Smoothed Particle Hydrodynamics

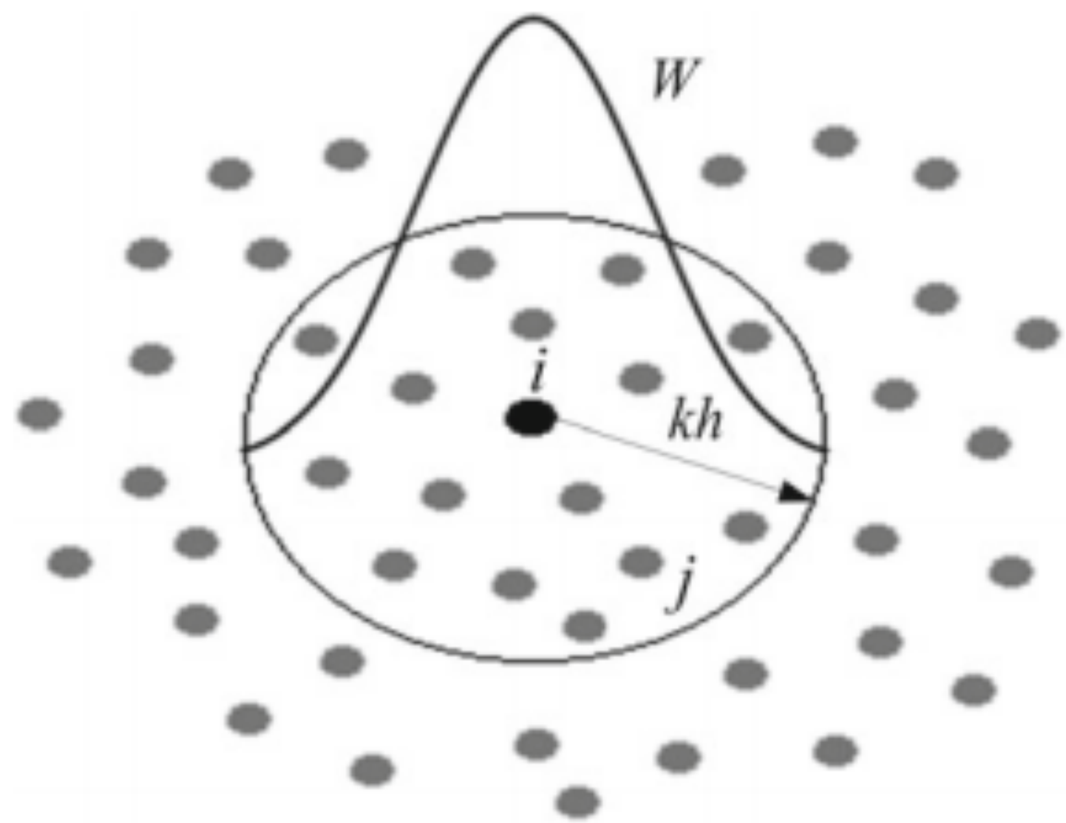
# Kernel

Alsteens Louis, Kumar Sanjay, Leyssens Thomas, Mathiazhagan Akilan

- Properties of the Kernel function
- Analytical development
    - Gradient
    - Divergent
    - Laplacian
- Different kernel functions
- Pros and cons of each Kernel function

# Properties of the kernel

According to you, what are the important properties of a kernel function?



# Properties of the kernel

- Positivity $\boldsymbol{W(|X - X'|) > 0}$
- Symmetry $\boldsymbol{W(|X - X'|) = W(|X' - X|)}$
- Unity $\boldsymbol{\int W(X - X')dX' = 1}$
- Compact support $\boldsymbol{W(|X - X'|) = 0, if\ |X - X'| > kh}$
- Convergence $\lim\limits_{h \to 0} \boldsymbol{\left(W(|X - X'|, h)\right) = \delta(|X - X'|)}$
- Decay $\boldsymbol{if\ |X - X'| \nearrow \quad then\, W(|X - X'|) \searrow}$
- Smoothness

# Discretisation of the Continuum Domain

$$< f(x) >= \int_{\Omega} f(X')W(X - X', h)dX'$$

$$f_i = \sum_{j=1}^{n} f_j W(X_i - X_j, h)\frac{m_j}{\rho_j}$$

Notation :

- subscript *i* : fixed particle
- subscript *j* : neighboring particle
- $m_k$ : mass of particle *k*
- $\rho_k$density of particle *k*
- $f_k$value of function in position of particle *k*

# Approximation of Divergent of a Vectorial Function

$$< \nabla \cdot \mathbf{f}(X) >= \int_\Omega [\nabla \cdot \mathbf{f}(X')] W(X - X', h) dX'$$

- $\nabla \cdot [\mathbf{f}W] = [\nabla \cdot \mathbf{f}]W + \mathbf{f} \cdot [\nabla W]$
- compact support property ($W(q) = 0$ if $q \in \partial\Omega$)

$$< \nabla \cdot \mathbf{f}(X) >= -\int_\Omega \mathbf{f}(X') \cdot \nabla W(X - X', h) dX'$$

- Discretisation

$$\nabla \cdot \mathbf{f_i} = -\sum_j \mathbf{f_j} \cdot \nabla W(X_i - X_j, h) \frac{m_j}{\rho_j}$$

BUT : poor results due to sensitivity to particle distribution

# Approximation of Divergent of a Vectorial Function

$$< \nabla \cdot \mathbf{f}(X) >= \int_{\Omega} [\nabla \cdot \mathbf{f}(X')] W(X - X', h) dX'$$

→ Using : $\nabla \cdot [\rho \mathbf{f}] = \rho[\nabla \cdot \mathbf{f}] + [\nabla \rho] \cdot \mathbf{f}$

$$\boxed{\nabla \cdot \mathbf{f_i} = -\frac{1}{\rho_i} \sum_j (\mathbf{f_j} - \mathbf{f_i}) \cdot \nabla W(X_i - X_j, h) m_j}$$

# Approximation of the Gradient of a scalar Function

$$< \nabla f(X) >= \int_{\Omega} [\nabla f(X')] W(X - X', h) dX'$$

Similarly to the divergent :

$$< \nabla f(X) >= -\int_{\Omega} f(X') \nabla W(X - X', h) dX'$$

Discretisation :

$$\nabla f_i = -\frac{1}{\rho_i} \sum_j (f_j - f_i) \nabla W(X_i - X_j, h) m_j$$

BUT : does not give symmetric results between 2 particles
(necessary for pressure gradient)

# Approximation of the Gradient of a scalar Function

$$< \nabla f(X) >= \int_{\Omega} [\nabla f(X')] W(X - X', h) dX'$$

→ to ensure symmetry, use of :

$$\nabla (\frac{f(X}{\rho}) = \frac{\nabla f(x)}{\rho} - \frac{f(X)}{\rho^2} \nabla \rho \Leftrightarrow \nabla f(X) = \rho \nabla (\frac{f(X)}{\rho}) + \frac{f(X)}{\rho} \nabla \rho$$

After discretising both terms :

$$\boxed{\nabla f_i = -\rho_i \sum_j m_j (\frac{f_i}{\rho_i^2} + \frac{f_j}{\rho_j^2}) \nabla W(X_i - X_j, h)}$$

# Approximation of the Laplacian of a scalar function

Obtained through Taylor expansion around a fixed spatial point :

$$f(x',y') = f(x,y) + (x'-x)\frac{\partial f}{\partial x} + (y'-y)\frac{\partial f}{\partial y} + \frac{1}{2}(x'^2 - x^2)\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}(y'^2 - y^2)\frac{\partial^2 f}{\partial y^2} + \frac{1}{2}(x'-x)(y'-y)\frac{\partial^2 f}{\partial x \partial y} + h.o.t.$$

Assuming 3rd order error, multiplying by , and integrating :

$$LHS : \int_\Omega f(x',y')\Delta X \cdot \nabla W dX' =$$

$$RHS : \int_\Omega f(x,y)\Delta X \cdot \nabla W dX' + \underbrace{\int_\Omega (x'-x)\frac{\partial f}{\partial x}\Delta X \cdot \nabla W dX' + \int_\Omega (y'-y)\frac{\partial f}{\partial y}\Delta X \cdot \nabla W dX'}_{= 0}$$

$$+ \underbrace{\int_\Omega \frac{1}{2}(x'-x)^2\frac{\partial^2 f}{\partial x^2}\Delta X \cdot \nabla W dX' + \int_\Omega \frac{1}{2}(y'-y)^2\frac{\partial^2 f}{\partial y^2}\Delta X \cdot \nabla W dX'}_{= -\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \nabla^2 f|_{x,y}}$$

$$+ \underbrace{\int_\Omega (x'-x)(y'-y)\frac{\partial^2 f}{\partial x \partial y}\Delta X \cdot \nabla W dX'}_{= 0}$$

$$\Delta X = \frac{(X - X')}{|X - X'|^2}$$

# Approximation of the Laplacian of a scalar function

$$LHS : \int_\Omega f(x', y')\Delta X \cdot \nabla W dX' =$$

$$RHS : \int_\Omega f(x, y)\Delta X \cdot \nabla W dX' + \underbrace{\int_\Omega (x' - x)\frac{\partial f}{\partial x}\Delta X \cdot \nabla W dX' + \int_\Omega (y' - y)\frac{\partial f}{\partial y}\Delta X \cdot \nabla W dX'}_{= 0}$$

$$+ \underbrace{\int_\Omega \frac{1}{2}(x' - x)^2\frac{\partial^2 f}{\partial x^2}\Delta X \cdot \nabla W dX' + \int_\Omega \frac{1}{2}(y' - y)^2\frac{\partial^2 f}{\partial y^2}\Delta X \cdot \nabla W dX'}_{= -\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \nabla^2 f|_{x,y}}$$

$$+ \underbrace{\int_\Omega (x' - x)(y' - y)\frac{\partial^2 f}{\partial x \partial y}\Delta X \cdot \nabla W dX'}_{= 0}$$

$$\Delta X = \frac{(X - X')}{|X - X'|^2}$$

Finally,

$$\nabla^2 f|_{x,y} = -2\int_\Omega [f(x', y') - f(x, y)]\Delta X \cdot \nabla W dX'$$

# Approximation of the Laplacian of a scalar function

$$\nabla^2 f|_{x,y} = -2\int_{\Omega}[f(x',y') - f(x,y)]\Delta X \cdot \nabla W dX'$$

Discretisation :

Cartesian coordinates system :

$$\boxed{\Delta f_i = 2\sum_j \frac{m_j}{\rho_j}(f_i - f_j)\frac{(X_i - X_j)}{|X_i - X_j|^2} \cdot \nabla W(X_i - X_j, h)}$$

Polar coordinates system :

$$\boxed{\Delta f_i = 2\sum_j \frac{m_j}{\rho_j}(f_i - f_j)\frac{\partial W(\mathbf{r_i} - \mathbf{r_j})}{\partial r}\frac{1}{|\mathbf{r}_{ij}|}}$$

# Error estimation in SPH Methods

Scheme 1 ( Standard Scheme) :

$$\langle \nabla f \rangle_i = \sum_j \frac{m_j}{\rho_j} f_j \nabla W_{ij} \,, where \quad W_{ij} = W\left(X_i - X_j \,, h\right)$$

$$\langle \nabla f \rangle_i = \sum_j \frac{m_j}{\rho_j} \left( f_i - X_{ij} \nabla f|_i + \frac{1}{2} X_{ij} X_{ij} : \nabla\nabla f|_i + .. \right) \nabla W_{ij}$$

$$\langle \nabla f \rangle_i = f_i \sum_j \frac{m_j}{\rho_j} \nabla W_{ij} - \nabla f|_i \sum_j X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \nabla\nabla f|_i : \sum_j \frac{1}{2} X_{ij} X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \ \ldots$$

# Error estimation in SPH Methods

Scheme 1 ( Standard Scheme) :

$$\langle \nabla f \rangle_i = f_i \sum_j \frac{m_j}{\rho_j} \nabla W_{ij} - \nabla f|_i \sum_j X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \nabla\nabla f|_i : \sum_j \frac{1}{2} X_{ij} X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \ \ldots$$

Scheme 1 Truncation Error :

$$Er_i = \langle \nabla f \rangle_i - \nabla f|_i \ = f_i \sum_j \frac{m_j}{\rho_j} \nabla W_{ij} - \nabla f|_i \cdot \left( I + \sum_j \frac{m_j}{\rho_j} X_{ij} \nabla W_{ij} \right) + \frac{1}{2} \nabla\nabla f|_i : \sum_j \frac{m_j}{\rho_j} X_{ij} X_{ij} \nabla W_{ij} + ..$$

$$\boxed{= \ f_i \sum_j \frac{m_j}{\rho_j} \nabla W_{ij} - \nabla f|_i \cdot \left( I + B_i^{-1} \right) + \cdots} \text{, where } B_i = \left[ \sum_j \frac{m_j}{\rho_j} X_{ij} \nabla W_{ij} \right]^{-1}$$

## Error estimation in SPH Methods

Scheme 2 ( First derivative of a constant Function is 0) :

$$\langle \nabla u \rangle_i = \sum_j \omega_j (u_j - u_i) \nabla W_{ij} \,, where \quad W_{ij} = W(r_i - r_j \,, h)$$

$$\langle \nabla u \rangle_i = \sum_j \omega_j \left( -r_{ij} \cdot \nabla u|_i + \frac{1}{2} r_{ij} r_{ij} : \nabla\nabla u|_i + .. \right) \nabla W_{ij}$$

$$\langle \nabla u \rangle_i = -\nabla u|_i \sum_j r_{ij} \omega_j \nabla W_{ij} + \nabla\nabla u|_i : \sum_j \frac{1}{2} r_{ij} r_{ij} \omega_j \nabla W_{ij} + \ \ldots$$

# Error estimation in SPH Methods

$$\langle \nabla f \rangle_i = -\nabla f|_i \sum_j X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \nabla\nabla f|_i : \sum_j \frac{1}{2} X_{ij} X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \ \dots$$

Scheme 2 Truncation Error :

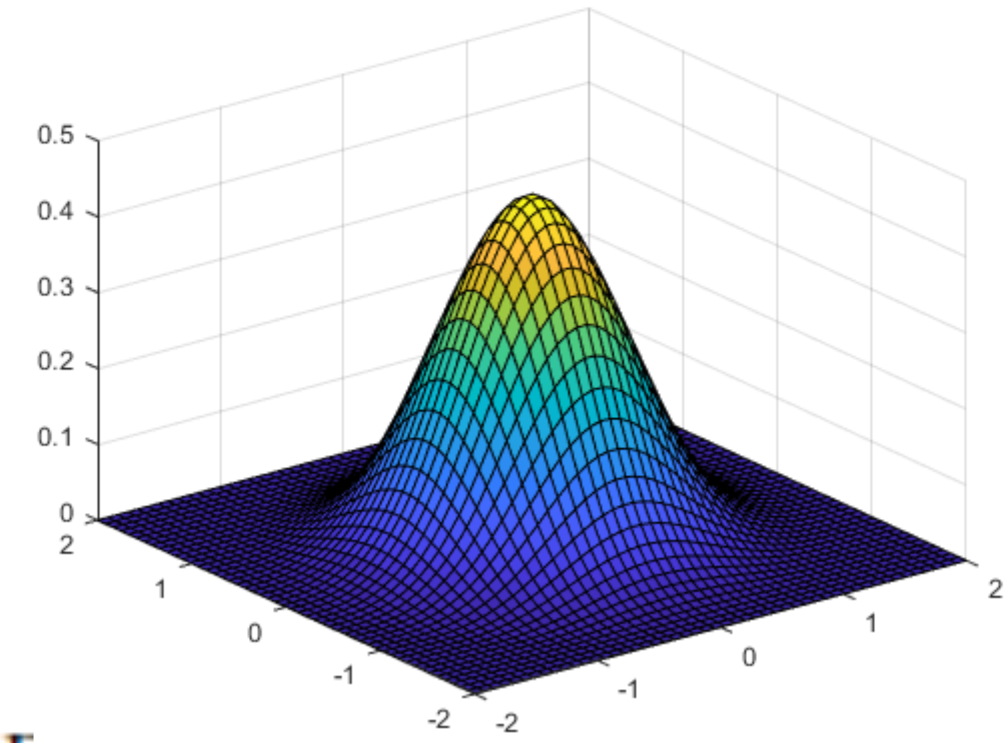
$$Er_i = \langle \nabla f \rangle_i - \ \nabla f|_i \ = 0 + \nabla f|_i \cdot \left( \sum_j \frac{m_j}{\rho_j} X_{ij} \nabla W_{ij} - I \right) + \frac{1}{2} \nabla\nabla f|_i : \sum_j \frac{m_j}{\rho_j} X_{ij} X_{ij} \nabla W_{ij} + \dots$$

$$\boxed{= 0 + \nabla f|_i \cdot \left( I + B_i^{-1} \right) + \frac{1}{2} \nabla\nabla f|_i : \sum_j \frac{m_j}{\rho_j} \mathrm{X}_{ij} X_{ij} \nabla W_{ij} + \cdots}$$

# 1). Cubic Spline Kernel

$$W(X-X',h)=\alpha_D\begin{cases}\left(\frac{2}{3}-q^2+\frac{1}{2}q^3\right),0\le q\le 1\\ \left(\frac{1}{6}(2-q)^3\right),1<q\le 2\\ 0,\text{ in the other case.}\end{cases}$$

**Where,** $\alpha_D=\dfrac{15}{7\pi h^2}$ and $\dfrac{3}{2\pi h^3}$ in 2-D and 3-D



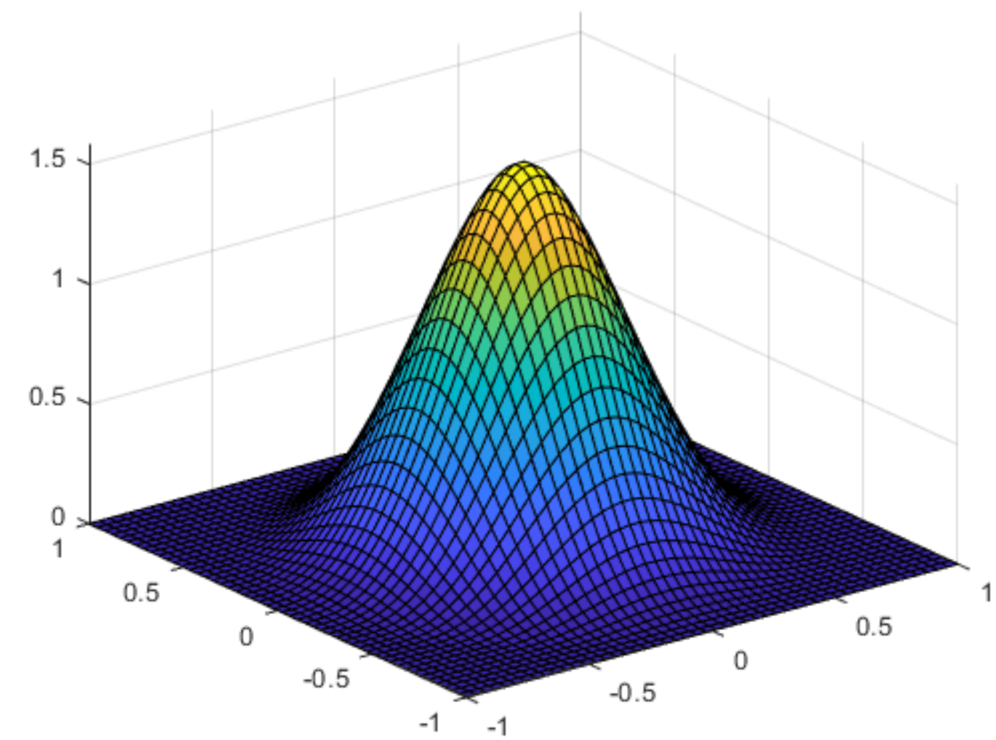

**For 2-D Domain**

# 2). Lucy's Quartric Kernel

$$W(X-X',h)=\alpha_D\begin{cases}(1+3q)\,(1-q)^3,0\le q\le 1\\ 0,\text{ in the other case.}\end{cases}$$

**Where,** $q=\dfrac{|(X-X',h)|}{h}$

$\alpha_D$ = $\dfrac{5}{\pi h^2}$ **and** $\dfrac{105}{16\pi h^3}$ **in 2-D & 3D Domain**

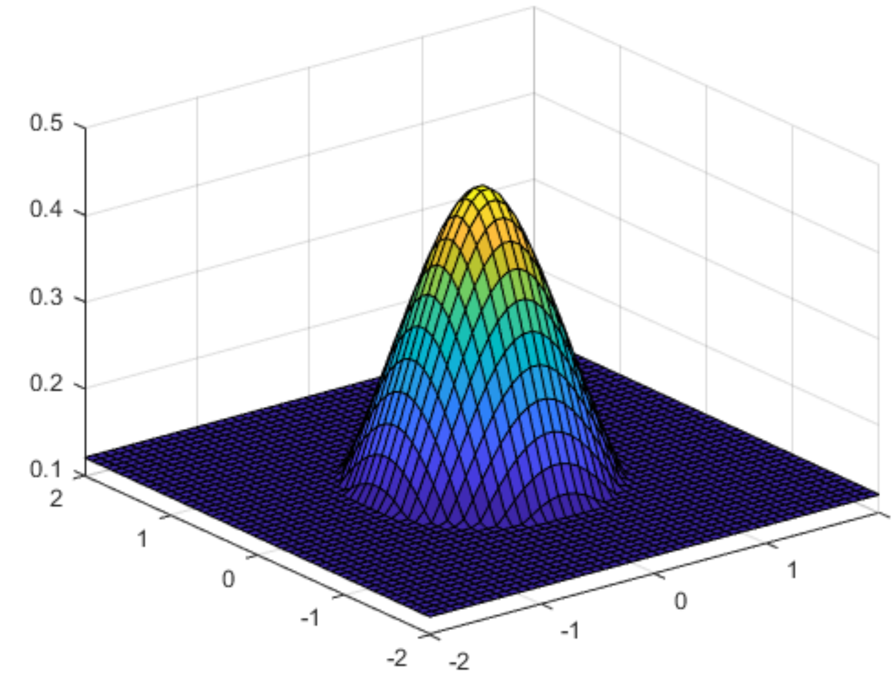



**For 2-D Domain**

# 3). New Quartic Kernel

$$W(X-X',h)=\alpha_D\begin{cases}\left(\frac{2}{3}-\frac{9}{8}q^2+\frac{19}{24}q^3-\frac{5}{32}q^4\right),0\le q\le 2\\ 0,\text{ in the other case.}\end{cases}$$

**Where,** $\alpha_D=\dfrac{15}{7\pi h^2}$ and $\dfrac{315}{208\pi h^3}$ in 2-D and 3-D domain



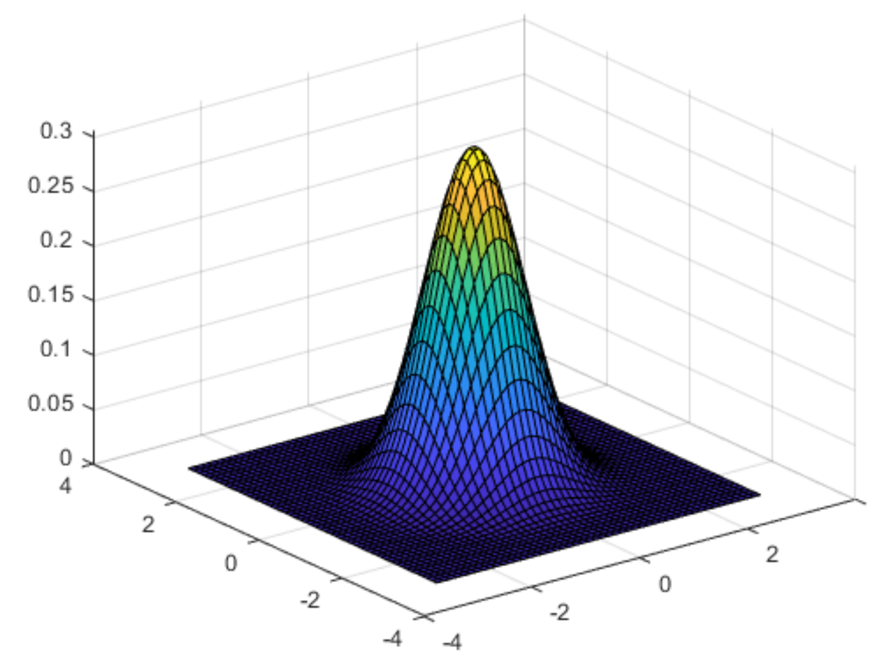

**For 2-D Domain**

# 4). Quintic Spline Kernel

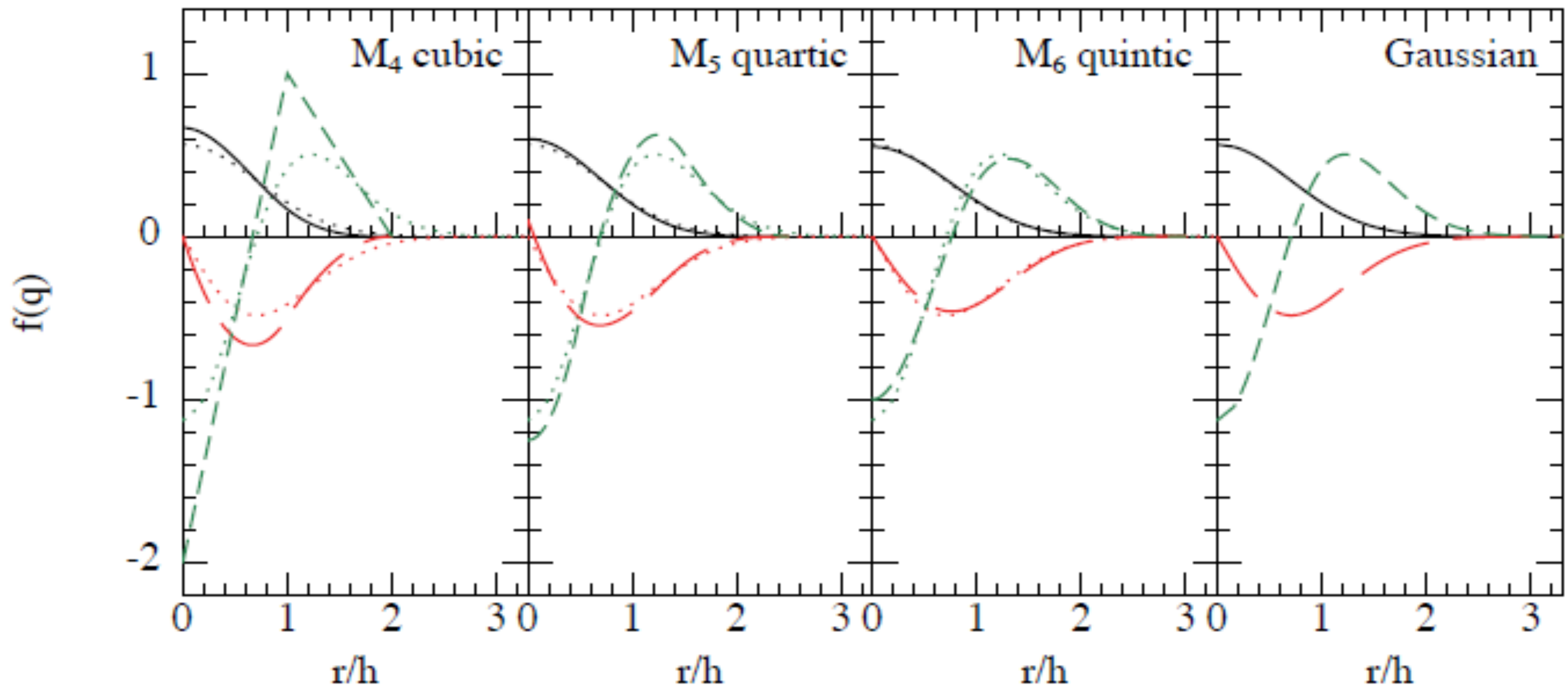
$$W(X-X',h)=\alpha_D\begin{cases}(3-q)^5-6(2-q)^5+15(1-q)^5,0\le q\le 1\\ (3-q)^5-6(2-q)^5,1<q\le 2\\ (3-q)^5,2<q\le 3\\ 0,\text{ in the other case.}\end{cases}$$

where

$\alpha_D=\dfrac{7}{478\pi h^2}$ and $\dfrac{1}{120\pi h^3}$ in 2-D and 3-D domain




**For 2-D Domain**




- B-spline kernel functions (solid lines) and their first (long-dashed) and second (short-dashed) derivatives, compared to the Gaussian (right panel and dotted lines).
- The number of neighbours increases in quartic and quintic spline functions compared to cubic spline, the smoothing scale h retains the same meaning with respect to the Gaussian.
- The higher order B-splines is a way to increase the smoothness of the kernel summations without altering the resolution length, and is very different to simply increasing the number of neighbours under the cubic spline

Thank you for your attention!

# Back-up slides

# Error estimation in SPH Methods

Scheme 3 ( Symmetric Form ) :

$$\langle \nabla f \rangle_i = \sum_j \frac{m_j}{\rho_j}\left(f_j + f_i\right)\nabla W_{ij}\,, where \quad W_{ij} = W\left(X_i - X_j\,, h\right)$$

$$\langle \nabla f \rangle_i = \sum_j \frac{m_j}{\rho_j}\left(2f_i - X_{ij}\cdot \nabla f|_i + \frac{1}{2}X_{ij}X_{ij} : \nabla\nabla f|_i + ..\right)\nabla W_{ij}$$

$$\langle \nabla f \rangle_i = 2f_i \sum_j \frac{m_j}{\rho_j}\nabla W_{ij} - \nabla f|_i \sum_j X_{ij}\frac{m_j}{\rho_j}\nabla W_{ij} + \nabla\nabla f|_i : \sum_j \frac{1}{2}X_{ij}X_{ij}\frac{m_j}{\rho_j}\nabla W_{ij} + \ \ldots$$

# Error estimation in SPH Methods

$$\langle \nabla f \rangle_i = 2f_i \sum_j \frac{m_j}{\rho_j} \nabla W_{ij} - \nabla f|_i \sum_j X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \nabla\nabla f|_i : \sum_j \frac{1}{2} X_{ij} X_{ij} \frac{m_j}{\rho_j} \nabla W_{ij} + \ \dots$$

Scheme 3 Truncation Error :

$$Er_i = \langle \nabla f \rangle_i - \nabla f|_i = 2f_i \sum_j \frac{m_j}{\rho_j} \nabla W_{ij} + \nabla f|_i \cdot \left( -I + \sum_j \frac{m_j}{\rho_j} X_{ij} \nabla W_{ij} \right) + \frac{1}{2} \nabla\nabla f|_i : \sum_j \frac{m_j}{\rho_j} X_{ij} X_{ij} \nabla W_{ij} + \dots$$

$$\boxed{= 2f_i \sum_j \frac{m_j}{\rho_j} \nabla W_{ij} + \nabla f|_i \cdot \left( B_i^{-1} - I \right) + \cdots}$$

# Error estimation in SPH Methods

Scheme 4 :

$$\langle \nabla u \rangle_i = \sum_j \omega_j (u_j - u_i) B_i \cdot \nabla W_{ij} \, ,$$

$$where \quad B_i = \left[ \sum_j \omega_j r_{ij} \nabla W_{ij} \right]^{-1}$$

Scheme 4 Truncation Error :

$$\boxed{\begin{gathered} Er_i = \langle \nabla u \rangle_i - \nabla u|_i \\ = \frac{1}{2} \nabla\nabla u|_i : \sum_j \omega_j r_{ij} r_{ij} B_i \nabla W_{ij} - \frac{1}{6} \nabla\nabla\nabla u|_i : \sum_j \omega_j r_{ij} r_{ij} r_{ij} B_i \nabla W_{ij} + .. \end{gathered}}$$

$Note$: *The second derivate is the lowest order term that can be identified in the truncation error. This is the only scheme capable of predicting the error of the first derivative of a linear fucntion.*