Three proofs of the converse to Clairaut's theorem

If $P(x,y)$ and $Q(x,y)$ are two functions on a convex domain that satisfy

$$(\star) \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

then there is a function $f(x,y)$ whose partial derivatives are

$$\frac{\partial f}{\partial x} = P(x,y) \text{ and } \frac{\partial f}{\partial y} = Q(x,y).$$

1. First proof – differentiating under the integral

We claim that the function given by

$$(\dagger) \qquad f(x,y) = \int_0^x P(u,0)du + \int_0^y Q(x,v)dv$$

does the trick.

To verify this we compute the partial derivatives of this function. The y derivative follows directly from the fundamental theorem of calculus:

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}\left\{\int_0^x P(u,0)du + \int_0^y Q(x,v)dv\right\} \\ &= 0 + \frac{\partial}{\partial y}\int_0^y Q(x,v)dv \\ &= Q(x,y). \end{aligned}$$

For the x-derivative we argue as follows:

$$\begin{aligned} f_x(x,y) &= P(x,0) + \frac{\partial}{\partial x}\int_0^y Q(x,v)dv \\ &= P(x,0) + \int_0^y Q_x(x,v)dv \\ &= P(x,0) + \int_0^y P_y(x,v)dv \\ &= P(x,0) + \big[P(x,y) - P(x,0)\big] \\ &= P(x,y). \end{aligned}$$

To go from the first to the second line in this computation we used

$$\frac{\partial}{\partial x}\int_0^y Q(x,v)dv = \int_0^y \frac{\partial Q}{\partial x}(x,v)dv,$$

which is known as "differentiating under the integral." This is justified, but the proof that differentiating under the integral is allowed is not simple.

2. Second proof – switching a double integral

Our second proof starts again with the claim that

$$f(x,y) = \int_0^x P(u,0)du + \int_0^y Q(x,v)dv$$

is the function we are looking for. The y derivative again follows as in the first proof. To compute the x derivative we use a double integral to rewrite our definition of $f(x, y)$ as follows

$$\begin{aligned}
f(x,y) &= \int_0^x P(u,0)du + \int_0^y Q(x,v)dv \\
&= \int_0^x P(u,0)du + \int_0^y \left\{ Q(0,v)dv + \int_0^x \frac{\partial Q}{\partial x}(u,v)du \right\} dv \\
&= \int_0^x P(u,0)du + \int_0^y Q(0,v)dv + \int_0^y \int_0^x \frac{\partial P}{\partial y}(u,v)du\,dv && \textit{(switch order} \\
&= \int_0^x P(u,0)du + \int_0^y Q(0,v)dv + \int_0^x \int_0^y \frac{\partial P}{\partial y}(u,v)dv\,du && \textit{of integration)} \\
&= \int_0^x P(u,0)du + \int_0^y Q(0,v)dv + \int_0^x \int_0^y \big[P(u,v)\big]_{v=0}^{y} du \\
&= \int_0^x P(u,0)du + \int_0^y Q(0,v)dv + \int_0^x \int_0^y \big[P(u,y) - P(u,0)\big] du \\
&= \int_0^x P(u,y)du + \int_0^y Q(0,v)dv.
\end{aligned}$$

This new form is easy to differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left\{ \int_0^x P(u,y)du + \int_0^y Q(0,v)dv \right\} = P(x,y).$$

3. The third proof – using Green's theorem

Our last proof also rewrites the definition of $f(x, y)$ to make it easier to differentiate with respect to x. By applying Green's theorem to the rectangle

$$\mathcal{R} = \{(u,v) : 0 \le u \le x, 0 \le v \le y\}$$

we get

$$\oint_{\mathcal{C}} P(x,y)dx + Q(x,y)dy = \iint_{\mathcal{R}} \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dA = 0$$

where $\mathcal{C}$ is the counterclockwise traversed boundary of the rectangle $\mathcal{R}$. Parametrizing the edges of $\mathcal{R}$ we find that the line integral is

$$\begin{aligned}
&\oint_{\mathcal{C}} P(x,y)dx + Q(x,y)dy \\
&\qquad = \int_0^x P(u,0)du + \int_0^y Q(x,v)dv - \int_0^x P(u,y)du - \int_0^y Q(0,v)dv.
\end{aligned}$$

Therefore we have

$$f(x,y) = \int_0^x P(u,0)du + \int_0^y Q(x,v)dv = \int_0^x P(u,y)du + \int_0^y Q(0,v)dv.$$

To find f_x we differentiate the second form, and to find f_y we differentiate the first.