VECTOR CALCULUS PROBLEMS

1. Let $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$ be two constant vectors, with components

$$\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{ and } \vec{\boldsymbol{m}} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}.$$

Let $\vec{\boldsymbol{v}}(x, y, z)$ be the vector field $\vec{\boldsymbol{v}} = (\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{x}})\vec{\boldsymbol{a}}$.
(*i*) Write $\vec{\boldsymbol{v}}$ in terms of its components:

$$\vec{\boldsymbol{v}} = \begin{pmatrix} \cdots ? \cdots \\ \cdots ? \cdots \\ \cdots ? \cdots \end{pmatrix}.$$

(*ii*) Compute $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$.
(*iii*) Compute $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$.
(*iv*) If $\vec{\boldsymbol{v}}$ is the gradient of some function f, what can you say about the vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$?
(*v*) If $\vec{\boldsymbol{v}}$ is the curl of some vector field $\vec{\boldsymbol{w}}$, what can you say about the vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$?

2. Let $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$ be as in the previous problem. Consider the vector field

$$\vec{\boldsymbol{v}}(x, y, z) = e^{\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{x}}} \vec{\boldsymbol{a}} = e^{m_1 x + m_2 y + m_3 z} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

(*i*) Show by computing the derivatives that $\vec{\boldsymbol{\nabla}}\left(e^{\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{x}}}\right) = e^{\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{x}}} \vec{\boldsymbol{m}}$.
(*ii*) Compute $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$. (Find the shortest way to write the answer.)
(*iii*) Compute $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$. Again, simplify your answer.
(*iv*) Which condition must the vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$ satisfy if $\vec{\boldsymbol{v}}$ is to be "divergence free," i.e. if $\operatorname{div} \vec{\boldsymbol{v}} = 0$?
(*v*) Suppose that $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\nabla}}\phi$ for some function. What do you know about $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$?

3. If $\vec{\boldsymbol{v}} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ is a vector field and f is a function, then what is $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}} f$?

4. ***Product rules.*** Let f be a function of three variables, and let $\vec{\boldsymbol{v}}$ be a three dimensional vector field.
(*i*) $\vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{v}}) = (\vec{\boldsymbol{\nabla}} f) \cdot \vec{\boldsymbol{v}} + f\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$
(*ii*) Guess a product rule for $\vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{v}})$ and prove it.

5. Check the following formulas

$$\vec{\boldsymbol{\nabla}}\rho = \frac{\vec{\boldsymbol{x}}}{\rho}, \text{ and } \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{x}} = 3.$$

by direct computation. Here $\rho = \sqrt{x^2 + y^2 + z^2}$ is the radius from spherical coordinates.

6. Use the product rule from Problem **??** and the formulas from problem **??** to compute the following quantities

(*i*) $\vec{\boldsymbol{\nabla}} \cdot (\rho^2 \vec{\boldsymbol{x}})$ (*ii*) $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\nabla}}\rho$ (*iii*) $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\nabla}}\left(\|\vec{\boldsymbol{x}}\|\right)$ (*iv*) $\operatorname{div} \dfrac{\vec{\boldsymbol{x}}}{\|\vec{\boldsymbol{x}}\|^3}$.

7. In this problem, as in all the problems in this section, $\rho = \sqrt{x^2+y^2+z^2} = \|\vec{\boldsymbol{x}}\|$ is the radius in spherical coordinates.
(***i***) Show that $\vec{\boldsymbol{x}} = \frac{1}{2}\vec{\boldsymbol{\nabla}}(\rho^2)$.
(***ii***) Compute $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{x}}$ without doing any derivatives.
(***iii***) Compute $\vec{\boldsymbol{\nabla}} \times (\rho\vec{\boldsymbol{x}})$ using the product rule from problem **??**.

8. Compute $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$ for the vector field $\vec{\boldsymbol{v}}(x,y,z) = \vec{\boldsymbol{k}} \times \vec{\boldsymbol{x}}$.

9. Consider the vector field

$$\vec{\boldsymbol{v}}(x,y,z) = \rho^n\vec{\boldsymbol{x}},$$

where n is a constant. (Both Newton's law of gravitation and Coulomb's law have this vector field with $n=-3$.)
(***i***) Write $\vec{\boldsymbol{v}}(x,y,z)$ in the form $\left(\begin{smallmatrix}\cdots\\ \cdots\\ \cdots\end{smallmatrix}\right)$, using only Cartesian coordinates x,y,z.
(***ii***) Compute $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$. (Use one of the product rules from Problem **??**; you already computed the derivatives of ρ in problem **??**.)
(***iii***) For which value(s) of n does one have $\operatorname{div}\vec{\boldsymbol{v}} = 0$?

10. A function of three variables is called ***radially symmetric*** if it only depends on the radius $\rho = \sqrt{x^2+y^2+z^2}$, i.e. if it can be written as $F(\rho)$ for some function F of one variable. E.g. $f(x,y,z) = \rho^{-2}$, or $g(x,y,z) = e^{-\rho}$ are radially symmetric functions.
(***i***) Find the gradient of a radially symmetric function $F(\rho)$.
(***ii***) Let $\vec{\boldsymbol{v}} = \rho^n\vec{\boldsymbol{x}}$, as in problem **??**. Does there exist a function $f(x,y,z)$ such that $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\nabla}} f$? (Hint: try a radially symmetric function, and use problem **??**.)

(2i) $\left(e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}\right)_{x_1} = m_1 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}$, and the same for the x_2 and x_3 derivatives. Therefore

$$\vec{\boldsymbol{\nabla}}\left(e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}\right) = \begin{pmatrix} m_1 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \\ m_2 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \\ m_3 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \end{pmatrix} = e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \left(\begin{smallmatrix} m_1 \\ m_2 \\ m_3 \end{smallmatrix}\right).$$

(2ii) After simplifying you get $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = \vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{a}} e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}$.

(2iii) $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}} = \vec{\boldsymbol{m}} \times \vec{\boldsymbol{a}} e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}$.

(2iv) $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$ must be perpendicular.

(2v) If $\vec{\boldsymbol{v}}$ is the gradient of some function, then its curl must vanish. Therefore $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{m}} = \vec{\boldsymbol{0}}$ in view of part **??** of this problem. The conclusion is that $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$ must be parallel.

(3) $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}} f = Pf_x + Qf_y + Rf_z$.

(4i) By definition,

$$\vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{v}}) = \vec{\boldsymbol{\nabla}} \cdot \begin{pmatrix} fP \\ fQ \\ fR \end{pmatrix} = \frac{\partial fP}{\partial x} + \frac{\partial fQ}{\partial y} + \frac{\partial fR}{\partial z}$$
$$= f_x P + f P_x + f_y Q + f Q_y + f_z R + f R_z$$
$$= f_x P + f_y Q + f_z R + f\big(P_x + Q_y + R_z\big)$$
$$= \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + f\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$$
$$= \vec{\boldsymbol{\nabla}} f \cdot \vec{\boldsymbol{v}} + f\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}},$$

as claimed.

(4ii) $\vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{v}}) = (\vec{\boldsymbol{\nabla}} f) \times \vec{\boldsymbol{v}} + f\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$ is the rule. The derivation goes along the same lines as in the previous product rule.

(7ii) Since $\vec{\boldsymbol{x}}$ is the gradient of some function its curl must vanish.

(7iii) $\vec{\boldsymbol{\nabla}} \times (\rho\vec{\boldsymbol{x}}) = (\vec{\boldsymbol{\nabla}}\rho) \times \vec{\boldsymbol{x}} + \rho\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$

(8) $\vec{\boldsymbol{v}}(x, y, z) = \left(\begin{smallmatrix} -y \\ x \\ 0 \end{smallmatrix}\right)$ so $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}} = \left(\begin{smallmatrix} 0 \\ 0 \\ 2 \end{smallmatrix}\right) = 2\vec{\boldsymbol{k}}$.

(9i) $\vec{\boldsymbol{v}}(x, y, z) = \begin{pmatrix} x(x^2 + y^2 + z^2)^{n/2} \\ y(x^2 + y^2 + z^2)^{n/2} \\ z(x^2 + y^2 + z^2)^{n/2} \end{pmatrix}$.

(9ii) Using the product rule, you get

$$\vec{\boldsymbol{\nabla}}(\rho^n\vec{\boldsymbol{x}}) = (\vec{\boldsymbol{\nabla}}\rho^n) \cdot \vec{\boldsymbol{x}} + \rho^n\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{x}} = n\rho^{n-1}(\vec{\boldsymbol{\nabla}}\rho) \cdot \vec{\boldsymbol{x}} + \rho^n\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{x}}.$$

Now recall (or compute again):

$$\vec{\boldsymbol{\nabla}}\rho = \frac{\vec{\boldsymbol{x}}}{\rho}, \text{ and } \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{x}} = 3.$$

This leads to

$$\vec{\boldsymbol{\nabla}}(\rho^n\vec{\boldsymbol{x}}) = -n\rho^{n-1}\frac{\vec{\boldsymbol{x}}}{\rho} \cdot \vec{\boldsymbol{x}} + 3\rho^n = -n\rho^{n-2}\|\vec{\boldsymbol{x}}\|^2 + 3\rho^n = (n+3)\rho^{-n})$$

(9iii) $n = -3$.

(10i) There are a long and a short answer. The long(er) computation goes likes this:

$$\vec{\boldsymbol{\nabla}} F(\rho) = \begin{pmatrix} F(\rho)_x \\ F(\rho)_y \\ F(\rho)_z \end{pmatrix} = \begin{pmatrix} F'(\rho)\rho_x \\ F'(\rho)\rho_y \\ F'(\rho)\rho_z \end{pmatrix} = F'(\rho)\begin{pmatrix} \rho_x \\ \rho_y \\ \rho_z \end{pmatrix}.$$

Now recall problem **??**, and you find

$$\vec{\boldsymbol{\nabla}} F(\rho) = F'(\rho)\begin{pmatrix} x/\rho \\ y/\rho \\ z/\rho \end{pmatrix} = \frac{1}{\rho}F'(\rho)\vec{\boldsymbol{x}}.$$

The short computation is essentially the same, but you never write the components of the vectors:

$$\vec{\boldsymbol{\nabla}}F(\rho) = F'(\rho)\vec{\boldsymbol{\nabla}}\rho = \frac{1}{\rho}F'(\rho)\vec{\boldsymbol{x}}.$$

(10ii) If $f(x, y, z) = F(\rho)$, then by the previous problem we have $\vec{\boldsymbol{\nabla}}f = \rho^{-1}F'(\rho)\vec{\boldsymbol{x}}$. We want this to be equal to $\rho^{-n}\vec{\boldsymbol{x}}$, so $F(\rho)$ must satisfy

$$\rho^{-1}F'(rho) = \rho^n \implies F'(\rho) = \rho^{1+n} \implies F(\rho) = \frac{\rho^{2+n}}{2+n} + C$$

for some constant C. We are only asked to find on function f, so we find that the given vector field is indeed the gradient of a radially symmetric function:

$$\vec{\boldsymbol{v}} = \rho^n\vec{\boldsymbol{x}} = \vec{\boldsymbol{\nabla}}\Big(\frac{\rho^{2+n}}{2+n}\Big).$$

The exceptional case is when $n = -2$, in which case you get $F(\rho) = \ln\rho$.