}\right)}{\ln\left(1 + \frac{0.0375}{12}\right)}\\ \Rightarrow t &= \frac{1}{12}\cdot\frac{\ln\left(\frac{3}{2}\right)}{\ln\left(1 + \frac{0.0375}{12}\right)}\\ &\approx 10.8293\end{aligned}$$

If the interest is compounded *monthly*, it will take just under 11 years for the initial investment to reach \$3000.

(c) $A(t) = 2000\left(1 + \frac{0.0375}{365}\right)^{365t}$; to calculate when $A(t) = 3000$ we need to solve the equation

$$
\begin{aligned}
3000 = 2000\left(1+\frac{0.0375}{365}\right)^{365t} &\Rightarrow \frac{3}{2} = \left(1+\frac{0.0375}{365}\right)^{365t} \\
&\Rightarrow \ln\left(\frac{3}{2}\right) = \ln\left(1+\frac{0.0375}{365}\right)^{365t} \\
&\Rightarrow \ln\left(\frac{3}{2}\right) = 365t\ln\left(1+\frac{0.0375}{365}\right) \\
&\Rightarrow 365t = \frac{\ln\left(\frac{3}{2}\right)}{\left(1+\frac{0.0375}{365}\right)} \\
&\Rightarrow t = \frac{1}{365}\cdot\frac{\ln\left(\frac{3}{2}\right)}{\left(1+\frac{0.0375}{365}\right)} \\
&\approx 10.8130
\end{aligned}
$$

If the interest is compounded *daily*, it will take just under 11 years for the initial investment to reach \$3000.

(d) $A(t) = 2000e^{0.0375t}$; to calculate when $A(t) = 3000$ we need to solve the equation

$$
\begin{aligned}
3000 = 2000e^{0.0375t} &\Rightarrow \frac{3}{2} = e^{0.0375t} \\
&\Rightarrow \ln\left(\frac{3}{2}\right) = 0.0375t \\
&\Rightarrow t = \frac{1}{0.0375}\ln\left(\frac{3}{2}\right) \\
&\approx 10.8124
\end{aligned}
$$

If the interest is compounded *continuously*, it will take just under 11 years for the initial investment to reach \$3000. ■

Example 5 – A cautionary tale: Tyrell and Latisha are studying the equation

$$\ln(x^2) = 3 \tag{4.3}$$

Tyrell uses property (l_3) to solve the equation

$$
\begin{aligned}
\ln(x^2) = 3 &\Rightarrow 2\ln(x) = 3 \\
&\Rightarrow \ln(x) = \frac{3}{2} \\
&\Rightarrow x = e^{\frac{3}{2}}
\end{aligned}
$$

Latisha takes a different approach:

$$
\begin{aligned}
\ln(x^2) = 3 &\Rightarrow x^2 = e^3 \\
&\Rightarrow x = \pm\sqrt{e^3} \\
&= \pm e^{\frac{3}{2}}
\end{aligned}
$$

Note that Latisha has 2 solutions, and Tyrell only has 1! Who has the correct solution set?

Solution Let's begin by exploring property (l_3); it is certainly true to say

$$\ln(3^2) = 2\ln(3)$$

but it is not true to say that

$$\ln((-3)^2) = 2\ln(-3)$$

since we can not take the logarithm of negative number. These two examples illustrate that Tyrell's application of property (l_3) was not appropriate, since we can input both positive *and* negative numbers into Equation (4.3).

We conclude that Latisha has the correct solution set. ■

The properties of logarithms may seem a little mysterious. Remembering that logarithmic expressions are closely related to exponential expressions (see Definition 18), it should sound reasonable that the properties of logarithms are somewhat related to the properties of exponents. Let's see if we can tie the two ideas together, and prove property (l_1).

Example 6 – Proving that $\log_b(xy) = \log_b(x) + \log_b(y)$: When proving such an identify, we have a few options:

- We could start with one side of the identity, and try to work toward the other side of it.
- We could start with one side of the identity, simplify it, and then try to reach the same expression by working with the other side of the identity.

We will demonstrate a proof using the second of these options.

Solution We'll start by writing

$$m = \log_b(x), \qquad n = \log_b(y) \tag{4.4}$$

We can write the equations in Equation (4.4) in their equivalent exponential form

$$b^m = x, \qquad b^n = y$$

We clearly see that $b^m \cdot b^n = xy$ and so

$$xy = b^{m+n}$$

Let's write an equivalent logarithmic equation to the exponential equation $xy = b^{m+n}$

$$\log_b(xy) = m + n \tag{4.5}$$

Notice that this equation contains the left hand side of property (l_1); we are at the half-way point of our proof| let's see if we can meet here using the right hand side of property (l_1).

We can write $\log_b(x) + \log_b(y)$ in terms of m and n

$$\log_b(x) + \log_b(y) = m + n \tag{4.6}$$

Combining Equations (4.5) and (4.6) gives the desired result

$$\begin{aligned} \log_b(x) + \log_b(y) &= m + n \\ &= \log_b(xy) \end{aligned}$$

■

★ try it yourself ★

Problem 1
Use Example 6 to help guide you in prooving property (l_2).

make sure you try it!

Investigations

Problem 2 (True or false?)

2.1 Use the change of base formula and a calculator to help plot the function f that has formula $f(x) = \log_b(x)$ for each of the following values of b

(a) $b = 2$ (b) $b = \frac{1}{4}$ (c) $b = 5$ (d) $b = \frac{1}{3}$

Use your answer to Problem 2.1 to help you determine if each of the following statements are true or false for all values of b; if you believe that the statement is false, provide an example that supports it.

2.2 The function f is increasing.

2.3 The function f is decreasing.

2.4 The function f has a vertical asymptote at 0.

2.5 The function f is concave up.

2.6 The function f has a zero at 1.

2.7 The function f has a vertical intercept.

2.8 The function f is concave down.

Problem 3 (Properties (l_1) and (l_2))

3.1 Use properties (l_1) and (l_2) to help you complete Table 4.4; note that you should be able to do so without using a calculator.

Table 4.4: Properties (l_1) and (l_2)

A	B	b	$\log_b(A)$	$\log_b(B)$	$\log_b(AB)$	$\log_b\left(\frac{A}{B}\right)$
1	2	2				
e^5	e^3	e				
36	$\sqrt[3]{6}$	6				
0.001	10000	10				
4	$1/16$	$1/4$				

Use your answer to Problem 3.1 to help you decide if the following properties of logarithms are true or false.

3.2 $\log_b(AB) = \log_b(A) \cdot \log_b(B)$

3.3 $\log_b(A+B) = \log_b(A) + \log_b(B)$

3.4 $\log_b(AB) = \log_b(A) + \log_b(B)$

3.5 $\log_b\left(\frac{A}{B}\right) = \frac{\log_b(A)}{\log_b(B)}$

3.6 $\log_b\left(\frac{A}{B}\right) = \log_b(A) - \log_b(B)$

Exercises

Problem 4 (Change of base)
Use the change of base formula, Equation (4.2), and a calculator to approximate each of the following (if possible).

4.1 $\log_2(3)$

4.2 $\log_{23}(-2)$

4.3 $\log_3(7)$

4.4 $\log_{\frac{1}{2}}(13)$

4.5 $\log_8(2)$

4.6 $\log_{-1}(5)$

4.7 $\log_\pi(5)$

4.8 $\log_2(0)$

Problem 5 (Expand logarithmic expressions)
Use the properties of logarithms to write each of the following expressions as the sum and/or difference of logarithms; leave your answer in exact form.

5.1 $\log(2x)$

5.2 $\log_3\left(\frac{4}{x}\right)$

5.3 $\log_5(x^7)$

5.4 $\log_9(4x^3)$

5.5 $\ln(\sqrt{x})$

5.6 $\ln\left(\sqrt[7]{\frac{x^3}{x+2}}\right)$

5.7 $\log_\pi\left(\frac{x^2}{4}\right)$

5.8 $3\log(10x)$

Problem 6 (Condense logarithmic expressions)
Use the properties of logarithms to write each of the following expressions as a single logarithm.

FIX

Problem 7 (Solving equations involving logarithms)
Use the properties of logarithms to help you solve the following equations.

FIX

7.1 $\log_2(x) + \log_2(7) = 3$

7.2 $\log_4(x) - \log_4(3) = -1$

7.3 $\log_3(x) + \log_3(9) = -2$

7.4 $\ln(2x) - \ln(9) = 0$

Problem 8 (Solving equations involving logarithms)
Use the properties of logarithms to help you solve the following equations.

8.1 $\log_2(x-2)+\log_2(x+9)=\log_2(12)$

8.2 $\ln(x+6)-\ln(x-2)=\ln(5)$

8.3 $\log(x+2)+\log(x-4)=\log(7)$

8.4 $\log_5(x+32)-\log_5(x)=\log_5(5)$

8.5 $\log_3(x-5)+\log_3(x)=\log_3(24)$

8.6 $\ln(x+76)-\ln(x+4)=\ln(9)$

8.7 $\log_{13}(x+3)+\log_{13}(x+1)=\log_{13}(24)$

8.8 $\log_\pi(x+58)-\log_\pi(x+7)=\log_\pi(4)$

Problem 9 (Solving equations involving logarithms)
Use the properties of logarithms to help you solve the following equations.

9.1 $\log_2(x^2)+\log_2(7)=3$

9.2 $\log(\sqrt[3]{x+1})+\log(x+1)=\log_2(16)$

9.3 $\log_3(\sqrt{x})-\log_3(5)=-2$

9.4 $\frac{1}{4}\ln(x+1)+\frac{1}{4}\ln(x)=1$

Problem 10 (Piecewise logarithmic functions)
Consider the function f that has formula

$$f(x)=\begin{cases}\log(x^2) & x<-2\\ \ln(x+2) & x>-2\end{cases}$$

Evaluate each of the following (if possible), giving both the exact and an approximate answer.

10.1 $f(-5)$ **10.2** $f(-1)$ **10.3** $f(0)$ **10.4** $f(-2)$

Problem 11 (Composition of logarithmic functions)
Let f and g be functions that have the following formulas

$$f(x)=\log(x+5),\qquad g(x)=\ln(x)$$

Evaluate each of the following (if possible), giving both the exact and an approximate answer.

11.1 $(f\circ g)(1)$

11.2 $(g\circ f)(1)$

11.3 $(f\circ g)\left(e^{-5}\right)$

11.4 $(g\circ f)(-4)$

11.5 $(f\circ g)\left(\frac{1}{2}\right)$

11.6 $(f\circ g)(-3)$

11.7 $(f\circ g)(x)$

11.8 $(g\circ f)(x)$

Problem 12 (Decomposition)
In each of the following problems, you are given a formula for function h. Decompose h into two functions f and g such that $h=f\circ g$.

12.1 $h(x)=\log(3x^2)$ **12.2** $h(x)=-2\ln(5-x)$ **12.3** $h(x)=\log_3\left(\sqrt[3]{x}\right)$ **12.4** $h(x)=\log_5(x^2)+7^{x^2}$

Problem 13 (Function algebra)
Let f and g be the functions that have formulas

$$f(x)=\log(x),\qquad g(x)=\ln(x)$$

Evaluate each of the following (if possible), giving the exact and an approximate solution (where appropriate).

13.1 $(f+g)(1)$

13.2 $(f-g)(1)$

13.3 $(f\cdot g)(1)$

13.4 $\left(\frac{f}{g}\right)(1)$

13.5 $(f+g)(5)$

13.6 $(f-g)(\pi)$

13.7 $(f\cdot g)(e)$

13.8 $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$

POLYNOMIAL AND RATIONAL FUNCTIONS 5

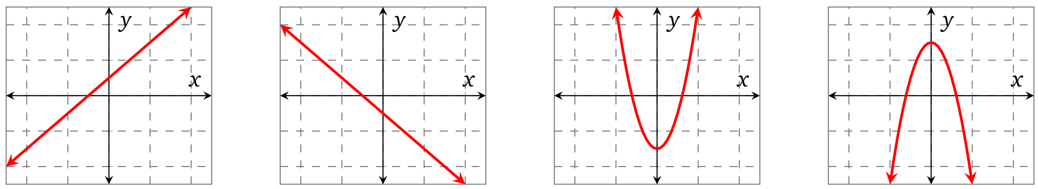
Contents

5.1 Polynomial functions

In your previous mathematics classes you have studied *linear* and *quadratic* functions. The most general forms of these types of functions can be represented (respectively) by the functions f and g that have formulas

$$f(x) = mx + b, \qquad g(x) = ax^2 + bx + c \tag{5.1}$$

We know that m is the slope of f, and that a is the *leading coefficient* of g. We also know that the *signs* of m and a completely determine the behavior of the functions f and g. For example, if $m > 0$ then f is an *increasing* function, and if $m < 0$ then f is a *decreasing* function. Similarly, if $a > 0$ then g is *concave up* and if $a < 0$ then g is *concave down*. Graphical representations of these statements are given in Figure 5.1.



(a) $m > 0$ (b) $m < 0$ (c) $a > 0$ (d) $a < 0$

Figure 5.1: Typical graphs of linear and quadratic functions.

Let's look a little more closely at the formulas for f and g in Equation (5.1). Note that the *degree* of f is 1 since the highest power of x that is present in the formula for $f(x)$ is 1. Since f has 2 terms, we may call it a *bi*nomial function. Similarly, the degree of g is 2 since the highest power of x that is present in the formula for $g(x)$ is 2. Since g has 3 terms, we may call it a *tri*nomial function.

In this section we will build upon our knowledge of these elementary functions. In particular, we will generalize our knowledege of the functions f and g to the study of a *poly*nomial function p that has any degree (and any number of terms) that we wish. The only restriction that we will enforce is that the degree of p must be an integer.

essential skills

The following problems contain prerequisite skills that are essential for success. Make sure that you can complete them before moving on!

Problem 1 (Quadratic functions)
Every quadratic function has the form $y = ax^2 + bx + c$; state the value of a for each of the following functions, and hence decide if the parabola that represents the function opens upward or downward.

1.1 $F(x) = x^2 + 3$

1.2 $G(t) = 4 - 5t^2$

1.3 $H(y) = 4y^2 - 96y + 8$

1.4 $K(z) = -19z^2$

Now let's generalize our findings for the most general quadratic function g that has formula $g(x) = a_2x^2 + a_1x + a_0$. Complete the following sentences.

1.5 When $a_2 > 0$, the parabola that represents $y = g(x)$ opens ...

1.6 When $a_2 < 0$, the parabola that represents $y = g(x)$ opens ...

Power functions with positive exponents

The study of polynomials will rely upon a good knowledge of power functions| you may reasonably ask, what is a power function?

Definition 20 (Power functions)
The most general formula for a power functions is

$$f(x) = a_n x^n$$

where n can be any real number.

Note that for this section we will only be concerned with the case when n is a positive integer.

You may find assurance in the fact that you are already very comfortable with power functions that have $n = 1$ (linear) and $n = 2$ (quadratic). Let's explore some power functions that you might not be so familiar with. As you read Examples 1 and 2, try and spot as many patterns and similarities as you can.

Example 1 – Power functions with odd positive exponents: Graph each the functions f, g, and h that have formulas

$$f(x) = x^3, \qquad g(x) = x^5, \qquad h(x) = x^7$$

and state their domain, and their long-run behavior as $x \to \pm\infty$

Solution The functions f, g, and h are plotted in Figure 5.2. The domain of each of the functions f, g, and h is $(-\infty, \infty)$. Note that the long-run behavior of each of the functions is the same, and in particular

$$f(x) \to \infty \text{ as } x \to \infty$$
$$\text{and } f(x) \to -\infty \text{ as } x \to -\infty$$

The same results hold for g and h. Note that the range of each of the functions f, g, and h is $(-\infty, \infty)$.

FIX

It appears from Figure 5.2 that each of the functions f, g, and h are symmetric about the origin. Remember from REF that a function that exhibits this behavior is called *odd*. We can test a function algebraically to see if it is odd by evaluating $f(-x)$; let's do that for each of the functions f, g, and h:

$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x) \end{aligned} \qquad \begin{aligned} g(-x) &= (-x)^5 \\ &= -x^5 \\ &= -g(x) \end{aligned} \qquad \begin{aligned} h(-x) &= (-x)^7 \\ &= -x^7 \\ &= -h(x) \end{aligned}$$

We conclude that each of the functions f, g, and h are odd. ■

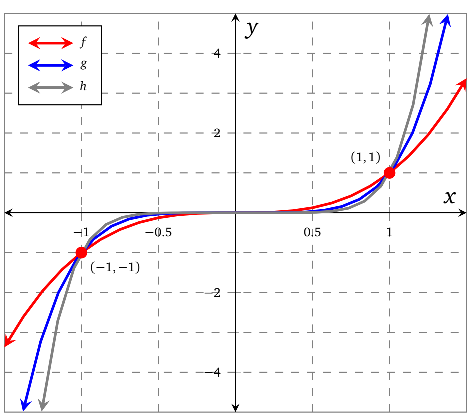


Figure 5.2: Odd power functions

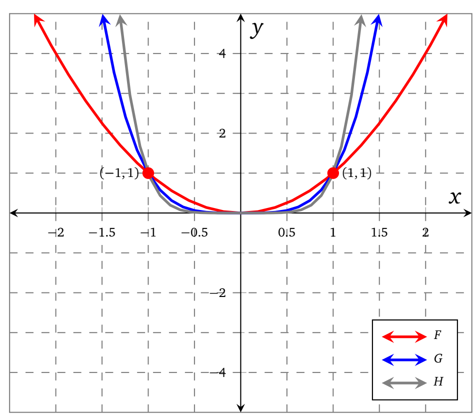


Figure 5.3: Even power functions

Example 2 – Power functions with even positive exponents: Graph each the functions F, G, and H that have formulas

$$F(x) = x^2, \qquad G(x) = x^4, \qquad H(x) = x^6$$

and state their domain, and their long-run behavior as $x \to \pm\infty$

Solution The functions F, G, and H are plotted in Figure 5.3. The domain of each of the functions is $(-\infty, \infty)$. Note that the long-run behavior of each of the functions is the same, and in particular

$$F(x) \to \infty \text{ as } x \to \infty$$
$$\text{and } F(x) \to \infty \text{ as } x \to -\infty$$

The same result holds for G and H. Note that the range of each of the functions F, G, and H is $[0, \infty)$.

FIX

It appears from Figure 5.3 that each of the functions F, G, and H are symmetric across the vertical axis. Remember from REF that a function that exhibits this behavior is called *even*. We can test a function algebraically to see if it is even by evaluating $f(-x)$; let;s do that for each of the functions F, G, and H:

$$\begin{aligned} F(-x) &= (-x)^2 \\ &= x^2 \\ &= F(x) \end{aligned} \qquad \begin{aligned} G(-x) &= (-x)^4 \\ &= x^4 \\ &= G(x) \end{aligned} \qquad \begin{aligned} H(-x) &= (-x)^6 \\ &= x^6 \\ &= H(x) \end{aligned}$$

We conclude that each of the functions F, G, and H are even. ■

★ try it yourself ★

Problem 2
Repeat Examples 1 and 2 using (respectively) the functions that have the following formulas.

2.1 $f(x) = -x^3$, $g(x) = -x^5$, $h(x) = -x^7$

2.2 $F(x) = -x^2$, $G(x) = -x^4$, $H(x) = -x^6$

make sure you try it!

Polynomial functions

Now that we have a little more familiarity with power functions, we can define polynomial functions. Provided that you were comfortable with our opening discussion about linear and quadratic functions (see f and g in Equation (5.1)) then there is every chance that you'll be able to master polynomial functions as well; just remember that polynomial functions are a natural generalization of linear and quadratic functions. Once you've studied the examples and problems in this section, you'll hopefully agree that polynomial functions are remarkably predictable.

Definition 21 (Polynomial functions)
The most general formula for a polynomial function, p, is

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where a_n, a_{n-1}, a_{n-2}, …, a_0 are real numbers.

- We call n the degree of the polynomial, and require that n is a non-negative integer;
- a_n, a_{n-1}, a_{n-2}, …, a_0 are called the coefficients;
- We typically write polynomial functions in descending powers of x.

In particular, we call a_n the *leading* coefficient, and $a_n x^n$ the *leading term*.

Note that if a polynomial is given in factored form, then the degree can be found by counting the number of linear factors.

Example 3 – Polynomial or not: Decide if the following formulas correspond to polynomial functions or not; if so, state the degree of the polynomial.

(a) $p(x) = x^2 - 3$ (d) $s(x) = x^{-2} + x^{23}$ (g) $h(x) = \sqrt[3]{x^7} - x^2 + x$

(b) $q(x) = -4x^{1/2} + 10$ (e) $f(x) = -8$ (h) $k(x) = 4x(x+2)(x-3)$

(c) $r(x) = 10x^5$ (f) $g(x) = 3^x$ (i) $j(x) = x^2(x-4)(5-x)$

Solution (a) p is a polynomial, and its degree is 2.

(b) q is *not* a polynomial, because $\frac{1}{2}$ is not an integer.

(c) r is a polynomial, and its degree is 5.

(d) s is *not* a polynomial, because -2 is not a positive integer.

(e) f is a polynomial, and its degree is 0.

(f) g is *not* a polynomial, because the independent variable, x, is in the exponent.

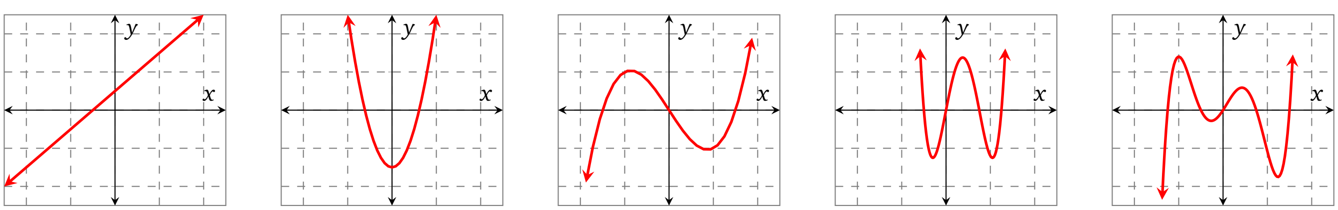
(g) h is *not* a polynomial, because $\frac{7}{3}$ is not an integer.

(h) k is a polynomial, and its degree is 3.

(i) j is a polynomial, and its degree is 4. ■

Example 4 – Typical graphs: Figure 5.4 shows graphs of some polynomial functions; the ticks have deliberately been left off the axis to allow us to concentrate on the features of each graph. Note in particular that:

- Figure 5.4a shows a degree-1 polynomial (you might also classify the function as linear) whose leading coefficient, a_1, is positive.
- Figure 5.4b shows a degree-2 polynomial (you might also classify the function as quadratic) whose leading coefficient, a_2, is positive.
- Figure 5.4c shows a degree-3 polynomial whose leading coefficient, a_3, is positive| compare its overall shape and long-run behavior to the functions described in Example 1.
- Figure 5.4d shows a degree-4 polynomial whose leading coefficient, a_4, is positive|compare its overall shape and long-run behavior to the functions described in Example 2.
- Figure 5.4e shows a degree-5 polynomial whose leading coefficient, a_5, is positive| compare its overall shape and long-run behavior to the functions described in Example 1. ■



(a) $a_1 > 0$ (b) $a_2 > 0$ (c) $a_3 > 0$ (d) $a_4 > 0$ (e) $a_5 > 0$

Figure 5.4: Graphs to illustrate typical curves of polynomial functions.

★ try it yourself ★

Problem 3

Use Example 4 and Figure 5.4 to help you sketch the graphs of polynomial functions that have negative leading coefficients| note that there are many ways to do this! The intention with this problem is to use your knowledge of transformations- in particular, *reflections*- to guide you.

make sure you try it!

The main intention behind Example 4 was to provide sketches of some typical polynomial functions. The graphs in Figure 5.4 do not have much detail| in Example 5 we study two polynomial functions in much more depth.

Example 5 Study the graphs of the polynomial functions p and q defined by the following formulas:

(a) $p(x) = \frac{1}{8}(x+6)(x+1)(x-5)$ (b) $q(x) = \frac{x}{20}(x+4)(x-3)(x-6)$

Describe the long-run behavior, the intervals of increase and decrease, and the intervals of concavity of each function. Determine if each function is odd, even, or neither.

Solution (a) The first observation we note about the function p is that since it has three linear factors, the degree of p is 3. We can illustrate this further by expanding the formula for $p(x)$

$$p(x) = \frac{x^3}{8} + \frac{x^2}{4} - \frac{29x}{8} - \frac{15}{4}$$

The curve $y = p(x)$ is graphed in Figure 5.5a. There are three zeros of p: -6, -1, and 5.

In order to determine the long-rung behavior of p, we examine the leading term of $p(x)$ which is $\frac{x^3}{8}$. If we view Figure 5.5a on a larger viewing window (imagine zooming out), then we can visualize that the overall shape of the curve $y = p(x)$ will look like the curve $y = \frac{x^3}{8}$ (see Figure 5.2).

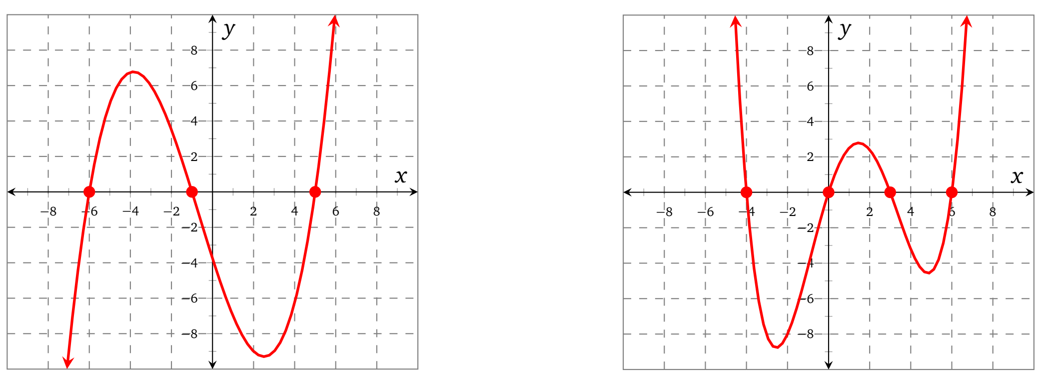
We can approximate the intervals of increase and decrease using Figure 5.5a. p is increasing on (approximately) the interval $(-\infty, -3.9) \cup (2.2, \infty)$ and decreasing on (approximately) the interval $(-3.9, 2.2)$.

We may similarly approximate the intervals of concavity: p is concave down on (approximately) the interval $(-\infty, -0.5)$ and is concave up on (approximately) the interval $(-0.5, \infty)$.

Remember that all of the power functions in Example 1 have *odd* exponents and are *odd* functions. Does it therefore follow that since p is a degree-3 polynomial and 3 is an odd number, that p is an odd function? Let's evaluate $p(-x)$ to find out

$$\begin{aligned} p(-x) &= \frac{1}{8}(-x+6)(-x+1)(-x-5) \\ &\neq -p(x) \text{ or } p(x) \end{aligned}$$

We therefore conclude that p is neither odd nor even; this is confirmed visually in Figure 5.5a since the curve $y = p(x)$ is not symmetric about the origin nor about the vertical axis.



(a) $y = \frac{1}{8}(x+6)(x+1)(x-5)$ (b) $y = \frac{x}{20}(x+4)(x-3)(x-6)$

Figure 5.5: The functions p and q.

(b) The degree of q is 4 since it has four linear factors. q has four zeros: -4, 0, 3, and 6. Furthermore, me may expand the formula for $q(x)$

$$q(x) = \frac{x^4}{20} - \frac{x^3}{4} - \frac{9x^2}{10} + \frac{18x}{5}$$

which allows to see that the leading term of q is $\frac{x^4}{20}$. If we imagine viewing Figure 5.5b on a larger viewing window, then we can visualize that the overall shape of the curve $y = q(x)$ will look like $y = \frac{x^4}{20}$ (see Figure 5.3).

Using Figure 5.5b as a guide, we see that q is increasing on (approximately) the interval $(-2.2, 1.5) \cup (4.8, \infty)$ and decreasing on (approximately) the interval $(-\infty, -2.2) \cup (1.5, 4.8)$.

We can also approximate the intervals of concavity: q is concave up on (approximately) the interval $(-\infty, -1) \cup (3.1, \infty)$, and is concave down on (approximately) the interval $(-1, 3.1)$.

The power functions in Example 2 have *even* exponents and are *even* functions. Since q has degree 4, which is an even number, does it therefore follow that q is an even function? Let's evaluate $q(-x)$ to find out

$$
\begin{aligned}
q(-x) &= -\frac{x}{20}(-x+4)(-x-3)(-x-6) \\
&\neq q(x) \text{ or } -q(x)
\end{aligned}
$$

We conclude that q is neither even nor odd; this is confirmed visually in Figure 5.5b, since the curve $y = q(x)$ is not symmetric about the vertical axis nor about the origin. ■

The polynomial functions in Example 5 had many differences, but they also had one feature in common| at each of their zeros, the curve of the function *crossed through* the horizontal axis. Not all polynomial functions exhibit this behavior as we shall see in the next example.

Example 6 – Multiple zeros: Consider the polynomial functions p, q, and r which are graphed in Figure 5.6. The formulas for p, q, and r are as follows

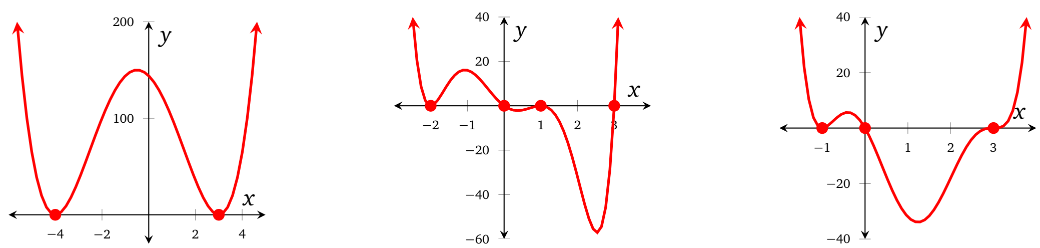
$$
\begin{aligned}
p(x) &= (x-3)^2(x+4)^2 \\
q(x) &= x(x+2)^2(x-1)^2(x-3) \\
r(x) &= x(x-3)^3(x+1)^2
\end{aligned}
$$

Find the degree of p, q, and r, and decide if the functions bounce off or cut through the horizontal axis at each of their zeros.

Solution The degree of p is 4. Referring to Figure 5.6a, the curve bounces off the horizontal axis at both zeros, 3 and 4.

The degree of q is 6. Referring to Figure 5.6b, the curve bounces off the horizontal axis at -2 and 1, and cuts through the horizontal axis at 0 and 3.

The degree of r is 6. Referring to Figure 5.6c, the curve bounces off the horizontal axis at -1, and cuts through the horizontal axis at 0 and at 3, although is flattened immediately to the left and right of 3. ■



(a) $y = p(x)$ (b) $y = q(x)$ (c) $y = r(x)$

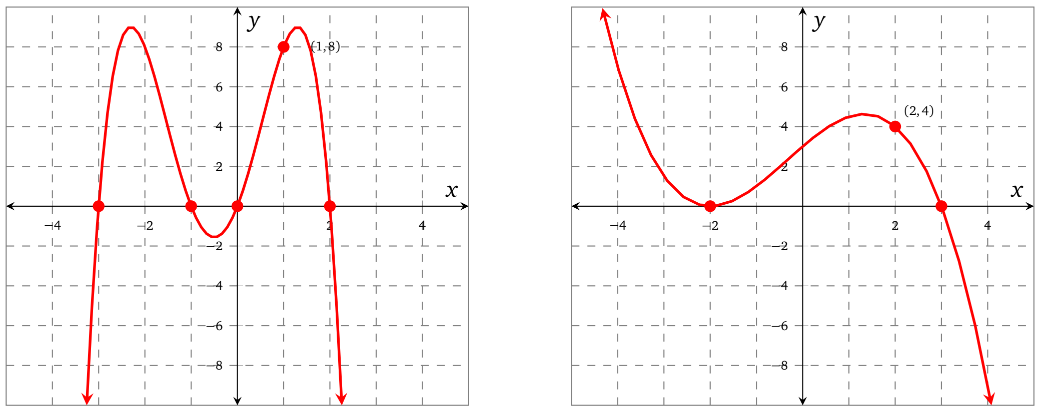
Figure 5.6

Definition 22 (Multiple zeros)
Let p be a polynomial that has a repeated linear factor $(x-a)^n$. Then we say that p has a multiple zero at a of multiplicity n and

- if the factor $(x-a)$ is repeated an even number of times, the graph of $y = p(x)$ does not cross the x axis at a, but 'bounces' off the horizontal axis at a.
- if the factor $(x-a)$ is repeated an odd number of times, the graph of $y = p(x)$ crosses the horizontal axis at a, but it looks 'flattened' there

If $n = 1$, then we say that p has a *simple* zero at a.

Example 7 – Find a formula: Find formulas for the polynomial functions, p and q, graphed in Figure 5.7.



(a) p (b) q

Figure 5.7

Solution (a) We begin by noting that the horizontal intercepts of p are $(-3, 0)$, $(-1, 0)$, $(0, 0)$ and $(2, 0)$. We also note that each zero is simple (multiplicity 1). If we assume that p has no other zeros, then we can start by writing

$$\begin{aligned} p(x) &= (x+3)(x+1)(x-0)(x-2) \\ &= x(x+3)(x+1)(x-2) \end{aligned}$$

According to Figure 5.7a, the point $(1, 8)$ lies on the curve $y = p(x)$. Let's check if the formula we have written satisfies this requirement

$$\begin{aligned} p(1) &= (1)(4)(2)(-1) \\ &= -8 \end{aligned}$$

which is clearly not correct| it is close though. We can correct this by multiplying p by a constant k; so let's assume that

$$p(x) = kx(x+3)(x+1)(x-2)$$

Then $p(1) = -8k$, and if this is to equal 8, then $k = -1$. Therefore the formula for $p(x)$ is

$$p(x) = -x(x+3)(x+1)(x-2)$$

(b) The function q has a zero at -2 of multiplicity 2, and zero of multiplicity 1 at 3 (so 3 is a simple zero of q); we can therefore assume that q has the form

$$q(x) = k(x+2)^2(x-3)$$

where k is some real number. In order to find k, we use the given ordered pair, $(2, 4)$, and evaluate $p(2)$

$$\begin{aligned} p(2) &= k(4)^2(-1) \\ &= -16k \end{aligned}$$

We solve the equation $4 = -8k$ and obtain $k = -\frac{1}{4}$ and conclude that the formula for $q(x)$ is

$$q(x) = -\frac{1}{4}(x+2)^2(x-3)$$

■

FIX

Sketching polynomial functions

In the examples that we have considered so far, we have been provided with a formula for a polynomial function and its corresponding graph. Of course, we may not always be fortunate enough to have access to the graph of the relevant function, and may need to construct it ourselves. In such a scenario, we can use steps (P_1)–(P_3) to guide us.

Steps to follow when sketching polynomial functions

(P_1) Determine the degree of the polynomial, its leading term and leading coefficient, and hence determine the long-run behavior of the polynomial| does it behave like $\pm x^2$ or $\pm x^3$ as $x \to \pm\infty$?

(P_2) Determine the zeros and their multiplicity. Mark all zeros and the vertical intercept on the graph using solid circles •.

(P_3) Deduce the overall shape of the curve, and sketch it. If there isn't enough information from the previous steps, then construct a table of values.

Remember that until we have the tools of calculus, we won't be able to find the exact coordinates of local minimums, local maximums, and points of inflection.

Before we demonstrate some examples, it is important to remember the following:

- our sketches will give a good representation of the overall shape of the graph, but until we have the tools of calculus (from MTH 251) we can not find local minimums, local maximums, and inflection points algebraically. This means that we will make our best guess as to where these points are.
- we will not concern ourselves too much with the vertical scale (because of our previous point)| we will, however, mark the vertical intercept (assuming there is one), and any horizontal asymptotes.

Example 8 Use steps (P_1)–(P_3) to sketch a graph of the function p that has formula

$$p(x) = \frac{1}{2}(x-4)(x-1)(x+3)$$

Solution (P_1) p has degree 3. The leading term of p is $\frac{1}{2}x^3$, so the leading coefficient of p is $\frac{1}{2}$. The long-run behavior of p is therefore similar to that of x^3.

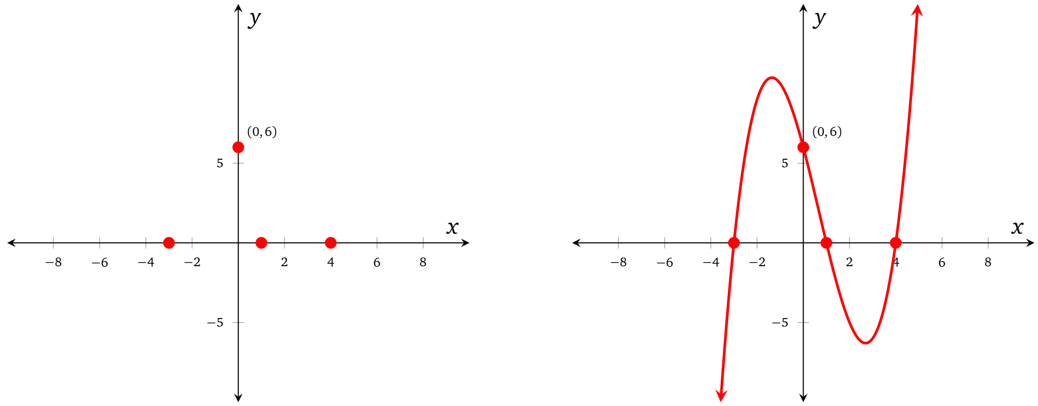
(P_2) The zeros of p are -3, 1, and 4; each zero is simple (i.e, it has multiplicity 1). This means that the curve of p cuts the horizontal axis at each zero. The vertical intercept of p is $(0, 6)$.

(P_3) We draw the details we have obtained so far on Figure 5.8a. Given that the curve of p looks like the curve of x^3 in the long-run, we are able to complete a sketch of the graph of p in Figure 5.8b.

Note that we can not find the coordinates of the local minimums, local maximums, and inflection points| for the moment we make reasonable guesses as to where these points are (you'll find how to do this in calculus). ■

Example 9 Use steps (P_1)–(P_3) to sketch a graph of the function q that has formula

$$q(x) = \frac{1}{200}(x+7)^2(2-x)(x-6)^2$$



(a) (b)

Figure 5.8: $y = \frac{1}{2}(x-4)(x-1)(x+3)$

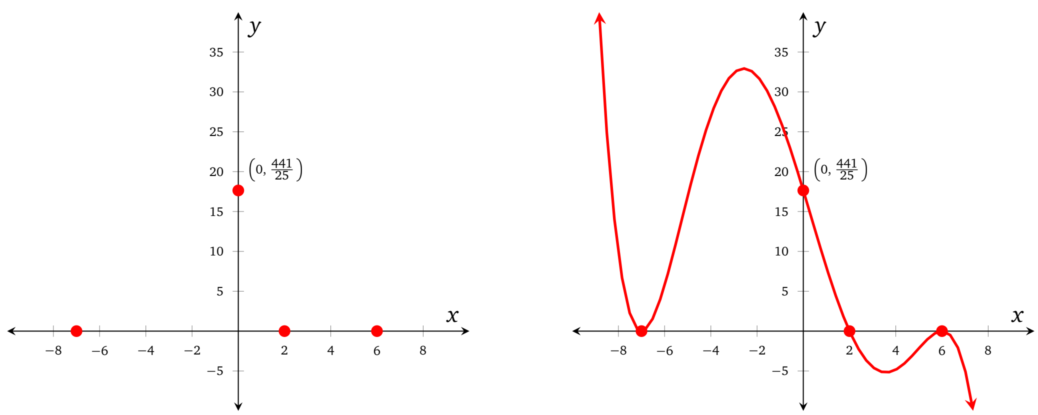
Solution (P_1) q has degree 4. The leading term of q is

$$-\frac{1}{200}x^5$$

so the leading coefficient of q is $-\frac{1}{200}$. The long-run behavior of q is therefore similar to that of $-x^5$.

(P_2) The zeros of q are -7 (multiplicity 2), 2 (simple), and 6 (multiplicity 2). The curve of q bounces off the horizontal axis at the zeros with multiplicity 2 and cuts the horizontal axis at the simple zeros. The vertical intercept of q is $\left(0, \frac{441}{25}\right)$.

(P_3) We mark the details we have found so far on Figure 5.9a. Given that the curve of q looks like the curve of $-x^5$ in the long-run, we can complete Figure 5.9b. ■



(a) (b)

Figure 5.9: $y = \frac{1}{200}(x+7)^2(2-x)(x-6)^2$

Example 10 Use steps (P_1)–(P_3) to sketch a graph of the function r that has formula

$$r(x) = \frac{1}{100}x^3(x+4)(x-4)(x-6)$$

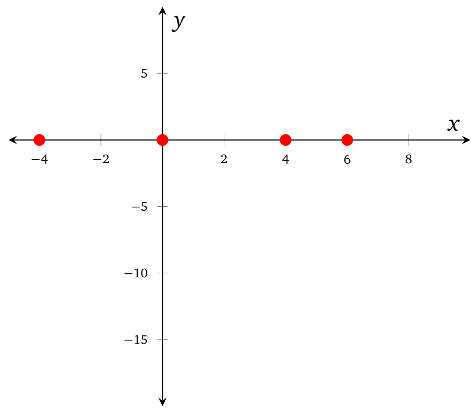
Solution (P_1) r has degree 6. The leading term of r is

$$\frac{1}{100}x^6$$

so the leading coefficient of r is $\frac{1}{100}$. The long-run behavior of r is therefore similar to that of x^6.

(P_2) The zeros of r are -4 (simple), 0 (multiplicity 3), 4 (simple), and 6 (simple). The vertical intercept of r is $(0,0)$. The curve of r cuts the horizontal axis at the simple zeros, and goes through the axis at $(0,0)$, but does so in a flattened way.

(P_3) We mark the zeros and vertical intercept on Figure 5.10a. Given that the curve of r looks like the curve of x^6 in the long-run, we complete the graph of r in Figure 5.10b. ■



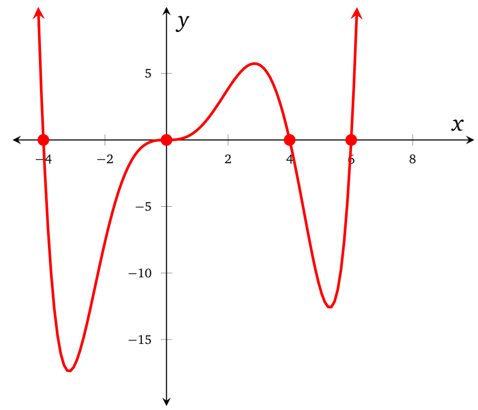


Figure 5.10: $y = \dfrac{1}{100}(x+4)x^3(x-4)(x-6)$

Example 11 – An open-topped box: A cardboard company makes open-topped boxes for their clients. The specifications dictate that the box must have a square base, and that it must be open-topped. The company uses sheets of cardboard that are 1200 cm^2. Assuming that the base of each box has side x (measured in cm), it can be shown that the volume of each box, $V(x)$, has formula

$$V(x) = \frac{x}{4}(1200 - x^2)$$

Find the dimensions of the box that maximize the volume.

Solution We graph $y = V(x)$ in Figure 5.11. Note that because x represents the length of a side, and $V(x)$ represents the volume of the box, we necessarily require both values to be positive; we illustrate the part of the curve that applies to this problem using a solid line.

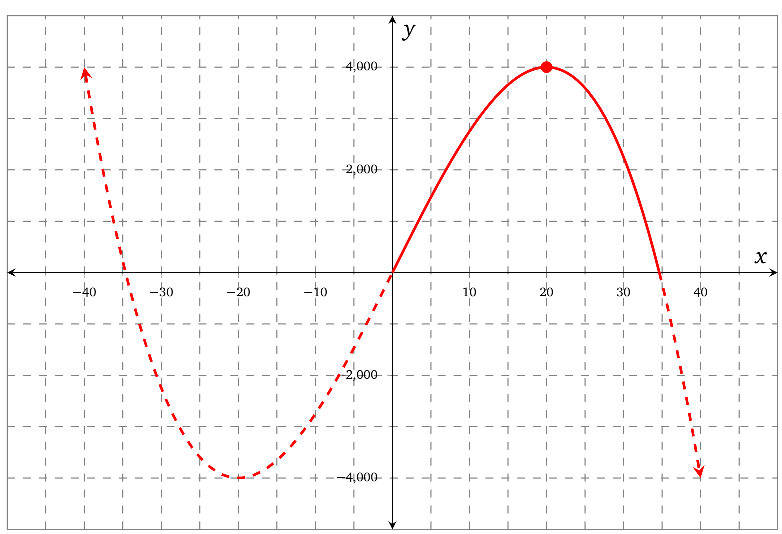


Figure 5.11: $y = V(x)$

According to Figure 5.11, the maximum volume of such a box is approximately 4000 cm^2, and we achieve it using a base of length approximately 20 cm. Since the base is square and each sheet of cardboard is $1200\,\text{cm}^2$, we conclude that the dimensions of each box are $20\,\text{cm} \times 20\,\text{cm} \times 30\,\text{cm}$. ■

Complex zeros

There has been a pattern to all of the examples that we have seen so far| the degree of the polynomial has dictated the number of *real* zeros that the polynomial has. For example, the function p in Example 8 has degree 3, and p has three real zeros; the function q in Example 9 has degree 5 and q has five real zeros.

You may wonder if this result can be generalized| does every polynomial that has degree n have n real zeros? Before we tackle the general result, let's consider an example that may help motivate it.

Example 12 Consider the polynomial function c that has formula

$$c(x) = x(x^2 + 1)$$

It is clear that c has degree 3, and that c has a (simple) zero at 0. Does c have any other zeros, i.e, can we find any values of x that satisfy the equation

$$x^2 + 1 = 0 \tag{5.2}$$

The solutions to Equation (5.2) are $\pm i$.

We conclude that c has three zeros: 0 and $\pm i$; we note that *not all of them are real*. ■

Example 12 shows that not every degree-3 polynomial has 3 *real* zeros; however, if we are prepared to venture into the complex numbers, then we can state the following theorem.

The fundamental theorem of algebra
Every polynomial function of degree n has n roots, some of which may be complex, and some may be repeated.

FIX

Example 13 Find all the zeros of the polynomial function p that has formula to solve it

$$p(x) = x^4 - 2x^3 + 5x^2$$

Solution We begin by factoring p

$$\begin{aligned} p(x) &= x^4 - 2x^3 + 5x^2 \\ &= x^2(x^2 - 2x + 5) \end{aligned}$$

We note that 0 is a zero of p with multiplicity 2. The other zeros of p can be found by solving the equation

$$x^2 - 2x + 5 = 0$$

This equation can not be factored, so we use the quadratic formula

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 20}}{2(1)} \\ &= \frac{2 \pm \sqrt{-16}}{2} \\ &= 1 \pm 2i \end{aligned}$$

We conclude that p has four zeros: 0 (multiplicity 2), and $1 \pm 2i$ (simple). ■

Example 14 Find a polynomial that has zeros at $2 \pm i\sqrt{2}$.

Solution We know that the zeros of a polynomial can be found by analyzing the linear factors. We are given the zeros, and have to work backwards to find the linear factors.

We begin by assuming that p has the form

$$\begin{aligned} p(x) &= (x-(2-i\sqrt{2}))(x-(2+i\sqrt{2})) \\ &= x^2 - x(2+i\sqrt{2}) - x(2-i\sqrt{2}) + (2-i\sqrt{2})(2+i\sqrt{2}) \\ &= x^2 - 4x + (4-2i^2) \\ &= x^2 - 4x + 6 \end{aligned}$$

We conclude that a possible formula for a polynomial function, p, that has zeros at $2 \pm i\sqrt{2}$ is

$$p(x) = x^2 - 4x + 6$$

Note that we could multiply p by any real number and still ensure that p has the same zeros. ∎

Investigations

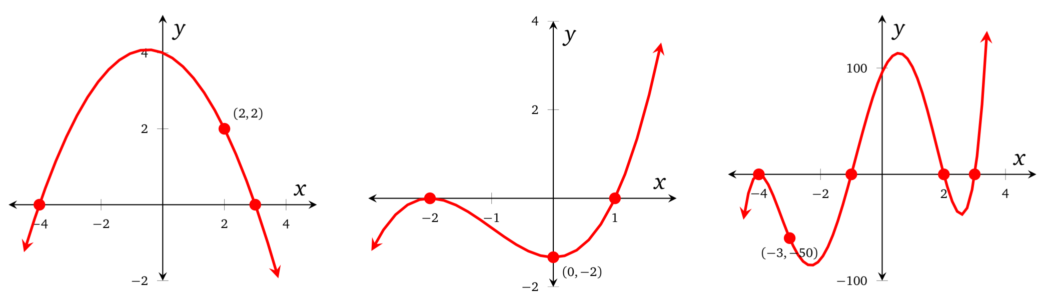
Problem 4 (Find a formula from a graph)
For each of the polynomial functions that are represented in Figure 5.12

(a) count the number of times the curve turns round, and cuts/bounces off the x axis;

(b) approximate the degree of the polynomial;

(c) use your information to find the linear factors of each polynomial, and therefore write a possible formula for each;

(d) make sure your polynomial goes through the given ordered pair.



(a) (b) (c)

Figure 5.12

Exercises

Problem 5 (Prerequisite classifacation skills)
Decide if the following formulas correspond to linear or quadratic functions.

5.1 $f(x) = 2x + 3$

5.2 $g(x) = 10 - 7x$

5.3 $h(x) = -x^2 + 3x - 9$

5.4 $k(x) = -17$

5.5 $l(x) = -82x^2 - 4$

5.6 $m(x) = 6^2 x - 8$

Problem 6 (Prerequisite slope identification)
The following formulas correspond to the linear functions α, β, γ, and δ. State the slope of each function, and hence decide if each function is increasing or decreasing.

6.1 $\alpha(x) = 4x + 1$

6.2 $\beta(x) = -9x$

6.3 $\gamma(t) = 18t + 100$

6.4 $\delta(y) = 23 - y$

Now let's generalize our findings for the most general linear function f that has formula $f(x) = mx + b$. Complete the following sentences.

6.5 When $m > 0$, the function f is ...

6.6 When $m < 0$, the function f is ...

Problem 7 (Polynomial or not?)
Decide if the following formulas correspond to polynomial functions or not; if so, state the degree of the polynomial.

7.1 $p(x) = 2x + 1$

7.2 $p(x) = 7x^2 + 4x$

7.3 $p(x) = \sqrt{x} + 2x + 1$

7.4 $p(x) = 2^x - 45$

7.5 $p(x) = 6x^4 - 5x^3 + 9$

7.6 $p(x) = -5x^{17} + 9x + 2$

7.7 $p(x) = 4x(x+7)^2(x-3)^3$

7.8 $p(x) = 4x^{-5} - x^2 + x$

7.9 $p(x) = -x^6(x^2+1)(x^3-2)$

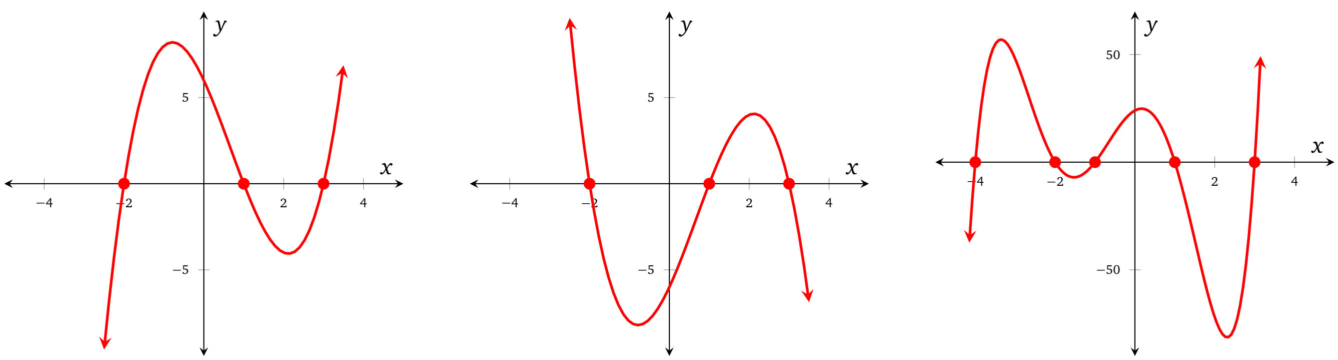
Problem 8 (Polynomial graphs)
Three polynomial functions p, m, and n are shown in Figures 5.13a–5.13c. The functions have the following formulas

$$\begin{aligned} p(x) &= (x-1)(x+2)(x-3) \\ m(x) &= -(x-1)(x+2)(x-3) \\ n(x) &= (x-1)(x+2)(x-3)(x+1)(x+4) \end{aligned}$$

Note that for our present purposes we are not concerned with the vertical scale of the graphs.

8.1 Identify both on the graph *and* algebraically, the zeros of each polynomial.

8.2 Write down the degree, how many times the curve of each function 'turns around', and how many zeros it has



(a) $y = p(x)$ (b) $y = m(x)$ (c) $y = n(x)$

Figure 5.13

Problem 9 (Horizontal intercepts)
The following formulas correspond to the polynomial functions p, q, r, and s. State the horizontal intercepts (as ordered pairs) of each function.

9.1 $p(x) = (x-1)(x+2)(x-3)(x+1)(x+4)$

9.2 $q(x) = -(x-1)(x+2)(x-3)$

9.3 $r(x) = (x-1)(x+2)(x-3)$

9.4 $s(x) = (x-2)(x+2)$

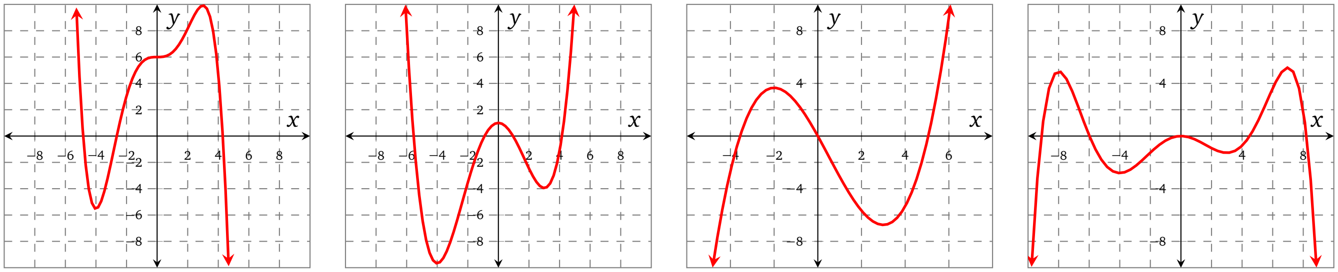
Problem 10 (Minimums, maximums, and concavity)
Four polynomial functions are graphed in Figure 5.14. The formulas for these functions are (not respectively)

$$p(x) = \frac{x^3}{6} - \frac{x^2}{4} - 3x, \qquad q(x) = \frac{x^4}{20} + \frac{x^3}{15} - \frac{6}{5}x^2 + 1$$
$$r(x) = -\frac{x^5}{50} - \frac{x^4}{40} + \frac{2x^3}{5} + 6, \qquad s(x) = -\frac{x^6}{6000} - \frac{x^5}{2500} + \frac{67x^4}{4000} + \frac{17x^3}{750} - \frac{42x^2}{125}$$

10.1 Match each of the formulas with one of the given graphs.

10.2 Approximate the zeros of each function using the appropriate graph.

10.3 Approximate the local maximums and minimums of each of the functions.



(a) (b) (c) (d)

Figure 5.14: Graphs for Problem 10.

10.4 Approximate the global maximums and minimums of each of the functions.

10.5 Approximate the intervals on which each function is increasing and decreasing.

10.6 Approximate the intervals on which each function is concave up and concave down.

10.7 The degree of q is 5. Assuming that all of the real zeros of q are shown in its graph, how many complex zeros does q have?

Problem 11 (Long-run behaviour of polynomials)
Describe the long-run behavior of each of polynomial functions in Problems 9.1–9.4.

Problem 12 (True of false?)
Let p be a polynomial function. Label each of the following statements as true (T) or false (F); if they are false, provide an example that supports your answer.

12.1 If p has degree 3, then p has 3 distinct zeros.

12.2 If p has degree 4, then $\lim_{x\to-\infty} p(x) = \infty$ and $\lim_{x\to\infty} p(x) = \infty$.

12.3 If p has even degree, then it is possible that p can have no real zeros.

12.4 If p has odd degree, then it is possible that p can have no real zeros.

Problem 13 (Find a formula from a description)
In each of the following problems, give a possible formula for a polynomial function that has the specified properties.

13.1 Degree 2 and has zeros at 4 and 5.

13.2 Degree 3 and has zeros at 4,5 and -3.

13.3 Degree 4 and has zeros at 0, 4, 5, -3.

13.4 Degree 4, with zeros that make the graph cut at 2, -5, and a zero that makes the graph touch at -2;

13.5 Degree 3, with only one zero at -1.

Problem 14 (Step (P_3))
Saheed is graphing a polynomial function, p. He is following steps (P_1)–(P_3) and has so far marked the zeros of p on Figure 5.15a. Saheed tells you that p has degree 3, but does *not* say if the leading coefficient of p is positive or negative.

14.1 Use the information in Figure 5.15a to help sketch p, assuming that the leading coefficient is positive.

14.2 Use the information in Figure 5.15a to help sketch p, assuming that the leading coefficient is negative.

Saheed now turns his attention to another polynomial function, q. He finds the zeros of q (there are only 2) and marks them on Figure 5.15b. Saheed knows that q has degree 3, but doesn't know if the leading coefficient is positive or negative.

14.3 Use the information in Figure 5.15b to help sketch q, assuming that the leading coefficient of q is positive. Hint: only one of the zeros is simple.

14.4 Use the information in Figure 5.15b to help sketch q, assuming that the leading coefficient of q is negative.

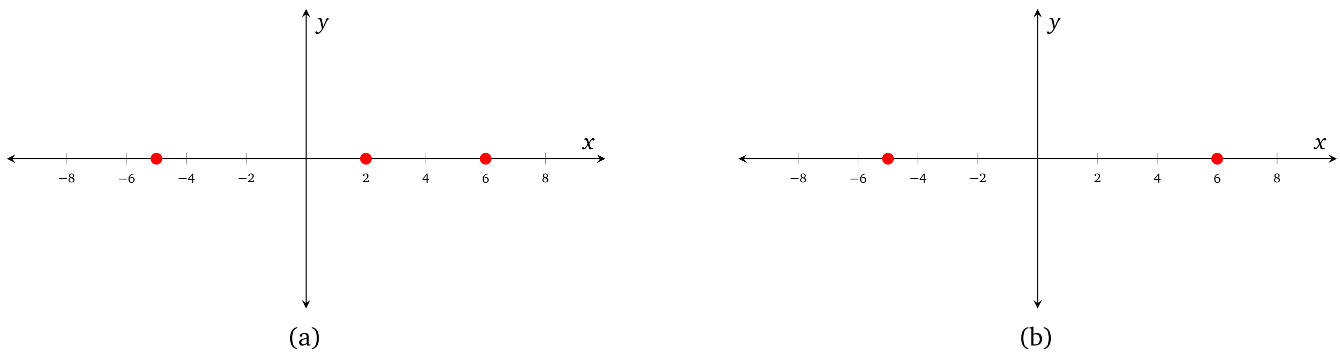


Figure 5.15

Problem 15 (Zeros)
Find all zeros of each of the following polynomial functions, making sure to detail their multiplicity. Note that you may need to use factoring, or the quadratic formula, or both! Also note that some zeros may be repeated, and some may be complex.

15.1 $p(x) = x^2 + 1$

15.2 $q(y) = (y^2 - 9)(y^2 - 7)$

15.3 $r(z) = -4z^3(z^2 + 3)(z^2 + 64)$

15.4 $a(x) = x^4 - 81$

15.5 $b(y) = y^3 - 8$

15.6 $c(m) = m^3 - m^2$

15.7 $h(n) = (n + 1)(n^2 + 4)$

15.8 $f(\alpha) = (\alpha^2 - 16)(\alpha^2 - 5\alpha + 4)$

15.9 $g(\beta) = (\beta^2 - 25)(\beta^2 - 5\beta - 4)$

Problem 16 (Given zeros, find a formula)
In each of the following problems you are given the zeros of a polynomial. Write a possible formula for each polynomial| you may leave your answer in factored form, but it may not contain complex numbers. Unless otherwise stated, assume that the zeros are simple.

16.1 1, 2

16.2 0, 5, 13

16.3 −7, 2 (multiplicity 3), 5

16.4 0, $\pm i$

16.5 $\pm 2i$, ± 7

16.6 $-2 \pm i\sqrt{6}$

FIX

Problem 17 (Composition of polynomials)
Let p and q be polynomial functions that have formulas

$$p(x) = (x + 1)(x + 2)(x + 5), \qquad q(x) = 3 - x^4$$

Evaluate each of the following.

17.1 $(p \circ q)(0)$ **17.2** $(q \circ p)(0)$ **17.3** $(p \circ q)(1)$ **17.4** $(p \circ p)(0)$

Problem 18 (Piecewise polynomial functions)
Let P be the piecewise-defined function with formula

$$P(x) = \begin{cases} (1 - x)(2x + 5)(x^2 + 1), & x \leq -3 \\ 4 - x^2, & -3 < x < 4 \\ x^3 & x \geq 4 \end{cases}$$

Evaluate each of the following

18.1 $P(-4)$ **18.2** $P(0)$ **18.3** $P(4)$ **18.4** $P(-3)$ **18.5** $(P \circ P)(0)$

Problem 19 (Function algebra)
Let p and q be the polynomial functions that have formulas

$$p(x) = x(x + 1)(x - 3)^2, \qquad q(x) = 7 - x^2$$

Evaluate each of the following (if possible).

19.1 $(p+q)(1)$ **19.2** $(p-q)(0)$ **19.3** $(p \cdot q)(\sqrt{7})$ **19.4** $\left(\frac{q}{p}\right)(1)$

19.5 What is the domain of the function $\frac{q}{p}$?

Problem 20 (Transformations: given the transformation, find the formula)
Let p be the polynomial function that has formula.

$$p(x) = 4x(x^2-1)(x+3)$$

In each of the following problems apply the given transformation to the function p and write a formula for the transformed version of p.

20.1 Shift p to the right by 5 units.

20.2 Shift p to the left by 6 units.

20.3 Shift p up by 12 units.

20.4 Shift p down by 2 units.

20.5 Reflect p over the horizontal axis.

20.6 Reflect p over the vertical axis.

Problem 21 (Find a formula from a table)
Tables 5.1a–5.1d show values of polynomial functions, p, q, r, and s.

Table 5.1: Tables for Problem 21

(a) $y = p(x)$

x	y
-4	-56
-3	-18
-2	0
-1	4
0	0
1	-6
2	-8
3	0
4	24

(b) $y = q(x)$

x	y
-4	-16
-3	-3
-2	0
-1	-1
0	0
1	9
2	32
3	75
4	144

(c) $y = r(x)$

x	y
-4	105
-3	0
-2	-15
-1	0
0	9
1	0
2	-15
3	0
4	105

(d) $y = s(x)$

x	y
-4	75
-3	0
-2	-9
-1	0
0	3
1	0
2	15
3	96
4	760

21.1 Assuming that all of the zeros of p are shown (in Table 5.1a), how many zeros does p have?

21.2 What is the degree of p?

21.3 Write a formula for $p(x)$.

21.4 Assuming that all of the zeros of q are shown (in Table 5.1b), how many zeros does q have?

21.5 Describe the difference in behavior of p and q at -2.

21.6 Given that q is a degree-3 polynomial, write a formula for $q(x)$.

21.7 Assuming that all of the zeros of r are shown (in Table 5.1c), find a formula for $r(x)$.

21.8 Assuming that all of the zeros of s are shown (in Table 5.1d), find a formula for $s(x)$.

5.2 Rational functions

Power functions with negative exponents

The study of rational functions will rely upon a good knowledge of power functions with negative exponents. Examples 1 and 2 are simple but fundamental to understanding the behavior of rational functions.

Example 1 – Power functions with odd negative exponents: Graph each the functions f, g, and h that have formulas

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{1}{x^3}, \qquad h(x) = \frac{1}{x^5}$$

and state their domain in interval notation, and their behavior as $x \to 0^-$ and $x \to 0^+$.

Solution The functions f, g, and k are plotted in Figure 5.16. The domain of each of the functions f, g, and h is $(-\infty, 0) \cup (0, \infty)$. Note that the long-run behavior of each of the functions is the same, and in particular

$$\begin{aligned} &f(x) \to 0 \text{ as } x \to \infty \\ \text{and } &f(x) \to 0 \text{ as } x \to -\infty \end{aligned}$$

The same results hold for g and h. Note also that each of the functions has a *vertical asymptote* at 0. We see that

$$\begin{aligned} &f(x) \to -\infty \text{ as } x \to 0^- \\ \text{and } &f(x) \to \infty \text{ as } x \to 0^+ \end{aligned}$$

The same results hold for g and h. Note that the range of each of the functions f, g, and h is $(-\infty, 0) \cup (0, \infty)$.

The curve of a function that has a vertical asymptote is necessarily separated into *branches*| each of the functions f, g, and h have two branches.

FIX

It appears from Figure 5.16 that each curve is symmetric about the origin| perhaps each function is *odd*. Let's test each function to see if they are odd or not:

$$\begin{aligned} f(-x) &= \frac{1}{(-x)^3} & g(-x) &= \frac{1}{(-x)^5} & h(-x) &= \frac{1}{(-x)^7} \\ &= \frac{1}{-x^3} & &= \frac{1}{-x^5} & &= \frac{1}{-x^7} \\ &= -\frac{1}{x^3} & &= -\frac{1}{x^5} & &= -\frac{1}{x^7} \\ &= -f(x) & &= -g(x) & &= -h(x) \end{aligned}$$

We conclude that each of the functions f, g, and h *are* odd. ∎

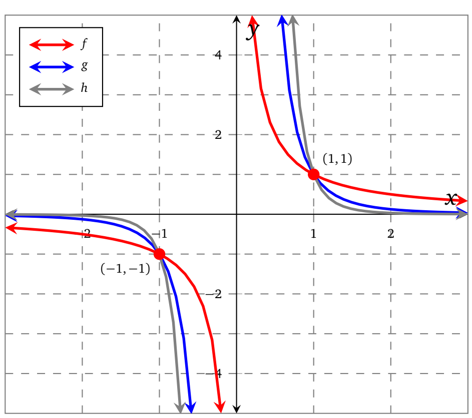


Figure 5.16

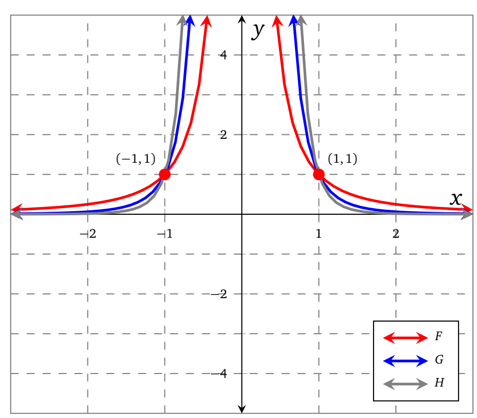


Figure 5.17

Example 2 – Power functions with even negative exponents: Graph each the functions F, G, and H that have formulas

$$F(x) = \frac{1}{x^2}, \qquad G(x) = \frac{1}{x^4}, \qquad H(x) = \frac{1}{x^6}$$

and state their domain, and their behavior as $x \to 0^-$ and $x \to 0^+$.

Solution The functions F, G, and H are plotted in Figure 5.17. The domain of each of the functions F, G, and H is $(-\infty, 0) \cup (0, \infty)$. Note that the long-run behavior of each of the functions is the same, and in particular

$$\begin{aligned} F(x) &\to 0 \text{ as } x \to \infty \\ \text{and } F(x) &\to 0 \text{ as } x \to -\infty \end{aligned}$$

As in Example 1, F has a horizontal asymptote that has equation $y = 0$. The same results hold for G and H. Note also that each of the functions has a *vertical asymptote* at 0. We see that

$$\begin{aligned} F(x) &\to \infty \text{ as } x \to 0^- \\ \text{and } F(x) &\to \infty \text{ as } x \to 0^+ \end{aligned}$$

The same results hold for G and H. Each of the functions F, G, and H have two branches, and the range of each function is $(0, \infty)$.

It appears from Figure 5.17 that each curve is symmetric about the vertical axis| perhaps each function is *even*. Let's test each function to see if they are even or not:

FIX

$$\begin{aligned} F(-x) &= \frac{1}{(-x)^2} \\ &= \frac{1}{x^2} \\ &= F(x) \end{aligned} \qquad \begin{aligned} G(-x) &= \frac{1}{(-x)^4} \\ &= \frac{1}{x^4} \\ &= G(x) \end{aligned} \qquad \begin{aligned} H(-x) &= \frac{1}{(-x)^6} \\ &= \frac{1}{x^6} \\ &= H(x) \end{aligned}$$

We conclude that each of the functions f, g, and h *are* even. ■

★ **try it yourself** ★

Problem 1

Repeat Examples 1 and 2 using (respectively) the functions that have the following formulas.

1.1 $k(x) = -\frac{1}{x}$, $m(x) = -\frac{1}{x^3}$, $n(x) = -\frac{1}{x^5}$

1.2 $K(x) = -\frac{1}{x^2}$, $M(x) = -\frac{1}{x^4}$, $N(x) = -\frac{1}{x^6}$

make sure you try it!

Rational functions

Definition 23 (Rational functions)

The most general formula for a rational function, r, is

$$r(x) = \frac{p(x)}{q(x)}$$

where both p and q are polynomial functions.

Note that

- the domain or r will be all real numbers, except those that make the *denominator*, $q(x)$, equal to 0;
- the zeros of r are the zeros of p, i.e the real numbers that make the *numerator*, $p(x)$, equal to 0.

Examples 1 and 2 are particularly important because r will behave like $\frac{1}{x}$, or $\frac{1}{x^2}$ around its vertical asymptotes, depending on the power that the relevant term is raised to| we will demonstrate this in what follows.

Example 3 – Rational or not: Decide if the following formulas correspond to rational functions or not; if the function is rational, state its domain.

(a) $r(x) = \dfrac{1}{x}$ (d) $h(x) = \dfrac{3+x}{4-x}$ (g) $m(x) = \dfrac{x+5}{(x-7)(x+9)}$

(b) $f(x) = 2^x + 3$ (e) $k(x) = \dfrac{x^3+2x}{x-15}$ (h) $n(x) = x^2 + 6x + 7$

(c) $g(x) = 19$ (f) $l(x) = 9 - 4x$ (i) $q(x) = 1 - \dfrac{3}{x+1}$

Solution (a) r is rational; the domain of r is $(-\infty, 0) \cup (0, \infty)$.

(b) f is not rational.

(c) g is not rational; g is constant.

(d) h is rational; the domain of h is $(-\infty, 4) \cup (4, \infty)$.

(e) k is rational; the domain of k is $(-\infty, 15) \cup (15, \infty)$.

(f) l is not rational; l is linear.

(g) m is rational; the domain of m is $(-\infty, -9) \cup (-9, 7) \cup (7, \infty)$.

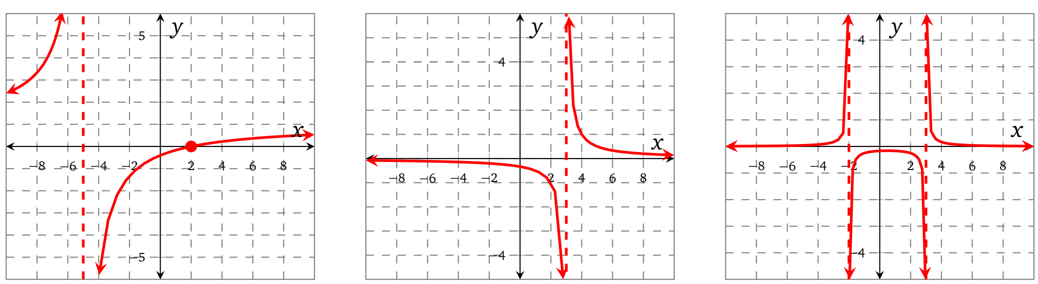
(h) n is not rational; n is quadratic (or you might describe n as a polynomial).

(i) q is rational; the domain of q is $(-\infty, -1) \cup (-1, \infty)$. ■

Example 4 – Match formula to graph: The functions r, q, and k that have formulas

$$r(x) = \frac{1}{x-3}, \qquad q(x) = \frac{x-2}{x+5}, \qquad k(x) = \frac{1}{(x+2)(x-3)}$$

are graphed in Figure 5.18. Match each formula to the appropriate graph.



(a) (b) (c)

Figure 5.18

Solution Let's start with the function r. Note that domain of r is $(-\infty, 3) \cup (0, 3)$, so we search for a function that has a vertical asymptote at 3. There are two possible choices: the functions graphed in Figures 5.18b and 5.18c, but note that the function in Figure 5.18c also has a vertical asymptote at -2 which is not consistent with the formula for $r(x)$. Therefore, $y = r(x)$ is graphed in Figure 5.18b.

The function q has domain $(-\infty, -5) \cup (-5, \infty)$, so we search for a function that has a vertical asymptote at -5. The only candidate is the curve shown in Figure 5.18a; note that the curve also goes through $(2, 0)$, which is consistent with the formula for $q(x)$, since $q(2) = 0$, i.e q has a zero at 2.

The function k has domain $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$, and has vertical asymptotes at -2 and 3. This is consistent with the graph in Figure 5.18c (and is the only curve that has 3 branches). ■

We note that each function in Example 4 behaves like $\frac{1}{x}$ around its vertical asymptotes, because each linear factor in each denominator is raised to the power 1; if (for example) the definition of r was instead

$$r(x) = \frac{1}{(x-3)^2}$$

then we would see that r behaves like $\frac{1}{x^2}$ around its vertical asymptote, and the graph of r would be very different. We will deal with these cases in the examples that follow.

Example 5 – Repeated factors in the denominator: Consider the functions f, g, and h that have formulas

$$f(x) = \frac{x-2}{(x-3)(x+2)}, \qquad g(x) = \frac{x-2}{(x-3)^2(x+2)}, \qquad h(x) = \frac{x-2}{(x-3)(x+2)^2}$$

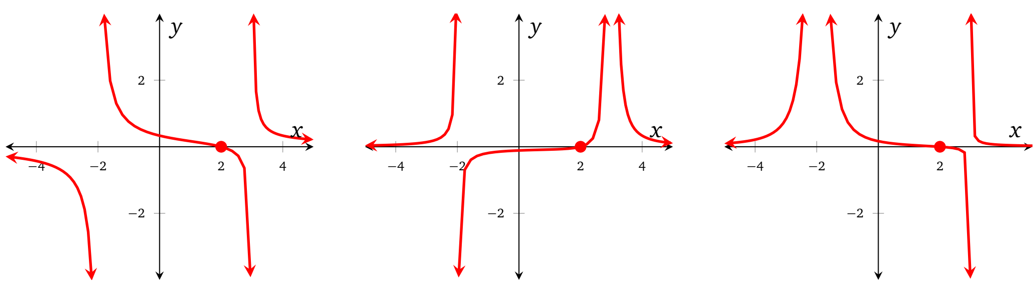
which are graphed in Figure 5.19. Note that each function has 2 vertical asymptotes, and the domain of each function is

$$(-\infty,-2)\cup(-2,3)\cup(3,\infty)$$

so we are not surprised to see that each curve has 3 branches. We also note that the numerator of each function is the same, which tells us that each function has only 1 zero at 2.

The functions g and h are different from those that we have considered previously, because they have a repeated factor in the denominator. Notice in particular the way that the functions behave around their asymptotes:

- f behaves like $\frac{1}{x}$ around both of its asymptotes;
- g behaves like $\frac{1}{x}$ around -2, and like $\frac{1}{x^2}$ around 3;
- h behaves like $\frac{1}{x^2}$ around -2, and like $\frac{1}{x}$ around 3. ■



(a) $y = \frac{x-2}{(x+2)(x-3)}$ (b) $y = \frac{x-2}{(x+2)(x-3)^2}$ (c) $y = \frac{x-2}{(x+2)^2(x-3)}$

Figure 5.19

Definition 23 says that the zeros of the rational function r that has formula $r(x) = \frac{p(x)}{q(x)}$ are the zeros of p. Let's explore this a little more.

Example 6 – Zeros: Find the zeros of the functions α, β, and γ that formulas

$$\alpha(x) = \frac{x+5}{3x-7}, \qquad \beta(x) = \frac{9-x}{x+1}, \qquad \gamma(x) = \frac{17x^2-10}{2x+1}$$

Solution We find the zeros of each function in turn by setting the numerator equal to 0. The zeros of α are found by solving

$$x+5=0$$

The zero of α is -5.

Similarly, we may solve $9-x=0$ to find the zero of β, which is clearly 9.

The zeros of γ satisfy the equation

$$17x^2-10=0$$

which we can solve using the square root property to obtain

$$x = \pm\frac{10}{17}$$

The zeros of γ are $\pm\frac{10}{17}$. ■

Long-run behavior

Our focus so far has been on the behavior of rational functions around their *vertical* asymptotes. In fact, rational functions also have interesting long-run behavior around their *horizontal* or *oblique* asymptotes. A rational function will always have either a horizontal or an oblique asymptote| the case is determined by the degree of the numerator and the degree of the denominator.

Definition 24 (Long-run behavior)
Let r be the rational function that has formula

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0}$$

We can classify the long-run behavior of the rational function r according to the following criteria:

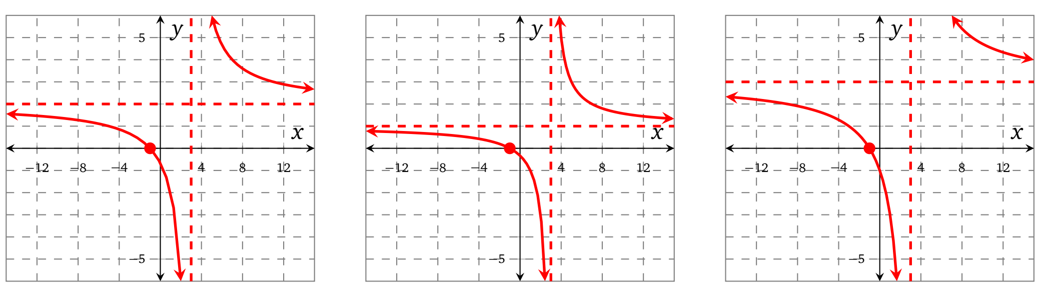
- if $n < m$ then r has a horizontal asymptote with equation $y = 0$;
- if $n = m$ then r has a horizontal asymptote with equation $y = \dfrac{a_n}{b_m}$;
- if $n > m$ then r will have an oblique asymptote as $x \to \pm\infty$ (more on this in Section 5.4)

We will concentrate on functions that have horizontal asymptotes until we reach Section 5.4.

Example 7 – Long-run behavior graphically: Kebede has graphed the functions r, s, and t that have formulas

$$r(x) = \frac{x+1}{x-3}, \qquad s(x) = \frac{2(x+1)}{x-3}, \qquad t(x) = \frac{3(x+1)}{x-3}$$

in his graphing calculator and obtained the curves shown in Figure 5.20. Kebede decides to test his knowledgeable friend Oscar, and asks him to match the formulas to the graphs.



(a) (b) (c)

Figure 5.20: Horizontal asymptotes

Oscar notices that each function has a vertical asymptote at 3 and a zero at -1. The main thing that catches Oscar's eye is that each function has a different coefficient in the numerator, and that each curve has a different horizontal asymptote. In particular, Oscar notes that:

- the curve shown in Figure 5.20a has a horizontal asymptote with equation $y = 2$;
- the curve shown in Figure 5.20b has a horizontal asymptote with equation $y = 1$;
- the curve shown in Figure 5.20c has a horizontal asymptote with equation $y = 3$.

Oscar is able to tie it all together for Kebede by referencing Definition 24. He says that since the degree of the numerator and the degree of the denominator is the same for each of the functions r, s, and t, the horizontal asymptote will be determined by evaluating the ratio of their leading coefficients.

Oscar therefore says that r should have a horizontal asymptote $y = \frac{1}{1} = 1$, s should have a horizontal asymptote $y = \frac{2}{1} = 2$, and t should have a horizontal asymptote $y = \frac{3}{1} = 3$. Kebede is able to finish the problem from here, and says that r is shown in Figure 5.20b, s is shown in Figure 5.20a, and t is shown in Figure 5.20c. ■

Example 8 – Long-run behavior numerically: Xiao and Dwayne saw Example 7 but are a little confused about horizontal asymptotes. What does it mean to say that a function r has a horizontal asymptote?

They decide to explore the concept by constructing a table of values for the rational functions R and S that have formulas

$$R(x) = \frac{-5(x+1)}{x-3}, \qquad S(x) = \frac{7(x-5)}{2(x+1)}$$

In Table 5.2 they model the behavior of R and S as $x \to \infty$, and in Table 5.3 they model the behavior of R and S as $x \to -\infty$ by substituting very large values of $|x|$ into each function.

Table 5.2: $R(x)$ and $S(x)$ as $x \to \infty$

x	$R(x)$	$S(x)$
1×10^2	-5.20619	3.29208
1×10^3	-5.02006	3.47902
1×10^4	-5.00200	3.49790
1×10^5	-5.00020	3.49979
1×10^6	-5.00002	3.49998

Table 5.3: $R(x)$ and $S(x)$ as $x \to -\infty$

x	$R(x)$	$S(x)$
-1×10^2	-4.80583	3.71212
-1×10^3	-4.98006	3.52102
-1×10^4	-4.99800	3.50210
-1×10^5	-4.99980	3.50021
-1×10^6	-4.99998	3.50002

Xiao and Dwayne study Tables 5.2 and 5.3 and decide that the functions R and S never actually touch their horizontal asymptotes, but they do get infinitely close. They also feel as if they have a better understanding of what it means to study the behavior of a function as $x \to \pm\infty$. ■

Example 9 – Repeated factors in the numerator: Consider the functions f, g, and h that have formulas

$$f(x) = \frac{(x-2)^2}{(x-3)(x+1)}, \qquad g(x) = \frac{x-2}{(x-3)(x+1)}, \qquad h(x) = \frac{(x-2)^3}{(x-3)(x+1)}$$

which are graphed in Figure 5.21. We note that each function has vertical asymptotes at -1 and 3, and so the domain of each function is

$$(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$$

We also notice that the numerators of each function are quite similar| indeed, each function has a zero at 2, but how does each function behave around their zero?

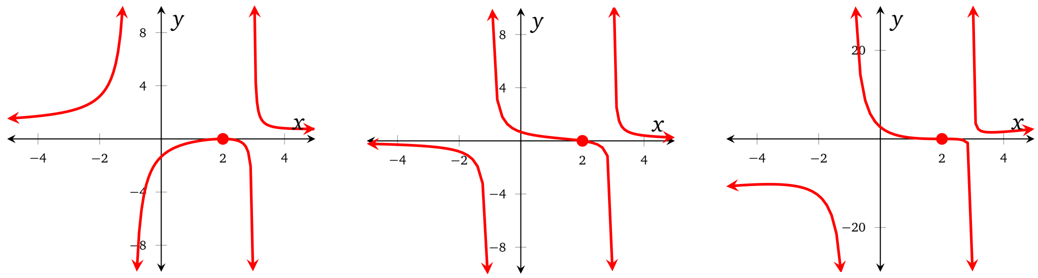
Using Figure 5.21 to guide us, we note that

- f has a horizontal intercept $(2, 0)$, but the curve of f does not cut the horizontal axis| it bounces off it;
- g also has a horizontal intercept $(2, 0)$, and the curve of g *does* cut the horizontal axis;
- h has a horizontal intercept $(2, 0)$, and the curve of h also cuts the axis, but appears flattened as it does so.

We can further enrich our study by discussing the long-run behavior of each function. Using the tools of Definition 24, we can deduce that

- f has a horizontal asymptote with equation $y = 1$;

- g has a horizontal asymptote with equation $y = 0$;
- h does *not* have a horizontal asymptote| it has an oblique asymptote (we'll study this more in Section 5.4). ■



(a) $y = \dfrac{(x-2)^2}{(x+1)(x-3)}$ (b) $y = \dfrac{x-2}{(x+1)(x-3)}$ (c) $y = \dfrac{(x-2)^3}{(x+1)(x-3)}$

Figure 5.21

Holes

Rational functions have a vertical asymptote at a if the denominator is 0 at a. What happens if the numerator is 0 at the same place? In this case, we say that the rational function has a *hole* at a.

Definition 25 (Holes)
The rational function

$$r(x) = \frac{p(x)}{q(x)}$$

has a hole at a if $p(a) = q(a) = 0$. Note that holes are different from a vertical asymptotes. We represent that r has a hole at the point $(a, r(a))$ on the curve $y = r(x)$ by using a hollow circle, $\circ$.

Example 10 Mohammed and Sue have graphed the function r that has formula

$$r(x) = \frac{x^2 + x - 6}{(x-2)}$$

in their calculators, and can not decide if the correct graph is Figure 5.22 or Figure 5.23.

Luckily for them, Oscar is nearby, and can help them settle the debate. Oscar demonstrates that

$$\begin{aligned} r(x) &= \frac{(x+3)(x-2)}{(x-2)} \\ &= x + 3 \end{aligned}$$

but only when $x \neq 2$, because the function is undefined at 2. Oscar says that this necessarily means that the domain or r is

$$(-\infty, 2) \cup (2, \infty)$$

and that r must have a hole at 2. ■

Mohammed and Sue are very grateful for the clarification, and conclude that the graph of r is shown in Figure 5.23.

Example 11 Consider the function f that has formula

$$f(x) = \frac{x(x+3)}{x^2 - 4x}$$

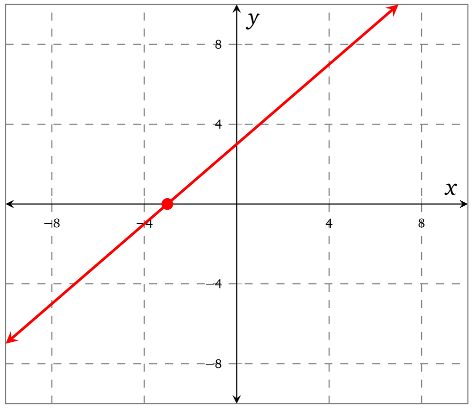


Figure 5.22

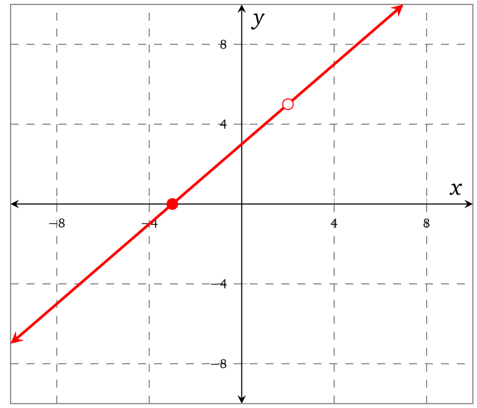


Figure 5.23

The domain of f is $(-\infty,0)\cup(0,4)\cup(4,\infty)$ because both 0 and 4 make the denominator equal to 0. Notice that

$$\begin{aligned} f(x) &= \frac{x(x+3)}{x(x-4)} \\ &= \frac{x+3}{x-4} \end{aligned}$$

provided that $x \neq 0$. Since 0 makes the numerator and the denominator 0 at the same time, we say that f has a hole at $(0, -3/4)$. Note that this necessarily means that f does not have a vertical intercept.

We also note f has a vertical asymptote at 4; the function is graphed in Figure 5.24.

Example 12 – Minimums and maximums: Seamus and Trang are discussing rational functions. Seamus says that if a rational function has a vertical asymptote, then it can not possibly have local minimums and maximums, nor can it have global minimums and maximums.

Trang says this statement is not always true. She plots the functions f and g that have formulas

$$f(x) = -\frac{32(x-1)(x+1)}{(x-2)^2(x+2)^2}, \qquad g(x) = \frac{32(x-1)(x+1)}{(x-2)^2(x+2)^2}$$

in Figures 5.25 and 5.26 and shows them to Seamus. On seeing the graphs, Seamus quickly corrects himself, and says that f has a local (and global) maximum of 2 at 0, and that g has a local (and global) minimum of -2 at 0.

Seamus also notes that (in its domain) the function f is always concave down, and that (in its domain) the function g is always concave up. Furthermore, Trang observes that each function behaves like $\frac{1}{x^2}$ around each of its vertical asymptotes, because each linear factor in the denominator is raised to the power 2.

Oscar stops by and reminds both students about the long-run behavior; according to Definition 24 since the degree of the denominator is greater than the degree of the numerator (in both functions), each function has a horizontal asymptote at $y = 0$.

Investigations

Problem 2 (The spaghetti incident)
The same Queen from Example 2 on page 32 has recovered from the rice experiments, and has called her loyal jester for another challenge.

The jester has an 11−in piece of uncooked spaghetti that he puts on a table; he uses a book to cover 1 in of it so that 10 in hang over the edge. The jester then produces a box of mg weights that can be hung from the spaghetti.

The jester says it will take y mg to break the spaghetti when hung x in from the edge, according to the rule $y = \frac{100}{x}$.

Figure 5.24: $y = \dfrac{x(x+3)}{x^2-4x}$

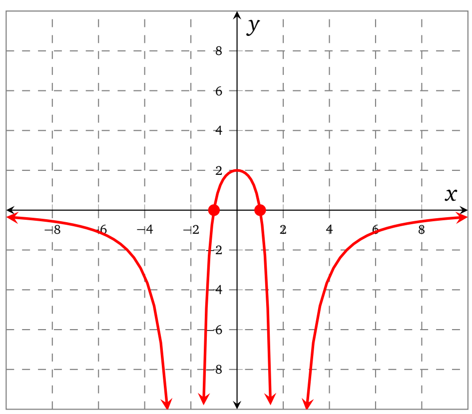


Figure 5.25: $y = f(x)$

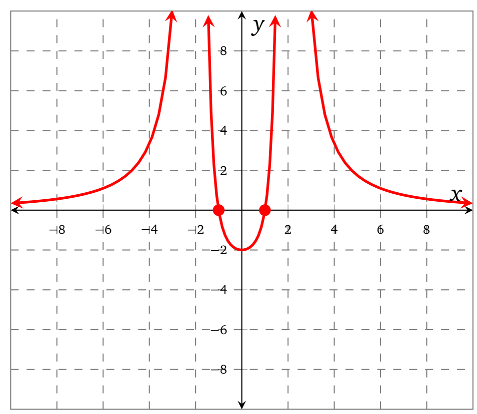


Figure 5.26: $y = g(x)$

2.1 Help the Queen complete Table 5.4, and use 2 digits after the decimal where appropriate.

2.2 What do you notice about the number of mg that it takes to break the spaghetti as x increases?

2.3 The Queen wonders what happens when x gets very small| help the Queen construct a table of values for x and y when $x = 0.0001, 0.001, 0.01, 0.1, 0.5, 1$.

2.4 What do you notice about the number of mg that it takes to break the spaghetti as $x \to 0$? Would it ever make sense to let $x = 0$?

2.5 Plot your results from Problems 2.1 and 2.3 on the same graph, and join the points using a smooth curve| set the maximum value of y as 200, and note that this necessarily means that

Table 5.4

x	y
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

you will not be able to plot all of the points.

2.6 Using your graph, observe what happens to y as x increases. If we could somehow construct a piece of uncooked spaghetti that was 101 in long, how many mg would it take to break the spaghetti?

The Queen looks forward to more food-related investigations from her jester.

Problem 3 (Debt Amortization)
To amortize a debt means to pay it off in a given length of time using equal periodic payments. The payments include interest on the unpaid balance. The following formula gives the monthly payment, M, in dollars that is necessary to amortize a debt of P dollars in n months at a monthly interest rate of i

$$M = \frac{P \cdot i}{1-(1+i)^{-n}}$$

Use this formula in each of the following problems.

3.1 What monthly payments are necessary on a credit card debt of \$2000 at 1.5 % monthly if you want to pay off the debt in 2 years? In one year? How much money will you save by paying off the debt in the shorter amount of time?

3.2 To purchase a home, a family needs a loan of \$300,000 at 5.2 % annual interest. Compare a 20 year loan to a 30 year loan and make a recommendation for the family. (Note: when given an annual interest rate, it is a common business practice to divide by 12 to get a monthly rate.)

3.3 Ellen wants to make monthly payments of \$100 to pay off a debt of \$3000 at 12 % annual interest. How long will it take her to pay off the debt?

3.4 Jake is going to buy a new car. He puts \$2000 down and wants to finance the remaining \$14,000. The dealer will offer him 4 % annual interest for 5 years, or a \$2000 rebate which he can use to reduce the amount of the loan and 8 % annual interest for 5 years. Which should he choose?

Exercises

Problem 4 (Rational or not)
Decide if the following formulas correspond to rational functions or not; if the function is rational, state its domain.

4.1 $r(x) = \dfrac{3}{x}$

4.2 $s(y) = \dfrac{y}{6}$

4.3 $t(z) = \dfrac{4-x}{7-8z}$

4.4 $u(w) = \dfrac{w^2}{(w-3)(w+4)}$

4.5 $v(x) = \dfrac{4}{(x-2)^2}$

4.6 $w(x) = \dfrac{9-x}{x+17}$

4.7 $a(x) = x^2 + 4$

4.8 $b(y) = 3^y$

4.9 $c(z) = \dfrac{z^2}{z^3}$

4.10 $d(x) = x^2(x+3)(5x-7)$

4.11 $e(\alpha) = \dfrac{\alpha^2}{\alpha^2-1}$

4.12 $f(\beta) = \dfrac{3}{4}$

Problem 5 (Function evaluation)
Let r be the function that has formula

$$r(x) = \frac{(x-2)(x+3)}{(x+5)(x-7)}$$

Evaluate each of the following (if possible); if the value is undefined, then state so.

5.1 $r(0)$

5.2 $r(1)$

5.3 $r(2)$

5.4 $r(4)$

5.5 $r(7)$

5.6 $r(-3)$

5.7 $r(-5)$

5.8 $r\left(\frac{1}{2}\right)$

Problem 6 (Holes or asymptotes?)
State the domain of each of the rational functions implied by the following formulas. Identify any holes or asymptotes.

6.1 $f(x) = \dfrac{12}{x-2}$

6.2 $g(x) = \dfrac{x^2+x}{(x+1)(x-2)}$

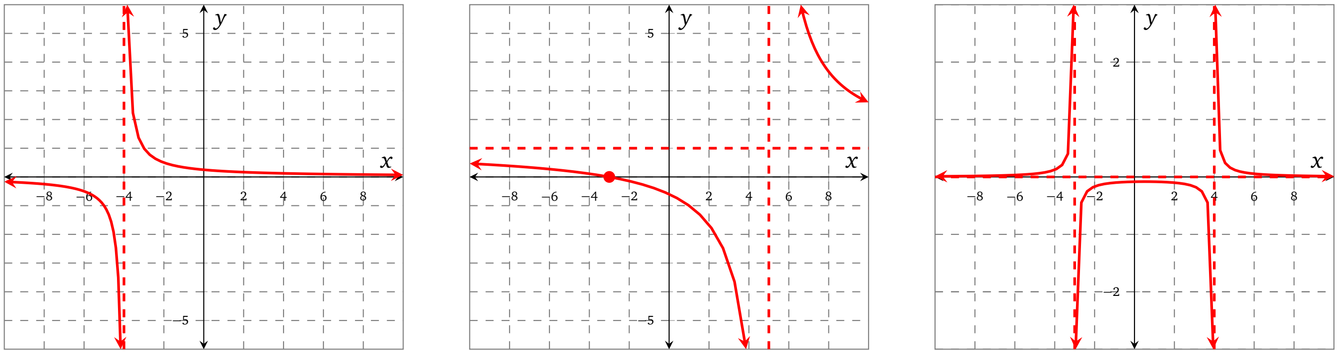
6.3 $h(x) = \dfrac{x^2+5x+4}{x^2+x-12}$

6.4 $k(z) = \dfrac{z+2}{2z-3}$

6.5 $l(w) = \dfrac{w}{w^2+1}$

6.6 $m(t) = \dfrac{14}{13-t^2}$

Problem 7 (Find a formula from a graph)
Consider the rational functions graphed in Figure 5.27. Find the vertical asymptotes for each function, together with any zeros, and give a possible formula for each.



(a) (b) (c)

Figure 5.27

Problem 8 (Find a formula from a description)
In each of the following problems, give a formula of a rational function that has the listed properties.

8.1 Vertical asymptote at 2.

8.2 Vertical asymptote at 5.

8.3 Vertical asymptote at -2, and zero at 6.

8.4 Zeros at 2 and -5 and vertical asymptotes at 1 and -7.

Problem 9 (Given formula, find horizontal asymptotes)
Each of the rational functions implied by the following formulas has a horizontal asymptote. Write the equation of the horizontal asymptote for each function.

9.1 $f(x) = \dfrac{1}{x}$

9.2 $g(x) = \dfrac{2x+3}{x}$

9.3 $h(x) = \dfrac{x^2+2x}{x^2+3}$

9.4 $k(x) = \dfrac{x^2+7}{x}$

9.5 $l(x) = \dfrac{3x-2}{5x+8}$

9.6 $m(x) = \dfrac{3x-2}{5x^2+8}$

9.7 $n(x) = \dfrac{(6x+1)(x-7)}{(11x-8)(x-5)}$

9.8 $p(x) = \dfrac{19x^3}{5-x^4}$

9.9 $q(x) = \dfrac{14x^2+x}{1-7x^2}$

Problem 10 (Given horizontal asymptotes, find formula)
In each of the following problems, give a formula for a rational function that has the given horizontal asymptote. Note that there may be more than one option.

10.1 $y = 7$

10.2 $y = -1$

10.3 $y = 53$

10.4 $y = -17$

10.5 $y = \dfrac{3}{2}$

10.6 $y = 0$

10.7 $y = -1$

10.8 $y = 2$

Problem 11 (Find a formula from a description)
In each of the following problems, give a formula for a rational function that has the prescribed properties. Note that there may be more than one option.

11.1 $f(x) \to 3$ as $x \to \pm\infty$.

11.2 $r(x) \to -4$ as $x \to \pm\infty$.

11.3 $k(x) \to 2$ as $x \to \pm\infty$, and k has vertical asymptotes at -3 and 5.

Problem 12
Let r be the rational function that has

$$r(x) = \frac{(x+2)(x-1)}{(x+3)(x-4)}$$

Each of the following questions are in relation to this function.

12.1 What is the vertical intercept of this function? State your answer as an ordered pair.

12.2 What values of x make the denominator equal to 0?

12.3 Use your answer to Problem 12.2 to write the domain of the function in both interval, and set builder notation.

12.4 What are the vertical asymptotes of the function? State your answers in the form $x =$

12.5 What values of x make the numerator equal to 0?

12.6 Use your answer to Problem 12.5 to write the horizontal intercepts of r as ordered pairs.

Problem 13 (Holes)
Josh and Pedro are discussing the function

$$r(x) = \frac{x^2-1}{(x+3)(x-1)}$$

13.1 What is the domain of r?

13.2 Josh notices that the numerator can be factored- can you see how?

13.3 Pedro asks, 'Doesn't that just mean that

$$r(x) = \frac{x+1}{x+3}$$

for all values of x?' Josh says, 'Nearly. . . but not for all values of x'. What does Josh mean?

13.4 Where does r have vertical asymptotes, and where does it have holes?

13.5 Sketch a graph of r.

Problem 14 (Function algebra)
Let r and s be the rational functions that have formulas

$$r(x) = \frac{2-x}{x+3}, \qquad s(x) = \frac{x^2}{x-4}$$

Evaluate each of the following (if possible).

14.1 $(r+s)(5)$

14.2 $(r-s)(3)$

14.3 $(r \cdot s)(4)$

14.4 $\left(\frac{r}{s}\right)(1)$

Problem 15 (Transformations: given the transformation, find the formula)
Let r be the rational function that has formula.

$$r(x) = \frac{x+5}{2x-3}$$

In each of the following problems apply the given transformation to the function r and write a formula for the transformed version of r.

15.1 Shift r to the right by 3 units.

15.2 Shift r to the left by 4 units.

15.3 Shift r up by π units.

15.4 Shift r down by 17 units.

15.5 Reflect r over the horizontal axis.

15.6 Reflect r over the vertical axis.

Table 5.5: Tables for Problem 16

(a) $y = r(x)$

x	y
-4	$7/2$
-3	-18
-2	X
-1	-4
0	$-3/2$
1	$-2/3$
2	$-1/4$
3	0
4	$1/6$

(b) $y = s(x)$

x	y
-4	$-2/21$
-3	$-1/12$
-2	0
-1	X
0	$-2/3$
1	$-3/4$
2	$-4/3$
3	X
4	$6/5$

(c) $y = t(x)$

x	y
-4	$3/5$
-3	0
-2	X
-1	3
0	3
1	X
2	0
3	$3/5$
4	$7/9$

(d) $y = u(x)$

x	y
-4	$16/7$
-3	X
-2	$-4/5$
-1	$-1/8$
0	0
1	$-1/8$
2	$-4/5$
3	X
4	$16/7$

Problem 16 (Find a formula from a table)
Tables 5.5a–5.5d show values of rational functions r, q, s, and t. Assume that any values marked with an X are undefined.

16.1 Given that the formula for $r(x)$ has the form $r(x) = \dfrac{x - A}{x - B}$, use Table 5.5a to find values of A and B.

16.2 Check your formula by computing $r(x)$ at the values specified in the table.

16.3 The function s in Table 5.5b has two vertical asymptotes and one zero. Can you find a formula for $s(x)$?

16.4 Check your formula by computing $s(x)$ at the values specified in the table.

16.5 Given that the formula for $t(x)$ has the form $t(x) = \dfrac{(x - A)(x - B)}{(x - C)(x - D)}$, use Table 5.5c to find the values of A, B, C, and D; hence write a formula for $t(x)$.

16.6 Given that the formula for $u(x)$ has the form $u(x) = \dfrac{(x - A)^2}{(x - B)(x - C)}$, use Table 5.5d to find the values of A, B, and C; hence write a formula for $u(x)$.

5.3 Graphing rational functions (horizontal asymptotes)

We studied rational functions in the previous section, but were not asked to graph them; in this section we will demonstrate the steps to be followed in order to sketch graphs of the functions.

Remember from Definition 23 on page 128 that rational functions have the form

$$r(x) = \frac{p(x)}{q(x)}$$

In this section we will restrict attention to the case when

$$\text{degree of } p \leq \text{degree of } q$$

Note that this necessarily means that each function that we consider in this section *will have a horizontal asymptote* (see Definition 24 on page 131). The cases in which the degree of p is greater than the degree of q is covered in the next section.

Before we begin, it is important to remember the following:

- Our sketches will give a good representation of the overall shape of the graph, but until we have the tools of calculus (from MTH 251) we can not find local minimums, local maximums, and inflection points algebraically. This means that we will make our best guess as to where these points are.

- We will not concern ourselves too much with the vertical scale (because of our previous point)| we will, however, mark the vertical intercept (assuming there is one), and any horizontal asymptotes.

FIX

Steps to follow when sketching rational functions

(R_1) Find all vertical asymptotes and holes, and mark them on the graph using dashed vertical lines and open circles $\circ$ respectively.

(R_2) Find any intercepts, and mark them using solid circles $\bullet$; determine if the curve cuts the axis, or bounces off it at each zero.

(R_3) Determine the behavior of the function around each asymptote| does it behave like $\frac{1}{x}$ or $\frac{1}{x^2}$?

(R_4) Determine the long-run behavior of the function, and mark the horizontal asymptote using a dashed horizontal line.

(R_5) Deduce the overall shape of the curve, and sketch it. If there isn't enough information from the previous steps, then construct a table of values including sample points from each branch.

Remember that until we have the tools of calculus, we won't be able to find the exact coordinates of local minimums, local maximums, and points of inflection.

The examples that follow show how steps (R_1)–(R_5) can be applied to a variety of different rational functions.

Example 1 Use steps (R_1)–(R_5) to sketch a graph of the function r that has formula

$$r(x) = \frac{1}{x-2}$$

Solution (R_1) r has a vertical asymptote at 2; r does not have any holes. The curve of r will have 2 branches.

(R_2) r does not have any zeros since the numerator is never equal to 0. The vertical intercept of r is $\left(0, -\frac{1}{2}\right)$.

(R_3) r behaves like $\frac{1}{x}$ around its vertical asymptote since $(x-2)$ is raised to the power 1.

(R_4) Since the degree of the numerator is less than the degree of the denominator, according to Definition 24 on page 131 the horizontal asymptote of r has equation $y = 0$.

(R_5) We put the details we have obtained so far on Figure 5.28a. Notice that there is only one way to complete the graph, which we have done in Figure 5.28b. ■

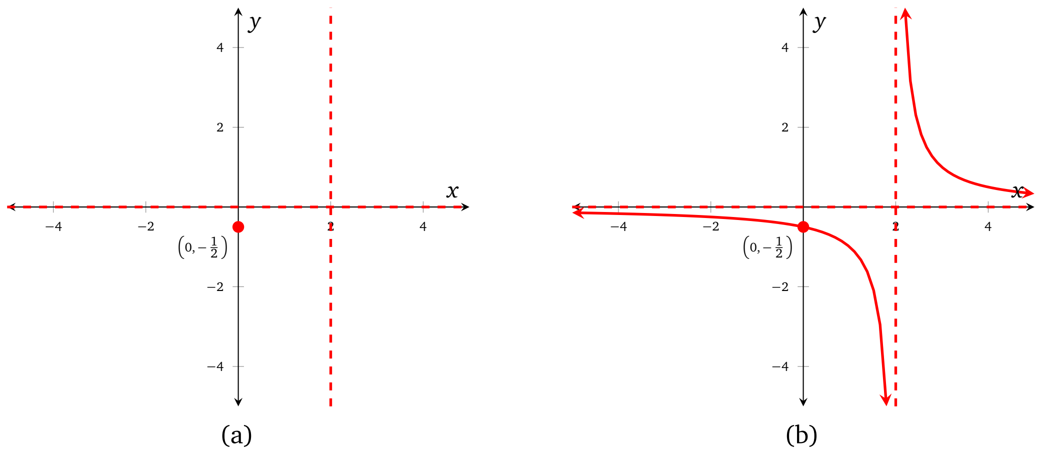


Figure 5.28: $y = \dfrac{1}{x-2}$

The function r in Example 1 has a horizontal asymptote which has equation $y = 0$. This asymptote lies on the horizontal axis, and you might (understandably) find it hard to distinguish between the two lines (Figure 5.28b). When faced with such a situation, it is perfectly acceptable to draw the horizontal axis as a dashed line| just make sure to label it correctly. We will demonstrate this in the next example.

Example 2 Use steps (R_1)–(R_5) to sketch a graph of the function v that has formula

$$v(x) = \frac{10}{x}$$

Solution (R_1) v has a vertical asymptote at 0. v does not have any holes. The curve of v will have 2 branches.

(R_2) v does not have any zeros (since $10 \neq 0$). Furthermore, v does not have a vertical intercept since $v(0)$ is undefined.

(R_3) v behaves like $\frac{1}{x}$ around its vertical asymptote.

(R_4) v has a horizontal asymptote with equation $y = 0$.

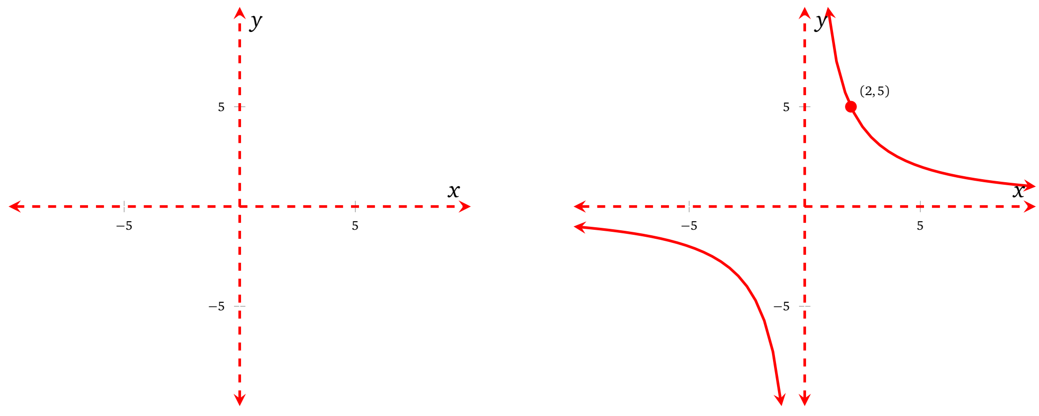
(R_5) We put the details we have obtained so far in Figure 5.29a. We do not have enough information to sketch v yet (because v does not have any intercepts), so let's pick a sample point in either of the 2 branches| it doesn't matter where our sample point is, because we know what the overall shape will be. Let's compute $v(2)$

$$\begin{aligned} v(2) &= \frac{10}{2} \\ &= 5 \end{aligned}$$

We therefore mark the point $(2, 5)$ on Figure 5.29b, and then complete the sketch using the details we found in the previous steps. ■

Example 3 Use steps (R_1)–(R_5) to sketch a graph of the function u that has formula

$$u(x) = \frac{-4(x^2 - 9)}{x^2 - 8x + 15}$$



(a) (b)

Figure 5.29: $y = \dfrac{10}{x}$

Solution (R_1) We begin by factoring both the numerator and denominator of u to help us find any vertical asymptotes or holes

$$\begin{aligned} u(x) &= \frac{-4(x^2-9)}{x^2-8x+15} \\ &= \frac{-4(x+3)(x-3)}{(x-5)(x-3)} \\ &= \frac{-4(x+3)}{x-5} \end{aligned}$$

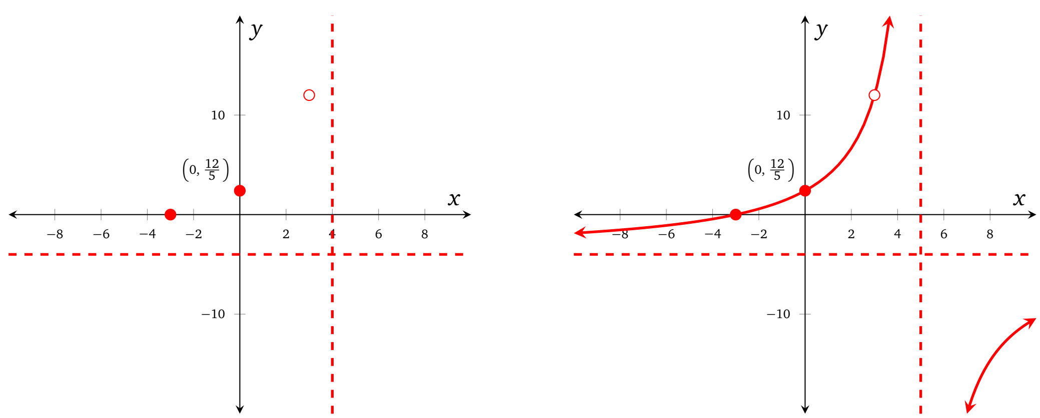
provided that $x \neq 3$. Therefore u has a vertical asymptote at 5 and a hole at 3. The curve of u has 2 branches.

(R_2) u has a simple zero at -3. The vertical intercept of u is $\left(0, \frac{12}{5}\right)$.

(R_3) u behaves like $\frac{1}{x}$ around its vertical asymptote at 4.

(R_4) Using Definition 24 on page 131 the equation of the horizontal asymptote of u is $y = -4$.

(R_5) We put the details we have obtained so far on Figure 5.28a. Notice that there is only one way to complete the graph, which we have done in Figure 5.28b. ■



(a) (b)

Figure 5.30: $y = \dfrac{-4(x+3)}{x-5}$

Examples 1–3 have focused on functions that only have one vertical asymptote; the remaining examples in this section concern functions that have more than one vertical asymptote. We will demonstrate that steps (R_1)–(R_5) still apply.

Example 4 Use steps (R_1)–(R_5) to sketch a graph of the function w that has formula

$$w(x) = \frac{2(x+3)(x-5)}{(x+5)(x-4)}$$

Solution (R_1) w has vertical asymptotes at -5 and 4. w does not have any holes. The curve of w will have 3 branches.

(R_2) w has simple zeros at -3 and 5. The vertical intercept of w is $\left(0, \frac{3}{2}\right)$.

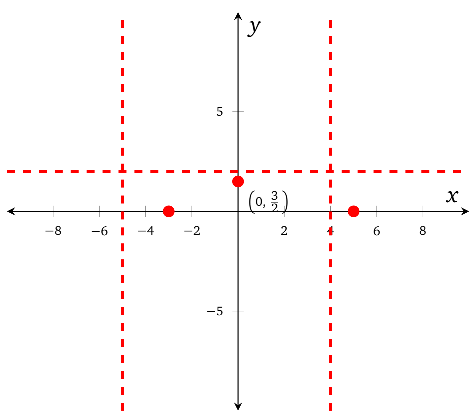
(R_3) w behaves like $\frac{1}{x}$ around both of its vertical asymptotes.

(R_4) The degree of the numerator of w is 2 and the degree of the denominator of w is also 2. Using the ratio of the leading coefficients of the numerator and denominator, we say that w has a horizontal asymptote with equation $y = \frac{2}{1} = 2$.

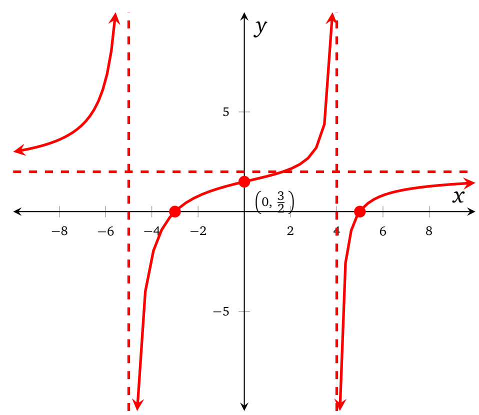
(R_5) We put the details we have obtained so far on Figure 5.31a.

The function w is a little more complicated than the functions that we have considered in the previous examples because the curve has 3 branches. When graphing such functions, it is generally a good idea to start with the branch for which you have the most information| in this case, that is the *middle* branch on the interval $(-5, 4)$.

Once we have drawn the middle branch, there is only one way to complete the graph (because of our observations about the behavior of w around its vertical asymptotes), which we have done in Figure 5.31b. ∎



(a)



(b)

Figure 5.31: $y = \dfrac{2(x+3)(x-5)}{(x+5)(x-4)}$

The rational functions that we have considered so far have had simple factors in the denominator; each function has behaved like $\frac{1}{x}$ around each of its vertical asymptotes. Examples 5 and 6 consider functions that have a repeated factor in the denominator.

Example 5 Use steps (R_1)–(R_5) to sketch a graph of the function f that has formula

$$f(x) = \frac{100}{(x+5)(x-4)^2}$$

Solution (R_1) f has vertical asymptotes at -5 and 4. f does not have any holes. The curve of f will have 3 branches.

(R_2) f does not have any zeros (since $100 \neq 0$). The vertical intercept of f is $\left(0, \frac{5}{4}\right)$.

(R_3) f behaves like $\frac{1}{x}$ around -5 and behaves like $\frac{1}{x^2}$ around 4.

(R_4) The degree of the numerator of f is 0 and the degree of the denominator of f is 2. f has a horizontal asymptote with equation $y = 0$.

(R_5) We put the details we have obtained so far on Figure 5.32a.

The function f is similar to the function w that we considered in Example 4| it has two vertical asymptotes and 3 branches, but in contrast to w it does not have any zeros.

We sketch f in Figure 5.32b, using the middle branch as our guide because we have the most information about the function on the interval $(-5, 4)$.

Once we have drawn the middle branch, there is only one way to complete the graph because of our observations about the behavior of f around its vertical asymptotes (it behaves like $\frac{1}{x}$), which we have done in Figure 5.32b.

Note that we are not yet able to find the local minimum of f algebraically on the interval $(-5, 4)$, so we make a reasonable guess as to where it is| we can be confident that it is above the horizontal axis since f has no zeros. You may think that this is unsatisfactory, but once we have the tools of calculus, we will be able to find local minimums more precisely. ■

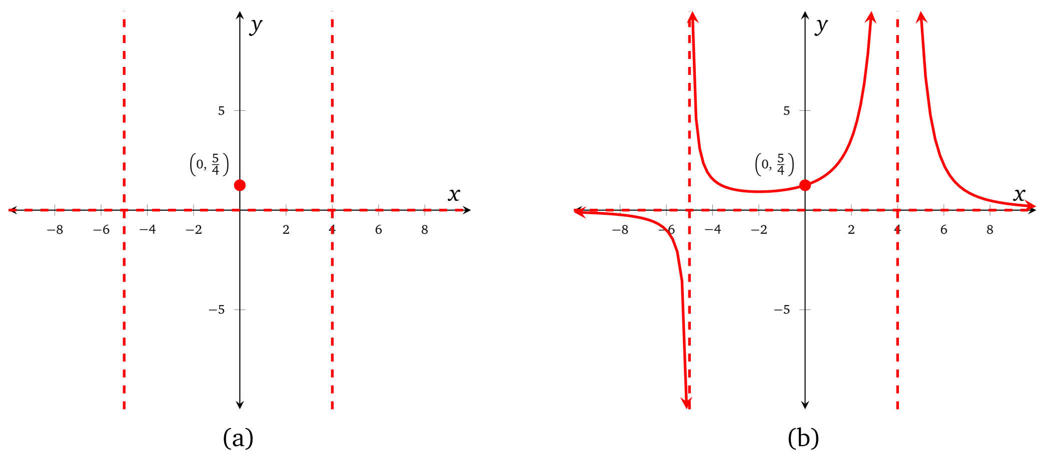


Figure 5.32: $y = \dfrac{100}{(x+5)(x-4)^2}$

Example 6 Use steps (R_1)–(R_5) to sketch a graph of the function g that has formula

$$g(x) = \frac{50(2-x)}{(x+3)^2(x-5)^2}$$

Solution (R_1) g has vertical asymptotes at -3 and 5. g does not have any holes. The curve of g will have 3 branches.

(R_2) g has a simple zero at 2. The vertical intercept of g is $\left(0, \frac{4}{9}\right)$.

(R_3) g behaves like $\frac{1}{x^2}$ around both of its vertical asymptotes.

(R_4) The degree of the numerator of g is 1 and the degree of the denominator of g is 4. Using Definition 24 on page 131, we calculate that the horizontal asymptote of g has equation $y = 0$.

(R_5) The details that we have found so far have been drawn in Figure 5.33a. The function g is similar to the functions we considered in Examples 4 and 5 because it has 2 vertical asymptotes and 3 branches.

We sketch g using the middle branch as our guide because we have the most information about g on the interval $(-3, 5)$. Note that there is no other way to draw this branch without introducing other zeros which g does not have.

Once we have drawn the middle branch, there is only one way to complete the graph because of our observations about the behavior of g around its vertical asymptotes| it behaves like $\frac{1}{x^2}$.

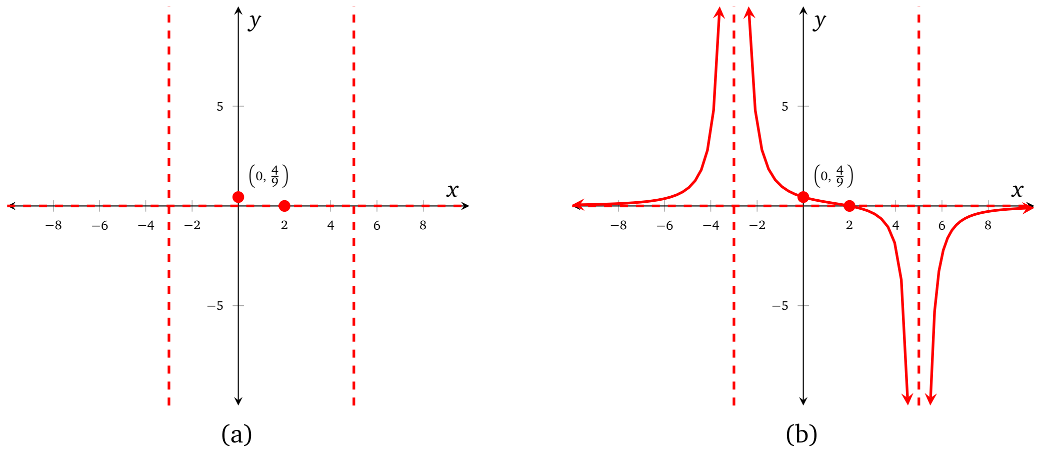


Figure 5.33: $y = \dfrac{50(2-x)}{(x+3)^2(x-5)^2}$

Each of the rational functions that we have considered so far has had either a *simple* zero, or no zeros at all. Remember from our work on polynomial functions, and particularly Definition 22 on page 116, that a *repeated* zero corresponds to the curve of the function behaving differently at the zero when compared to how the curve behaves at a simple zero. Example 7 details a function that has a non-simple zero.

Example 7 Use steps (R_1)–(R_5) to sketch a graph of the function g that has formula

$$h(x) = \frac{(x-3)^2}{(x+4)(x-6)}$$

Solution (R_1) h has vertical asymptotes at -4 and 6. h does not have any holes. The curve of h will have 3 branches.

(R_2) h has a zero at 3 that has *multiplicity* 2. The vertical intercept of h is $\left(0, -\frac{3}{8}\right)$.

(R_3) h behaves like $\frac{1}{x}$ around both of its vertical asymptotes.

(R_4) The degree of the numerator of h is 2 and the degree of the denominator of h is 2. Using Definition 24 on page 131, we calculate that the horizontal asymptote of h has equation $y = 1$.

(R_5) The details that we have found so far have been drawn in Figure 5.34a. The function h is different from the functions that we have considered in previous examples because of the multiplicity of the zero at 3.

We sketch h using the middle branch as our guide because we have the most information about h on the interval $(-4, 6)$. Note that there is no other way to draw this branch without introducing other zeros which h does not have| also note how the curve bounces off the horizontal axis at 3.

Once we have drawn the middle branch, there is only one way to complete the graph because of our observations about the behavior of h around its vertical asymptotes| it behaves like $\frac{1}{x}$.

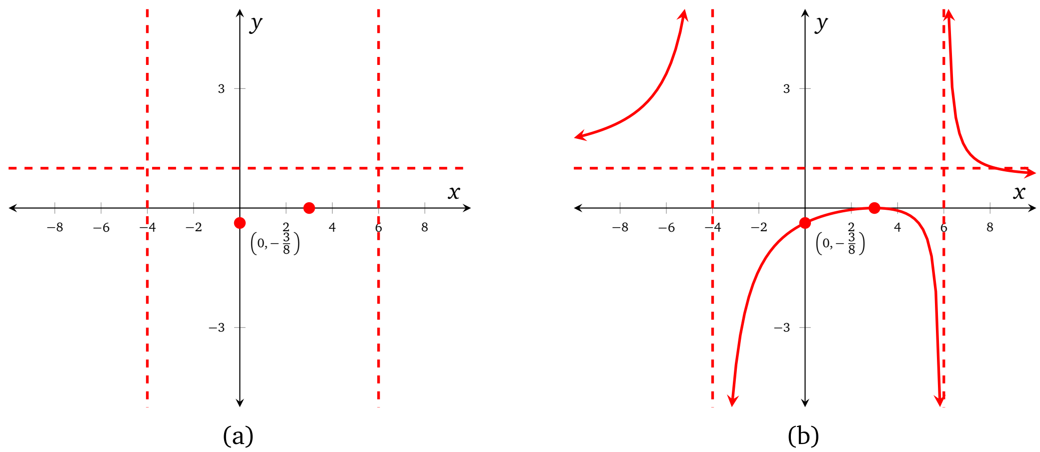


Figure 5.34: $y = \dfrac{(x-3)^2}{(x+4)(x-6)}$

Exercises

Problem 1 (Step (R_5))
Katie is working on graphing rational functions. She has been concentrating on functions f that have formula

$$f(x) = \frac{a(x-b)}{x-c} \tag{5.3}$$

Katie notes that functions with this type of formula have a zero at b, and a vertical asymptote at c. Furthermore, these functions behave like $\frac{1}{x}$ around their vertical asymptote, and the curve of each function will have 2 branches.

Katie has been working with 3 functions that have the form given in Equation (5.3), and has followed steps (R_1)–(R_4); her results are shown in Figure 5.35. There is just one more thing to do to complete the graphs| follow step (R_5). Help Katie finish each graph by deducing the curve of each function.

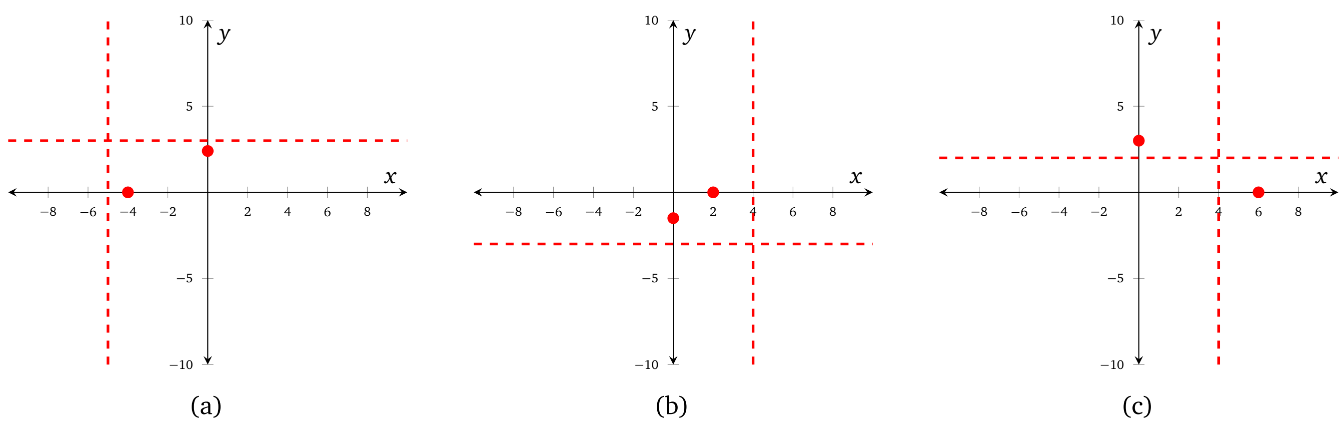


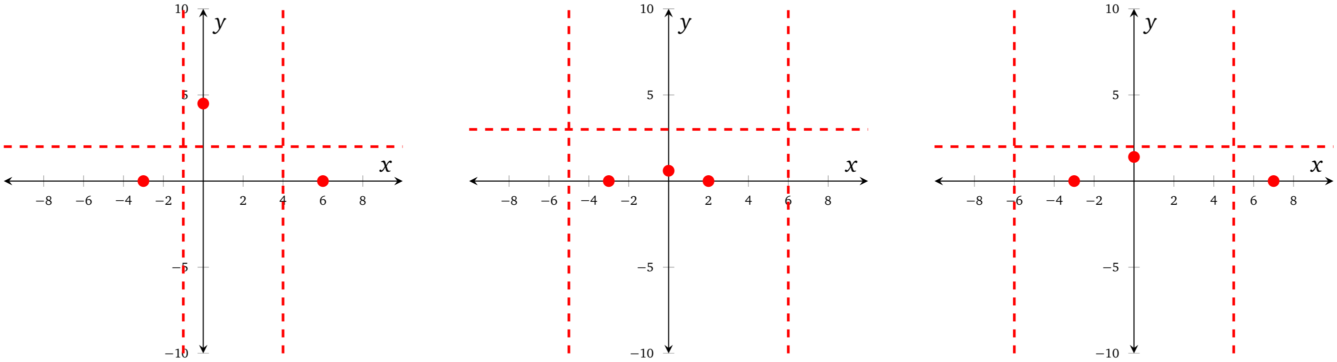
Figure 5.35: Graphs for Problem 1

Problem 2 (Step (R_5) for more complicated rational functions)
David is also working on graphing rational functions, and has been concentrating on functions r that have formula

$$r(x) = \frac{a(x-b)(x-c)}{(x-d)(x-e)}$$

David notices that functions with this type of formula have simple zeros at b and c, and vertical asymptotes at d and e. Furthermore, these functions behave like $\frac{1}{x}$ around both vertical asymptotes, and the curve of the function will have 3 branches.

David has followed steps (R_1)–(R_4) for 3 separate functions, and drawn the results in Figure 5.36. Help David finish each graph by deducing the curve of each function.



(a) (b) (c)

Figure 5.36: Graphs for Problem 2

Problem 3 (Steps (R_1)–(R_5))

FIX

Use steps (R_1)–(R_5) to sketch a graph of each of the following curves

3.1 $y = \dfrac{4}{x+2}$

3.2 $y = \dfrac{2x-1}{x^2-9}$

3.3 $y = \dfrac{x+3}{x-5}$

3.4 $y = \dfrac{2x+3}{3x-1}$

3.5 $y = \dfrac{4-x^2}{x^2-9}$

3.6 $y = \dfrac{(4x+5)(3x-4)}{(2x+5)(x-5)}$

Problem 4 (Inverse functions)

Each of the rational functions F and G are invertible; the functions have formulas

$$F(x) = \frac{2x+1}{x-3}, \qquad G(x) = \frac{1-4x}{x+3}$$

4.1 State the domain of each function.

4.2 Find the inverse of each function, and state its domain.

4.3 Hence state the range of the original functions.

4.4 State the range of each inverse function.

Problem 5 (Composition)

Let r and s be the rational functions that have formulas

$$r(x) = \frac{3}{x^2}, \qquad s(x) = \frac{4-x}{x+5}$$

Evaluate each of the following.

5.1 $(r \circ s)(0)$

5.2 $(s \circ r)(0)$

5.3 $(r \circ s)(2)$

5.4 $(s \circ r)(3)$

5.5 $(s \circ r)(4)$

5.6 $(s \circ r)(x)$

Problem 6 (Piecewise rational functions)

The function R has formula

$$R(x) = \begin{cases} \dfrac{2}{x+3}, & x < -5 \\ \dfrac{x-4}{x-10}, & x \geq -5 \end{cases}$$

Evaluate each of the following.

6.1 $R(-6)$

6.2 $R(-5)$

6.3 $R(-3)$

6.4 $R(5)$

6.5 What is the domain of R?

5.4 Graphing rational functions (oblique asymptotes)

0.6 $y = \dfrac{x^2 + 1}{x - 4}$

0.7 $y = \dfrac{x^3(x + 3)}{x - 5}$

Answers

Solutions for problems in Section 2.1

1. Answers will vary.

2. $f(0) = 0.5$, $f(3) = 3$, and $f(4) = 2$

3. $u(2011) \approx 9$; In 2011 the US unemployment rate was about 9%.

4. $t \approx 2003$ or $t \approx 2008$; The points at which unemployment was 6% were in early 2003 and early 2008.

5.1. Slope is 3, vertical intercept is $(0, -1)$.

5.2. Slope is $-\frac{1}{2}$, vertical intercept is $(0, 5)$.

5.3. Slope is -10, vertical intercept is $(0, \pi)$.

5.4. Slope is m, vertical intercept is $(0, b)$.

6.1. Linear.

6.2. Quadratic.

6.3. Linear.

6.4. Linear.

6.5. Quadratic.

6.6. Linear.

6.7. Quadratic.

6.8. Linear.

7.1. Vertex: $(3, 4)$, range: $[4, \infty)$.

7.2. Vertex: $(3, 4)$, range: $(-\infty, 4]$.

7.3. Vertex: $(5, 0)$, range: $[0, \infty)$.

7.4. Vertex: $\left(\frac{4}{3}, 7\right)$, range: $[7, \infty)$.

7.5. Vertex: $\left(-\frac{1}{2}, \frac{5}{4}\right)$, range: $\left[\frac{5}{4}, \infty\right)$

7.6. Vertex: $\left(-\frac{5}{6}, -\frac{25}{12}\right)$, range: $\left[-\frac{25}{12}, \infty\right)$

7.7. Vertex: $(0, 4)$, range: $(-\infty, 4]$.

7.8. Vertex: $\left(-\frac{5}{4}, \frac{1}{3}\right)$, range: $\left(-\infty, \frac{1}{3}\right]$.

Solutions for problems in Section 2.2

1. -8 would be a bad input value; it would lead to division by 0

2. Since sqrt can only be applied to nonnegative numbers, the domain is the set of all nonnegative numbers. That is, the domain is all numbers greater than or equal to 0.

3. The domain is $(-\infty, 2) \cup (2, \infty)$ and the range is $(\infty, 0) \cup (0, \infty)$.

4.1. $[0, \infty)$

4.2. $[-10, \infty)$

4.3. $(-\infty, \infty)$

4.4. $\left[-\frac{2}{5}, \infty\right)$

4.5. $(-\infty, \infty)$

4.6. $(-\infty, 2]$

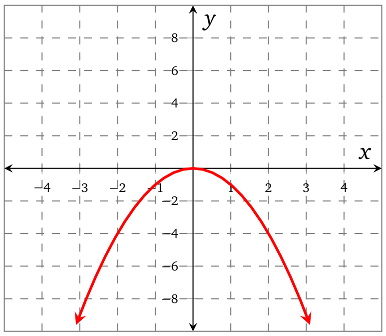
4.7. $(-\infty, \infty)$

4.8. $(-\infty, \infty)$

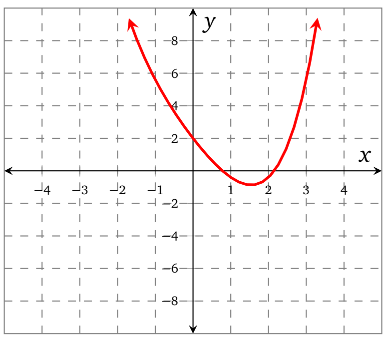
Solutions for problems in Section 2.3

1. f is increasng on $(-\infty, 3]$ and on $[7, \infty)$. We could also say that f is increasing on $(-\infty, 3] \cup [7, \infty)$

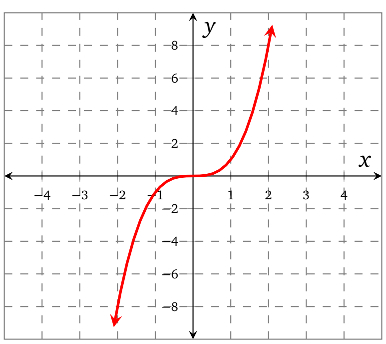
2. f is concave down on $(-\infty, \infty)$.



g is concave up on $(-\infty, \infty)$.



k is neither concave up on $(-\infty, \infty)$ nor concave down on on $(-\infty, \infty)$. It is concave up on on $[0, \infty)$ and concave down on $(-\infty, 0]$



3.1.

- p has zeros at about -3.8, 0, and 5.
- q has zeros at about -5.9, -1, 1, and 4.
- r has zeros at about -5, -2.9, and 4.1.
- s has zeros at about -9, -6, 4.2, 8.1, and 0.

3.2.

- p has a local maximum of approximately 3.9 at -2, and a local minimum of approximately -6.5 at 3.
- q has a local minimum of approximately -10 at -4, and -4 at 3; q has a local maximum of approximately 1 at 0.
- r has a local minimum of approximately -5.5 at -4, and a local maximum of approximately 10 at 3.

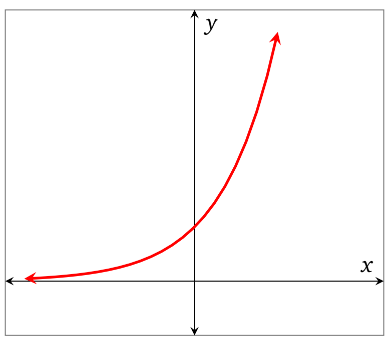
- s has a local maximum of approximately 5 at -8, 0 at 0, and 5 at 7; s has local minimums of approximately -3 at -4, and -1 at 3.

3.3.
- p does not have a global maximum, nor a global minimum.
- q has a global minimum of approximately -10; it does not have a global maximum.
- r does not have a global maximum, nor a global minimum.
- s has a global maximum of approximately 5; it does not have a global minimum.

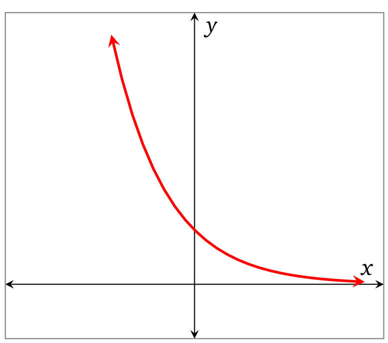
3.4.
- p is increasing on $(-\infty,-2)\cup(3,\infty)$, and decreasing on $(-2,3)$.
- q is increasing on $(-4,0)\cup(3,\infty)$, and decreasing on $(-\infty,-4)\cup(0,3)$.
- r is increasing on $(-4,3)$, and decreasing on $(-\infty,-4)\cup(3,\infty)$.
- s is increasing on $(-\infty,-8)\cup(-4,0)\cup(3,5)$, and decreasing on $(-8,-4)\cup(0,3)\cup(5,\infty)$.

3.5.
- p is concave up on $(1,\infty)$, and concave down on $(-\infty,1)$.
- q is concave up on $(-\infty,-1)\cup(1,\infty)$, and concave down on $(-1,1)$.
- r is concave up on $(-\infty,-3)\cup(0,2)$, and concave down on $(-3,0)\cup(2,\infty)$.
- s is concave up on $(-6,-2)\cup(2,5)$, and concave down on $(-\infty,-6)\cup(-2,2)\cup(5,\infty)$.

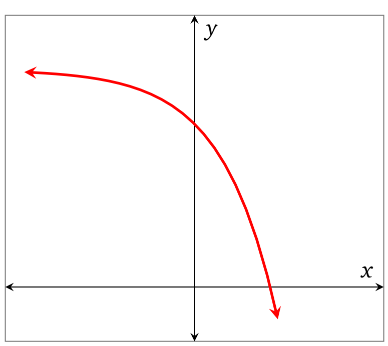
4.1.

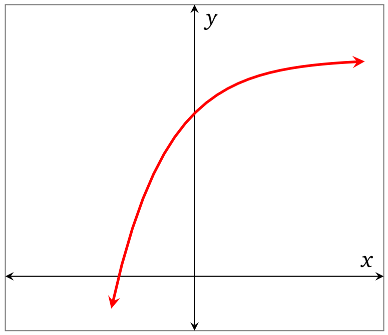


4.2.



4.3.



4.4. 

5. (a) f is decreasing.

(b) g is constant– neither increasing nor decreasing.

(c) h is increasing.

(d) j is increasing.

6. (a) f is concave up.

(b) g is neither concave up nor concave down.

(c) h is neither concave up nor concave down.

(d) j is concave up.

7.1. $f(x) = 2x$ or $g(x) = \sqrt{x}$; many other choices are available.

7.2. $g(x) = 2x$; many other choices are available.

7.3. $h(x) = -3x$; many other choices are available.

7.4. $j(x) = -5x$; many other choices are available.

7.5. Consider k that has formula $k(x) = x^2$ which decreases on $(-\infty, 0)$.

7.6. Consider l that has formula $l(x) = x^2$ which increases on $(0, \infty\infty\infty\infty\infty\infty\infty\infty)$.

Solutions for problems in Section 2.4
Solutions for problems in Section 2.5

1.1. 0

1.2. 5

1.3. 4

1.4. 13

1.5. 3

1.6. 13

1.7. 8

1.8. 6

2.1. 2

2.2. 0

2.3. 2

2.4. -4

2.5. Undefined; in fact, the function $q \circ r$ is not defined for any values of x.

2.6. 2

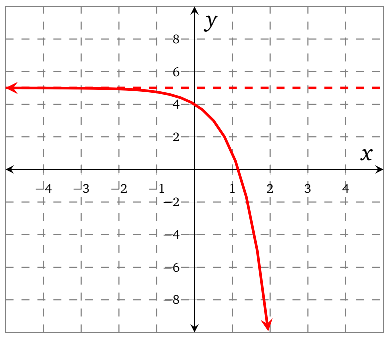
Solutions for problems in Section 2.6

Solutions for problems in Section 3.1

1. The equivalent of the function g is h and has formula $h(x) = 4^x$.

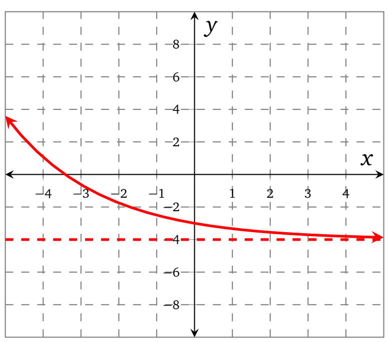
★ *2.1.* The graph of g is shown below. Note that

- g is decreasing
- g is concave down; in particular g is decreasing at a faster and faster rate;
- the line $y = 5$ is a horizontal asymptote of g as $x \to -\infty$;
- $g(x) \to -\infty$ as $x \to \infty$;
- the range of g is $(-\infty, 5)$.



2.2. The graph of h is shown below. Note that

- h is decreasing;
- h is concave up; in particular h is decreasing at a slower and slower rate;
- the line $y = -4$ is a horizontal asymptote of h as $x \to \infty$;
- $h(x) \to \infty$ as $x \to -\infty$;
- the range of h is $(-4, \infty)$.



★ **3.1.** There are approximately $7200 \cdot 3 = 21600$ grains of rice in a 1-lb bag.

3.2. $2^{15} = 32768 > 21600$. The 15th square.

★ **3.3.** Answers will vary.

3.4. The 16th square would give us $2^{16}/21600 \approx 3.03$ bags of rice. We would get approximately \$6 from the rice on the 16th square.

3.5. Using the previous value of \$6, we would obtain \$12 from the rice on the 17th square.

3.6. $6 \cdot 2^{18} \approx 1572864 > 1000000$. We would only have to get to the 34th square on the board in order to get \$1,000,000 worth of rice.

3.7. $\$6 \cdot 2^{48} \approx \$1.68884986 \times 10^{15}$. Quite a lot.

3.8. Answers will vary.

4.1. $m(x)$ in Figure 2.7 is increasing at a faster and faster rate, $n(x)$ in Figure 2.8 is increasing at a faster and faster rate, and $o(x)$ in Figure 2.9 is increasing at a slower and slower rate.

4.2. $p(x)$ is increasing at a constant rate, $q(x)$ is increasing at a slower and slower rate, and $r(x)$ is increasing at a faster and faster rate.

4.3. In (a.), the investment is increasing at a faster and faster rate. In (b.), the distance is increasing, but at a slower and slower rate. In (c.), the elevation is decreasing at a faster and faster rate, as the speed increases.

4.4. (a.) is increasing at a slower and slower rate, (b.) is increasing at a constant rate, and (c.) is increasing at a faster and faster rate.

5.1. $Q(t) = 4(0.915)^t$

5.2. The growth factor is 0.915, and the growth rate is $-8.5\,\%$. (We could also say that the decay rate is $8.5\,\%$.)

5.3. No; the model is a limited in this way.

6.1. $A(t) = 5.35(0.92)^t$, where t is the number of decades since 2005, and $A(t)$ is measured in million km^2.

6.2. A graph indicates that about 8.3 decades after 2005 (or in 2088), the Arctic sea ice cover will have reached half of its 2005 level. So the half-life of the ice is 8.3 decades.

6.3. $A(-1) = 5.35(0.92)^{-1} \approx 5.82$. According to the model, in 1995 the Arctic sea ice level was about 5.82 million km^2

6.4. Using $B(t)$ to represent a linear model for the ice cover, where t is measured in decades since 2005, B has slope of $2.675 - 5.815/8.3-(-1)$ million km^2/decade or -0.3377 million km^2/decade. Using the point-slope form of a line equation, $B(t) = -0.3377(t+1) + 5.815$.

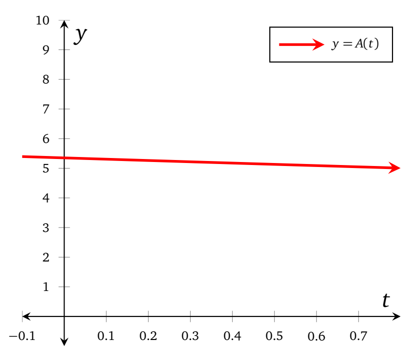
For the year 2010, $t = 0.5$. $A(0.5) \approx 5.13$ and $B(0.5) \approx 5.31$. So in 2010, the exponential model predicts lower ice cover than the linear model.

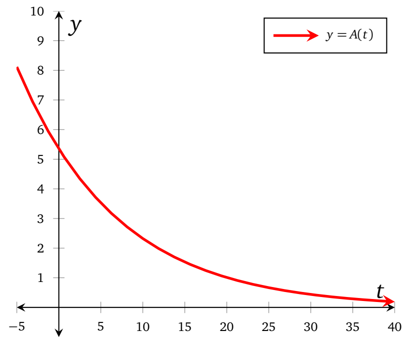
For the year 2030, $t = 2.5$. $A(2.5) \approx 4.34$ and $B(2.5) \approx 4.63$. So in 2030, the exponential model still predicts lower ice cover than the linear model, and the difference is larger than it was in 2010.

For the year 2050, $t = 4.5$. $A(4.5) \approx 3.67$ and $B(4.5) \approx 3.95$. So in 2050, the exponential model still predicts lower ice cover than the linear model.

6.5. Since $A(t)$ and $B(t)$ are measured in million km^2, we should work with the value 0.0001 million km^2. A graph shows that $B(16.2) \approx 0.0001$ and $A(130.5) \approx 0.0001$. So the linear model predicts that after 6.2 decades (in 2212) the Arctic sea ice cover will melt to less than 100 km^2. The exponential model predicts that this will not happen for 130.5 decades, or until the year 3310.

6.6.





One should consider the scale of a graph and location of the origin when reading graphs.

6.7. Responses will vary.

7.1. Exponential.

7.2. Not exponential.

7.3. Exponential.

7.4. Exponential.

7.5. Not exponential.

7.6. Exponential.

7.7. Not exponential.

7.8. Not exponential.

7.9. Exponential.

7.10. Not exponential.

7.11. Not exponential.

7.12. Not exponential.

8.1. $a = 2,\ b = 3$

8.2. $a = -4,\ b = 5$

8.3. $a = 1,\ b = \frac{2}{3}$

8.4. $a = -1,\ b = \frac{2}{3}$

8.5. $a = 1,\ b = \frac{1}{3}$

8.6. $a = 2,\ b = \frac{1}{3}$

8.7. $a = -\frac{1}{5},\ b = 4$

8.8. $a = -10,\ b = \frac{1}{5}$

9.1.
$$\begin{aligned} f(-10) &= \frac{1}{1024} \approx 0.00 \\ f(-5) &= \frac{1}{32} \approx 0.03 \\ f(0) &= \frac{1}{1} = 1 \\ f(5) &= 32 \\ f(10) &= 1024 \end{aligned}$$

9.2. $g(-10) = 59049$

$g(-5) = 243$

$g(0) = 1$

$g(5) = \dfrac{1}{243} \approx 0.00$

$g(10) = \dfrac{1}{59049} \approx 0.00$

9.3. $h(-10) = -\dfrac{1}{9765625} \approx -0.00$

$h(-5) = -\dfrac{1}{3125} \approx -0.00$

$h(0) = -1$

$h(5) = -3125$

$h(10) = -9765625$

9.4. $k(-10) = -\dfrac{9765625}{1024} \approx -9536.74$

$k(-5) = -\dfrac{3125}{32} \approx -97.66$

$k(0) = -1$

$k(5) = -\dfrac{32}{3125} \approx -0.01$

$k(10) = -\dfrac{1024}{9765625} \approx -0.00$

10.1. f is increasing, g is decreasing, h is decreasing, k is increasing.

10.2. f is concave up, g is concave up, h is concave down, and k is concave down.

10.3.
- f has domain $(-\infty, \infty)$, and range $(0, \infty)$.
- g has domain $(-\infty, \infty)$, and range $(0, \infty)$.
- h has domain $(-\infty, \infty)$, and range $(-\infty, 0)$.
- k has domain $(-\infty, \infty)$, and range $(-\infty), 0$.

11.2. $0.66 \cdot 180 = 118.80$. The person contains approximately 118 lb of water.

11.3. $0.20 \cdot 16000 = 3200$. There is approximately 3200 ft^3 of oxygen in the room.

11.4. $(1.15) \cdot 26 = 29.90$. The total bill is \$29.90.

11.5. $0.01 \cdot 100,000 \cdot 0.5 = 500$. You get \$500.

12.1. Growth factor is 3, initial value is 5. Growth rate is 200 %.

12.2. Decay factor is 0.5, initial value is 6. Growth rate is −50 %.

12.3. Decay factor is 0.25, initial value is 2. Growth rate is −25 %.

12.4. Growth factor is 2.5, initial value is 500. Growth rate is 150 %.

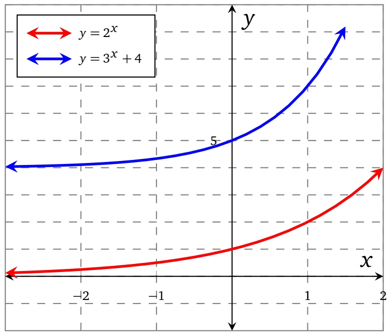
13.1. Growth rate is 10 % or 0.1, growth factor is 1.1.

13.2. Growth rate is −6 % or −0.06, growth factor is 0.94.

14.1.

x	$f(x)$	$g(x)$
-3	$1/8$	$109/27$
-2	$1/4$	$37/9$
-1	$1/2$	$13/3$
0	1	5
1	2	7
2	4	13

The functions f and g are graphed below.



14.2. f has asymptote $y=0$, and g has asymptote $y=4$.

14.3. g is increasing at a faster rate than f.

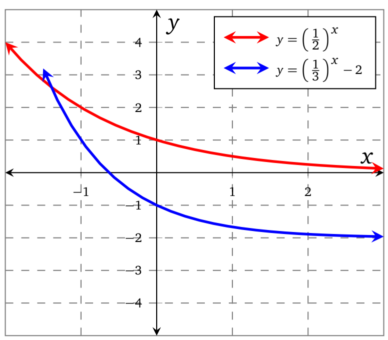
14.4. No. There is no number such that $2^x=0$, and no real number such that 2^x is negative.

14.5. Both have domain $(-\infty,\infty)$; the range of f is $(0,\infty)$ and the range of g is $(4,\infty)$.

15.1.

x	$m(x)$	$n(x)$
-3	8	25
-2	4	7
-1	2	1
0	1	-1
1	$1/2$	$-5/3$
2	$1/4$	$-17/9$

The functions m and n are graphed below.



15.2. m has asymptote $y=0$, and n has asymptote $y=-2$.

15.3. Both have domain $(-\infty,\infty)$; the range of m is $(0,\infty)$ and the range of n is $(-2,\infty)$.

15.4. Because $b<1$ and $a>0$.

16.1. Figure 2.10b on page 40 shows $f(x)=3^x-1$.
Figure 2.10d on page 40 shows $f(x)=-4^x-3$.
Figure 2.10a on page 40 shows $f(x)=2^x+1$.
Figure 2.10c on page 40 shows $f(x)=-5^x+2$.

16.2. Figure 2.10d on page 40, $g(x)=-4^x-3$:

- $g(x)\to -3$ as $x\to-\infty$;
- the horizontal asymptote as $x\to-\infty$ is the line $y=-3$;
- the range of g is range: $(-\infty,-3)$.

Figure 2.10a on page 40, $h(x)=2^x+1$:

- $h(x)\to 1$ as $x\to-\infty$;
- the horizontal asymptote as $x\to-\infty$ is the line $y=1$;
- the range of h is $(1,\infty)$.

Figure 2.10c on page 40, $j(x) = -5^x + 2$:

- $j(x) \to 2$ as $x \to -\infty$;
- the horizontal asymptote as $x \to -\infty$ is the line $y = 2$;
- the range of j is $(-\infty, 2)$.

16.3. $-4^x - 3 \to -\infty$ as $x \to \infty$.
$2^x + 1 \to \infty$ as $x \to \infty -$.
$-5^x + 2 \to -\infty$ as $x \to \infty$.

16.4. $\lim_{x\to-\infty}(-4^x - 3) = 3$; $\lim_{x\to\infty}(-4^x - 3) = -\infty$.
$\lim_{x\to-\infty}(2^x + 1) = 1$; $\lim_{x\to\infty}(2^x + 1) = \infty$.
$\lim_{x\to-\infty}(-5^x + 2) = 2$; $\lim_{x\to\infty}(-5^x + 2) = -\infty$.

17.1. Figure 2.11b on page 41 shows $F(x) = \left(\dfrac{1}{3}\right)^x - 1$.

Figure 2.11d on page 41 shows $G(x) = -\left(\dfrac{1}{4}\right)^x - 3$.

Figure 2.11c on page 41 shows $H(x) = -\left(\dfrac{1}{5}\right)^x + 2$.

Figure 2.11a on page 41 shows $J(x) = \left(\dfrac{1}{2}\right)^x + 1$.

17.2. $\left(\frac{1}{3}\right)^x - 1 \to -1$ as $x \to \infty$; $\left(\frac{1}{3}\right)^x - 1 \to \infty$ as $x \to -\infty$.
$-\left(\frac{1}{5}\right)^x + 2 \to 2$ as $x \to \infty$; $-\left(\frac{1}{5}\right)^x + 2 \to -\infty$ as $x \to -\infty$.
$-\left(\frac{1}{4}\right)^x - 3 \to -3$ as $x \to \infty$; $-\left(\frac{1}{4}\right)^x - 3 \to -\infty$ as $x \to -\infty$.

17.3. $\lim_{x\to\infty}\left(\left(\dfrac{1}{3}\right)^x - 1\right) = -1$; $\lim_{x\to-\infty}\left(\left(\dfrac{1}{3}\right)^x - 1\right) = \infty$.
$\lim_{x\to\infty} -\left(\left(\dfrac{1}{5}\right)^x + 2\right) = 2$; $\lim_{x\to-\infty}\left(-\left(\dfrac{1}{5}\right)^x + 2\right) = -\infty$.
$\lim_{x\to\infty}\left(-\left(\dfrac{1}{4}\right)^x - 3\right) = -3$; $\lim_{x\to-\infty}\left(-\left(\dfrac{1}{4}\right)^x - 3\right) = -\infty$.

18.1. $f(x) = 2^x$. There are infinitely many other choices available.

18.2. $f(x) = -\left(\frac{1}{3}\right)^x$. There are infinitely many other choices available.

18.3. $f(x) = \left(\frac{1}{4}\right)^x$. There are infinitely many other choices available.

18.4. $f(x) = -5^x$. There are infinitely many other choices available.

19.1. $\dfrac{f(x+1)}{f(x)} = \dfrac{ab^{x+1}}{ab^x} = \dfrac{b^{x+1}}{b^x} = b$

19.2. The function graphed in Figure 2.12 is exponential, $n(x) = 2^x$. The function graphed in Figure 2.13 is exponential, and $r(x) = \left(\frac{1}{3}\right)^x$. The function graphed in Figure 2.14 is not exponential.

19.3. The function tabulated in Table 2.11 is exponential, and $g(x) = 3^x$. The function tabulated in Table 2.12 is exponential, and $h(x) = \sqrt[3]{25}\cdot(\sqrt[3]{5})^x$. The function tabulated in Table 2.13 is not exponential.

20.1. False. Consider $f(x) = -2^x$.

20.2. False. Consider $f(x) = -\left(\frac{1}{9}^x\right)$.

20.3. True.

20.4. True.

20.5. False. Consider $f(x) = -8^x$.

20.6. False. Consider $f(x) = -\left(\frac{1}{8}\right)^x$.

20.7. True.

20.8. False. Consider $f(x) = 8^x$.

20.9. True.

20.10. True.

20.11. True.

20.12. False. Consider any exponential function. There are no vertical asymptotes.

Solutions for problems in Section 3.2

1.1. 4

1.2. -8

1.3. -16

1.4. 9

1.5. $\frac{4}{9}$

1.6. $\frac{4}{81}$

1.7. $-\frac{25}{6}$

1.8. $-\frac{49}{100}$

1.9. $\frac{49}{100}$

1.10. 1

1.11. -1

1.12. -3

2. $$\begin{aligned} f(t) &= 2^{t+1} \cdot 2^{3t} \\ &= 2^t \cdot 2 \cdot 2^{3t} \\ &= 2 \cdot 2^{4t} \\ &= 2 \cdot 16^t \end{aligned}$$

3. $x = 2$ (The solution is 2.)

4. $x = 0, 2$ (The solutions are 0, and 2)

5. $x \approx 3.32$ (The solution is approximately 3.32.)

6.1. $\frac{2^{12}}{12} \approx 341.33$; our elevation is approximately 341 ft above the ground when we are 1 ft from our front door.

6.2. $\frac{2^{24}}{63360} \approx 264.79$; our elevation is approximately 264.79 mi above the ground when we are 2 ft from our front door.

6.3. We need to solve the equation $2^x = 1.595 \times 10^{10}$. Using a graphing calculator, we obtain $x \approx 33.89$; we conclude that when we reach the Moon, our horizontal distance from our front door is approximately 34 in.

7.1. $$\begin{aligned} f(t) &= 4^{t+1} \cdot 2^{3t} \\ &= \left(2^2\right)^{t+1} \cdot 2^{3t} \\ &= 2^{2t} \cdot 2^2 \cdot 2^{3t} \\ &= 4 \cdot 2^{5t} \\ &= 4 \cdot 32^t \end{aligned}$$

7.2. $f(t) = 4^{\frac{t+1}{2}} \cdot 3^{2t}$
$= 4^{\frac{1}{2}(t+1)} \cdot 3^{2t}$
$= (4^{\frac{1}{2}})^{t+1} \cdot (3^2)^t$
$= 2^{t+1} \cdot 9^t$
$= 2^1 \cdot 2^t \cdot 9^t$
$= 2 \cdot (2 \cdot 9)^t$
$= 2 \cdot 18^t$

8.1. $x = 3$ (The solution is 3.)

8.2. $x = 3$ (The solution is 3.)

8.3. $x = 2$ (The solution is 2.)

8.4. $x = \pm 2$ (The solutions are ± 2.)

8.5. $x = 2, 3$ (The solutions are 2 and 3.)

8.6. $x = 2, 0$ (The solutions are 2 and 0.)

9.1. $x \approx 2.18$ (The solution is approximately 2.18.)

9.2. $x \approx 2.43$ (The solution is approximately 2.43.)

9.3. $x \approx 2.55$ (The solution is approximately 2.55.)

9.4. No solution.

9.5. $x \approx 1.92$ (The solution is approximately 1.92.)

9.6. $x \approx 0.53$ (The solution is approximately 0.53.)

9.7. $x \approx 2.32$ (The solution is approximately 2.32.)

9.8. $x \approx 2.01$ (The solution is approximately 2.01.)

10.1. $x = 0$ (The solution is 0.) Note that there are no solutions to $4^x + 3 = 0$.

10.2. $x = 1$ (The solution is 1.) Note that there are no solutions to $5^x = -1$.

10.3. $x = 0, 2$ (The solutions are 0 and 2.) Note that there are no solutions to $5^x = 0$.

11.
- Figure 2.16a represents $3^{x^2} = 3$
- Figure 2.16b represents $4^{-x^2} = \frac{1}{4}$
- Figure 2.16c represents $5^x = 5$
- Figure 2.16d represents $-\left(\frac{1}{3}\right)^x = -3$

12.
- f has 2 zeros
- g has 1 zero
- h has 1 zero
- k has 2 zeros

13.1. True.

13.2. False; consider $2^x = -3$.

13.3. False; there are no values of x that satisfy $2^x = 0$.

14.1. By counting the number of points of intersection, there are 10 solutions on the interval $[0, 9]$.

14.2. The function h has 10 zeros on the interval $[0, 9]$.

Solutions for problems in Section 3.3

1.1. b^3

1.2. b^6

1.3. $\frac{1}{b^6}$ or b^{-6}

1.4. $\frac{1}{b^8}$ or b^{-8}

2.1. 2

2.2. $\frac{1}{4}$

2.3. 32

2.4. 81

3.1. $x = 3$ or $x = -3$

3.2. $x = -2$

3.3. There are no real solutions for x.

3.4. $x = \sqrt[4]{19} = 19^{1/4} \approx 2.0878$ or $x = -\sqrt[4]{19} = -19^{1/4} \approx -2.0878$

4. $f(t) = 2 \cdot 4^t$; the growth factor is 4.

5. $f(x) = \frac{10^{11/9}}{23^{2/9}} \left(\frac{23}{10}\right)^{x/9} \approx 8.31(1.096963)^x$

6.1. $p(t) = 76.2\left(\frac{309}{76.2}\right)^{t/110} \approx 76.2(1.012808)^t$

6.2. $p(100) \approx 272.072992$; the population of the U.S.A. in the year 2000 was approximately 272 million people, according to the model.

6.3. The model underestimated the actual population by about 10 million people.

6.4. $P(t) = 282\left(\frac{309}{282}\right)^{t/10} \approx 282(1.009185)^t$

6.5. $P(-50) \approx 178.526320$ and $p(50) \approx 143.985979$. $P(-50)$ was an overestimate by about 27.5 million people, or 18.2 %. $p(50)$ was an underestimate by about 7 million people, or 4.6 %. So the prediction based off of the model that used the years 1900 and 2000 is better. Generally, it is better to interpolate than to extrapolate.

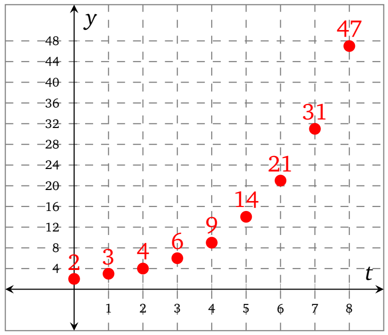
7.1. $c(t) = 18.1\left(\frac{16.3}{18.1}\right)^{t/21} \approx 18.1(0.995024)^t$

7.2. $c(-10) \approx 19.025713$. Each US citizen was responsible for approximately 19.03 tons of CO_2 in 2000.

7.3. $C(t) = P(t)c(t) = 282\left(\frac{309}{282}\right)^{t/10} 18.1\left(\frac{16.3}{18.1}\right)^{t/21} \approx 5104.2(1.00416412)^t$

8.1.

t	$F(t)$
0	2
1	3
2	4
3	6
4	9
5	14
6	21
7	31
8	47

8.2. 

8.3. Choosing $(0,2)$ and $(8,47)$,

$$\begin{aligned} A(t) &= 2\left(\left(\frac{47}{2}\right)^{1/8}\right)^t \\ &\approx 2(1.483828)^t \end{aligned}$$

8.4.

t	$F(t)$	$A(t)$
0	2	2.000 000
1	3	2.967 655
2	4	4.403 489
3	6	6.503 420
4	9	9.695 360
5	14	14.386 240
6	21	21.346 710
7	31	31.674 840
8	47	47.000 000

9.1. 12^{x+y}

9.2. a^{6x+3}

9.3. $9x^6y^4$

9.4. $10b^{10}c^7$

9.5. $\dfrac{7}{6}x^3$

9.6. $\dfrac{2}{9}x^{10}$

9.7. $\dfrac{64}{27}x^{18}$

9.8. 2^x

9.9. 10^x

9.10. $27^x b^{4x}$

9.11. 64^t

9.12. $81t^4$

10.1. $x = 4$ or $x = -4$

10.2. $y = 3/2$

10.3. $x = 5$ or $x = -5$

10.4. $x = 0$

10.5. $x = \sqrt{3}$ or $x = -\sqrt{3}$

10.6. $x = \sqrt[3]{7}$ or $x = -\sqrt[3]{5}$

11.1. $f(t) = -3 \cdot 5^t$; the growth factor is 5.

11.2. $f(t)=6^t$; the growth factor is 6.

11.3. $f(t)=2\left(\frac{1}{3}\right)^t$; the growth factor is $1/3$.

11.4. $f(t)=17\left(\frac{2}{5}\right)^t$; the growth factor is $2/5$.

11.5. $f(t)=186\,(2/3)^t$; the growth factor is $2/3$.

11.6. $f(t)=100\,(4/5)^t$; the growth factor is $4/5$.

12.1. $f(x)=4\left(\frac{5}{2}\right)^{3/8}\left(\frac{2}{5}\right)^{x/8}\approx 5.64(0.891780)^x$

12.2. $f(x)=\frac{49}{2}\left(\frac{2}{7}\right)^{x/4}\approx 24.5(0.731110)^x$

12.3. $f(x)=\frac{6^{13/10}}{29^{3/10}}\left(\frac{29}{6}\right)^{x/10}\approx 3.74(1.170644)^x$

12.4. $f(x)=6\left(\frac{10}{3}\right)^{5/12}\left(\frac{10}{3}\right)^{x/12}\approx 9.91(1.105537)^x$

13. $f(x)=2\cdot 3^x$, $g(x)=2\left(\frac{1}{2}\right)^x$, $h(x)=-5^x$, $k(x)=-4\left(\frac{1}{3}\right)^x$

14. The graphs of f and g are concave up; the graphs of h and k are concave down.

15. Table 2.16: $f(x)=10^x$; Table 2.17: $f(x)=-9\cdot 7^x$; Table 2.18: $f(x)=-6\left(\frac{1}{3}\right)^x$; Table 2.19: $f(x)=3\left(\frac{1}{5}\right)^x$.

16.1. False; consider g and h in Figure 2.22.

16.2. False; consider f and k in Figure 2.22.

16.3. True.

16.4. False; consider Tables 2.16–2.19.

16.5. True.

16.6. False; consider Tables 2.16–2.19.

17.1. $(0,877)$, $(7,8325)$.

17.2. $S(t)=877\left(\frac{8325}{877}\right)^{t/7}\approx 877(1.38)^t$

17.3. The growth factor is $\left(\frac{8325}{877}\right)^{1/7}\approx 1.379$. The growth rate is $\left(\frac{8325}{877}\right)^{1/7}-1\approx 0.379$, or about 37.9 % per year.

17.4. Yes; we found $b\approx 1.379$, which means the annual growth rate is approximately 37.9 %.

17.5. $S(10)\approx 21\,840.54$ MW. This is much larger than 13 729 MW, which indicates that the growth rate did not continue at 37.9 % after 2007.

Solutions for problems in Section 3.4

1.1. $0.1\cdot 100=10$

1.2. $0.2\cdot 10=2$

1.3. $0.13\cdot 28=3.64$

1.4. $0.81\cdot 3=2.43$

1.5. $17\cdot 1.28=21.76$

1.6. $42\cdot (1.67=70.14$

1.7. $107\cdot 0.9=96.3$

1.8. $243\cdot 0.24=58.32$

2. $P(t)=P_0(0.92)^t$

3. Let $P(t)$ be the population of Oregon (in millions of people) at time t in years since 2000. Since the initial population is 3.83 million people and $r = 0.120$,

$$P(t) = 3.83(1.12)^t$$

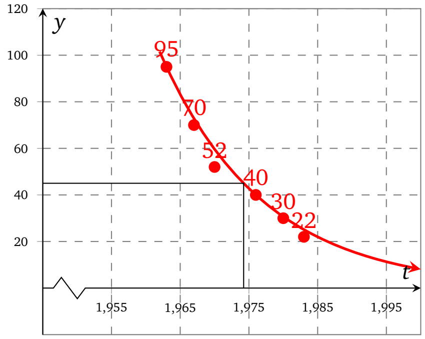
4.1. In 1963 the atmospheric concentration of C-14 in Austria was 95 % greater than the pre-nuclear age concentration.

4.2. Let $f(t)$ represent the percent of C-14 in the atmosphere compared to normal t years after 1963. Using the points $(0,95)$ and $(20,22)$, $f(t) \approx 95(0.93)^t$.

4.3. Using $(7,52)$ and $(13,40)$ we obtain $f(t) = \frac{52^{13/6}}{40^{7/6}}\left(\frac{40}{52}\right)^{t/6} \approx 70.62(0.96)^t$.

4.4. Based on visual inspection, the model in Problem 4.2 most closely fits each of the data points.

5. We will use our answer from Problem 4.2: $f(t) \approx 95(0.936)^t$ where t is years since 1963.



So this person was probably 6 or 7 years old in about 1974. The person was probably born in 1967 or 1968.

6.1.

t (years)	Q (barrels)
0	1000.00
1	800.00
2	640.00
3	512.00
4	409.60
5	327.68

6.2. $Q(t) = 1000(0.8)^t$

6.3. We need to solve the equation $1000(0.8)^t = 100$. Using a graph, we find that $t \approx 10.31$. We conclude that the first year that the amount of oil drilled will be less than 100 barrels is 2015.

6.4. Never. $Q(t)$ never crosses the horizontal axis if the model holds. However the amount that can be drilled eventually becomes small.

7.1. $$T(t) = 76.2\left(\frac{309}{76.2}\right)^{100/110} 18.1\left(\frac{16.3}{18.1}\right)^{-10/21}\left(\frac{309}{76.2}\right)^{t/110}\left(\frac{16.3}{18.1}\right)^{t/21}$$
$$\approx 5176.382674(1.007769)^t$$

7.2. The total amount of CO_2 produced in the US each year is increasing by approximately 0.7769 %.

7.3. $T(15) \approx 5813$. The approximation is only about 134 million metric tons of CO_2 off, which may seem large but is really only about 2 % off of the actual estimate.

8.1. $P(t) = 500(1.06)^t$

8.2. $P(t) = 1500(1.12)^t$

8.3. $P(t) = 2700(1.23)^t$

8.4. $P(t) = 3600(1.52)^t$

8.5. $P(t) = 700(0.94)^t$

8.6. $P(t) = 2405(0.88)^t$

8.7. $P(t) = 4302(0.77)^t$

8.8. $P(t) = 7300(0.48)^t$

9.1. $P_0 = 1000$, $r = 10\%$; the population is initially 1000 people, and increases at 10% per year.

9.2. $P_0 = 1800$, $r = 7\%$; the population is initially 1800 people, and increases at 7% per year.

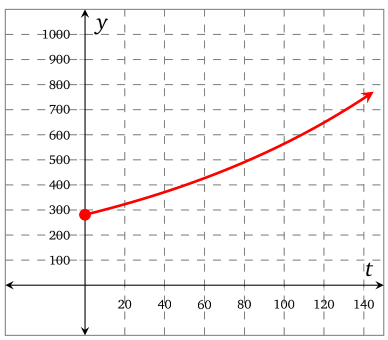
9.3. $P_0 = 200$, $r = -13\%$; the population is initially 200 people, and decreases at 13% per year.

9.4. $P_0 = 907$, $r = -24\%$; the population is initially 907 people, and decreases at 24% per year.

10.1. $P(t) = 281(1.007)^t$

10.2. $P(10) \approx 301$; in 2010 there were approximately 301 million people. $P(35) \approx 359$; in 2035 there will be approximately 359 million people.

10.3.



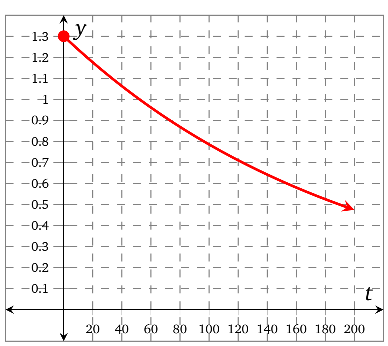
10.4. Answers will vary. Possible strengths: % increases are realistic in some sense – a family reproduces, then their children reproduce, and so on.

Possible weaknesses: doesn't allow for immigration or emigration. Does it account for death?

11.1. $P(t) = 1.3(0.995)^t$.

11.2. In 2025: $P(16) \approx 1.20$; the population of China in 2025 will be approximately 1.20 billion people. In 2050: $P(41) \approx 1.06$; the population of China in 2050 will be approximately 1.06 billion people.

11.3.



11.4. Using a graph or table of values, the population will be 700 million people in approximately the year 2132.

11.5. Answers will vary.

11.6. Answers will vary. Possible strengths: % decreases are realistic in some sense – a family has fewer children, then the next generation has fewer people who also have fewer children, and so on.

Possible weaknesses: doesn't allow for immigration or emigration. Does it account for death?

12.1. $N(t) = 142\left(\frac{1}{2}\right)^{\frac{1}{5730}t}$.

12.2. $N(3000) \approx 98.8$; about 98.8 billion atoms remain radioactive after 3000 years.

12.3. We need to solve the equation $93 = 142\left(\frac{1}{2}\right)^{\frac{t}{5730}}$. Using a graphing calculator, we obtain $t \approx 3499$. We conclude that the clothing is approximately 3500 years old.

13.1. $A(3) = 1000(1+0.04)^3 \approx 1124.86$

13.2. $A(4) = 2500(1+0.07)^4 \approx 3276.99$

13.3. $P = \frac{2500}{(1+0.07)^4} \approx 1907.24$

13.4. $P = \frac{3600}{(1+0.10)^6} \approx 2032.11$

13.5. $r = \left(\frac{3600}{1200}\right)^{1/6} - 1 \approx 0.20$

13.6. $r = \left(\frac{5000}{600}\right)^{1/20} - 1 \approx 0.11$

14.1. $A(t) = 15000(1.05)^t$

14.2. In 2020: $A(10) \approx 24433.42$; In 2030: $A(20) \approx 39799.47$; In 2040: $A(30) \approx 64829.14$

14.3. Using a graph or table of values, approximately 6 years.

15.1. $8000\left(1+\frac{0.05}{1}\right)^{4\cdot 1} \approx 9724.05$. There is \$9724.05 in the account.

15.2. $8000\left(1+\frac{0.05}{52}\right)^{4\cdot 52} \approx 9770.28$. There is \$9770.28 in the account.

15.3. $8000\left(1+\frac{0.05}{365}\right)^{4\cdot 365} \approx 9771.09$. There is \$9771.09 in the account.

15.4. A table of values says that the function $8000\left(1+\frac{0.05}{365}\right)^{365t}$ reaches 16000 when t is between 13 and 14 years. The function reaches 24000 between 21 and 22 years.

15.5. Trial and error gives $r = 9\,\%$.

16.1. $\left(1+\frac{0.06}{1}\right)^1 - 1 = 0.06000000$. The effective rate is 6 %.

16.2. $\left(1+\frac{0.06}{4}\right)^4 - 1 \approx 0.06136355$. The effective rate is approximately 6.136 355 %.

16.3. $\left(1+\frac{0.06}{365}\right)^{365} - 1 \approx 0.06183131$. The effective rate is approximately 6.183 131 %.

16.4. There are $365 \cdot 24 \cdot 60 \cdot 60 = 31536000$ seconds in a year.

$$\left(1+\frac{0.06}{31536000}\right)^{31536000} - 1 \approx 0.06183696$$

The effective rate is approximately 6.183 696 %.

16.5. No practical difference.

17. $\left(1+\frac{0.24}{12}\right)^{12} - 1 \approx 0.26824179$; the effective annual interest rate is about 26.82 %.

18.1. The interest charged on the \$375 loan was \$75. As $\frac{75}{375} = 0.2$, the 2-week percentage rate is 20 %.

18.2. $1.2^{26} - 1 \approx 113.475$; the effective annual rate is approximately 11 347.5 %.

19.1. We need to solve $(1.45)P_0 = P_0(1+r)^{10}$. This gives $r \approx 3.7855\,\%$.

19.2. We need to solve $(1.45)P_0 = P_0\left(1+\frac{r}{12}\right)^{120}$. This gives $r \approx 3.7214\,\%$.

Solutions for problems in Section 3.5

1. (a) Using Definition 14,

$$Q(t) = 4000\, e^{0.02t}$$

(b) We evaluate the function Q when $t = 4$

$$\begin{aligned} Q(4) &= 4000\, e^{0.02(4)} \\ &\approx 4333.15 \end{aligned}$$

The amount in the account after four years is \$4333.15.

(c) Not relevant to this problem.

(d) We calculate the effective annual rate of interest using Definition 14

$$e^{0.02} - 1 \approx 0.0202$$

The effective annual rate of interest is approximately 2.02 %.

2. Growth rates:

- Daily: $\left(1 + \frac{0.08}{365}\right)^{365 \cdot 1} - 1 \approx 0.083277572$; the effective annual rate is approximately 8.327 757 2 %.
- Continuous : $e^{0.08} - 1 \approx 0.08328707$; the effective annual rate is approximately 8.328 707 %.

Growth factors:

- Daily: $\left(1 + \frac{0.08}{365}\right)^{365 \cdot 1} \approx 8.3277572$; the annual growth factor is approximately 8.3277572%.
- Continuous: $e^{0.08} \approx 1.08328707$; the growth factor is approximately 1.08328707.

Balance after 10 years:

- Annually: $15000(1.08)^{10} \approx 32,383.87$; the balance will be \$32, 383.87.
- Daily: $15000\left(1 + \frac{0.08}{365}\right)^{10 \cdot 365} \approx 33380.19$; the balance will be \$33380.19.
- Continuously: $15000e^{0.08 \cdot 10} \approx 33383.11$; the balance will be \$33383.11.

3.1. $f(10) \approx 2.59374$, $f(100) \approx 2.70481$, $f(1000) \approx 2.71692$, $f(10000) \approx 2.71814$, $f(100000) \approx 2.71827$

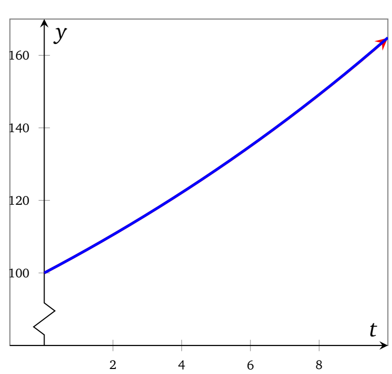
3.2. $e \approx 2.71828$; note that $f(100000)$ is closest to this.

3.3. Because ∞ is not a number – it is a concept that we use to represent that a variable is growing without bound.

4.1. $Q(1) = 100e^{0.05} \approx 105.13$. After 1 year there is approximately \$105.13 in the account.

4.2. The effective annual growth rate is $e^{0.05} - 1 \approx 0.05127$ or approximately 5.1271 %.

4.3. We know that $e^{0.05} \approx 1.05127$, so we are not surprised to see that Q and P are approximately the same function. The function Q is plotted below as a solid line, and the function P is plotted as dots.



5.1. $P(t) = 600(1.07)^t$

5.2. $P(t) = 1500e^{0.07t}$

5.3. $P(t) = 2300(1.27)^t$

5.4. $P(t) = 3600e^{0.52t}$

5.5. $P(t) = 450(0.94)^t$

5.6. $P(t) = 2405e^{-0.12t}$

5.7. $P(t) = 4402(0.81)^t$

5.8. $P(t) = 7203e^{-0.31t}$

6.1. $P_0 = 1000$, $r = 10\,\%$; the population is initially 1000 people, and increases at 10 % per year.

6.2. $P_0 = 1000$, $r = \mathrm{e}^{0.11}\,\% \approx 1.116\,278\,07\,\%$; the population is initially 1800 people, and increases at approximately 11.627 807 % per year.

6.3. $P_0 = 300$, $r = -17\,\%$; the population is initially 200 people, and decreases at 17 % per year.

6.4. $P_0 = 907$, $r = \mathrm{e}^{-0.08}\,\% \approx 0.923\,116\,346\,\%$; the population is initially 907 people, and decreases at approximately 7.69 % per year.

7. In ascending order: $1/e^2 < 1/4 < 1/3 < 1/e < 2 < e < 3 < e^2 < 9$

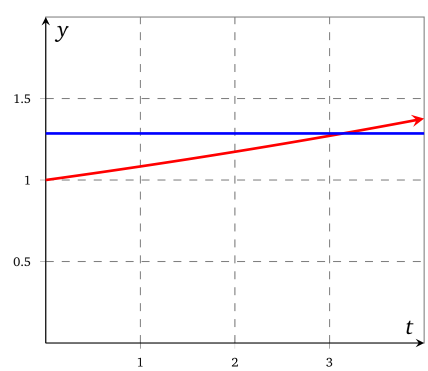
8.1. $A(t) = 2000(1.0375)^t$; the effective annual rate is 3.75 %.

8.2. $A(t) = 2000\left(1 + \frac{0.0375}{12}\right)^{12t}$; the effective annual rate is calculated using $\left(1 + \frac{0.0375}{12}\right)^{12} \approx$ 1.03815129. The effective annual rate is approximately 1.038 151 29 %.

8.3. $A(t) = 2000\left(1 + \frac{0.0375}{365}\right)^{365t}$; the effective annual rate is calculated using $\left(1 + \frac{0.0375}{365}\right)^{365} \approx$ 1.0382100. The effective annual rate is approximately 1.038 210 0 %.

8.4. $A(t) = 2000e^{0.0375t}$; the effective annual rate is calculated using $e^{0.0375} \approx 1.0382120$. The effective annual rate is approximately 1.038 212 0 %.

9. We have to solve the equation $9000 = 7000e^{0.08t}$ for t. This can be simplified to $9/7 = e^{0.08t}$. We can solve this using our graphing calculator as shown below – this gives that $t \approx 3.14$. So it would take just over 3 years for our account to reach \$9,000.



10.1. $A(t) = 2500(1.1)^t$.

10.2. We need to solve the equation $e^r = 1.1$; using a table of values, we find $r \approx 0.09531$, and the continuous growth rate is approximately 9.531 %.

11.1. $T(0) = 90$.

11.2. $T(t) \to 0$ as $t \to \infty$.

11.3. $T(t) = 90e^{-0.07t}$

11.4. $T(10) = 90e^{-0.07\cdot 10} \approx 45$; the temperature of the coffee is approximately 45 °C.

11.5. We need to solve the equation $5 = 90e^{-0.07t}$ for t. Using a graphing calculator, we find $t \approx 41.29$. The coffee will be 5 °C approximately 41 min after it was bought.

11.6. No – the model is limited in this way.

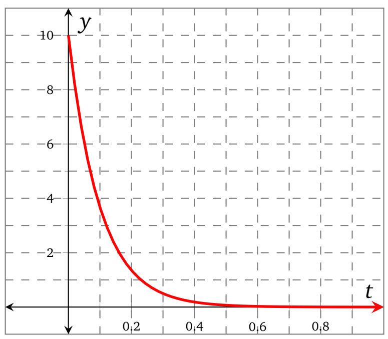
12.1. $V(t) = 10e^{-\frac{t}{0.1}} = 10e^{-10t}$

12.2.

t (s)	$V(t)$ (V)
0.0	10.000
0.1	3.369
0.2	1.353
0.3	0.498
0.4	0.183
0.5	0.067

12.3. Concave up, since the slopes between successive entries are increasing by becoming less negative.

12.4. $y = V(t)$ is shown below.



12.5. After 0.1 s, the voltage on the capacitor has decreased by about 63 %. After 0.2 s, the voltage on the capacitor has decreased by about 86 %.

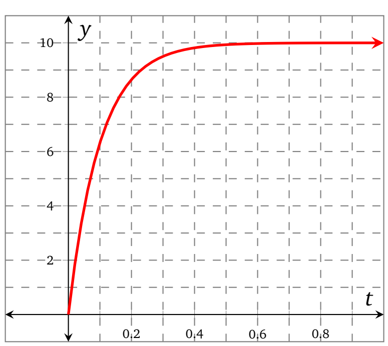
13.1. $V(t) = 10(1 - e^{-0.1t})$

13.2.

t (s)	$V(t)$ (V)
0.0	0.000
0.1	6.321
0.2	8.647
0.3	9.502
0.4	9.817
0.5	9.933

13.3. The graph will be concave down since the slopes between successive entries are decreasing by becoming less positive.

13.4. $y = V(t)$ is shown below.



13.5. After 0.1 s, the capacitor has charged up to about 63 % of the battery voltage. After 0.2 s, the capacitor has charged up to about 86 % of the battery voltage.

13.6. The value of the function can increase even though the exponential term is decaying because it is being subtracted from 1. As the exponential term decreases, the difference between its value and 1 increases, creating an increasing function.

14.1. The half-life is approximately 6931.47 years.

14.2. The half-life is approximately 13.86 years.

14.3. The half-life does not depend upon Q_0. It does depend upon k.

15.1. $e^t \approx 2.7182818^t$, so f has continuous growth rate 1 and growth rate about 1.7182818.

15.2. $e^{0.2t} \approx 1.2214028^t$, so f has continuous growth rate 20 % and growth rate about 22.140 28 %.

15.3. $e^{-0.1t} \approx 0.9048374^t$, so f has continuous decay rate 10 % and decay rate about 9.516 26 %.

16.1.

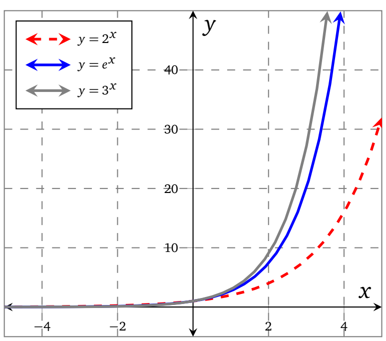
x	$f(x)$
-3	0.049 79
-2	0.135 34
-1	0.367 88
0	1.000 00
1	2.718 28
2	7.389 06
3	20.0855

16.2. f has domain $(-\infty, \infty)$, and range $(0, \infty)$; f is concave up.

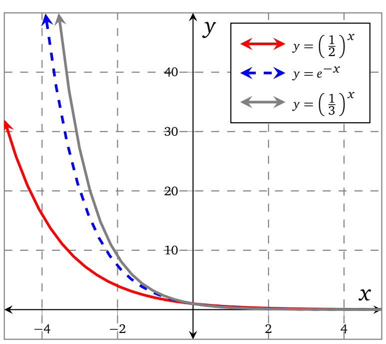
16.3.

- g has domain $(-\infty, \infty)$ and range $(4, \infty)$
- h has domain $(-\infty, \infty)$ and range $(-\infty, 0)$

16.4.



16.5.



17.1. Verify graphically using Figure 2.28 on page 69.

17.2. The slope is about 2.20. Verify graphically using Figure 2.29 on page 69.

18.

Exact value	Decimal value (to 5 d.p)
$1/0! + 1/1! + 1/2! + 1/3! + 1/4! + 1/5!$	2.71667
$1/0! + 1/1! + 1/2! + 1/3! + 1/4! + 1/5! + 1/6!$	2.71806
$1/0! + 1/1! + 1/2! + 1/3! + 1/4! + 1/5! + 1/6! + 1/7!$	2.71825
$1/0! + 1/1! + 1/2! + 1/3! + 1/4! + 1/5! + 1/6! + 1/7! + 1/8!$	2.71828

We observe that as we add more terms, the decimal value appears to approach e.

Solutions for problems in Section 3.6

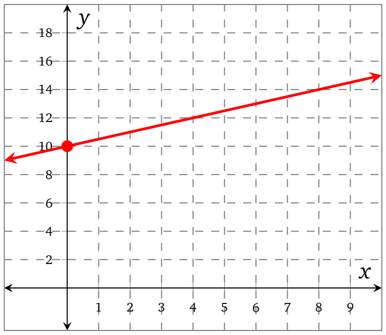
1.1. Exponential.

1.2. Linear.

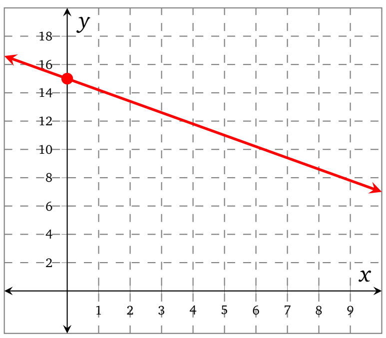
1.3. Linear.

1.4. Exponential.

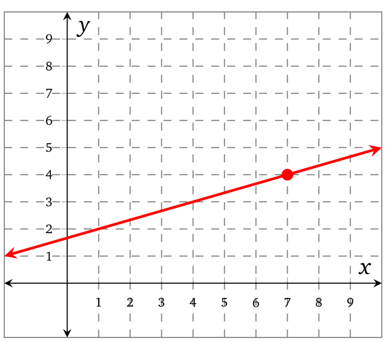
2.1. The graph of $y = \frac{1}{2}x + 10$ is shown below.



2.2. The graph of $y = 15 - 0.8x$ is shown below.



2.3. The graph of $y = \frac{1}{3}(x-7) + 4$ is shown below.



3.1. $m = \frac{28-30}{5-12} = \frac{2}{7}$

3.2. $m = \frac{15.3-18.4}{2012-2001} = \frac{-3.1}{11} = -\frac{31}{110}$

3.3. $m = \frac{y-118}{17-t}$

4. Table 2.29: this is linear data since y values increase by a constant 6 when x increases by 1. Table 2.30: this data is reasonably close to linear since y values increase by *almost* a constant 0.2 when x increases by 1. Table 2.31: this data is not linear.

5. Table 2.37: since successive differences are all equal to 2, a linear model would be appropriate. Successive ratios decrease steadily, so an exponential model would not be appropriate. Table 2.38: since successive ratios are all equal to 3, an exponential model would be appropriate. Successive differences increase, so a linear model would not be appropriate. Table 2.39: since successive differences are all quite close to 2.3, a linear model might be appropriate. Successive ratios drop steadily from about 1.13 to 1.09, so an exponential model would not be appropriate. Table 2.40: since successive ratios are all quite close to 0.92, an exponential model might be appropriate. Successive differences rise steadily from -44 to -31, so a linear model would not be appropriate.

6.1. $(3, 158)$ and $(6, 233)$.

$$\begin{aligned} m &= \frac{233-158}{6-3} \\ &= \frac{75}{3} \\ &= 25 \end{aligned}$$

The units in the numerator are millions and the units in the denominator are years. So the slope of 25 has meaning as a rate of 25 $^{\text{million}}/_{\text{year}}$. This means there are 25 million new cell phone subscriptions each year since 2000.

6.2. Let $f(t) = mt + b$, be the number of cell phone users in millions at time t in years since 2000. We have already found m; we need to find b.

$$\begin{aligned} 158 &= 25(3) + b \\ b &= 83 \end{aligned}$$

Therefore, $f(t) = 25t + 83$.

6.3. $f(t) = \frac{(158)^2}{233}\left(\frac{233}{158}\right)^{t/3} \approx 107.14(1.13824)^t$.

6.4. The linear model predicts that there were about 283 million cell phone users in 2008. The exponential model predicts that there were about 302 million cell phone users in 2008. The linear model does a better job of predicting the number of cell phone users in 2008.

7.1. An exponential function seems to be more appropriate. First of all the graph of the data has the basic shape of an exponential function. More by the numbers, we can examine the approximate successive ratios. It's difficult to read the chart on its left side with any relative precision. Starting from the right and using rough approximate readings from the chart:

$$\frac{195000}{160000} \approx 1.22$$

$$\frac{160000}{125000} \approx 1.28$$

$$\frac{125000}{95000} \approx 1.32$$

$$\frac{95000}{75000} \approx 1.27$$

$$\frac{75000}{60000} \approx 1.25$$

$$\frac{60000}{50000} \approx 1.20$$

The successive ratios are all fairly close to each other, so an exponential model would be appropriate.

7.2. Using a table of values, we find that $W(16) \approx 248470$, and $W(17) \approx 317980$. We conclude that, according to the model, the world's wind power capacity will be 250 000 MW in early 2012.

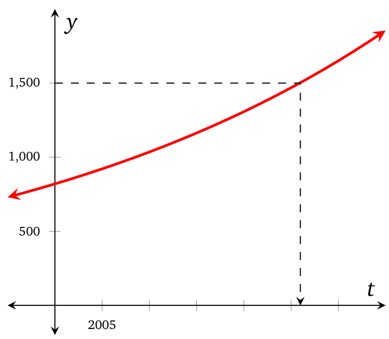
8.1. Answers will vary. By the numbers, the successive ratios are all between 1.0231 and 1.0242. Since they are so close to each other, an exponential model is appropriate. Alternatively, a plot of the data reveals a concave up trend.

8.2. Answers will vary. One solution uses the exponential curve that passes through $(2001, 839.0)$ and $(2007, 964.7)$: $P(t) \approx 819.7(1.023530)^t$. However it is valid to use other pairs of points and find slightly different functions.

An alternative solution would use the initial population of 819.5 and the average of the successive ratios: 1.023600. This would give $P(t) \approx 819.5(1.023600)^t$

8.3. The model suggests that in 2020 the population of Africa will be about 1310 million, or 1.31 billion.

8.4. According to the model, it appears that the population of Africa will reach 1.5 billion in the year 2026.



8.5. Answers will vary as models vary. Using the model $P(t) = 819.7(1.023530)^t$:

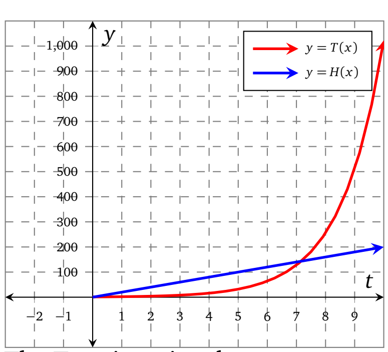
Year	Actual	Model
1999	800.2	800.9
2000	819.5	819.7
2001	839.0	839.0
2002	858.9	858.8
2003	879.2	879.0
2004	899.9	900.0
2005	921.1	920.8
2006	942.7	942.5
2007	964.7	964.7
2008	987.1	987.4
2009	1009.9	1010.6

The model seems quite accurate.

9.1. $H(t) = 20t$

9.2. $T(t) = 2^t$

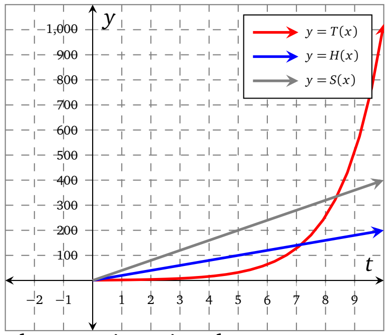
9.3. The graphs of H and T are shown below.



The Tortoise wins the race.

9.4. $S(t) = 40t$

9.5. The graphs of H, T, and S are shown below.



The Tortoise wins the race.

10.1. Exponential.

10.2. Linear.

10.3. Linear.

10.4. Exponential.

11.1. Linear.

11.2. Exponential.

11.3. Linear.

11.4. Exponential.

11.5. Linear.

12.1. $f(x) = \frac{3}{2}x - 3$, $g(x) = 16\left(\frac{1}{4}\right)^x$, $h(x) = 3\left(\frac{3}{2}\right)^x$, $k(x) = 16 - 4x$.

12.2. The data in Table 2.48 could be modeled with an exponential function, since all of the ratios are fairly close to 1.1. The model could be $y = f(x)$, where $f(x) = 5(1.1)^x$. The data in Table 2.49 could be modeled with a linear function, since all of the differences are fairly close to 1.12. The model could be $y = g(x)$, where $g(x) = 1.12x + 2.03$. The data in Table 2.50 should not be modeled with either a linear function or an exponential function, since the differences are increasing and the ratios are decreasing.

13.1. $f(1) = 1 \times 10^6$, $g(1) = 2$. Clearly $f(1) > g(1)$.

13.2. $f(10) = 1.01 \times 10^6$, $g(10) = 1024$. Clearly $f(10) > g(10)$.

13.3. Answers will vary.

13.4. $f(20) = 1.02 \times 10^6$, $g(20) \approx 1.05 \times 10^6$.

14.1. $f(x) = \frac{5}{4}x + \frac{25}{4}$

14.2. $g(x) = \frac{10}{(3/2)^{3/4}}\left(\sqrt[4]{\frac{3}{2}}\right)^x \approx 7.38(1.106682)^x$

14.3. $x = 8$ (perhaps obviously).

15.1. False; linear functions are neither concave up nor concave down.

15.2. False; linear functions are neither concave up nor concave down.

15.3. True.

15.4. False; exponential functions do not have a constant slope.

15.5. False; exponential functions do not decrease (nor increase) at a constant rate.

16.1. Linear, with $m = 3/2$.

16.2. Exponential, with $b = 5$.

16.3. Linear.

16.4. Linear.

16.5. Exponential.

17.1. $y = \frac{5}{9}(x-32)$

17.2. $y = 100(0.9)^x$

17.3. $y = 2\pi x$

17.4. $y = 2^x$

17.5. $y = \frac{9}{5}x + 32$

17.6. $y = 100 - 10x$

18.

t	$F(t)$	successive ratio
0	2	
1	3	1.5
2	4	1.33...
3	6	1.5
4	9	1.5
5	14	1.55...
6	21	1.5
7	31	1.48...
8	47	1.52...

The successive ratios are fairly close to 1.5, although there is some variation. This suggests that an exponential model might be appropriate with a base of 1.5.

19.1. $f(x) \to 0$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. $n(x) \to -\infty$ as $x \to \infty$ and $n(x) \to \infty$ as $x \to -\infty$.

19.2. $g(x) \to 0$ as $x \to \infty$ and $g(x) \to -\infty$ as $x \to -\infty$. $m(x) \to \infty$ as $x \to \infty$ and $m(x) \to -\infty$ as $x \to -\infty$.

20.

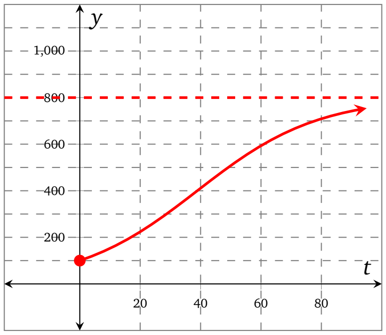
- Figure 2.39a: $y = 10^x$
- Figure 2.39b: $y = -\pi x$
- Figure 2.39c: $y = \left(\frac{1}{2}\right)^x$
- Figure 2.39d: $y = x$
- Figure 2.39e: $y = 4$
- Figure 2.39f: $y = -\left(\frac{1}{2}\right)^x$
- Figure 2.39g: $y = 10x + 2$
- Figure 2.39h: $y = -4^x$

Solutions for problems in Section 3.7

1.1. $P(t) = \frac{800\cdot 100}{100+(800-100)e^{-0.05t}} = \frac{800}{1+7e^{-0.05t}}$

1.2. $P(1) \approx 104.46$. We calculate $\frac{P(1)}{P(0)} \approx 1.0446$. The relative growth over the first year is approximately 4.46%.

1.3. $P(t) \to 800$ as $t \to \infty$, as shown below.



1.4. P is increasing.

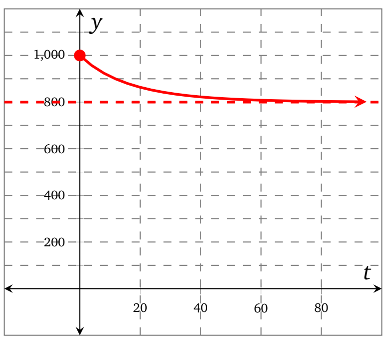
2.1. $P(t) \approx \frac{700000}{35+19965e^{-0.9t}}$.

2.2. About 8.3 years.

3.1. $P(t) \approx \frac{400000}{8+399992e^{-3.5t}}$.

3.2. About 3.1 years.

4.1. There are infinitely many choices available to us- we just need to choose $P_0 > 800$. With $P_0 = 1000$, we have $P(t) = \frac{4000}{5-e^{-0.05t}}$.



4.2. When $P_0 = 800$, $P(t) = 800$. P is neither increasing nor decreasing, it is constant!

★ **5.1.** 8

★ **5.2.** 81

★ **5.3.** 2

5.4. 3

5.5. 2^{3^x}

5.6. 3^{2^x}

6.1. $f(x) = 2^x$, $g(x) = x^2$

6.2. $f(x) = -4^x$, $g(x) = x^3 + 2x$

6.3. $f(x) = 2^x + 3^x$, $g(x) = x^2$

6.4. $f(x) = e^x$, $g(x) = -x^2 + 2$

7.1. 3

7.2. 4

7.3. -2

7.4. 0

8.1. 64

8.2. 10

8.3. -10

8.4. -10

8.5. 30

8.6. 36

8.7. $k(7)$ is undefined.

8.8. -65536

9.1. 1

9.2. $\frac{143}{16}$

9.3. $\frac{16}{9}$

9.4. 12

10.1. $f(x-2) = 7^{x-2}$

10.2. $f(x+5) = 7^{x+5}$

10.3. $f(x) + 11 = 7^x + 11$

10.4. $f(x) - 1 = 7^x - 1$

10.5. $-f(x) = -7^x$

10.6. $f(-x) = 7^{-x}$

11.1. g is the function f shifted to the left by 7 units.

11.2. h is the function f shifted to the right by 13 units.

11.3. j is the function f horizontally compressed by a factor of 2, and shifted to the left by 9 units.

11.4. k is the function f reflected across the vertical axis, and vertically stretched by a factor of 7.

Solutions for problems in Section 4.1

1.1. Domain of $f \cdot g$: $(-\infty, \infty)$; domain of $\frac{f}{g}$: $(-\infty, \infty)$.

1.2. Domain of $f \cdot g$: $[0, \infty)$; domain of $\frac{f}{g}$: $(0, \infty)$.

1.3. Domain of $f \cdot g$: $[1, \infty)$; domain of $\frac{f}{g}$: $[1, \infty)$

1.4. Domain of $f \cdot g$: $(-\infty, \infty)$; domain of $\frac{f}{g}$: $(-\infty, -1) \cup (-1, 10) \cup (10, \infty)$.

2.1.

x	$(f+g)(x)$
-4	-72
-3	-21
-2	0
-1	3
0	0
1	3
2	24
3	75
4	168

2.2.

x	$(f-g)(x)$
-4	-40
-3	-15
-2	0
-1	5
0	0
1	-15
2	-40
3	-75
4	-120

2.3.

x	$(g \cdot h)(x)$
-4	-32
-3	-12
-2	0
-1	-8
0	0
1	108
2	448
3	1200
4	2592

2.4.

x	$(h+j)(x)$
-4	32
-3	25
-2	18
-1	11
0	4
1	-3
2	29
3	112
4	778

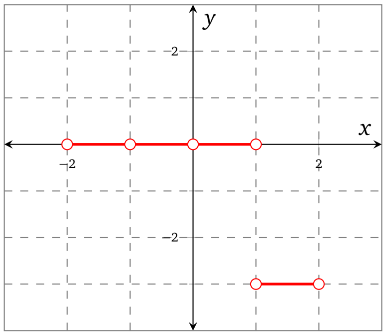
2.5.

x	$\left(\frac{j}{h}\right)(x)$
-4	15
-3	$21/4$
-2	2
-1	$3/8$
0	$-3/5$
1	$-5/4$
2	$15/14$
3	6
4	$380/9$

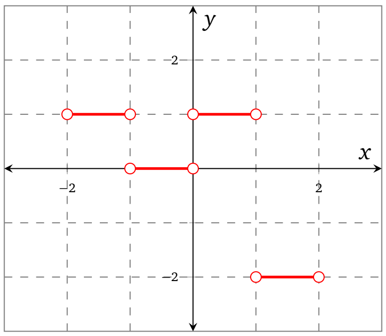
2.6.

x	$\left(\frac{j}{f}\right)(x)$
-4	$-15/28$
-3	$-7/6$
-2	X
-1	$3/4$
0	X
1	$5/2$
2	$-15/8$
3	X
4	$95/3$

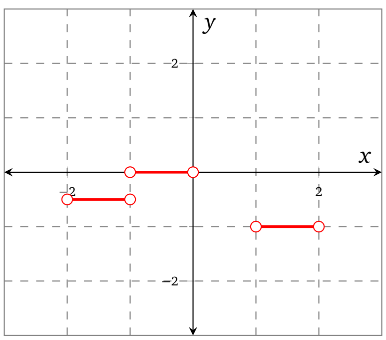
3.1. The function $F+G$ is shown below.



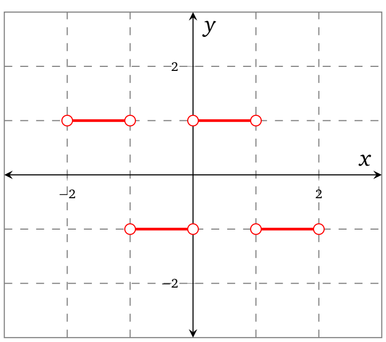
3.2. The function $G \cdot H$ is shown below.



3.3. The function $\frac{H}{J}$ is shown below; note that this function is undefined on the interval $(0, 1)$.



3.4. The function $J - F$ is shown below.



4.

x	-6	-4	-2	0	2	4	6
$f(x)$	2	1	3	0	2	π	12
$g(x)$	8	1	3	5	-1	-1	2
$(f+g)(x)$	10	2	6	5	1	$\pi - 1$	14
$(f-g)(x)$	-6	0	0	-5	3	$\pi + 1$	10
$(f \cdot g)(x)$	16	1	9	0	-2	$-\pi$	24
$\left(\frac{f}{g}\right)(x)$	$1/4$	1	1	0	-2	$-\pi$	6

5.1. -2

5.2. 5

5.3. -2

5.4. $\frac{1}{2}$

Solutions for problems in Section 4.2

1.1. $F(x) = \begin{cases} 1, & -2 < x < -1 \\ 2, & -1 < x < 0 \\ -1, & 0 < x < 1 \\ -2, & 1 < x < 2 \end{cases}$

1.2. $G(x) = \begin{cases} -1, & -2 < x < -1 \\ -2, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ -1, & 1 < x < 2 \end{cases}$

1.3. $H(x) = \begin{cases} -1, & -2 < x < -1 \\ 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$

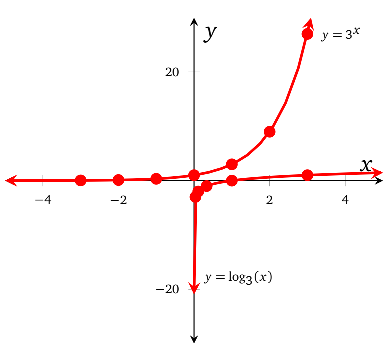
1.4. $J(x) = \begin{cases} 2, & -2 < x < -1 \\ 1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ -2, & 1 < x < 2 \end{cases}$

Solutions for problems in Section 5.1

1.

x	$f(x)$
-3	$1/27$
-2	$1/9$
-1	$1/3$
0	1
1	3
2	9
3	27

x	$f^{-1}(x)$
$1/27$	-3
$1/9$	-2
$1/3$	-1
1	0
3	1
9	2
27	3



2.1. $(-7, \infty)$

2.2. $(2, \infty)$

2.3. $(0, \infty)$

2.4. $\left(-\frac{3}{4}, \infty\right)$

2.5. $(-\infty, -3) \cup (3, \infty)$

2.6. $(-\sqrt{2}, \sqrt{2})$

2.7. $(-\infty, \infty)$

2.8. $(0, \infty)$

3.1. g is the function f shifted to the left by 3 units. The domain of g is $(-3, \infty)$.

3.2. h is the function f shifted to the right by 5 units. The domain of h is $(5, \infty)$.

3.3. j is the function f horizontally compressed by a factor of 2, and shifted to the left by 7 units. The domain of j is $(-7, \infty)$.

3.4. k is the function f reflected across the vertical axis, and vertically stretched by a factor of 5. The domain of k is $(-\infty, 0)$.

4.1. $f(x-2) = \log(x-2)$

4.2. $f(x+5) = \log(x+5)$

4.3. $f(x) + 11 = \log(x) + 11$

4.4. $f(x) - 1 = \log(x) - 1$

5.
- Figure 4.3a on page 100: $b = 2$, so $y = \log_2(x)$;
- Figure 4.3b on page 100: $b = 4$, so $y = \log_4(x+3)$;
- Figure 4.3c on page 100: $b = \frac{1}{2}$, so $y = \log_{\frac{1}{2}}(x-4)$;
- Figure 4.3d on page 100: $b = \frac{1}{3}$, so $y = \log_{\frac{1}{3}}(x+2)$.

6.1. $x = \ln(7) \approx 1.95$

6.2. $x = \ln(5) \approx 1.61$

6.3. $x = \ln(10) - 5 \approx -2.70$

6.4. $x = \frac{1}{5}(\ln(6) - 7) \approx -1.04$

6.5. $x = 0$

6.6. $x = \log(11) - 1 \approx 0.04$

6.7. $x = \frac{\log(4)}{2} \approx 0.30$

6.8. $x = 4 - \log(21) \approx 2.68$

6.9. $x = \frac{1}{2}\ln\left(\frac{9}{5}\right) \approx 0.29$

6.10. $x = -\frac{1}{3}\ln\left(\frac{18}{7}\right) \approx -0.31$

6.11. $x = 5 - \ln\left(\frac{1}{9}\right) \approx 7.20$

6.12. $x = \frac{1}{3}\ln\left(\frac{2}{3}\right) \approx -0.14$

7.1. $x = e^7 \approx 1096.63$

7.2. $x = e^{-3/2} \approx 0.22$

7.3. $x = \frac{e^{-2}}{2} \approx 0.07$

7.4. $x = e^{\sqrt{5}} \approx 9.36$ and $x = e^{-\sqrt{5}} \approx 0.11$

7.5. $x = 10^7$

7.6. $x = 1000$

7.7. $x = \frac{1}{5}\left(10^{-3} - 2\right) \approx -0.40$

7.8. $x = -995$

8.
- Table 4.3a on page 101: $b = 2$, so $y = \log_2(3x)$;
- Table 4.3b on page 101: $b = 3$, so $y = \log_3(5x)$;
- Table 4.3c on page 101: $b = \frac{1}{4}$, so $y = \log_{\frac{1}{4}}(x)$;
- Table 4.3d on page 101: $b = \frac{1}{3}$, so $y = \log_{\frac{2}{3}}(-2x)$.

9.1.

x	$f(x)$
-3	$1/64$
-2	$1/16$
-1	$1/4$
0	1
1	4
2	16
3	64

9.2.

x	$f^{-1}(x)$
$1/64$	-3
$1/16$	-2
$1/4$	-1
1	0
4	1
16	2
64	3

9.3. (a) $f^{-1}(4) = 1$

(b) $f^{-1}(16) = 2$

(c) $f^{-1}\left(\frac{1}{4}\right) = -1$

(d) $f^{-1}\left(\frac{1}{16}\right) = -2$

9.4. $f^{-1}(x) = \log_4(x)$

10.1. (a) domain: $(-\infty, \infty)$, range: $(0, \infty)$.

(b) $f^{-1}(x) = \frac{1}{5}\log_2(x)$

(c) domain: $(0, \infty)$, range: $(-\infty, \infty)$.

10.2. (a) domain: $(-\infty, \infty)$, range: $(0, \infty)$.

(b) $g^{-1}(t) = \frac{1}{3}(\ln(t) - 4)$

(c) domain: $(0, \infty)$, range: $(-\infty, \infty)$.

10.3. (a) domain: $(-\infty, \infty)$, range: $(-\infty, 5)$.

(b) $h^{-1}(s) = 7 + \log_4(5 - s)$

(c) domain: $(-\infty, 5)$, range: $(-\infty, \infty)$.

10.4. (a) domain: $(-\infty, \infty)$, range: $(0, \infty)$.

(b) $j^{-1}(u) = \log_5\left(\frac{3}{u}\right)$

(c) domain: $(0, \infty)$, range: $(-\infty, \infty)$.

10.5. (a) domain: $\left(-\frac{2}{3}, \infty\right)$, range: $(-\infty, \infty)$.

(b) $k^{-1}(v) = \frac{1}{3}(e^v - 2)$

(c) domain: $(-\infty, \infty)$, range: $\left(-\frac{2}{3}, \infty\right)$.

10.6. (a) domain: $\left(-\infty, \frac{2}{7}\right)$, range: $(-\infty, \infty)$.

(b) $l^{-1}(w) = \frac{1}{7}\left(2 - 10^{\frac{w}{5}}\right)$

(c) domain: $(-\infty, \infty)$, range: $\left(-\infty, \frac{2}{7}\right)$.

10.7. (a) domain: $\left(\frac{1}{2}, \infty\right)$, range: $(-\infty, \infty)$.

(b) $m^{-1}(\alpha) = \frac{1}{2}(8^\alpha + 1)$

(c) domain: $(-\infty, \infty)$, range: $\left(\frac{1}{2}, \infty\right)$.

10.8. (a) domain: $\left(\frac{7}{4}, \infty\right)$, range: $(-\infty, \infty)$.

(b) $n^{-1}(\beta) = \frac{1}{4}\left(3^{\frac{3\beta}{2}} + 7\right)$

(c) domain: $(-\infty, \infty)$, range: $\left(\frac{7}{4}, \infty\right)$.

11.1. $I(t) = \frac{3}{50}\left(1 - e^{-40t}\right)$

11.2.
- $I(0.025) \approx 0.038$; the current in the circuit after 0.025 s is approximately 0.038 A;
- $I(0.05) \approx 0.052$ A; the current in the circuit after 0.05 s is approximately 0.052 A.

11.3. The maximum value the current could reach is $3/50$ A $= 0.06$ A.

11.4. We need to solve the equation $0.95 \cdot 0.06 = I(t)$; so $t \approx 0.075$. The current reaches 95 % of its maximum value after about 0.075 s.

12.1. $x = \ln(2) \approx 0.69$ and $x = \ln(4) \approx 1.39$.

12.2. $x = \ln(3) \approx 1.10$ and $x = \ln(1) = 0$.

12.3. $x = \ln(10) \approx 2.30$ (there are no solutions to the equation $e^x = -2$).

12.4. There are no solutions to this equation.

12.5. $x = \log(2) \approx 0.30$ and $x = \log(4) \approx 0.60$.

12.6. $x = \log(3) \approx 0.48$ and $x = \log(1) = 0$.

12.7. $x = \log(10) = 1$ (there are no solutions to the equation $10^x = -2$).

12.8. There are no solutions to this equation.

13.1. $x = e^{-2} \approx 0.14$ and $x = e^{-1} \approx 0.37$.

13.2. $x = e^2 \approx 7.39$ and $x = e \approx 2.72$.

13.3. $x = e^4 \approx 54.60$ and $x = e^{-4} \approx 0.02$.

13.4. $x = 10^{-7}$ and $x = 10$.

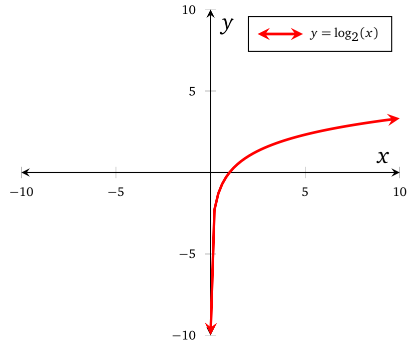
13.5. $x = 10^{-4}$ and $x = 10^{-3}$.

13.6. $x = 10$ and $x = 10^{-1}$.

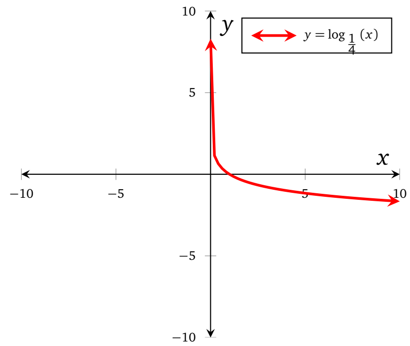
Solutions for problems in Section 5.2

1. Put $m = \log_b(x)$ and $n = \log_b(y)$ so that $b^m = x$ and $b^n = y$. Therefore $b^{m-n} = \frac{x}{y}$, and equivalently $\log_b\left(\frac{x}{y}\right) = m - n$. Also, $\log_b(x) - \log_b(y) = m - n$. The result follows.

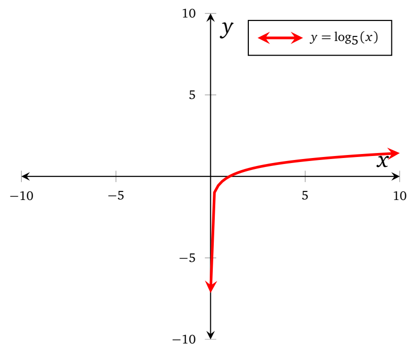
2.1. (a) $b = 2$



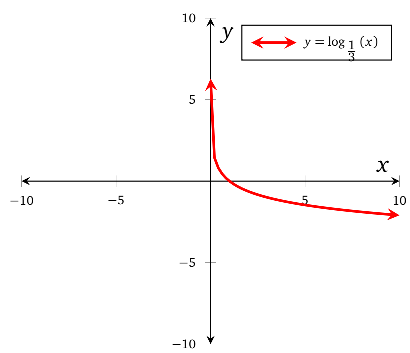
(b) $b = \frac{1}{4}$



(c) $b = 5$



(d) $b = \frac{1}{3}$



2.2. False; consider $b = \frac{1}{4}$ or $b = \frac{1}{3}$, or any other value of b such that $0 < b < 1$.

2.3. False; consider $b = 2$ or $b = 5$, or any other value of b such that $b > 1$.

2.4. True.

2.5. False; consider $b = 2$ or $b = 5$, or any other value of b such that $b > 1$.

2.6. True.

2.7. False; consider any value of b.

2.8. False; consider $b = \frac{1}{4}$ or $b = \frac{1}{3}$, or any other value of b such that $0 < b < 1$.

3.1.

A	B	b	$\log_b(A)$	$\log_b(B)$	$\log_b(AB)$	$\log_b\left(\frac{A}{B}\right)$
1	2	2	0	1	1	−1
e^5	e^3	e	5	3	8	2
36	$\sqrt[3]{6}$	6	2	1/3	7/3	5/3
0.001	10000	10	−3	4	1	−7
4	1/16	1/4	−1	2	1	−3

3.2. False.

3.3. False.

3.4. True.

3.5. False.

3.6. True.

4.1. $\frac{\ln(3)}{\ln(2)} \approx 1.58$

4.2. Undefined since the argument is negative.

4.3. $\frac{\ln(7)}{\ln(3)} \approx 1.77$

4.4. $\frac{\ln(13)}{\ln\left(\frac{1}{2}\right)} \approx -3.70$

4.5. $\frac{\ln(2)}{\ln(8)} \approx .33$

4.6. Undefined since the base is negative.

4.7. $\frac{\ln(5)}{\ln(\pi)} \approx 1.41$

4.8. Undefined since the argument is 0.

5.1. $\log(2) + \log(x)$

5.2. $\log_3(4) - \log_3(x)$

5.3. $7\log_5(x)$

5.4. $\log_9(4) + 3\log_9(x)$

5.5. $\frac{1}{2}\ln(x)$

5.6. $\frac{3}{7}\ln(x) - \frac{1}{7}\ln(x+2)$

5.7. $2\log_\pi(x) - \log_\pi(4)$

5.8. $3 + 3\log(x)$

7.1. $\frac{8}{7}$

7.2. $\frac{3}{4}$

7.3. $\frac{1}{81}$

7.4. $\frac{9}{2}$

8.1. 3

8.2. 4

8.3. 5

8.4. 8

8.5. 8

8.6. 5

8.7. 3

8.8. 10

9.1. $\pm\frac{8}{7}$

9.2. 999

9.3. $\frac{25}{81}$

9.4. $\frac{-1+\sqrt{1+4e^4}}{2} \approx 6.91$

10.1. $\log(25) \approx 1.40$

10.2. $\ln(1) = 0$

10.3. $\ln(2) \approx 0.69$

10.4. Undefined.

11.1. $\log(5) \approx 0.70$

11.2. -0.25

11.3. Undefined.

11.4. Undefined.

11.5. $\log\left(\ln\left(\frac{1}{2}\right)+5\right) \approx .63$

11.6. $\ln(\log(2)) \approx -1.20$

11.7. $\log(\ln(x)+5)$

11.8. $\ln(\log(x+5))$

12.1. $f(x) = \log(x)$, $g(x) = 3x^2$

12.2. $f(x) = -2\ln(x)$, $g(x) = 5-x$

12.3. $f(x) = \log_3(x)$, $g(x) = \sqrt[3]{x}$

12.4. $f(x) = \log_5(x) + 7^x$, $g(x) = x^2$

13.1. 2

13.2. 0

13.3. 1

13.4. Undefined.

13.5. $\log(5)+\ln(5) \approx 2.31$

13.6. $\log(\pi)-\ln(\pi) \approx -0.65$

13.7. $\log(e) \approx .43$

13.8. $\frac{\log\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{2}\right)} \approx 0.43$

Solutions for problems in Section 6.1

1.1. $a = 1$; the parabola opens upward.

1.2. $a = -5$; the parabola opens downward.

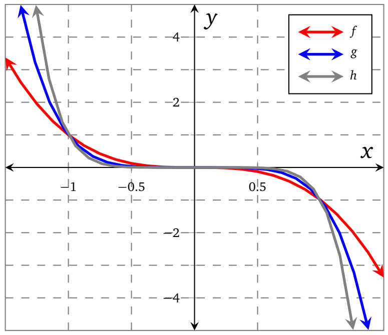
1.3. $a = 4$; the parabola opens upward.

1.4. $m = -19$; the parabola opens downward.

1.5. When $a_2 > 0$, the parabola that represents the function opens upward.

1.6. When $a_2 < 0$, the parabola that represents the function opens downward.

2.1. The functions f, g, and h have domain $(-\infty, \infty)$ and are graphed below.



Note that

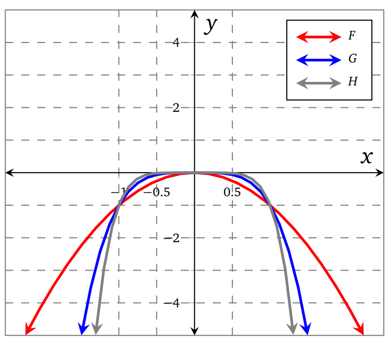
$$f(x) \to -\infty \text{ as } x \to \infty$$
$$\text{and } f(x) \to \infty \text{ as } x \to -\infty$$

The same is true for g and h. The range of f, g, and h is $(-\infty, \infty)$.

Each of the functions f, g, and h are odd

$$\begin{aligned} f(-x) &= -(-x)^3 \\ &= x^3 \\ &= -f(x) \end{aligned} \qquad \begin{aligned} g(-x) &= -(-x)^5 \\ &= x^5 \\ &= -g(x) \end{aligned} \qquad \begin{aligned} h(-x) &= -(-x)^7 \\ &= x^7 \\ &= -h(x) \end{aligned}$$

2.2. The functions F, G, and H have domain $(-\infty, \infty)$ and are graphed below.



Note that

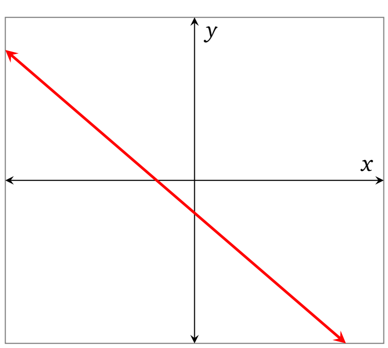
$$F(x) \to -\infty \text{ as } x \to \infty$$
$$\text{and } F(x) \to -\infty \text{ as } x \to -\infty$$

The same is true for G and H. The range of F, G, and H is $(-\infty, 0]$.

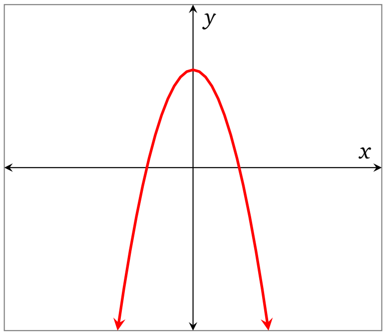
Each of the functions F, G, and H are even

$$\begin{aligned} F(-x) &= -(-x)^2 \\ &= -x^2 \\ &= F(x) \end{aligned} \qquad \begin{aligned} G(-x) &= -(-x)^4 \\ &= -x^4 \\ &= G(x) \end{aligned} \qquad \begin{aligned} H(-x) &= -(-x)^6 \\ &= -x^6 \\ &= H(x) \end{aligned}$$

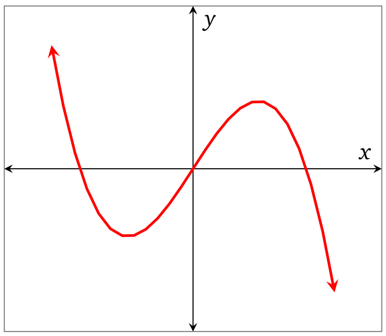
3. $a_1 < 0$:



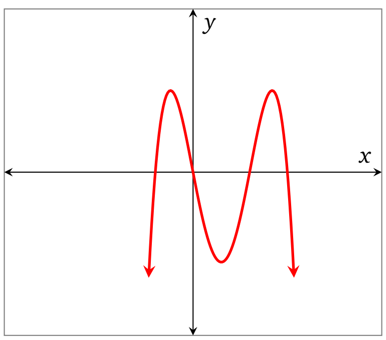
$a_2 < 0$



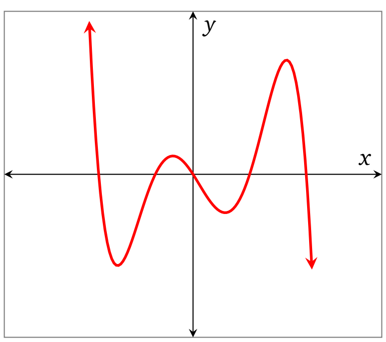
$a_3 < 0$



$a_4 < 0$



$a_5 < 0$



4. Figure 5.12a on page 122:

(a) the curve turns round once;

(b) the degree could be 2;

(c) based on the zeros, the linear factors are $(x + 5)$ and $(x - 3)$; since the graph opens downwards, we will assume the leading coefficient is negative: $p(x) = -k(x + 5)(x - 3)$;

(d) p goes through $(2, 2)$, so we need to solve $2 = -k(7)(-1)$ and therefore $k = {}^2/_7$, so

$$p(x) = -\frac{2}{7}(x+5)(x-3)$$

Figure 5.12b on page 122:

(a) the curve turns around twice;

(b) the degree could be 3;

(c) based on the zeros, the linear factors are $(x+2)^2$, and $(x-1)$; based on the behavior of p, we assume that the leading coefficient is positive, and try $p(x) = k(x+2)^2(x-1)$;

(d) p goes through $(0,-2)$, so we need to solve $-2 = k(4)(-1)$ and therefore $k = 1/2$, so

$$p(x) = \frac{1}{2}(x+2)^2(x-1)$$

Figure 5.12c on page 122:

(a) the curve turns around 4 times;

(b) the degree could be 5;

(c) based on the zeros, the linear factors are $(x+5)^2$, $(x+1)$, $(x-2)$, $(x-3)$; based on the behavior of p, we assume that the leading coefficient is positive, and try $p(x) = k(x+5)^2(x+1)(x-2)(x-3)$;

(d) p goes through $(-3,-50)$, so we need to solve $-50 = k(64)(-2)(-5)(-6)$ and therefore $k = 5/384$, so

$$p(x) = \frac{5}{384}(x+5)^2(x+1)(x-2)(x-3)$$

5.1. f is linear.

5.2. g is linear

5.3. h is quadratic.

5.4. k is linear.

5.5. l is quadratic

5.6. m is linear.

6.1. $m = 4$; α is increasing.

6.2. $m = -9$; β is decreasing.

6.3. $m = 18$; γ is increasing.

6.4. $m = -1$; δ is decreasing.

6.5. When $m > 0$, the function f is … *increasing*.

6.6. When $m < 0$, the function f is … *decreasing*.

7.1. p is a polynomial (you might also describe p as linear). The degree of p is 1.

7.2. p is a polynomial (you might also describe p as quadratic). The degree of p is 2.

7.3. p is not a polynomial; we require the powers of x to be integer values.

7.4. p is not a polynomial; the 2^x term is exponential.

7.5. p is a polynomial, and the degree of p is 6.

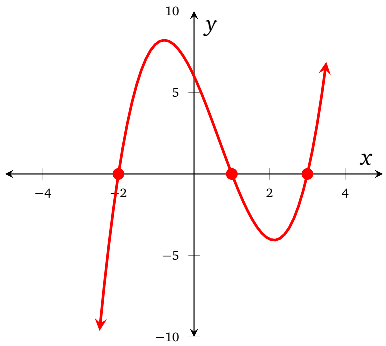
7.6. p is a polynomial, and the degree of p is 17.

7.7. p is a polynomial, and the degree of p is 6.

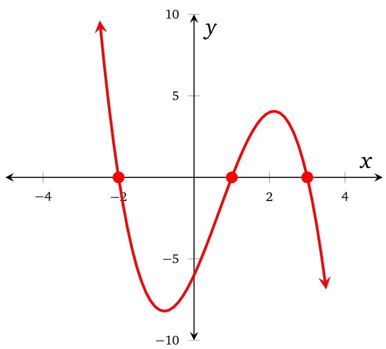
7.8. p is not a polynomial because -5 is not a positive integer.

7.9. p is a polynomial, and the degree of p is 11.

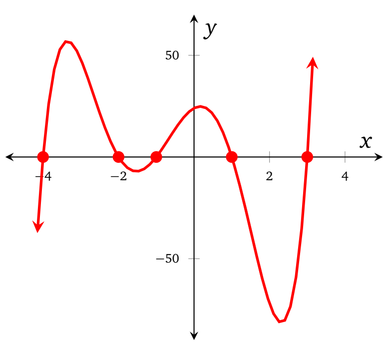
8.1. $y = p(x)$ is shown below.



$y = m(x)$ is shown below.



$y = n(x)$ is shown below.



The zeros of p are -2, 1, and 3; the zeros of m are -2, 1, and 3; the zeros of n are -4, -2, -1, and 3.

8.2.
- The degree of p is 3, and the curve $y = p(x)$ turns around twice.
- The degree of q is also 3, and the curve $y = q(x)$ turns around twice.
- The degree of n is 5, and the curve $y = n(x)$ turns around 4 times.

9.1. $(-4,0)$, $(-2,0)$, $(-1,0)$, $(1,0)$, $(3,0)$

9.2. $(-2,0)$, $(1,0)$, $(3,0)$

9.3. $(-2,0)$, $(1,0)$, $(3,0)$

9.4. $(-2,0)$, $(2,0)$

10.1.
- p is graphed in Figure 5.14c on page 124;
- q is graphed in Figure 5.14b on page 124;
- r is graphed in Figure 5.14a on page 124;

- s is graphed in Figure 5.14d on page 124.

10.2.
- p has simple zeros at about -3.8, 0, and 5.
- q has simple zeros at about -5.9, -1, 1, and 4.
- r has simple zeros at about -5, -2.9, and 4.1.
- s has simple zeros at about -9, -6, 4.2, 8.1, and a zero of multiplicity 2 at 0.

10.3.
- p has a local maximum of approximately 3.9 at -2, and a local minimum of approximately -6.5 at 3.
- q has a local minimum of approximately -10 at -4, and -4 at 3; q has a local maximum of approximately 1 at 0.
- r has a local minimum of approximately -5.5 at -4, and a local maximum of approximately 10 at 3.
- s has a local maximum of approximately 5 at -8, 0 at 0, and 5 at 7; s has local minimums of approximately -3 at -4, and -1 at 3.

10.4.
- p does not have a global maximum, nor a global minimum.
- q has a global minimum of approximately -10; it does not have a global maximum.
- r does not have a global maximum, nor a global minimum.
- s has a global maximum of approximately 5; it does not have a global minimum.

10.5.
- p is increasing on $(-\infty,-2)\cup(3,\infty)$, and decreasing on $(-2,3)$.
- q is increasing on $(-4,0)\cup(3,\infty)$, and decreasing on $(-\infty,-4)\cup(0,3)$.
- r is increasing on $(-4,3)$, and decreasing on $(-\infty,-4)\cup(3,\infty)$.
- s is increasing on $(-\infty,-8)\cup(-4,0)\cup(3,5)$, and decreasing on $(-8,-4)\cup(0,3)\cup(5,\infty)$.

10.6.
- p is concave up on $(1,\infty)$, and concave down on $(-\infty,1)$.
- q is concave up on $(-\infty,-1)\cup(1,\infty)$, and concave down on $(-1,1)$.
- r is concave up on $(-\infty,-3)\cup(0,2)$, and concave down on $(-3,0)\cup(2,\infty)$.
- s is concave up on $(-6,-2)\cup(2,5)$, and concave down on $(-\infty,-6)\cup(-2,2)\cup(5,\infty)$.

10.7. Figure 5.14b on page 124 shows that q has 3 real zeros since the curve of q cuts the horizontal axis 3 times. Since q has degree 5, q must have 2 complex zeros.

11. $\lim_{x\to-\infty} p(x)=-\infty$, $\lim_{x\to\infty} p(x)=\infty$, $\lim_{x\to-\infty} q(x)=\infty$, $\lim_{x\to\infty} q(x)=-\infty$, $\lim_{x\to-\infty} r(x)=-\infty$, $\lim_{x\to\infty} r(x)=\infty$, $\lim_{x\to-\infty} s(x)=\infty$, $\lim_{x\to\infty} s(x)=\infty$,

12.1. False. Consider $p(x)=x^2(x+1)$ which has only 2 distinct zeros.

12.2. False. Consider $p(x)=-x^4$.

12.3. True.

12.4. False. All odd degree polynomials will cut the horizontal axis at least once.

13.1. Possible option: $p(x)=(x-4)(x-5)$. Note we could multiply p by any real number, and still meet the requirements.

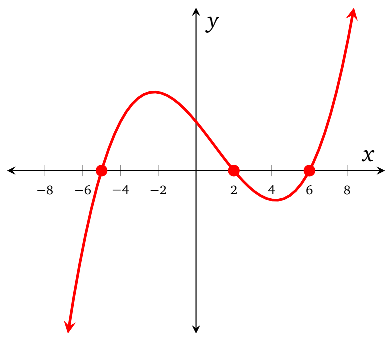
13.2. Possible option: $p(x)=(x-4)(x-5)(x+3)$. Note we could multiply p by any real number, and still meet the requirements.

13.3. Possible option: $p(x)=x(x-4)(x-5)(x+3)$. Note we could multiply p by any real number, and still meet the requirements.

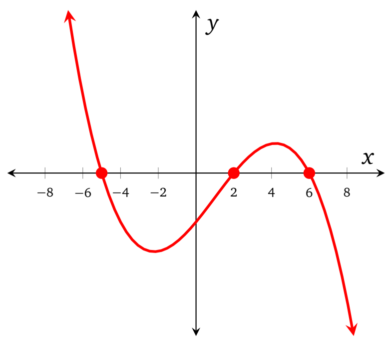
13.4. Possible option: $p(x) = (x-2)(x+5)(x+2)^2$. Note we could multiply p by any real number, and still meet the requirements.

13.5. Possible option: $p(x) = (x+1)^3$. Note we could multiply p by any real number, and still meet the requirements.

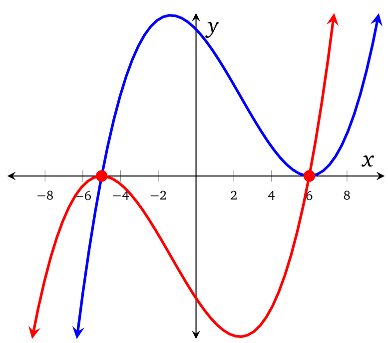
14.1. Assuming that $a_3 > 0$:



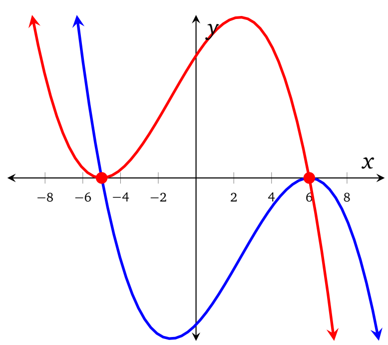
14.2. Assuming that $a_3 < 0$:



14.3. Assuming that $a_4 > 0$ there are 2 different options:



14.4. Assuming that $a_4 < 0$ there are 2 different options:



15.1. $\pm i$ (simple).

15.2. ± 3, $\pm\sqrt{7}$ (all are simple).

15.3. 0 (multiplicity 3), $\pm\sqrt{3}$ (simple), $\pm\sqrt{8}$ (simple).

15.4. ± 3, $\pm 3i$ (all are simple).

15.5. 2, $-1 \pm i\sqrt{3}$ (all are simple).

15.6. 0 (multiplicity 2), 1 (simple).

15.7. -1, $\pm 2i$ (all are simple).

15.8. -4 (simple), 4 (multiplicity 2), 1 (simple).

15.9. ± 5, $\dfrac{5 \pm \sqrt{41}}{2}$ (all are simple).

16.1. $p(x) = (x-1)(x-2)$

16.2. $p(x) = x(x-5)(x-13)$

16.3. $p(x) = (x+7)(x-2)^3(x-5)$

16.4. $p(x) = x(x^2+1)$

16.5. $p(x) = (x^2+4)(x^2-49)$

17.1. 160

17.2. -9997

17.3. 84

17.4. 1980

18.1. -255

18.2. 4

18.3. 64

18.4. -40

18.5. 64

19.1. 14

19.2. 7

19.3. 0

19.4. $\frac{3}{4}$

19.5. $(-\infty, -1) \cup (-1, 0) \cup (0, 3) \cup (3, \infty)$

20.1. $p(x-5) = 4(x-5)(x-2)(x^2-10x+24)$

20.2. $p(x+6) = 4(x+6)(x+9)(x^2+12x+35)$

20.3. $p(x)+12 = 4x(x^2-1)(x+3)+12$

20.4. $p(x)-2 = 4x(x^2-1)(x+3)-2$

20.5. $-p(x) = -4x(x^2-1)(x+3)$

20.6. $p(-x) = -4x(x^2-1)(3-x)$

21.1. p has 3 zeros.

21.2. p is degree 3.

21.3. $p(x) = x(x+2)(x-3)$

21.4. q has 2 zeros.

21.5. p changes sign at -2, and q does not change sign at -2.

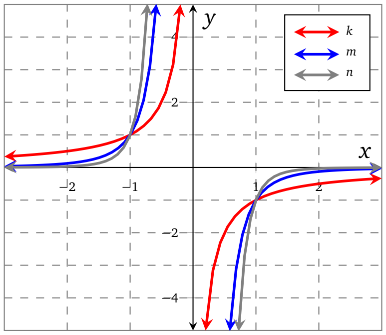
21.6. $q(x) = x(x+2)^2$

21.7. $r(x) = (x+3)(x+1)(x-1)(x-3)$

21.8. $s(x) = (x+3)(x+1)(x-1)^2$

Solutions for problems in Section 6.2

1.1. The functions k, m, and n have domain $(-\infty, 0) \cup (0, \infty)$, and range $(-\infty, 0) \cup (0, \infty)$; the functions are graphed below.



Note that

$$\begin{aligned} k(x) &\to 0 \text{ as } x \to \infty \\ \text{and } k(x) &\to 0 \text{ as } x \to -\infty \end{aligned}$$

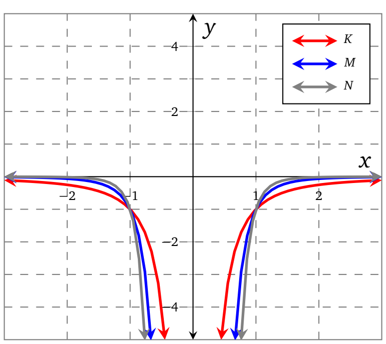
and also

$$\begin{aligned} k(x) &\to \infty \text{ as } x \to 0^- \\ \text{and } k(x) &\to -\infty \text{ as } x \to 0^+ \end{aligned}$$

The same are true for m and n. Note that each function is odd:

$$\begin{aligned} k(-x) &= -\frac{1}{(-x)^3} \\ &= -\frac{1}{-x^3} \\ &= \frac{1}{x^3} \\ &= -k(x) \end{aligned} \qquad \begin{aligned} m(-x) &= -\frac{1}{(-x)^5} \\ &= -\frac{1}{-x^5} \\ &= \frac{1}{x^5} \\ &= -m(x) \end{aligned} \qquad \begin{aligned} n(-x) &= -\frac{1}{(-x)^7} \\ &= -\frac{1}{-x^7} \\ &= \frac{1}{x^7} \\ &= -n(x) \end{aligned}$$

1.2. The functions K, M, and N have domain $(-\infty, 0) \cup (0, \infty)$; the range of each function is $(-\infty, 0)$. The functions are are graphed below.



Note that

$$\begin{aligned} K(x) &\to 0 \text{ as } x \to \infty \\ \text{and } K(x) &\to 0 \text{ as } x \to -\infty \end{aligned}$$

and also

$$\begin{aligned} K(x) &\to -\infty \text{ as } x \to 0^- \\ \text{and } K(x) &\to -\infty \text{ as } x \to 0^+ \end{aligned}$$

The same are true for M and N. Note that each function is even:

$$K(-x) = -\frac{1}{(-x)^2} = -\frac{1}{x^2} = K(x) \qquad M(-x) = -\frac{1}{(-x)^4} = -\frac{1}{x^4} = M(x) \qquad N(-x) = -\frac{1}{(-x)^6} = -\frac{1}{x^6} = N(x)$$

2.1.

x	y
1	100
2	50
3	33.33
4	25
5	20
6	16.67
7	14.29
8	12.50
9	11.11
10	10

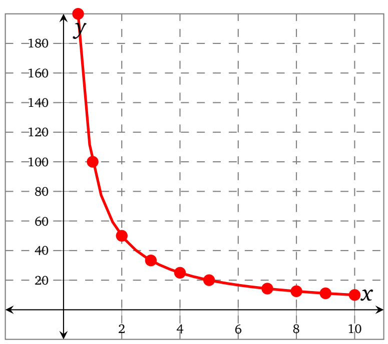
2.2. It seems that the number of mg that it takes to break the spaghetti decreases as x increases.

2.3.

x	y
0.0001	1 000 000
0.001	100 000
0.01	10 000
0.1	1000
0.5	200
1	100

2.4. The number of mg required to break the spaghetti increases as $x \to 0$. We can not allow x to be 0, as we can not divide by 0, and we can not be 0 inches from the edge of the table.

2.5. The graph of $y = \frac{100}{x}$ is shown below.



2.6. As x increases, $y \to 0$. If we could construct a piece of spaghetti 101 in long, it would only take 1 mg to break it $\left(\frac{100}{100} = 1\right)$. Of course, the weight of spaghetti would probably cause it to break without the weight.

3.1. Paying off the debt in 2 years, we use

$$M = \frac{2000 \cdot 0.015}{1 - (1 + 0.015)^{-24}} \approx 99.85$$

The monthly payments are \$99.85.

Paying off the debt in 1 year, we use

$$\begin{aligned}M &= \frac{2000 \cdot 0.015}{1-(1+0.015)^{-12}} \\ &\approx 183.36\end{aligned}$$

The monthly payments are $183.36

In the 2-year model we would pay a total of $\$99.85 \cdot 12 = \$2396.40$. In the 1-year model we would pay a total of $\$183.36 \cdot 12 = \$2200.32$. We would therefore save $196.08 if we went with the 1-year model instead of the 2-year model.

3.2. For the 20-year loan we use

$$\begin{aligned}M &= \frac{300000 \cdot \frac{0.052}{12}}{1-\left(1+\frac{0.052}{12}\right)^{-12\cdot 20}} \\ &\approx 2013.16\end{aligned}$$

The monthly payments are $2013.16.

For the 30-year loan we use

$$\begin{aligned}M &= \frac{300000 \cdot \frac{0.052}{12}}{1-\left(1+\frac{0.052}{12}\right)^{-12\cdot 30}} \\ &\approx 1647.33\end{aligned}$$

The monthly payments are $1647.33.

The total amount paid during the 20-year loan is $\$2013.16 \cdot 12 \cdot 20 = \$483,158.40$. The total amount paid during the 30-year loan is $\$1647.33 \cdot 12 \cdot 30 = \$593,038.80$.

Recommendation: if you can afford the payments, choose the 20-year loan.

3.3. We are given $M = 100$, $P = 3000$, $i = 0.01$, and we need to find n in the equation

$$100 = \frac{3000 \cdot 0.01}{1-(1+0.01)^{-n}}$$

Using logarithms, we find that $n \approx 36$. It will take Ellen about 3 years to pay off the debt.

3.4. **Option 1:** 4% annual interest for 5 years on $14,000. This means that the monthly payments will be calculated using

$$\begin{aligned}M &= \frac{14000 \cdot \frac{0.04}{12}}{1-\left(1+\frac{0.04}{12}\right)^{-12\cdot 5}} \\ &\approx 257.83\end{aligned}$$

The monthly payments will be $257.83. The total amount paid will be $\$257.83 \cdot 5 \cdot 12 = \$15,469.80$, of which $1469.80 is interest.

Option 2: 8% annual interest for 5 years on $12,000. This means that the monthly payments will be calculated using

$$\begin{aligned}M &= \frac{12000 \cdot \frac{0.08}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-12\cdot 5}} \\ &\approx 243.32\end{aligned}$$

The monthly payments will be $243.32. The total amount paid will be $\$243.32 \cdot 5 \cdot 12 = \$14,599.20$, of which $2599.2 is interest.

Jake should choose option 1 to minimize the amount of interest he has to pay.

4.1. r is rational; the domain of r is $(-\infty,0)\cup(0,\infty)$.

4.2. s is not rational (s is linear).

4.3. t is rational; the domain of t is $\left(-\infty,\frac{7}{8}\right)\cup\left(\frac{7}{8},\infty\right)$.

4.4. u is rational; the domain of w is $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$.

4.5. v is rational; the domain of v is $(-\infty,2)\cup(2,\infty)$.

4.6. w is rational; the domain of w is $(-\infty,-17)\cup(-17,\infty)$.

4.7. a is not rational (a is quadratic, or a polynomial of degree 2).

4.8. b is not rational (b is exponential).

4.9. c is rational; the domain of c is $(-\infty,0)\cup(0,\infty)$.

4.10. d is not rational (d is a polynomial).

4.11. e is rational; the domain of e is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

4.12. f is not rational (f is constant).

5.1. $$\begin{aligned} r(0) &= \frac{(0-2)(0+3)}{(0+5)(0-7)} \\ &= \frac{-6}{-35} \\ &= \frac{6}{35} \end{aligned}$$

5.2. $$\begin{aligned} r(1) &= \frac{(1-2)(1+3)}{(1+5)(1-7)} \\ &= \frac{-4}{-36} \\ &= \frac{1}{9} \end{aligned}$$

5.3. $$\begin{aligned} r(2) &= \frac{(2-2)(2+3)}{(2+5)(2-7)} \\ &= \frac{0}{-50} \\ &= 0 \end{aligned}$$

5.4. $$\begin{aligned} r(4) &= \frac{(4-2)(4+3)}{(4+5)(4-7)} \\ &= \frac{14}{-27} \\ &= -\frac{14}{27} \end{aligned}$$

5.5. $$\begin{aligned} r(7) &= \frac{(7-2)(7+3)}{(7+5)(7-7)} \\ &= \frac{50}{0} \end{aligned}$$

$r(7)$ is undefined.

5.6. $$\begin{aligned} r(-3) &= \frac{(-3-2)(-3+3)}{(-3+5)(-3-7)} \\ &= \frac{0}{-20} \\ &= 0 \end{aligned}$$

5.7. $$r(-5) = \frac{(-5-2)(-5+3)}{(-5+5)(-5-7)} = \frac{14}{0}$$

$r(-5)$ is undefined.

5.8. $$r\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}-2\right)\left(\frac{1}{2}+3\right)}{\left(\frac{1}{2}+5\right)\left(\frac{1}{2}-7\right)} = \frac{-\frac{3}{2}\cdot\frac{7}{2}}{\frac{11}{2}\left(-\frac{13}{2}\right)} = \frac{-\frac{21}{4}}{-\frac{143}{4}} = \frac{37}{143}$$

6.1. f has a vertical asymptote at 2; the domain of f is $(-\infty,2)\cup(2,\infty)$.

6.2. g has a vertical asymptote at 2, and a hole at -1; the domain of g is $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$.

6.3. h has a vertical asymptote at 3, and a whole at -4; the domain of h is $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$.

6.4. k has a vertical asymptote at $\frac{3}{2}$; the domain of k is $\left(-\infty,\frac{3}{2}\right)\cup\left(\frac{3}{2},\infty\right)$.

6.5. l does not have any vertical asymptotes nor holes; the domain of w is $(-\infty,\infty)$.

6.6. m has vertical asymptotes at $\pm\sqrt{13}$; the domain of m is $(-\infty,\sqrt{13})\cup(-\sqrt{13},\sqrt{13})\cup(\sqrt{13},\infty)$.

7.
- Figure 5.27a on page 137: possible formula is $r(x) = \dfrac{1}{x+5}$
- Figure 5.27b on page 137: possible formula is $r(x) = \dfrac{(x+3)}{(x-5)}$
- Figure 5.27c on page 137: possible formula is $r(x) = \dfrac{1}{(x-4)(x+3)}$.

8.1. Possible option: $r(x) = \dfrac{1}{x-2}$. Note that we could multiply the numerator or denominator by any real number and still have the desired properties.

8.2. Possible option: $r(x) = \dfrac{1}{x-5}$. Note that we could multiply the numerator or denominator by any real number and still have the desired properties.

8.3. Possible option: $r(x) = \dfrac{x-6}{x+2}$. Note that we could multiply the numerator or denominator by any real number and still have the desired properties.

8.4. Possible option: $r(x) = \dfrac{(x-2)(x+5)}{(x-1)(x+7)}$. Note that we could multiply the numerator or denominator by any real number and still have the desired properties.

9.1. $y = 0$

9.2. $y = 2$

9.3. $y = 1$

9.4. $y = 1$

9.5. $y = \dfrac{3}{5}$

9.6. $y = 0$

9.7. $y = \dfrac{6}{11}$

9.8. $y = 0$

9.9. $y = -2$

10.1. Possible option: $f(x) = \dfrac{7(x-2)}{x+1}$. Note that there are other options, provided that the degree of the numerator is the same as the degree of the denominator, and that the ratio of the leading coefficients is 7.

10.2. Possible option: $f(x) = \dfrac{5-x^2}{x^2+10}$. Note that there are other options, provided that the degree of the numerator is the same as the degree of the denominator, and that the ratio of the leading coefficients is 10.

10.3. Possible option: $f(x) = \dfrac{53x^3}{x^3+4x^2-7}$. Note that there are other options, provided that the degree of the numerator is the same as the degree of the denominator, and that the ratio of the leading coefficients is 53.

10.4. Possible option: $f(x) = \dfrac{34(x+2)}{7-2x}$. Note that there are other options, provided that the degree of the numerator is the same as the degree of the denominator, and that the ratio of the leading coefficients is -17.

10.5. Possible option: $f(x) = \dfrac{3x+4}{2(x+1)}$. Note that there are other options, provided that the degree of the numerator is the same as the degree of the denominator, and that the ratio of the leading coefficients is $\dfrac{3}{2}$.

10.6. Possible option: $f(x) = \dfrac{4}{x}$. Note that there are other options, provided that the degree of the numerator is less than the degree of the denominator.

10.7. Possible option: $f(x) = \dfrac{10x}{5-10x}$. Note that there are other options, provided that the degree of the numerator is the same as the degree of the denominator, and that the ratio of the leading coefficients is -1.

10.8. Possible option: $f(x) = \dfrac{8x-3}{4x+1}$. Note that there are other options, provided that the degree of the numerator is the same as the degree of the denominator, and that the ratio of the leading coefficients is 2.

11.1. Possible option: $f(x) = \dfrac{3(x-2)}{x+7}$. Note that the zero and asymptote of f could be changed, and f would still have the desired properties.

11.2. Possible option: $r(x) = \dfrac{-4(x-2)}{x+7}$. Note that the zero and asymptote of r could be changed, and r would still have the desired properties.

11.3. Possible option: $k(x) = \dfrac{2x^2}{(x+3)(x-5)}$. Note that the denominator must have the given factors; the numerator could be any degree 2 polynomial, provided the leading coefficient is 2.

12.1. $\left(0, \frac{1}{6}\right)$

12.2. $-3, 4$

12.3. Interval notation: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$. Set builder: $\{x \mid x \neq -3, \text{and } x \neq 4\}$

12.4. $x = -3$ and $x = 4$

12.5. $-2, 1$

12.6. $(-2, 0)$ and $(1, 0)$

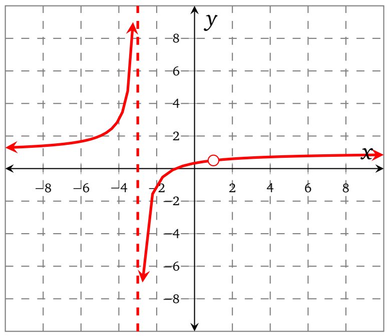
13.1. The domain of r is $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$.

13.2. $(x^2 - 1) = (x - 1)(x + 1)$

13.3. $r(x) = \dfrac{x+1}{x+3}$ provided that $x \neq -1$.

13.4. The function r has a vertical asymptote at -3, and a hole at 1.

13.5. A graph of r is shown below.



14.1. $\frac{197}{8}$

14.2. $\frac{53}{6}$

14.3. Undefined.

14.4. $-\frac{3}{4}$

15.1. $r(x-3) = \frac{x+2}{2x-9}$

15.2. $r(x+4) = \frac{x+9}{2x+5}$

15.3. $r(x) + \pi = \frac{x+5}{2x-3} + \pi$

15.4. $r(x) - 17 = \frac{x+5}{2x-3} - 17$

15.5. $-r(x) = -\frac{x+5}{2x-3}$

15.6. $r(-x) = \frac{x-5}{2x+3}$

16.1. $A = 3$ and $B = -2$, so $r(x) = \dfrac{x-3}{x+2}$.

16.2.
$$\begin{aligned} r(-4) &= \frac{-4-3}{-4+2} \\ &= \frac{7}{2} \end{aligned}$$
$r(-3) = \ldots$ etc

16.3. $s(x) = \dfrac{x+2}{(x-3)(x+1)}$

16.4.
$$\begin{aligned} s(-4) &= \frac{-4+2}{(-4-3)(-4+1)} \\ &= -\frac{2}{21} \end{aligned}$$
$s(-3) = \ldots$ etc

16.5. $t(x) = \dfrac{(x+3)(x-2)}{(x+2)(x+1)}$

16.6. $u(x) = \dfrac{x^2}{(x+3)(x-3)}$

Solutions for problems in Section 6.3

1. Figure 5.35a on page 146

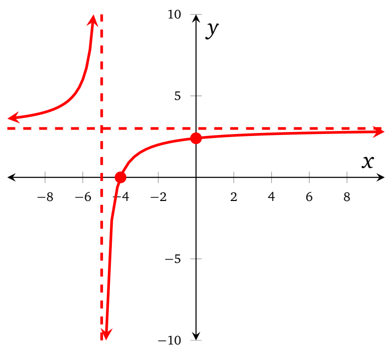


Figure 5.35b on page 146

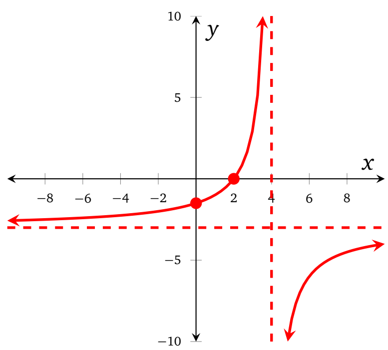
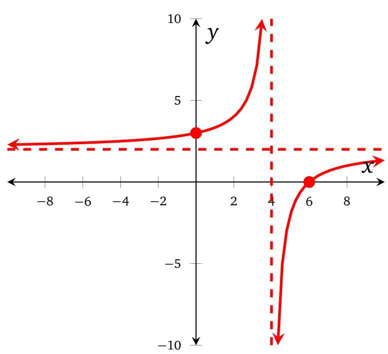


Figure 5.35c on page 146



2. Figure 5.36a on page 147

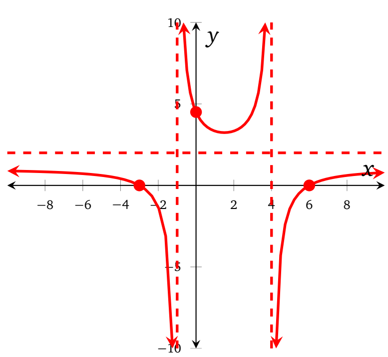


Figure 5.36b on page 147

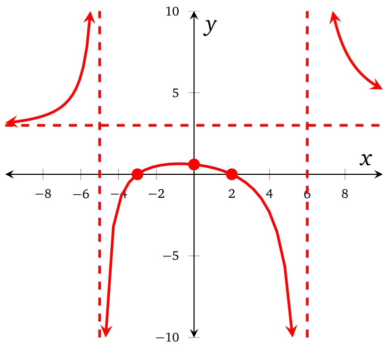
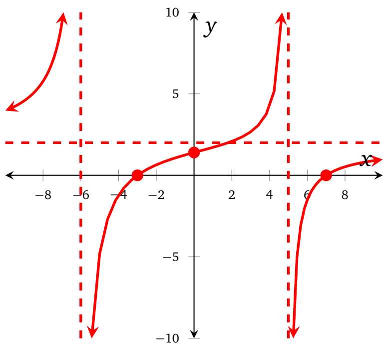
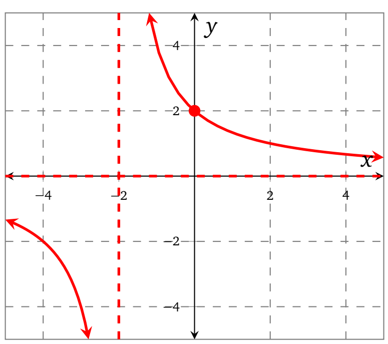


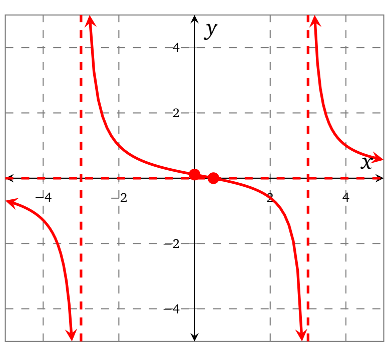
Figure 5.36c on page 147



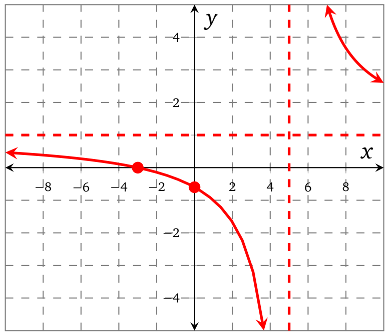
3.1. Vertical intercept: $(0,2)$; vertical asymptote: $x=-2$, horizontal asymptote: $y=0$.



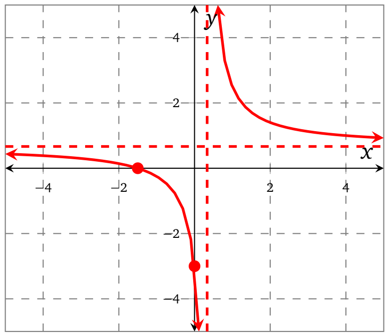
3.2. Vertical intercept: $\left(0,\frac{1}{9}\right)$; horizontal intercept: $\left(\frac{1}{2},0\right)$; vertical asymptotes: $x=-3$, $x=3$, horizontal asymptote: $y=0$.



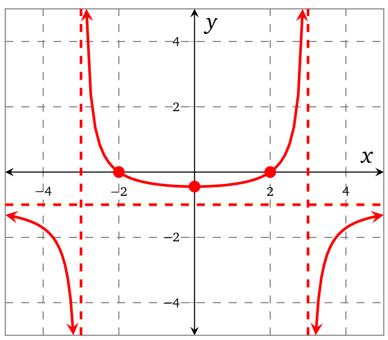
3.3. Vertical intercept $\left(0,-\frac{3}{5}\right)$; horizontal intercept: $(-3,0)$; vertical asymptote: $x=5$; horizontal asymptote: $y=1$.



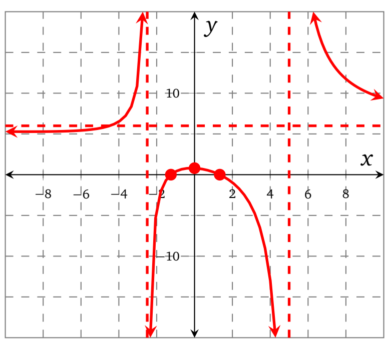
3.4. Vertical intercept: $(0,-3)$; horizontal intercept: $\left(-\frac{3}{2},0\right)$; vertical asymptote: $x=\frac{1}{3}$, horizontal asymptote: $y=\frac{2}{3}$.



3.5. Vertical intercept: $\left(0,-\frac{4}{9}\right)$; horizontal intercepts: $(2,0)$, $(-2,0)$; vertical asymptotes: $x=-3$, $x=3$; horizontal asymptote: $y=-1$.



3.6. Vertical intercept: $\left(0,\frac{4}{5}\right)$; horizontal intercepts: $\left(-\frac{5}{4},0\right)$, $\left(\frac{4}{3},0\right)$; vertical asymptotes: $x=-\frac{5}{2}$, $x=5$; horizontal asymptote: $y=6$.



4.1.
- The domain of F is $(-\infty,3)\cup(3,\infty)$.
- The domain of G is $(-\infty,-3)\cup(-3,\infty)$.

4.2.
- $F^{-1}(x)=\frac{3x+1}{x-2}$; the domain of F^{-1} is $(-\infty,2)\cup(2,\infty)$.
- $G^{-1}(x)=\frac{3x+1}{x+4}$; the domain of G^{-1} is $(-\infty,-4)\cup(-4,\infty)$.

4.3.
- The range of F is the domain of F^{-1}, which is $(-\infty,2)\cup(2,\infty)$.
- The range of G is the domain of G^{-1}, which is $(-\infty,-4)\cup(-4,\infty)$.

4.4.
- The range of F^{-1} is the domain of F, which is $(-\infty,3)\cup(3,\infty)$.
- The range of G^{-1} is the domain of G, which is $(-\infty,-3)\cup(-3,\infty)$.

5.1. $\frac{75}{16}$

5.2. $(s\circ r)(0)$ is undefined.

5.3. $\frac{147}{4}$

5.4. 192

5.5. $(s\circ r)(4)$ is undefined.

5.6. $\dfrac{4x^2-3}{1+5x^2}$

6.1. $-\frac{2}{3}$

6.2. $\frac{3}{5}$

6.3. $\frac{7}{13}$

6.4. $-\frac{1}{5}$

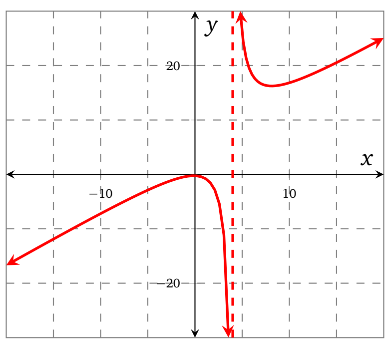
6.5. $(-\infty,10)\cup(10,\infty)$

Solutions for problems in Section 6.4

0.6. (a) $\left(0,-\frac{1}{4}\right)$

(b) Vertical asymptote: $x=4$.

(c) A graph of the function is shown below



0.7. (a) $(0,0)$, $(-3,0)$

(b) Vertical asymptote: $x=5$, horizontal asymptote: none.

(c) A graph of the function is shown below