Iterated Integrals

An iterated integral is an expression of this form:

$$\int_a^b \left\{ \int_{c(x)}^{d(x)} f(x,y)\, dy \right\} dx$$

Usually the big braces "{" and "}" are omitted and the iterated integral is written as

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx.$$

The "inner integral" is with respect to y, and there the x variable is to be considered "frozen."

1. Compute these iterated integrals:

(***i***) $\displaystyle\int_0^1 \int_0^4 x\, dy\, dx$

(***ii***) $\displaystyle\int_0^1 \int_0^4 x\, dx\, dy$

(***iii***) $\displaystyle\int_{-1}^1 \int_0^{x^2} dy\, dx$

(***iv***) $\displaystyle\int_0^1 \int_0^y \frac{\sin y}{y}\, dx\, dy$

2. What is wrong with the iterated integral $\displaystyle\int_x^1 \left\{ \int_0^1 \sin(\pi x) dx \right\} dy$?

Is the answer a number – does it depend on x or y?

3. Is the following true or false?

For any two functions $f(x)$ and $g(y)$ one has

$$\int_0^1 \int_0^2 f(x)g(y)\, dx\, dy = \int_0^1 f(x)\, dx \times \int_0^2 g(y)\, dy\ .$$

Explain your answer (if you claim "true" give a proof, if you claim "false" give a counterexample.)

Answers

(1i) 2

(1ii) 8

(1iii) 2/3

(1iv) $\displaystyle\int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy = \int_0^\pi \frac{\sin y}{y} \cdot y\, dy = \int_0^\pi \sin y\, dy = 2.$

(2) Once you compute the inner integral

$$\int_0^1 \sin(\pi x) dx = [-\cos \pi x]_{x=0}^1 = -\cos \pi - (-\cos 0) = 2,$$

you get

$$\int_x^1 \Big\{ \int_0^1 \sin(\pi x) dx \Big\} dy = \int_x^1 2dy = [2y]_{y=x}^1 = 2(1-x).$$

The result depends on x. The x in the answer and the two x-es in the inner integral refer to different quantities. This is at best confusing, and should really never be done.

(3) This is almost true, but in fact false. The correct statement which looks like the one in the problem is

For any two functions $f(x)$ and $g(y)$ one has

$$\int_0^1 \int_0^2 f(x) g(y)\, dx\, dy = \int_0^2 f(x)\, dx \ \times \ \int_0^1 g(y)\, dy\ .$$

(what's the difference? Look at the integration bounds!) To give a counterexample for the statement in the problem, almost any two functions f and g will do, as long as f is not a constant multiple of g. For instance, if you choose $f(x) = x$, $g(y) = 1$, then you get

$$\int_0^1 \int_0^2 f(x) g(y)\, dx\, dy = \int_0^1 \int_0^2 x dx dy = 2.$$

but

$$\int_0^1 f(x)\, dx \ \times \ \int_0^2 g(y)\, dy = \int_0^1 x\, dx \ \times \ \int_0^2 dy = \frac{1}{2} \times 2 = 1.$$