MATH 221 PROJECT

Functions and Mathematical Models

1. Purpose of this project

An understanding of calculus can only be achieved if you, the student, have a firm grasp on the concept of a function. Literally everything in this, and future, courses depends on it. Therefore, in this project we will explore what is meant by the term *function*, and see how functions arise naturally in a number of application areas. Further, we will think about what a "mathematical model" is, what they are used for, and the limits of their use.

2. Background on Functions

We recall from the class notes the following definition.

2.1. Definition. *A function f is a rule giving a value, denote $f(x)$, for each x. The set of x for which f is defined (i.e. for which the rule can be applied) is called the **domain** of f, and the set of all possible values, $f(x)$, is called the **range**. That is, the range of f is the set*

$$\{f(x) \mid x \text{ is in the domain of } f\}.$$

All of the functions encountered in math 221 will have a domain and range that are *subsets* of the real numbers. Note that the definition of a function is very broad and so particular functions can be defined in a number of different ways as the next few examples demonstrate.

2.2. Example. Consider a vendor selling pizza by the slice on State Street. Suppose you can purchase 1, 2, 3, or 4 slices of pizza at a cost of \$2, \$3.75, \$5.25, and \$6.50, respectively. The vendor does not take any other orders (that is, he will not allow you to purchase 5 slices since that is greedy).

If we denote by s the number of slices purchased, and by $p(s)$ the price of that order, in dollars, then p is a function with domain $\{1, 2, 3, 4\}$ and range $\{2, 3.75, 5.25, 6.50\}$. The rule defining p is then given by the following list

$$p(1) = \$2, \quad p(2) = \$3.75, \quad p(3) = \$5.25, \quad p(4) = \$6.50.$$

□

2.3. Example. A scientist at the Wisconsin Institute for Discovery believes that for every virus particle that infects a group of cells, one thousand new virus particles will be made. If we let v denote the number of virus particles that infect a group of cells, and by $f(v)$ the number of resulting virus particles, then $f(v)$ is a function with domain $\{0, 1, 2, \ldots\}$ and range $\{0, 1000, 2000, \ldots\}$ defined by the rule

$$f(v) = 1000 \cdot v.$$

Note that because the domain is no longer finite, it is no longer possible to give a list detailing the rule.□

2.4. Example. A car is moving away from you at a speed of 30 miles per hour. At time zero, the car was already 2 miles away. Let $t \geq 0$ denote time (in hours), and let $d(t)$ denote the distance between you and the car. Then $d(t)$ is a function with domain $[0, \infty)$ and range $[2, \infty)$ defined by the rule

$$d(t) = 2 + 30t$$

□

Of course, each of the functions above was derived from some (made up) "real world" application. In a math course we will typically consider functions with no overt connection to the "real world." For example, as in the class notes on page 9, we can define $f(x)$ to be the function defined piecewise via

$$f(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}.$$

Having such flexibility allows us to play with an infinite number of examples.

2.5. Functions and Mathematical models. A *mathematical model* is a mathematical description of some real world phenomenon. Usually the model is described via an equation or a system of equations. The purpose of a mathematical model is to gain insight into the phenomenon being studied and, hopefully, to make predictions.

2.6. Example. Consider a large manufacturer of Bucky Badger shirts. Every year, the company spends $200,000 on fixed operational costs (wages, electricity, mortgage, etc.). Further, every shirt made costs an addition $5. Let q denote the number of shirts produced by the company in a given year, and let $C(q)$ denote the cost to manufacture q shirts. Assume that if zero shirts are ever produced, they will fire all their employees and close their factory yielding a manufacturing cost of zero. □

3. Problems

1. What is the domain of the function C? What is the range?

2. A quantity of great interest to many manufacturers is the *marginal cost*, defined as the cost required to produce one addition unit. That is, the marginal cost at 5 units is the difference in the cost of 6 and 5 units. Define a function giving the marginal cost for our manufacturer. Be clear in specifying a domain, range, and rule.

3. Our manufacturer does not only care about costs, but also wants to know about revenue. Suppose they sell each unit for $13. Write down a function for revenue, $R(q)$, generated by selling the Bucky Badger shirts, being careful to specify an appropriate domain and range. How many shirts would need to be sold to make it worthwhile to manufacture them? Be explicit in how you solved this problem. A graph would certainly help.

4. Relative risk and the effects of exercise on rates of breast cancer

According to the Susan G. Komen foundation for the Cure[1], we define the *absolute risk* as a person's chance of developing a certain disease over a certain period of time. Next, a *relative risk* is calculated by comparing two absolute risks. The numerator (the top number in a fraction) is the absolute risk among those with the risk factor. The denominator (the bottom number) is the absolute risk among those without the risk factor. Hence,

$$\text{Relative risk} = \frac{\text{Absolute risk with factor}}{\text{Absolute risk without factor}}.$$

Note therefore, that if the relative risk of an activity is above 1.0, then the risk is higher for those people performing that activity. If, on the other hand, the relative risk is below 1.0, then the risk is lower for the portion of the population performing that activity is lower than the population at large.

[1] See, http://ww5.komen.org/BreastCancer/UnderstandingRisk.html

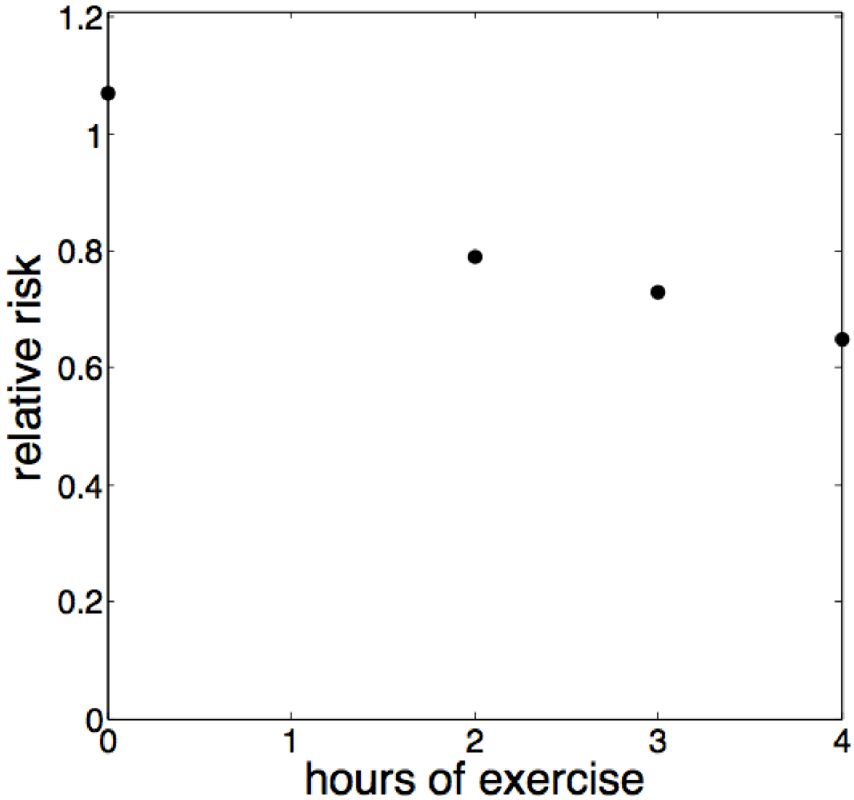

Figure 1. Relative risk of breast cancer vs. hours of exercise.

We present the following case study, that is based on a similar study performed in Duke University's Laboratory Calculus course. For reference, the data used in this study is taken from a study of women, age 50 - 64, that were residents of King County in northwestern Washington State.[2]

One risk factor considered in the King County study was exercise level. The researchers asked each participant in the study "During the 2-year period prior to (date of study), with what frequency did you do any strenuous physical activities, exercise, or sports?" The following data was collected:

Average number of exercise hours per week	0	2.0	3.0	4.0
Relative Risk	1.07	0.79	0.73	0.65

This data in the table above is plotted in Figure 1

[2] *Occurrence of Breast Cancer in Relation to Recreational Exercise in Women Age 50 - 64 Years*, Epidemiology, Nov. 1996, Vol. 7, No. 6, 598 - 604

5. Problems

1. **(a)** Draw a straight line in Figure 1 that fits the data well (do not use the statistical features of your calculator). Denote the relative risk by the variable R, and hours of exercise by the variable E. Determine the slope of the line you drew. Also determine the R intercept. What is the equation of the line?

(b) What does the R intercept tell you about exercise and the relative risk of breast cancer? is your intercept the same as your data point on the axis? Which do you think is more accurate? Why?

(c) The slope of your line should be negative. Describe what this means in terms of relative risk of breast cancer and exercise.

(d) What is the absolute value of the slope? Use this number and the negativity of the slope of the line to finish the following sentence: "If a woman increases her exercise by one hour per week, then her relative risk of breast cancer will be (approximately) ...."

(e) Estimate the relative risk of breast cancer if a woman exercises for 3.3 hours per week.

(f) Find the E intercept (the place where the line intersects the horizontal axis), and denote it by M. Interpret M.

(g) For what values of E do you believe your linear model to be a decent approximation to the true relative risk?

Report Instructions

Using complete sentences (including punctuation!), write out your solutions to the exercises found in Example 2.6. Next, carefully write up your answers to the questions found in Example 4. You should include not only the formulas used, but also all the derivations of the formulas. General instructions:

(1) Do not turn in your first draft. Instead, solve all the problems first, and then write or type up a clean report.
(2) Only one report is required per group, but all members must take an active role in the solution and writing of the report. If a member does not participate, his or her name should not be included on the final report.