MATH 221 PROJECT

Riemann Sums

1. Purpose of this project

We will attempt to gain an understanding of the definite integral

$$\int_a^b f(x)dx,$$

by computing different types of Riemann sums. We will learn that there is more than one such sum, but that one of them is much more accurate than the others.

2. Approximations

Consider the definite integral

$$\int_0^4 \sqrt{x+1}\,dx.$$

We will use a variety of Riemann sums to approximate this integral. First, we discretize $[0,4]$ into N equally spaced intervals, $[x_{k-1}, x_k]$, with

$$\Delta x = x_k - x_{k-1} = \frac{4}{N}.$$

For example, in the case $N = 3$, we have $\{x_0, x_1, x_2, x_3\} = \{0, \frac{4}{3}, \frac{8}{3}, 4\}$ with associated intervals

$$[x_0, x_1] = [0, \tfrac{4}{3}], \quad [x_1, x_2] = [\tfrac{4}{3}, \tfrac{8}{3}], \quad [x_2, x_3] = [\tfrac{8}{3}, 4].$$

Letting

$$f(x) = \sqrt{x+1},$$

we will now compute sums of the form

$$\sum_{k=1}^{N} f(c_k)\Delta x,$$

for different values of N, and where $c_k \in [x_{k-1}, x_k]$ are the *sample points*.

See the lecture notes Chapter VII, Sections 1 and 2; also see the website http://mathworld.wolfram.com/RiemannSum.html for interactive pictures of Riemann sums.

3. Left sum

For the "Left sum", we will choose $c_k = x_{k-1}$, i.e. the left endpoint of the interval $[x_{k-1}, x_k]$, for all $k = 1, \ldots, N$. For example, when $N = 3$, the left sum is

$$f(0)\tfrac{4}{3} + f(\tfrac{4}{3})\tfrac{4}{3} + f(\tfrac{8}{3})\tfrac{4}{3} = 1 \cdot \tfrac{4}{3} + \sqrt{\tfrac{7}{3}}\,\tfrac{4}{3} + \sqrt{\tfrac{11}{3}}\,\tfrac{4}{3} \approx 5.9231,$$

and when $N = 6$ the sum is

$$\sum_{k=1}^{6} f\left(\frac{4(k-1)}{6}\right)\frac{4}{6} \approx 6.3647. \tag{1}$$

Using your calculator, software such as Excel, or the Riemann-sum website http://mathworld.wolfram.com/RiemannSum.html, fill in the table below with the associated left sums.

N	$\sum_{k=1}^{N} f(x_{k-1})\Delta x$
6	6.3647
8	
12	
15	

4. Right sum

For the "Right sum", we will choose $c_k = x_k$, i.e. the right endpoint of the interval $[x_{k-1}, x_k]$, for all $k = 1, \ldots, N$. For example, when $N = 3$, the right sum is

$$f(\tfrac{4}{3})\tfrac{4}{3} + f(\tfrac{8}{3})\tfrac{4}{3} + f(\tfrac{12}{3})\tfrac{4}{3} = \sqrt{\tfrac{7}{3}}\tfrac{4}{3} + \sqrt{\tfrac{11}{3}}\tfrac{4}{3} + \sqrt{\tfrac{15}{3}}\tfrac{4}{3} \approx 7.57126. \tag{2}$$

Using your calculator, or software, such as Excel, fill in the table below with the associated right sums.

N	$\sum_{k=1}^{N} f(x_k)\Delta x$
6	7.18877
8	
12	
15	

5. Midpoint sum

The actual value of the integral is

$$\int_0^4 \sqrt{x+1}\, dx = \frac{2}{3}\left(5\sqrt{5} - 1\right) \approx 6.786893. \tag{3}$$

In the previous sections of this project you should have found that the left sums always underestimated the correct solution for this problem while the right sum always overestimated. This should lead you to believe that perhaps it would be best to choose c_k to be the midpoint of x_{k-1} and x_k. That is, we suspect that a good approximation method would be

$$\sum_{k=1}^{N} f\left(\frac{x_{k-1} + x_k}{2}\right) \Delta x.$$

For example, when $N = 3$, the sum above sum is

$$f(\tfrac{2}{3})\ \tfrac{4}{3} + f(\tfrac{6}{3})\ \tfrac{4}{3} + f(\tfrac{10}{3})\ \tfrac{4}{3} \approx 6.80628.$$

Using your calculator, or Excel, fill in the table below with the associated midpoint sums.

N	$\sum_{k=1}^{N} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x$
6	6.79193
8	
12	
15	

6. Report instructions

Your project should include

(1) Summary of ideas in project including all computations performed and conclusions reached. *Why did the left hand sum underestimate the true solution for this problem, while the right hand sum overestimated?* (Hint: draw some pictures!)

(2) Find formulas for x_k and c_k in the Left-sum (1). What is $\frac{4(k-1)}{6}$ doing there? Write the Right-sum (2) with $N = 3$ using summation notation in the manner of Equation (1).

(3) Do the integral that leads to the actual value stated in Equation (3).

(4) A function that is central to the study of probability is

$$f(x) = e^{-x^2/2}.$$

Compute the left hand, right hand, and midpoint sums for this function integrated over the interval $[0, 2]$ for several increasing values of N. *Do the left hand sums underestimate the integral, or do they overestimate the integral?* Stop each computation when you are confident that your N is large enough so that your value is accurate to two decimal places. How do you know you can stop when you did?

N	left sum	right sum	midpoint sum
6			
8			
12			
15			
...			
...			