Math 234 Page 123 problem 6(e)

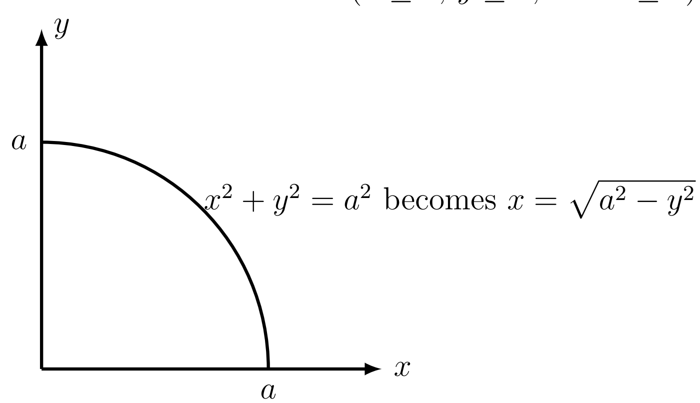
Cartesian coordinates solution

6. Find the volumes of the following regions by computing a double integral.

(e). The region in the first octant bounded by $x^2 + y^2 = a^2$ and $z = x + y$.

Solution: When doing problems like these, we want to find 2 dimensions which are relatively easy to express their bounds. These will become the integration limits. The last dimension will tell us how far we have to integrate (this becomes the integrand; recall, the integrand is the function that comes before the $\mathrm{d}x$ and/or $\mathrm{d}y$ and/or $\mathrm{d}z$.

In this problem, we have one function ($z = x + y$) which uses all three variables. This may be difficult to turn into an integration bound (since we only integrate over 2 variables). So for now let's focus on the remaining equation: $x^2 + y^2 = a^2$. Since we are in the first octant ($x \geq 0$, $y \geq 0$, and $z \geq 0$), we can draw this region:



This is a relatively easy region to convert into a double integral:

$$\int_{y=0}^{y=a} \int_{x=0}^{x=\sqrt{a^2-y^2}} ???\, \mathrm{d}x\, \mathrm{d}y\,,$$

you could just as easily change the order in terms of $\mathrm{d}y\, \mathrm{d}x$. Our final dimension to look at is z. We already know z is bounded by the equation (plane) $z = x + y$. But we are in the first octant and thus $z \geq 0$. With a little bit of thinking, we see that z is bounded from below by $z = 0$ and bounded above by $z = x + y$. This is precisely the height of the function or depth of the volume; so this becomes the integrand:

$$\int_{y=0}^{y=a}\int_{x=0}^{x=\sqrt{a^2-y^2}} (x+y)-(0)\,\mathrm{d}x\,\mathrm{d}y\,.$$

All that is left now is to do compute this double integral,

$$\begin{aligned}\int_{y=0}^{y=a}\int_{x=0}^{x=\sqrt{a^2-y^2}} (x+y)-(0)\,\mathrm{d}x\,\mathrm{d}y &= \int_{y=0}^{y=a}\left[\int_{x=0}^{x=\sqrt{a^2-y^2}} x+y\,\mathrm{d}x\right]\mathrm{d}y\\ =\int_{y=0}^{y=a}\left[\tfrac{1}{2}x^2+yx\right]_{x=0}^{x=\sqrt{a^2-y^2}}\mathrm{d}y &= \int_{y=0}^{y=a}\left[\tfrac{1}{2}(a^2-y^2)+y\sqrt{a^2-y^2}\right]\mathrm{d}y\\ =\left[\tfrac{1}{2}a^2y-\tfrac{1}{6}y^3-\tfrac{1}{3}(a^2-y^2)^{3/2}\right]_{y=0}^{y=a} &= \left[\tfrac{1}{2}a^3-\tfrac{1}{6}a^3-0\right]-\left[0-0-\tfrac{1}{3}(a^2)^{3/2}\right]\\ &= \tfrac{1}{2}a^3-\tfrac{1}{6}a^3+\tfrac{1}{3}a^3=\tfrac{2}{3}a^3\,,\end{aligned}$$

where we used the following u-substitution above:

$$\int y\sqrt{a^2-y^2}\,\mathrm{d}y=\int -\tfrac{1}{2}\sqrt{u}\,\mathrm{d}u=-\tfrac{1}{3}u^{3/2}=-\tfrac{1}{3}(a^2-y^2)^{3/2}\,,$$

where $u=a^2-y^2$ and $\mathrm{d}u=-2y\,\mathrm{d}y$ (so $y\,\mathrm{d}y=-\frac{1}{2}\,\mathrm{d}u$).

Polar coordinates solution

Alternatively, you are more than welcome to use polar coordinates. Using the same region of $x^2+y^2=a^2$ above with $x\geq 0$ and $y\geq 0$, this tells us: $0\leq r\leq a$ and $0\leq\theta\leq\pi/2$. Thus our double integration bounds are

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a}[???]\,r\,\mathrm{d}r\,\mathrm{d}\theta\,,$$

where we have to remember that in polar coordinates, we end up with an extra r in our integrand (consequence of the change in coordinate system from cartesian to polar). The rest of our integrand (the ??? part) is still $(x+y)$, however, in order to use this in polar coordinates we use the substitutions

$$x=r\cos\theta\qquad y=r\sin\theta\,.$$

Thus we have

$$\begin{aligned}
&\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a} [r\cos\theta + r\sin\theta]\, r\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a} r^2\cos\theta + r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta \\
&\qquad = \int_{\theta=0}^{\theta=\pi/2} \left[\tfrac{1}{3}r^3\cos\theta + \tfrac{1}{3}r^3\sin\theta\right]_{r=0}^{r=a}\,\mathrm{d}\theta = \int_{\theta=0}^{\theta=\pi/2} \tfrac{1}{3}a^3\cos\theta + \tfrac{1}{3}a^3\sin\theta\,\mathrm{d}\theta \\
&\quad = \left[\tfrac{1}{3}a^3\sin\theta - \tfrac{1}{3}a^3\cos\theta\right]_{\theta=0}^{\theta=\pi/2} = \left[\tfrac{1}{3}a^3(1) - \tfrac{1}{3}a^3(0)\right] - \left[\tfrac{1}{3}a^3(0) - \tfrac{1}{3}a^3(1)\right] = \tfrac{2}{3}a^3\,.
\end{aligned}$$