MATH 221 PROJECT

Models for the dynamics of populations

1. Purpose of this project

We will build mathematical models for the rates of change of different populations from first principle arguments. We will see that the same mathematical model can be used to describe such disparate things as populations of living organisms and chemical concentrations.

2. Malthusian Population Growth

Consider a population living in isolation with an abundance of resources. Let $P(t)$ denote the size of the population at some time, t. Note that $P(0)$ is the size of the population at time zero (where we can define "time zero" to be whenever we want: 0 A.D., January 1st 2012, yesterday, etc.). We now make the *assumption* that for any small $h > 0$, representing a small window of time, the growth of the population from time t to time $t + h$ should be proportional to both

(1) the size of the population, and
(2) the size of h.

Mathematically, we assume that

$$P(t+h) \approx P(t) + \lambda P(t)h, \tag{1}$$

for some $\lambda > 0$ (the constant of proportionality). Rearranging terms yields

$$\frac{P(t+h) - P(t)}{h} \approx \lambda P(t). \tag{2}$$

The term λ can often be approximated from experimental data (How? The project for Chapter 6 will explain this in detail.). Note that the right hand side of equation (2) does not depend upon h. Further, the above is assumed to hold for *all* $h > 0$, and so letting h get very small we see that the left hand side is well approximated by the derivative of P at t (see page 20 of the notes):

$$P'(t) = \lim_{h \to 0} \frac{P(t+h) - P(t)}{h}.$$

Thus, we are led to the conclusion that the rate of change of $P(t)$ equals $\lambda P(t)$ for some $\lambda > 0$. Mathematically, we have that P satisfies the *differential equation*

$$P'(t) = \lambda P(t).$$

Note that this conclusion was reached logically from our assumptions above, which we consider to be our "first principles" for this model. If you do not believe one of the assumptions holds for a particular population, then you can not trust the conclusion!

We will not consider how to solve differential equations here (you will see these again in Math 222), however we observe that the above equation makes a startling claim:

the population will increase without bound forever!

Let's see why by returning to equation (1). For ease, we suppose that $P(0) = 1$, $\lambda = 1$, and we take $h = 1/2$. Using these values, fill in the following chart:

t	$P_{1/2}(t)$
0	1.00
0.5	1.50
1.0	2.25
1.5	
2.0	
2.5	
3.0	

where we are writing $P_{1/2}(t)$ for the size of the population since we are explicitly choosing an h of size $1/2$. Note that the population is not just growing, but the *rate* at which the population is growing is even increasing.

3. Problems

1. Make a similar chart as that above with $h = 1/4$. That is, calculate $P_{1/4}(t)$ for $t \in \{0, 0.25, 0.5, 0.75, \ldots, 3.0\}$. How do the solutions $P_{1/4}(t)$ and $P_{1/2}(t)$ compare for

$$t \in \{0.5, 1, 1.5, 2, 2.5, 3.0\}?$$

Why do you think this is happening?

4. The Logistic Growth Model

The fact that the previous model predicts continued growth, without bound, is a serious shortcoming for many reasonable situations. For example, if there is a limited amount of natural resources, unsustained growth is clearly not realistic. The following twist on the previous model fixes the problem.

We still consider a small time window $[t, t+h]$ and ask how the population changes over that time period. Now we make the following assumptions:

(1) There is a "carrying capacity" $K > 0$ such that if $X(t) > K$, the population should decrease and if $X(t) < K$, then the population should increase.
(2) The further from K the population is, the stronger the gain/decrease.

One such model satisfying these reasonable ideas is

$$X(t+h) \approx X(t) + \lambda X(t)\left(1 - \frac{X(t)}{K}\right)h. \tag{3}$$

Similarly to the Malthusian model, the associated *differential equation* satisfied by X is

$$X'(t) = \lambda X(t)\left(1 - \frac{X(t)}{K}\right). \tag{4}$$

The model described here is called the *logistical growth model.*

Returning to equation (3), complete the following charts for the different $X(0)$ assuming that $\lambda = 1$, $K = 4$, and $h = 0.5$.

t	$X(t)$
0	1.00
0.5	1.375
1.0	1.826
1.5	
2.0	
2.5	
3.0	

t	$X(t)$
0	7.00
0.5	4.375
1.0	4.170
1.5	
2.0	
2.5	
3.0	

What do you think each $X(t)$ converge to for larger and larger t?

1. What do you think each $X(t)$ converge to for larger and larger t? Why?

2. More generally, consider $X(t)$ that satisfies (4) for some $\lambda > 0$ and $K > 0$. What is the sign of $X'(t)$ when $X(t) > K$? What about when $X(t) < K$? Using these facts, argue what will happen to $X(t)$ as t gets very large. Try to be persuasive. (Note: proving this predicted behavior is a subject for a future mathematic course on *dynamical systems*.)

3. Check that that if X satisfies (3), then it satisfies the assumptions (a) and (b) above.

4. If X satisfies (3) for all $h > 0$, why should it satisfy the differential equation (4)?

5. Chemistry

We now change topics and consider the chemical reaction

$$A + B \to 2A.$$

Compare this with the chemical reaction found on page 21 of the notes. Note that every time the above chemical reaction occurs, we gain a molecule of A and lose a molecule of B. This implies that the total number of molecules, $A + B$, is *conserved*. Thus, if $[A](t)$ and $[B](t)$ represent the concentration of A and B at time t, then

$$[A](t) + [B](t) = M$$

implying

$$[B](t) = M - [A](t). \tag{5}$$

We now make the following assumptions, which are our *first principles*. The number of times the reaction happens in the the time window $[t, t+h]$ is proportional to

(1) the product of $[A]$ and $[B]$, and
(2) the size of h.

6. Problems

1. Use the first principles above to find a difference equation, similar to (3), for $[A](t)$. Note that, due to (5), $[B](t)$ should not appear in the equation.

$$[A](t+h) = [A](t) +$$

2. Using your result from Exercise 6.1, derive a differential equation for $[A](t)$:

$$\frac{d[A](t)}{dt} =$$

Compare the differential equation to (4).

Report Instructions

Using complete sentences, write a few paragraphs summarizing the project. Next, write up your solutions to all of the exercises. Be sure to include all pertinent information from the project itself. That is, a reader should be able to sit down with only your report and be able to understand what you are doing, and the conclusions you have drawn.