Applications of Representation Theory in McKay-Cartan Quivers

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1 Representation Theory

I consulted Bruce Sagan's *The Symmetric Group*[4] for the use of several definitions in this section.

1.1 Definition of a matrix representation of a finite group

Let G be a group written multiplicatively with identity ϵ. We assume G to be finite. Let $GL_n(\mathbb{C})$ be the set of $n \times n$ invertible matrices with $\{a_{11}, a_{12}, ..., a_{nn}\} \in \mathbb{C}$. Let Mat_n be the set of all $n \times n$ matrices with entries in $\mathbb{C}$. A *matrix representation* of G is a group homomorphism

$$X : G \to GL_n(\mathbb{C}).$$

Equivalently, to each g $\in$ G is assigned X(g) $\in \mathrm{Mat}_n$ such that

1. X(ϵ) = I, the identity matrix, and
2. X(gh) = X(g)X(h) for all g,h $\in$ G.

This definition allows us to make an abstract group object more concrete by describing its elements by matrices and their algebraic operations, such as matrix multiplication. Since the theory of matrices and linear operations is well understood, representations of group objects in this way helps us to learn properties and simplify calculations on group theory.

1.2 The permutation representation

Let S_n be the symmetric group of degree n.
If a permutation $\sigma \in S_n$, then we let X(σ) = $(x_{i,j})_{1 \le i,j \le n}$, where

$$x_{i,j} = \begin{cases} 1 & if\ \sigma(j) = i, \\ 0 & otherwise \end{cases}$$

We define X(σ) to be a *permutation matrix*, as it contains only zeros and ones, with a unique one in every row and column. In accordance with the definition of a matrix representation, the permutation representation can be thought of as the homomorphism sending a permutation to its corresponding permutation matrix

$$X : S_n \mapsto GL_n(\mathbb{R})$$
$$\sigma \mapsto X(\sigma)$$

For example:

$$X(\epsilon) \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad X(2,1,3) \Longleftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.3 Character of a representation

Let X(g), g $\in$ G, be a matrix representation. Then the character of X(g) is

$$\chi(g) = \mathrm{tr}(X(g)),$$

where tr denotes the trace of the matrix. An important property of trace is that, for all A,B$\in$G, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. If two permutations x,y$\in$G are conjugate to each other, then $\mathrm{tr}(X(x)) = \mathrm{tr}(X(y))$.

Proof:

Let x= gyg^{-1} for some g$\in$G.

$$\begin{aligned} \text{Then } \mathrm{tr}(X(x)) = \mathrm{tr}(X(gyg^{-1})) &= \mathrm{tr}(X(g)X(y)X(g)^{-1}) \\ &= \mathrm{tr}(X(g)X(g)^{-1}X(y)) \text{ by property of trace.} \\ &= \mathrm{tr}(X(y)) \quad \square \end{aligned}$$

However, the converse is not true.

Example:

Consider the trivial representation

$$\text{triv: } S_3 \mapsto GL_1(\mathbb{C})$$

which maps any permutation in S_3 to 1. Hence, for all $\sigma \in S_3$, X(σ)=1. The permutations (12) and (123) are not conjugate, but have the same character (since triv(a) $\cdot$ triv(b) = triv(ab)).

Proposition: The character of the permutation representation is simply the number of fixed points in the permutation.

Proof. Let the permutation matrix $p(\sigma)_{i,j} = \begin{cases} 1 & \text{if} j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$
Then a point j is fixed if $j = i$.
Then $j = i$ will be at the diagonal of any $n \times n$ matrix.
Hence, for each fixed point, there will exist a 1 in the diagonal.
Then tr(p(σ)) will sum all of the 1s in the diagonal, or in other words, all of the fixed points.
Then $\chi(p(\sigma)) = tr(p(\sigma))$ is the number of fixed points. □

1.4 Stable sub-spaces

Let S be a set and T a function on S. We say that S is a stable subspace of T if T(S)⊆S. When working to find "S_3-stable" sub-spaces, we look to find sub-spaces in $\mathbb{R}^3$ that are closed under multiplication by the permutation matrix representations. Here, I will give an example and a non-example:
Example:

$$M = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix} : a \in \mathbb{R} \right\}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ a \\ a \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$$

This vector is closed under multiplication with every matrix in S_3. Hence, M is a stable subspace.
Non-example:

$$W = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \text{for all } k \in \mathbb{R} \right\}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix} \notin W$$

This vector is **not** closed under multiplication with every matrix in S_3. Therefore, W is not a stable subspace of S_3.

Definition 1.4.1. A sub-representation of a representation M of a group G is a representation R such that R is a stable subspace of M.

1.5 The reflection representation

Theorem 1.5.1. *(Maschke's Theorem) Every representation of a finite group can be decomposed uniquely into irreducible representations.*

Recall the vector $X = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$.
Consider the vector $X^{-1} = \langle v \in \mathbb{R}^3 : v \cdot x = 0 \text{ for all } x \in X \rangle$.
Then $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $v \cdot x = v_1 + v_2 + v_3$.

Any v such that $v_1 + v_2 + v_3 = 0$ can always be written in terms of $w_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

The vector $X^\perp = \langle w_1, w_2 \rangle$ is also S_3-stable, since any multiplication by a permutation matrix will either result in $\pm w_1$, $\pm w_2$, or some vector v that can be represented in terms of w_1 and w_2.

Furthermore, let us define vectors e_1, e_2, e_3 as follows:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $w_1 = e_1 - e_2$ and $w_2 = e_2 - e_3$.

Suppose for some $\sigma \in S_3$, we take $\sigma \cdot e_i = e_{\sigma_i}$. Then for $w_1 = e_1 - e_2$ and $w_2 = e_2 - e_3$, we can calculate $\sigma \cdot w_1 = e_{\sigma_1} - e_{\sigma_2}$ and $\sigma \cdot w_2 = e_{\sigma_2} - e_{\sigma_3}$, which can again be represented in terms of w_1 and w_2.

Example:

$(2,1,3) \cdot (e_1 - e_2) = e_2 - e_1 = -w_1$

$(2,1,3) \cdot (e_2 - e_3) = e_1 - e_3 = w_1 + w_2$

Our goal is to construct a representation ρ such that

$$\rho : S_3 \mapsto GL_2(\mathbb{R}).$$

We refer to $\rho(\sigma)$ as the reflection representation, where $\rho_j = \text{sgn}(\sigma \cdot w_j)$. To continue off the previous example of $(2,1,3)$:

Example:

$(2,1,3) \mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.

Note that the trace of this matrix is 0, but $\chi((2,1,3)) = 1$.

Proposition: The character of an element of the reflection representation is the number of fixed points -1.

1.6 Irreducible representations

Representations that are reducible can be broken down into smaller representations. Hence, an irreducible representation is one that can not be broken down further. Irreducibles are found by finding stable sub-spaces of reducible representations.

Definition 1.6.1. A representation X is *irreducible* if it contains no non-trivial, stable subspace.

Here, I will demonstrate how the permutation representation is not irreducible (i.e., it *is* reducible):

Consider the S_3-stable sub-space X $= \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \subset \mathbb{R}^3$.

Because X is S_3-stable, it is a sub-representation of the permutation representation. We know this is true because $X(\pi) \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for all $\pi \in S_3$. Additionally, X is an irreducible representation; it can be thought of as the homomorphism

$$X : S_3 \to GL_1(\mathbb{R}), \text{ or } X : \sigma \to [1].$$

Since there exists an irreducible sub-representation of the permutation representation, the permutation representation must be reducible (not irreducible). *Note: X is actually the trivial representation.*

1.6.1 Inner Character Product

The inner character product can be used to determine if a representation is irreducible. Let χ and Ψ be any two characters from a group G to the complex numbers $\mathbb{C}$. The *inner character product* of χ and Ψ is

$$\langle \chi, \Psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\Psi(g)}.[4]$$

This is useful because if we take a representation X,

$$\langle X, X \rangle_G = 1 \iff X \text{ irreducible}.$$

Take the permutation representation, for example:

$$\begin{aligned} \langle \rho, \rho \rangle_{S_3} &= \frac{1}{6} \sum_{\sigma \in S_3} \rho(\sigma)\rho(\sigma) \\ &= \frac{1}{6} \sum_{\sigma \in S_3} \rho(\sigma)^2 \;\; (\rho(\sigma) \text{ indicating the number of fixed points in each permutation}) \\ &= \frac{1}{6}(9 + 1 + 1 + 1 + 0 + 0) \\ &= \frac{1}{6} \cdot 12 \neq 1, \text{ therefore not irreducible} \end{aligned}$$

Thus showing again that the permutation representation isn't irreducible. Rather, it can be reduced.

2 Chip Firing

I consulted Scott Corry and David Perkinson's *Divisors and Sandpiles* [2] for the use of several definitions in this section.

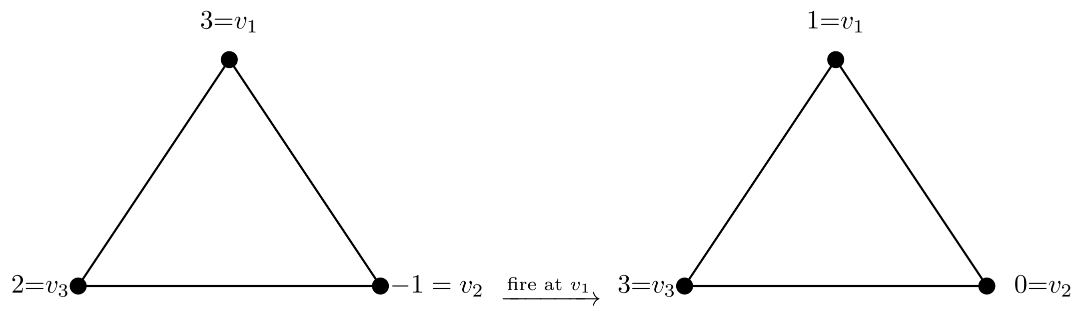
2.1 Firing on a graph

Definition 2.1.1. A graph is an ordered pair $G = (V, E)$ comprising of V, a set of vertices, and E, a set of edges such that $E \subseteq \{\{x, y\} | x, y \in V \text{ and } x \neq y\}$.

We define the free abelian group on V to be $\mathbb{Z}V$, the abelian group generated by all possible linear combinations of elements of V. Then we define a divisor on a graph G to be an element of the free abelian group on the vertices:

$$Div(G) = \mathbb{Z}V = \left\{ \sum_{v \in V} D(v)v : D(v) \in \mathbb{Z} \right\} [2]$$

When we say "fire" at v_n, it is to say that a chip is being sent along each edge incident to v_n. For example, consider the following divisor with chip values for v_1, v_2, v_3 being 3,1, and 2, respectively:



A **quiver** is similar to a divisor, with the difference being that the edges on quivers have direction, and loops and multiple edges between two vertices are allowed.

Note: The divisors above are **weighted**, meaning that all vertices are assigned a value.

2.2 The Laplacian of a graph

The *adjacency matrix* A_{ij} of a graph $G \in Div(G)$ is defined as the number of edges between v_i and v_j. The adjacency matrix A has $i - j$-th entry A_{ij} as the number of edges between v_i and v_j. The formal definition of this adjacency matrix is:

$$A_{ij} = \begin{cases} 1 & \text{if there exists an edge i} \longrightarrow \text{j} \\ 0 & \text{otherwise} \end{cases}$$

The *degree matrix* of a graph G shows how many vertices each vertex is connected to by an edge. The degree matrix D has $i - j$-th entry D_{ij} as the number of vertices v_i is connected to by an edge. To be more clear, the entry D_{ij} can also be expressed as

$$D_{ij} = \begin{cases} \deg_G(V_i) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \quad [2]$$

If a graph G has n vertices, then its Laplacian L is an $n \times n$ matrix, where

$$\text{L} = (\text{degree matrix}) - (\text{adjacency matrix}).$$

If we take the example of G from 2.1, we can compute the Laplacian as follows:

$$\begin{aligned} L &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

Interestingly, given a divisor $D \in Div(G)$, firing at the vertex v_i is equivalent to subtracting the i-th column of L from D. For example, if we use L to represent the divisor from 2.1, we can show that subtracting the first column of L from D will yield {1,0,3}:

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

2.3 McKay-Cartan quivers

Let $X : G \mapsto GL_n(\mathbb{C})$ be a representation of a finite group with character χ_X. Let $X_1, ..., X_n$ be the irreducible representations of G, with X_1 being the trivial representation, with respective characters $\chi_1, ..., \chi_n$. For each i, denote the character of the tensor product $p \otimes p_i$ by

$$\chi_p \cdot \chi_i = \sum_{j=1}^{n} m_{ij} x_i.$$

We define the $n \times n$ matrix $M = (m_{ij})$ and the extended McKay-Cartan matrix $\tilde{C} = nI_n - M$. The McKay Cartan matrix is the submatrix C formed by removing the first row and first column of $\tilde{C}$. [1]

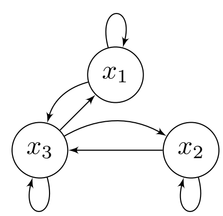
A **McKay-Cartan** quiver is a directed graph with vertices $\{X_1, ..., X_j\}$ and m_{ij} directed edges from X_i to X_j for each i,j. Similar to how we could chip fire on divisors using the Laplacian matrix L, we can fire on a McKay-Cartan quiver using the Extended McKay-Cartan matrix $\tilde{C}$

$$\tilde{C} = nI_n - M,$$

with M representing the adjacency matrix. Say we take the reflection representation in S_3 to represent M for our quiver:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

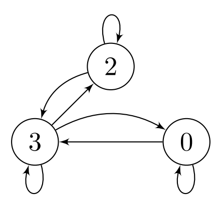
Then we can visually represent this matrix with the following McKay quiver:



Because our matrix M is representing the reflection representation in S_3, we know that $n = 3$. Thus, our Extended McKay-Cartan matrix $\tilde{C}$ will be calculated as such:

$$\tilde{C} = 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

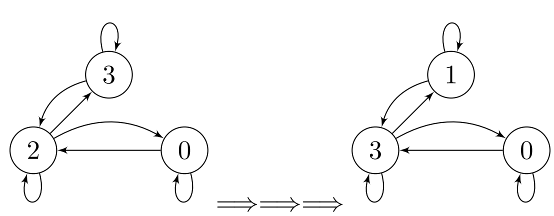
Much like how we were able to use the Laplacian matrix to calculate values of a weighted divisor when firing at different vertices, we can similarly use the Extended McKay-Cartan matrix to calculate values of a weighted quiver when firing at different vertices. Suppose I assign the following values to the quiver representing M:



Then if we want to fire at $v_1 = 3$, we subtract the first column of $\tilde{(}C)$ from the values of the weighted quiver. Like so:

$$\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Therefore, when we fire at v_1 with this quiver, it results in the following transformation:



3 Code

This code was made in Sage to compute matrices of $\tilde{C}$ associated to the reflection representation of S_n. By $[3, Prop.8]$, these matrices are computed as such:
The irreducible representations of S_n can be indexed by partitions λ of n. Let s_λ be the irreducible representation corresponding to λ.
Let $c(\lambda)$ be the number of indexes i in the partition of n such that the value of i is greater than $i+1$.
Proposition 8. Let $n \in \mathbb{Z}$ and λ a partition of n. Then

$$s_{n-1,1} \cdot s_\lambda = c(\lambda)s_\lambda + \sum s_\mu,$$

where $c(\lambda) = |\{i|\lambda_i > \lambda_{i+1}, 1 \leq i \leq l(\lambda) - 1\}|$ and the sum is over all partitions different from λ that can be obtained by removing one box and then adding a box to λ. [3]

```
def diffpartitions(L):
    X = []
    for i in Partition(L).down_list():
        M =Partition(i).up_list()
        for m in M:
            if m != L:
                X.append(m)
    return (X)

def matrow(L,n):
    F = diffpartitions(L)
    d = Partitions(n).list()
    H = [0]*len(d)
    for f in F:
        for i in range(len(d)):
            if f == d[i]:
                H[i]+=1
    return H

def strictdecrease(l):
    count = 0
    for i in range(0,len(l)-1):
        if l[i] - l[i+1] > 0:
            count+=1
    return count

def matrixgen(n):
    P = Partitions(n).list()
    Mat = []
    for i in P:
```

```
        Mat.append(matrow(i,n))
    return Mat

def fullmatrixgen(n):
    P = Partitions(n).list()
    Mat = matrixgen(n)
    for i in range(len(P)):
        Mat[i][i] = strictdecrease(P[i])
    return Mat
```

References

[1] S. Brauner, F. Glebe, and D. Perkinson. Enumerating linear systems on graphs. *Mathematische Zeitschrift*, pages 1–34, 2020.

[2] S. Corry and D. Perkinson. *Divisors and sandpiles*, volume 114. American Mathematical Soc., 2018.

[3] C. Gaetz. Critical groups of mckay–cartan matrices. 2016.

[4] B. E. Sagan. *The Symmetric Group*. Springer Sciences+Business Media, 2001.