DECOMPOSITION OF SEMIGROUP RINGS

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Abstract. We show how a semigroup ring $R[B]$ can be decomposed into a direct sum of submodules in $R[G(A)]$ over $R[A]$, where $A \subseteq B$ are cancellative abelian semigroups and R is an integral domain. In the case of a finite extension of positive affine semigroup rings we obtain an algorithm computing the decomposition. This can be applied to characterize ring theoretic properties in the simplicial case. Moreover, the regularity of homogeneous affine semigroup rings can be computed in terms of the decomposition. This leads to a new fast algorithm, which we use to confirm the Eisenbud-Goto conjecture for some cases. All algorithms are implemented in the Macaulay2 package MonomialAlgebras.

1. Introduction

Let R be an integral domain and $(B, +)$ a cancellative abelian semigroup. Denote by $G(B)$ the group generated by B, and by $R[B]$ the semigroup ring associated to B, that is the free R-module with basis consisting of the symbols t^a for $a \in B$, and multiplication given by the R-bilinear extension of $t^a \cdot t^b = t^{a+b}$. Hoa and Stückrad presented in [12] a decomposition of homogeneous simplicial affine semigroup rings into a direct sum of certain monomial ideals. They used this to bound the Castelnuovo-Mumford regularity of the semigroup ring. We will generalize this result in Theorem 2.1 showing that for a subsemigroup $A \subseteq B$ the $R[A]$-module $R[B]$ is isomorphic to a direct sum of submodules $I_g \subseteq R[G(A)]$ indexed by elements $g \in G := G(B)/G(A)$. Here we consider the $R[A]$-module structure on $R[B]$ given by inclusion.

We then focus on the case that K is a field and $A \subseteq B \subseteq \mathbb{N}^m$ are affine semigroups such that $K[B]$ is a finite $K[A]$-module. In this case the number of submodules I_g is finite, moreover, we can always choose them to be in $K[A]$. This allows us to give an algorithm computing the decomposition, which is implemented in our Macaulay2 [9] package MonomialAlgebras [3]. In case that B is simplicial, that is, B generates a simplicial cone, many ring theoretic properties of $K[B]$ can be described in terms of the semigroup B, such as being Cohen-Macaulay, Buchsbaum, Gorenstein, normal, or seminormal, see [7, 14, 15, 17, 21]. If A is chosen to be generated by elements on the extremal rays of B, all these properties can be characterized in terms of the decomposition, see Proposition 3.1. Using this we can provide functions to test those properties efficiently.

In Section 4 we consider the case that B is homogeneous, that is, there is a grading on $K[B]$ in which every element in the minimal generating set $\mathrm{Hilb}(B)$ of B has degree 1. The Castelnuovo-Mumford regularity of $K[B]$ (see Section 4) is usually computed from a minimal

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graded free resolution of $K[B]$ as a canonical $K[x_1,\dots,x_n]$-module, where n is the cardinality of $\mathrm{Hilb}(B)$. The main problem is that this free resolution could have length n, and if n is large this computation becomes nearly impossible, say, for $n \geq 15$. By using the decomposition of $K[B]$ we can compute its regularity in terms of the regularity of the (monomial) ideals I_g in $K[A]$, under the assumption that A is generated by degree one elements. Thus, the problem of computing the regularity of $K[B]$ comes down to computing a minimal graded free resolution of I_g which could have length d, where d is the cardinality of $\mathrm{Hilb}(A)$. Since the decomposition can be obtained very efficiently by our package also in high codimension, our regularity computation is typically much faster than the usual one, see Section 4 for timings. This enables us to test the Eisenbud-Goto conjecture [6] for certain affine semigroup rings in Theorem 4.3.

2. Decomposition

For a subset X in $G(B)$ we denote $t^X := \{t^x \mid x \in X\}$.

Theorem 2.1. *Let $A \subseteq B$ be cancellative abelian semigroups, and let R be an integral domain. The $R[A]$-module $R[B]$ is isomorphic to the direct sum of submodules $I_g \subseteq R[G(A)]$ indexed by elements $g \in G := G(B)/G(A)$.*

Proof. Let

$$\Gamma'_g := \{b \in B \mid b \equiv g \bmod G(A)\}.$$

By construction, we have

$$R[B] = \bigoplus_{g \in G} R \cdot t^{\Gamma'_g}.$$

For each $g \in G$, choose an element $h_g \in G(B)$ such that $h_g \equiv g \bmod G(A)$. The module $R \cdot t^{\Gamma'_g}$ is an $R[A]$-submodule of $R[B]$ and, as such, it is isomorphic to

$$I_g := R \cdot \{t^{b-h_g} \mid b \in \Gamma'_g\} \subseteq R[G(A)].$$

□

Note also that the collection of indexed ideals $\{I_g \subseteq R[G(A)]\}_{g \in G}$, together with the elements $h_g \in G(B)$ are sufficient to determine the ring structure of $R[B]$; indeed, if $x \in I_{g_1}$ and $y \in I_{g_2}$ and $xy = z$ as elements of $R[G(A)]$ then as elements in the decomposition of $R[B]$

$$x \cdot_{R[B]} y = \frac{t^{h_{g_1}} t^{h_{g_2}}}{t^{h_{g_1+g_2}}} z \in I_{g_1+g_2}.$$

In the following we consider the case that the coefficient ring is a field, denoted by K, and $A \subseteq B \subseteq \mathbb{N}^m$ are affine semigroups, that is, A and B are finitely generated submonoids of $(\mathbb{N}^m, +)$. We can always choose a minimal subset B_A of B such that $K[B] = K[A] \cdot t^{B_A}$ as $K[A]$-modules. Note that

$$B_A = \{x \in B \mid x - a \notin B \ \forall a \in A \setminus \{0\}\}.$$

Let $g \in G$ and $\Gamma_g = \{b \in B_A \mid b \equiv g \bmod G(A)\}$. By construction t^{Γ_g} is a minimal generating set of $K \cdot t^{\Gamma'_g}$ as a $K[A]$-module. Thus, to compute the decomposition we want the set B_A to be finite. This is clearly equivalent to $K[B]$ being a finite $K[A]$-module. Moreover, it is equivalent to $C(A) = C(B)$, where $C(X)$ denotes the cone spanned by X in $\mathbb{Q}^m$. Note that if B_A is finite, then $G(B)/G(A)$ is also finite.

The second claim follows from the observation that in case $C(A) \subsetneq C(B)$ we can choose an element $x \in B$ on a ray of $C(B)$ not in $C(A)$. So $nx \in B_A$ for all $n \in \mathbb{N}^+$, hence B_A is not finite. Conversely, if $C(A) = C(B)$ we get that B_A is finite, since for all $b \in B$ there is an $n \in \mathbb{N}^+$ such that $nb \in A$.

From these observations we obtain Algorithm 1 computing the set B_A and the decomposition.

Algorithm 1 Decompose monomial algebra

Input: A homogeneous monomorphism

$$\psi : K[y_1, \ldots, y_d] \to K[x_1, \ldots, x_n]$$

of $\mathbb{N}^m$-graded polynomial rings with $\deg x_i = b_i$ and $\deg y_j = e_j$ specifying the affine semigroups $A = \langle e_1, \ldots, e_d \rangle \subseteq B = \langle b_1, \ldots, b_n \rangle \subseteq \mathbb{N}^m$ with $C(A) = C(B)$.

Output: Ideals $I_g \subseteq K[A]$ and shifts $h_g \in G(B)$ with

$$K[B] \cong \bigoplus\nolimits_{g \in G} I_g(-h_g)$$

as $\mathbb{Z}^m$-graded $K[A]$-modules, where $G = G(B)/G(A)$.

1: Compute the toric ideal $I_B = \ker\varphi$ associated to B, where

$$\varphi : K[x_1, \ldots, x_n] \to K[B], \quad x_i \mapsto t^{b_i}.$$

2: Compute a monomial K-basis $v_1, \ldots, v_r$ of

$$K[x_1, \ldots, x_n]/(I_B + \psi(\langle y_1, \ldots, y_d \rangle)).$$

3: Reduce $\deg v_i$ modulo $G(A)$ to obtain $g_1, \ldots, g_f \in G(B)$ representing the elements of G.

4: We have

$$B_A = \{\deg v_1, \ldots, \deg v_r\} = \bigcup\nolimits_{i=1}^{f} \Gamma_{\bar{g}_i},$$

where $\Gamma_{\bar{g}_i}$ are the equivalence classes on B_A modulo $G(A)$.

5: **for** $i = 1$ to f **do**

6:   **for** $v \in \Gamma_{\bar{g}_i}$ **do**

7:     Fix some elements $c_{vj} \in \mathbb{Z}$ such that

$$v - g_i = \sum\nolimits_{j=1}^{d} c_{vj} e_j.$$

8:   **end for**

9:   Let $h_{\bar{g}_i} = g_i + \sum_{j=1}^{d} \min\{c_{vj} \mid v \in \Gamma_{\bar{g}_i}\} e_j$ and $I_{\bar{g}_i} = \left\langle t^{v - h_{\bar{g}_i}} \mid v \in \Gamma_{\bar{g}_i} \right\rangle$.

10: **end for**

11: **return** $I_{\bar{g}_i}$ and $h_{\bar{g}_i}$ for $i = 1, \ldots, f$.

Note that for $x \in \Gamma_{\bar{g}_i}$ we always have

$$x - h_{\bar{g}_i} = \sum\nolimits_{j=1}^{d} (c_{xj} - \min\{c_{vj} \mid v \in \Gamma_{\bar{g}_i}\}) e_j = \sum\nolimits_{j=1}^{d} n_j e_j \in A,$$

for some $n_j \in \mathbb{N}$, hence $I_{\bar{g}_i}$ are monomial ideals in $K[A]$.

Example 2.2. Consider $B = \langle (2,0,3),(4,0,1),(0,2,3),(1,3,1),(1,2,2)\rangle \subset \mathbb{N}^3$ and the submonoid $A = \langle (2,0,3),(4,0,1),(0,2,3),(1,3,1)\rangle$. We get the decomposition of B_A into equivalence classes $B_A = \{0,(2,4,4)\} \cup \{(1,2,2),(3,6,6)\}$. Choosing $h_1 = (-4,-2,-4)$ and $h_2 = (-3,0,-2)$ in $G(B)$ we have

$$\begin{aligned} K[B] &\cong K[A]\{t^{(4,2,4)},t^{(6,6,8)}\}(-h_1) \oplus K[A]\{t^{(4,2,4)},t^{(6,6,8)}\}(-h_2) \\ &\cong \langle xy, w^2z^2\rangle(-h_1) \oplus \langle xy, w^2z^2\rangle(-h_2), \end{aligned}$$

where $K[A] \cong K[w,x,y,z]/\langle w^3z^2 - x^2y^3\rangle$.

Example 2.3. Using our implementation of Algorithm 1 in the Macaulay2 package MonomialAlgebras we compute the decomposition of $K[B]$ over $K[A]$ for the following example:

```
i1:  loadPackage "MonomialAlgebras";
i2:  A = {{5,0},{0,5}};
i3:  B = {{5,0},{0,5},{1,4},{4,1}};
i4:  S = QQ[x_0 ..  x_3, Degrees=>B];
i5:  P = QQ[x_0, x_1, Degrees=>A];
i6:  f = map(S,P);
i7:  decomposeMonomialAlgebra f
o7:  HashTable{ {0,0}   => { ideal 1,            {0,0} }
                {1,-1}  => { ideal 1,            {1,4} }
                {-1,1}  => { ideal 1,            {4,1} }
                {2,-2}  => { ideal ( x_0, x_1^2 ),  {2,3} }
                {-2,2}  => { ideal ( x_0^2, x_1 ),  {3,2} }
```

The keys of the hash table represent the elements of G.

3. Ring theoretic properties

Recall that an affine semigroup $B \subseteq \mathbb{N}^m$ is called simplicial if it spans a simplicial cone, or equivalently, there are linearly independent elements $e_1,\ldots,e_d \in B$ with $C(B) = C(\{e_1,\ldots,e_d\})$.

Proposition 3.1. *Let K be a field, $B \subseteq \mathbb{N}^m$ a simplicial affine semigroup, and let A be the submonoid of B which is generated by linearly independent elements $e_1,\ldots,e_d$ of B with $C(A) = C(B)$. Let B_A be as above, and $K[B] \cong \bigoplus_{g\in G} I_g(-h_g)$ be the output of Algorithm 1 with respect to $A \subseteq B$. We have:*

1. *The depth of $K[B]$ is the minimum of the depths of the ideals I_g.*
2. *$K[B]$ is Cohen-Macaulay if and only if every ideal I_g is equal to $K[A]$.*
3. *$K[B]$ is Gorenstein if and only if $K[B]$ is Cohen-Macaulay and the set of shifts $\{h_g\}_{g\in G}$ has exactly one maximal element with respect to $\le$ given by $x \le y$ if there is an element $z \in B$ such that $x + z = y$.*
4. *$K[B]$ is Buchsbaum if and only if every proper ideal I_g is equal to the homogeneous maximal ideal of $K[A]$ and $h_g + b \in B$ for all $b \in B \setminus \{0\}$.*
5. *$K[B]$ is normal if and only if for every element x in B_A there exist $\lambda_1,\ldots,\lambda_d \in \mathbb{Q}$ with $0 \le \lambda_i < 1$ such that $x = \sum_{i=1}^d \lambda_i e_i$.*
6. *$K[B]$ is seminormal if and only if for every element x in B_A there exist $\lambda_1,\ldots,\lambda_d \in \mathbb{Q}$ with $0 \le \lambda_i \le 1$ such that $x = \sum_{i=1}^d \lambda_i e_i$.*

Proof. For every $x \in G(B)$ there are uniquely determined elements $\lambda_1^x, \ldots, \lambda_d^x \in \mathbb{Q}$ such that $x = \sum_{j=1}^d \lambda_j^x e_j$. Then by construction

$$h_g = \sum\nolimits_{j=1}^d \min\left\{\lambda_j^v \mid v \in \Gamma_g\right\} e_j.$$

Assertion (1) and (2) follow immediately; (2) was already proved in [21, Theorem 6.4]. Assertion (3) can be found in [21, Corollary 6.5].

(4) Let I_g be a proper ideal, equivalently, $\#\Gamma_g \geq 2$. The ideal I_g is equal to the homogeneous maximal ideal of $K[A]$ and $h_g+b \in B$ for all $b \in B\setminus\{0\}$ if and only if $\Gamma_g = \{m+e_1, \ldots, m+e_d\}$ for some m with $m+b \in B$ for all $b \in B \setminus \{0\}$ and this is equivalent to $K[B]$ is Buchsbaum, by [7, Theorem 9].

(5) We set $D_A = \{x \in G(B) \mid x = \sum_{i=1}^d \lambda_i e_i, \lambda_i \in \mathbb{Q} \text{ and } 0 \leq \lambda_i < 1 \ \forall i\}$. The ring $K[B]$ is normal if and only if $B = C(B) \cap G(B)$ by [14, Proposition 1]. We need to show that $C(B) \cap G(B) \subseteq B$ if and only if $B_A \subseteq D_A$. We have $B_A \subseteq D_A$ if and only if $D_A \subseteq B_A$, since B_A has $\#G = \#D_A$ equivalence classes and by definition of B_A. Note that $D_A \subseteq C(B) \cap G(B)$ and $D_A \cap B \subseteq B_A$. The assertion follows from the fact that every element $x \in C(B) \cap G(B)$ can be written as $x = x' + \sum_{i=1}^d n_i e_i$ for some $x' \in D_A$ and $n_i \in \mathbb{N}$.

(6) We set $\bar{D}_A := \{x \in B \mid x = \sum_{i=1}^d \lambda_i e_i, \lambda_i \in \mathbb{Q} \text{ and } 0 \leq \lambda_i \leq 1 \ \forall i\}$. By [15, Proposition 5.32] and [17, Theorem 4.1.1] $K[B]$ is seminormal if and only if $B_A \subseteq \bar{D}_A$, provided that $e_1, \ldots, e_d \in \operatorname{Hilb}(B)$. Otherwise there is a $k \in \{1, \ldots, d\}$ with $e_k = e_k' + e_k''$ and $e_k', e_k'' \in B \setminus \{0\}$. We set $A' = \langle e_1, \ldots, e_k', \ldots, e_d\rangle$ and $A'' = \langle e_1, \ldots, e_k'', \ldots, e_d\rangle$. Clearly $C(A) = C(A') = C(A'')$. We need to show that $B_A \subseteq \bar{D}_A$ if and only if $B_{A'} \subseteq \bar{D}_{A'}$. Let $x \in B_A \setminus \bar{D}_A$. If $x - e_k' \notin B$, then $x \in B_{A'} \setminus \bar{D}_{A'}$. If $x - e_k' \in B$, then $x - e_k' \in B_{A''} \setminus \bar{D}_{A''}$. Let $x \in B_{A'} \setminus \bar{D}_{A'}$, say $x = \sum_{j \neq k} \lambda_j e_j + \lambda_k e_k'$ and $\lambda_j > 1$ for some j. If $j \neq k$, then $x \in B_A \setminus \bar{D}_A$. Let $j = k$; consider the element $y = x + e_k'' - \sum_{j\neq k} n_j e_j \in B$ for some $n_j \in \mathbb{N}$ such that $\sum_{j \neq k} n_j$ is maximal. It follows that $y \in B_A \setminus \bar{D}_A$ and we are done. □

Example 3.2 (Macaulay-Curves)**.** Consider $B = \langle(\alpha, 0), (\alpha-1, 1), (1, \alpha-1), (0, \alpha)\rangle \subseteq \mathbb{N}^2$ and set $A = \langle(\alpha, 0), (0, \alpha)\rangle$, say $K[A] = K[x, y]$. Note that we have α equivalence classes. We get

$$K[B] \cong K[x,y]^3 \oplus \langle x^{\alpha-3}, y\rangle \oplus \langle x^{\alpha-4}, y^2\rangle \oplus \ldots \oplus \langle x, y^{\alpha-3}\rangle,$$

as $K[x,y]$-modules, where the shifts are omitted. Hence each ideal of the form $\langle x^i, y^j\rangle$, $1 \leq i, j \leq \alpha - 3$ with $i + j = \alpha - 2$ appears exactly once in the decomposition. Hence $K[B]$ is not Buchsbaum for $\alpha > 4$, since $\langle x^{\alpha-3}, y\rangle$ is a direct summand. In case that $\alpha = 4$ there is only one proper ideal $I_4 = \langle x, y\rangle$ and $h_4 = (2, 2)$; in fact $(2,2) + B \setminus \{0\} \subseteq B$ and therefore $K[B]$ is Buchsbaum. It follows immediately that $K[B]$ is Cohen-Macaulay for $\alpha \leq 3$, Gorenstein for $\alpha \leq 2$, seminormal for $\alpha \leq 3$, and normal for $\alpha \leq 3$. Note that we could also decompose $K[B]$ over the subring $K[A]$, where $A = \langle(2\alpha, 0), (0, 2\alpha)\rangle = K[x,y]$, for $\alpha = 4$ we would get

$$K[B] \cong K[x,y]^{15} \oplus \langle x, y\rangle$$

and the corresponding shift of $\langle x, y\rangle$ is again $(2, 2)$.

Example 3.3 ([17])**.** Let $B = \langle(1,0,0), (0,1,0), (0,0,2), (1,0,1), (0,1,1)\rangle \subset \mathbb{N}^3$, moreover, let $A = \langle(1,0,0), (0,1,0), (0,0,2)\rangle$, say $K[A] = K[x,y,z]$. We have

$$K[B] \cong K[B] \oplus \langle x, y\rangle(-(0,0,1)),$$

as $\mathbb{Z}^3$-graded $K[A]$-modules. Hence $K[B]$ is not Buchsbaum, since $\langle x, y\rangle$ is not maximal; moreover, $K[B]$ is seminormal, but not normal.

Example 3.4. Consider the semigroup $B = \langle (1,0,0),(0,2,0),(0,0,2),(1,0,1),(0,1,1)\rangle \subset \mathbb{N}^3$, and set $A = \langle (1,0,0),(0,2,0),(0,0,2)\rangle$. We get

$$K[B] \cong K[A] \oplus K[A](-(1,0,1)) \oplus K[A](-(0,1,1)) \oplus K[A](-(1,1,2)).$$

Hence $K[B]$ is Gorenstein, since $(1,0,1)+(0,1,1) = (1,1,2)$. Moreover, $K[B]$ is not normal, since $(1,0,1) = (1,0,0) + \frac{1}{2}(0,0,2)$, but seminormal.

Example 3.5. We illustrate our implementation of the characterizations given in Proposition 3.1 at Example 3.4:

```
i1:  B = {{1,0,0},{0,2,0},{0,0,2},{1,0,1},{0,1,1}};
i2:  isGorensteinMA B
o2:  true
i3:  isNormalMA B
o3:  false
i4:  isSeminormalMA B
o4:  true
```

Note that there are also commands `isCohenMacaulayMA` and `isBuchsbaumMA` available testing the Cohen-Macaulay and the Buchsbaum property, respectively.

4. Regularity

Let K be a field and $R = K[x_1, \dots, x_n]$ be a standard graded polynomial ring, that is, $\deg x_i = 1$ for all $i = 1, \dots, n$. Moreover, let R_+ be the homogeneous maximal ideal of R, and let M be a finitely generated graded R-module. We define the Castelnuovo-Mumford regularity $\mathrm{reg} M$ of M by

$$\mathrm{reg} M := \max \left\{ a(H^i_{R_+}(M)) + i \mid i \geq 0 \right\},$$

where $a(H^i_{R_+}(M)) := \max \left\{ n \mid [H^i_{R_+}(M)]_n \neq 0 \right\}$ and $a(0) = -\infty$; $H^i_{R_+}(M)$ denotes the i-th local cohomology module of M with respect to R_+. Note that $\mathrm{reg} M$ can also be computed in terms of the shifts in a minimal graded free resolution of M. An important application of the regularity is that it can be used to bound the degrees in certain minimal Gröbner bases by [1]. Thus, it is of interest to compute or bound the regularity of a homogeneous ideal. The following conjecture (Eisenbud-Goto) was made in [6]: If K is algebraically closed and I is a homogeneous prime ideal of R then for $S = R/I$

$$\mathrm{reg} S \leq \deg S - \mathrm{codim} S,$$

where $\deg S$ denotes the degree of S and $\mathrm{codim} S := \dim_K [S]_1 - \dim S$. This has been proved for several cases including dimension 2 by Gruson, Lazarsfeld, and Peskine [10], the Buchsbaum case by Stückrad and Vogel [22] (see also [23]), for $\deg S \leq \mathrm{codim} S + 2$ by Hoa, Stückrad, and Vogel [13], and for smooth surfaces in characteristic zero by Lazarsfeld [16]. There is also a stronger version in which S is only required to be reduced and connected in codimension 1; this version has been proved for curves by Giaimo in [8]. For homogeneous affine semigroup rings the conjecture holds in codimension 2 by a result of Peeva and Sturmfels [20]. Even in the simplicial setting the conjecture is open except for the isolated singularity case by Herzog and Hibi [11], if the ring is seminormal by [18], and for some other cases by [12, 19].

We now focus on computing the regularity in the homogeneous affine semigroup case. Homogeneity of an affine semigroup B in $\mathbb{N}^m$ with $\operatorname{Hilb}(B) = \{b_1, \ldots, b_n\}$ is equivalent to the existence of a group homomorphism $\deg : G(B) \to \mathbb{Z}$ with $\deg b_i = 1$ for all i. We always consider the R-module structure on $K[B]$ given by the homogeneous surjective K-algebra homomorphism $R \twoheadrightarrow K[B], x_i \mapsto t^{b_i}$. In terms of the decomposition the regularity can be computed as follows:

Proposition 4.1. *Let K be an arbitrary field and let $B \subseteq \mathbb{N}^m$ be a homogeneous affine semigroup. Fix a group homomorphism $\deg : G(B) \to \mathbb{Z}$ with $\deg b = 1$ for all $b \in \operatorname{Hilb}(B)$. Moreover, let A be a submonoid of B with $\operatorname{Hilb}(A) = \{e_1, \ldots, e_d\}$, $\deg e_i = 1$ for all i, and $C(A) = C(B)$. Let $K[B] \cong \bigoplus_{g \in G} I_g(-h_g)$ be the output of Algorithm 1 with respect to $A \subseteq B$. Then*

(1) $\operatorname{reg} K[B] = \max\{\operatorname{reg} I_g + \deg h_g \mid g \in G\}$; *where* $\operatorname{reg} I_g$ *denotes the regularity of the ideal $I_g \subseteq K[A]$ as a canonical $K[x_1, \ldots, x_d]$-module.*

(2) $\deg K[B] = \#G \cdot \deg K[A]$.

Proof. (1) Consider the $T = K[x_1, \ldots, x_d]$-module structure on $K[B]$ which is given by $T \twoheadrightarrow K[A] \subseteq K[B]$. Since $C(A) = C(B)$ we get by [4, Theorem 13.1.6]

$$a(H^i_{R_+}(K[B])) = a(H^i_{K[B]_+}(K[B])) = a(H^i_{T_+}(K[B])),$$

where $K[B]_+$ is the homogeneous maximal ideal of $K[B]$. Then the assertion follows from the fact $K[B] \cong \bigoplus_{g \in G} I_g(-\deg h_g)$ as $\mathbb{Z}$-graded T-modules.

(2) Follows from $\deg I_g = \deg K[A]$ for all $g \in G$. □

Using Proposition 4.1 we obtain Algorithm 2.

Algorithm 2 The regularity algorithm

Input: The Hilbert basis $\operatorname{Hilb}(B)$ of a homogeneous affine semigroup $B \subseteq \mathbb{N}^m$ and a field K.
Output: The Castelnuovo-Mumford regularity $\operatorname{reg} K[B]$.

1: Choose a minimal subset $\{e_1, \ldots, e_d\}$ of $\operatorname{Hilb}(B)$ with $C(\{e_1, \ldots, e_d\}) = C(B)$, and set $A = \langle e_1, \ldots, e_d \rangle$.
2: Compute the decomposition $K[B] \cong \bigoplus_{g \in G} I_g(-h_g)$ over $K[A]$.
3: Compute a hyperplane $H = \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid \sum_{j=1}^m a_j t_j = c\}$ with $c \neq 0$ such that $\operatorname{Hilb}(B) \subseteq H$. Define $\deg : \mathbb{R}^m \to \mathbb{R}$ by $\deg(t_1, \ldots, t_m) = (\sum_{j=1}^m a_j t_j)/c$.
4: **return** $\operatorname{reg} K[B] = \max\{\operatorname{reg} I_g + \deg h_g \mid g \in G\}$.

By Algorithm 2 the computation of $\operatorname{reg} K[B]$ reduces to computing minimal graded free resolutions of the monomial ideals I_g in $K[A]$ as a $K[x_1, \ldots, x_d]$-module.

Example 4.2. We apply Algorithm 2 using the decomposition computed in Example 2.2. A resolution of $I = \left\langle xy, w^2 z^2 \right\rangle$ as a $T = K[w, x, y, z]$-module is

$$0 \longrightarrow T(-5) \oplus T(-6) \xrightarrow{d} T(-2) \oplus T(-4) \longrightarrow I \longrightarrow 0$$

with

$$d = \begin{pmatrix} xy^2 & z^2 w^2 \\ -w & -xy \end{pmatrix},$$

hence $\mathrm{reg} I = 5$. The group homomorphism is given by $\deg b = (\sum_{j=1}^{4} b_j)/5$ and therefore $\mathrm{reg} K[B] = \max\{5-1, 5-2\} = 4$.

In case of monomial curves the ideals I_g are monomial ideals in two variables. Hence we can read off $\mathrm{reg} I_g$ by ordering the monomials with respect to the lexicographic order (see for example [19, Proposition 4.1]). This further improves the performance of the algorithm.

With respect to timings, we focus on dimension 3 comparing our implementation of Algorithm 2 in the Macaulay2 package MonomialAlgebras (marked in the tables by MA) with other methods. Here we consider the computation of the regularity via a minimal graded free resolution both in Macaulay2 (M2) and Singular [5] (S). Furthermore, we compare with the implementation of Bermejo, Gimenez, and Greuel in the Singular package mregular.lib [2] (MREG) which does not require the computation of a free resolution. We give the average computation times over n examples generated by the function `randomSemigroups(`α`,d,c,n)`. Starting with the standard random seed, this function generates n random semigroups in $\mathbb{N}^d$ with full dimension, coordinate sum α, and codimension c. All timings are in seconds on a single 2 GHz core and 4 GB of Ram. In the cases marked by a star at least one of the computations ran out of memory or did not finish within 1200 seconds. Note that the computation of $\mathrm{reg} I_g$ in step 4 of Algorithm 2 can be done fully parallel. This is not available in our Macaulay2 implementation so far.

The next table shows the comparison for $d = 3$, $\alpha = 5$, and $n = 15$ examples. For the coefficient field we will always choose $K = \mathbb{Q}$ as mregular.lib does not perform well over finite fields.

c	1	2	3	4	5	6	7	8	9
MA	.09	.12	.13	.15	.17	.18	.19	.27	.25
M2	.02	.02	.03	.04	.07	.14	.53	3.4	23
S	.01	.01	.01	.01	.03	.07	.23	.95	6.0
MREG	.01	.11	.57	4.0	32	41	130	$*$	$*$

c	10	11	12	13	14	15	16	17	18
MA	.34	.47	.42	.44	.54	.53	.67	.75	.86
M2	177	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
S	26	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
MREG	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

For small codimension c the decomposition approach has slightly higher overhead than the traditional algorithms. For larger codimensions, however, both the resolution approach and the Bermejo-Gimenez-Greuel implementation fail, whereas the average computation time for Algorithm 2 stays under one second.

To illustrate the performance of Algorithm 2 we present the computation times ($K = \mathbb{Q}$, $n = 1$) of our implementation for $d = 3$ and various α and c:

$\alpha\backslash c$	4	8	12	16	20	24	28	32	36	40	44	48	52
3	.11												
4	.10	.14	.41										
5	.15	.37	.24	.44									
6	.15	.78	.43	.66	.80	2.0							
7	.18	.45	.58	1.2	1.7	3.2	3.6	6.0					
8	.14	.60	1.1	1.6	2.0	3.7	4.1	6.4	11	22			
9	.24	.90	1.3	4.7	4.1	4.4	13	15	16	32	39	55	81

The following table is based on a similar setup for $d = 4$:

$\alpha\backslash c$	4	8	12	16	20	24	28	32	36	40	44	48	52
2	.16												
3	.15	.26	.29	.87									
4	.15	.72	1.1	1.1	1.3	2.1	4.1						
5	.24	.85	13	44	6.7	14	20	16	17	33	42	58	81
6	.48	20	220	510	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Due to the good performance of Algorithm 2 we can actually do the regularity computation for all possible semigroups B in $\mathbb{N}^d$ such that the generators have coordinate sum α for some α and d. This confirms the Eisenbud-Goto conjecture for some cases.

Proposition 4.3. *The regularity of* $\mathbb{Q}[B]$ *is bounded by* $\deg\mathbb{Q}[B] - \operatorname{codim}\mathbb{Q}[B]$*, provided that the minimal generators of* B *in* $\mathbb{N}^d$ *have fixed coordinate sum* α *for* $d = 3$ *and* $\alpha \leq 5$*, for* $d = 4$ *and* $\alpha \leq 3$*, as well as for* $d = 5$ *and* $\alpha = 2$*.*

Proof. The list of all generating sets $\operatorname{Hilb}(B)$ together with $\operatorname{reg}\mathbb{Q}[B]$, $\deg\mathbb{Q}[B]$, and $\operatorname{codim}\mathbb{Q}[B]$ can be found under the link given in [3]. $\square$

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