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CONTENTS

This paper presents some algorithmic techniques for computing explicitly the Noetherian operators associated with a class of ideals and modules over a polynomial ring. The procedures we include in this work can be easily encoded in computer algebra packages such as CoCoA [CoCoATeam 05].

1. INTRODUCTION

The Ehrenpreis–Palamodov fundamental principle [Ehrenpreis 70, Palamodov 70] states the following:

Theorem 1.1. *Let $p_1(D), \dots, p_r(D)$ be linear constant-coefficient partial differential operators in n variables. Then there are algebraic varieties $V_1, \dots, V_t$ in $\mathbb{C}^n$ and differential operators $\partial_1, \dots, \partial_t$ with polynomial coefficients such that every function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfying*

$$p_1(D)f = \cdots = p_r(D)f = 0$$

can be represented as

$$f(x) = \sum_{j=1}^{t} \int_{V_j} \partial_j(e^{ix\cdot z}) d\nu_j(z), \tag{1–1}$$

for suitable Radon measures $d\nu_j$.

The collection

$$V = \{(V_1, \partial_1); (V_2, \partial_2); \dots; (V_t, \partial_t)\}$$

is said to be a multiplicity variety, and Theorem 1.1 is equivalent to the following strengthening of the classic Nullstellensatz:

Theorem 1.2. *Let I be an ideal of $R = \mathbb{C}[x_1, \dots, x_n]$. There exists a multiplicity variety V such that a polynomial f belongs to I if and only if $\partial_j f_{|V_j} = 0$ for every $j = 1, \dots, t$.*

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The operators $\partial_1, \ldots, \partial_t$ are called, in Palamodov's terminology, Noetherian operators because their construction relies essentially on a theorem of M. Noether on a membership criterion for polynomial submodules (see, for example, [Palamodov 70, pp. 161–162]). The nature of the original proof of the fundamental principle is essentially existential, and therefore the question of the explicit construction of such operators is of great interest whenever we consider a concrete application of the fundamental principle. Note that if I is the ideal generated by the polynomials $p_1, \ldots, p_r$ and if

$$I = Q_1 \cap \cdots \cap Q_t$$

is its primary decomposition, then the varieties V_j that appear in Theorem 1.1 are simply given by the algebraic sets $\mathcal{V}(Q_j)$. The information on the multiplicity of each of them is left to the operators ∂_j.

In this paper we build on some recent results in the construction of Noetherian operators [Heiß et al. 06, Marinari et al. 98, Oberst 90, Oberst 99, Sturmfels 02] and provide some new algorithms that allow the automatic construction of these operators at least in a rather large class of cases. We include several experiments using algorithms implemented in CoCoA .

In Section 2 we quickly review the fundamental tools from computational algebra (mostly the theory of Gröbner bases). The core of the paper is Section 3, in which we deal with the case of zero-dimensional ideals and present several explicit algorithms. A final section deals with the case of ideals of positive dimension.

Executable versions of the algorithms discussed in this paper have been explicitly written for CoCoA and are freely available at http://www.tlc185.com/coala.

2. COMPUTATIONAL ALGEBRA TOOLS

Throughout this paper, we will work in the ring $R = \mathbb{C}[x_1, \ldots, x_n]$ of polynomials in n variables with complex coefficients; we will think of R as the ring of symbols for the differential operators we are studying. Even though we consider differential operators with constant coefficients, the fundamental principle shows that Noetherian operators have, in general, polynomial coefficients; we will use the symbol A_n to denote the Weyl algebra $\mathbb{C}[x_1, \ldots, x_n, \partial x_1, \ldots, \partial x_n]$ of such operators. Here and throughout the paper, the symbol ∂x will be used as an abbreviation for $\frac{\partial}{\partial x}$.

Using the notation introduced in [Kreuzer and Robbiano 00], we will denote the monoid of power products in R by $\mathbb{T}^n$ and the module monoid of power products in R^s by

$$\mathbb{T}^n\langle e_1, \ldots, e_s\rangle = \{te_i \mid t \in \mathbb{T}^n,\ i = 1, \ldots, s\},$$

where e_i is the ith element of the canonical basis of R^s. All the definitions to follow are given for ideals but can be extended in a straightforward fashion to the case of modules [Kreuzer and Robbiano 00]. A term ordering σ on $\mathbb{T}^n$ is a total ordering on power products with the following two properties:

(I) If $t_1 >_\sigma t_2$ and $t \in \mathbb{T}^n$ then $t \cdot t_1 >_\sigma t \cdot t_2$.

(II) If $t \in \mathbb{T}^n$ and $s \in \mathbb{T}^n$ then $s \cdot t >_\sigma t$.

The *leading-term ideal* associated with I with respect to σ is the ideal generated by all the leading terms of elements of I, and will be denoted by

$$\mathrm{LT}_\sigma(I) = (\{\mathrm{LT}_\sigma(f) | f \in I\}).$$

More generally, the *leading-term ideal* associated with a subset G of R will be written as $\mathrm{LT}_\sigma(G) = (\{\mathrm{LT}_\sigma(f) | f \in G\})$. Note that $\mathrm{LT}_\sigma(G) = \mathrm{LT}_\sigma(I)$ if and only if the set G is a Gröbner basis for the ideal I, this being the main characterization of a Gröbner basis.

The algorithm that associates with an ideal I of R its Gröbner basis $\mathcal{G}_\sigma(I)$ is the core algorithm of the theory of Gröbner bases and can be found, for example, in [Kreuzer and Robbiano 00, Theorem 2.5.5]. Another key tool in computational algebra is the division algorithm (see again [Kreuzer and Robbiano 00, Theorem 1.6.4]), which can be performed to generate the remainder of a polynomial with respect to a set of generators of I.

Note that the remainder of a polynomial depends on the set of generators chosen for I (in fact, it even depends on their order). The fundamental property of Gröbner bases is that such a remainder is zero if and only if the polynomial belongs to the ideal. For this reason the remainder calculated with respect to a Gröbner basis is called the *normal form* of a polynomial.

Given a polynomial $f \in I$ and a term ordering σ, we will denote by $\mathrm{NF}_\sigma(f)$ the normal form of f with respect to the σ-Gröbner basis of I (the same notation is used for modules). An equivalent way to compute a remainder is using *rewrite rules* [Kreuzer and Robbiano 00, Section 2.2]. Given a polynomial $g \in R$, we say that a polynomial f_1 rewrites to f_2 with respect to the rewrite rule $\xrightarrow{g}$ (and this is indicated by $f_1 \xrightarrow{g} f_2$) if there exists a monomial m in R such that $f_2 = f_1 - mg$ and $\mathrm{LT}_\sigma(mg)$ is not in the support of f_2. This is also called

a one-step reduction. We can rewrite a polynomial using a set of elements $\mathcal{G} = \{g_1, \ldots, g_s\}$ by performing a one-step reduction with each of the g_i's, in that order. We will denote by $\xrightarrow{\mathcal{G}}$ the transitive closure of the relations $\xrightarrow{g_1}, \ldots, \xrightarrow{g_s}$. This relation is called a *rewrite relation* or *rewrite rule*. By applying a sequence of one-step reductions to a polynomial f using the elements in $\mathcal{G}$ we then obtain a remainder of f with respect to $\{g_1, \ldots, g_s\}$. In particular, if $\mathcal{G}$ is a Gröbner basis we have that f rewrites to its normal form, i.e., $f \xrightarrow{\mathcal{G}} \mathrm{NF}_\sigma(f)$.

We now introduce some definitions about elimination theory and term orderings (see [Kreuzer and Robbiano 00, Section 3.4] for details on this topic) that will be necessary in the last section.

Definition 2.1. Let $R = \mathbb{C}[x,t]$, where $x = (x_1, \ldots, x_{n-d})$, $t = (t_1, \ldots, t_d)$. A term ordering σ on $\mathbb{T}^n$ is called an elimination ordering with respect to x if every element $f \in R$ whose leading term is contained in $\mathbb{C}[t]$ is such that $f \in \mathbb{C}[t]$. In other words,

$$\forall f \in R, \quad \mathrm{LT}_\sigma(f) \in \mathbb{C}[t] \;\Rightarrow\; f \in \mathbb{C}[t].$$

The reason such a term ordering is called an *elimination* ordering is that it allows one to eliminate the variables x from an ideal, i.e., it allows one to compute $I \cap \mathbb{C}[t]$. To do this, it suffices to compute a Gröbner basis with respect to any elimination ordering as in Definition 2.1 and then keep only the elements that do not contain any monomials in x. Such elements actually form a Gröbner basis for the ideal $I \cap \mathbb{C}[t]$. It can be easily checked that `Lex`, the lexicographic term ordering on $\mathbb{T}^n$, is an elimination ordering with respect to any "initial" subset of variables, i.e., with respect to any subset of the type $\{x_1, \ldots, x_k\}$ in $\mathbb{C}[x_1, \ldots, x_n]$, with $k \leq n$. A class of term orderings that satisfy the elimination property that we are going to use for our goal of computing the Noetherian operators in $\mathbb{C}(t)[x]$ is that of the so-called *product orderings*.

Definition 2.2. Let $R = \mathbb{C}[x,t]$ as before and let σ_x and σ_t be two term orderings on the set of terms $\mathbb{T}_x = \{x^a \mid a \in \mathbb{N}^{n-d}\}$ and $\mathbb{T}_t = \{t^b \mid b \in \mathbb{N}^d\}$ respectively. The product ordering $\sigma_x \cdot \sigma_t$ is defined by

$$x^a t^b >_{\sigma_x \cdot \sigma_t} x^c t^d \;\Leftrightarrow\; x^a >_{\sigma_x} x^c \,\text{or}\, (x^a = x^c \,\text{and}\, t^b >_{\sigma_t} t^d).$$

It is immediate to show that the product ordering defined above is an elimination ordering with respect to x, no matter what the choice of σ_x and σ_t is. Elimination orderings are usually slow when it comes to Gröbner basis computations; in particular, `Lex` is known to be one of the slowest. Product orderings are then introduced to achieve better performance. One can in fact define a "fast" term ordering (such as `DegRevLex`) on each of the two subsets of variables, and then take the product. The following lemma will be useful later in the paper (see also [Vasconcelos 98, p. 220]).

Lemma 2.3. *Let $R = \mathbb{C}[x,t]$ be a polynomial ring equipped with a product ordering σ of the type $\sigma_x \cdot \sigma_t$ as in Definition 2.2. Let I be an ideal of R and let $\mathcal{G} = (g_1, \ldots, g_s)$ be a σ-Gröbner basis for I. Consider the extended ideal IR_d in $R_d = \mathbb{C}(t)[x]$ endowed with the term ordering σ_x. Then $\mathcal{G}$ forms a Gröbner basis for IR_d with respect to σ_x.*

Proof: Denote by $x^{a_i} t^{c_i}$ the leading term of g_i, where $a_i \in \mathbb{N}^{n-d}$ and $c_i \in \mathbb{N}^d$, $i = 1, \ldots, s$. From the fact that we chose a product ordering σ, it follows that once we view g_i as an element of IR_d, its leading term is x^{a_i}. In other words, $\mathrm{LT}_{\sigma_x}(g_i) = x^{a_i}$ in R_d. Consider a polynomial f in IR_d. The set $\mathcal{G}$ still forms a set of generators for the extended ideal, so f can be written as an R_d-linear combination of the g_i's. Moreover, supposing f monic, we can write f as

$$f = x^a + \sum_b p_b(t) x^b, \;\text{where}\, b \in \mathbb{N}^{n-d} \,\text{and}\, x^a >_{\sigma_x} x^b \,\forall\, b.$$

Consider the product $D(t)$ of all the denominators of the coefficients $p_b(t)$ in f. Then $D(t)f$ is a polynomial in R, and it is still a combination of the elements of $\mathcal{G}$, so $D(t)f \in I$. Because of the fact that σ is a product order, the leading term of $D(t)f$ is simply the leading term of f multiplied by some power of t, i.e., $\mathrm{LT}_\sigma(D(t)f) = x^a t^c$ for some $c \in \mathbb{N}^d$. Hence since $\mathcal{G}$ is a Gröbner basis for I, $x^a t^c$ is a multiple of one of the leading terms of its elements, say $x^{a_1} t^{c_1}$ modulo a change in the order in $\mathcal{G}$. This means that there exist $\alpha \in \mathbb{N}^{n-d}$ and $\gamma \in \mathbb{N}^d$ such that

$$x^a t^c = x^\alpha t^\gamma x^{a_1} t^{c_1},$$

which means that x^a is a multiple of x^{a_1}, and this concludes the proof. □

3. THE ZERO-DIMENSIONAL CASE

In this section, I is a primary zero-dimensional ideal, i.e., the algebraic set $\mathcal{V}(I)$ is a finite union of points in $\mathbb{C}^n$. Since a zero-dimensional primary ideal is associated with a single point of the variety $\mathcal{V}(I)$, we can always assume,

with a change of coordinates, that $\mathcal{V}(I) = \{(0,\dots,0)\}$, or equivalently that $\sqrt{I} = (x_1,\dots,x_n)$.

3.1 Closed Differential Conditions

A first complete description of the differential condition characterizing a zero-dimensional primary ideal centered at zero has been given in [Marinari et al. 98]. We briefly recall the main notation and definitions of that paper. We will denote by $D(i_1,\dots,i_n) : R \to R$ the differential operator defined by

$$D(i_1,\dots,i_n) = \frac{1}{i_1!\cdots i_n!}\partial x_1^{i_1}\cdots\partial x_n^{i_n}, \quad i_j \in \mathbb{N},$$

for all $j = 1,\dots,n$, or alternatively, if $t = x_1^{i_1}\cdots x_n^{i_n} \in \mathbb{T}^n$, we will use the symbol $D(t)$ to mean $D(i_1,\dots,i_n)$. Moreover, we write $\mathcal{D} = \{D(t) | t \in \mathbb{T}^n\}$ and denote by $\mathrm{Span}_{\mathbb{C}}(\mathcal{D})$ the $\mathbb{C}$-vector space generated by $\mathcal{D}$. We now introduce some morphisms on $\mathcal{D}$ that act as "derivative" and "integral":

$$\sigma_{x_j}\left(D(i_1,\dots,i_n)\right) = \begin{cases} D(i_1,\dots,i_j-1,\dots,i_n) & \text{if } i_j > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3–1}$$

and

$$\rho_{x_j}\left(D(i_1,\dots,i_n)\right) = D(i_1,\dots,i_j+1,\dots,i_n). \tag{3–2}$$

Such operators extend trivially to $\mathrm{Span}_{\mathbb{C}}(\mathcal{D})$ by linearity, and one can easily define σ_t and ρ_t for any $t \in \mathbb{T}^n$ by composition.

Definition 3.1. A subspace L of $\mathrm{Span}_{\mathbb{C}}(\mathcal{D})$ is said to be *closed* if

$$\sigma_{x_j}(L) \subseteq L, \text{ for all } j = 1,\dots,n.$$

Definition 3.2. Let I be a primary ideal in R such that $\sqrt{I} = (x_1,\dots,x_n)$. We define the subspace of differential operators associated with I as

$$\Delta(I) := \{L \in \mathrm{Span}_{\mathbb{C}}(\mathcal{D}) \,|\, L(f)(0,\dots,0) = 0 \text{ for all } f \in I\}.$$

Similarly, we associate with each subset $V \subseteq \mathrm{Span}_{\mathbb{C}}(\mathcal{D})$ an ideal

$$\mathcal{I}(V) := \{f \in R \,|\, L(f)(0,\dots,0) = 0 \text{ for all } L \in V\}.$$

Theorem 3.3. *Let $\mathfrak{m}$ be the maximal ideal $(x_1,\dots,x_n)$ of R. There is a bijective correspondence between $\mathfrak{m}$-primary ideals of R and closed subspaces of $\mathrm{Span}_{\mathbb{C}}(\mathcal{D})$:*

$$\{\mathfrak{m}\text{-primary ideals in } R\} \underset{\mathcal{I}}{\overset{\Delta}{\rightleftarrows}} \{\text{closed subspaces of } \mathrm{Span}_{\mathbb{C}}(\mathcal{D})\},$$

so that $I = \mathcal{I}\Delta(V)$ and $V = \Delta\mathcal{I}(I)$ for every I and V. Moreover, for a zero-dimensional $\mathfrak{m}$-primary ideal of R whose multiplicity is μ, we have $\dim_{\mathbb{C}}(\Delta(I)) = \mu$.

Theorem 3.3 shows that the Noetherian operators associated with a zero-dimensional primary ideal form a closed subspace of $\mathrm{Span}_{\mathbb{C}}(\mathcal{D})$. In addition, when considering a zero-dimensional primary ideal, since the dimension of $\Delta(I)$ is finite, we can view a basis of $\Delta(I)$ as a set of Noetherian operators that in this particular case happen to be operators with constant coefficients. Moreover, such a vector space has the nice property of being closed, a fact that has been used by the authors of [Marinari et al. 98] to construct a procedure that, given I, computes $\Delta(I)$. The algorithm is described below.

Algorithm 3.4. Let I be a zero-dimensional primary ideal of R such that $V(I) = \{(0,\dots,0)\}$ and let $\mu = \dim_{\mathbb{C}}(R/I)$ be its multiplicity. The following procedure computes the Noetherian operators associated with I:

Input: $\mathcal{G} = \{g_1,\dots,g_t\}$ a Gröbner basis for I.

Output: $\Delta(I) = \{L_0,\dots,L_{\mu-1}\}$.

1. **Initialization**: $i = 1$, $L_0 = 1 = \mathrm{Id}_{\mathrm{Span}_{\mathbb{C}}(\mathcal{D})}$.
2. **If** $\mu > 1$, construct a linear operator $L_1 = \sum_{j=1}^{n} c_j \partial x_j$ with a suitable choice of the c_j's such that $L_1(f)(0,\dots,0) = 0$ is satisfied for each generator f of I.
3. Put $i = 2$.
4. **While** $i < \mu$ **do**

   Define L_{i+1} as a linear combination of $\rho_{x_{j_0}}(L_0),\dots,\rho_{x_{j_i}}(L_i)$ such that $\langle L_0,\dots,L_{i+1}\rangle$ is closed and $L_{i+1}(f)(0) = 0$ for each generator f of I.

Corollary 3.5. *Let L be an operator of $\Delta(I)$, where I is as in Algorithm 3.4 and μ is its multiplicity. Then $\deg L < \mu$ as an element of A_n.*

Proof: The construction of $\Delta(I)$ starts with $L_0 = 1$, and at each step the degree of L_{i+1} increases at most 1, so that the last element $L_{\mu-1}$ has degree at most μ. □

Remark 3.6. Algorithm 3.4 is basically the solution of a system of linear equations in the coefficients c_j of the linear combinations $L_{i+1} = c_0\rho_{j_0}L_0 + \cdots + c_i\rho_{j_i}L_i$. Since the

system can have more than one solution, one may simply pick the one with minimal norm. An implementation for a simplified version of 3.4 has been coded for CoCoA and is available through the CoAlA web page [Damiano and Mannino 05].

Example 3.7. The following example is taken from [Ehrenpreis 70, Example 4, p. 37]. Here we show how to study it using Algorithm 3.4. Let us consider the primary ideal at the origin $I = (y^2, x^2 - y) \subset \mathbb{C}[x, y]$, whose multiplicity is 4. We start with $L_0 = 1$, and an obvious choice for a linear operator is $L_1 = \partial x$. This has also a geometric interpretation: the origin is the intersection of the two curves given by the generators y^2 (the x-axis twice) and $x^2 - y$ (a parabola with vertex at the origin). Not only do two such curves intersect at the origin, they are also tangent along the direction of the x-axis.

Therefore $L_1 = \partial x$ must be a Noetherian operator. The higher-degree operators describe a higher contact of the line and the parabola at zero. We can try to find the next one as a combination $L_2 = a\partial x + b\partial xy$.

However, this operator L_2 does not respect the closure condition, since $\sigma_x(L_2) = a + b\partial y$, which is not in the subspace $\langle L_0, L_1\rangle = \langle 1, \partial x\rangle$. Instead, a different choice for the morphisms ρ_{x_j} gives $L_2 = a\rho_y(1) + b\rho_x(\partial x) = a\partial y + b\partial x^2$, which respects closure and annihilates the generators of I at zero with $a = 1$ and $b = \frac{1}{2}$.

Again, this operator could have been foreseen in advance, since it is the global annihilator of $x^2 - y$, and it annihilates y^2 at the origin. As a last operator, one can choose $L_3 = \rho_x(L_2) = \partial xy + \frac{1}{6}\partial x^3$. Of course, the choice $\rho_y(L_2) = \frac{1}{2}\partial y^2 + \frac{1}{2}\partial x^2 y$ would have been possible as far as the annihilation of I is concerned, but it would have violated closure, since $\sigma_x(L_2) = \partial xy$ is not a combination of the previous operators. The iteration ends here, since we have found four differential operators.

3.2 Forward Reduction

We are now going to present an alternative procedure to compute the Noetherian operators associated with I that makes no use of linear algebra and utilizes the power of Gröbner bases.

Algorithm 3.8. (Computation of Noetherian operators for zero-dimensional ideals.) Let I be a zero-dimensional primary ideal of R such that $\mathcal{V}(I) = \{(0, \ldots, 0)\}$. The following procedure computes the Noetherian operators associated with I:

Input: $\mathcal{G} = \{g_1, \ldots, g_t\}$ a Gröbner basis for I.

Output: $\Delta(I) = \{L_1, \ldots, L_\mu\}$.

1. Compute $\mu(I) = \dim_{\mathbb{C}}(R/I)$.
2. Write the Taylor expansion at the origin of a polynomial $h \in R$ up to the degree $\mu - 1$ with coefficients $c_\alpha \in \mathbb{C}$:
$$T_{\mu-1}h(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n,\, |\alpha| < \mu} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$
3. Write the normal form of $T_{\mu-1}h$ with respect to $\mathcal{G}$ as
$$\mathrm{NF}_\sigma T_{\mu-1}h(x_1, \ldots, x_n) = \sum_\beta d_\beta x_1^{\beta_1} \cdots x_n^{\beta_n} \quad (3\text{–}3)$$
and find scalars $a_{\beta\alpha} \in \mathbb{C}$ such that $d_\beta = \sum_\alpha a_{\beta\alpha} c_\alpha$.
4. For each β such that $d_\beta \neq 0$, return the operator
$$\begin{aligned} L_\beta &= \sum_\alpha a_{\beta\alpha} \frac{1}{\alpha_1! \cdots \alpha_n!} \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \\ &= \sum_\alpha a_{\beta\alpha} D(\alpha_1 \cdots \alpha_n). \end{aligned}$$

Proof: Let $h(x_1, \ldots, x_n) = \sum_{|\alpha|=0}^{\deg(h)} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be the Taylor expansion centered at the origin of a polynomial $h \in R$ and let $\mathcal{G}$ be the Gröbner basis of I. From the theory of Gröbner bases we know that the normal form with respect to $\mathcal{G}$ of h is zero if and only if $h \in I$, so the condition $\mathrm{NF}_\sigma h = 0$ is the one that we want to characterize. It suffices to write
$$\mathrm{NF}_\sigma\Big(\sum_{|\alpha|=0}^{\deg(h)} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}\Big) = \sum_{|\beta|=0}^{\deg(h)} d_\beta x_1^{\beta_1} \cdots x_n^{\beta_n} = 0 \quad (3\text{–}4)$$
and deduce from the annihilation of each coefficient d_β in (3–4) a differential condition on h. This completely characterizes the membership of a polynomial h to I. The only thing to observe is that we do not need to work with terms up to $\deg(h)$ for the Taylor expansion. In fact, the number of differential conditions we need is precisely μ, and so from Corollary 3.5 it follows that the derivatives to be considered are, in the worst case, those of order $\mu - 1$ (see also [Mourrain 97]). Those differential conditions arise by using coefficients c_α up to $|\alpha| = \mu - 1$. Therefore the Taylor expansion can be truncated at $\mu - 1$. □

Remark 3.9. It is crucial to observe that we do not need to characterize the membership of a polynomial h of undetermined degree $\deg(h)$, since we have the bound $\mu - 1$

on its degree. Thus Algorithm 3.8 is a procedure that is implementable in any computer algebra software package. Moreover, the computation of the normal form (3–3) can be done degree by degree, so that we can stop the reduction process whenever the normal form of a particular degree is zero. This actually speeds up the computations in most cases we studied (up-to-date CPU times for several examples are available from [Damiano and Mannino 05]).

Example 3.10. Consider again Example 3.7 to show the substantial difference between Algorithms 3.4 and 3.8. Since $\mu(I) = 4$, we start by writing the truncated Taylor expansion of a polynomial $h \in \mathbb{C}[x, y]$,

$$T_3h(x,y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3,$$

and perform the normal-form computation using $x^2 \mapsto y$ and $y^2 \mapsto 0$ as rewrite rules. Grouping like terms, we can write the remainder of T_3h as a linear combination of the generators $1, x, y, xy$ of R/I as follows:

$$\mathrm{NF}_\sigma(T_3h) = [c_{00}] + [c_{10}]x + [c_{01} + c_{20}]y + [c_{11} + c_{30}]xy. \tag{3–5}$$

We call these four terms a *Macaulay basis* for the ideal I, although this name is also used by some authors for a generalization of a Gröbner basis. Note that the terms y^2, x^2y, xy^2, and y^3 disappeared since they all rewrote to zero. The computation ends by expressing the coefficients written in square brackets in (3–5) as operators according to their meaning as Taylor coefficients, namely $[c_{00}] \to 1, [c_{10}] \mapsto \partial x, [c_{01}+c_{20}] \mapsto \partial y + \frac{1}{2}\partial x^2, [c_{11}+c_{30}] \mapsto \partial xy + \frac{1}{6}\partial x^3$. This gives the same result obtained in Example 3.7 as expected. This is not surprising, since Theorem 3.3 states that the correspondence $I \leftrightarrow \Delta(I)$ is one-to-one.

Algorithm 3.8 does not take directly into account the closure of the space of Noetherian operators, as does Algorithm 3.4. The fact that $\Delta(I)$ is closed is a general fact that follows from a Leibniz formula for the morphisms σ_{x_j} and the fact that I is an ideal [Marinari et al. 93, Proposition 2.4]. This is true not only for zero-dimensional ideals but also in positive dimension, as we will see in Section 4. We want to show that the closure of $\Delta(I)$ is also a direct consequence of Algorithm 3.8 and of the following property of Macaulay bases.

Lemma 3.11. *Let $I \subset R$ be an ideal and let $\mathcal{M}$ be the Macaulay basis of R/I, i.e., the generators of R/I as a $\mathbb{C}$-vector space. Let $s_{x_j} : \mathbb{T}^n \to \mathbb{T}^n$ be the "derivative" morphism*

$$s_j(x_1^{i_1} \cdots x_n^{i_n}) = \begin{cases} x_1^{i_1} \cdots x_j^{i_j - 1} \cdots x_{i_n}^n & \text{if } i_j > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3–6}$$

Then $\mathcal{M}$ is s_j-closed for each j.

Proof: It is known that the Macaulay basis for R/I can be computed through a Gröbner basis $\mathcal{G}$ of I. In fact, it is [Kreuzer and Robbiano 00, theorem 1.5.7]

$$\mathcal{M} = \mathbb{T}^n \backslash \mathrm{LT}_\sigma(\mathcal{G}),$$

where σ is any term ordering on $\mathbb{T}^n$. Since $\mathcal{G}$ is a Gröbner basis for I, the leading-term ideal $\mathrm{LT}_\sigma(I)$ coincides with $\mathrm{LT}_\sigma(\mathcal{G})$. Let $t \neq 0$ be a term of $\mathcal{M}$. Suppose that there exists an index j such that $0 \neq s_j(t) \notin \mathcal{M}$. Then $s_j(t) \in \mathrm{LT}_\sigma(\mathcal{G})$. The latter being an ideal, we have $t = x_j \cdot s_j(t) \in \mathrm{LT}_\sigma(\mathcal{G})$, which is a contradiction. Note that if $s_j(t) \notin \mathcal{M}$ for all j, this simply says that $t = 0$, which is again a contradiction. □

The morphism s_j introduced in the above lemma is the analogue of σ_{x_j} defined in Section 3.1, and we will show in the next proposition that the s_j-closure of $\mathcal{M}$ is equivalent to the σ_{x_j}-closure of the space of Noetherian operators associated with I.

Proposition 3.12. *Let I be a zero-dimensional primary ideal of R such that $\mathcal{V}(I) = \{(0, \ldots, 0)\}$ and let $\mathcal{O} = \{L_\beta\}$ be the set of operators computed with Algorithm 3.8. Then $\mathrm{Span}_\mathbb{C}(\{L_\beta\})$ is a closed subspace of $\mathrm{Span}_\mathbb{C}(\mathcal{D})$.*

Proof: Let $L_\beta \in \mathcal{O}$, and let d_β be the corresponding coefficient of the normal form $\mathrm{NF}_\sigma(h)$ as computed with the algorithm. Let $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ be the term whose coefficient is d_β. It is clear that such a term is part of the Macaulay basis of R/I, since it appears in the expression of $\mathrm{NF}_\sigma(h)$, which is a representation of the class of h in the quotient R/I. Denote by F_β the set of operators of $\mathcal{O}$ such that the corresponding term in the expression of $\mathrm{NF}_\sigma(h)$ divides x^β,

$$F_\beta = \{L_\gamma \in \mathcal{O} \,\text{such that}\, x^\gamma | x^\beta\},$$

and for each $L_\gamma \in F_\beta$ consider $t_\gamma = x^{\beta-\gamma}$. Since each L_γ has been computed from the Taylor expansion that results from using a division algorithm that uses a Gröbner basis $\mathcal{G}$ of I, we have that [Kreuzer and Robbiano 00, Proposition 2.2.2] if h' is such that

$$x^\gamma = \mathrm{NF}_\sigma(h') \text{ and } \mathrm{supp}(h') \subseteq \mathrm{supp}(h),$$

then

$$x^\beta = t_\gamma x^\gamma = \mathrm{NF}_\sigma(t_\gamma)\mathrm{NF}_\sigma(h') = \mathrm{NF}_\sigma(t_\gamma h'),$$

i.e., the term in x^β is obtained by rewriting a multiple of that part of the polynomial h that rewrites to x^γ. By looking at the expression of L_β it is then obvious that

$$\sigma_{t_\gamma}(L_\beta) = L_\gamma,$$

since L_β is written as a combination of Taylor coefficients corresponding to the terms of $t_\gamma h'$. It now suffices to prove that such t_γ's are enough to conclude that $\mathcal{O}$ is closed. This is a consequence of the previous lemma, since all the d_γ in F_β are associated with those terms x^γ of the Macaulay basis $\mathcal{M}$ that divide x^β. Hence from the s_j-closure of $\mathcal{M}$ we deduce that $\{x^\gamma = s_{t_\gamma}(x^\beta)\} = \{s_j(x^\beta), j = 1, \ldots, n\}$. □

3.3 Backward Reduction

We could think of performing the reduction step of the algorithm for the computation of Noetherian operators for zero-dimensional ideals "backward." Instead of writing the full Taylor expansion and then using the Gröbner basis of I to rewrite it, we start from the residual monomials, which are easily calculated, for example, with CoCoA. We then "pull back" each monomial using the generators of I as "antirewrite rules." Let us explain what we mean by this. In general, when using a polynomial f to rewrite another polynomial g, we use its leading monomial $\mathrm{LT}(f)$ to divide the polynomials g and then we substitute each $\mathrm{LT}(f)$ in g with the *tail* of f, $\mathrm{LT}(f) - f$. For instance, we rewrite $g = x^3$ to xy using $f = x^2 - y$, by replacing x^2 in x^3 with the tail $x^2 - (x^2 - y) = y$. This operation, when performed using the elements of a Gröbner basis for I, does not alter the class of g in R/I and leads to the normal form $\mathrm{NF}_\sigma(g)$. What we mean by "antirewriting" is, roughly speaking, to use the smallest monomial $in(f)$ of f and replace it with the head of the polynomial $in(f) - f$. This way, from $in(f)$ we "climb up" to find all the other monomials that are equivalent to $in(f)$ modulo (f). Here is a more precise definition.

Definition 3.13. Let f be a polynomial of R, let g be a monomial, and let $m = in(f)$ be the smallest term of f with respect to a given term ordering on $\mathbb{T}^n$. We say that g *rewrites backward* to g' in one step, using f, if m divides g and

$$g' = \frac{g}{m}(m - f).$$

Example 3.14. With this terminology, $g = xy$ rewrites backward to x^3 using $x^2 - y$, which is exactly the opposite of the standard rewrite process that leads from x^3 to xy. If we use $f = x^2 + xy - 2y$ instead, $g = xy$ rewrites to $g' = \frac{1}{2}x^3 + \frac{1}{2}x^2y$. Finally, g could not be rewritten backward using $x^2 - y^2$, since y^2 does not divide g. Notice that in general, if we perform a one-step backward reduction and then a one-step reduction in the usual way, we again obtain g.

We can now apply an iteration of this procedure of rewriting a monomial backward using a Gröbner basis for I. We start from a residual monomial and we rewrite it backward using one generator. Then we rewrite backward each monomial obtained after this step, if possible, using any element of the Gröbner basis. Technically, this procedure never ends, since we can imagine obtaining a new polynomial of higher degree at each step, as for example with $g = x$ and $f = x^2 - x$. However, for the purpose of computing Noetherian operators, we know from Section 3 that as polynomials in $\mathbb{C}[\partial x_1, \ldots, \partial x_n]$, they have degree at most $\mu - 1$. Therefore we can stop the iteration once we have reached a polynomial of such a degree. Let us illustrate this idea with an example before we present the algorithm in general.

Example 3.15. Consider the ideal $J = (x^2 - z, y^2 - z, z^2)$ in $\mathbb{C}[x, y, z]$. It represents the origin in $\mathbb{C}^3$ with multiplicity 8. Its generators are a `DegLex` Gröbner basis. The residual monomials for R/J are

$$\{1, x, y, xy, z, xz, yz, xyz\}.$$

First, let us reconstruct the Noetherian operators associated with xyz. By rewriting it using $x^2 - z$ we obtain the new monomial x^3y. This cannot be rewritten further. However, the term xy^3 is another monomial that is "attracted" by xyz via the other generator $y^2 - z$ of J. Summing up the residual monomial and all the results of the backward reduction, we then obtain $g' = x^3y + xy^3 + xyz$, whose dual $D(g') = \frac{1}{6}\partial x^3\partial y + \frac{1}{6}\partial x\partial y^3 + \partial x\partial y\partial z$ is actually the Noetherian operator of J relative to xyz.

The choice of the residual monomial xyz in Example 3.15 is not random. Indeed, it is maximal among all the residual monomials with respect to the derivative morphisms (3–6).

Definition 3.16. Let m be a residual monomial of R/I. We say that m is a *corner monomial* if it is maximal with respect to the monoid structure of $\mathbb{T}^n$, i.e., if

$$x_i \cdot m \in \mathrm{LT}(I), \text{ for all } i = 1, \ldots, n.$$

If we represent R/I as a subset of $\mathbb{N}^n$, the corner monomials are exactly in corner position. Proposition 3.12 says that the Noetherian operators are generated by those corresponding to the corner monomials by taking the closure with respect to the morphisms (3–1). This fact allows one to come up with a general procedure that constructs the Noetherian operators starting with the corner monomials and then generates the entire space of Noetherian operators.

Algorithm 3.17. Let $I \subset R$ be a zero-dimensional primary ideal of multiplicity μ centered at the origin. The following list of instructions constructs the Noetherian operators associated with I:

Input: A Gröbner basis $\mathcal{G}$ of I and the residual monomials of R/I.

Output: The space of Noetherian operators associated with I.

1. Construct the set $\mathcal{C}$ of corner monomials using Definition 3.16.
2. For each corner monomial $m \in \mathcal{C}$ find the associated Noetherian operators by rewriting it backward with respect to $\mathcal{G}$ using Definition 3.13. Stop when the backward reduction is no longer possible or when the degree of the polynomial obtained is $\mu - 1$.
3. Collect all the polynomials obtained in the set $\mathcal{D}$.
4. Compute the closure of $\mathcal{D}$ by applying the morphism σ_{x_i}, $i = 1, \ldots, n$, to all its elements.
5. For each element L in the closure of $\mathcal{D}$ calculate $D(L)$.

3.4 Extension to Modules

All the results in the previous subsections can be extended in a straightforward fashion to the case of zero-dimensional primary modules. Rather than giving the details, we use the CoCoA version of the algorithm for modules to look at a couple of examples.

Example 3.18. Let A be the matrix

$$A = \begin{pmatrix} x & 1 \\ y & x \\ 0 & y \end{pmatrix}$$

and let M be the module generated by the rows of A, i.e., $M = \langle xe_1 + e_2,\, xe_2 + ye_1,\, ye_2 \rangle$. The module term ordering we choose is `Lex-Pos`, meaning that to compare two terms we first look at the power product, using `Lex`, and then at the position. The way we just wrote the generators of M reflects this choice. It is clear that $J_M = (x^2 - y, y^2, xy)$, and using, for example, CoCoA, we obtain the following:

1. $\mu(M) = 3$.
2. The `Lex`-Gröbner basis of M is $\mathcal{G} = \{xe_1 + e_2,\, xe_2 + ye_1,\, ye_2,\, y^2e_1\}$.
3. A Macaulay basis for M is the set $\{e_1, e_2, ye_1\}$.

We begin by writing explicitly the vectorial Taylor expansion of a vector $w(x, y) \in R^s$ up to degree 2:

$$\begin{aligned} T_2 w(x,y) = {} & c^1_{00}e_1 + c^2_{00}e_2 + c^1_{10}xe_1 + c^2_{10}xe_2 + c^1_{01}ye_1 \\ & + c^2_{01}ye_2 + c^1_{20}x^2e_1 + c^2_{20}x^2e_2 + c^1_{11}xye_1 \\ & + c^2_{11}xye_2 + c^1_{02}y^2e_1 + c^2_{02}y^2e_2. \end{aligned}$$

Only few terms survive after we compute the normal form relative to the Gröbner basis $\mathcal{G}$, leading to

$$\mathrm{NF}_\sigma(w) = [c^1_{00}]e_1 + [c^2_{00} - c^1_{10}]e_2 + [c^1_{20} + c^1_{01} - c^2_{10}]ye_1.$$

We conclude that the Noetherian operators associated with M, written in vectorial form, are

$$\begin{aligned} D^1_{00} &= (1, 0), \quad D^2_{00} = (-\partial x, 1), \\ D^1_{01} &= \left(\frac{1}{2}\partial x^2 + \partial y, -\partial x \right), \end{aligned}$$

and it is easy to check that they generate a closed subspace, since $\sigma_x(D^2_{00}) = \sigma_y(D^1_{01}) = D^1_{00}$ and $\sigma_x(D^1_{01}) = D^2_{00}$.

Example 3.19. (Solution of a system of PDEs.) In the introduction we saw that the fundamental principle can be used to write an integral representation of the solution of a system of linear constant-coefficient partial differential equations. We will show how this can be applied, now that we know how to compute Noetherian operators. Consider the overdetermined PDE system given by

$$\begin{aligned} f_{zz} - f_z + f_t + 2g_z &= g, \\ f_{zt} + g_t &= 0, \\ f_{tt} + g_{zt} - g_t &= 0, \\ f_t - g_{zz} + g_z + g_t &= 0, \end{aligned} \tag{3–7}$$

where $f, g \in C^\infty(\mathbb{R}^2)$ and we use indices to denote derivatives. The general solution to (3–7) can be written using

a generalization of (1–1). We consider the rectangular operator $P(D)$ defined by

$$P = \begin{pmatrix} x^2 - x + y & 2x - 1 \\ xy & y \\ y^2 & xy - y \\ -y & x^2 - x - y \end{pmatrix},$$

where x and y are the dual variables of z and t, respectively. Note that we are choosing a particular Fourier transform to write $P(D)$, so that it does not take into account the factor $-\sqrt{-1}$. The module M associated with the matrix P is not primary; hence we can use Singular [Greuel et al. 01] to get a primary decomposition (using the function `modDec` from the library `mprimdec.lib`).The module M is the intersection of the two zero-dimensional modules

$$M_1 = \langle (x,1), (y,x), (0,y) \rangle, \quad J_1 = \sqrt{M_1} = (x,y)$$

and

$$M_2 = \langle (x-1,1), (y,0), (y,x-1) \rangle, \quad J_2 = \sqrt{M_2} = (x-1,y)$$

of multiplicities 3 and 2. We already computed the operators associated with the module M_1 in the previous example. To compute the operators associated with M_2 we need to shift the variety to the origin using the change of coordinates ($X = x - 1, Y = y$). Then, using the new variables X and Y, we can apply the module version of Algorithm 3.8 and obtain the Noetherian operators $\{(1,0), (\partial X, -1)\}$. Returning to the variables x, y, we have the set $\{(1,0), (\partial x, -1)\}$. Therefore, it is possible to write explicitly the solutions to (3–7) as

$$\begin{aligned} \begin{pmatrix} f(z,t) \\ g(z,t) \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{|(0,0)} & e^{zx+ty} \\ + B \begin{pmatrix} -\partial x \\ 1 \end{pmatrix}_{|(0,0)} & e^{zx+ty} \\ + C \begin{pmatrix} \frac{1}{2}\partial x^2 + \partial y \\ -\partial x \end{pmatrix}_{|(0,0)} & e^{zx+ty} \\ + D \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{|(1,0)} & e^{zx+ty} \\ + E \begin{pmatrix} \partial x \\ -1 \end{pmatrix}_{|(1,0)} & e^{zx+ty} \\ = \begin{pmatrix} A - Bz + \frac{1}{2}Cz^2 + Ct + De^z + Eze^z \\ B - Cz - Ee^z \end{pmatrix}. & \end{aligned}$$

4. THE CASE OF POSITIVE DIMENSION

In dealing with ideals and modules whose dimension is positive, in general one may not expect the associated Noetherian operators to be constant-coefficient linear operators. In fact, this is the case for some of the examples from the literature (see [Ehrenpreis 70, Palamodov 70]). For instance, in considering the ideal $I = (x^2, y^2, -xz+y) \subset \mathbb{C}[x,y,z]$ one has that a set of Noetherian operators associated with I is $\{1, \partial x + z\partial y\}$, and it can be proved that there exists no set of Noetherian operators with constant coefficients associated with I (see [Palamodov 70, Example 4, p. 183]). However, an interesting property that we notice in this case is that the set of "differential" variables forms the set of variables appearing in the polynomial coefficients (in this case such sets are respectively $\{x, y\}$ and $\{z\}$). This is actually valid whenever we can put the algebraic variety in a particular position, through an opportune change of coordinates, called *normal position*. To do this, one can apply the procedure of Noether normalization to the ideal I. This algorithm comes from the so-called Noether normalization theorem [Björk 79, p. 116]. We now state a version of the theorem that we will need for our computations.

Theorem 4.1. (Noether normalization theorem.) *Let I be a primary ideal of $\mathbb{C}[z_1, \ldots, z_n]$. There exist a nonnegative integer d and a (linear) change of coordinates*

$$\varphi : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[x_1, \ldots, x_{n-d}, t_1, \ldots, t_d]$$

such that:

(a) $\varphi(I) \cap \mathbb{C}[t_1, \ldots, t_d] = (0)$.

(b) $\mathbb{C}[z_1, \ldots, z_n]/I$ *is a finitely generated* $\mathbb{C}[t_1, \ldots, t_d]$*-module.*

(c) *For each $i = 1, \ldots, n-d$, $\varphi(I)$ contains a polynomial of the form*

$$\begin{aligned} Q_i(t_1, \ldots, t_d, x_i) = x_i^{e_i} + p_1(t_1, \ldots, t_d) x_i^{e_i - 1} + \cdots \\ + p_{e_i}(t_1, \ldots, t_d), \end{aligned}$$

where e_i is the degree of the polynomial Q_i.

The ideal $\varphi(I)$ is said to be in normal position *with respect to the variables $x_1, \ldots, x_{n-d}$.*

Remark 4.2. The proof of the normalization theorem can be found, for example, in [Björk 79], in the case of prime ideals. However, as shown in [Greuel and Pfister 02], the result holds for the general case with the exception of condition (a), which requires I to be primary. If the ideal I is prime, the polynomials Q_i in condition (c) can be chosen

to be irreducible. The proof of the theorem provides an algorithm to achieve the normal position. Basically, at each step one constructs the polynomial Q_i, performing a *generic* coordinate change such that Q_i has a monic leading term of the form $x_i^{e_i}$, and then one eliminates the variable x_i. A procedure to compute the Noether normalization of an ideal has also been studied in [Logar 88], and it is available in Singular through the library `algebra.lib` (see [Greuel et al. 01] and its manual). We coded a version of the algorithm for CoCoA as well.

Theorem 4.1 basically states that it is possible to find a new system of coordinates for which the x variables act as "variables" and the t variables act as "coordinates," and for which the integer d appearing in Theorem 4.1 is nothing but the dimension of the ideal I. Hence, if we make the variables t invertible, i.e., if we extend the ideal to the ring $\mathbb{C}(t)[x]$, where $\mathbb{C}(t)$ is the ring of quotients of $\mathbb{C}[t]$, we end up with a zero-dimensional ideal. Furthermore, since we are interested only in primary ideals, we may expect that the extension of the ideal to $\mathbb{C}(t)[x]$ is still primary. The following proposition ensures that such facts hold if I is in normal position.

Proposition 4.3. *Let $I = (f_1, \ldots, f_r)$ be a primary ideal of dimension d in the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_{n-d}, t_1, \ldots, t_d]$, in normal position with respect to $x_1, \ldots, x_{n-d}$. Denote by $R_d = \mathbb{C}(t_1, \ldots, t_d)[x_1, \ldots, x_{n-d}]$ the ring of polynomials in the x variables with coefficients in the field of fractions $\mathbb{C}(t_1, \ldots, t_d) = \mathrm{Frac}(\mathbb{C}[t_1, \ldots, t_d])$. We have the following facts:*

1. *the inclusion map $\varphi_{|_I} : I \hookrightarrow IR_d$ is injective and $IR_d \cap R = I$,*
2. *the extended ideal IR_d is primary,*
3. *the extended ideal IR_d is zero-dimensional.*

Proof: The fact that the inclusion is injective is trivial. To prove (1), let us consider a polynomial f in $R \cap IR_d$. As an element of IR_d it can be written in the form

$$f = \sum_{i=1}^{r} \frac{a_i(x,t)}{b_i(t)} f_i(x,t),$$

where $x = (x_1, \ldots, x_{n-d})$, $t = (t_1, \ldots, t_d)$, and a_i and b_i are just polynomials in the set of variables indicated in parentheses. Let $b(t) = \prod_{i=1}^{r} b_i(t)$ and consider the product bf. Both b and f are polynomials in R, and their product is an R-linear combination of the generators of I, so $bf \in I$. Since I is primary, it follows that either $b^m \in I$ for some positive integer m or $f \in I$. The first possibility is in contradiction to condition (a) of the Noether normalization, whence $f \in I$. This proves that $IR_d \cap R \subseteq I$.

The opposite inclusion is trivial, so we conclude that $IR_d \cap R = I$. The same type of argument can be used to prove that IR_d is primary: consider two fractions

$$f(x,t) = \frac{a(x,t)}{b(t)}, \quad g(x,t) = \frac{c(x,t)}{e(t)}$$

such that $fg \in IR_d$. Then $(bf)\cdot(eg)$ is a polynomial in I, and since I is primary we have either $bf \in I$ or $e^m g^m \in I$ for some positive integer m. In the first case, using again that I is primary and using condition (a) of Theorem 4.1, we get that f is in I. In the second case we have that g^m is in I. Therefore either $f \in IR_d$ or $g^m \in IR_d$. Finally, statement (3) follows from the theory of the dimension of an ideal, since $(\bar{t_1}, \ldots, \bar{t_d})$ is a maximal regular sequence in R/I that reduces to just constants when the ideal is extended to R_d. □

Before we move on and present an equivalent version of Algorithm 3.8 for positive-dimensional ideals, there is still one more step. Previously, when treating the zero-dimensional case, we chose to start with a Gröbner basis for the ideal I, computed with respect to any term ordering. This is no longer possible if we want to extend the procedure to the positive-dimensional case. In fact, after we perform the normalization, the variables t play the role of "constants" once we extend I to $R_d = \mathbb{C}(t)[x]$. The following example illustrates a problem that may occur if we do not choose the term ordering on R carefully.

Example 4.4. Consider the ideal $I = (x^2 - t, xt - 1)$ in $\mathbb{C}[x,t]$. A `DegLex` Gröbner basis for I (with $x > t$) is given by $\mathcal{G} = \{\boldsymbol{x^2} - t, \boldsymbol{xt} - 1, \boldsymbol{t^2} - x\}$, where the leading terms are highlighted in bold. When we look at such polynomials in R_d, however, we see that the leading terms change. In fact, the last polynomial would be better written as $-\boldsymbol{x} + t^2$. Note that in this case the extended ideal IR_d happens to be the whole ring R_d, since the polynomial $t^3 - 1$ belongs to IR_d, and such a polynomial is a constant in $\mathbb{C}(t)[x]$. It is a necessary and sufficient condition for an ideal to be the whole ring that any Gröbner basis with respect to any ordering contain a constant polynomial, but if we look at $\mathcal{G}$, we see that there is no such constant, meaning a polynomial only in the variable t. Therefore we conclude that the set $\mathcal{G}$ does

not form a Gröbner basis for IR_d with respect to the ordering `DegLex` restricted to the terms in x. If we choose instead the term ordering `Lex`, a Gröbner basis for I is given by $\mathcal{G} = \{-\boldsymbol{x}+t^2, \boldsymbol{t^3}-1\}$, and in this case it contains a polynomial in t, making $\mathcal{G}$ a Gröbner basis for IR_d as well.

As the example shows, we really want the variables x to be the main variables with respect to which the Gröbner basis needs to be computed. This can be achieved using `Lex`, or any other elimination ordering with respect to the variables x. Lemma 2.3 then ensures that after extending the ideal to R_d, Gröbner bases are preserved. We now have all the ingredients to generalize Algorithm 3.8 to the case of an ideal of dimension greater than zero. As in Section 3, we will suppose that a primary decomposition of the ideal has already been calculated.

Algorithm 4.5. (Noetherian operators for positive-dimensional ideals.) Let d be a positive integer, $x = (x_1, \ldots, x_{n-d})$ and $t = (t_1, \ldots, t_d)$ be variables and let $\sigma = \sigma_x \cdot \sigma_t$ be a product ordering. Let I be a primary ideal in $R = \mathbb{C}[x, t]$. Suppose that I is in normal position with respect to x. Moreover, let IR_d be the extended ideal in $R_d = \mathbb{C}(t)[x]$ and suppose that the characteristic variety of IR_d in $\mathbb{C}(t)^d$ is the origin. The following procedure computes the Noetherian operators associated with I:

Input: $\mathcal{G} = \{g_1, \ldots, g_r\}$ a σ-Gröbner basis for I.

Output: A set of Noetherian operators for I.

1. Compute the multiplicity of the ideal, $\mu(I)$.
2. Write the Taylor expansion at the origin of a polynomial $h \in \mathbb{C}[x]$.
3. Up to the degree $\mu - 1$ with variable coefficients c_α:
$$\hat{h} := T_{\mu-1}h(x_1, \ldots, x_{n-d}) = \sum_{\alpha \in \mathbb{N}^{n-d}}^{|\alpha|<\mu} c_\alpha x_1^{\alpha_1} \cdots x_{n-d}^{\alpha_{n-d}}. \tag{4–1}$$
4. Let $x^{a_i}t^{b_i}$ be the leading term of g_i and define $t^\gamma := t^{b_1} \cdots t^{b_r}$.
5. **Repeat**
   (a) Multiply $\hat{h}$ by t^γ and compute its normal form with respect to $\mathcal{G}$.
   (b) Rename that as $\hat{h}$:
$$\hat{h} := \mathrm{NF}_\sigma(t^\gamma \hat{h}) = \sum_\beta d_\beta(t) x_1^{\beta_1} \cdots x_{n-d}^{\beta_{n-d}}. \tag{4–2}$$
   Until the number of nonzero d_β is exactly μ.
6. For each β such that $d_\beta \neq 0$, find polynomials $a_{\beta\alpha}(t)$ such that $d_\beta(t) = \sum_\alpha a_{\beta\alpha}(t)c_\alpha$ and return the operator
$$\begin{aligned} L_\beta &= \sum_\alpha a_{\beta\alpha}(t) \frac{1}{\alpha_1! \cdots \alpha_{n-d}!} \partial x_1^{\alpha_1} \cdots \partial x_{n-d}^{\alpha_{n-d}} \\ &= \sum_\alpha a_{\beta\alpha}(t) D(\alpha_1, \ldots, \alpha_{n-d}, 0, \ldots, 0). \end{aligned}$$

Proof: Let h be a polynomial of R. We want to characterize the membership of h in I. Since we are assuming that I is in normal position, by condition (1) of Proposition 4.3 this is equivalent to the membership of h in IR_d. Since the latter is a zero-dimensional ideal of multiplicity μ, it follows that $h \in IR_d$ if and only if the Taylor polynomial of degree $\mu - 1$ of h, with coefficients in $\mathbb{C}(t)$, reduces to zero when it is rewritten using a Gröbner basis for IR_d. This follows from the the same proof as that for Algorithm 3.8. By Lemma 2.3, a σ_x-Gröbner basis for IR_d is given by the same elements of the Gröbner basis of I. This means that computing a normal form in I and computing one in IR_d are equivalent.

However, when writing the Taylor expansion (4–1), we need to consider that the coefficients c_α also depend on t. In order to be able to perform a one-step reduction, we need each term in (4–1) to be multiplied by at least t^γ. This does not affect the membership of $T_{\mu-1}h$ as a polynomial in R_d, since it is just a multiplication by a constant. Also, in considering the expression (4–1) in $\mathbb{C}[x, t]$, the effect of the multiplication does not change the annihilation of $\mathrm{NF}_\sigma(T_{\mu-1}h)$, since obviously

$$\mathrm{NF}_\sigma(T_{\mu-1}h) = 0 \Leftrightarrow \mathrm{NF}_\sigma(t^\gamma T_{\mu-1}h) = 0.$$

The one-step reduction is then iterated enough times in (4–2) until we reach a sufficiently small number of nonzero terms (namely μ). By what we have proved so far, it is then clear that at the end of the process the polynomial $\hat{h}$ is exactly the normal form of $T_{\mu-1}h$ as a polynomial in R_d, and hence the annihilation of its coefficients is equivalent to the condition $h \in IR_d$. □

Remark 4.6. The main difference with respect to the algorithm for zero-dimensional ideals is that in this case,

we do not know whether after just one step of reduction we have achieved the normal form of the polynomial $h(x,t)$, since the multiplication by t^γ might not be enough to ensure that h has been rewritten to a sum that runs over just the Macaulay basis terms for IR_d. Multiplying $T_{\mu-1}h$ once by t^γ is definitely enough for a one-step reduction of each term of the Taylor expansion. That is, each term is being rewritten using at most one of the elements of the Gröbner basis. However, further reductions might occur if we multiply again by t^γ. Also, note that such an iteration has to terminate because σ_x is a well-ordering.

Remark 4.7. The reduction step (4–2) for ideals with few generators is not very difficult, but performing it multiple times could slow down the procedure by a significant amount. We believe that it is possible to find an exponent γ_1 large enough so that we need to multiply by t^{γ_1} just once, allowing the reduction to bring $\hat{h}$ all the way down to its final expression. For example, choosing $\gamma_1 = \mu \cdot \gamma$ seems to work fine at least in the cases we tested, without the need of further iteration.

When Algorithm 4.5 is applied to an ideal I in normal position, some redundant factors in t could appear as an effect of the iterative multiplication by t^γ at each step. Since such factors are constants in R_d, they are actually not needed to characterize the membership of a polynomial in R_d. It is then possible to eliminate these factors from the final expression of the Noetherian operators. The next example will clarify what we mean.

Example 4.8. Consider the system of partial differential equations in three variables given by

$$f_{xx} = 0, \quad f_{yy} = 0, \quad f_y = f_{xt}.$$

Its solutions are differentiable functions of the form $f(x,y,t) = A(t) + B(t)x + B'(t)y$, where A and B are arbitrary functions of t. We want to derive this last statement using the fundamental principle. The primary ideal associated with the system is $I = (x^2, y^2, -xt + y)$ in $\mathbb{C}[x,y,t]$ (see [Palamodov 70]). If we consider the `Lex` ordering whereby $x > y > t$, a Gröbner basis for I is given by $(x^2, xy, y^2, -xt + y)$. Let us compute the associated Noetherian operators using Algorithm 4.5.

It is immediate to check that I is in normal position with respect to x and y and that after inverting t, the variety associated with $I\mathbb{C}(t)[x,y]$ is the origin in $\mathbb{C}(t)^2$. The multiplicity of I can be computed with CoCoA, and it is $\mu = 2$. So we just need to write a linear polynomial h with variable coefficients and multiply it by t, which is the only term in t appearing in the leading terms of the Gröbner basis:

$$T_1\hat{h} = t \cdot T_1 h = tc_{00} + tc_{10}x + tc_{01}y.$$

The only rewrite rule that we need to use to reduce h is hence $xt \mapsto y$, which leads to the final expression for the normal form

$$\mathrm{NF}_\sigma(\hat{h}) = [tc_{00}] + [c_{10} + tc_{01}]y.$$

Since the terms in x and y of the last expression are exactly $\mu = 2$, we do not need to proceed further, and then we conclude that the Noetherian operators are $\{t, \partial x + t\partial y\}$. Since the first is a multiple of t, we can divide it by t and get the final set $\{1, \partial x + t\partial y\}$.

Now we can write the integral formula for the general solution of the system, using ζ, η, τ as dual variables:

$$\begin{aligned} f(x,y,t) &= \int_{\zeta=\eta=0} e^{i(x\zeta+y\eta+t\tau)} d\mu_1(\zeta,\eta,\tau) \\ &\quad + \int_{\zeta=\eta=0} (\partial\zeta + \tau\partial\eta) e^{i(x\zeta+y\eta+t\tau)} d\mu_2(\zeta,\eta,\tau) \\ &= \int_{\mathbb{R}} e^{it\tau} d\mu_1(\tau) + \int_{\mathbb{R}} i(x+y\tau) e^{it\tau} d\mu_2(\tau) \\ &= \int_{\mathbb{R}} e^{it\tau} d\mu_1(\tau) + x\int_{\mathbb{R}} ie^{it\tau} d\mu_2(\tau) \\ &\quad + y\int_{\mathbb{R}} i\tau e^{it\tau} d\mu_2(\tau). \end{aligned}$$

The last expression gives exactly the general solution as anticipated above. One just has to consider arbitrary Radon measures $d\mu_1(\tau) = \hat{A}(\tau)d\tau$ and $d\mu_2(\tau) = \hat{B}(\tau)d\tau$, where $\hat{A}$ and $\hat{B}$ are the Fourier transforms of the two arbitrary functions A and B.

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