Noetherian operators

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1 Intro

Definition 1.1. Let $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a ideal. The set $N \subseteq \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ is a set of *Noetherian operators* for I if

$$f \in I \iff D \bullet f \in \sqrt{I} \, \forall D \in N.$$

The article [1] has algorithms to compute Noetherian operators in the zero dimensional case. These algorithms are implemented in `nops.m2`. The caveat is that we assume that the variety corresponding to the ideal is the origin.

For higher dimensions, we assume the ideal is in normal position. If not, use Noether Normalization to find a linear change of coordinates that puts the ideal in normal position. Then we can compute Noetherian operators with the changed coordinates, and change coordinates back to get Noetherian operators for the original ideal. TODO: details on this procedure.

2 Macaulay matrices

We can attempt to compute annihilators of an ideal $I = \langle f_1, \dots, f_n \rangle$ by using Macaulay matrix based approach. Fix a limit for the degrees of monomials: n_x is the highest degree of a x monomial, and n_∂ is the highest degree of a ∂ monomial. Then, create a (large) vector $R = \{x^\alpha f_i\}$ for all $|\alpha| \leq n_x$ and all generators f_i of I. Next create a vector $C = \{\partial^\beta\}$ for all $|\beta| \leq n_\partial$. We will create a matrix M of dimensions $|R| \times |C|$ indexed by elements of R, C. The entry corresponding to row $x^\alpha f_i$ and column ∂^β will be

$$M_{\alpha,i;\beta} = \partial^\beta \bullet x^\alpha f_i \pmod{\sqrt{I}} \tag{1}$$

Let $v \in \ker M \subset (\mathbb{C}[x])^{|C|}$. This means that for all α, i we have

$$\left(\sum_{|\beta| \leq n_\partial} v_\beta \partial^\beta \right) \bullet x^\alpha f_i = 0 \pmod{\sqrt{I}}. \tag{2}$$

Since $x^\alpha f_i \in I$, the property above is a necessary condition of a set of Noetherian operators.

Questions:

- Does the kernel of M give Noetherian operators? Basis of the kernel?
- How to choose n_x, n_∂? Ideally we want them to be small, but not too small
  - It is clear by construction that n_∂ gives an upper bound on the degree of the potential Noetherian operator that is found. I.e. if we know that an ideal has
- Can we increment n_x, n_∂ until stabilization?

3 Examples and experiments

In the following examples, all ideals are primary.

Example 3.1 (Point at origin in $\mathbb{R}^2$)**.** Let $I = \langle x^2 - y, y^2 \rangle$. This is zero dimensional, and the correct Noetherian operators are $\{1, dx, dx^2 + 2 * dy, dx^3 + 6 * dx * dy\}$. We get these by running the Macaulay matrix method with $n_x = 20, n_\partial = 5$.

We see that by incrementing, we eventually get all operators. Also we notice that by taking a large n_x and incrementing, we get all operators. However, a large n_∂ and small n_x gives the wrong answer. The smallest pair that gives the right answer is (1,3).

(n_x, n_∂)	Nops
(1,1)	1
(2,2)	$1, \partial_x$
(3,3)	$1, \partial_x, \partial_x^2 + 2\partial_y$
(4,4)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y$
(5,5)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y$
(6,6)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y$
(7,7)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y$
(10,1)	1
(10,2)	$1, \partial_x$
(10,3)	$1, \partial_x, \partial_x^2 + 2\partial_y$
(10,4)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y$
(10,5)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y$
(6,10)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y, \partial_y^9, \partial_x\partial_y^8, \partial_x^2\partial_y^7, \partial_x^3\partial_y^6, \partial_x^4\partial_y^5, \partial_x^5\partial_y^4, \partial_x^6\partial_y^3, \partial_x^7\partial_y^2, \partial_x^8\partial_y, \partial_x^9, \partial_y^{10}, \partial_x\partial_y^9, \partial_x^2\partial_y^8, \partial_x^3\partial_y^7, \partial_x^4\partial_y^6, \ldots$
(1,3)	$1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y$

Table 1: Example 3.1

Example 3.2 (Point in origin of $\mathbb{R}^3$)**.** Let $I = \langle x^2 - z, y^2 - z, z^2 \rangle$. We get the right answer with $(n_x, n_\partial) = (10,4)$, and the smallest value of n_x that gives the right answer is 2.

(n_x, n_∂)	Nops
(10,4)	$1, \partial_y, \partial_x, \partial_x\partial_y, \partial_x^2 + \partial_y^2 + 2\partial_z, 3\partial_x^2\partial_y + \partial_y^3 + 6\partial_y\partial_z, \partial_x^3 + 3\partial_x\partial_y^2 + 6\partial_x\partial_z$
(5,4)	$1, \partial_y, \partial_x, \partial_x\partial_y, \partial_x^2 + \partial_y^2 + 2\partial_z, 3\partial_x^2\partial_y + \partial_y^3 + 6\partial_y\partial_z, \partial_x^3 + 3\partial_x\partial_y^2 + 6\partial_x\partial_z$
(2,4)	$1, \partial_y, \partial_x, \partial_x\partial_y, \partial_x^2 + \partial_y^2 + 2\partial_z, 3\partial_x^2\partial_y + \partial_y^3 + 6\partial_y\partial_z, \partial_x^3 + 3\partial_x\partial_y^2 + 6\partial_x\partial_z$
(1,4)	$1, \partial_y, \partial_x, \partial_x\partial_y, \partial_x^2 + \partial_y^2 + 2\partial_z, \partial_x\partial_y\partial_z, 3\partial_x^2\partial_y + \partial_y^3 + 6\partial_y\partial_z, \partial_x^3 + 3\partial_x\partial_y^2 + 6\partial_x\partial_z, \partial_z^4, \partial_y\partial_z^3, \partial_x\partial_z^3, \partial_y^2\partial_z^2, \partial_x\partial_y\partial_z^2, \partial_x^2\partial_z^2, \ldots$

Table 2: Example 3.2

Example 3.3 (Point not in origin, $\mathbb{R}^2$)**.** Consider the ideal $\langle (x-2)^2 - (y+1), (y+1)^2$. Note that this is the same ideal as in Example 3.1, but the variety is shifted from the origin to $(2,-1)$. Hence the Noetherian operators should be the same as in Example 3.1, i.e. we expect $\{1, \partial_x, \partial_x^2 + 2\partial_y, \partial_x^3 + 6\partial_x\partial_y\}$. Table 3 shows that we get constant multiples of the operators. Also, the right answer comes at the same time (n_x, n_∂) as in Example 3.1.

Example 3.4 (Positive dimensional in $\mathbb{R}^3$, centered)**.** Consider the ideal $\langle x^2 - ty, y^2 \rangle \subseteq \mathbb{C}[x,y,t]$. This is in normal position with respect to x, y, i.e. the image of I in $C(t)[x,y]$ is zero-dimensional. The algorithm in [1] gives us the Noetherian operators $1, \partial_x, t\partial_x + 2\partial_y, t\partial_x^3 + 6\partial_x\partial_y$.

Example 3.5 (Positive dimensional in $\mathbb{R}^2$, not centered)**.** Consider the ideal from the previous example, but translate the variety by substituting $x \mapsto x + t, y \mapsto y - t$. Then setting $(n_x, n_\partial) = (5,5)$, we recover the same Noetherian operators as before: $t\partial_x^3 + 6\partial_x\partial_y, \partial_x, t\partial_x^2 + 2\partial_y, 1$.

(n_x, n_∂)	Nops
$(10,5)$	$1, \partial_x, 43200\,\partial_x^2 + 86400\,\partial_y, 720\,\partial_x^3 + 4320\,\partial_x\,\partial_y$
$(5,5)$	$1, \partial_x, 1800\,\partial_x^2 + 3600\,\partial_y, 120\,\partial_x^3 + 720\,\partial_x\,\partial_y$
$(3,3)$	$1, \partial_x, 9\,\partial_x^2 + 18\,\partial_y, \partial_x^3 + 6\,\partial_x\,\partial_y$
$(2,3)$	$1, \partial_x, 6\,\partial_x^2 + 12\,\partial_y, \partial_x^3 + 6\,\partial_x\,\partial_y$
$(1,3)$	$1, \partial_x, 3\,\partial_x^2 + 6\,\partial_y, \partial_x^3 + 6\,\partial_x\,\partial_y$
$(0,3)$	$1, \partial_x, \partial_x\,\partial_y, \partial_x^2 + 2\,\partial_y, \partial_y^3, \partial_x\,\partial_y^2, \partial_x^2\partial_y, \partial_x^3$

Table 3: Example 3.3

(n_x, n_∂)	Nops
$(10,10)$	$1, \partial_x, t\,\partial_x^2 + 2\,\partial_y, t\,\partial_x^3 + 6\,\partial_x\,\partial_y$
$(0,5)$	$1, \partial_x, \partial_x\,\partial_y, \partial_y^3, \partial_x\,\partial_y^2, \partial_x^2\partial_y, \partial_x^3, t\,\partial_x^2 + 2\,\partial_y, \partial_y^4, \partial_x\,\partial_y^3, \partial_x^2\partial_y^2, \partial_x^3\partial_y, \partial_x^4, \partial_y^5, \partial_x\,\partial_y^4, \partial_x^2\partial_y^3, \partial_x^3\partial_y^2, \partial_x^4\partial_y, \partial_x^5$
$(1,5)$	$1, \partial_x, t\,\partial_x^2 + 2\,\partial_y, \partial_y^4, \partial_x\,\partial_y^3, \partial_x^2\partial_y^2, \partial_x^3\partial_y, \partial_x^4, t\,\partial_x^3 + 6\,\partial_x\,\partial_y, \partial_y^5, \partial_x\,\partial_y^4, \partial_x^2\partial_y^3, \partial_x^3\partial_y^2, \partial_x^4\partial_y, \partial_x^5$
$(2,5)$	$1, \partial_x, t\,\partial_x^2 + 2\,\partial_y, t\,\partial_x^3 + 6\,\partial_x\,\partial_y, \partial_y^5, \partial_x\,\partial_y^4, \partial_x^2\partial_y^3, \partial_x^3\partial_y^2, \partial_x^4\partial_y, \partial_x^5$
$(3,5)$	$1, \partial_x, t\,\partial_x^2 + 2\,\partial_y, t\,\partial_x^3 + 6\,\partial_x\,\partial_y$
$(4,5)$	$1, \partial_x, t\,\partial_x^2 + 2\,\partial_y, t\,\partial_x^3 + 6\,\partial_x\,\partial_y$
$(5,5)$	$1, \partial_x, t\,\partial_x^2 + 2\,\partial_y, t\,\partial_x^3 + 6\,\partial_x\,\partial_y$

Table 4: Example 3.4

Example 3.6 (Non-primary 0-dimensional ideal)**.** Let $I = \langle x^2, y\rangle \cap \langle x-1, y-1\rangle$. We get Table 5. Modulo a constant these are all the same, except that $(5,5)$ has one extra element.

(n_x, n_∂)	Nops	Nops modulo constant
$(10,5)$	$1, -641520000y\partial_x + 641520000\partial_x, -641520000y\partial_x + 641520000\partial_x$	$1, -y\partial_x + \partial_x, -y\partial_x + \partial_x$
$(5,5)$	$1, -28224000y\partial_x + 28224000\partial_x, -705600y\partial_x + 705600\partial_x, -705600y\partial_x + 705600\partial_x$	$1, -y\partial_x + \partial_x, -y\partial_x + \partial_x, -y\partial_x + \partial_x$
$(3,3)$	$1, 2160y\partial_x - 2160\partial_x, 2160y\partial_x - 2160\partial_x$	$1, y\partial_x - \partial_x, y\partial_x - \partial_x$

Table 5: Example 3.6

4 Numerical approaches

4.1 Zero-dimensional case

Assume the ideal I is zero-dimensional and primary, i.e. the variety is a point p. Then taking modulo $\sqrt{I}$ in (1) corresponds to evaluation at the point, so we get a numerical matrix whose kernel will possibly give us Noetherian operators.

Questions:

- What happens if the point is not exact?
- Does numerical kernel give any meaningful results? ("numerical Noetherian operators"?)

If the ideal is not primary, the numerical approach will give us correct Noetherian operators for each primary component.

Example 4.1. Consider the ideal $I = \langle x^2, y\rangle \cap \langle x-1, (y-1)^2\rangle$. We know that the Noetherian operators should be $1, \partial_x$ for the first component and $1, \partial_y$ for the second one. The first component corresponds to the point $(0,0)$, and if we substitute this point in our Macaulay matrix with $(n_x, n_\partial) = (3,3)$, its kernel will correspond to the operator $1, \partial_x$. Likewise, if we substitute $(1,1)$, the kernel will be $1, \partial_y$.

We note that in the example above, we do not need to know the primary decomposition a priori: we only need the ideal, and points in different irreducible components.

4.2 Positive dimensions

For primary ideals, we may again evaluate at some points, but we may also have to interpolate in addition.

Example 4.2. Let $I = \langle (x+t)^2 - t(y-t), (y-t)^2 \rangle \subseteq \mathbb{C}[x,y,t]$, which is in normal position with respect to x, y. We know also that the correct Noetherian operators are

$$1, \partial_x, \frac{1}{2}t\partial_x^2 + \partial_y, \frac{1}{6}t\partial_x^3 + \partial_x\partial_y$$

Say we are given three (exact) points on the variety: $(-1,1,1)$, $(-2,2,2)$, $(-3,3,3)$. When we substitute these values in the Macaulay matrix (with $(n_x, n_\partial) = (3,3)$) and compute the kernel, we get the Table 6. We also know that the Noetherian operators should be polynomials in ∂_x, ∂_y and t, so we interpolate to find the general form $1, \partial_x, \frac{1}{2}t\partial_x^2 + \partial_y, \frac{1}{6}t\partial_x^3 + \partial_x\partial_y$, which is the correct answer. Also, we can match

(x,y,t)	t	Nop 1	Nop 2	Nop 3	Nop 4
$(-1,1,1)$	1	1	∂_x	$\frac{1}{6}\partial_x^3 + \partial_x\partial_y$	$\frac{1}{2}\partial_x^2 + \partial_y$
$(-2,2,2)$	2	1	∂_x	$\frac{1}{3}\partial_x^3 + \partial_x\partial_y$	$\partial_x^2 + \partial_y$
$(-3,3,3)$	3	1	∂_x	$\frac{1}{2}\partial_x^3 + \partial_x\partial_y$	$\frac{3}{2}\partial_x^2 + \partial_y$

Table 6: Operators corresponding to kernel of Macaulay matrix when points are substituted. From these, we can conclude that the coeficcient

Next, we will look at a non-primary, positive dimensional example

Example 4.3. Let $l_1 = \langle x^2 - zy, y^2 \rangle$ and $l_2 = \langle x+y+z, x-y+z \rangle$ be two ideals in $\mathbb{C}[x,y,z]$, corresponding to two lines. Let $I = l_1 \cap l_2$.

We will focus on l_1, but the procedure is the same for l_2. Assume we are given points on l_1 (that are not also on l_2). The results of evaluating the Macaulay matrix are summarized below. Hence we can interpolate

(x,y,z)	Nop 1	Nop 2	Nop 3	Nop 4
$(0,0,1)$	1	∂_x	$\partial_x^3 + 6\partial_x\partial_y$	$\partial_x^2 + 2\partial_y$
$(0,0,2)$	1	∂_x	$\partial_x^3 + 3\partial_x\partial_y$	$\partial_x^2 + \partial_y$
$(0,0,3)$	1	∂_x	$\partial_x^3 + 2\partial_x\partial_y$	$-3\partial_x^2 - 2\partial_y$
$(0,0,4)$	1	∂_x	$2\partial_x^3 + 3\partial_x\partial_y$	$-2\partial_x^2 - \partial_y$

to conclude that the Noetherian operators for line 1 are

$$1, \partial_x, z\partial_x^3 + 6\partial_x\partial_y, z\partial_x^2 + 2\partial_y,$$

up to scaling by a constant.

If we run `primaryDecomposition I` in Macaulay2, we get $I = \langle y, x+z \rangle \cap \langle y^2, x^2 - yz \rangle$. Assuming that the primary decomposition is known, using Macaulay matrices, we see that the Noetherian operators corresponding to the first primary component $\langle y, x+z \rangle$ (line 1) are the same as what we found above.

Questions:

- What if exact points are not available?
- How to deal with embedded components?
  - The procedure in Example 4.3 does not work for some examples in `demo3.m2`

Here we give a construction for a valid set of Noetherian operators for an unmixed ideal:

Proposition 4.4. *Let I be an unmixed ideal and let $I = q_1 \cap \ldots \cap q_r$ be a minimal primary decomposition of I. Choose $h_i \in \left(\bigcap_{j \neq i} \sqrt{q_j}\right) \setminus \sqrt{q_i}$ for each i. If N_i is a set of Noetherian operators for q_i, then $N := \bigcup_i h_i N_i$ is a set of Noetherian operators for I (where $h_i N_i := \{h_i D \mid D \in N_i\}$).*

Proof. Suppose $f \in I$. Then $f \in q_i$ for all i, so $D \bullet f \in \sqrt{q_i}$ for every $D \in N_i$. By choice of h_i, this implies $h_i D \bullet f \in \sqrt{q_i} \cap \left(\bigcap_{j \neq i} \sqrt{q_j}\right) = \sqrt{I}$.

Conversely, suppose $f \notin I$. Then WLOG $f \notin q_1$, so there exists $D \in N_1$ such that $D \bullet f \notin \sqrt{q_1}$. Since also $h_1 \notin \sqrt{q_1}$ and $\sqrt{q_1}$ is prime, this means $h_1 D \bullet f \notin \sqrt{q_1}$, and thus $h_1 D \bullet f \notin \sqrt{I}$. □

4.3 Linear coordinate changes

Here we investigate how Noetherian operators behave with respect to a linear change of coordinates:

Proposition 4.5. *Let $R := k[x_1, \ldots, x_n]$ be a polynomial ring, and $\varphi : R \to R$ a k-linear automorphism of R, given by $\varphi(x) := Ax$ for some matrix $A \in GL_n(k)$. Define a k-linear automorphism of the Weyl algebra $W := R[\partial_{x_1}, \ldots, \partial_{x_n}]$ by*

$$\psi : \begin{pmatrix} x \\ \partial \end{pmatrix} \mapsto \begin{pmatrix} Ax \\ (A^{-1})^T \partial \end{pmatrix}.$$

Then, if $D_1, \ldots, D_r$ is a set of Noetherian operators for an ideal $I \subseteq R$, then $\psi(D_1), \ldots, \psi(D_r)$ is a set of Noetherian operators for $\varphi(I) \subseteq R$.

Proof. For $f \in R$, one has

$$\begin{aligned} f \in \varphi(I) &\iff \varphi^{-1}(f) \in I \iff D_i \bullet \varphi^{-1}(f) \in \sqrt{I} \quad \forall i = 1, \ldots, r \\ &\iff \varphi(D_i \bullet \varphi^{-1}(f)) \in \sqrt{\varphi(I)} \quad \forall i = 1, \ldots, r, \end{aligned}$$

since $\sqrt{\varphi(I)} = \varphi(\sqrt{I})$, as φ is a k-linear automorphism of R. Writing $D_i = \sum_\alpha p_\alpha \partial^\alpha$, we have $\varphi(D_i \bullet \varphi^{-1}(f)) = \varphi((\sum_\alpha p_\alpha \partial^\alpha) \bullet \varphi^{-1}(f)) = \sum_\alpha \varphi(p_\alpha) \varphi(\partial^\alpha \bullet \varphi^{-1}(f))$, so it suffices to show that $\varphi(\partial^\alpha \bullet \varphi^{-1}(f)) = \psi(\partial^\alpha) \bullet f$ for any $f \in R$. By linearity, it suffices to check this when $f = x^\beta$ is a monomial, i.e. we must show $\varphi(\partial^\alpha \bullet \varphi^{-1}(x^\beta)) = \psi(\partial^\alpha) \bullet x^\beta$ for all $\alpha, \beta \in \mathbb{N}^n$.

We first consider the case where α, β are standard basis vectors, i.e. $\partial^\alpha = \partial_{x_j}$ and $x^\beta = x_i$ for some $i, j \in \{1, \ldots, n\}$. Then $\varphi\left(\partial_{x_j} \bullet \varphi^{-1}(x_i)\right) = \varphi\left(\partial_{x_j} \bullet \sum_{k=1}^n (A^{-1})_{i,k} x_k\right) = \varphi\left((A^{-1})_{i,j}\right) = (A^{-1})_{i,j} = \left(\sum_{k=1}^n (A^{-1})_{k,j} \partial_{x_k}\right) \bullet x_i = \psi(\partial_{x_j}) \bullet x_i$.

To show that this extends to arbitrary β, note that both $\varphi\left(\partial_{x_j} \bullet \varphi^{-1}(\_)\right)$ and $\psi(\partial_{x_j}) \bullet (\_)$ are both differential operators, which must satisfy the product rule, so if these agree on any variable x_i then they agree on any monomial x^β. To extend to arbitrary α, note that ψ preserves multiplication in W by definition, so

$$\begin{aligned} \varphi\left(\partial_{x_j} \partial_{x_k} \bullet \varphi^{-1}(\_)\right) &= \varphi\left(\partial_{x_j} \bullet \varphi^{-1} \varphi\left(\partial_{x_k} \bullet \varphi^{-1}(\_)\right)\right) \\ &= \varphi\left(\partial_{x_j} \bullet \varphi^{-1}(\psi(\partial_{x_k}) \bullet (\_))\right) \\ &= \psi(\partial_{x_j}) \bullet \psi(\partial_{x_k} \bullet (\_)) \\ &= \psi(\partial_{x_j}) \psi(\partial_{x_k} \bullet (\_)) \\ &= \psi(\partial_{x_j} \partial_{x_k}) \bullet (\_) \end{aligned}$$

and thus inductively $\varphi\left(\partial^\alpha \bullet \varphi^{-1}(\_)\right) = \psi(\partial^\alpha) \bullet (\_)$ for any α. □

References

[1] Alberto Damiano, Irene Sabadini, and Daniele C. Struppa. Computational methods for the construction of a class of Noetherian operators. *Experiment. Math.*, 16(1):41–53, 2007.