Answers and Hints

(I2.1) The decimal expansion of

$$1/7 = 0.\overline{142857}\,142857\,142857\,\cdots$$

repeats after 6 digits. Since $2007 = 334 \times 6 + 3$ the 2007^{th} digit is the same as the 3^{rd}, which happens to be a 2.

(I2.5) Yes these are the same sets. Both sets consist of all positive real numbers: since they contain exactly the same numbers, they are the same sets.

(I2.6) $100x = 31.313131\cdots = 31 + x \implies 99x = 31 \implies x = \frac{31}{99}$.
Similarly, $1000y = 273 + y$ so $y = \frac{273}{999}$.
In z the initial "2" is not part of the repeating pattern, so subtract it:
$z = 0.2 + 0.0154154154\cdots$. Now let $w = 0.0154154154\cdots$. You get
$1000w = 15.4 + w = 15\frac{2}{5} + w = \frac{77}{5} + w$. Therefore $w = \frac{77}{5\times 999}$. From this you get

$$z = \tfrac{1}{5} + w = \tfrac{1}{5} + \frac{77}{5\times 999} = \frac{1076}{4995}.$$

(I7.1) They are the same function. Both are defined for all real numbers, and both will square whatever number you give them, so they are the same function.

(I7.4) Let x be any number. Then, $f(x)$, if it is defined, is the largest

(I7.6) The domain of k^{-1} is $(0, \infty)$, and $k^{-1}(x) = -\sqrt{x}$.

(I7.8a) False: Since $\arcsin x$ is only defined if $-1 \le x \le 1$ and hence not for *all* x, it is not true that $\sin\bigl(\arcsin x\bigr) = x$ for *all* real numbers x. However, it is true that $\sin(\arcsin x) = x$ for all x in the interval $[-1, 1]$.

(I7.8b) $\arcsin(\sin x)$ is defined for all x since $\sin x$ is defined for all x, and $\sin x$ is always between -1 and 1. However the arcsine function always returns a number (angle) between $-\pi/2$ and $\pi/2$, so $\arcsin(\sin x) = x$ can't be true when $x > \pi/2$ or $x < -\pi/2$. For $|x| \le \pi/2$ it is true that $\arcsin \sin x = x$.

(I7.8c) Again, not true: if $x = \pi/2$ then $\tan x$ is not defined and therefore $\arctan(\tan x)$ is not defined either.

Apart from that, arctan(anything) always lies between $-\pi/2$ and $+\pi/2$, so $\arctan(\tan x)$ cannot be the same as x if either $x > \pi/2$ or $x < -\pi/2$.

(I7.8d) True.

(I7.14a) Set $x = -3/2$ in $f(2x+3) = x^2$ and you find $f(0) = (-3/2)^2 = \frac{9}{4}$.

(I7.14b) Set $x = 0$ in $f(2x+3) = x^2$ and you find $f(3) = 0^2 = 0$.

(I7.14c) Solve $2x + 3 = t$ for x: $x = \frac{t-3}{2}$. Substitute this in $f(2x+3) = x^2$ and you find $f(t) = \bigl(\frac{t-3}{2}\bigr)^2$.

(I7.14d) From the previous problem we know what $f(t)$ is for any t so just substitute $t = x$: $f(x) = b\bigl(\frac{x-3}{2}\bigr)^2$.

(I7.14e) $f(2) = \big((2-3)/2\big)^2 = \frac{1}{4}$.

(I7.14f) $f(2f(x)) = \big(\frac{2f(x)-3}{2}\big)^2 = \Big\{\frac{2\big(\frac{x-3}{2}\big)^2-3}{2}\Big\}^2$.

(I7.15a) We know $f\big(\frac{1}{x+1}\big) = 2x - 12$ for all x, so if we want to know $f(1)$ then we have to find an x with $\frac{1}{x+1} = 1$. Solving $\frac{1}{x+1} = 1$ for x you find $x = 0$. Substitute $x = 0$ in $f\big(\frac{1}{x+1}\big) = 2x - 12$ and you get $f(1) = 2 \times 0 - 12 = -12$.

(I7.15b) To find $f(0)$ you proceed as above, this time solving $\frac{1}{x+1} = 0$ for x. In this case there is no solution x, and therefore the equation $f\big(\frac{1}{x+1}\big) = 2x - 12$ does not tell us what $f(0)$ is. Conclusion: either 0 is not in the domain of f, or we cannot tell what $f(0)$ is from the information provided in the problem.

(I7.15c) To find $f(t)$ you do the same as when you want to find $f(1)$. We know $f\big(\frac{1}{x+1}\big) = 2x - 12$ for all x, so if we want to know $f(t)$ then we have to find an x with $\frac{1}{x+1} = t$. Solving $\frac{1}{x+1} = t$ for x you find $x = \frac{1}{t} - 1$. Substitute $x = \frac{1}{t} - 1$ in $f\big(\frac{1}{x+1}\big) = 2x - 12$ and you get $f(t) = 2 \times \big(\frac{1}{t} - 1\big) - 12 = \frac{2}{t} - 14$.

(I7.15d) $f(2f(x)) = \frac{2}{2f(x)} - 14 = \frac{1}{f(x)} - 14 = \frac{1}{\frac{2}{x} - 14} - 14$. You could simplify this if you wanted to, but that was not part of the question.

(I7.15e) After finding $f(t) = \frac{2}{t} - 14$ you can substitute $t = x$ and you find $f(x) = \frac{2}{x} - 14$.

(I7.15f) $f(2) = \frac{2}{2} - 14 = -13$ and therefore $f(f(2)) = f(-13) = \frac{2}{-13} - 14 = -14\frac{2}{13}$.

(I7.16) No. For instance if you set $x = 1$ you get $f(1) = 1 + 1 = 2$, and if you set $x = -1$ then you get $f((-1)^2) = (-1) + 1$, i.e. $f(1) = 0$. But $f(1)$ can't be equal to both 2 and 0, the formula $f(x^2) = x + 1$ cannot be true for all real numbers x.

(I7.18) $g(x) = -2\big(x^2 - 2x\big) = -2\big(x^2 - 2x + 1 - 1\big) = -2\big[(x-1)^2 - 1\big] = -2(x-1)^2 + 2$, so the range of g is $(-\infty, 2]$.

Alternatively:

$y = g(x) \iff y = -2x^2 + 4x \iff 2x^2 - 4x + y = 0$. The quadratic formula says that the solutions are

$$x = \frac{4 \pm \sqrt{16 - 8y}}{4}.$$

If $16 - 8y < 0$ then there are no solutions and y does not belong to the range of g.

If $16 - 8y \geq 0$ then there is at least one solution and y does belong to the range of g.

Conclusion, the range of g consists of all y with $16 - 8y \geq 0$, i.e. all $y \leq 2$.

(II6.3) **(a)**

$$\begin{aligned} \Delta y &= (x + \Delta x)^2 - 2(x + \Delta x) + 1 - [x^2 - 2x + 1] \\ &= (2x - 2)\Delta x + (\Delta x)^2 \text{ so that} \\ \frac{\Delta y}{\Delta x} &= 2x - 2 + \Delta x \end{aligned}$$

(II6.4a) In this picture $s(t)$ is on the horizontal axis and t is on the vertical axis, so horizontal and vertical have been swapped. This curve should pass the *horizontal line test*, which it does.

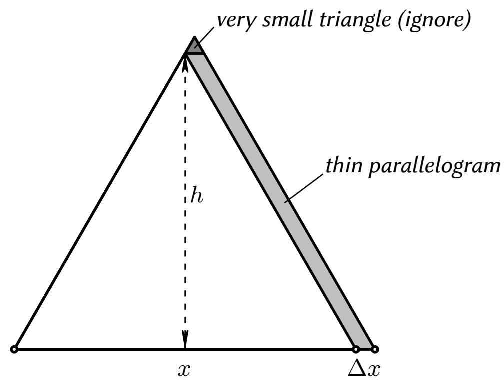
(II6.4b) With a ruler I tried to draw the closest tangent lines at the four different times. Then I measured the slope of those four lines using the grid.

(II6.5) At A and B the graph of f is tangent to the drawn lines, so the derivative at A is -1 and ther derivative at B is $+1$.

(II6.6) Δx : feet. Δy pounds. $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$ are measured in pounds per feet.

(II6.7) Gallons per second.

(II6.8b) (a) $A(x)$ is an area so it has units square inch and x is measured in inches, so $\dfrac{dA}{dx}$ is measured in $\dfrac{\text{inch}^2}{\text{inch}} = \text{inch}$.



(b) Hint: The extra area ΔA that you get when the side of an equilateral triangle grows from x to $x + \Delta x$ can be split into a thin parallelogram and a very tiny triangle. Ignore the area of the tiny triangle since the area of the parallelogram will be much larger. What is the area of this parallelogram?

The area of a parallelogram is "base time height" so here it is $h \times \Delta x$, where h is the height of the triangle.

Conclusion: $\dfrac{\Delta A}{\Delta x} \approx \dfrac{h\Delta x}{\Delta x} = h.$

The derivative is therefore the height of the triangle.

(III4.3) The equation (??) already contains a function f, but that is not the right function. In (??) Δx is the variable, and $g(\Delta x) = (f(x + \Delta x) - f(x))/\Delta x$ is the function; we want $\lim_{\Delta x \to 0} g(\Delta x)$.

(III4.4) $\delta = \varepsilon/2$.

(III4.5) $\delta = \min\{1, \frac{1}{6}\varepsilon\}$

(III4.6) $|f(x) - (-7)| = |x^2 - 7x + 10| = |x - 2| \cdot |x - 5|$. If you choose $\delta \leq 1$ then $|x - 2| < \delta$ implies $1 < x < 3$, so that $|x - 5|$ is at most $|1 - 5| = 4$.

So, choosing $\delta \leq 1$ we always have $|f(x) - L| < 4|x - 2|$ and $|f(x) - L| < \varepsilon$ will follow from $|x - 2| < \frac{1}{4}\varepsilon$.

Our choice is then: $\delta = \min\{1, \frac{1}{4}\varepsilon\}$.

(III4.7) $f(x) = x^3$, $a = 3$, $L = 27$.

When $x = 3$ one has $x^3 = 27$, so $x^3 - 27 = 0$ for $x = 3$. Therefore you can factor out $x - 3$ from $x^3 - 27$ by doing a long division. You get $x^3 - 27 = (x-3)(x^2 + 3x + 9)$, and thus

$$|f(x) - L| = |x^3 - 27| = |x^2 + 3x + 9| \cdot |x - 3|.$$

Never choose $\delta > 1$. Then $|x - 3| < \delta$ will imply $2 < x < 4$ and therefore

$$|x^2 + 3x + 9| \leq 4^2 + 3 \cdot 4 + 9 = 37.$$

So if we always choose $\delta \leq 1$, then we will always have

$$|x^3 - 27| \leq 37\delta \quad \text{for} \quad |x - 3| < \delta.$$

Hence, if we choose $\delta = \min\left\{1, \frac{1}{37}\varepsilon\right\}$ then $|x - 3| < \delta$ guarantees $|x^3 - 27| < \varepsilon$.

(III4.9) $f(x) = \sqrt{x}$, $a = 4$, $L = 2$.

You have

$$\sqrt{x} - 2 = \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} + 2} = \frac{x - 4}{\sqrt{x} + 2}$$

and therefore

$$|f(x) - L| = \frac{1}{\sqrt{x} + 2}|x - 4|. \tag{1}$$

Once again it would be nice if we could replace $1/(\sqrt{x} + 2)$ by a constant, and we achieve this by always choosing $\delta \leq 1$. If we do that then for $|x - 4| < \delta$ we always have $3 < x < 5$ and hence

$$\frac{1}{\sqrt{x} + 2} < \frac{1}{\sqrt{3} + 2},$$

since $1/(\sqrt{x} + 2)$ increases as you decrease x.

So, if we always choose $\delta \leq 1$ then $|x - 4| < \delta$ guarantees

$$|f(x) - 2| < \frac{1}{\sqrt{3} + 2}|x - 4|,$$

which prompts us to choose $\delta = \min\left\{1, (\sqrt{3} + 2)\varepsilon\right\}$.

A smarter solution: We *can* replace $1/(\sqrt{x} + 2)$ by a constant in (??), because for all x in the domain of f we have $\sqrt{x} \geq 0$, which implies

$$\frac{1}{\sqrt{x} + 2} \leq \frac{1}{2}.$$

Therefore $|\sqrt{x} - 2| \leq \frac{1}{2}|x - 4|$, and we could choose $\delta = 2\varepsilon$.

(III4.10) Hints:

$$\sqrt{x + 6} - 3 = \frac{x + 6 - 9}{\sqrt{x + 6} + 3} = \frac{x - 3}{\sqrt{x + 6} + 3}$$

so

$$|\sqrt{x + 6} - 3| \leq \tfrac{1}{3}|x - 3|.$$

(III4.11) We have

$$\left|\frac{1 + x}{4 + x} - \frac{1}{2}\right| = \left|\frac{x - 2}{4 + x}\right|.$$

If we choose $\delta \leq 1$ then $|x - 2| < \delta$ implies $1 < x < 3$ so that

$$\underline{\frac{1}{7} <} \text{ we don't care } \frac{1}{4 + x} < \frac{1}{5}.$$

Therefore

$$\left|\frac{x - 2}{4 + x}\right| < \tfrac{1}{5}|x - 2|,$$

so if we want $|f(x) - \frac{1}{2}| < \varepsilon$ then we must require $|x-2| < 5\varepsilon$. This leads us to choose
$$\delta = \min\{1, 5\varepsilon\}.$$

(III14.16) $A(\frac{2}{3}, -1)$; $B(\frac{2}{5}, 1)$; $C(\frac{2}{7}, -1)$; $D(-1, 0)$; $E(-\frac{2}{5}, -1)$.

(III14.17) False! The limit must not only exist *but also be equal to $f(a)$*!

(III14.18) There are of course many examples. Here are two: $f(x) = 1/x$ and $f(x) = \sin(\pi/x)$ (see §??)

(III14.19) False! Here's an example: $f(x) = \frac{1}{x}$ and $g(x) = x - \frac{1}{x}$. Then f and g don't have limits at $x = 0$, but $f(x) + g(x) = x$ *does* have a limit as $x \to 0$.

(III14.20) False again, as shown by the example $f(x) = g(x) = \frac{1}{x}$.

(III14.21a) False, for the following reason: $g(x)$ is the difference of $f(x) + g(x)$ and $f(x)$. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) + g(x)$ also exists, then
$$\begin{aligned}\lim_{x\to a} g(x) &= \lim_{x\to a}\bigl\{f(x) + g(x) - f(x)\bigr\}\\ &= \lim_{x\to a}\bigl\{f(x) + g(x)\bigr\} - \lim_{x\to a} f(x)\end{aligned}$$
also has to exist.

(III14.21b) True, as shown by the example $f(x) = x$, $g(x) = \frac{1}{x}$, and $a = 0$. For these two functions we have
$$\begin{aligned}\lim_{x\to 0} f(x) &= 0 \text{ (i.e. exists)}\\ \lim_{x\to 0} g(x) &= \text{ does not exist}\\ \lim_{x\to 0} f(x)g(x) &= \lim_{x\to 0} x \times \frac{1}{x} = 1 \text{ (i.e. exists)}\end{aligned}$$
You can make up other examples, but to show that this statement is true you only need one example.

(III14.21c) True, as shown by the same example $f(x) = x$, $g(x) = \frac{1}{x}$, $a = 0$. This time we have
$$\begin{aligned}\lim_{x\to 0} f(x) &= 0 \text{ (i.e. exists)}\\ \lim_{x\to 0} g(x) &= \text{ does not exist}\\ \lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} \frac{x}{1/x} = \lim_{x\to 0} x^2 = 0 \text{ (i.e. exists)}\end{aligned}$$
You can make up other examples, but to show that this statement is true you only need one example.

(III14.21d) False: If $\lim_{x\to a} g(x)$ and $\lim_{x\to a} f(x)/g(x)$ both exist then
$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} g(x) \times \frac{f(x)}{g(x)}\\ &\Bigl(\lim_{x\to a} g(x)\Bigr) \times \Bigl(\lim_{x\to a} \frac{f(x)}{g(x)}\Bigr)\end{aligned}$$
and therefore $\lim_{x\to a} f(x)$ would also have to exist.

(III16.1) the limit is 1.

(III16.2) The limit is 1. Use : $\frac{\theta}{\sin\theta} = \frac{1}{\frac{\sin\theta}{\theta}}$.

(III16.4) $\sin 2\alpha = 2\sin\alpha\cos\alpha$ so the limit is $\lim_{\alpha\to 0}\frac{2\sin\alpha\cos\alpha}{\sin\alpha} = \lim_{\alpha\to 0} 2\cos\alpha = 2$.

Other approach: $\dfrac{\sin 2\alpha}{\sin\alpha} = \dfrac{\frac{\sin 2\alpha}{2\alpha}}{\frac{\sin\alpha}{\alpha}} \cdot \dfrac{2\alpha}{\alpha}$. Take the limit and you get 2.

(III16.5) $\frac{3}{2}$.

(III16.6) $\frac{\tan 4\alpha}{\sin 2\alpha} = \frac{\tan 4\alpha}{4\alpha}\cdot\frac{4\alpha}{2\alpha}\cdot\frac{2\alpha}{\sin 2\alpha}$.

Take the limit and you get $\ldots = 1\cdot 1\cdot 2 = 2$.

(III16.7) Hint: multiply top and bottom with $1+\cos x$.

(III16.8) Hint: substitute $\theta = \frac{\pi}{2} - \varphi$, and let $\varphi \to 0$. Answer: -1.

(III16.9) Multiply top and bottom with $1+\cos x$. The answer is 2.

(III16.10) Substitute $x^2 = u$ and let $u \to 0$. Answer: 1.

(III16.11) Multiply and divide by $1+\cos x$. Write $\tan x$ as $\frac{\sin x}{\cos x}$. Answer is $\frac{1}{2}$.

(III16.12) $\dfrac{\sin(x^2)}{1-\cos x} = \dfrac{\sin(x^2)}{x^2}\,\dfrac{x^2}{1-\cos x}$. The answer is 2.

(III16.13) Substitute $\theta = x - \pi/2$ and remember that $\cos x = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$. You get

$$\lim_{x\to\pi/2}\frac{x-\frac{\pi}{2}}{\cos x} = \lim_{\theta\to 0}\frac{\theta}{-\sin\theta} = -1.$$

(III16.14) Similar to the previous problem, once you use $\tan x = \frac{\sin x}{\cos x}$. The answer is again -1.

(III16.15) $1/9$

(III16.16) Substitute $\theta = x - \pi$. Then $\lim_{x\to\pi}\theta = 0$, so

$$\lim_{x\to\pi}\frac{\sin x}{x-\pi} = \lim_{\theta\to 0}\frac{\sin(\pi+\theta)}{\theta} = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = -1.$$

Here you have to remember from trigonometry that $\sin(\pi+\theta) = -\sin\theta$.

(III16.17) Divide top and bottom by x. The answer is $1/2$.

(III16.18) Note that the limit is for $x \to \infty$! As x goes to infinity $\sin x$ oscillates up and down between -1 and $+1$. Dividing by x then gives you a quantity which goes to zero. To give a good proof you use the Sandwich Theorem like this:

Since $-1 \le \sin x \le 1$ for all x you have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Since both $-1/x$ and $1/x$ go to zero as $x \to \infty$ the function in the middle must also go to zero. Hence

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(III16.19) zero again.

(III16.23) This is not a rational function, but you can use the same trick: factor out the highest power of x from numerator and denominator. You get

$$\frac{x}{\cos x + x^2} = \frac{x}{x^2}\frac{1}{\frac{\cos x}{x^2}+1}.$$

Using the Sandwich Theorem as in the previous problems you get $\lim_{x\to\infty} \frac{\cos x}{x^2} = 0$. With the limit properties you then get

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{\cos x + x^2} &= \lim_{x\to\infty} \frac{x}{x^2}\frac{1}{\frac{\cos x}{x^2}+1}\\ &= 0 \times \frac{1}{0+1}\\ &= 0.\end{aligned}$$

(III16.24) 2.

(III16.25a) $\lim_{\theta\to 0} \frac{\tan\theta - \sin\theta}{\theta^3} = \frac{1}{2}$

(III16.25b) $\tan 0.1 - \sin 0.1 \approx \frac{1}{2}(0.1)^3 = 0.0005$, which is really a lot smaller than 0.1.

(III16.26) $\sin 0.2 \approx 0.2$,
$\cos 0.2 \approx 1 - \frac{1}{2}(0.2)^2 = 0.98$,
$\tan 0.2 = (\sin 0.2)/(\cos 0.2) \approx 0.2$.
$\sin\bigl(\pi/2 - 0.2\bigr) = \cos 0.2 \approx 0.98$.
$\cos\bigl(\pi/2 + 0.2\bigr) = -\sin 0.2 \approx -0.2$.
$\tan\bigl(\pi/2 - 0.2\bigr) = \dfrac{1}{\tan 0.2} \approx \dfrac{1}{0.2} = 50$.

(III16.27) Same approach as before, but in this problem you first have to convert 10° to radians:

$$10^\circ = \frac{10}{360} \times 2\pi\text{radians} = \frac{\pi}{18}.$$

You get

$$\begin{aligned}\sin 10^\circ &\approx \frac{\pi}{18},\\ \cos 10^\circ &\approx 1 - \frac{\pi^2}{2\times 18^2}.\end{aligned}$$

You don't have a calculator, so, had this been 1965, you would have enthusiastically computed these numbers by hand (to two decimals).

For a really rough estimate assume $\pi \approx 3$, to get

$$10^\circ \approx \frac{3}{18} = \frac{1}{6} \approx 0.17,$$

$$\tan 10^\circ \approx \sin 10^\circ \approx \frac{3}{18} = \frac{1}{6} \approx 0.17,$$

$$\begin{aligned}\cos 10^\circ &\approx 1 - \frac{1}{2}\left(\frac{1}{6}\right)^2 \\ &= 1 - \frac{1}{72} \\ &\approx 1 - 0.014 = 0.986\end{aligned}$$

To find the other expressions, use $\sin(\frac{\pi}{2} + \theta) = \cos\theta$ and $100^\circ = 90^\circ + 10^\circ$.
$\sin 100^\circ = \cos 10^\circ \approx -0.986$
$\cos 190^\circ = -\cos 10^\circ \approx -0.986$
$\tan 80^\circ = \left(\tan 10^\circ\right)^{-1} \approx 6$.

(III16.29) No. As $x \to 0$ the quantity $\sin \frac{1}{x}$ oscillates between -1 and $+1$ and does not converge to any particular value. Therefore, no matter how you choose k, it will never be true that $\lim_{x\to 0} \sin \frac{1}{x} = k$, because the limit doesn't exist.

(III16.30) The function $f(x) = (\sin x)/x$ is continuous at all $x \neq 0$, so we only have to check that $\lim_{x\to 0} f(x) = f(0)$, i.e. $\lim_{x\to 0} \frac{\sin x}{2x} = A$. This only happens if you choose $A = \frac{1}{2}$.

(III18.2a) No vertical asymptote. No horizontal asymptote. If there were a slanted asymptote then $m = \lim_{x\to\infty} \frac{\sqrt{x}}{x} = 0$. But $n = \lim_{x\to\infty} f(x) - mx = \lim_{x\to\infty} \sqrt{x}$ does not exist.

(III18.5) We are given that

$$\lim_{x\to\infty} f(x) - mx - n = 0.$$

Adding n to both sides gives us

$$\lim_{x\to\infty} f(x) - mx = n,$$

which is the formula for n we had to prove.
To get the formula for m we multiply with

$$\lim_{x\to\infty} 1/x = 0$$

and use the limit properties:

$$\lim_{x\to\infty} \frac{f(x) - mx - n}{x} = \left(\lim_{x\to\infty} f(x) - mx - n\right) \times \left(\lim_{x\to\infty} \frac{1}{x}\right) = 0 \times 0 = 0.$$

Work out the left hand side:

$$0 = \lim_{x\to\infty} \frac{f(x)}{x} - m - \frac{n}{x}.$$

This implies

$$0 = \lim_{x\to\infty} \frac{f(x)}{x} - m$$

and thus

$$\lim_{x\to\infty} \frac{f(x)}{x} = m.$$

(IV10.6d) The derivative of $x/(x+2)$ is $2/(x+2)^2$, so the derivative at $x=1$ is $A=\frac{2}{9}$. On the other hand $1/(1+2)=\frac{1}{3}$ is constant, so its derivative is $B=0$.

(IV10.6e) Simplicio is mistaken. The mistake is that he assumes that setting x equal to some constant and then differentiating gives the same result as first differentiating w.r.t. x and then setting x equal to some constant. This example shows that is not true.

(IV12.15) $f'(x)=2\tan x/\cos^2 x$
$f''(x)=2/\cos^4 x+4\tan x\sin x/\cos^3 x$
$f'''(x)=-8\sin x/\cos^8 x+4\sin x/\cos^5 x+4\tan x/\cos^2 x-12\tan x\sin^2 x/\cos^6 x.$
Since $\tan^2 x=\frac{1}{\cos^2 x}-1$ one has $g'(x)=f'(x)$ and $g''(x)=f''(x)$.

(IV14.6a) $f'(x)=2\cos 2x+3\sin 3x.$

(IV14.6b) $f'(x)=-\dfrac{\pi}{x^2}\cos\dfrac{\pi}{x}$

(IV14.6c) $f'(x)=\cos(\cos 3x)\cdot(-\sin 3x)\cdot 3=-3\sin 3x\cos\cos 3x.$

(IV14.6d) $f'(x)=\dfrac{x^2\cdot 2x\sin x^2-2x\sin x^2}{x^4}$

(IV14.6e) $f'(x)=\dfrac{1}{\left(\cos\sqrt{1+x^2}\right)^2}\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x$

(IV14.6f) $f'(x)=2(\cos x)(-\sin x)-2(\cos x)\cdot 2x$

(IV14.8) $f'(x)=\cos\frac{\pi}{x}+\frac{\pi}{x}\sin\frac{\pi}{x}$. At C one has $x=-\frac{2}{3}$, so $\cos\frac{\pi}{x}=0$ and $\sin\frac{\pi}{x}=-1$. So at C one has $f'(x)=-\frac{3}{2}\pi$.

(IV14.9) $v(x)=f(g(x))=(x+5)^2+1=x^2+10\,x+26$
$w(x)=g(f(x))=(x^2+1)+5=x^2+6$
$p(x)=f(x)g(x)=(x^2+1)(x+5)=x^3+5x^2+x+5$
$q(x)=g(x)f(x)=f(x)g(x)=p(x).$

(IV14.12b) **(a)** If $f(x)=\sin ax$, then $f''(x)=-a^2\sin ax$, so $f''(x)=-64f(x)$ holds if $a^2=64$, i.e. $a=\pm 8$. So $\sin 8x$ and $\sin(-8x)=-\sin 8x$ are the two solutions you find this way.

(b) $a=\pm 8$, but A and b can have any value. All functions of the form $f(x)=A\sin(8x+b)$ satisfy (†).

(IV14.13d) **(a)** $V=S^3$, so the function f for which $V(t)=f(S(t))$ is the function $f(x)=x^3$.

(b) $S'(t)$ is the rate with which Bob's side grows with time. $V'(t)$ is the rate with which the Bob's volume grows with time.

Quantity	Units
t	minutes
$S(t)$	inch
$V(t)$	inch^3
$S'(t)$	inch/minute
$V'(t)$	inch^3/minute

(c) Three versions of the same answer:
$V(t) = f(S(t))$ so the chain rule says $V'(t) = f'(S(t))S'(t)$
$V(t) = S(t)^3$ so the chain rule says $V'(t) = 3S(t)^2 S'(t)$
$V = S^3$ so the chain rule says $\dfrac{dV}{dt} = 3S^2 \dfrac{dS}{dt}$.
(d) We are given $V(t) = 8$, and $V'(t) = 2$. Since $V = S^3$ we get $S = 2$. From (c) we know $V'(t) = 3S(t)^2 S'(t)$, so $2 = 3 \cdot 2^2 \cdot S'(t)$, whence $S'(t) = \frac{1}{6}$ inch per minute.

(V3.1) At $x = 3$.

(V3.2) At $x = a/2$.

(V3.3) At $x = a + 2a^3$.

(V3.5) At $x = a + \frac{1}{2}$.

(V3.11) False. If you try to solve $f(x) = 0$, then you get the equation $\frac{x^2+|x|}{x} = 0$. If $x \neq 0$ then this is the same as $x^2 + |x| = 0$, which has no solutions (both terms are positive when $x \neq 0$). If $x = 0$ then $f(x)$ isn't even defined. So there is no solution to $f(x) = 0$.

This doesn't contradict the IVT, because the function isn't continuous, in fact it isn't even defined at $x = 0$, so the IVT doesn't have to apply.

(V12.6) Not necessarily true, and therefore false. Consider the example $f(x) = x^4$, and see the next problem.

(V12.7) An inflection point is a point on the graph of a function where the second derivative changes its sign. At such a point you must have $f''(x) = 0$, but by itself that it is no enough.

(V12.10) The first is possible, e.g. $f(x) = x$ satisfies $f'(x) > 0$ and $f''(x) = 0$ for all x.

The second is impossible, since f'' is the derivative of f', so $f'(x) = 0$ for all x implies that $f''(x) = 0$ for all x.

(V12.15) $y = 0$ at $x = -1, 0, 0$. Only sign change at $x = -1$, not at $x = 0$.
$x = 0$ loc min; $x = -\frac{4}{3}$ loc max; $x = -2/3$ inflection point. No global max or min.

(V12.16) zero at $x = 0, 4$; sign change at $x = 4$; loc min at $x = \frac{8}{3}$; loc max at $x = 0$; inflection point at $x = 4/3$. No global max or min.

(V12.17) sign changes at $x = 0, -3$; global min at $x = -3/4^{1/3}$; no inflection points, the graph is convex.

(V12.18) mirror image of previous problem.

(V12.19) $x^4 + 2x^2 - 3 = \big(x^2 - 1\big)\big(x^2 + 3\big)$ so sign changes at $x = \pm 1$. Global min at $x = 0$; graph is convex, no inflection points.

(V12.20) Sign changes at $\pm 2, \pm 1$; *two* global minima, at $\pm\sqrt{5/2}$; one local max at x=0; two inflection points, at $x = \pm\sqrt{5/6}$.

(V12.21) Sign change at $x = 0$; function is always increasing so no stationary points; inflection point at $x = 0$.

(V12.22) sign change at $x = 0, \pm 2$; loc max at $x = 2/5^{1/4}$; loc min at $x = -2/5^{1/4}$. Inflection point at $x = 0$.

(V12.23) Function not defined at $x = -1$. For $x > -1$ sign change at $x = 0$, no stationary points, no inflection points (graph is concave). Horizontal asymptote $\lim_{x\to\infty} f(x) = 1$.

For $x < -1$ no sign change , function is increasing and convex, horizontal asymptote with $\lim_{x\to-\infty} f(x) = 1$.

(V12.24) global max (min) at $x = 1$ (x=-1), inflection points at $x = \pm\sqrt{3}$; horizontal asymptotes $\lim_{x\to\pm\infty} f(x) = 0$.

(V12.25) $y = 0$ at $x = 0$ but no sign changes anywhere; $x = 0$ is a global min; there's no local or global max; two inflection points at $x = \pm\frac{1}{3}\sqrt{3}$; horizontal asymptotes at height $y = 1$.

(V12.26) Not defined at $x = -1$. For $x > -1$ the graph is convex and has a minimum at $x = -1 + \sqrt{2}$; for $x < -1$ the graph is concave with a maximum at $x = -1 - \sqrt{2}$. No horizontal asymptotes.

(V12.27) Not def'd at $x = 0$. No sign changes (except at $x = 0$). For $x > 0$ convex with minimum at $x = 1$, for $x < 0$ concave with maximum at $x = -1$.

(V12.28) Not def'd at $x = 0$. Sign changes at $x = \pm 1$ and also at $x = 0$. No stationary points. Both branches ($x > 0$ and $x < 0$) are increasing. Non inflection points, no horizontal asymptotes.

(V12.29) Zero at $x = 0, -1$ sign only changes at -1; loc min at $x = -\frac{1}{3}$; loc max at $x = -1$. Inflection point at $x = -2/3$.

(V12.30) Changes sign at $x = -1 \pm \sqrt{2}$ and $x = 0$; loc min at $(-2 + \sqrt{7})/3$, loc max at $(-2 - \sqrt{7})/3$; inflection point at $x = -\frac{2}{3}$.

(V12.31) Factor $y = x^4 - x^3 - x = x(x^3 - x^2 - 1)$. One zero is obvious, namely at $x = 0$. For the other(s) you must solve $x^3 - x^2 - 1 = 0$ which is beyond what's expected in this course.
The derivative is $y' = 4x^3 - 3x^2 - 1$. A cubic function whose coefficients add up to 0 so $x = 1$ is a root, and you can factor $y' = 4x^3 - 3x^2 - 1 = (x - 1)(4x^2 + x + 1)$ from which you see that $x = 1$ is the only root. So: one stationary point at $x = 1$, which is a global minimum
The second derivative is $y'' = 12x^2 - 6x$; there are two inflection points, at $x = \frac{1}{2}$ and at $x = 0$.

(V12.32) Again one obvious solution to $y = 0$, namely $x = 0$. The other require solving a cubic equation.
The derivative is $y' = 4x^3 - 6x^2 + 2$ which is also cubic, but the coefficients add up to 0, so $x = 1$ is a root. You can then factor $y' = 4x^3 - 6x^2 + 2 = (x - 1)(4x^2 - 2x - 2)$. There are three stationary points: local minima at $x = 1$, $x = -\frac{1}{4} - \frac{1}{2}\sqrt{3}$, local max at $x = -\frac{1}{4} + \frac{1}{2}\sqrt{3}$. One of the two loc min is a global minimum.

(V12.33) Global min at $x = 0$, no other stationary points; function is convex, no inflection points. No horizontal asymptotes.

(V12.34) The graph is the upper half of the unit circle.

(V12.35) Always positive, so no sign changes; global minimum at $x = 0$, no other stationary points; two inflection points at $\pm\sqrt{2}$. No horizontal asymptotes since $\lim_{x\to\pm\infty} \sqrt[4]{1 + x^2} = \infty$(DNE).

(V12.36) Always positive hence no sign changes; global max at $x = 0$, no other stationary points; two inflection points at $x = \pm\sqrt[4]{3/5}$; second derivative also vanishes at $x = 0$ but this is not an inflection point.

(V12.38) Zeroes at $x = 3\pi/4, 7\pi/4$. Absolute max at $x = \pi/4$, abs min at $x = 5\pi/4$. Inflection points and zeroes coincide. Note that $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$.

(V12.39) Zeroes at $x = 0, \pi, 3\pi/2$ but no sign change at $3\pi/2$. Global max at $x = \pi/2$, local max at $x = 3\pi/2$, global min at $x = 7\pi/6, 11\pi/6$.

(V14.1) If the length of one side is x and the other y, then the perimeter is $2x + 2y = 1$, so $y = \frac{1}{2} - x$. Thus the area enclosed is $A(x) = x(\frac{1}{2} - x)$, and we're only interested in values of x between 0 and $\frac{1}{2}$.

The maximal area occurs when $x = \frac{1}{4}$ (and it is $A(1/4) = 1/16$.) The minimal area occurs when either $x = 0$ or $x = 1/2$. In either case the "rectangle" is a line segment of length $\frac{1}{2}$ and width 0, or the other way around. So the minimal area is 0.

(V14.2) If the sides are x and y, then the area is $xy = 100$, so $y = 1/x$. Therefore the height plus twice the width is $f(x) = x + 2y = x + 2/x$. This is extremal when $f'(x) = 0$, i.e. when $f'(x) = 1 - 2/x^2 = 0$. This happens for $x = \sqrt{2}$.

(V14.3) Perimeter is $2R + R\theta = 1$ (given), so if you choose the angle to be θ then the radius is $R = 1/(2 + \theta)$. The area is then $A(\theta) = \theta R^2 = \theta/(2 + \theta)^2$, which is maximal when $\theta = 2$ (radians). The smallest area arises when you choose $\theta = 0$. Choosing $\theta \geq 2\pi$ doesn't make sense (why? Draw the corresponding wedge!)

You could also say that for any given radius $R > 0$ "perimeter $= 1$" implies that one has $\theta = (1/R) - 2$. Hence the area will be $A(R) = \theta R^2 = R^2\big((1/R) - 2\big) = R - 2R^2$. Thus the area is maximal when $R = \frac{1}{4}$, and hence $\theta = 2$radians. Again we note that this answer is reasonable because values of $\theta > 2\pi$ don't make sense, but $\theta = 2$ does.

(V14.4b) **(a)** The intensity at x is a function of x. Let's call it $I(x)$. Then at x the distance to the big light is x, and the distance to the smaller light is $1000 - x$. Therefore

$$I(x) = \frac{1000}{x^2} + \frac{125}{(1000 - x)^2}$$

(b) Find the minimum of $I(x)$ for $0 < x < 1000$.

$$I'(x) = -2000x^{-3} + 250(1000 - x)^{-3}.$$

$I'(x) = 0$ has one solution, namely, $x = \frac{1000}{3}$. By looking at the signs of $I'(x)$ you see that $I(x)$ must have a minimum. If you don't like looking at signs, you could instead look at the second derivative

$$I''(x) = 6000x^{-4} + 750(1000 - x)^{-4}$$

which is always positive.

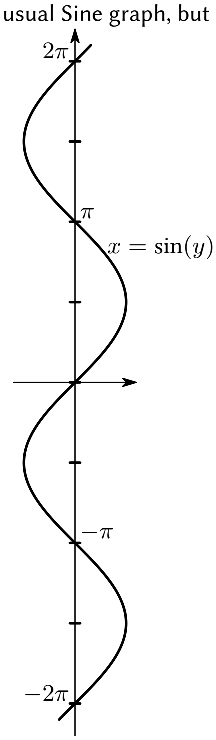
(V14.5c) The optimal radius is $r = \sqrt{50/3\pi}$, and the corresponding height is $h = 100/(3\pi r) = 100/\sqrt{150\pi}$.

(V16.1a) The straight line $y = x + 1$, traversed from the top right to the bottom left as t increases from $-\infty$ to $+\infty$.

(V16.1b) The diagonal $y = x$ traversed from left to right.

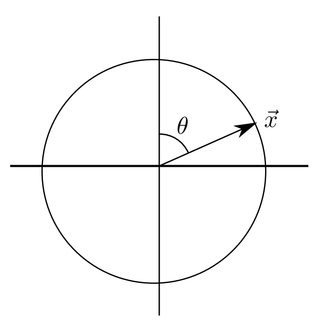
(V16.1c) The standard parabola $y = x^2$, from left to right.

(V16.1d) The graph $x = \sin y$. This is the usual Sine graph, but on its side.

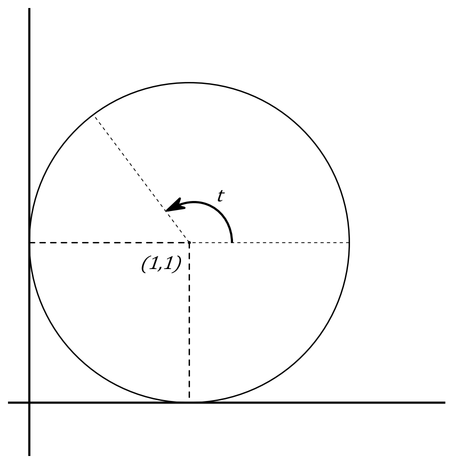


(V16.1e) We remember that $\cos 2\alpha = 1 - 2\sin^2\alpha$, so that $x(t), y(t)$ traces out a part of the parabola $y = 1 - x^2$. Looking at $x(t) = \sin t$ we see the point $(x(t), y(t))$ goes back and forth on the part of the parabola $y = 1 - 2x^2$ between $x = -1$ and $x = +1$.

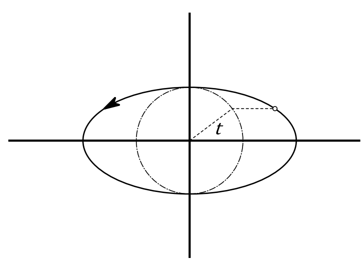
(V16.1f) The unit circle, traversed *clockwise*, 25 times every 2π time units. Note that the angle $\theta = 25t$ is measured from the y-axis instead of from the x-axis.



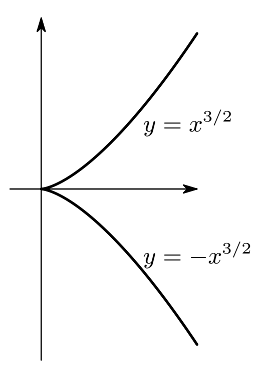
(V16.1g) Circle with radius 1 and center $(1, 1)$ (it touches the x and y axes). Traversed infinitely often in counterclockwise fashion.



(V16.1h) Without the 2 this would be the standard unit circle (dashed curve below). Multiplying the x component by 2 stretches this circle to an ellipse. So $(x(t), y(t))$ traces out an ellipse, infinitely often, counterclockwise.



(V16.1i) For each $y = t^3$ there is exactly one t, namely, $t = y^{1/3}$. So the curve is a graph (with x as a function of y instead of the other way around). It is the graph of $x = y^{2/3} = \sqrt[3]{y^2}$.



The curve is called *Neil's parabola*.

(V16.6) Since $\sin^2 t + \cos^2 t = 1$ we have $y(t) = 1 - x(t)$ on this curve. The curve is a straight line and therefore its curvature is zero.