NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

CZ3005
Artificial Intelligence

Markov Decision Process

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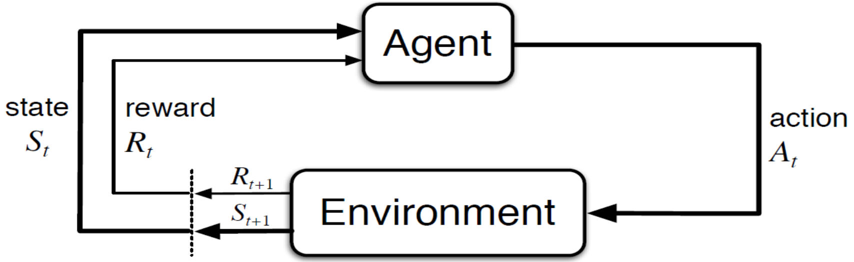
Lesson Outline

- Introduction
- Markov Decision Process
- Two methods for solving MDP
  - Value iteration
  - Policy iteration

Introduction

- We consider a framework for decision making under uncertainty
- Markov decision processes (MDPs) and their extensions provide an extremely general way to think about how we can act optimally under uncertainty
- For many medium-sized problems, we can use the techniques from this lecture to compute an optimal decision policy
- For large-scale problems, approximate techniques are often needed (more on these in later lectures), but the paradigm often forms the basis for these approximate methods

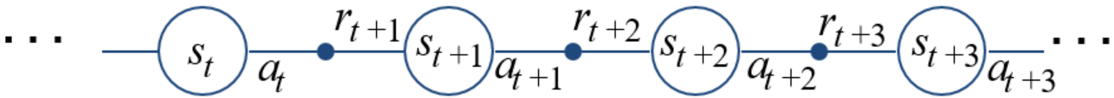
The Agent-Environment Interface



Agent and environment interact at discrete time steps: $t = 0,1,2, \ldots$

Agent:

1. observes state at step t: $s_t \in S$
2. Produces action at step t: $a_t \in A(s_t)$
3. Gets resulting reward: r_{t+1} and the next state: $s_{t+1} \in S$



Making Complex Decisions

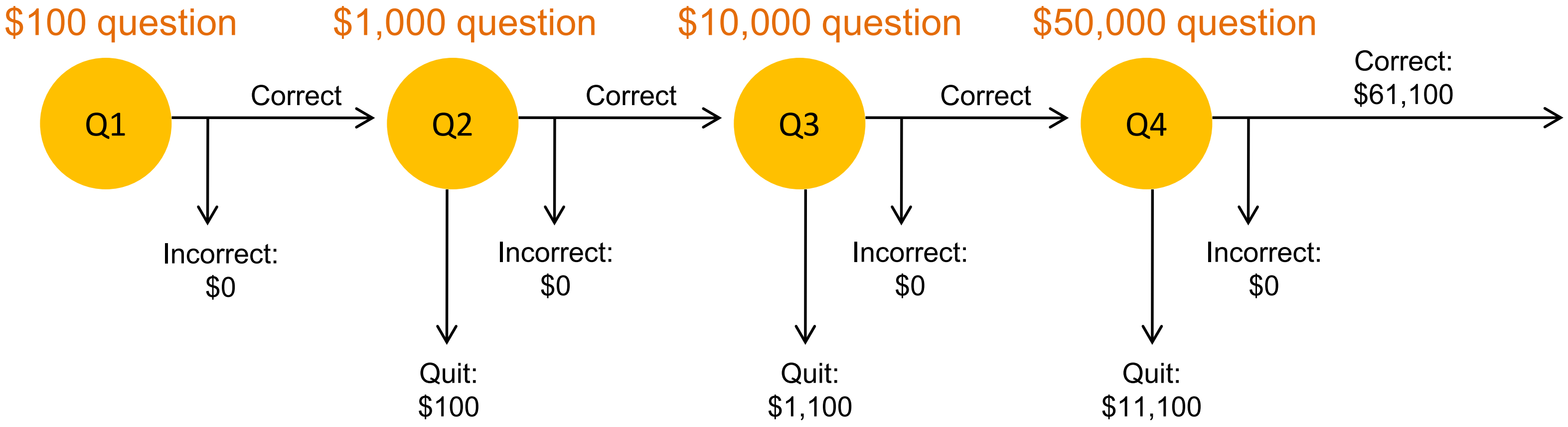
- Make a sequence of decisions
  - Agent’s utility depends on a sequence of decisions
  - Sequential Decision Making
- Markov Property
  - Transition properties depend only on the current state, not on previous history (how that state was reached)
  - Markov Decision Processes

Markov Decision Processes

- Formulate the agent-environment interaction as an MDP
- Components:
  - *Markov* **States** s, beginning with initial state s_0
  - **Actions** a
    - Each state s has actions $A(s)$ available from it
  - **Transition model** $P(s' \mid s, a)$
    - *assumption*: the probability of going to s' from s depends only on s and a and not on any other past actions or states
  - **Reward function** $R(s)$, or r(s)
- **Policy** $\pi(s)$: the action that an agent takes in any given state
  - The "solution" to an MDP

Game Show

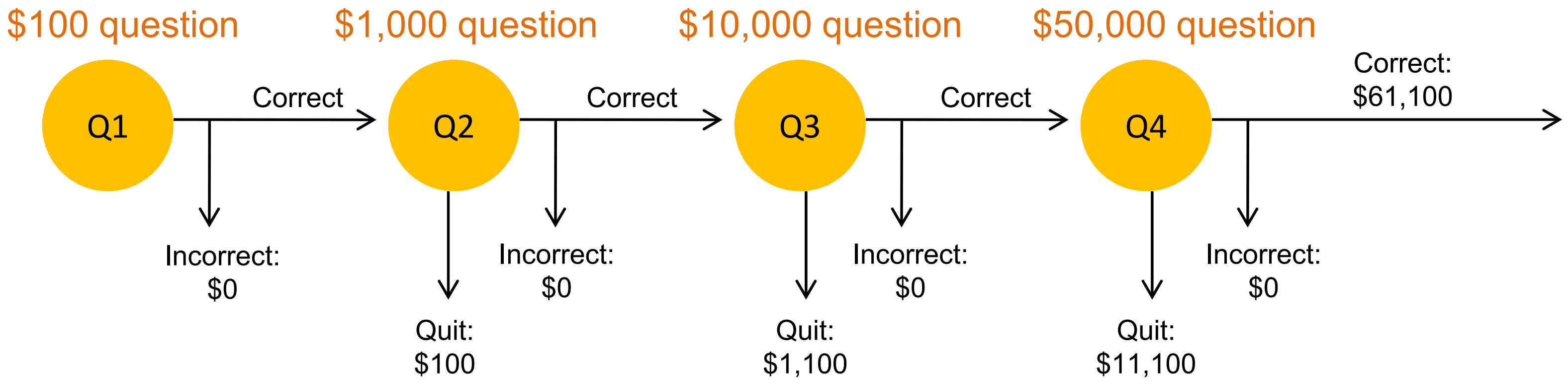
- A series of questions with increasing level of difficulty and increasing payoff
- Decision: at each step, take your earnings and quit, or go for the next question
  - If you answer wrong, you lose everything



Game Show

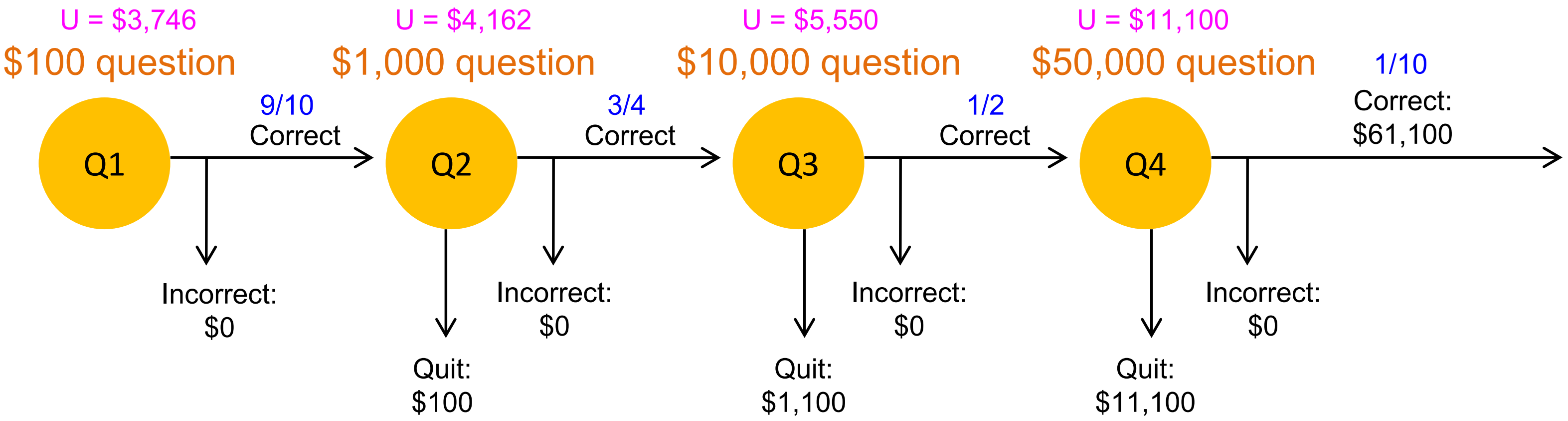
- Consider $50,000 question
  - Probability of guessing correctly: 1/10
  - Quit or go for the question?
- What is the expected payoff for continuing?

  0.1 * 61,100 + 0.9 * 0 = 6,110
- What is the optimal decision?

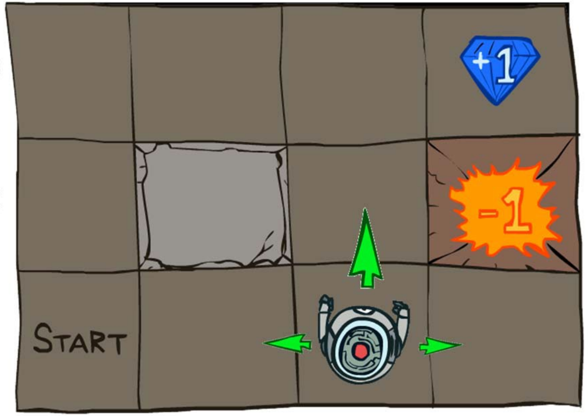


Game Show

- What should we do in Q3?
  - Payoff for quitting: $1,100
  - Payoff for continuing: 0.5 * $11,100 = $5,550
- What about Q2?
  - $100 for quitting vs. $4,162 for continuing
- What about Q1?

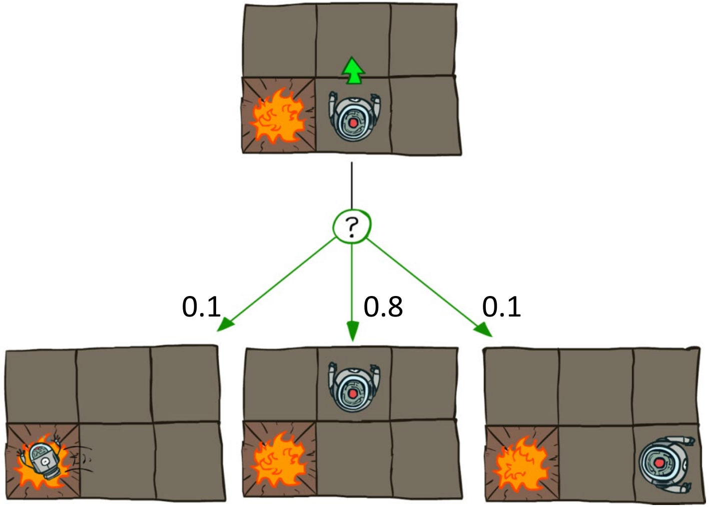


Grid World



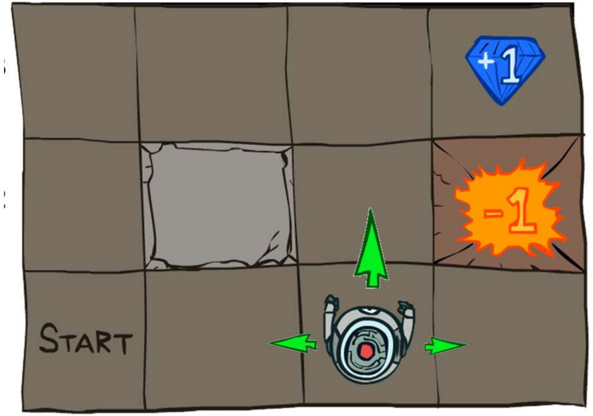
R(s) = -0.04 for every non-terminal state

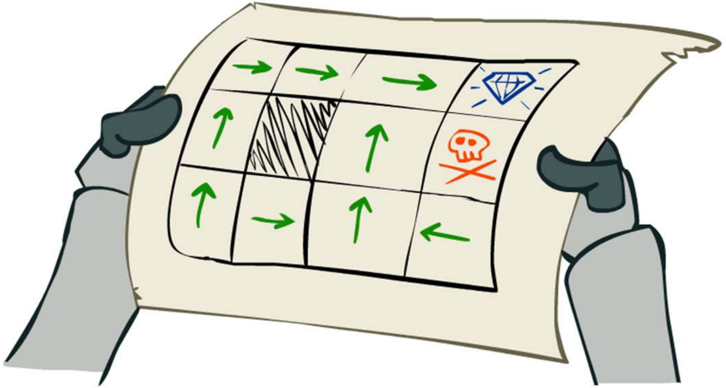
Transition model:



Source: P. Abbeel and D. Klein

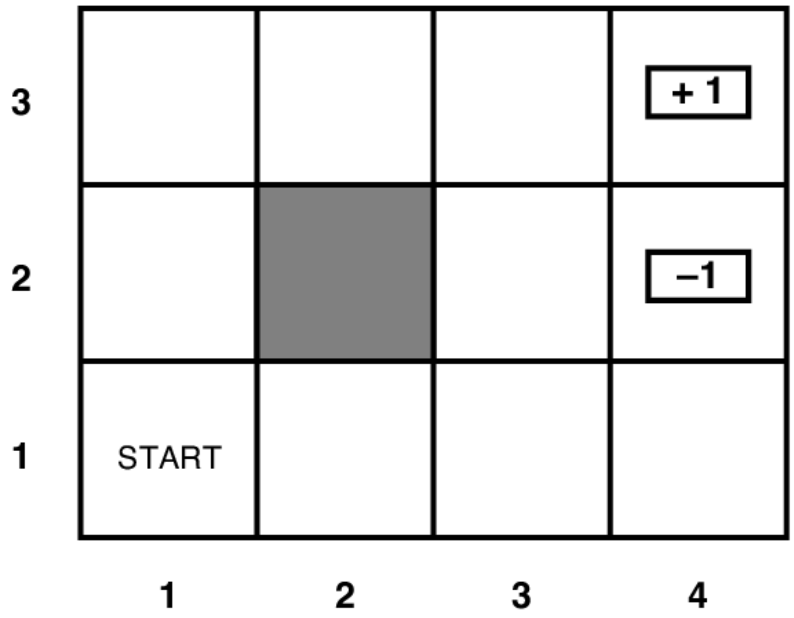
Goal: Policy



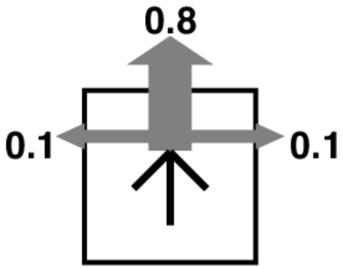


Source: P. Abbeel and D. Klein

Grid World

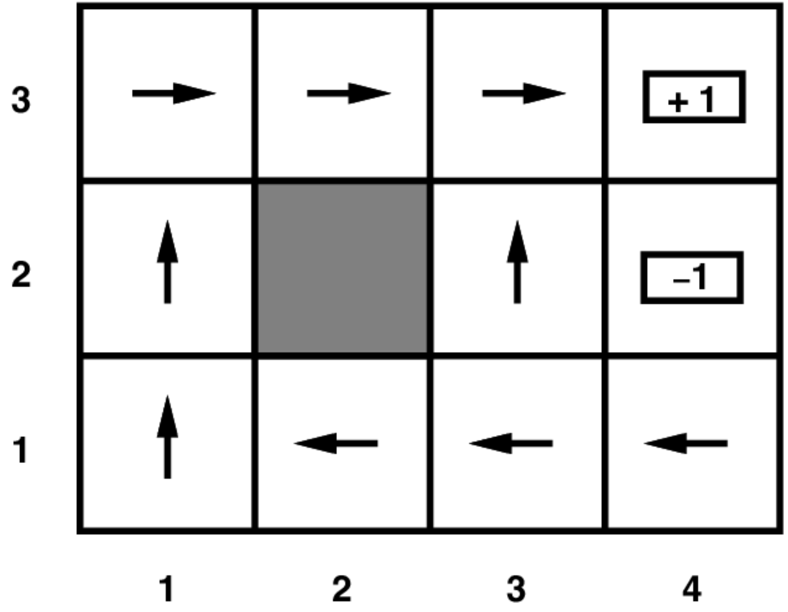


Transition model:



R(s) = -0.04 for every non-terminal state

Grid World



Optimal policy when R(s) = -0.04 for every non-terminal state

Atari Video Games

Execute actions to get the maximum accumulated rewards

Solving MDPs

- MDP components:
  - **States** s
  - **Actions** a
  - **Transition model** $P(s' \mid s, a)$
  - **Reward function** $R(s)$
- The solution:
  - **Policy** $\pi(s)$mapping from states to actions
  - How to find the optimal policy?

Maximizing Accumulated Rewards

- The optimal policy should maximise the accumulated *rewards over* given a trajectories like $\tau =< S_1, A_1, R_1, \ldots, S_T, A_T, R_T >$ under some policies:

$$G_t = R_t + \gamma R_{t+1} + \gamma^2 R_{t+2} + \cdots \gamma^K R_{t+K} = \sum_{k=0}^{K} \gamma^k R_{t+k}$$

- How to define the accumulated rewards of a state sequence?
  - Discounted sum of rewards of individual states
  - Problem: infinite state sequences
  - If finite, LP can be applied

Accumulated Rewards

- Normally, we would define the *accumulated rewards* of trajectories as the discounted sum of the rewards
- **Problem:** infinite time horizon
- **Solution:** *discount* the individual rewards by a factor γ between 0 and 1:

$$
\begin{aligned}
G_t &= R_t + \gamma R_{t+1} + \gamma^2 R_{t+2} + \cdots \\
&= \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \leq \frac{R_{max}}{1-\gamma} \quad (0 < \gamma < 1)
\end{aligned}
$$

  - Sooner rewards count more than later rewards
  - Makes sure the total *accumulated rewards* stays bounded
  - Helps algorithms converge

Value Function

- The "true" value of a state, denoted $V(s)$, is the expected sum of discounted rewards if the agent executes an *optimal* policy starting in state s

$$V^{\pi}(s) = \mathbb{E}_{\pi}[G_t \mid S_t = s]$$

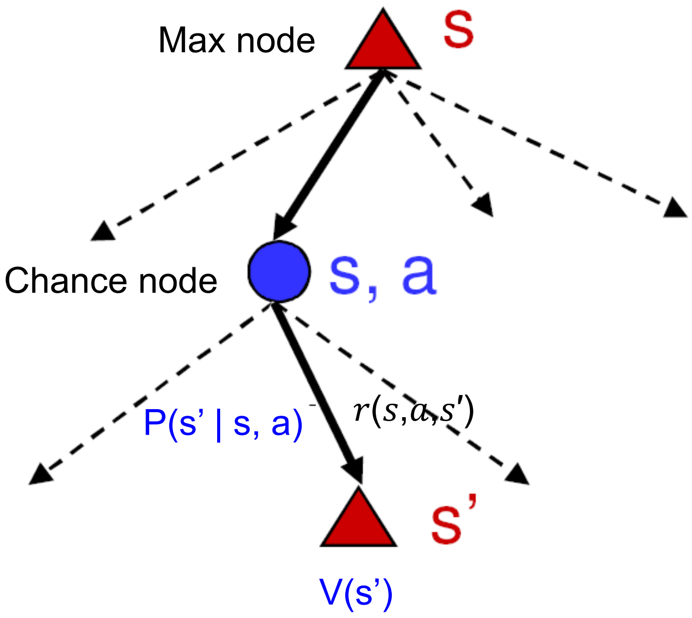
- Similarly, we define the action-value of a state-action pair as

$$Q^{\pi}(s, a) = \mathbb{E}_{\pi}[G_t \mid S_t = s, A_t = a]$$

- The relationship between Q and V

$$V^{\pi}(s) = \sum_{a \in A} Q^{\pi}(s, a) \pi(a|s)$$

Finding the Value Function of States



- What is the expected value of taking action $\boldsymbol{a}$ in state $\boldsymbol{s}$?

$$\sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma * V(s')]$$

- How do we choose the optimal action?

$$\pi^*(s) = \operatorname{argmax}_{a\in A} \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V(s')]$$

- What is the recursive expression for $V(s)$ in terms of the utilities of its successor states?

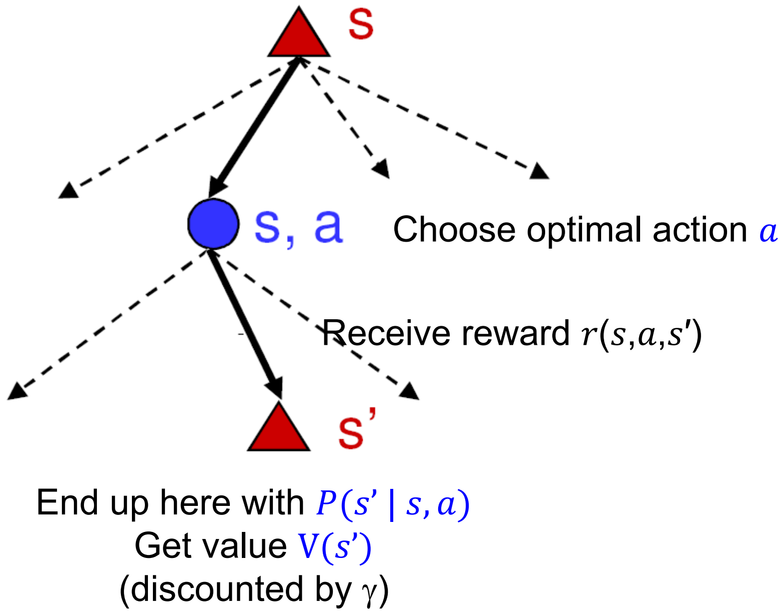
$$\mathrm{V}(s) = \max_{a\in A} \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V(s')]$$

The Bellman Equation

- Recursive relationship between the accumulated rewards of successive states:

$$V(s) = \max_{a \in A} \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V(s')]$$

- For *N* states, we get *N* equations in *N* unknowns
  - Solving them solves the MDP
  - We could try to solve them through expectimax search, but that would run into trouble with infinite sequences
  - Instead, we solve them algebraically
  - Two methods: **value iteration** and **policy iteration**



Method 1: Value Iteration

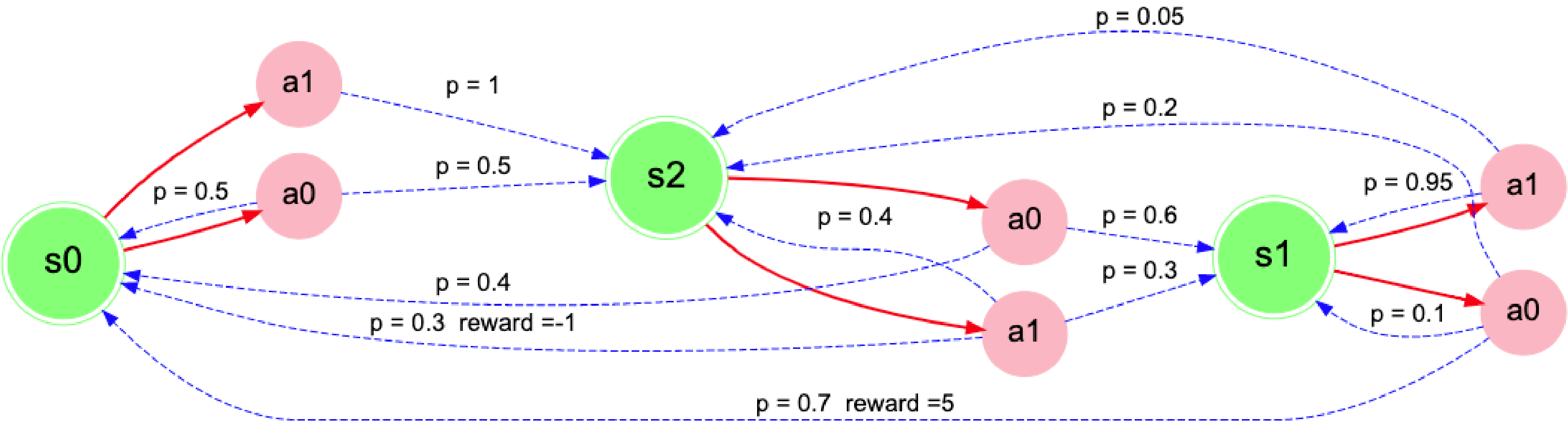
- Start out with every $V(s) = 0$
- Iterate until convergence
  - During the *i*th iteration, update the value of each state according to this rule:

$$V_{i+1}(s) = \max_a \sum_{s' \in S} P(s'|s,a)\,[R(s,a,s') + \gamma V_i(s')]$$

- In the limit of infinitely many iterations, guaranteed to find the correct values
  - In practice, don't need an infinite number of iterations…

Value Iteration: Example

- A simple example to show how VI works
  - State, action, reward (non-zero) and transition probability are shown in the figure
  - We use this example as an MDP and solve it using VI and PI



These structure can be easily implemented by `dict` of Python or `HashMap` of Java

Value Iteration: Example (cont'd)

- Given the states and actions are finite, we can use matrices to represent the value function $V(s)$
- Pseudo-code of VI

1. Initialize $\mathrm{V}_0(s)$, for all s
2. For $i = 0, 1, 2 \ldots$
3. $V_{i+1}(S) = \max\limits_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_i(s')]$ for all state s

Value Iteration: Example (cont'd)

- How VI works in an iteration? Given iteration $i = 0$

For all state find

$$V_{i+1}(S) = \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_i(s')]$$

We can instead calculate Q(s, a) values for each s and *a* and get the best V(s)

$$Q_{i+1}(s,a) = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_i(s')]$$

Finally

$$V_{i+1}(S) = \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_i(s')] = \max_a Q_{i+1}(s,a)$$

Value Iteration: Example (cont'd)

- We use the previous example and solve it using VI
- Construct a Q table, Q(s, a)
  - 3 rows (3 states) and 2 columns (2 actions)
  - $V_0 = [0, 0, 0]$ # 3 states

$$Q_{i+1}(s,a) = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_i(s')]$$

Q_0

	a_0	a_1
s_0	0	0
s_1	0	0
s_2	0	0

V_0

0
0
0

The initial Q and V table

Q_1

	a_0	a_1
s_0	0	0
s_1	3.5	0
s_2	0	-0.3

V_1

0
3.5
0

$$\begin{aligned} Q_1(s_1,a_0) = {} & P(s_0|s_1,a_0)[r(s_1,a_0,s_0) + \gamma V_o(s_0)] + \\ & P(s_1|s_1,a_0)\,[r(s_1,a_0,s_1) + \gamma V_o(s_1)]\; + \\ & P(s_2|s_1,a_0)\,[r(s_1,a_0,s_2) + \gamma V_o(s_2)] \end{aligned}$$

$$\mathrm{Q}_1(\mathrm{s}_1,\mathrm{a}_0) = 0.7 * [5+0] + 0.1 * [0+0] + 0.2 * [0+0] = 0.35$$

Repeat this process until it converges

Value Iteration: Example (cont'd)

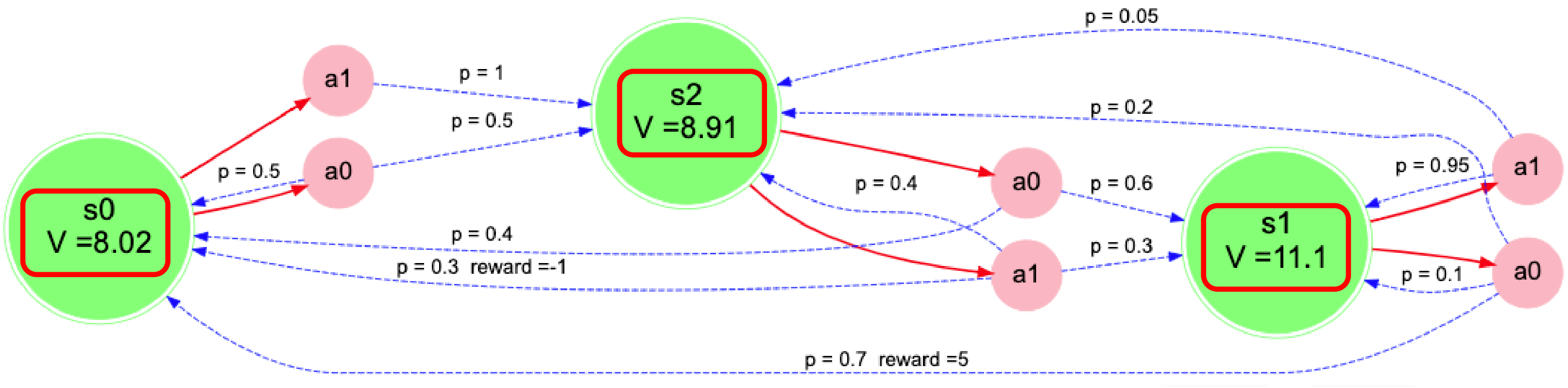
- Iterating the process (code available on later slides)

```
iter   0  |  V(s0) = 0.000   V(s1) = 0.000   V(s2) = 0.000

iter   1  |  V(s0) = 0.000   V(s1) = 3.500   V(s2) = 0.000

iter   2  |  V(s0) = 0.000   V(s1) = 3.815   V(s2) = 1.890

iter   3  |  V(s0) = 1.701   V(s1) = 4.184   V(s2) = 2.060

...  ...

iter  63  |  V(s0) = 8.020   V(s1) = 11.160   V(s2) = 8.912

iter  64  |  V(s0) = 8.021   V(s1) = 11.161   V(s2) = 8.913

iter  65  |  V(s0) = 8.022   V(s1) = 11.162   V(s2) = 8.915
```

Very good, the values converge now

Value Iteration: Example (cont'd)

- Put the optimal values on the graph



Value Iteration: Example (cont'd)

- Use V^* to find optimal policy
  - aka optimal actions in each state

$$\pi^*(s) = \underset{a}{\operatorname{argmax}} \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_i(s')] = \underset{a}{\operatorname{argmax}}\, Q_i(s,a)$$

π^*

	a^*
s_0	a_1
s_1	a_0
s_2	a_0

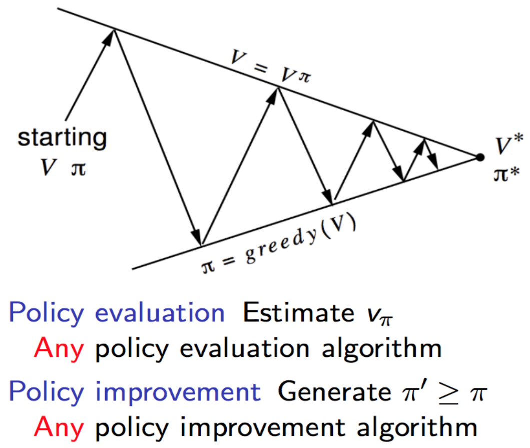
Done! We get the optimal policy of the example

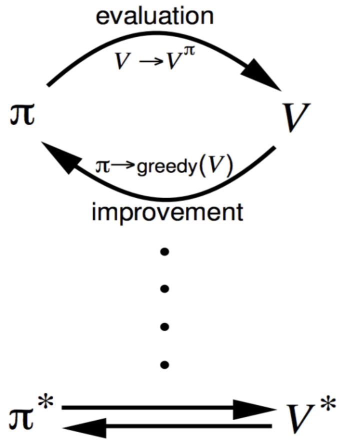
Python Code on Colab: https://colab.research.google.com/drive/1DnYIr3QJxpfs_rR_jAAUrqHZMjvsjGSx?usp=sharing

Method 2: Policy Iteration

- Start with some initial policy π_0 and alternate between the following steps:
  - **Policy evaluation:** calculate $V^{\pi_i}(s)$ for every state *s, like VI*
  - **Policy improvement:** calculate a new policy π_{i+1} based on the updated utilities

$$\pi_{i+1}(s) = \operatorname{argmax}_{a \in A} \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^{\pi_i}(s')]$$





Policy Iteration: Main Steps

- Unlike VI, policy iteration has to maintain a policy - chosen actions from all states - and estimate V^{π_i} based on this policy.
  - Iterate this process until convergence (like VI)
- Steps of PI
  1. Initialization
  2. Policy Evaluation (calculating the V)
  3. Policy Improvement (calculate the policy π)

Policy Iteration: Detailed Steps

1. Initialization
   a. Initialize $V(s)$ and $\pi(s)$ for all state s

2. Policy Evaluation (calculate the V)
   a. Repeat
   b. $\quad \Delta \leftarrow 0$
   c. $\quad$ For each state s:
   d. $\quad\quad v \leftarrow V(s)$
   e. $\quad\quad V(s) \leftarrow \sum_{s'} p(s'|s,\pi(s))[R(s,\pi(s),s') + \gamma V(s')]$
   f. $\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
   g. until $\Delta < \omega$ (a small positive number)

3. Policy Improvement (calculate the new policy π)
   a. $policystable \leftarrow True$
   b. For each state s:
   c. $\quad b \leftarrow \pi(s)$
   d. $\quad \pi(s) \leftarrow \mathrm{argmax}_{\pi(s)} \sum_{s'} p(s'|s,\pi(s))[R(s,\pi(s),s') + \gamma V(s')]$
   e. $\quad$ If $b \neq \pi(s)$, then $policystable \leftarrow False$
   f. If $policystable \leftarrow True$, then stop; else go to step 2;

Policy Iteration: Policy Evaluation

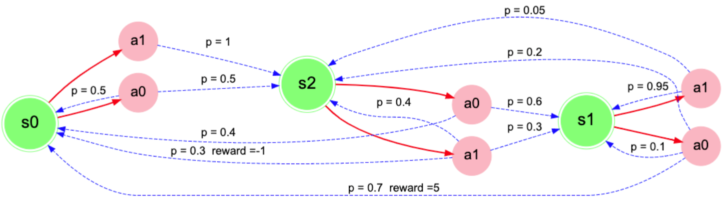
- Use the example used in VI
- Start iteration $i = 0$, initialize random π , $\gamma = 0.99$, and $V(s) = 0$ for all state s

$$V^{\pi_i}(s) = \sum_{s'} P(s, \pi_i(s), s')[R(s, \pi_i(s), s') + \gamma V^{\pi_i}(s')]$$

$$\begin{aligned} V(s_0) \leftarrow\ & P(s_0, \pi(s_0), s_0)[R(s_0, \pi(s_0), s_0) + \gamma V(s_0)] + \\ & P(s_0, \pi(s_0), s_1)[R(s_0, \pi(s_0), s_1) + \gamma V(s_1)] + \\ & P(s_0, \pi(s_0), s_2)[R(s_0, \pi(s_0), s_2) + \gamma V(s_2)] \end{aligned}$$

$$\begin{aligned} V(s_1) \leftarrow\ & P(s_1, \pi(s_1), s_0)[R(s_1, \pi(s_1), s_0) + \gamma V(s_0)] + \\ & P(s_1, \pi(s_1), s_1)[R(s_1, \pi(s_1), s_1) + \gamma V(s_1)] + \\ & P(s_1, \pi(s_1), s_2)[R(s_1, \pi(s_1), s_2) + \gamma V(s_2)] \end{aligned}$$

$$\begin{aligned} V(s_2) \leftarrow\ & P(s_2, \pi(s_2), s_0)[R(s_2, \pi(s_2), s_0) + \gamma V(s_0)] + \\ & P(s_2, \pi(s_2), s_1)[R(s_2, \pi(s_2), s_1) + \gamma V(s_1)] + \\ & P(s_2, \pi(s_2), s_2)[R(s_2, \pi(s_2), s_2) + \gamma V(s_2)] \end{aligned}$$



Policy Iteration: Policy Evaluation (cont'd)

- Use the example used in VI
- Start iteration $i = 0$, $\gamma = 0.99$, initialize random $\pi = [a_1, a_0, a_1]$ and $V(s) = 0$ for states s_0, s_1 and s_2

$$V(s_0) \leftarrow 0.0[0.0 + \gamma V(s_0)] + 0.0[0.0 + \gamma V(s_1)] + 1.0[0.0 + \gamma V(s_2)]$$

$$V(s_1) \leftarrow 0.7[5.0 + \gamma V(s_0)] + 0.1[0.0 + \gamma V(s_1)] + 0.2[0.0 + \gamma V(s_2)]$$

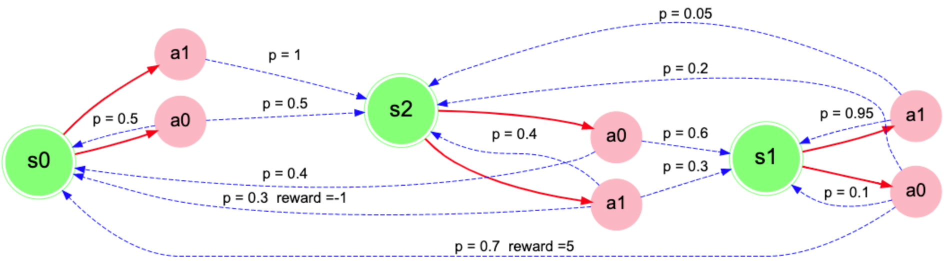
$$V(s_2) \leftarrow 0.3[-1 + \gamma V(s_0)] + 0.3[0.0 + \gamma V(s_1)] + 0.4[0.0 + \gamma V(s_2)]$$

- Values are calculated asynchronously
- The converged values are used for policy improvement

$$V(s_0) = 0$$

$$V(s_1) = 0.7 * 5.0 = 3.5$$

$$V(s_2) = 0.3 * (-1) + 0.3 * (0.99 * 3.5) = 0.7395$$

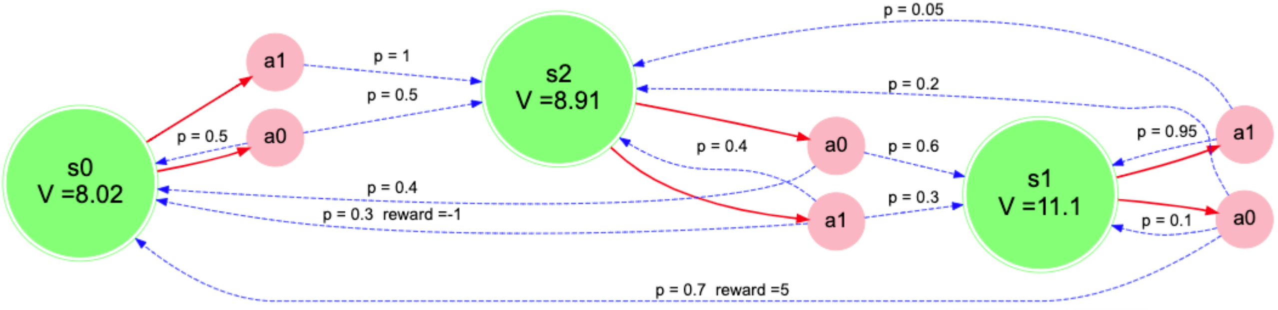


… loop this process as instructed in step 2 …

Policy Iteration: Example (cont'd)

- The optimal Q values are
  - Nearly the same as that of VI (as shown in the figure below)

	V^*
s_0	8.03
s_1	11.2
s_2	8.9



  - We can easily calculate the optimal policy. Can you try it?

π^*

	a^*
s_0	a_1
s_1	a_0
s_2	a_0

Further Reading [AAAI’18: http://www.ntu.edu.sg/home/boan/papers/AAAI18_Malmo.pdf]

We Won 2017 Microsoft Collaborative AI Challenge

- Collaborative AI
  - *How can AI agents learn to recognise someone’s intent (that is, what they are trying to achieve)?*
  - *How can AI agents learn what behaviours are helpful when working toward a common goal?*
  - *How can they coordinate or communicate with another agent to agree on a shared strategy for problem-solving?*

Further Reading [AAAI'18: http://www.ntu.edu.sg/home/boan/papers/AAAI18_Malmo.pdf]

- Microsoft Malmo Collaborative AI Challenge
  - *Collaborative mini-game, based on an extension "stag hunt"*
  - *Uncertainty of pig movement*
  - *Unknown type of the other agent*
  - *Detection noise (frequency 25%)*
- Our team HogRider won the challenge (out of more than 80 teams from 26 countries)
  - *learning + game theoretic reasoning + sequential decision making + optimisation*