NANYANG TECHNOLOGICAL UNIVERSITY
School of Computer Science and Engineering

CZ3005 – Artificial Intelligence

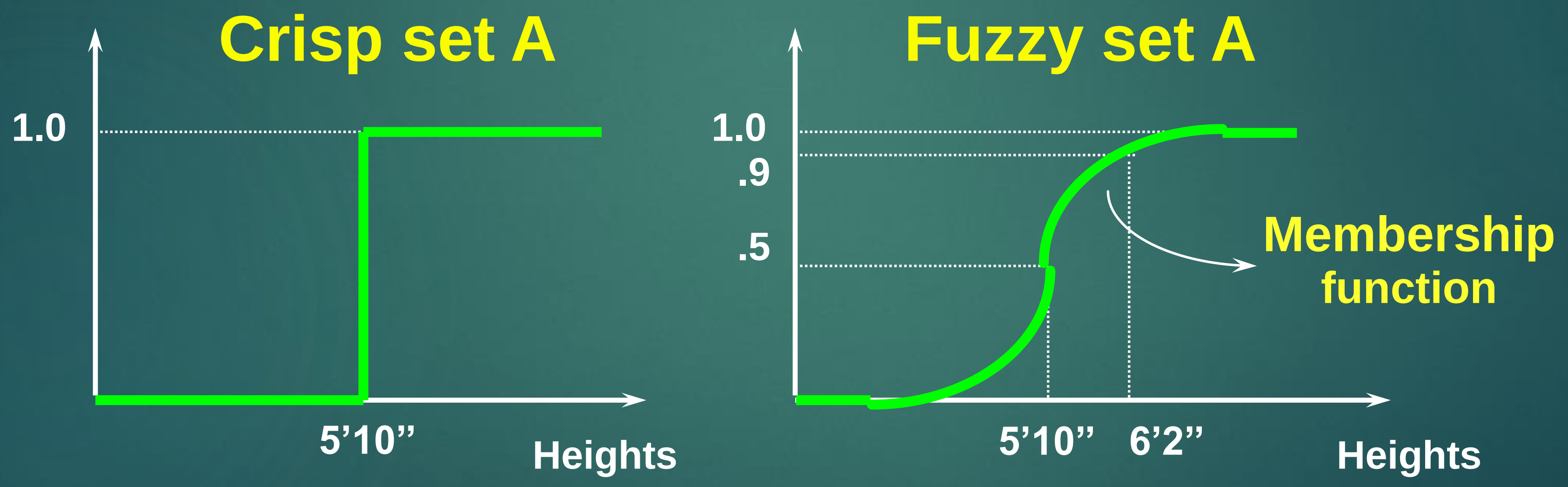
FUZZY LOGIC

Asst Prof. Mahardhika Pratama – mpratama@ntu.edu.sg

Fuzzy Sets

Sets with fuzzy boundaries

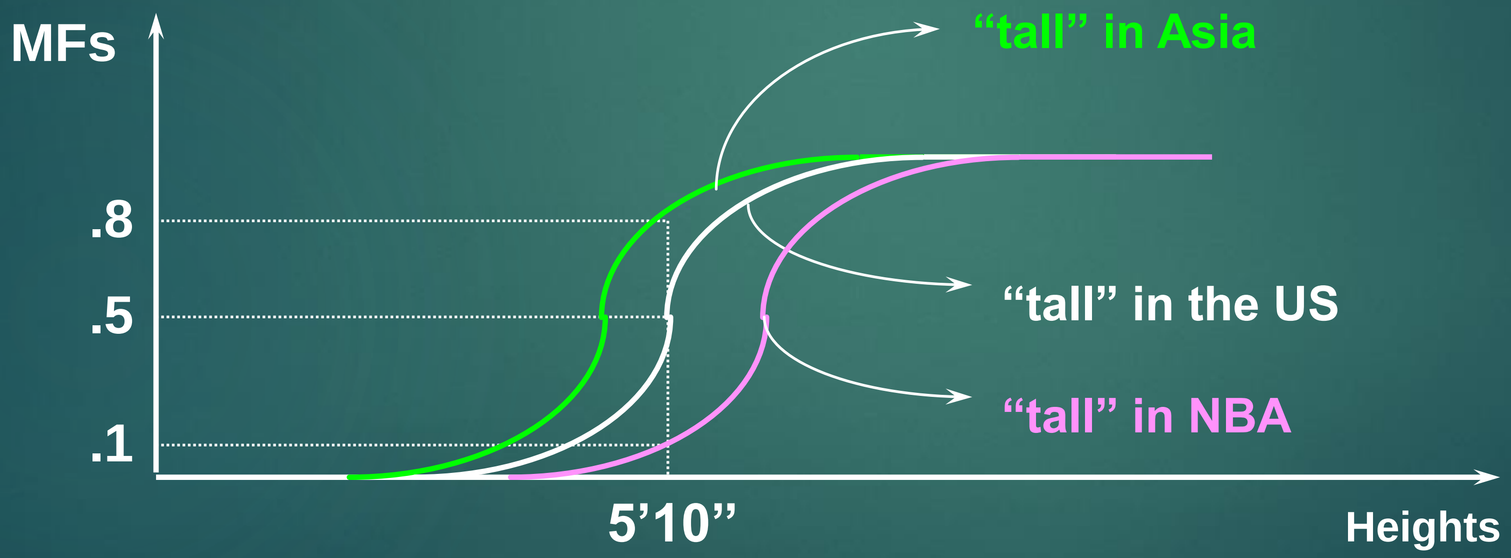
A = Set of tall people



Membership Functions (MFs)

Characteristics of MFs:

- **Subjective measures**
- **Not probability functions**



Fuzzy Sets

Formal definition:

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$\underbrace{A}_{\text{Fuzzy set}} = \{(x, \underbrace{\mu_A(x)}_{\text{Membership function (MF)}}) \mid x \in \underbrace{X}_{\text{Universe or universe of discourse}}\}$$

A fuzzy set is totally characterized by a membership function (MF).

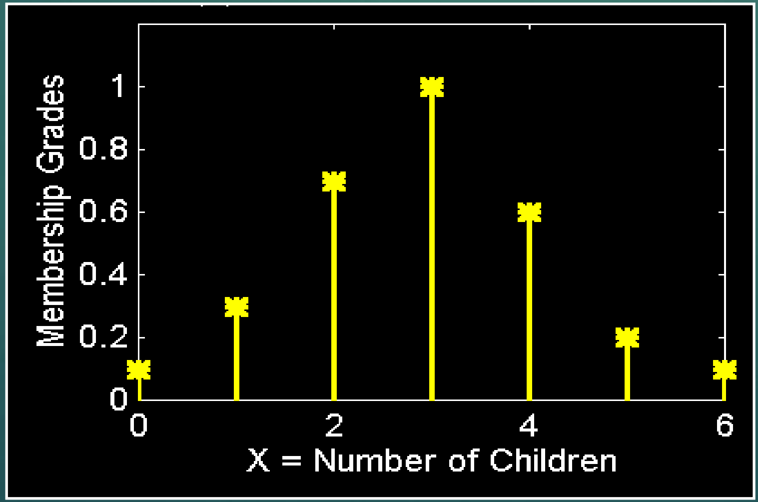
Fuzzy Sets with Discrete Universe

Fuzzy set C = "desirable city to live in"

- X = {SF, Boston, LA} (discrete and nonordered)
- C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

Fuzzy set A = "sensible number of children"

- X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)
- A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



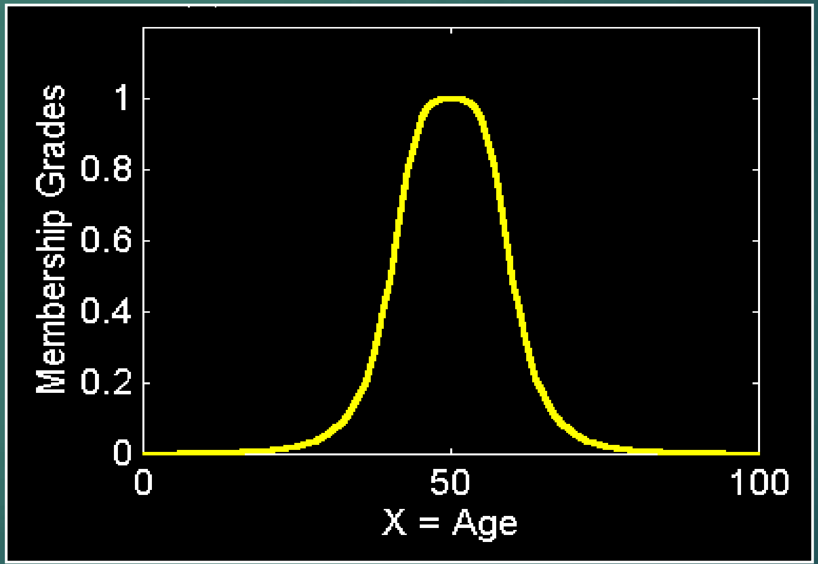
Fuzzy Sets with Continuous Universe

Fuzzy set B = "about 50 years old"

X = Set of positive real numbers (continuous)

B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



Alternative Notation

A fuzzy set A can be alternatively denoted as follows:

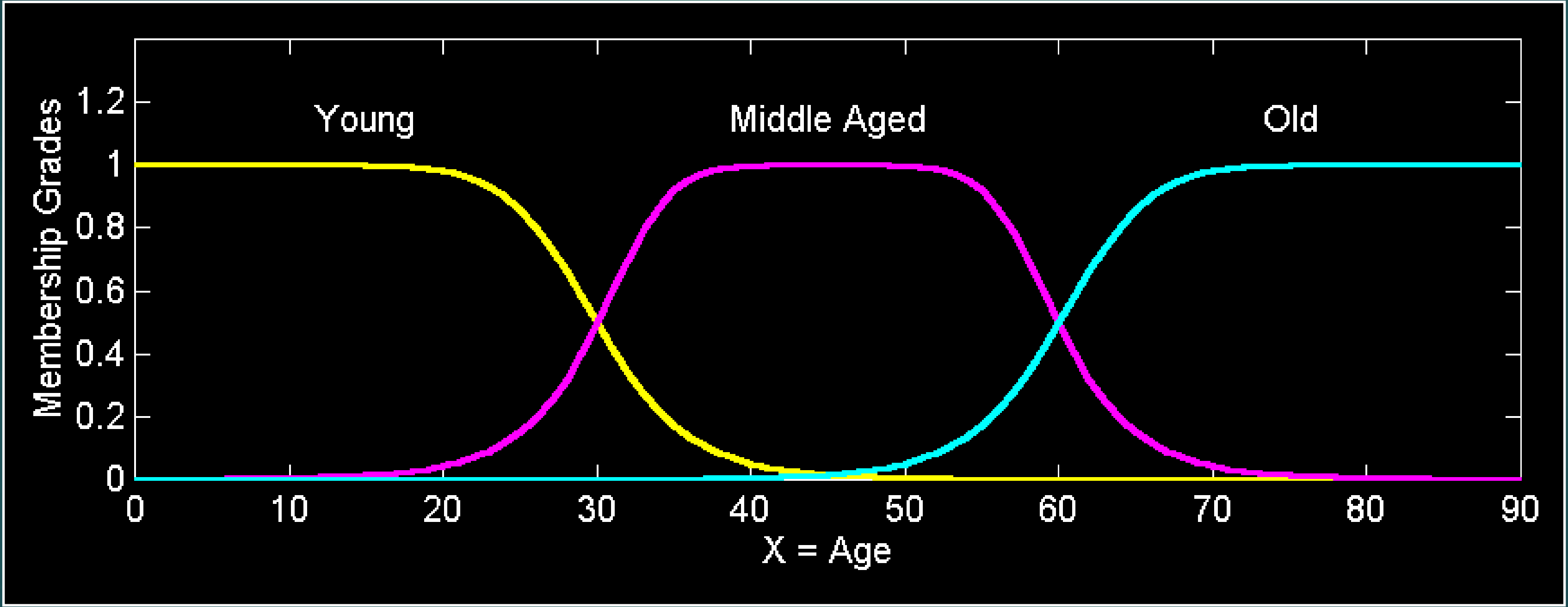
X is discrete → $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

X is continuous → $$A = \int_X \mu_A(x) / x$$

Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

Fuzzy Partition

Fuzzy partitions formed by the linguistic values "young", "middle aged", and "old":



lingmf.m

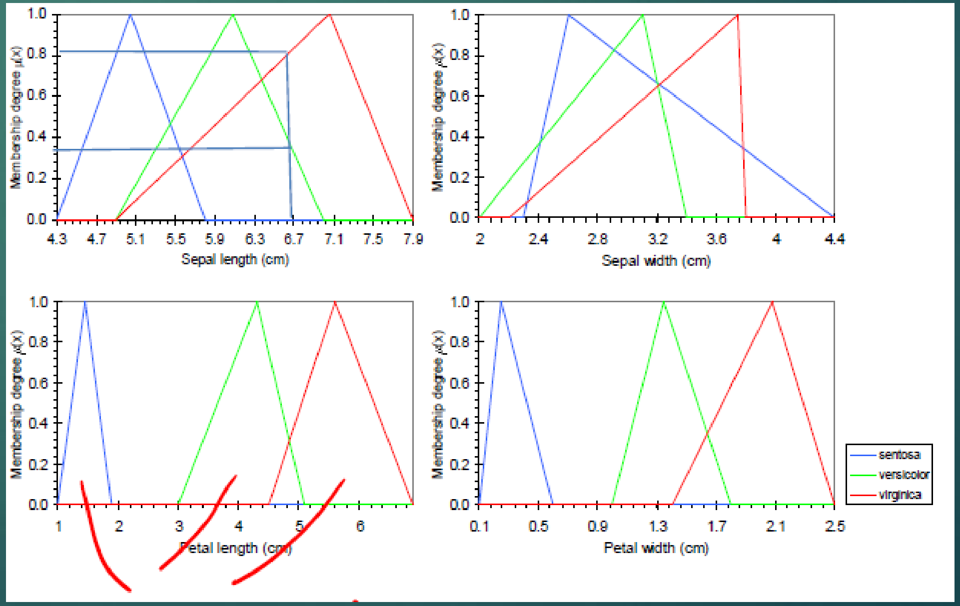
Non-Pseudo Partitioning

- Let c the set of membership functions that fuzzy partition the space of x.
- This fuzzy space is non Pseudo-ly partitioned when:

Each fn is normal and convex $sup_x\left(\mu_{i,i\epsilon c}(X)\right) = 1$

Summation of mf value at X is 1

$$\sum_{i=1}^{c} \mu_{i,i\epsilon c} \neq 1$$



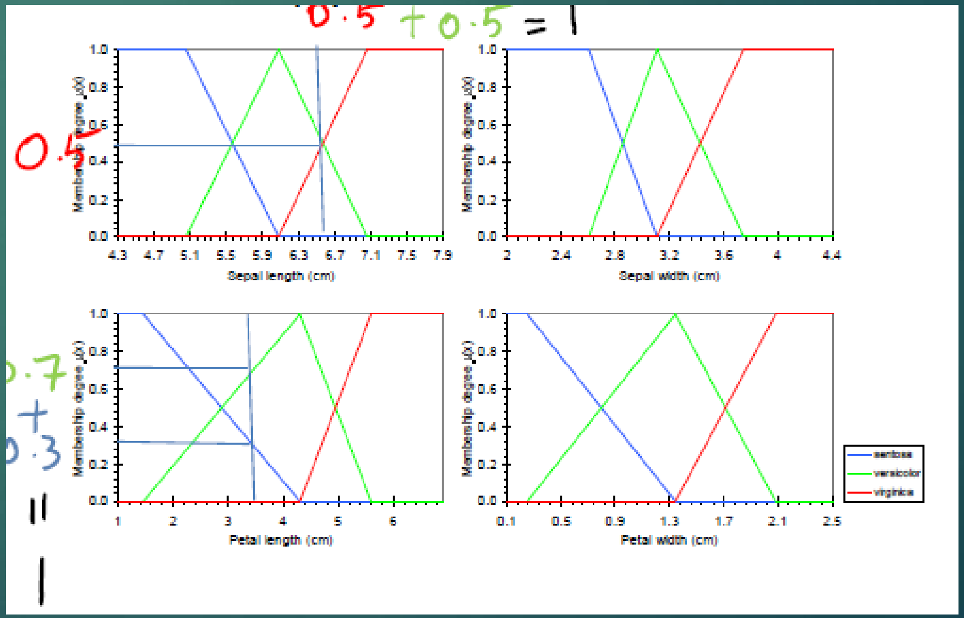
Pseudo Partitioning

- Let c the set of membership functions that fuzzy partition the space of x.
- This fuzzy space is Pseudo-ly partitioned when:

Each fn is normal and convex $sup_x\left(\mu_{i,i\epsilon c}(X)\right) = 1$

Summation of mf value at X is 1

$$\sum_{i=1}^{c} \mu_{i,i\epsilon c} = 1$$



More Definitions

Support

Core

Normality

Crossover points

Fuzzy singleton

α-cut, strong α-cut

Convexity

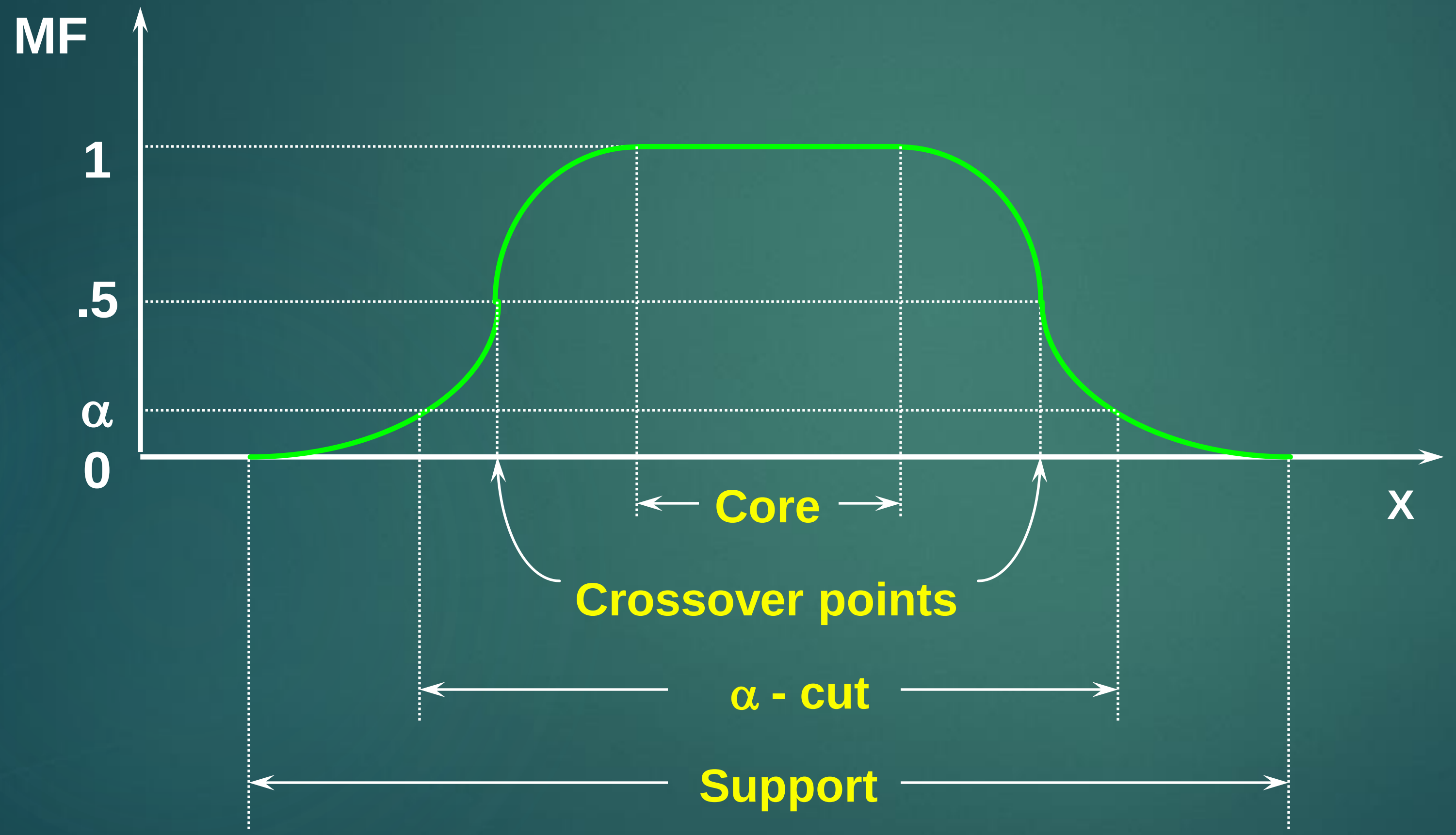
Fuzzy numbers

Bandwidth

Symmetricity

Open left or right, closed

MF terminology



Set-Theoretic Operations

Subset:

$$A \subseteq B \Leftrightarrow \mu_A \le \mu_B$$

Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

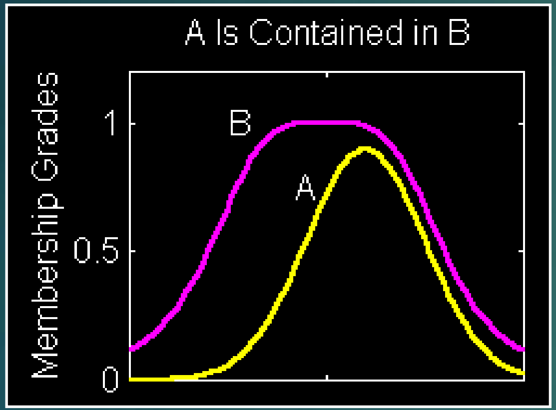
Union:

$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Set-Theoretic Operations



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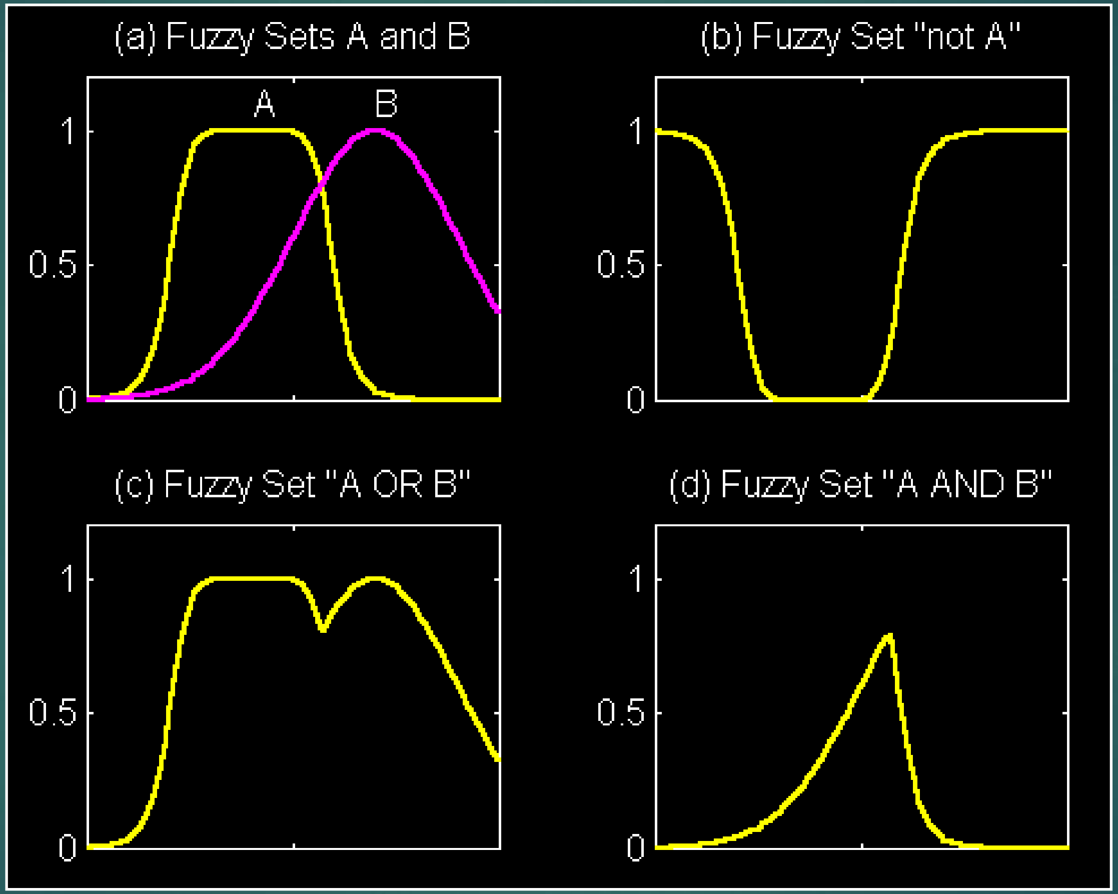

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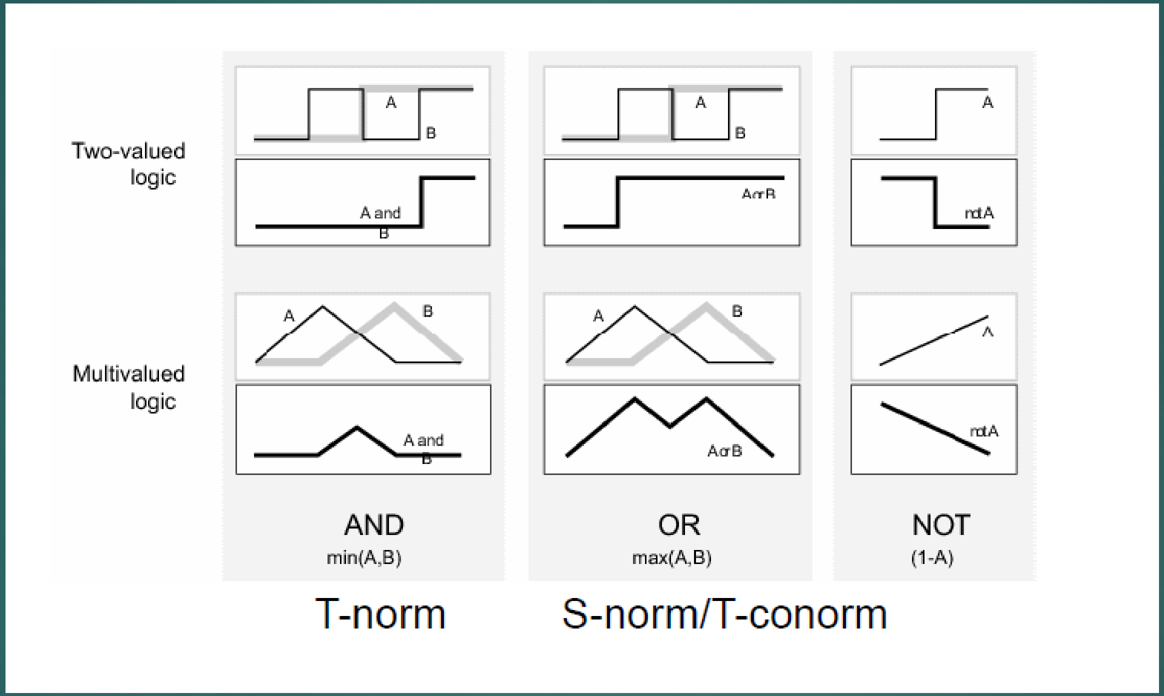
Fuzzy Logical Operation



MF Formulation

Triangular MF:

$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:

$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

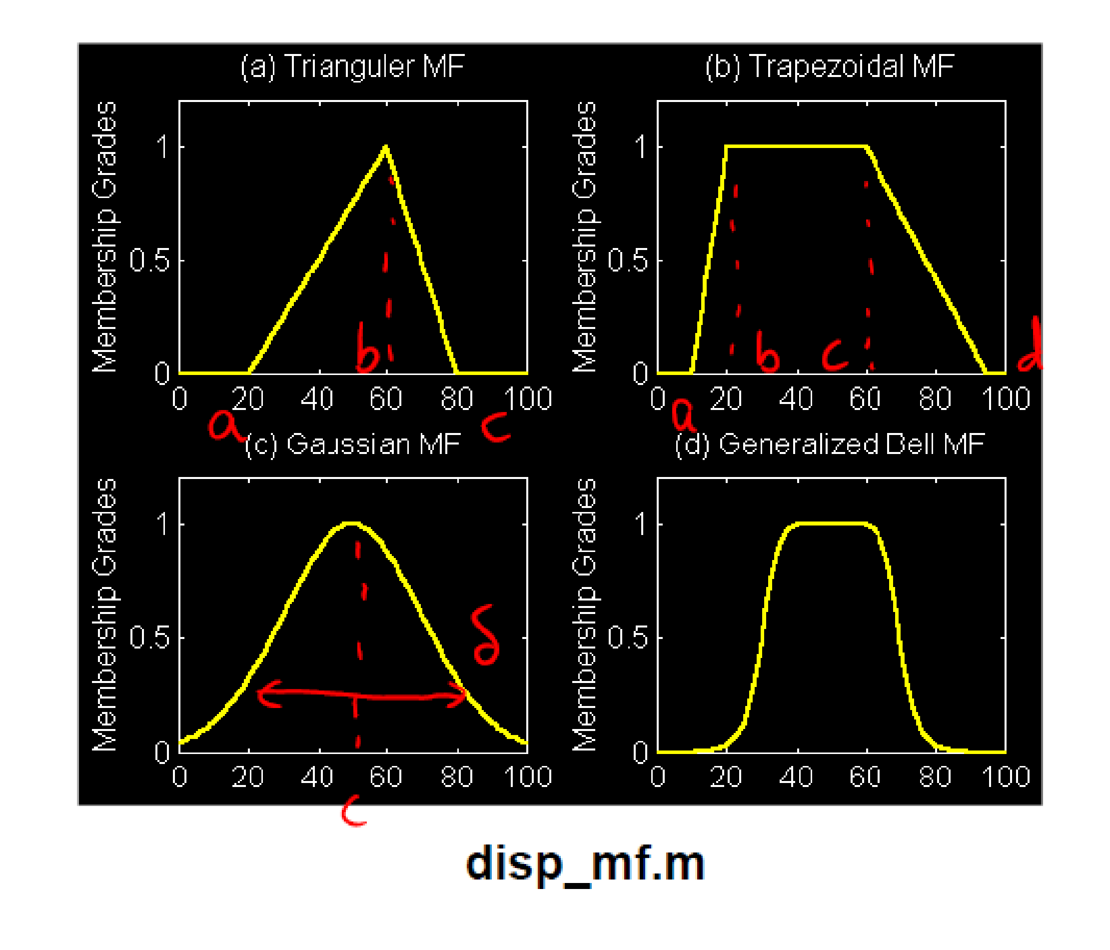
Gaussian MF:

$$gaussmf(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Generalized bell MF:

$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

MF Formulation



Linguistic Hedge - Modifiers

- Linguistic hedge/modifiers are operations that modify the meaning of a term – fuzzy label (fuzzy set).
  - "very tall", the word very modifies "Tall" which is a fuzzy set

- Others modifiers are:
  - "more or less", "possibly" and "definetely"

Linguistic Hedge - Modifiers

- Very $a = a^2$
- More or less $a = a^{0.5}$
- Extremely $a = a^3$
- Slightly $a = a^{0.333}$
- Somewhat a =morl a and not slightly a

Example

$$young = [{}^{1}/_{0}, {}^{0.6}/_{20}, {}^{0.1}/_{40}, {}^{0}/_{60}, {}^{0}/_{80}]$$

$$\text{very } young = [{}^{1}/_{0}, {}^{0.36}/_{20}, {}^{0.01}/_{40}, {}^{0}/_{60}, {}^{0}/_{80}]$$

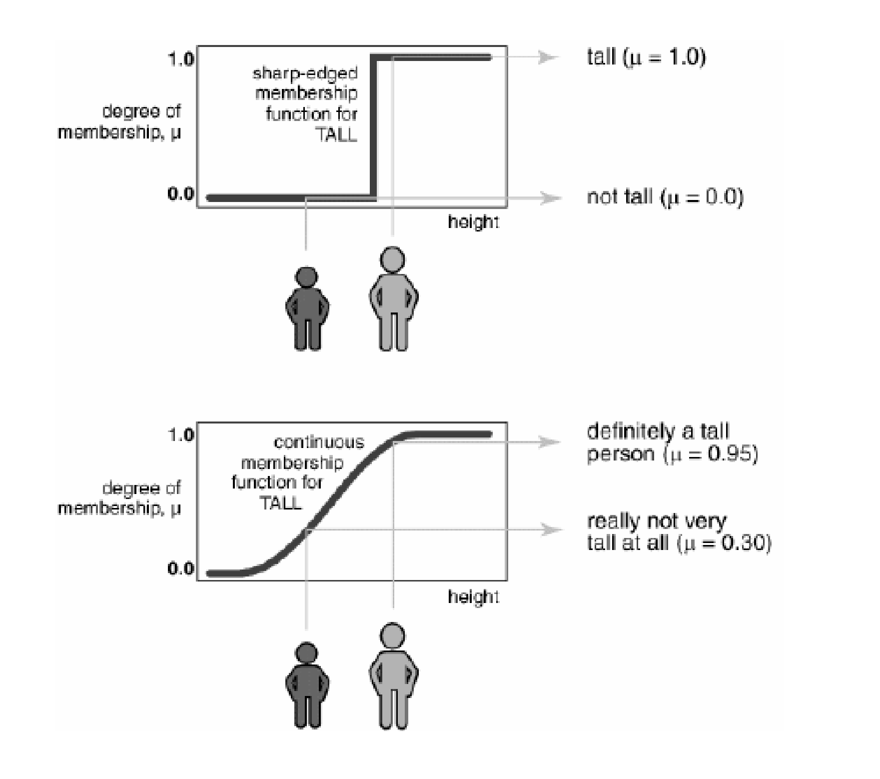
Summary of Membership Function

- Fuzzy sets allow the description of vague concepts (eg SLOW, MEDIUM and FAST) for a fuzzy variable (eg SPEED)
- This provides the semantics (concepts) to linguistic rules involving fuzzy variables.

eg. The SPEED is FAST.

- The fuzzy set admits the possibility of partial memberships in it. (eg. Friday is sort of a weekend day, the weather is rather hot).

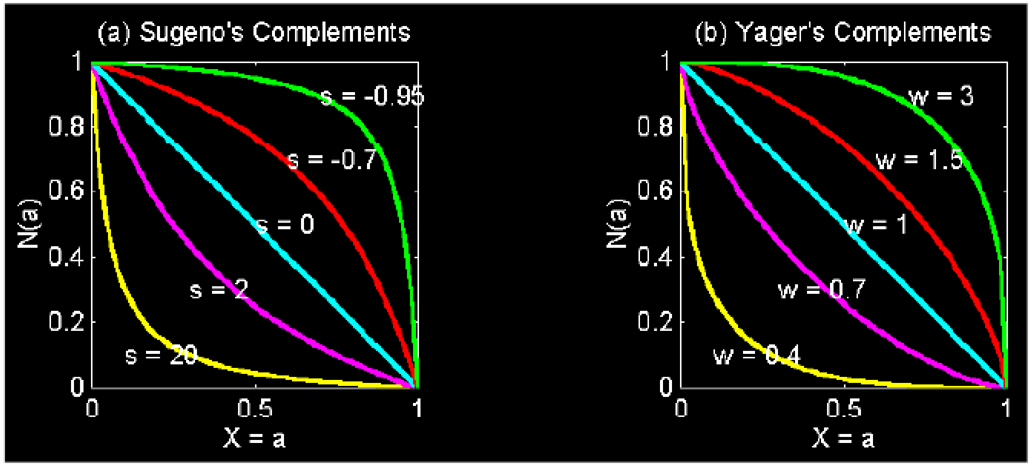
Semantic Meaning and Partial Membership



Fuzzy Complement

- Fuzzy Complement: The complement of A denoted by $\overline{A}$ or NOT A and is defined by the membership function $\mu_{\overline{A}} = 1 - \mu_A(x)$
- The general requirements
  - Boundary: N(0)=1 and N(1)=0
  - Monotonicity: N(a)>N(b) if a<b
  - Involution: N(N(a))=a
- Two types of fuzzy complements:
  - Sugeno's complement: $N_s(a) = \frac{1-a}{1+sa}$
  - Yager's complement: $N_w(a) = (1 - a^w)^{1/w}$

Fuzzy Complement



negation.m

Fuzzy Intersection: T-norm

Basic requirements:

- **Boundary: $T(0, 0) = 0$, $T(a, 1) = T(1, a) = a$**
- **Monotonicity: $T(a, b) < T(c, d)$ if $a < c$ and $b < d$**
- **Commutativity: $T(a, b) = T(b, a)$**
- **Associativity: $T(a, T(b, c)) = T(T(a, b), c)$**

Four examples:

- **Minimum: $T_m(a, b)$**
- **Algebraic product: $T_a(a, b)$**
- **Bounded product: $T_b(a, b)$**
- **Drastic product: $T_d(a, b)$**

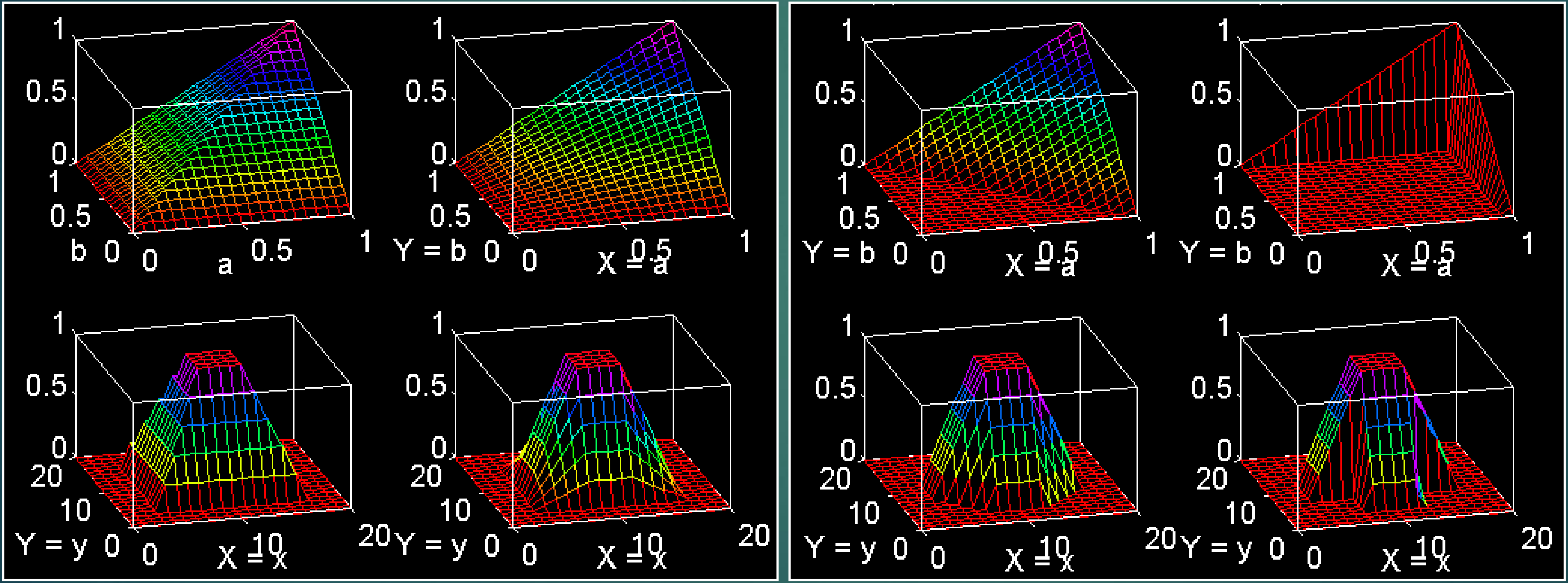
T-norm Operator

Minimum: $T_m(a, b)$

Algebraic product: $T_a(a, b)$

Bounded product: $T_b(a, b)$

Drastic product: $T_d(a, b)$



tnorm.m

Fuzzy Union: T-conorm or S-norm

Basic requirements:

- **Boundary: $S(1, 1) = 1$, $S(a, 0) = S(0, a) = a$**
- **Monotonicity: $S(a, b) < S(c, d)$ if $a < c$ and $b < d$**
- **Commutativity: $S(a, b) = S(b, a)$**
- **Associativity: $S(a, S(b, c)) = S(S(a, b), c)$**

Four examples (page 38):

- **Maximum: $S_m(a, b)$**
- **Algebraic sum: $S_a(a, b)$**
- **Bounded sum: $S_b(a, b)$**
- **Drastic sum: $S_d(a, b)$**

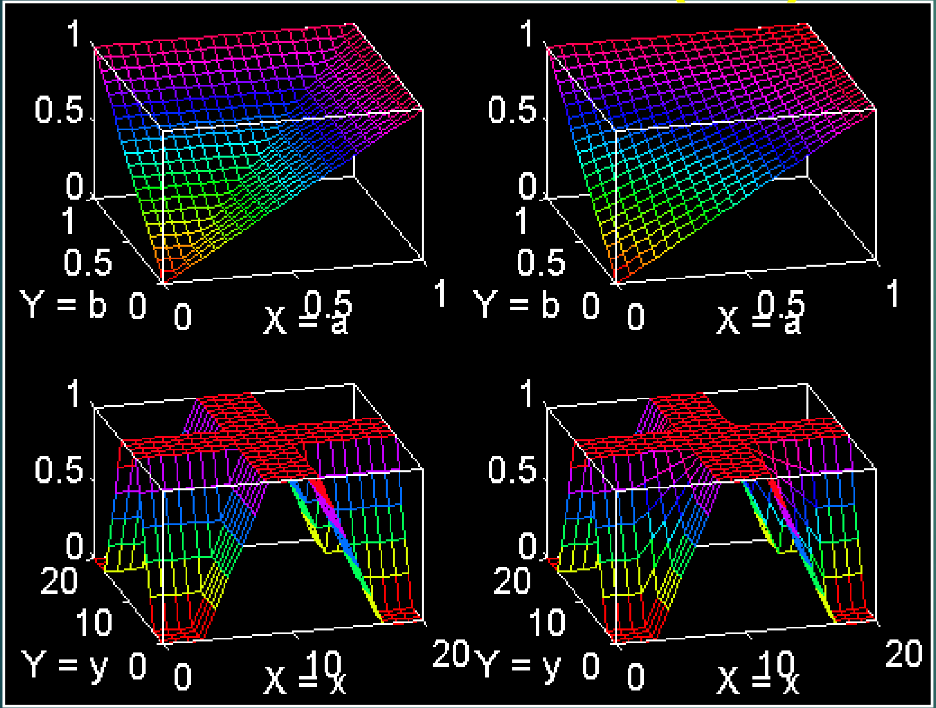
T-conorm or S-norm

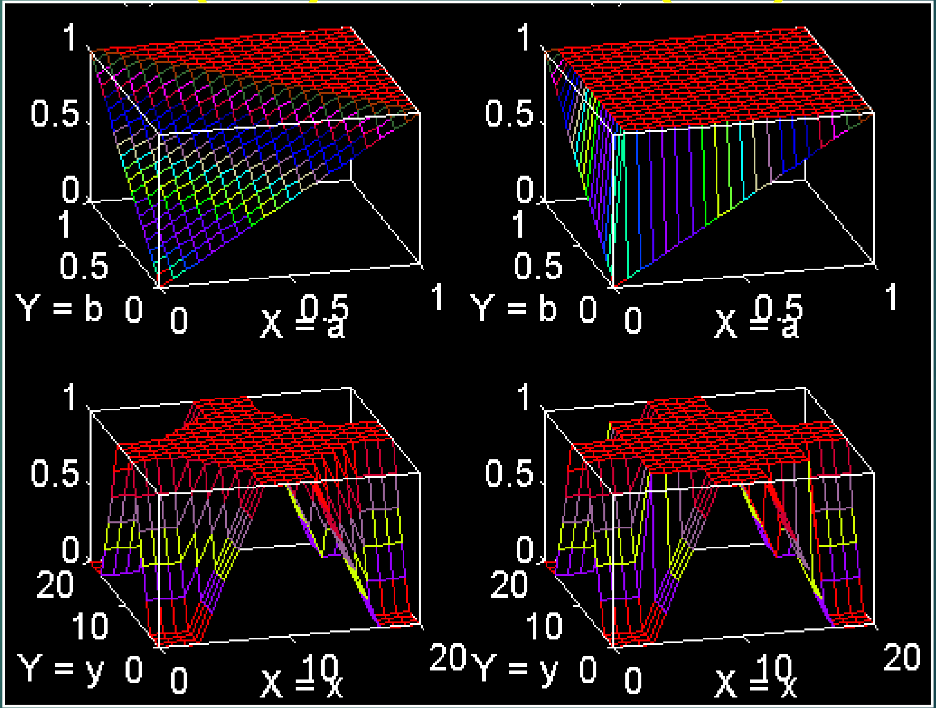
Maximum: $S_m(a, b)$

Algebraic sum: $S_a(a, b)$

Bounded sum: $S_b(a, b)$

Drastic sum: $S_d(a, b)$





tconorm.m

Fuzzy IF-Then Rules

General format:

If x is A then y is B

Examples:

- If pressure is high, then volume is small.
- If the road is slippery, then driving is dangerous.
- If a tomato is red, then it is ripe.
- If the speed is high, then apply the brake a little.

Fuzzy Reasoning

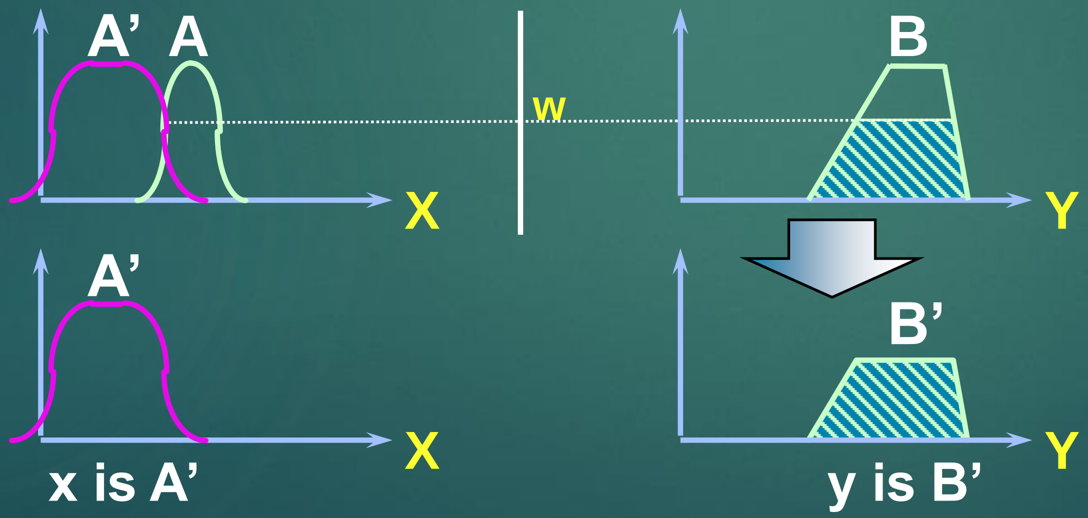
Single rule with single antecedent

Rule: if x is A then y is B

Fact: x is A’

Conclusion: y is B’

Graphic Representation:



Fuzzy Reasoning

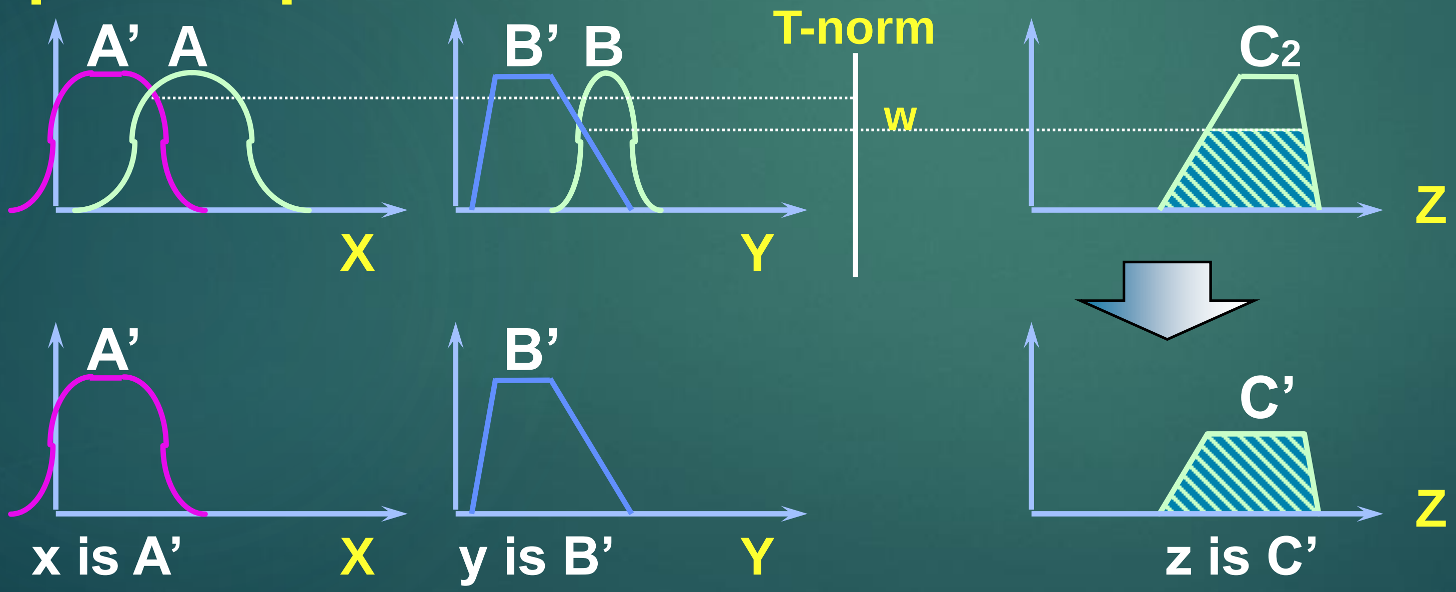
Single rule with multiple antecedent

Rule: if x is A and y is B then z is C

Fact: x is A' and y is B'

Conclusion: z is C'

Graphic Representation:



Fuzzy Reasoning

Multiple rules with multiple antecedent

Rule 1: if x is A_1 and y is B_1 then z is C_1

Rule 2: if x is A_2 and y is B_2 then z is C_2

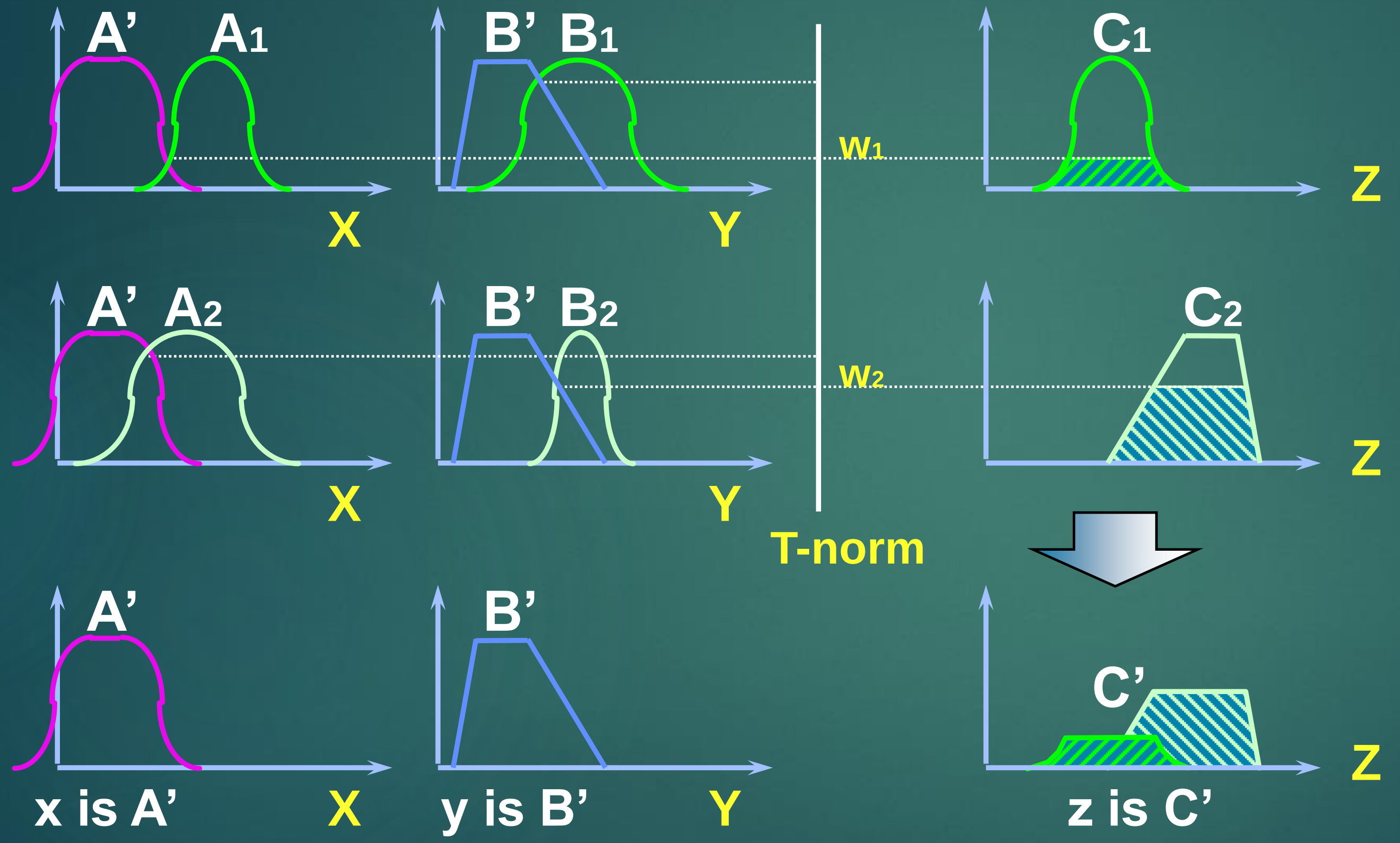
Fact: x is A' and y is B'

Conclusion: z is C'

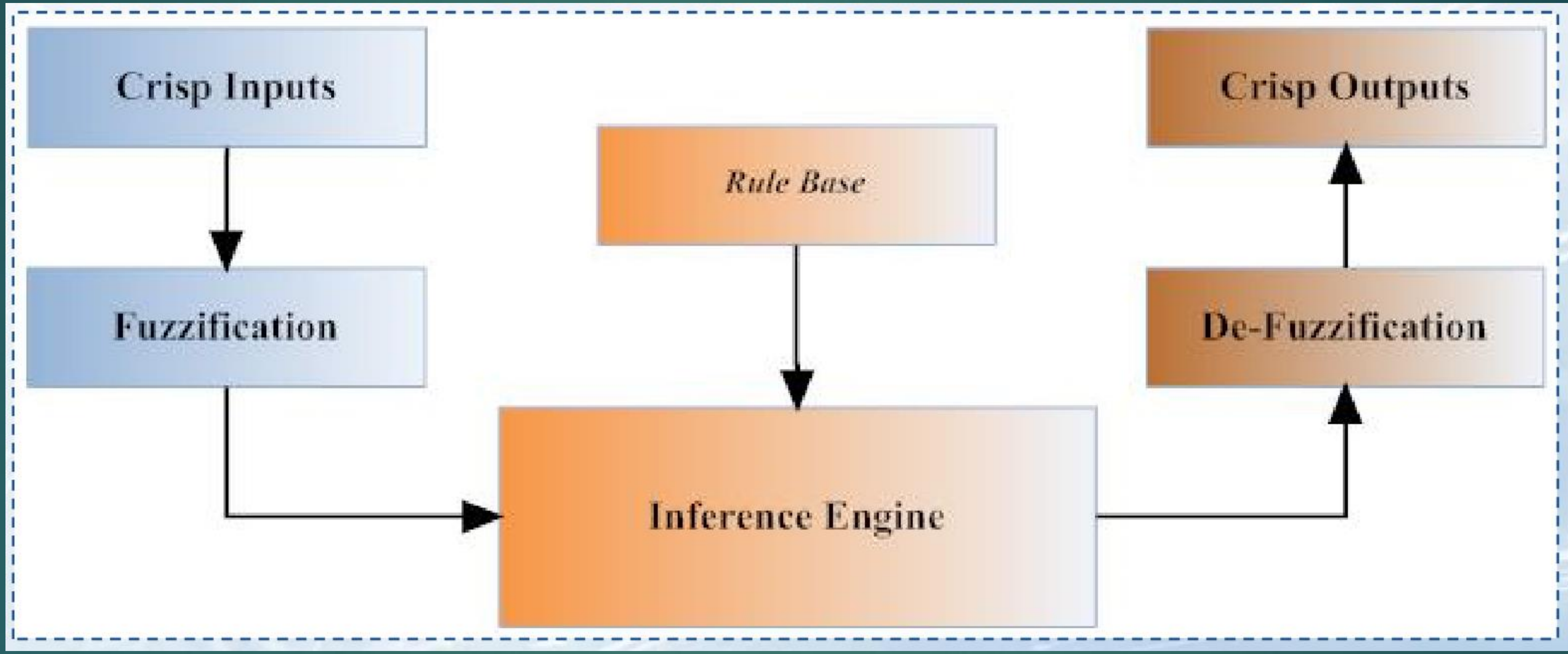
Graphic Representation: (next slide)

Fuzzy Reasoning

Graphics representation:



Fuzzy Inference System

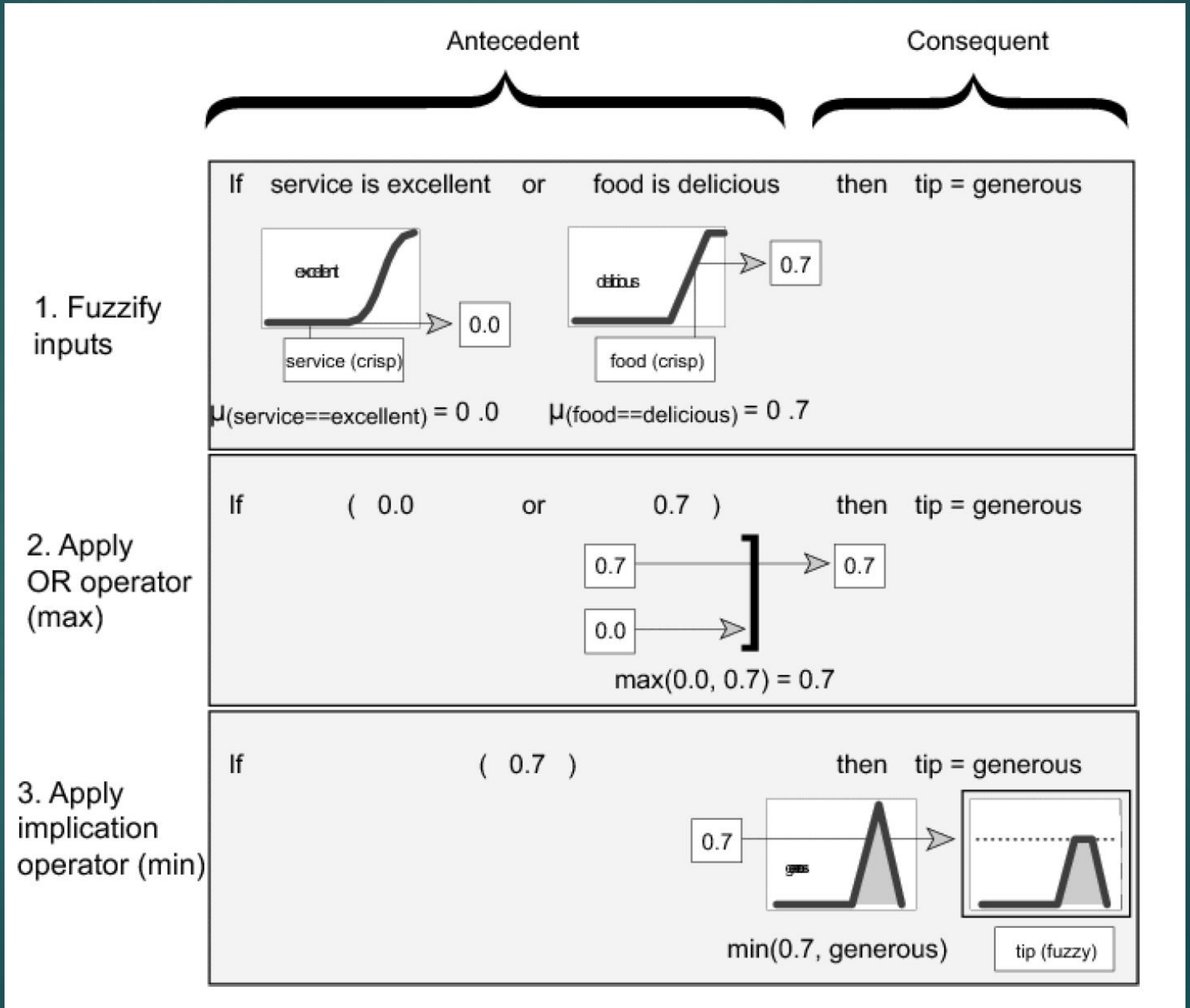


- The input of fuzzy rule-based system should be given by fuzzy inputs and therefore we have to fuzzify Crisp inputs
- The output of fuzzy system is always a fuzzy set, so we have to defuzzify to obtain crisp outputs

Operations of FRBS

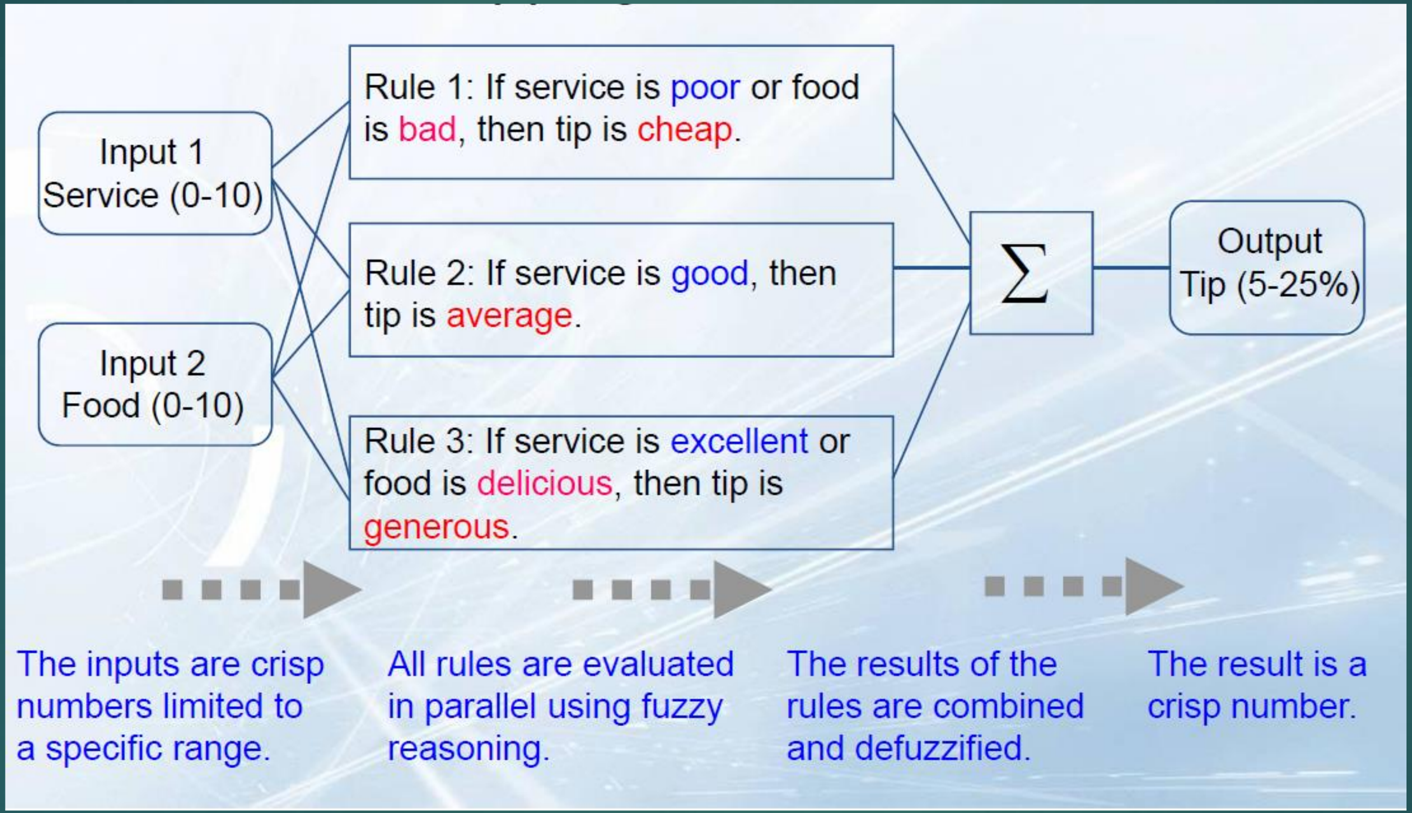
1. First, measurements are taken of all variables from the process.
2. Next, these measurements are converted into appropriate fuzzy sets to express measurement uncertainties - fuzzification.
3. The fuzzified measurements are then used by the inference engine to evaluate the control rules stored in the fuzzy rule base. - (fuzzy rules defined with fuzzy (linguistic) variables using fuzzy labels – fuzzy sets)
4. The result of this evaluation is one or several fuzzy sets defined on the universe of possible control actions. This fuzzy set is then converted, in the final step of the cycle, into a single crisp value or a vector of values which best represents the resultant fuzzy set or sets – defuzzification.

FIS - Example



Antecedent
Consequent
If service is excellent or food is delicious then tip = generous
1. Fuzzify inputs
0.0
0.7
service (crisp)
food (crisp)
μ(service==excellent) = 0 .0
μ(food==delicious) = 0 .7
If (0.0 or 0.7) then tip = generous
2. Apply OR operator (max)
0.7
0.0
0.7
max(0.0, 0.7) = 0.7
If (0.7) then tip = generous
3. Apply implication operator (min)
0.7
min(0.7, generous)
tip (fuzzy)

Example – A Tipping Problem

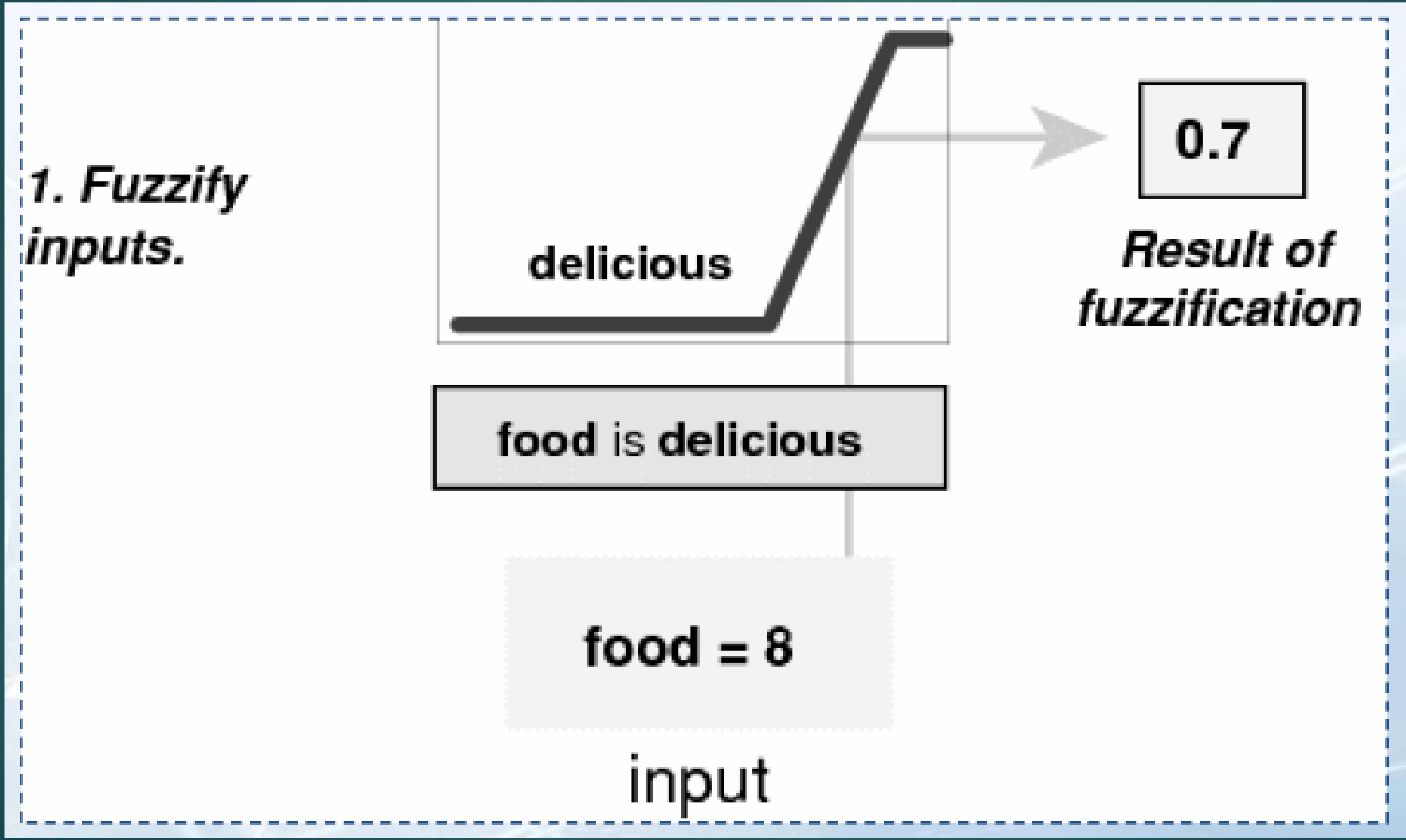


- Recall 5 parts of FIS
  - Fuzzification of the input attributes
  - Application of the fuzzy operator (AND or OR) in the antecedent
  - Implication from the antecedent to the consequent
  - Aggregation of the consequents across the rules
  - Defuzzification

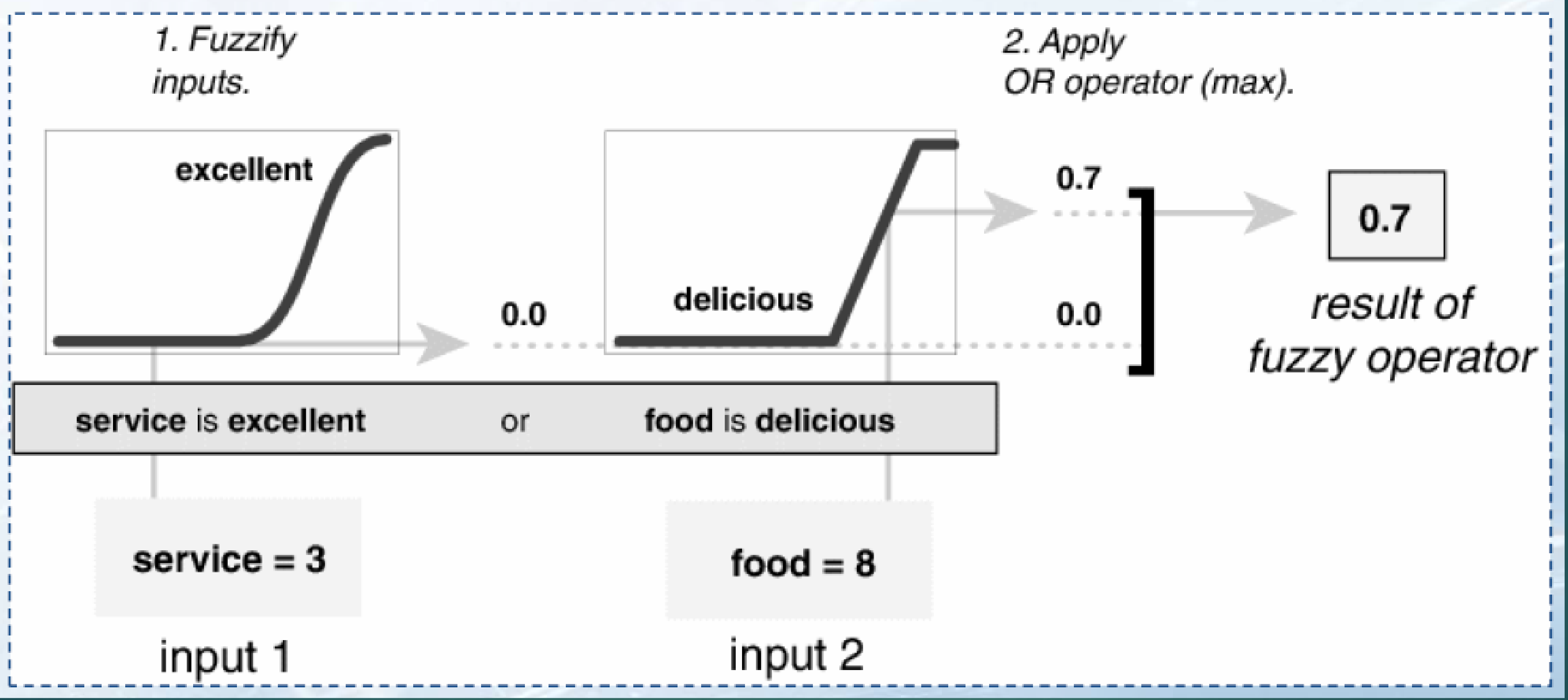
Fuzzification using a table lookup

Service	Poor	Good	Excellent
0	1	0	0
1	1	0	0
2	0.8	0.2	0
3	0.6	0.6	0
4	0.4	0.8	0.1
5	0.2	1	0.2
6	0.1	0.8	0.4
7	0	0.6	0.6
8	0	0.2	0.8
9	0	0	1
10	0	0	1

Fuzzification using a membership function



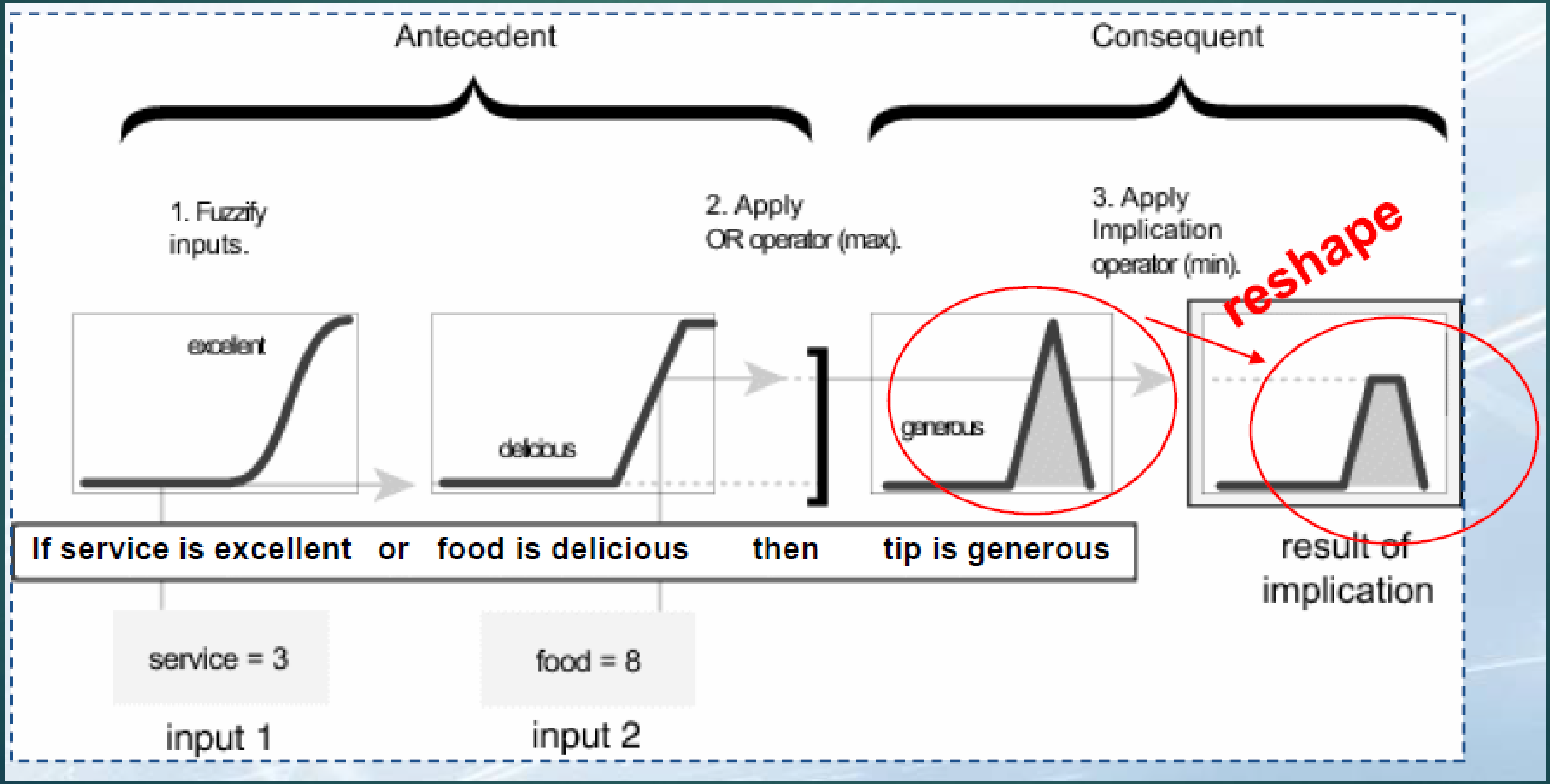
Applying Fuzzy Operator

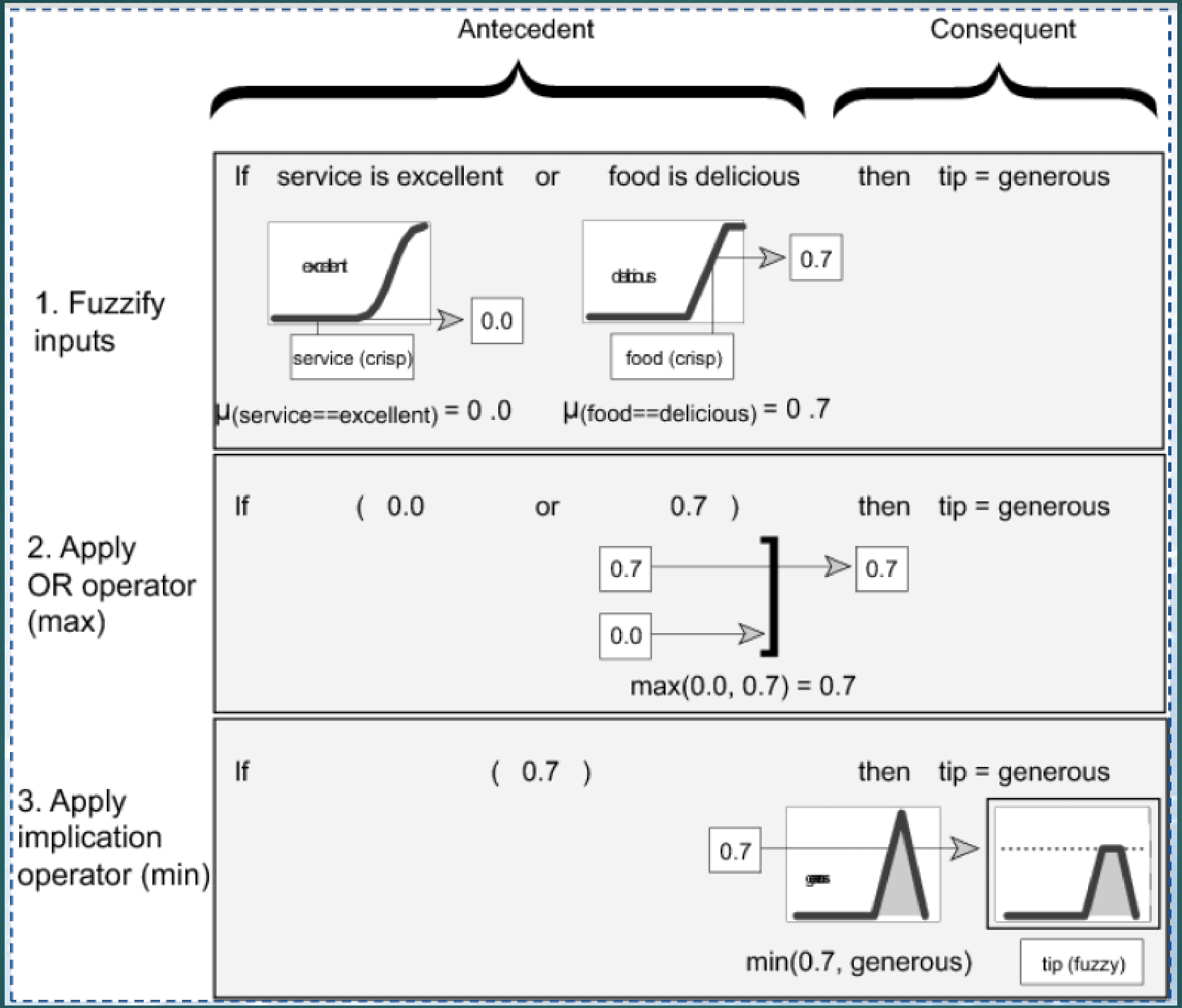


- OR – max(A,B) – S norm
- Example using fuzzy rule 3

Applying Implication Method

- T-norm – min(A,B)

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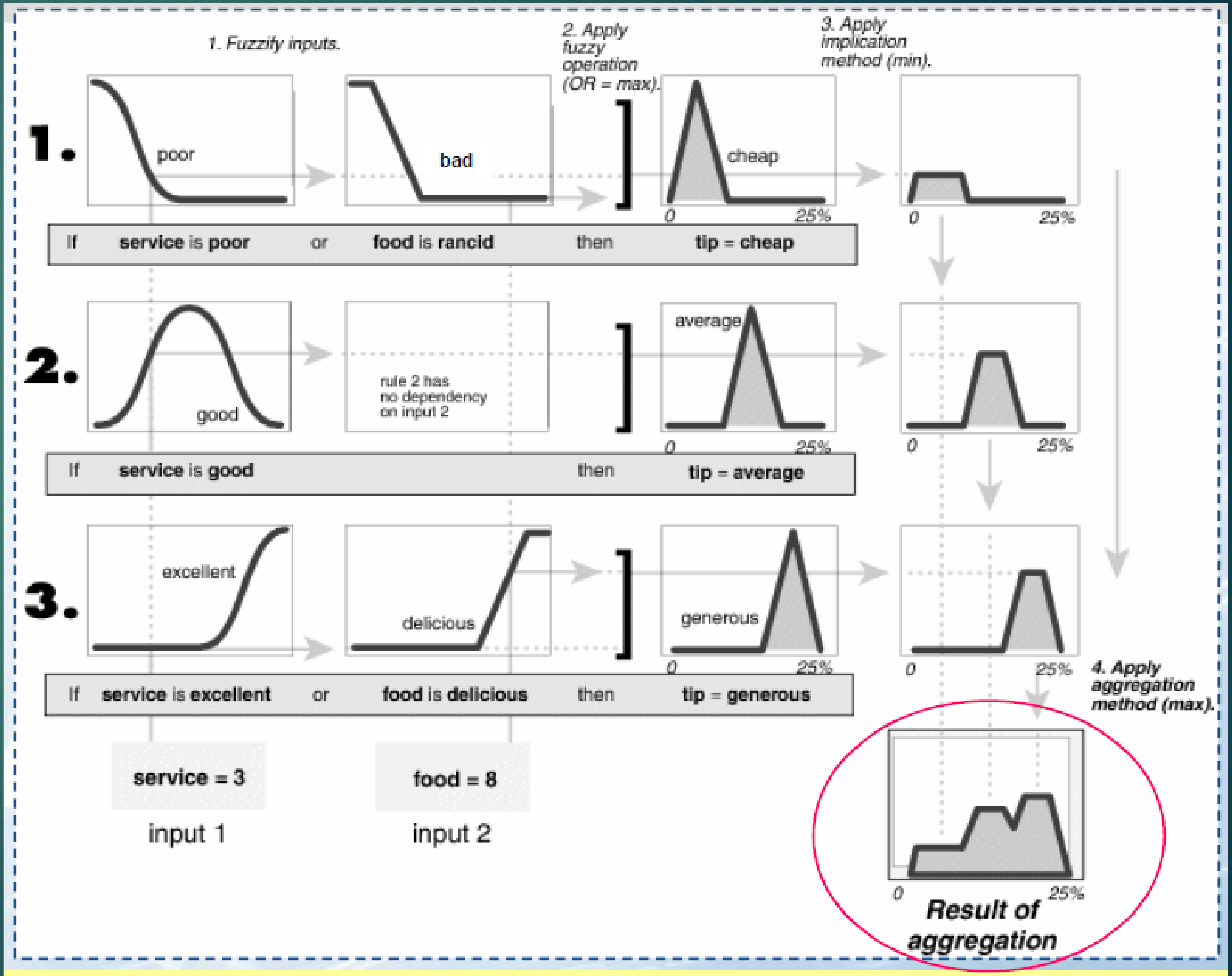
2. Apply OR operator (max)
If (0.0 or 0.7) then tip = generous
0.7
0.0
0.7
max(0.0, 0.7) = 0.7
3. Apply implication operator (min)
If (0.7) then tip = generous
0.7
min(0.7, generous)
tip (fuzzy)

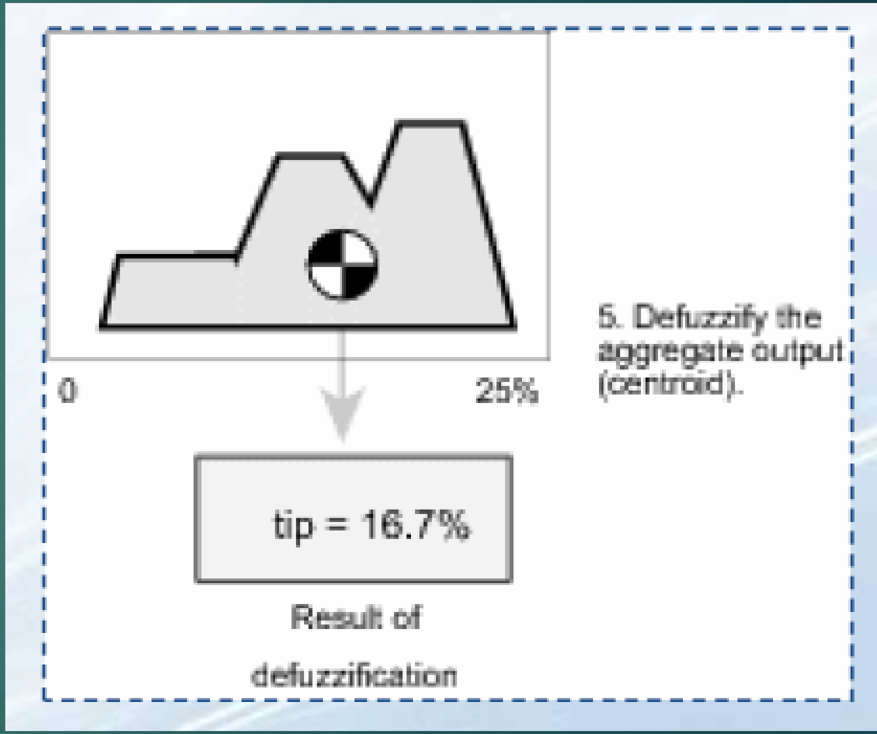
Aggregating All Outputs

- Aggregation is the process by which the fuzzy sets that represent the outputs of each rule are combined into a single fuzzy set
- Max operation



Defuzzification

- The input for the defuzzification process is a fuzzy set (the aggregate output fuzzy set) and the output is a single number
- The output is a single number



Types of Defuzzification

- Max-membership

$$z_0 \in \{z \mid C(z) = \max_w C(w)\}.$$

- Centroid

$$z_0 = \frac{\int_W zC(z)\,dz}{\int_W C(z)\,dz}.$$

$$z_0 = \frac{\sum z_j C(z_j)}{\sum C(z_j)}.$$

DEMO

- Fuzzy Pendulum
  - http://www.cs.dartmouth.edu/~spl/publications/fuzzy%20talk/FuzzyPendulum.html
- Open Source Fuzzy Logic
  - http://jfuzzylogic.sourceforge.net/html/index.html