RATIONALMAPS, A PACKAGE FOR MACAULAY2

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Abstract.

1. Introduction

[DHS12]

2. Base Loci

We begin with the problem of computing the base locus of a map to projective space. Let X be a projective variety over k and let $F : X \longrightarrow \mathbb{P}^m_k$ be a rational map of X to projective space. Then we can choose some representative $(f_0, \cdots, f_m)$ of F, where each f_i is the i^{th} coordinate of F. A priori, each f_i is in $K = \operatorname{frac} R$, where R is the coordinate ring of X. However, we can get another representative of F by clearing denominators. (Note this does not enlarge the base locus of F since F is undefined wherever the denominator of the of the f_is vanishes.) Thus we can assume that $f_i \in R$ for all i, and that all the f_i are homogenous of the same degree.

In this setting, one might naively think that the map F is undefined exactly when all of the f_i vanish, and thus the base locus is the vanishing set of the ideal $(f_0, \cdots, f_m)$. However, this can yield a base locus that's too big. For example, suppose $X = \operatorname{Proj} k[x, y, z]$ and F is the rational map represented by (x^2y, x^2z, xyz). Then the vanishing locus of the ideal (x^2y, x^2z, xyz) is the union of the line $\{x = 0\}$ with the point $(1 : 0 : 0)$. However, the same rational map is given by (xy, xz, yz) since these two representations agree where $\{x \neq 0\}$ (see [**?**, I.4]), and the vanishing locus of (xy, xz, yz) is just the three points $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$.

Thus, to find the base locus of a rational map, we must consider all possible representations of the map and find where none of them are defined. To do this, we use a lemma of Aron Simis [Sim04, Proposition 1.1]. We restate the lemma here for convenience's sake:

Lemma 2.1. *Let $F : X \dashrightarrow \mathbb{P}^m$ be a rational map and let $\boldsymbol{f} = \{f_0, \ldots, f_m\}$ be a representative of F with $f_i \in R$ homogenous of degree d for all i. Set $I = (f_0, \cdots, f_m)$. Then the set of such representatives of F corresponds bijectively to the homogenous vectors in the rank 1 graded R-module $\operatorname{Hom}_R(I, R)$.*

Proof. Suppose that $(f'_0, \cdots, f'_m)$ is another such representative of F. Then there is some $h \in K$ such that $hf_i = f'_i$ for all i. In particular, we have that each $hf_i \in R$ for each i, and so $h \in (R :_K I)$. Further, it's clear that each homogenous element of $(R :_K I)$ gives another

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representative of F. Thus the set of representatives of F is $\{(hf_0, \cdots, hf_m) \mid h \in (R :_K I)\}$. It's a standard fact that $\operatorname{Hom}_R(I, R) = \{x \mapsto hx \mid h \in (R :_K I)\}$, giving us the desired result. $\square$

Now, in the setting of 2.1, let

$$\bigoplus_s R(d_s) \xrightarrow{\varphi} R(-d)^m \xrightarrow{[f_0, \cdots, f_m]} I \longrightarrow 0$$

be a free resolution of I. Then we get

$$0 \longrightarrow \operatorname{Hom}_R(I, R) \longrightarrow \left(R(-d)^{m+1}\right)^\vee \xrightarrow{\varphi^t} \left(\bigoplus_s R(d_s)\right)^\vee$$

where φ^t is the transpose of φ and $R^\vee$ is the dual module of R. Thus, we get that $\operatorname{Hom}_R(I, R) \cong \ker \varphi^t$, and so each representative of F corresponds to a vector in $\ker \varphi^t$. The correspondence takes a representative $(hf_0, \cdots, hf_m)$ to the map that multiplies vectors in R^{m+1} by $[hf_0, \cdots, hf_m]$ on the left.

The base locus of F is the intersection of the sets $V(f_0^i, \cdots, f_m^i)$ as $\mathbf{f}^i = (f_0^i, \cdots, f_m^i)$ ranges over all the representatives of F. The above implies that this is the same as the intersection of the sets $V(w_0^i, \cdots, w_m^i)$ as $\mathbf{w}^i = (w_0^i, \cdots, w_m^i)$ ranges over the vectors in $\ker \varphi^t$. Now, given any $a, f, g \in R$, we have $V(af) \supseteq V(f)$ and $V(f+g) \supseteq V(f) \cap V(g)$. Thus, it's enough to take a generating set $\mathbf{w}^1, \cdots, \mathbf{w}^n$ of $\ker \varphi^t$ and take the intersection over this generating set.

In summary, if we're given a representation $(f_0, \cdots, f_m)$ of a rational map to projective space, we compute its base locus by:

(a) finding $M = \ker \varphi^t$, where φ is a presentation matrix for I,
(b) fixing a generating set $\mathbf{w}^1, \cdots, \mathbf{w}^n$ for M as an R-module, and
(c) taking the ideal generated by all the entries of all of the $\mathbf{w}^i$.

The base locus of F is then the variety cut out by this ideal, though our function `baseLocusOfMap` just returns this ideal:

```
i1 : loadPackage "RationalMaps"

o1 = RationalMaps

o1 : Package

i2 : R = QQ[x,y,z]

o2 = R

o2 : PolynomialRing

i3 : f = {x^2*y, x^2*z, x*y*z}

      2    2
o3 = {x y, x z, x*y*z}
```

```
o3 : List

i4 : baseLocusOfMap(f)

o4 = ideal (y*z, x*z, x*y)

o4 : Ideal of R
```

If the SaturateOutput option is left on, our function will return the saturation of this ideal:

```
i5 : baseLocusOfMap({x,y,z})

o5 = ideal 1

o5 : Ideal of R

i6 : baseLocusOfMap({x,y,z}, SaturateOutput=>false)

o6 = ideal (x, y, z)

o6 : Ideal of R
```

This is desirable because, for any ideal I, the saturation of I is the largest ideal cutting out the same projective variety as I [**?**, II.5 exercise whatever]. Thus, by saturating the output, we get a canonical ideal that cuts out the base locus.

3. Birationality and Inverse Maps

4. Embeddings

References

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