Answers and Hints

(I2.1) The decimal expansion of

$$1/7 = 0.\overline{142857}\,142857\,142857\,\cdots$$

repeats after 6 digits. Since $2007 = 334 \times 6 + 3$ the 2007^{th} digit is the same as the 3^{rd}, which happens to be a 2.

(I2.5) Yes these are the same sets. Both sets consist of all positive real numbers: since they contain exactly the same numbers, they are the same sets.

(I2.6) $100x = 31.313131\cdots = 31 + x \implies 99x = 31 \implies x = \frac{31}{99}$.
Similarly, $1000y = 273 + y$ so $y = \frac{273}{999}$.
In z the initial "2" is not part of the repeating pattern, so subtract it:
$z = 0.2 + 0.0154154154\cdots$. Now let $w = 0.0154154154\cdots$. You get
$1000w = 15.4 + w = 15\frac{2}{5} + w = \frac{77}{5} + w$. Therefore $w = \frac{77}{5\times 999}$. From this you get

$$z = \tfrac{1}{5} + w = \tfrac{1}{5} + \frac{77}{5 \times 999} = \frac{1076}{4995}.$$

(I7.1) They are the same function. Both are defined for all real numbers, and both will square whatever number you give them, so they are the same function.

(I7.4) Let x be any number. Then, $f(x)$, if it is defined, is the largest

(I7.6) The domain of k^{-1} is $(0, \infty)$, and $k^{-1}(x) = -\sqrt{x}$.

(I7.8a) False: Since $\arcsin x$ is only defined if $-1 \le x \le 1$ and hence not for *all* x, it is not true that $\sin\big(\arcsin x\big) = x$ for *all* real numbers x. However, it is true that $\sin(\arcsin x) = x$ for all x in the interval $[-1, 1]$.

(I7.8b) $\arcsin(\sin x)$ is defined for all x since $\sin x$ is defined for all x, and $\sin x$ is always between -1 and 1. However the arcsine function always returns a number (angle) between $-\pi/2$ and $\pi/2$, so $\arcsin(\sin x) = x$ can't be true when $x > \pi/2$ or $x < -\pi/2$. For $|x| \le \pi/2$ it is true that $\arcsin \sin x = x$.

(I7.8c) Again, not true: if $x = \pi/2$ then $\tan x$ is not defined and therefore $\arctan(\tan x)$ is not defined either.
Apart from that, arctan(anything) always lies between $-\pi/2$ and $+\pi/2$, so $\arctan(\tan x)$ cannot be the same as x if either $x > \pi/2$ or $x < -\pi/2$.

(I7.8d) True.

(I7.14a) Set $x = -3/2$ in $f(2x + 3) = x^2$ and you find $f(0) = (-3/2)^2 = \frac{9}{4}$.

(I7.14b) Set $x = 0$ in $f(2x+3) = x^2$ and you find $f(3) = 0^2 = 0$.

(I7.14c) Solve $2x + 3 = t$ for x: $x = \frac{t-3}{2}$. Substitute this in $f(2x+3) = x^2$ and you find $f(t) = \left(\frac{t-3}{2}\right)^2$.

(I7.14d) From the previous problem we know what $f(t)$ is for any t so just substitute $t = x$: $f(x) = b\left(\frac{x-3}{2}\right)^2$.

(I7.14e) $f(2) = \left((2-3)/2\right)^2 = \frac{1}{4}$.

(I7.14f) $f(2f(x)) = \left(\frac{2f(x)-3}{2}\right)^2 = \left\{\frac{2\left(\frac{x-3}{2}\right)^2 - 3}{2}\right\}^2$.

(I7.15a) We know $f\left(\frac{1}{x+1}\right) = 2x - 12$ for all x, so if we want to know $f(1)$ then we have to find an x with $\frac{1}{x+1} = 1$. Solving $\frac{1}{x+1} = 1$ for x you find $x = 0$. Substitute $x = 0$ in $f\left(\frac{1}{x+1}\right) = 2x - 12$ and you get $f(1) = 2 \times 0 - 12 = -12$.

(I7.15b) To find $f(0)$ you proceed as above, this time solving $\frac{1}{x+1} = 0$ for x. In this case there is no solution x, and therefore the equation $f\left(\frac{1}{x+1}\right) = 2x - 12$ does not tell us what $f(0)$ is. Conclusion: either 0 is not in the domain of f, or we cannot tell what $f(0)$ is from the information provided in the problem.

(I7.15c) To find $f(t)$ you do the same as when you want to find $f(1)$. We know $f\left(\frac{1}{x+1}\right) = 2x - 12$ for all x, so if we want to know $f(t)$ then we have to find an x with $\frac{1}{x+1} = t$. Solving $\frac{1}{x+1} = t$ for x you find $x = \frac{1}{t} - 1$. Substitute $x = \frac{1}{t} - 1$ in $f\left(\frac{1}{x+1}\right) = 2x - 12$ and you get $f(t) = 2 \times \left(\frac{1}{t} - 1\right) - 12 = \frac{2}{t} - 14$.

(I7.15d) $f(2f(x)) = \frac{2}{2f(x)} - 14 = \frac{1}{f(x)} - 14 = \frac{1}{\frac{2}{x} - 14} - 14$. You could simplify this if you wanted to, but that was not part of the question.

(I7.15e) After finding $f(t) = \frac{2}{t} - 14$ you can substitute $t = x$ and you find $f(x) = \frac{2}{x} - 14$.

(I7.15f) $f(2) = \frac{2}{2} - 14 = -13$ and therefore $f(f(2)) = f(-13) = \frac{2}{-13} - 14 = -14\frac{2}{13}$.

(I7.16) No. For instance if you set $x = 1$ you get $f(1) = 1 + 1 = 2$, and if you set $x = -1$ then you get $f((-1)^2) = (-1) + 1$, i.e. $f(1) = 0$. But $f(1)$ can't be equal to both 2 and 0, the formula $f(x^2) = x + 1$ cannot be true for all real numbers x.

(I7.18) $g(x) = -2\left(x^2 - 2x\right) = -2\left(x^2 - 2x + 1 - 1\right) = -2\left[(x-1)^2 - 1\right] = -2(x-1)^2 + 2$, so the range of g is $(-\infty, 2]$.

Alternatively:

$y = g(x) \iff y = -2x^2 + 4x \iff 2x^2 - 4x + y = 0$. The quadratic formula says that the solutions are

$$x = \frac{4 \pm \sqrt{16 - 8y}}{4}.$$

If $16 - 8y < 0$ then there are no solutions and y does not belong to the range of g.

If $16 - 8y \geq 0$ then there is at least one solution and y does belong to the range of g.

Conclusion, the range of g consists of all y with $16 - 8y \geq 0$, i.e. all $y \leq 2$.

(II6.3) **(a)**

$$\begin{aligned}\Delta y &= (x+\Delta x)^2 - 2(x+\Delta x) + 1 - [x^2 - 2x + 1] \\ &= (2x-2)\Delta x + (\Delta x)^2 \text{ so that} \\ \frac{\Delta y}{\Delta x} &= 2x - 2 + \Delta x\end{aligned}$$

(II6.4a) In this picture $s(t)$ is on the horizontal axis and t is on the vertical axis, so horizontal and vertical have been swapped. This curve should pass the *horizontal line test*, which it does.

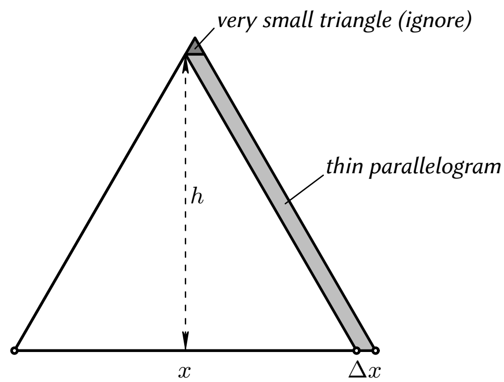
(II6.4b) With a ruler I tried to draw the closest tangent lines at the four different times. Then I measured the slope of those four lines using the grid.

(II6.5) At A and B the graph of f is tangent to the drawn lines, so the derivative at A is -1 and ther derivative at B is $+1$.

(II6.6) Δx : feet. Δy pounds. $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$ are measured in pounds per feet.

(II6.7) Gallons per second.

(II6.8b) (a) $A(x)$ is an area so it has units square inch and x is measured in inches, so $\dfrac{dA}{dx}$ is measured in $\dfrac{\text{inch}^2}{\text{inch}} = \text{inch}$.



(b) Hint: The extra area ΔA that you get when the side of an equilateral triangle grows from x to $x + \Delta x$ can be split into a thin parallelogram and a very tiny triangle. Ignore the area of the tiny triangle since the area of the parallelogram will be much larger. What is the area of this parallelogram?

The area of a parallelogram is "base time height" so here it is $h \times \Delta x$, where h is the height of the triangle.

Conclusion: $\dfrac{\Delta A}{\Delta x} \approx \dfrac{h\Delta x}{\Delta x} = h.$

The derivative is therefore the height of the triangle.

(III4.3) The equation (??) already contains a function f, but that is not the right function. In (??) Δx is the variable, and $g(\Delta x) = (f(x+\Delta x) - f(x))/\Delta x$ is the function; we want $\lim_{\Delta x \to 0} g(\Delta x)$.

(III4.4) $\delta = \varepsilon/2$.

(III4.5) $\delta = \min\{1, \frac{1}{6}\varepsilon\}$

(III4.6) $|f(x) - (-7)| = |x^2 - 7x + 10| = |x-2| \cdot |x-5|$. If you choose $\delta \le 1$ then $|x-2| < \delta$ implies $1 < x < 3$, so that $|x-5|$ is at most $|1-5| = 4$.

So, choosing $\delta \le 1$ we always have $|f(x) - L| < 4|x-2|$ and $|f(x) - L| < \varepsilon$ will follow from $|x-2| < \frac{1}{4}\varepsilon$.

Our choice is then: $\delta = \min\{1, \frac{1}{4}\varepsilon\}$.

(III4.7) $f(x) = x^3$, $a = 3$, $L = 27$.

When $x = 3$ one has $x^3 = 27$, so $x^3 - 27 = 0$ for $x = 3$. Therefore you can factor out $x - 3$ from $x^3 - 27$ by doing a long division. You get $x^3 - 27 = (x-3)(x^2 + 3x + 9)$, and thus

$$|f(x) - L| = |x^3 - 27| = |x^2 + 3x + 9| \cdot |x - 3|.$$

Never choose $\delta > 1$. Then $|x-3| < \delta$ will imply $2 < x < 4$ and therefore

$$|x^2 + 3x + 9| \le 4^2 + 3 \cdot 4 + 9 = 37.$$

So if we always choose $\delta \le 1$, then we will always have

$$|x^3 - 27| \le 37\delta \quad \text{for} \quad |x - 3| < \delta.$$

Hence, if we choose $\delta = \min\left\{1, \frac{1}{37}\varepsilon\right\}$ then $|x-3| < \delta$ guarantees $|x^3 - 27| < \varepsilon$.

(III4.9) $f(x) = \sqrt{x}$, $a = 4$, $L = 2$.

You have

$$\sqrt{x} - 2 = \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2} = \frac{x-4}{\sqrt{x}+2}$$

and therefore

$$|f(x) - L| = \frac{1}{\sqrt{x}+2}|x-4|. \tag{1}$$

Once again it would be nice if we could replace $1/(\sqrt{x}+2)$ by a constant, and we achieve this by always choosing $\delta \le 1$. If we do that then for $|x-4| < \delta$ we always have $3 < x < 5$ and hence

$$\frac{1}{\sqrt{x}+2} < \frac{1}{\sqrt{3}+2},$$

since $1/(\sqrt{x}+2)$ increases as you decrease x.

So, if we always choose $\delta \le 1$ then $|x-4| < \delta$ guarantees

$$|f(x) - 2| < \frac{1}{\sqrt{3}+2}|x-4|,$$

which prompts us to choose $\delta = \min\left\{1, (\sqrt{3}+2)\varepsilon\right\}$.

A smarter solution: We *can* replace $1/(\sqrt{x}+2)$ by a constant in (1), because for all x in the domain of f we have $\sqrt{x} \ge 0$, which implies

$$\frac{1}{\sqrt{x}+2} \le \frac{1}{2}.$$

Therefore $|\sqrt{x} - 2| \le \frac{1}{2}|x-4|$, and we could choose $\delta = 2\varepsilon$.

(III4.10) Hints:

$$\sqrt{x+6} - 3 = \frac{x+6-9}{\sqrt{x+6}+3} = \frac{x-3}{\sqrt{x+6}+3}$$

so

$$|\sqrt{x+6} - 3| \le \tfrac{1}{3}|x-3|.$$

(III4.11) We have
$$\left|\frac{1+x}{4+x}-\frac{1}{2}\right| = \left|\frac{x-2}{4+x}\right|.$$
If we choose $\delta \le 1$ then $|x-2| < \delta$ implies $1 < x < 3$ so that
$$\underline{\frac{1}{7} <} \text{ we don't care}\frac{1}{4+x} < \frac{1}{5}.$$
Therefore
$$\left|\frac{x-2}{4+x}\right| < \tfrac{1}{5}|x-2|,$$
so if we want $|f(x)-\frac{1}{2}| < \varepsilon$ then we must require $|x-2| < 5\varepsilon$. This leads us to choose
$$\delta = \min\{1, 5\varepsilon\}.$$

(III14.16) $A(\frac{2}{3},-1)$; $B(\frac{2}{5},1)$; $C(\frac{2}{7},-1)$; $D(-1,0)$; $E(-\frac{2}{5},-1)$.

(III14.17) False! The limit must not only exist *but also be equal to* $f(a)$!

(III14.18) There are of course many examples. Here are two: $f(x) = 1/x$ and $f(x) = \sin(\pi/x)$ (see §??)

(III14.19) False! Here's an example: $f(x) = \frac{1}{x}$ and $g(x) = x - \frac{1}{x}$. Then f and g don't have limits at $x = 0$, but $f(x) + g(x) = x$ *does* have a limit as $x \to 0$.

(III14.20) False again, as shown by the example $f(x) = g(x) = \frac{1}{x}$.

(III14.21a) False, for the following reason: $g(x)$ is the difference of $f(x)+g(x)$ and $f(x)$. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)+g(x)$ also exists, then
$$\begin{aligned}\lim_{x\to a} g(x) &= \lim_{x\to a}\big\{f(x)+g(x)-f(x)\big\}\\ &= \lim_{x\to a}\big\{f(x)+g(x)\big\} - \lim_{x\to a} f(x)\end{aligned}$$
also has to exist.

(III14.21b) True, as shown by the example $f(x) = x$, $g(x) = \frac{1}{x}$, and $a = 0$. For these two functions we have
$$\lim_{x\to 0} f(x) = 0 \text{ (i.e. exists)}$$
$$\lim_{x\to 0} g(x) = \text{ does not exist}$$
$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} x \times \frac{1}{x} = 1 \text{ (i.e. exists)}$$
You can make up other examples, but to show that this statement is true you only need one example.

(III14.21c) True, as shown by the same example $f(x) = x$, $g(x) = \frac{1}{x}$, $a = 0$. This time we have
$$\lim_{x\to 0} f(x) = 0 \text{ (i.e. exists)}$$
$$\lim_{x\to 0} g(x) = \text{ does not exist}$$
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x}{1/x} = \lim_{x\to 0} x^2 = 0 \text{ (i.e. exists)}$$
You can make up other examples, but to show that this statement is true you only need one example.

(III14.21d) False: If $\lim_{x\to a} g(x)$ and $\lim_{x\to a} f(x)/g(x)$ both exist then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) \times \frac{f(x)}{g(x)}$$
$$\Bigl(\lim_{x\to a} g(x)\Bigr) \times \Bigl(\lim_{x\to a} \frac{f(x)}{g(x)}\Bigr)$$

and therefore $\lim_{x\to a} f(x)$ would also have to exist.

(III16.1) the limit is 1.

(III16.2) The limit is 1. Use : $\frac{\theta}{\sin\theta} = \frac{1}{\frac{\sin\theta}{\theta}}$.

(III16.4) $\sin 2\alpha = 2\sin\alpha\cos\alpha$ so the limit is $\lim_{\alpha\to 0} \frac{2\sin\alpha\cos\alpha}{\sin\alpha} = \lim_{\alpha\to 0} 2\cos\alpha = 2$.
Other approach: $\dfrac{\sin 2\alpha}{\sin\alpha} = \dfrac{\frac{\sin 2\alpha}{2\alpha}}{\frac{\sin\alpha}{\alpha}} \cdot \dfrac{2\alpha}{\alpha}$. Take the limit and you get 2.

(III16.5) $\frac{3}{2}$.

(III16.6) $\frac{\tan 4\alpha}{\sin 2\alpha} = \frac{\tan 4\alpha}{4\alpha} \cdot \frac{4\alpha}{2\alpha} \cdot \frac{2\alpha}{\sin 2\alpha}$.
Take the limit and you get $\ldots = 1 \cdot 1 \cdot 2 = 2$.

(III16.7) Hint: multiply top and bottom with $1 + \cos x$.

(III16.8) Hint: substitute $\theta = \frac{\pi}{2} - \varphi$, and let $\varphi \to 0$. Answer: -1.

(III16.9) Multiply top and bottom with $1 + \cos x$. The answer is 2.

(III16.10) Substitute $x^2 = u$ and let $u \to 0$. Answer: 1.

(III16.11) Multiply and divide by $1 + \cos x$. Write $\tan x$ as $\frac{\sin x}{\cos x}$. Answer is $\frac{1}{2}$.

(III16.12) $\dfrac{\sin(x^2)}{1-\cos x} = \dfrac{\sin(x^2)}{x^2}\,\dfrac{x^2}{1-\cos x}$. The answer is 2.

(III16.13) Substitute $\theta = x - \pi/2$ and remember that $\cos x = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$. You get

$$\lim_{x\to\pi/2} \frac{x - \frac{\pi}{2}}{\cos x} = \lim_{\theta\to 0} \frac{\theta}{-\sin\theta} = -1.$$

(III16.14) Similar to the previous problem, once you use $\tan x = \frac{\sin x}{\cos x}$. The answer is again -1.

(III16.15) $1/9$

(III16.16) Substitute $\theta = x - \pi$. Then $\lim_{x\to\pi} \theta = 0$, so

$$\lim_{x\to\pi} \frac{\sin x}{x-\pi} = \lim_{\theta\to 0} \frac{\sin(\pi+\theta)}{\theta} = -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = -1.$$

Here you have to remember from trigonometry that $\sin(\pi + \theta) = -\sin\theta$.

(III16.17) Divide top and bottom by x. The answer is $1/2$.

(III16.18) Note that the limit is for $x \to \infty$! As x goes to infinity $\sin x$ oscillates up and down between -1 and $+1$. Dividing by x then gives you a quantity which goes to zero. To give a good proof you use the Sandwich Theorem like this:

Since $-1 \le \sin x \le 1$ for all x you have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Since both $-1/x$ and $1/x$ go to zero as $x \to \infty$ the function in the middle must also go to zero. Hence

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(III16.19) zero again.

(III16.23) This is not a rational function, but you can use the same trick: factor out the highest power of x from numerator and denominator. You get

$$\frac{x}{\cos x + x^2} = \frac{x}{x^2} \frac{1}{\frac{\cos x}{x^2} + 1}.$$

Using the Sandwich Theorem as in the previous problems you get $\lim_{x\to\infty} \frac{\cos x}{x^2} = 0$. With the limit properties you then get

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{\cos x + x^2} &= \lim_{x\to\infty} \frac{x}{x^2} \frac{1}{\frac{\cos x}{x^2} + 1} \\ &= 0 \times \frac{1}{0+1} \\ &= 0.\end{aligned}$$

(III16.24) 2.

(III16.25a) $\lim_{\theta\to 0} \dfrac{\tan\theta - \sin\theta}{\theta^3} = \dfrac{1}{2}$

(III16.25b) $\tan 0.1 - \sin 0.1 \approx \frac{1}{2}(0.1)^3 = 0.0005$, which is really a lot smaller than 0.1.

(III16.26) $\sin 0.2 \approx 0.2$,
$\cos 0.2 \approx 1 - \frac{1}{2}(0.2)^2 = 0.98$,
$\tan 0.2 = (\sin 0.2)/(\cos 0.2) \approx 0.2$.
$\sin\big(\pi/2 - 0.2\big) = \cos 0.2 \approx 0.98$.
$\cos\big(\pi/2 + 0.2\big) = -\sin 0.2 \approx -0.2$.
$\tan\big(\pi/2 - 0.2\big) = \dfrac{1}{\tan 0.2} \approx \dfrac{1}{0.2} = 50$.

(III16.27) Same approach as before, but in this problem you first have to convert 10° to radians:

$$10^\circ = \frac{10}{360} \times 2\pi \text{radians} = \frac{\pi}{18}.$$

You get

$$\begin{aligned}\sin 10^\circ &\approx \frac{\pi}{18}, \\ \cos 10^\circ &\approx 1 - \frac{\pi^2}{2 \times 18^2}.\end{aligned}$$

You don't have a calculator, so, had this been 1965, you would have enthusiastically computed these numbers by hand (to two decimals).

For a really rough estimate assume $\pi \approx 3$, to get

$$10^\circ \approx \frac{3}{18} = \frac{1}{6} \approx 0.17,$$

$$\tan 10^\circ \approx \sin 10^\circ \approx \frac{3}{18} = \frac{1}{6} \approx 0.17,$$

$$\begin{aligned}\cos 10^\circ &\approx 1 - \frac{1}{2}\big(\frac{1}{6}\big)^2 \\ &= 1 - \frac{1}{72} \\ &\approx 1 - 0.014 = 0.986\end{aligned}$$

To find the other expressions, use $\sin(\frac{\pi}{2} + \theta) = \cos\theta$ and $100^\circ = 90^\circ + 10^\circ$.
$\sin 100^\circ = \cos 10^\circ \approx -0.986$
$\cos 190^\circ = -\cos 10^\circ \approx -0.986$
$\tan 80^\circ = \big(\tan 10^\circ\big)^{-1} \approx 6$.

(III16.29) No. As $x \to 0$ the quantity $\sin\frac{1}{x}$ oscillates between -1 and $+1$ and does not converge to any particular value. Therefore, no matter how you choose k, it will never be true that $\lim_{x\to 0} \sin\frac{1}{x} = k$, because the limit doesn't exist.

(III16.30) The function $f(x) = (\sin x)/x$ is continuous at all $x \neq 0$, so we only have to check that $\lim_{x\to 0} f(x) = f(0)$, i.e. $\lim_{x\to 0} \frac{\sin x}{2x} = A$. This only happens if you choose $A = \frac{1}{2}$.

(III18.2a) No vertical asymptote. No horizontal asymptote. If there were a slanted asymptote then $m = \lim_{x\to\infty} \frac{\sqrt{x}}{x} = 0$. But $n = \lim_{x\to\infty} f(x) - mx = \lim_{x\to\infty} \sqrt{x}$ does not exist.

(III18.5) We are given that

$$\lim_{x\to\infty} f(x) - mx - n = 0.$$

Adding n to both sides gives us

$$\lim_{x\to\infty} f(x) - mx = n,$$

which is the formula for n we had to prove.
To get the formula for m we multiply with

$$\lim_{x\to\infty} 1/x = 0$$

and use the limit properties:

$$\lim_{x\to\infty} \frac{f(x) - mx - n}{x} = \Big(\lim_{x\to\infty} f(x) - mx - n\Big) \times \Big(\lim_{x\to\infty} \frac{1}{x}\Big) = 0 \times 0 = 0.$$

Work out the left hand side:

$$0 = \lim_{x\to\infty} \frac{f(x)}{x} - m - \frac{n}{x}.$$

This implies

$$0 = \lim_{x\to\infty} \frac{f(x)}{x} - m$$

and thus

$$\lim_{x\to\infty} \frac{f(x)}{x} = m.$$

(IV10.6d) The derivative of $x/(x+2)$ is $2/(x+2)^2$, so the derivative at $x=1$ is $A=\frac{2}{9}$. On the other hand $1/(1+2)=\frac{1}{3}$ is constant, so its derivative is $B=0$.

(IV10.6e) Simplicio is mistaken. The mistake is that he assumes that setting x equal to some constant and then differentiating gives the same result as first differentiating w.r.t. x and then setting x equal to some constant. This example shows that is not true.