NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

CZ3005 Artificial Intelligence

Inference of Propositional Logic

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Example

Express the following English statements in the language of propositional logic

1. It rains in July
2. The book is not costly
3. If it rains today and one does not carry umbrella, he will be drenched

Solution

1. A
2. $\neg A$
3. $A \wedge \neg B \Rightarrow C$

Example

If P is true and Q is true, are the following true or false?

1. $P \Rightarrow Q$
2. $(\neg P \vee Q) \Rightarrow Q$
3. $(\neg P \vee Q) \Rightarrow P$
4. $P \vee \neg P \Rightarrow T$

Solution

1. True
2. True
3. True
4. True

Logical Equivalent

Two sentences are logically equivalent (≡) if they are true in the same set of model

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) \quad \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) \quad \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) \quad \text{de Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) \quad \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

Example

$\neg(A \wedge B) \equiv \neg A \vee \neg B$

A	B	A ∧ B	¬(A ∧ B)	¬A∨¬B	LHS = RHS
1	1	1			
1	1	1			
1	0	0			
1	0	0			
0	1	0			
0	1	0			
0	0	0			
0	0	0			

Solution

¬(A∧B)	¬A∨¬B	LHS=RHS
0	0	Yes
0	0	Yes
1	1	Yes
1	1	Yes
1	1	Yes
1	1	Yes
1	1	Yes
1	1	Yes

Example

$$P \Leftrightarrow Q \equiv (P \land Q) \lor (\neg P \land \neg Q)$$

$$
\begin{aligned}
&= (P \Rightarrow Q) \land (Q \Rightarrow P) \\
&= (\neg P \lor Q) \land (\neg Q \lor P) \\
&= (\neg P \land \neg Q) \lor (\neg P \land P) \lor (Q \land \neg Q) \lor (P \land Q) \\
&= (P \land Q) \lor (\neg P \land \neg Q)
\end{aligned}
$$

Exam Question

$$\neg(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B)$$

- Using logical law, prove the equivalence of the two propositions
- Using truth table, prove the equivalence of the two propositions

Solution

$$\neg(A \Rightarrow B) \Leftrightarrow (A \land \neg B)$$

$$= \neg(\neg A \lor B) = A \land \neg B$$

Solution

$\neg(A \Rightarrow B)$	$(A \wedge \neg B)$	LHS=RHS
0	0	Yes
0	0	Yes
1	1	Yes
0	0	Yes

Entailment

The relation of logical **entailment** between sentences – a sentence follows logically from another sentence.

$$\alpha \models \beta$$

- α entails the sentence β.
- If α is true, then β must be true.

Inference

$$KB \models \alpha$$

KB is a knowledge base consisting of a set of sentences expressed in a knowledge representation language

Rule of Inference

- Classic rules of inference
  - Implication-Elimination, or *Modus Ponens*
    - $$\frac{\alpha \Rightarrow \beta, \ \alpha}{\beta}$$

e.g. Cloudy Λ Humid $\Rightarrow$ Rain

Cloudy Λ Humid

|= Rain

Classic rules of inference

- And-Elimination
  - $$\frac{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}{\alpha_i}$$

  e.g. Cloudy $\wedge$ Humid

  Cloudy $\Rightarrow$ NoSun

- And-Introduction
  - $$\frac{\alpha_1, \alpha_2, \ldots, \alpha_n}{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}$$

  e.g. Cloudy, Humid

  Cloudy $\wedge$ Humid $\Rightarrow$ Rain

- Or-Introduction
  - $$\frac{\alpha_i}{\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n}$$

- Double-Negation-Elimination
  - $$\frac{\neg\neg\alpha}{\alpha}$$

Example

- When it is raining, the ground is wet. (1)
- When the ground is wet, it is slippery.(2)
- It is raining. (3)

Prove that “it is slippery”.

Solution

Rule Base:

1) $A \Rightarrow B$

2) $B \Rightarrow C$

3) A

Using modus ponens, we can prove C

$(1)\&(3) \vdash B$ (4)

$(4)\&(2) \vdash C$

Exam Question

(i) If Andy works hard and Andy is smart, Andy passes the subject

(ii) Andy works hard

(iii) Andy is smart

- Using modus ponens, prove that Andy passes the subject

Solution

Rule Base

(1) $A \wedge B \Rightarrow C$

(2) A

(3) B

Using modus ponens, we can prove C

(2) & (3) $\vdash A \wedge B$ (4)

(4) & (1) $\vdash C$

Resolution

- Unit Resolution $$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k}$$

ℓ_i and m are complementary literals (i.e., one is the negation of the other).

  - Takes a clause (a disjunction of literals) and a literal and produce a new clause.

- Full Resolution
  - Input: $P_1 \vee P_2 \vee \ldots \vee P_n$, $\neg P_1 \vee Q_2 \vee \ldots \vee Q_m$
  - Output: $P_2 \vee \ldots \vee P_n \vee Q_2 \vee \ldots \vee Q_m$

Conversion to Clausal Form

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) && \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) && \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) && \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) && \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) && \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) && \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) && \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) && \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

Conversion to Clausal Form

Consider the following sentence

$$\neg(A \rightarrow B) \vee (C \rightarrow A)$$

1. eliminate implication sign

$$\neg(\neg A \vee B) \vee (\neg C \vee A)$$

2. eliminate double negation and reduce scope of not signs (De Morgan Law)

$$(A \wedge \neg B) \vee (\neg C \vee A)$$

3. convert to conjunctive normal form by using distributive and associative law

$$(A \wedge \neg B) \vee (\neg C \vee A) ==>$$
$$(A \vee \neg C \vee A) \wedge (\neg B \vee \neg C \vee A) ==>$$
$$(A \vee \neg C) \wedge (\neg B \vee \neg C \vee A)$$

4. Get Set of Clauses

$(A \vee \neg C)$

$(\neg B \vee \neg C \vee A)$

Example

- If it rains, Joe brings his umbrella.
- If Joe brings an umbrella, he does not get wet.
- If it does not rain, Joe does not get wet

Prove that Joe does not get wet.

Solution

(1) $\neg A \vee B$

(2) $\neg B \vee \neg C$

(3) $A \vee \neg C$

(4) C

Using Resolution by Refutation

(3) & (4) $\vdash A$ (5)

(2) & (4) $\vdash \neg B$ (6)

(1) & (6) $\vdash \neg A$ (7)

(5) & (7) $\vdash \emptyset$

Exam Question

(i) If Andy works hard and Andy is smart, Andy passes the subject

(ii) Andy works hard

(iii) Andy is smart

- Using resolution by refutation, prove that Andy passes the subject

Solution

(1) $\neg A \vee \neg B \vee C$

(2) A

(3) B

(4) $\neg C$

Using resolution by refutation, we can prove C

(1)&(2)&(3)&(4) $\vdash \emptyset$