NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

CZ3005
Artificial Intelligence

Inference of First Order Logic

Asst/P Mahardhika Pratama

Email: mpratama@ntu.edu.sg

Office: N4-02a-08

Existential Quantifier $\exists$

- **Express properties of some particular objects**
  - Make a statement about *one* object without naming it
    - e.g. “King John has a brother who is king”

$$\exists\, x,\ Brother(x, KingJohn) \land King(x)$$

instead of

$$Brother(Henry, KingJohn) \land King(Henry) \lor$$
$$Brother(KingJohn, KingJohn) \land King(KingJohn) \lor$$
$$Brother(Mary, KingJohn) \land King(Mary) \lor$$
$$Brother(London, KingJohn) \land King(London) \lor$$
$$Brother(Richard, KingJohn) \land King(Richard) \lor$$
$$\ldots$$

Using Existential Quantifier

- ***The conjunction (Λ) is the natural connective to use with the existential quantifier ($\exists$)***

Example

- General form: $\exists x\ P(x) \land Q(x)$ e.g. $\exists x\ Dog(x) \land Owns(John, x)$
  "John owns a dog"

- Use Implication? $\exists x\ P(x) \Rightarrow Q(x)$ e.g. $\exists x\ Dog(x) \Rightarrow Owns(John, x)$

  true for all x such that P(x) is false
  e.g. $Dog(Garfield) \Rightarrow Owns(John, Garfield)$

  –> yields a very weak statement (too weak! i.e. *useless*)

Example

- Mammals drink milk

$\forall x$, Mammal(x)$\Rightarrow$Drink(x,milk)

- Man is mortal

$\forall x$, Man(x)$\Rightarrow$Mortal(x)

Example

- Man is a mammal

$\forall x \; Man(x) \Rightarrow Mammal(x)$

- Tom is a man

Man(Tom)

❑ Combining ∀ and ∃

- Express more complex sentences
  - e.g. "if x is the parent of y then y is the child of x":

    $\forall$ x, $\forall$ y Parent(x, y) $\Rightarrow$ Child(y, x)

    "every person has a parent": $\forall$ x Person(x) $\Rightarrow$ $\exists$ y Parent(y, x)
- Semantics depends on quantifiers ordering
  - e.g. $\exists$ y, $\forall$ x Parent(y, x)

    "there is someone who is everybody's parent" ?!?
- Choosing variables to avoid confusion
  - e.g. $\forall$ x King(x) $\vee$ $\exists$ x Brother(Richard, x) is better written:

    $\forall$ x King(x) $\vee$ $\exists$ z Brother(Richard, z)

❑ Well-formed formula (WFF)

- Sentences with all variables properly quantified

Connections between Quantifiers

❑ Equivalences

- Using the negation *(hence only one quantifier is needed)*

  $$\forall x \ P(x) \Leftrightarrow \neg \exists x \neg P(x)$$

  - e.g. "everyone is mortal":

    $$\forall x \ Mortal(x) \Leftrightarrow \neg \exists x \neg Mortal(x)$$

- De Morgan's Laws

  - $\forall x\ P \Leftrightarrow \neg\exists x\ \neg P$ $\qquad$ $P \land Q \Leftrightarrow \neg(\neg P \lor \neg Q)$

    $\forall x\ \neg P \Leftrightarrow \neg\exists x\ P$ $\qquad$ $\neg P \land \neg Q \Leftrightarrow \neg(P \lor Q)$

    $\neg\forall x\ P \Leftrightarrow \exists x\ \neg P$ $\qquad$ $\neg (P \land Q) \Leftrightarrow \neg P \lor \neg Q$

    $\neg\forall x\ \neg P \Leftrightarrow \exists x\ P$ $\qquad$ $\neg(\neg P \land \neg Q) \Leftrightarrow P \lor Q$

Grammar of FOL

Sentence	$\rightarrow$	AtomicSentence | (Sentence)
		| Sentence Connective Sentence
		| ¬Sentence
		| Quantifier Variable, … Sentence
AtomicSentence	$\rightarrow$	Predicate(Term, …) | Term = Term
Term	$\rightarrow$	Function(Term, …) | Constant | Variable
Connective	$\rightarrow$	$\wedge$ | $\vee$ | $\Leftrightarrow$ | $\Rightarrow$
Quantifier	$\rightarrow$	$\forall$ | $\exists$
Constant	$\rightarrow$	A | X_1 | John | …
Variable	$\rightarrow$	a | x | person | …
Predicate	$\rightarrow$	P() | Colour() | Before() | …
Function	$\rightarrow$	F() | MotherOf() | SquareRootOf() | …

Example

1. Some dogs bark : $\exists x\ dog(x) \wedge bark(x)$
2. All dogs have four legs: $\forall x\ dog(x) \Rightarrow leg(x,4)$
3. All barking dogs are irritating: $\forall x\ dog(x) \wedge bark(x) \Rightarrow irritating(x)$
4. Not all dogs purr: $\neg(\forall x\ dog(x) \Rightarrow purr(x))$
5. Fathers are male parents with Children: $\forall x\ Father(x) \Rightarrow \exists y\ male(x) \wedge Children(y,x)$
6. Students are people who are enrolled in courses:

   $\forall x\ Student(x) \Rightarrow \exists y\ people(x) \wedge courses(y) \wedge enroll(x,y)$

Inference of FOL

“In which we define inference mechanisms that can efficiently answer questions posed in first-order logic.”

❑ Inference rules from Propositional Logic

- *Modus Ponens*
  - $$\frac{\alpha \Rightarrow \beta,\ \alpha}{\beta}$$
- Double-Negation-Elimination
  - $$\frac{\neg\neg\alpha}{\alpha}$$
- And-Elimination
  - $$\frac{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}{\alpha_i}$$
- And-Introduction
  - $$\frac{\alpha_1, \alpha_2, \ldots, \alpha_n}{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}$$
- Or-Introduction
  - $$\frac{\alpha_i}{\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n}$$
- Resolution
  - $$\frac{\alpha \vee \beta,\ \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

Inferences Rules with Quantifiers

$\forall$ x Dog(x) $\Rightarrow$ Friendly(x)

|– Dog(Snoopy) $\Rightarrow$ Friendly(Snoopy)

- **Substitutions**
  - SUBST(θ , α): binding list θ applied to a sentence α
    - e.g.: SUBST({x / John, y / Richard}, Brother(x, y)) = Brother(John, Richard)
- **Inference rules**
  - Universal Elimination
    - $$\frac{\forall\, x \;\; \alpha}{\text{SUBST}(\,\{x/g\},\, \alpha)}$$
  - Existential Elimination
    - $$\frac{\exists \;\; x \;\; \alpha}{\text{SUBST}(\,\{x/\ K\},\, \alpha)}$$

    (Skolemization)
  - Existential Introduction
    - $$\frac{\alpha}{\exists\; x\ \text{SUBST}(\,\{g/v\},\, \alpha)}$$

$\exists$x Dog (x) Λ Owns(John, x)

|– Dog (Lassie), Owns(John,Lassie)

❑ Proof procedure

- Analysis of the problem description (Natural Language)
- Translation from NL to first-order logic
- Application of inference rules (proof)

❑ Problem statement

- *"It is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold by Col. West, who is American."*

Translation in FOL

- "It is a crime for an American to sell weapons to hostile nations …"
  - **(1)** $\forall x,y,z\ American(x) \land Weapon(y) \land Nation(z) \land Hostile(z) \land Sells(x,z,y) \Rightarrow Criminal(x)$
- "The country Nono […] has some missiles, …"
  - **(2)** $\exists x\ Owns(Nono, x) \land Missile(x)$
- "… all of its missiles were sold by Col. West, …"
  - **(3)** $\forall x\ Owns(Nono,x) \land Missile(x) \Rightarrow Sells(West,Nono,x)$
- A missile is a weapon.
  - **(4)** $\forall x\ Missile(x) \Rightarrow Weapon(x)$
- An enemy of America is hostile.
  - **(5)** $\forall x\ Enemy(x,America) \Rightarrow Hostile(x)$
- "… West, who is American."
  - **(6)** $American(West)$
- "The country Nono …"
  - **(7)** $Nation(Nono)$
- "Nono, an enemy of America …"
  - **(8)** $Enemy(Nono,America)$
  - **(9)** $Nation(America)$

Knowledge Base	Inferences
(1) $\forall x,y,z\ American(x) \wedge Weapon(y) \wedge Nation(z) \wedge Hostile(z) \wedge Sells(x,z,y) \Rightarrow Criminal(x)$	From (2) and Existential-Elimination:
(2) $\exists x\ Owns(Nono, x) \wedge Missile(x)$	**(10)** $Owns(Nono, M1) \wedge Missile(M1)$
(3) $\forall x\ Owns(Nono,x) \wedge Missile(x) \Rightarrow Sells(West,Nono,x)$	From (10) and And-Elimination:
(4) $\forall x\ Missile(x) \Rightarrow Weapon(x)$	**(11)** $Owns(Nono, M1)$
(5) $\forall x\ Enemy(x,America) \Rightarrow Hostile(x)$	**(12)** $Missile(M1)$
(6) $American(West)$	From (4) and Universal-Elimination:
(7) $Nation(Nono)$	**(13)** $Missile(M1) \Rightarrow Weapon(M1)$
(8) $Enemy(Nono,America)$	From (12,13) and Modus Ponens:
(9) $Nation(America)$	**(14)** $Weapon(M1)$

Knowledge Base	Inferences
(1) $\forall x,y,z$ American(x) $\land$ Weapon(y) $\land$ Nation (z) $\land$ Hostile(z) $\land$ Sells(x,z,y) $\Rightarrow$ Criminal(x)	From (3) and Universal-Elimination:
…	**(15)** Owns(Nono, M1) $\land$ Missile(M1) $\Rightarrow$ Sells(West,Nono,M1)
(3) $\forall x$ Owns(Nono,x) $\land$ Missile(x) $\Rightarrow$ Sells(West,Nono,x)	From (15,10) and Modus Ponens:
…	**(16)** Sells(West,Nono,M1)
(6) American(West)	From (1) and Universal-Elimination (three times):
…	**(17)** American(West) $\land$ Weapon(M1) $\land$ Nation (Nono) $\land$ Hostile(Nono) $\land$ Sells(West,Nono,M1) $\Rightarrow$ Criminal(West)
(10) Owns(Nono, M1) $\land$ Missile(M1)	
…	
(14) Weapon(M1)	

Knowledge Base

…

(5) $\forall x$ Enemy(x,America) $\Rightarrow$ Hostile(x)

(6) American(West)

(7) Nation(Nono)

(8) Enemy(Nono,America)

…

(14) Weapon(M1)

(16) Sells(West,Nono,M1)

(17) American(West) $\wedge$ Weapon(M1) $\wedge$ Nation (Nono) $\wedge$ Hostile(Nono) $\wedge$ Sells(West,Nono,M1) $\Rightarrow$ Criminal(West)

Inferences

From (5) and Universal-Elimination:

(18) Enemy(Nono,America) $\Rightarrow$ Hostile(Nono)

From (8,18) and Modus Ponens:

(19) Hostile(Nono)

From (6,7,14,16,19) and And-Intro.:

(20) American(West) $\wedge$ Weapon(M1) $\wedge$ Nation (Nono) $\wedge$ Hostile(Nono) $\wedge$ Sells(West,Nono,M1)

From (17,20) and Modus Ponens:

(21) **Criminal(West)**