

# RATIONALMAPS, A PACKAGE FOR MACAULAY2

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ABSTRACT. This paper describes the `RationalMaps` package for Macaulay2. This package provides functionality for computing several aspects of rational maps.

## 1. INTRODUCTION

This package aims to compute several things about rational maps between varieties. In particular, this package will compute

- The base locus of a rational map.
- Whether a rational map is birational.
- The inverse of a birational map.
- Whether a map is a closed embedding.
- And more!

Our functions have numerous options which allow them to run much more quickly in certain examples if configured correctly. Setting the option `Verbosity` to a value  $\geq 1$  will mean that functions will provide hints as to the best ways to run them.

A rational map  $\mathfrak{F} : X \subseteq \mathbb{P}^n \dashrightarrow Y \subseteq \mathbb{P}^m$  between projective varieties is presented by  $m + 1$  forms  $\mathbf{f} = \{f_0, \dots, f_m\}$  of the same degree in the coordinate ring of  $X$ , denoted by  $R$ . The idea of looking at the syzygies of the forms  $\mathbf{f}$  to detect the geometric properties of  $\mathfrak{F}$  goes back at least to [HKS92] in the case where  $X = \mathbb{P}^n$ ,  $Y = \mathbb{P}^m$  and  $m = n$  (see also [ST69]). In [RS01] this method was developed by Russo and Simis to handle the case  $X = \mathbb{P}^n$  and  $m \geq n$ . Simis pushed the method further to the study of general rational maps between two integral projective schemes in arbitrary characteristic by an extended ideal-theoretic method emphasizing the role of the Rees algebra associated to the ideal generated by  $\mathbf{f}$  [Sim04]. Recently, Doria, Hassanzadeh, and Simis applied these Rees algebra techniques to study the birationality of  $\mathfrak{F}$  [DHS12]. Our core functions, particularly those related to computing inverse maps, rely heavily on this work.

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## 2. BASE LOCI

We begin with the problem of computing the base locus of a map to projective space. Let  $X$  be a projective variety over any field  $k$  and let  $\mathfrak{F} : X \rightarrow \mathbb{P}_k^m$  be a rational map from  $X$  to projective space. Then we choose a representative  $(f_0, \dots, f_m)$  of  $\mathfrak{F}$ , where each  $f_i$  is the  $i^{\text{th}}$  coordinate of  $\mathfrak{F}$ .

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A priori, each  $f_i$  is in  $K = \text{frac } R$ , where  $R$  is the coordinate ring of  $X$ . However, we can get another representative of  $\mathfrak{F}$  by clearing denominators. (Note that this does not enlarge the base locus of  $\mathfrak{F}$  since  $\mathfrak{F}$  is undefined whenever the denominator of any of the  $f_i$  vanishes.) Thus we assume that  $f_i \in R$  for all  $i$ , and that all the  $f_i$  are homogeneous of the same degree.

In this setting, one might naively think that the map  $\mathfrak{F}$  is undefined exactly when all of the  $f_i$  vanish, and thus the base locus is the vanishing set of the ideal  $(f_0, \dots, f_m)$ . However, this yields a base locus that's too big. Indeed, to find the base locus of a rational map, we must consider all possible representatives of the map and find where none of them are defined. To do this, we use the following result.

**Proposition 2.1.** [Sim04, Proposition 1.1] *Let  $\mathfrak{F} : X \dashrightarrow \mathbb{P}^m$  be a rational map and let  $\mathbf{f} = \{f_0, \dots, f_m\}$  be a representative of  $\mathfrak{F}$  with  $f_i \in R$  homogeneous of degree  $d$  for all  $i$ . Set  $I = (f_0, \dots, f_m)$ . Then the set of such representatives of  $\mathfrak{F}$  corresponds bijectively to the homogeneous vectors in the rank 1 graded  $R$ -module  $\text{Hom}_R(I, R) \cong (R :_K I)$ .*

The bijection comes from multiplying our fixed representative  $\mathbf{f}$  of  $\mathfrak{F}$  by  $h \in (R :_K I)$ . Now, in the setting of Proposition 2.1, let

$$\bigoplus_s R(-d_s) \xrightarrow{\varphi} R(-d)^{m+1} \xrightarrow{[f_0, \dots, f_m]} I \rightarrow 0$$

be a free resolution of  $I$ . Then we get

$$0 \rightarrow \text{Hom}_R(I, R) \rightarrow (R(-d)^{m+1})^\vee \xrightarrow{\varphi^t} \left( \bigoplus_s R(d_s) \right)^\vee$$

where  $\varphi^t$  is the transpose of  $\varphi$  and  $R^\vee$  is the dual module of  $R$ . Thus, we get that  $\text{Hom}_R(I, R) \cong \ker \varphi^t$ , and so each representative of  $\mathfrak{F}$  corresponds to a vector in  $\ker \varphi^t$ . The correspondence takes a representative  $(hf_0, \dots, hf_m)$  to the map that multiplies vectors in  $R^{m+1}$  by  $[hf_0, \dots, hf_m]$  on the left.

The base locus of  $\mathfrak{F}$  is the intersection of the sets  $V(f_0^i, \dots, f_m^i)$  as  $\mathbf{f}^i = (f_0^i, \dots, f_m^i)$  ranges over all the representatives of  $\mathfrak{F}$ . The above implies that this is the same as the intersection of the sets  $V(w_0^i, \dots, w_m^i)$  as  $\mathbf{w}^i = (w_0^i, \dots, w_m^i)$  ranges over the vectors in  $\ker \varphi^t$ . Now, given any  $a, f, g \in R$ , we have  $V(af) \supseteq V(f)$  and  $V(f+g) \supseteq V(f) \cap V(g)$ . Thus, it's enough to take a generating set  $\mathbf{w}^1, \dots, \mathbf{w}^n$  of  $\ker \varphi^t$  and take the intersection over this generating set.

The base locus of  $\mathfrak{F}$  is then the variety cut out by the ideal generated by all the entries of all of the  $\mathbf{w}^i$ . Our function `baseLocusOfMap` returns this ideal. It can be applied either to our new Type `RationalMapping` or to a `RingMap` between the homogeneous coordinate rings which represents the rational map.

```
i1 : loadPackage "RationalMaps";

i2 : R = QQ[x,y,z];

i3 : f = rationalMapping(R, R, {x^2*y, x^2*z, x*y*z})

o3 = Proj R - - - > Proj R      2      2
      {x y, x z, x*y*z}

o3 : RationalMapping

i4 : baseLocusOfMap(f)

o4 = ideal (y*z, x*z, x*y)

o4 : Ideal of R
```

If the `SaturateOutput` option is set to `false`, our function `baseLocusOfMap` will not saturate the output.

### 3. BIRATIONALITY AND INVERSE MAPS

Again, a rational map  $\mathfrak{F} : X \subseteq \mathbb{P}^n \dashrightarrow Y \subseteq \mathbb{P}^m$  between projective spaces is defined by  $m + 1$  forms  $\mathbf{f} = \{f_0, \dots, f_m\}$  of the same degree in the coordinate ring of  $X$ , denoted by  $R$ .  $R$  is a standard graded ring in  $n + 1$  variables. Here we assumed the varieties are defined over a field  $k$  and  $\dim R \geq 1$ . Our goal is to find a ring theoretic criterion for birationality and, on top of that, to find the inverse of a rational map. To do this, we study the Rees algebra of the ideal  $I = (\mathbf{f})$  in  $R$ . To that end set  $R \simeq k[x_0, \dots, x_n] = k[\mathbf{X}]/\mathfrak{a}$  with  $k[\mathbf{X}] = k[X_0, \dots, X_n]$  and  $\mathfrak{a}$  a homogeneous ideal. The Rees algebra is defined by the polynomial relations among  $\{f_0, \dots, f_m\}$  in  $R$ . To this end, we consider the polynomial extension  $R[\mathbf{Y}] = R[Y_0, \dots, Y_m]$ . To keep track of the variables by degrees, we set the standard bigrading  $\deg(X_i) = (1, 0)$  and  $\deg(Y_j) = (0, 1)$ . Mapping  $Y_j \mapsto f_j t$  yields a presentation  $R[\mathbf{Y}]/\mathcal{J} \simeq \mathcal{R}_R((\mathbf{f}))$ , with  $\mathcal{J}$  a bihomogeneous *presentation ideal*.  $\mathcal{J}$  is a bigraded ideal that depends only on the rational map defined by  $\mathbf{f}$  and not on this particular representative.

$$\mathcal{J} = \bigoplus_{(p,q) \in \mathbb{N}^2} \mathcal{J}_{(p,q)},$$

where  $\mathcal{J}_{(p,q)}$  denotes the  $k$ -vector space of forms of bidegree  $(p, q)$ . Every piece of this ideal contains information about the rational map. For example  $\mathcal{J}_{0,*}$  determines the dimension of the image of the map. For birationality, the following bihomogeneous piece is important:

$$\mathcal{J}_{1,*} := \bigoplus_{q \in \mathbb{N}} \mathcal{J}_{1,q}$$

with  $\mathcal{J}_{1,q}$  denoting the bigraded piece of  $\mathcal{J}$  spanned by the forms of bidegree  $(1, q)$  for all  $q \geq 0$ . Now, a form of bidegree  $(1, *)$  can be written as  $\sum_{i=0}^n Q_i(\mathbf{Y}) x_i$ , for suitable homogeneous  $Q_i(\mathbf{Y}) \in R[\mathbf{Y}]$  of the same degree.

One then goes to construct a matrix that measures the birationality of the map. The first step is to lift the polynomials  $Q_i(\mathbf{Y}) \in R[\mathbf{Y}]$  into  $k[\mathbf{X}, \mathbf{Y}]$ . Since the  $\{y_0, \dots, y_m\}$  are indeterminates over  $R$ , each pair of such representations of the same form gives a syzygy of  $\{x_0, \dots, x_n\}$  with coefficients in  $k$ . This is where one must take into attention whether  $X \subseteq \mathbb{P}^n$  is minimally embedded or not. To measure this one can easily check the vector space dimension of  $\mathfrak{a}_1$ , the degree-1 part of  $\mathfrak{a}$ ; if it is zero then  $X \subseteq \mathbb{P}^n$  is non-degenerated.

Next, one can pick a minimal set of generators of the ideal  $(\mathcal{J}_{1,*})$  consisting of a finite number of forms of bidegree  $(1, q)$ , for various  $q$ 's. Let's assume  $X \subseteq \mathbb{P}^n$  is non-degenerated. Let  $\{P_1, \dots, P_s\} \subset k[\mathbf{X}, \mathbf{Y}]$  denote liftings of these biforms, consider the Jacobian matrix of the polynomials  $\{P_1, \dots, P_s\}$  with respect to  $\{x_0, \dots, x_n\}$ . This is a matrix with entries in  $k[\mathbf{Y}]$ . Write  $\psi$  for the corresponding matrix over  $S = k[\mathbf{Y}]/\mathfrak{b}$ , the coordinate ring of  $Y$ . This matrix is called the *weak Jacobian dual matrix* associated to the given set of generators of  $(\mathcal{J}_{1,*})$ . Note that a weak Jacobian matrix  $\psi$  is not uniquely defined due to the lack of uniqueness in the expression of an individual form and to the choice of bihomogeneous generators. However, it is shown in [DHS12, Lemma 2.13] that if the weak Jacobian matrix associated to one set of bihomogeneous minimal generators of  $(\mathcal{J}_{1,*})$  has rank over  $S$  then the weak Jacobian matrix associated to any other set of bihomogeneous minimal generators of  $(\mathcal{J}_{1,*})$  has rank over  $S$  and the two ranks coincide.

The following criterion is [DHS12, Theorem 2.18]. In the package, we consider only the cases where  $\mathbf{X}$  is irreducible i.e.  $R$  is a domain.

**Theorem 3.1.** *Let  $X \subseteq \mathbb{P}^n$  be non-degenerate. Then  $\mathfrak{F}$  is birational onto  $\mathbf{Y}$  if and only if  $\text{rank}(\psi) = \text{edim}(R) - 1 (= n)$ . Moreover*

- (i) *We get a representative for the inverse of  $\mathfrak{F}$  by taking the coordinates of any homogeneous vector of positive degree in the (rank one) null space of  $\psi$  over  $S$  for which these coordinates generate an ideal containing a regular element.*
- (ii) *If, further,  $R$  is a domain, the representative of  $\mathfrak{F}$  in (i) can be taken to be the set of the (ordered, signed)  $(\text{edim}(R) - 1)$ -minors of an arbitrary  $(\text{edim}(R) - 1) \times \text{edim}(R)$  submatrix of  $\psi$  of rank  $\text{edim}(R) - 1$ .*

As expected, the most expensive part of applying this theorem is computing the Rees ideal  $\mathcal{J}$ . In the package `RationalMaps`, we use `ReesStrategy` to compute the Rees equations. The algorithm is the standard elimination technique. However, we do not use the `ReesAlgebra` package, since verifying birationality according to Theorem 3.1 only requires computing a small part of the Rees ideal, namely elements of first-degree 1. This idea is applied in the `SimisStrategy`. More precisely, if the given map  $\mathfrak{F}$  is birational, then the Jacobian dual rank will attain its maximum value of  $\text{edim}(R) - 1$  after computing the Rees equations up to degree  $(1, N)$  for  $N$  sufficiently large. This allows us to compute the inverse map. The downside of `SimisStrategy` is that if  $\mathfrak{F}$  is not birational, the desired number  $N$  cannot be found and the process never terminates. To provide a definitive answer for birationality, we use `HybridStrategy`, which is a hybrid of `ReesStrategy` and `SimisStrategy`. The default strategy is `HybridStrategy`.

`HybridLimit` is an option to switch `SimisStrategy` to `ReesStrategy`, if the computations up to degree  $(1, \text{HybridLimit})$  do not lead to  $\text{rank}(\psi) = \text{edim}(R) - 1$ . The default value for `HybridLimit` is 15. The change from `SimisStrategy` to `ReesStrategy` is done in such a way that the generators of the Rees ideal computed in the `SimisStrategy` phase are not lost; the program computes other generators of the Rees ideal while keeping the generators it found before attaining `HybridLimit`.

There is yet another method for computing the Rees ideal called `SaturationStrategy`. In this option, the whole Rees ideal is computed by saturating the defining ideal of the symmetric algebra with respect to a non-zero element in  $R$  (we assume  $R$  to be a domain). This strategy appears to be slower in some examples, though one might be able to improve this option in the future by stopping the computation of the saturation at a certain step.

Computing inverse maps is the most important function of this package and is done by the function `inverseOfMap` (or by running `RationalMap^-1`). According to Theorem 3.1, there are two ways to compute the inverse of a map: (1) by finding any syzygy of the Jacobian dual matrix, and (2) by finding a sub-matrix of  $\psi$  of rank  $\text{edim}(R) - 1$ . Each way has its benefits. Method (1) is quite fast in many cases, however, method (2) is very useful if the rank of the Jacobian dual matrix  $\psi$  is relatively small compared to the degrees of the entries of  $\psi$ . Our function `inverseOfMap` starts by using the second method and later switches to the first method if the second method didn't work. The timing of this transition from the first method to the second method is controlled by the option `MinorsLimit`. Setting `MinorsLimit` to zero will mean that no minors are checked and the inverse map is computed just by looking at the syzygies of  $\psi$ . If `MinorsLimit` is left as null (the default value), these functions will determine a value using a heuristic that depends on the varieties involved.

In addition, to improve the speed of the function `inverseOfMap`, we have two other options, `AssumeDominant` and `CheckBirational`. If `AssumeDominant` is set to be `true`, then `inverseOfMap` assumes that the map from  $X$  to  $Y$  is dominant and does not compute the image of the map; this is time-consuming in certain cases as it computes the kernel of a ring map. However, this function goes through a call to `idealOfImageOfMap` which first checks whether the ring map is injective (at least if the target is a polynomial ring) using the method described in [Sim03, Proposition 1.1]. Similarly, if `CheckBirational` sets `false`, `inverseOfMap` will not check birationality although it

still computes the Jacobian dual matrix. The option `QuickRank` is available to many functions. At various points, the rank of a matrix is computed, and sometimes it is faster to compute the rank of an interesting-looking submatrix (using the tools of the package `FastMinors`, [MRSY20]). Turning `QuickRank` off will make showing that certain maps are birational slower, but will make showing that certain maps are *not* birational faster. There is a certain amount of randomness in the functions of `FastMinors`, and so occasionally rerunning a slow example will result in a massive speedup.

In general, as long as `Verbosity` is  $\geq 1$ , the function will make suggestions as to how to run it more quickly. For example.

```
i1 : loadPackage "RationalMaps";

i2 : Q=QQ[x,y,z,t,u];

i3 : f = map(Q,Q,matrix{{x^5,y*x^4,z*x^4+y^5,t*x^4+z^5,u*x^4+t^5}});

o3 : RingMap

i4 : phi=rationalMapping(f)
      5   4   5   4   5   4   5   4
o4 = Proj Q - - -> Proj Q   {x , x y, y  + x z, z  + x t, t  + x u}

o4 : RationalMapping

i5 : time inverseOfMap(phi, CheckBirational=>false, Verbosity => 1);
inverseOfMapSimis: About to find the image of the map.  If you know the image,
  you may want to use the AssumeDominant option if this is slow.
inverseOfMapSimis: About to check rank, if this is very slow, you may want to try turning QuickRank=>false.
inverseOfMapSimis: About to check rank, if this is very slow, you may want to try turning QuickRank=>false.
inverseOfMapSimis: About to check rank, if this is very slow, you may want to try turning QuickRank=>false.
inverseOfMapSimis: About to check rank, if this is very slow, you may want to try turning QuickRank=>false.
inverseOfMapSimis: About to check rank, if this is very slow, you may want to try turning QuickRank=>false.
inverseOfMapSimis: We give up. Using the previous computations, we compute the whole
  Groebner basis of the Rees ideal.  Increase HybridLimit and rerun to avoid this.
inverseOfMapSimis: Looking for a nonzero minor.
  If this fails, you may increase the attempts with MinorsLimit => #
inverseOfMapSimis: We found a nonzero minor.
  -- used 0.189563 seconds

o5 : RationalMapping

i6 : ident = rationalMapping map(Q,Q);

o6 : RationalMapping

i7 : o5*phi == ident

o7 = true
```

Using the `RationalMap-1` syntax to compute inverses of maps will always suppress such output.

```
i6 : time phi^-1;
  -- used 0.192791 seconds

o6 : RationalMapping

i7 : o4 == o7

o7 = true
```

## 4. EMBEDDINGS

Our package also checks whether a rational map  $\mathfrak{F} : X \rightarrow Y$  is a closed embedding. The strategy is quite simple.

- (a) We first check whether  $\mathfrak{F}$  is regular (by checking if its base locus is empty).
- (b) We next invert the map (if possible).
- (c) Finally, we check whether the inverse map is also regular.

If all three conditions are met, then the map is a closed embedding and the function returns `true`. Otherwise, `isEmbedding` returns false. In the following example which illustrates this, we take a plane quartic, choose a point  $Q$  on it, and take the map associated with the divisor  $12Q$ . This map is an embedding by [Har77, Chapter IV, Corollary 3.2], which we now verify.

```
i1 : needsPackage "Divisor"; --used to quickly define a map

i2 : C = ZZ/101[x,y,z]/(x^4+x^2*y*z+y^4+z^3*x);

i3 : Q = ideal(y,x+z);

o3 : Ideal of C

i4 : f2 = mapToProjectiveSpace(12*divisor(Q));

      ZZ
o4 : RingMap C <--- ---[YY ..YY ]
      101  1    10

i5 : needsPackage "RationalMaps";

i6 : time isEmbedding(f2)
isEmbedding: About to find the image of the map. If you know the image,
you may want to use the AssumeDominant option if this is slow.
inverseOfMapSimis: About to check rank, if this is very slow, you may want to try turning QuickRank=>false
inverseOfMapSimis: rank found, we computed enough of the Groebner basis.
-- used 0.140107 seconds

o6 = true
```

Notice that `MinorsLimit => 0` by default for `isEmbedding`. This is because the expressions defining the inverse map obtained from an appropriate minor frequently are more complicated than the expressions for the inverse map obtained via the syzygies. Complicated expressions can sometimes slow down the checking of whether the inverse map is regular.

## 5. FUNCTIONALITY OVERLAP WITH OTHER PACKAGES

We note that our package has some overlaps in functionality with other packages.

While the `Parametrization` package [Boe10] focuses mostly on curves, it also includes a function called `invertBirationalMap` that has the same functionality as `inverseOfMap`. On the other hand, these two functions were implemented differently and so sometimes one function can be substantially faster than the other.

The package `Cremona` [Sta, Sta18] focuses on very fast probabilistic computation in general cases and very fast deterministic computation for special kinds of maps from projective space. In particular, in `Cremona`,

- `isBirational` gives a probabilistic answer to the question of whether a map between varieties is birational. Furthermore, if the source is projective space, then `degreeOfRationalMap` with `MathMode=>true` gives a deterministic answer that can be faster than what our package provides with `isBirationalMap`.

- `inverseMap` gives a very fast computation of the inverse of a birational map if the source is projective space and the map has maximal linear rank. If this function is passed a map where the domain is not projective space, then it calls a modified, improved version of `invertBirationalMap` originally from `Parametrization`. Even in some cases with maximal linear rank, our `inverseOfMap` function appears to be quite competitive, however.

The package `ReesAlgebra` [ETPS, ETPS18] includes a function `jacobianDual` which computes the jacobian dual matrix. We also have a function `jacobianDualMatrix` which computes a weak form of this same matrix.

## 6. COMMENTS AND COMPARISONS ON FUNCTION SPEEDS

We begin with a comparison using examples with maximal linear rank where `Cremona` excels. These examples were executed using version 5.1 of `Cremona` and version 1.0 of `RationalMaps` running Macaulay2 1.19.1.1 on Ubuntu 20.04.

Indeed, in this example (taken from `Cremona`'s documentation), `Cremona` is substantially faster.

```
i1 : loadPackage "Cremona"; loadPackage "RationalMaps";

i3 : ringP20=QQ[t_0..t_20];

i4 : phi=map(ringP20,ringP20,{t_10*t_15-t_9*t_16+t_6*t_20,t_10*t_14-t_8*t_16+t_5*t_20,t_9*t_14-t_8*t_15+t_4*t_20,
t_6*t_14-t_5*t_15+t_4*t_16,t_11*t_13-t_16*t_17+t_15*t_18-t_14*t_19+t_12*t_20,t_3*t_13-t_10*t_17+t_9*t_18-t_8*t_19
+t_7*t_20,t_10*t_12-t_2*t_13-t_7*t_16-t_6*t_18+t_5*t_19,t_9*t_12-t_1*t_13-t_7*t_15-t_6*t_17+t_4*t_19,t_8*t_12
-t_0*t_13-t_7*t_14-t_5*t_17+t_4*t_18,t_10*t_11-t_3*t_16+t_2*t_20,t_9*t_11-t_3*t_15+t_1*t_20,t_8*t_11-t_3*t_14
+t_0*t_20,t_7*t_11-t_3*t_12+t_2*t_17-t_1*t_18+t_0*t_19,t_6*t_11-t_2*t_15+t_1*t_16,t_5*t_11-t_2*t_14+t_0*t_16,
t_4*t_11-t_1*t_14+t_0*t_15,t_6*t_8-t_5*t_9+t_4*t_10,t_3*t_6-t_2*t_9+t_1*t_10,t_3*t_5-t_2*t_8+t_0*t_10,t_3*t_4
-t_1*t_8+t_0*t_9,t_2*t_4-t_1*t_5+t_0*t_6});

o4 : RingMap ringP20 <--- ringP20

i5 : time inverseOfMap(phi, Verbosity=>0);-- Function from "RationalMaps"
-- used 0.118508 seconds

o5 : RationalMapping

i6 : time inverseMap phi;
-- used 0.0370978 seconds

o6 : RingMap ringP20 <--- ringP20

i7 : o5 == rationalMapping o6

o7 = true
```

However, sometimes the `RationalMaps` function is faster, even in examples with maximal linear rank (a good source of examples where different behaviors can be seen can be found in the documentation of `Cremona`). We now include an example where the map does not have the maximal linear rank.

```
i1 : loadPackage "Cremona"; loadPackage "RationalMaps";

i3 : Q=QQ[x,y,z,t,u];

i4 : phi=map(Q,Q,matrix{{x^5,y*x^4,z*x^4+y^5,t*x^4+z^5,u*x^4+t^5}});

o4 : RingMap Q <--- Q

i5 : (time g = inverseOfMap(phi, Verbosity=>0));
-- used 0.233111 seconds

i6 : (time f = inverseOfMap(phi, Verbosity=>0, MinorsLimit=>0));
```

```

-- used 60.1969 seconds

i7 : (time h = inverseMap(phi)); -- Function from "Cremona"
-- used 49.2842 seconds

o7 : RingMap Q <--- Q

i8 : f == rationalMapping h

o8 = true

i9 : g == rationalMapping h

o9 = true

```

In the previous example, setting `MinorsLimit=>0` makes `inverseOfMap` much slower – approximately the same speed as the corresponding command from `Cremona`. The takeaway for the user should be that changing the options `Strategy`, `HybridLimit`, `MinorSize`, and `QuickRank`, can make a large difference in performance.

We conclude with discussions of the limits of this package. A work of O. Gabber shows that if  $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$  is defined by forms of degree  $d$ , then its inverse can be defined by forms of degree  $d^{n-1}$ , [BCW82]. This bound is sharp, as the map

$$(x_0^d : x_1 x_0^{d-1} : x_2 x_0^{d-1} - x_1^d : \dots : x_n x_0^{d-1} - x_{n-1}^d)$$

has inverse given by forms of degree  $d^{n-1}$ , see [HS17]. Thus we might expect that this family of maps would be good to explore to see the limits of `RationalMaps`. We ran these examples with the following code.

```

R = ZZ/101[x_0..x_n];
L = {x_0^d, x_1*x_0^(d-1)} | toList(apply(2..n, i -> (x_i*x_0^(d-1) + x_(i-1)^d)));
psi = map(R, R, L);
time inv = inverseOfMap(psi, AssumeDominant=>true, CheckBirational=>false, Verbosity=>0);

```

When  $n = 3$  (we are working on  $\mathbb{P}^3$ ) here is a table showing the computation time, in seconds, to find the inverse map for various values of  $d$ . The degrees are those we would expect in this example (when  $d = 100$ , the degree of the forms in the inverse is 10000). Note that `Cremona` has very similar performance for these examples in  $\mathbb{P}^3$  ( $n = 3$ ), but seems substantially slower than `RationalMaps` as we increase the dimension.

$d$	5	10	20	40	60	80	100
seconds	0.0925	0.0958	0.1402	1.0667	7.2652	37.4577	135.915

However, as the size of projective space increases, this becomes much slower. Here is a table when  $n = 4$ .

$d$	5	8	10	11	12	13	14	15
seconds	0.1523	1.3115	7.4682	14.9912	28.8554	57.1229	120.778	217.706

We conclude with a table when  $n = 5$ .

$d$	3	4	5	6
seconds	0.2619	4.8770	134.424	2713.56

Note the  $d = 6$  case took more than 45 minutes.

Finally, Zhuang He and Lei Yang, working under the direction of Ana-Maria Castravet, communicated to us that they used `RationalMaps` to help understand and compute the inverse of a rational map from  $\mathbb{P}^3$  to  $\mathbb{P}^3$ , see [HL19]. Quoting Zhuang He, this rational map is “induced by a degree 13 linear system with the base locus at 6 very general points in  $\mathbb{P}^3$  and 9 lines through them”. From a computational perspective, this map was given by 4 degree 13 forms, with 485, 467, 467, and 467 terms respectively. Computing the inverse of this map took several hours, but it was successful.



# REFERENCES

- [BCW82] H. BASS, E. H. CONNELL, AND D. WRIGHT: *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 287–330. 663785
- [Boe10] J. BOEHM: *Parametrization*, Package for Macaulay2, see <http://www.math.uiuc.edu/Macaulay2/doc/Macaulay2-1.9.2/share/doc/Macaulay2/Parametrization/html/>.
- [DHS12] A. V. DORIA, S. H. HASSANZADEH, AND A. SIMIS: *A characteristic-free criterion of birationality*, Adv. Math. **230** (2012), no. 1, 390–413. 2900548
- [ETPS] D. EISENBUD, A. TAYLOR, S. POPESCU, AND M. E. STILLMAN: *ReesAlgebra: A Macaulay2 package. Version 2.2*.
- [ETPS18] D. EISENBUD, A. TAYLOR, S. POPESCU, AND M. E. STILLMAN: *The ReesAlgebra package in Macaulay2*, The Journal of Software for Algebra and Geometry **8** (2018).
- [Har77] R. HARTSHORNE: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)
- [HS17] S. H. HASSANZADEH AND A. SIMIS: *Bounds on degrees of birational maps with arithmetically Cohen-Macaulay graphs*, J. Algebra **478** (2017), 220–236. 3621670
- [HL19] Z. HE AND Y. LEI: *The Mori dream space property of blow-ups of projective spaces at points and lines*, Work in preparation.
- [HKS92] K. HULEK, S. KATZ, AND F.-O. SCHREYER: *Cremona transformations and syzygies*, Math. Z. **209** (1992), no. 3, 419–443. 1152265
- [MRSY20] B. MARTINOVA, M. ROBINSON, K. SCHWEDE, AND Y. W. YAO: *FastMinors package for Macaulay2*, arXiv:2002.05758.
- [RS01] F. RUSSO AND A. SIMIS: *On birational maps and Jacobian matrices*, Compositio Math. **126** (2001), no. 3, 335–358. 1834742
- [ST69] J. G. SEMPLE AND J. A. TYRRELL: *The Cremona transformation of  $S_6$  by quadrics through a normal elliptic septic scroll  $^1R^7$* , Mathematika **16** (1969), 89–97. 0249431
- [Sim03] A. SIMIS: *Two differential themes in characteristic zero*, Topics in algebraic and noncommutative geometry (Luminy/Annapolis, MD, 2001), Contemp. Math., vol. 324, Amer. Math. Soc., Providence, RI, 2003, pp. 195–204. 1986124
- [Sim04] A. SIMIS: *Cremona transformations and some related algebras*, J. Algebra **280** (2004), no. 1, 162–179. 2081926
- [Sta] G. STAGLIANÒ: *Cremona: Some computations for rational maps between projective varieties. Version 4.3*.
- [Sta18] G. STAGLIANÒ: *A Macaulay2 package for computations with rational maps*, The Journal of Software for Algebra and Geometry **8** (2018).

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