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ON THE DERIVED CATEGORY AND K-FUNCTOR OF COHERENT SHEAVES ON INTERSECTIONS OF QUADRICS

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ABSTRACT. A graded Clifford algebra connected with the complete intersection of several quadrics is considered. In terms of modules over this algebra, a description is given of the derived category of coherent sheaves and the Quillen K-functor of the intersection of quadrics, which generalizes the results of I. N. Bernshteĭn, I. M. Gel′fand, S. I. Gel′fand, and R. G. Swan. Here, ramified two-sheeted coverings of the parameter space arise in a natural way, the consideration of which is traditional for intersections of two or three quadrics.

Bibliography: 12 titles.

In this paper we consider only complete intersections of quadrics. Let E be an N-dimensional vector space over a field k, $\operatorname{char} k \neq 2$, and let $X \subset P(E)$ be the complete intersection of m quadrics defined over k, which are given by the inner products $\langle\ ,\ \rangle_i$ on E, $i = 1, \dots, m$. We consider the graded Clifford algebra $A = \bigoplus A_j$ generated by $\xi \in E$, $\deg \xi = 1$, and $h_1, \dots, h_m$, $\deg h_i = 2$, with the relations

$$\xi\eta + \eta\xi = 2\sum_{i=1}^{m}\langle \xi, \eta\rangle_i h_i, \qquad \xi h_j = h_j \xi$$

for all $\xi, \eta \in E$ and $j = 1, \dots, m$. Also let $B \subset \bigoplus_{i\geq 0} H^0(X, \mathcal{O}(i))$ be the projective coordinate algebra of X. The aim of this paper is to describe some algebro-geometric invariants of X in terms of modules over the algebra A. This description is based on a "duality" between the algebras B and A, analogous to the duality between the symmetric and exterior algebras, whose value for the classification of vector bundles on projective spaces has been demonstrated previously [1], [2]. This duality appears, for example, in the algebra isomorphisms $A \simeq \operatorname{Ext}^{\cdot}_B(k, k)$ and $B \simeq \operatorname{Ext}^{\cdot}_A(k, k)$, and in the availability of an analogue of the Koszul complex which is found by the Tate resolution [11] of the field k. Our approach, using tensoring by the Koszul complex, is based on [2].

In §1 we give a description of the derived category of coherent sheaves on X in terms of A-modules. In §2 this connection is established on the level of algebraic K-theory, using Waldhausen's definition [5]. We note that the K-theory for smooth relative quadrics was computed by Swan [10]; §3 connects the preceding considerations with the classical approach in the theory of intersections of quadrics [4]. This

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approach, going back to Weil, associates to X the following "dual object": the projective space $\mathscr{L} \simeq P^{m-1}$, parametrizing the quadrics passing through X, with the filtration by the dimension of the kernel $\mathscr{L} \supset \mathscr{L}_1 \supset \cdots$ and two-sheeted coverings $\tilde{V}_i \to \mathscr{L}_i - \mathscr{L}_{i+1}$ for $i \equiv N \pmod 2$, which arise from the presence of the two systems of straight-line generators on an even-dimensional quadric. The coverings $\tilde{V}_i$ are the spectra of sheaves of the centers of sheaves of algebras on $\mathscr{L}_i - \mathscr{L}_{i+1}$ naturally associated with A. The connection of (complexes of) sheaves on X and on $\tilde{V}_i$ that arises as a result is analogous to the "Fourier transform" in the derived category of coherent sheaves on abelian varieties that was introduced by Mukai [7].

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§1. Description of the derived category

1.1. Notation and definitions. X is not assumed to be smooth or reduced, and is considered as a scheme. $\delta = \sum \xi_i \otimes x_i \in A \otimes B$, where $\{\xi_i\}$ and $\{x_i\}$ are dual bases in $E = A_1$ and $E^* = B_1$; $\delta^2 = 0$.

If a is an element of a ring R, then we denote by l_a and r_a the operators $x \mapsto ax$ and $x \mapsto xa$, respectively.

$\mathfrak{G} = \{\underset{1}{E}; \underset{2}{k^m}\}$ is a graded Lie algebra, the only nontrivial component of whose supercommutator is $E \otimes E \to k^m$, given by the set of $\langle\ ,\ \rangle_i$. A is identified with $U(\mathfrak{G})$.

The complex $\cdots \to A_2^* \otimes B \to A_1^* \otimes B \to B$ with differential $\sum l^*_{\xi_i} \otimes l_{x_i}$ has the structure of a differential graded algebra because $A = U(\mathfrak{G})$ is a (graded cocommutative) Hopf algebra, and $\xi_i \in A$ are the primitive elements. This is the Tate resolution [11] of the field k: the only nonzero cohomology space is the cokernel of the last differential, isomorphic to k.

If $M = \bigoplus M_i$ is a graded left A-module, then the sequence of sheaves on X

$$\cdots \to M_i \otimes \mathscr{O}_X(i) \to M_{i+1} \otimes \mathscr{O}_X(i+1) \to \cdots$$

is a complex, which we denote by $\mathscr{O}(M)$.

If $\mathscr{E}$ is an abelian category, then $C^b(\mathscr{E})$, $C^+(\mathscr{E})$, and $C^-(\mathscr{E})$ are the corresponding categories of complexes; $C^{b+}(\mathscr{E})$ and $C^{b-}(\mathscr{E})$ are the categories of complexes over $\mathscr{E}$ bounded from the left or the right and having only a finite number of nonzero cohomology spaces; and $D^b(\mathscr{E})$, $D^\pm(\mathscr{E})$, and $D^{b\pm}(\mathscr{E})$ are the corresponding derived categories [12]. The embeddings $D^b(\mathscr{E}) \subset D^{b\pm}(\mathscr{E})$ are equivalences of categories.

$S = S^\cdot(E^*)$ and $\Lambda = \Lambda^\cdot(E)$ are the symmetric and exterior algebras.

$\operatorname{Sh} X$ is the category of coherent sheaves on X; $\mathscr{M}(A)$ is the category of finitely generated graded left A-modules; analogously for B, Λ, and S. $\mathscr{M}^*(B)$ is the category of graded modules over B, dual over k to the modules from $\mathscr{M}(B)$ (under dualization the grading is inverted). Analogously for S and A.

$\mathscr{F} \subset \mathscr{M}^*(B)$ is the category of finite-dimensional modules over k. By a theorem of Serre, $\mathscr{M}^*(B)/\mathscr{F} \sim \operatorname{Sh}(X)^{\mathrm{op}}$ (dual category).

$\overline{\mathscr{F}} \subset D^b(\mathscr{M}^*(B))$ is the full subcategory consisting of the objects isomorphic to finite complexes of finite-dimensional modules (or, what is the same, of complexes all of whose cohomology is finite-dimensional).

$\mathscr{M}F(A) \subset \mathscr{M}(A)$ is the full subcategory consisting of modules that are free over $k[h_1, \ldots, h_m] \subset A$; $I \subset D^b(\mathscr{M}(A))$ is the full subcategory consisting of the objects that are isomorphic to finite complexes of free A-modules.

The aim of this section is to prove the following assertion.

1.2. THEOREM. *I is a thick (in the sense of [6]) subcategory of $D^b(\mathcal{M}(A))$, and the corresponding factor-category $D^b(\mathcal{M}(A))/I$ is equivalent to $D^b(\operatorname{Sh} X)^{\mathrm{op}}$ as a triangulated category.*

(If X is a smooth variety, then $D^b(\operatorname{Sh} X) \sim D^b(\operatorname{Sh} X)^{\mathrm{op}}$, and we obtain a description of $D^b(\operatorname{Sh} X)$.)

For the proof we define covariant functors

$$F\colon C^b(\mathcal{M}^*(B)) \to C^-(\mathcal{M}(A)), \qquad G\colon C^b(\mathcal{M}(A)) \to C^+(\mathcal{M}^*(B)),$$

by setting, for $V \in \mathcal{M}^*(B)$,

$$F(V) = \{\cdots \to V_i \otimes_i A[-i] \to V_{i+1} \otimes_{i+1} A[-i-1] \to \cdots\}$$

(the subscripts indicate the ordinal number of the terms in the complex), $d = \sum l_{x_j} \otimes l_{\xi_j}$ and, for $W \in \mathcal{M}(A)$,

$$G(W) = \{\cdots \to W_i \otimes_i B^*[-i] \to W_{i+1} \otimes_{i+1} B^*[-i-1] \to \cdots\},$$

$d = \sum l_{\xi_j} \otimes l^*_{x_j}$, and extending these functors to complexes in the natural way.

Also let F_S and G_S be the analogous functors

$$C^b(\mathcal{M}^*(S)) \to C^-(\mathcal{M}(\Lambda)), \qquad C^b(\mathcal{M}(\Lambda)) \to C^b(\mathcal{M}^*(S)),$$

considered in [2] (more precisely, F_S and G_S are different from the functors of [2] by dualization). Let $i\colon S \to B$ and $j\colon A \to \Lambda$ be the natural epimorphisms. If V is a module over B, then i^*V denotes the same module considered as an S-mdoule; analogously for j^*.

1.3. PROPOSITION. *For any $V^\cdot \in C^b(\mathcal{M}^*(B))$ and $W^\cdot \in C^b(\mathcal{M}(A))$ there is a canonical isomorphism*

$$\operatorname{Hom}_{C^-(\mathcal{M}(A))}(F(V^\cdot), W^\cdot) \simeq \operatorname{Hom}_{C^+(\mathcal{M}^*(B))}(V^\cdot, G(W^\cdot)).$$

The proof is the same as the proof of the analogous assertion in [2].

1.4. PROPOSITION. *For any $V^\cdot \in C^b(\mathcal{M}^*(B))$ the complex $F_S(i^*V^\cdot)$ is naturally isomorphic in $C^-(\mathcal{M}(\Lambda))$ to the complex $F(V^\cdot) \otimes_A \Lambda$.*

Each A-module has a natural filtration given by powers of the ideal $(h_1, \ldots, h_m) \subset A$. Its quotients are Λ-modules. Therefore $G(W)$ for $W \in \mathcal{M}(A)$ also has a filtration by subcomplexes $G(W) \supset G(W)^{(1)} \supset G(W)^{(2)} \supset \cdots$. We consider the morphism Φ of complexes of S-modules which is the composition

$$i^*G(W) \to i^*(G(W)/G(W)^{(1)}) \to G_S(W \otimes_A \Lambda),$$

where the left arrow is the quotient morphism and the right arrow is induced by the embedding $B^* \subset S^*$ dual to $i\colon S \to B$.

1.5. PROPOSITION. *Φ extends to a natural transformation of functors $i^*G \to G_S(? \otimes_A \Lambda)$ acting from $C^b(\mathcal{M}(A))$ to $C^+(\mathcal{M}^*(S))$. Under this transformation Φ gives a quasi-isomorphism for objects from $C^b(\mathcal{M}F(A))$.*

PROOF. The first assertion is obvious. It suffices to prove the second for a single module $W \in \mathcal{M}F(A)$. We construct resolutions of the complexes $i^*G(W)$ and $G_S(W \otimes_A \Lambda)$ that consist of injective (dual to free) S-modules. Each term $W_i \otimes B^*$ of the complex $G(W)$ has a right Koszul resolution (since X is a complete intersection)

$$W_i \otimes S^* \to W_i \otimes S^*[\varepsilon_l] \to W_i \otimes S^*[\varepsilon_p \wedge \varepsilon_l] \to \cdots$$

with differential $d_0 = \sum_1^m 1 \otimes q_l \varepsilon_l$, where the q_l are the quadratic equations of X corresponding to $\langle\ ,\ \rangle_l$. The morphisms $d_{1,i}\colon W_i \otimes B^* \to W_{i+1} \otimes B^*$ extend to morphisms of their Koszul resolutions by the same formula: $d_1 = \sum_1^N \xi_i \otimes x_i$. However d_1^2 is now equal to $\sum_1^m h_l \otimes q_l$ and is not zero. Therefore we need to introduce another differential $d_2 = \sum_1^m h_l \otimes \partial/\partial\varepsilon_l$ and set the differential of the total complex equal to $d_0 + d_1 + d_2$ (with the required choice of signs in places):

$$\begin{tikzcd}
W_0 \otimes S^* \arrow[r, "d_0"] \arrow[d, "d_1"'] & W_0 \otimes S^*[\varepsilon_l] \arrow[r, "d_0"] \arrow[d, "d_1"] \arrow[ddl, "d_2"] & W_0 \otimes S^*[\varepsilon_p \wedge \varepsilon_l] \arrow[r] \arrow[d, "d_1"] \arrow[ddl, "d_2"] & \cdots \\
W_1 \otimes S^* \arrow[r] \arrow[d, "d_1"'] & W_1 \otimes S^*[\varepsilon_l] \arrow[r] \arrow[d] & W_1 \otimes S^*[\varepsilon_p \wedge \varepsilon_l] \arrow[r] \arrow[d, "d_1"] & \cdots \\
W_2 \otimes S^* \arrow[r] \arrow[d] & W_2 \otimes S^*[\varepsilon_l] \arrow[r] \arrow[d] & W_2 \otimes S^*[\varepsilon_p \wedge \varepsilon_l] \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

We denote this complex by $R^\cdot$.

Further, each term $(W/(h))_i \otimes S^*$ of the complex $G_S(W \otimes_A \Lambda)$ has a left Koszul resolution

$$\cdots \to W_{i-4} \otimes S^*[e_p \wedge e_l] \to W_{i-2} \otimes S^*[e_l] \to W_l \otimes S^*$$

with differential $d_0' = \sum_1^m h_l \otimes \partial/\partial e_l$. These resolutions are mapped into each other by means of $d_1' = \sum_1^N \xi_i \otimes x_i$ and again we need to introduce $d_2' = \sum_1^m 1 \otimes q_l e_l$:

$$\begin{tikzcd}
 & & W_0 \otimes S^* \arrow[d, "d_1'"] \arrow[ddl, "d_2'"'] \\
 & & W_1 \otimes S^* \arrow[d] \arrow[ddl] \\
 & W_0 \otimes S^*[e_l] \arrow[r, "d_0'"] \arrow[d] \arrow[ddl, "d_2'"'] & W_2 \otimes S^* \arrow[d] \arrow[ddl] \\
 & W_1 \otimes S^*[e_l] \arrow[r] \arrow[d] & W_3 \otimes S^* \arrow[d] \\
W_0 \times S^*[e_p \wedge e_l] \arrow[r, "d_0'"] \arrow[d] & W_2 \otimes S^*[e_l] \arrow[r, "d_0'"] \arrow[d] & W_4 \otimes S^* \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

We denote this complex by $R'^\cdot$, and we note that there is an isomorphism of complexes $R^\cdot \to R'^\cdot$, taking $d_0 \to d_2'$, $d_1 \to d_1'$, and $d_2 \to d_0'$, the horizontal arrows in

$R^{\cdot}$, to "lines with slope (2, 1)" in $R'^{\cdot}$, etc. This isomorphism covers our morphism Φ: $i^*G(W) \to G_S(W \otimes_A \Lambda)$, so that Φ is a quasi-isomorphism.

COROLLARY. *The functors F and G take values in $C^{b-}(\mathcal{M}(A))$ and $C^{b+}(\mathcal{M}^*(B))$ respectively.*

PROOF. For G the assertion follows from Proposition 1.5 and Hilbert's syzygy theorem. In order to prove it for F we consider for $V \in \mathcal{M}^*(B)$ the filtration $F(V) \supset F(V)^{(1)} \supset \cdots$ by subcomplexes induced by the filtration $A \supset (h_1, \dots, h_m)A \supset (h_1, \dots, h_m)^2 A \supset \cdots$ of the algebra A. The quotients $F(V)^{(l)}/F(V)^{(l+1)}$ are isomorphic to $S^l(k^m) \otimes F_S(i^*V)$ and thus (according to [2]) can have nonzero cohomology only for all l in a finite interval, and the interval is the same for all l. Consequently, $F(V)$ also can have nonzero cohomology only in this interval. □

It is clear that F and G take quasi-isomorphisms into quasi-isomorphisms. Therefore they give exact functors

$$F_D \colon D^b(\mathcal{M}^*(B)) \to D^{b-}(\mathcal{M}(A)), \qquad G_D \colon D^b(\mathcal{M}(A)) \to D^{b+}(\mathcal{M}^*(B)).$$

If $C^{\cdot}$ is a complex, then let $\tau_{\leq n}C^{\cdot} = \{\cdots \to C^{n-2} \to C^{n-1} \to \operatorname{Ker} d \to 0 \to \cdots\}$ and $\tau_{\geq n}C^{\cdot} = \{\cdots \to 0 \to \operatorname{Im} d \to C^{n+1} \to C^{n+2} \to \cdots\}$ be its truncations, included in the exact sequence $0 \to \tau_{\leq n}C^{\cdot} \to C^{\cdot} \to \tau_{\geq n}C^{\cdot} \to 0$. If $W \in C^b(\mathcal{M}(A))$, then the morphism $\tau_{\geq n}G(W) \to G(W)$ corresponds, by Proposition 1.3, to a morphism $\alpha_{W,n}\colon F(\tau_{\leq n}G(W)) \to W$. If $V \in C^b(\mathcal{M}^*(B))$, then the morphism $F(V) \to \tau_{\geq n}F(V)$ corresponds to $\beta_{V,n}\colon V \to G(\tau_{\geq n}F(V))$.

1.6. PROPOSITION. a) *If $V \in C^b(\mathcal{M}^*(B))$, then $\beta_{V,n}$ is a quasi-isomorphism for sufficiently small n.*

b) *If $W \in C^b(\mathcal{M}(A))$, then $\alpha_{W,n}$ is a quasi-isomorphism for sufficiently large n.*

PROOF. We may assume that V and W are ordinary modules. In case a), for sufficiently small n the analogous mapping

$$i^*V \to G_S(\tau_{\geq n}F_S(V)) = G_S(\tau_{\geq n}(F(V) \otimes_A \Lambda))$$

is a quasi-isomorphism, according to [2]. Since $F(V)$ has finite cohomology and Λ has finite Tor-dimension over A, then for sufficiently small n the natural morphism

$$(\tau_{\geq n}F(V)) \otimes_A \Lambda \to \tau_{\geq n}(F(V) \otimes_A \Lambda)$$

is a quasi-isomorphism. Therefore we have a commutative diagram

$$\begin{array}{ccc}
i^*V & \longrightarrow & G_S(\tau_{\geq n}F_S(V)) \\
 & & \| \\
\beta_{V,n}\big\downarrow & & G_S(\tau_{\geq n}(F(V) \otimes_A \Lambda)) \\
 & & \big\uparrow \\
i^*G(\tau_{\geq n}F(V)) & \longrightarrow & G_S((\tau_{\geq n}F(V)) \otimes_A \Lambda)
\end{array}$$

in which all the arrows except for $\beta_{V,n}$ are quasi-isomorphisms.

In case b) we may assume that W is an A-module that is free over $k[h_1, \dots, h_m]$. Since $G_S(W \otimes_A \Lambda)$ is a finite complex of S-modules, the composition

$$i^*\tau_{\leq n}G(W) \to i^*G(W) \xrightarrow{\Phi} G_S(W \otimes_A \Lambda)$$

is a quasi-isomorphism for sufficiently large n. The morphism $\alpha_{W,n} \otimes \Lambda$ can be split into the composition

$$F(\tau_{\leq n}G(W)) \otimes_A \Lambda = F_S(i^*\tau_{\leq n}G(W)) \to F_S(G_S(W \otimes_A \Lambda)) \xrightarrow{\sim} W \otimes_A \Lambda$$

and thus is a quasi-isomorphism.

LEMMA. *Let $M^\cdot$ be a complex of free graded $k[h_1, \dots, h_m]$-modules that is bounded on the right, such that for each i the number of free terms with generators in degrees $\leq i$ among the terms of $M^\cdot$ is finite. If $M^\cdot \otimes k$ is exact, then $M^\cdot$ is also exact.*

PROOF. We consider the filtration $\{M_j^\cdot\}$ of the complex $M^\cdot$, where M_j^i is the sum of free terms of M^i with generators in degrees $\leq j$. By hypothesis, the quotients of this filtration are exact. Therefore $M^\cdot$ is also exact. □

It follows from the lemma that the cone of the morphism $\alpha_{W,n}$, since it is exact after tensoring by Λ over A, is exact *ab initio*. This completes the proof of the proposition.

1.7. COROLLARY. *F_D and G_D are equivalences of categories.* □

Let $\overline{\mathscr{F}}_+ \subset D^{b+}(\mathscr{M}^*(B))$ and $I_- \subset D^{b-}(\mathscr{M}(A))$ be the full subcategories consisting of the objects isomorphic to the objects $\overline{\mathscr{F}} \subset D^b(\mathscr{M}^*(B))$ and $I \subset D^b(\mathscr{M}(A))$ respectively.

1.8. PROPOSITION. *The functor $F_D\colon D^b(\mathscr{M}^*(B)) \to D^{b-}(\mathscr{M}(A))$ realizes the equivalence of $\overline{\mathscr{F}}$ and I_-. The functor $G_D\colon D^b(\mathscr{M}(A)) \to D^{b+}(\mathscr{M}^*(B))$ realizes the equivalence of I and $\overline{\mathscr{F}}_+$.*

PROOF. It is clear that $F_D(\overline{\mathscr{F}}) \subset I_-$. Therefore F_D realizes the equivalence of $\overline{\mathscr{F}}$ with some full triangulated subcategory of I_-. However, the image $F_D(\overline{\mathscr{F}})$ contains all the complexes consisting of a single free A-module. Therefore $F_D\colon \overline{\mathscr{F}} \to I_-$ is surjective on the objects and is an equivalence. Further, if W is a one-dimensional free A-module, then $G(W) \sim k[j] \in \operatorname{Ob}\overline{\mathscr{F}}_+$ (Tate resolution). Consequently, $G_D(I) \subset \overline{\mathscr{F}}_+$. The image $G_D(I)$ contains the objects $k[j]$, $j \in \mathbf{Z}$, generating $\overline{\mathscr{F}}_+$. Therefore $G_D\colon I \to \overline{\mathscr{F}}_+$ is an equivalence.

Theorem 1.2 now follows from the fact that $\overline{\mathscr{F}} \subset D^b(\mathscr{M}^*(B))$ is a thick subcategory, the quotient modulo which is equivalent to $D^b(\operatorname{Sh} X)^{\mathrm{op}}$.

1.9. The remainder of this section is devoted to a description of $D^b(\operatorname{Sh} X)$ in terms of A-modules themselves and not complexes of them. We introduce the following notation.

$\varepsilon\colon k \to \Lambda^N E$ is a fixed isomorphism.

If $f\colon L \to L'$ is a morphism of A-modules, then

$$Cf = \operatorname{Coker}\{L \to L' \otimes (L \otimes_k \Lambda'(E)[-N])\},$$

where the second component of the morphism is

$$L \xrightarrow{1\otimes\varepsilon} L \otimes_k \Lambda^N E \subset L \otimes_k \Lambda^1(E)[-N],$$

and tensoring by an A-module over k is produced using the Hopf algebra structure on A.

As the fundamental functor we shall consider $\mathscr{O}(-)$, acting from $\mathscr{M}^*(A)$ to $D^{b-}(\operatorname{Sh} X)$. (If X is a smooth variety, we can consider $\mathscr{M}(A)$ and $D^{b+}(\operatorname{Sh} X)$.) If $L \in \mathscr{M}^*(A)$, then let $C(L \to 0) = \Sigma L$ and $\operatorname{Ker}(L \otimes_k \Lambda^1(E) \to L \otimes \Lambda^0(E) = L) = \Omega L$.

The functor $\mathscr{O}(-)\colon \mathscr{M}^*(A) \to D^{b-}(\operatorname{Sh} X)$ takes these operations into the usual operations of cone and translation (up to quasi-isomorphism). They allow us to define the "convolution" of a finite complex over $\mathscr{M}^*(A)$. If $0 \to M_i \to \cdots \to M_j \to 0$ is such a complex, then we form the complex $0 \to C(\mathscr{M}_i \to \mathscr{M}_{i+1}) \to \mathscr{M}_{i+2} \to \cdots \to M_j \to 0$ and repeat this procedure further. The convolution defined in this way does not depend on where we start counting but on where we finish, i.e., on the addition of a zero to the right. Let $C_{ij}(\mathscr{M}^*(A))$ be the category of complexes over $\mathscr{M}^*(A)$ with terms numbered from i to j. "Convolution" gives a functor $\operatorname{Tot}_{i,j}\colon C_{ij}(\mathscr{M}^*(A)) \to \mathscr{M}^*(A)$. If $i \le 0 \le j$ and $L^\cdot \in C_{ij}(\mathscr{M}^*(A))$ consists of a single module in the zeroth place, then $\operatorname{Tot}_{i,j}(L) = \Sigma^j L$.

1.10. PROPERTIES OF THE FUNCTOR. $\mathscr{O}(-)\colon \mathscr{M}^*(A) \to D^{b-}(\operatorname{Sh} X)$. a) $\mathscr{O}(M) \sim 0$ *if and only if M has a finite injective resolution (the injective objects in $\mathscr{M}^*(A)$ are the modules that are dual to free modules).*

b) *For every $\mathscr{G}^\cdot \in D^b(\operatorname{Sh} X)$ there exists an $M \in \mathscr{M}^*(A)$ such that $\mathscr{O}(M) \sim \mathscr{G}^\cdot$.*

c) *For every $L, M \in \mathscr{M}^*(A)$ and any morphism $\varphi\colon \mathscr{O}(L) \to \mathscr{O}(M)$ in $D^{b-}(\operatorname{Sh} X)$ there are a $j \ge 0$, $L' \in \mathscr{M}^*(A)$, and morphisms $\alpha\colon \Sigma^j L \to L'$ and $\beta\colon \Sigma^j M \to L'$ such that $\mathscr{O}(\beta)$ is a quasi-isomorphism and the diagram*

$$\mathscr{O}(\Sigma^j L) \xrightarrow{\mathscr{O}(\alpha)} \mathscr{O}(L') \xleftarrow{\mathscr{O}(\beta)} \mathscr{O}(\Sigma^j M)$$

represents $\varphi[j]\colon \mathscr{O}(L)[j] \to \mathscr{O}(M)[j]$.

d) *For $L, M \in \mathscr{M}^*(A)$ let $J_{LM} \subset \operatorname{Hom}_{\mathscr{M}^*(A)}(L, M)$ consist of those φ for which there exist complexes $\Phi^\cdot$, $\Psi^\cdot \in C^+(\mathscr{M}^*(A))$ consisting of injective modules, quasi-isomorphisms $M \to \Psi^\cdot$ and $L \to \Phi^\cdot$, and a morphism $\tilde\varphi\colon \Phi^\cdot \to \Psi^\cdot$ covering φ such that all its components $\tilde\varphi^i\colon \Phi^i \to \Psi^i$ except for a finite number are equal to* 0. *Then $\mathscr{O}(\varphi)$ is the zero morphism in $D^{b-}(\operatorname{Sh} X)$ if and only if $\varphi \in J_{LM}$.*

PROOF. a) follows from Proposition 1.8. *We shall prove Property* b) *first for the case when $\mathscr{G}^\cdot$ is a single sheaf $\mathscr{G}$ at the zeroth place.* Let $V = \bigoplus_{i\ge 0} H^0(X, \mathscr{G}(i))$ be the corresponding module over B and let $V^* \in \mathscr{M}^*(B)$ be its dual. There is a quasi-isomorphism $V^* \sim G(W^*)$, where $W^* = \tau_{\ge n} F(V^*)$ is a finite complex over $\mathscr{M}(A)$ and n is sufficiently small. Let $\tilde W = \operatorname{Tot}_{i,j}(W^*) \in \mathscr{M}^*(A)$, where $i, j \in \mathbf{Z}$ are such that $W^* \in C_{ij}(\mathscr{M}^*(A))$. Then $\mathscr{O}(\tilde W) \sim \mathscr{G}[j]$ in $D^{b-}(\operatorname{Sh} X)$ and it is easy to get rid of the shift by using the operation Ω. Now if $\mathscr{G}^\cdot$ is a finite complex of sheaves, then, by choosing the numbers n, i, and j suitably in the preceding argument, we associate to $\mathscr{G}^\cdot$ a complex $\{\cdots \to \tilde W^p \to \tilde W^{p+1} \to \cdots\}$, where $\tilde W^p$ corresponds to $\mathscr{G}^p[j]$. We apply the operation Tot to this complex, and then we make a shift using Ω.

c) Suppose we are given L, $M \in \operatorname{Ob}\mathscr{M}^*(A)$ and $\varphi\colon \mathscr{O}(L) \to \mathscr{O}(M) \in \operatorname{Mor} D^{b-}(\operatorname{Sh} X)$. The morphism φ corresponds to a diagram $G(L^*) \xleftarrow{\gamma} V \xrightarrow{\delta} G(M^*)$ in the category $C^{b+}(\mathscr{M}^*(B))$, in which $\operatorname{Cone}(\delta) \in \overline{\mathscr{F}}$. Applying the functor $F(\tau_{\le n}(-))$ for sufficiently large n to this diagram, we obtain the diagram

$$L^* \leftarrow F(\tau_{\le n} G(L^*)) \leftarrow F(\tau_{\le n} V) \to F(\tau_{\le n} G(M^*)) \to M^*.$$

Making another truncation, we obtain

$$L^* \xleftarrow{\gamma'} \tau_{\ge r} F(\tau_{\le n} V) \xrightarrow{\delta'} M^*,$$

where the cone of δ' has a finite free resolution. Suppose $(\tau_{\ge r} F(\tau_{\le n} V))^*$ lies in some $C_{r,j}(\mathscr{M}^*(A))$, $j \ge 0$. Applying dualization and "convolution", we obtain the desired diagram

$$\Sigma^j L = \mathrm{Tot}_{r,j}(L) \xrightarrow{\gamma''} \mathrm{Tot}_{r,j}((\tau_{\geq r} F(\tau_{\leq n} V))^*) \xleftarrow{\delta''} \mathrm{Tot}_{r,j}(M) = \Sigma^j M.$$

d) It is clear that if $\varphi \in J_{LM}$, then $\mathscr{O}(\varphi)$ is the zero morphism in $D^{b-}(\mathrm{Sh}\,X)$. Conversely, suppose that $\mathscr{O}(\varphi)$ is the zero morphism, i.e., $G(\varphi^*)\colon G(M^*) \to G(L^*)$ is equal to zero in $D^{b+}(\mathscr{M}^*(B))/\overline{\mathscr{F}}$. This means that there is a commutative diagram in $D^{b+}(\mathscr{M}^*(B))$:

$$\begin{tikzcd} & N \arrow[dl, "0"'] \arrow[d, "i"] \\ G(L^*) & G(M^*) \arrow[l, "G(\varphi^*)"] \end{tikzcd}$$

in which $\mathrm{Cone}(i) \in \overline{\mathscr{F}}$. Since the functor $\mathrm{Hom}(-, G(L^*))$ is cohomological, the morphism $G(\varphi^*)$ passes in $D^b(\mathscr{M}^*(B))$ through the morphism $\mathrm{Cone}(i) \to G(L^*)$. Here $\mathrm{Cone}(i)$ is quasi-isomorphic to a finite complex of finite-dimensional modules, say $D^\cdot$. Therefore $G(\varphi^*)$ can be decomposed (in $D^{b+}(\mathscr{M}^*(B))$) into the composition $G(M^*) \to D \to G(L^*)$. Hence, for sufficiently small n we have the commutative diagram

$$\begin{tikzcd} F(\tau_{\geq n} G(M^*) \arrow[r] \arrow[d, "\wr"'] & F(D) \arrow[r] & F(\tau_{\geq n} G(L^*)) \\ M^* \arrow[rr, "\varphi^*"] & & L^* \arrow[u, "\wr"'] \end{tikzcd}$$

where the morphisms in the upper row are a priori morphisms in the derived cateogry. However, the complexes in the upper row consist of free modules, so that the morphisms between them can be represented by genuine morphisms of complexes. Since $F(D)$ is a finite complex, we get what we required. □

1.11. Remark. In general, for given $L, M \in \mathscr{M}^*(A)$ the mapping

$$\mathrm{Hom}_{\mathscr{M}^*(A)}(L, M) \to \mathrm{Hom}_{D^{b-}(\mathrm{Sh}\,X)}(\mathscr{O}(L), \mathscr{O}(M))$$

is not surjective. As an example we consider the case when X is a smooth curve, and, hence,

$$\mathrm{Hom}_{D^b(\mathrm{Sh}\,X)}(\mathscr{O}, \mathscr{O}[1]) = \mathrm{Ext}^1_X(\mathscr{O}, \mathscr{O}) = H^1(X, \mathscr{O}) \neq 0.$$

On the other hand,

$$\mathscr{O}_X \sim \mathscr{O}\{\cdots 0; \underset{0}{k}; 0; \cdots\}, \qquad \mathscr{O}_X[1] \sim \mathscr{O}\{\underset{-N}{\Lambda^0 E}; \cdots; \underset{-1}{\Lambda^{N-1} E}; \underset{0}{0}; \cdots\},$$

and there are no morphisms between the indicated morphisms.

Let $\overline{\mathscr{M}^*}(A) = \mathscr{M}(A)/\{J_{LM}\}$ be the factor-category modulo the ideal defined in 1.10d). We consider the "suspended" category $\mathscr{M}^*\Sigma(A)$, setting $\mathrm{Ob}\,\mathscr{M}^*\Sigma(A) = \mathrm{Ob}\,\mathscr{M}^*(A) \times \mathbf{Z}$ and

$$\mathrm{Hom}_{\mathscr{M}^*\Sigma(A)}((L, i), (M, j)) = \varinjlim_n \mathrm{Hom}_{\mathscr{M}^*(A)}(\Sigma^{n+i} L, \Sigma^{n+j} M),$$

and the factorized suspended category $\overline{\mathscr{M}^*\Sigma}(A)$, in which $\mathrm{Ob}\,\overline{\mathscr{M}^*\Sigma}(A) = \mathrm{Ob}\,\mathscr{M}^*\Sigma(A)$ and

$$\mathrm{Hom}_{\overline{\mathscr{M}^*\Sigma}(A)}((L, i), (M, j)) = \varinjlim_n \mathrm{Hom}_{\overline{\mathscr{M}^*}(A)}(\Sigma^{n+i} L, \Sigma^{n+j} M).$$

There is a natural functor $\rho_0: \overline{\mathscr{M}^*\Sigma}(A) \to D^{b-}(\operatorname{Sh} X)$, taking (L, i) to $\mathscr{O}(L)[i]$, and which is injective on morphisms. Suppose $S \subset \operatorname{Mor}\overline{\mathscr{M}^*\Sigma}(A)$ is generated by those morphisms $\varphi \in \operatorname{Mor}\mathscr{M}^*(A)$ whose cones (in the sense of 1.9) have a finite injective resolution.

1.12. THEOREM. *The family of morphisms S admits a calculus of left fractions, and the category $\overline{\mathscr{M}^*\Sigma}(A)[S^{-1}]$ is equivalent to $D^b(\operatorname{Sh} X)$.*

PROOF. In order to establish the first assertion, we need to verify two properties ([3], Chapter I, §2).

a) Every right fraction $L \xleftarrow{t} L' \xrightarrow{v} M$, $t \in S$, is equal to some left fraction $L \xrightarrow{u} M' \xleftarrow{s} M$, $s \in S$, i.e., $ut = sv$.

b) For every diagram $L' \xrightarrow{t} L \overset{f}{\underset{g}{\rightrightarrows}} M$, where $t \in S$ and $ft = gt$, there exists a morphism $s: M \to M'$, $s \in S$, such that $sf = sg$.

We may assume that the given morphisms lie in $\overline{\mathscr{M}^*}(A)$, and construct the required ones in $\overline{\mathscr{M}^*\Sigma}(A)$. We prove a). A right fraction gives a morphism $\mathscr{O}(L) \to \mathscr{O}(M)$ in $D^{b-}(\operatorname{Sh} X)$, which, by 1.10c), corresponds to the diagram

$$\Sigma^j L \xrightarrow{u} L' \xleftarrow{s} \Sigma^j M, \qquad s \in S,$$

as required. We prove b). If we are given a diagram as indicated, then $\mathscr{O}(f) = \mathscr{O}(g)$ in $D^{b-}(\operatorname{Sh} X)$, i.e., $f = g$ in $\operatorname{Hom}_{\overline{\mathscr{M}^*}(A)}(L, M)$, and we can take $s = \mathrm{id}$.

Further, the functor $\rho_0: \overline{\mathscr{M}^*\Sigma}(A) \to D^{b-}(\operatorname{Sh} X)$ is factorized to a functor

$$\rho: \overline{\mathscr{M}^*\Sigma}(A)[S^{-1}] \to D^{b-}(\operatorname{Sh} X).$$

The functor ρ is surjective on objects and morphisms (this follows from 1.10). It remains to show that it is injective on morphisms. If the left fraction $L \xrightarrow{u} M \xleftarrow{s} M'$, $s \in S$, over $\overline{\mathscr{M}^*\Sigma}(A)$ goes to 0, then, since s goes to an isomorphism, $u \mapsto 0$, so that the fraction is the zero fraction. Hence ρ is an equivalence of categories.

§2. Isomorphism of K-functors

2.1. We shall use the definition of algebraic K-theory given by Waldhausen (see [5] and [8]). It consists in the following. Let $[n]$ be the ordered set $\{0 < 1 < \cdots < n\}$, considered as a category, and $\operatorname{Mor}[n]$ its category of morphisms; $\operatorname{Ob}\operatorname{Mor}[n] = \{(i, j): 0 \leq i \leq j \leq n\}$, and between (i, j) and (k, l) there is a unique morphism if $i \leq k$ and $j \leq l$. If $\mathscr{E}$ is an abelian category, then the category $S_n\mathscr{E}$ by definition has as objects the functors x: $\operatorname{Mor}[n] \to C^b(\mathscr{E})$ such that $x_{ij} = 0$ for all i, and, for each triple $i < j < k$, $0 \to x_{ij} \to x_{ik} \to x_{jk} \to 0$ is an exact sequence of complexes. The morphisms in $S_n\mathscr{E}$ are the natural transformations of functors, giving quasi-isomorphisms on each object of $\operatorname{Mor}[n]$. The categories $S_n\mathscr{E}$ have as union the simplicial category $S.\mathscr{E}$. The bisimplicial set $\operatorname{Nerv} S.\mathscr{E}$ is weakly homotopy equivalent (after realization) to the simplicial set $\operatorname{Nerv} Q\mathscr{E}$, where $Q\mathscr{E}$ is the Quillen category [9]. Sometimes the space of algebraic K-theory $\mathscr{E}$ will be abbreviated $\mathscr{K}(\mathscr{E})$; $\mathscr{K}(\mathscr{E}) \sim \Omega(|\operatorname{Nerv} Q\mathscr{E}|)$, where $|\cdot|$ is the realization. We also introduce the following notation.

$S.^{\pm}(\mathscr{E})$ is the simplicial category analogous to $S.(\mathscr{E})$, but defined using the complexes belonging to $C^{b\pm}(\mathscr{E})$ (with a finite number of nonzero cohomology sets).

$I. \subset S.(\mathscr{M}(A))$ is the simplicial subcategory defined using (finite) complexes of free A-modules; $[Y, Z]$ is the set of homotopy classes of mappings between the spaces Y and Z.

2.2. LEMMA. *For any abelian category $\mathscr{E}$ the embeddings $S.(\mathscr{E}) \to S.^{\pm}(\mathscr{E})$ are weak equivalences.*

The proof consists of applying the truncation operation τ. It preserves exact sequences of complexes if this is done where the complexes are acyclic. □

The functors F and G from §1 preserve exact sequences of complexes and take quasi-isomorphisms to quasi-isomorphisms. Therefore they give morphisms of simplicial categories

$$\overline{F}: S.(\mathscr{M}^*(B)) \to S.^-(\mathscr{M}(A)), \qquad \overline{G}: S.(\mathscr{M}(A)) \to S.^+(M^*(B)).$$

2.3. THEOREM. a) *$\overline{F}$ and $\overline{G}$ are weak equivalences.*

b) *$\overline{F}$ realizes a weak equivalence between $S.(\mathscr{F})$ and I.*

PROOF. a) It suffices to prove that $\overline{F}$ and $\overline{G}$ give weak equivalences on each level n. We shall prove this for F and $h = 1$ (in the other cases the proof is the same). We must prove that for any finite category J the mapping

$$[\operatorname{Nerv} J, \operatorname{Nerv} S_1(\mathscr{M}^*(B))] \xrightarrow[\overline{F}_*]{} [\operatorname{Nerv} J, \operatorname{Nerv} S_1^-(\mathscr{M}(A))]$$

is a bijection. For example, we shall prove that it is surjective. Suppose we are given a morphism of topological spaces ψ: $\operatorname{Nerv} J \to \operatorname{Nerv} S_1^-(\mathscr{M}(A))$. Using the simplicial approximation theorem, we can find a finite category J', a functor $J' \to J$ which is a homotopy equivalence, and a functor $\psi'\colon J' \to S_1(\mathscr{M}(A))$ (not simply into $S_1^-(\mathscr{M}(A))$), representing the same homotopy class as ψ. For any $W \in C^b(\mathscr{M}(A))$ there exists a ν_0 such that for $\nu \geq \nu_0$ we have that $\alpha_{W,\nu}\colon F(\tau_{\leq\nu} G(W)) \to W$ is a quasi-isomorphism. Suppose ν_0 is such that the above fact is true for all W belonging to the image of ψ'. Then $\tau_{\leq \nu_0} \circ G \circ \psi'\colon J' \to S_1(\mathscr{M}^*(B))$ gives the desired lifting of the functor ψ'. Similar arguments also prove the injectivity.

b) Let $\Phi.^{(\pm)} \subset S.^{(\pm)}(\mathscr{M}^*(B))$ be the simplicial subcategory defined using the complexes having all cohomology modules finite-dimensional (over k).

LEMMA. *The embeddings $S.(\mathscr{F}) \subset \Phi. \subset \Phi.^{\pm}$ are weak equivalences.*

PROOF. In order to establish that $S_n(\mathscr{F}) \to \Phi_n$ is a weak equivalence for all n, we need to apply truncation with respect to the grading of the modules included in the complex. The analogous fact for $\Phi_n \subset \Phi_n^{\pm}$ is proved using truncation of complexes. □

The functor G gives a morphism $I. \to \Phi.^+$, and the composition

$$S_n(\mathscr{F}) \xrightarrow{\overline{F}} I_n \xrightarrow{\overline{G}} \Phi_n^+$$

is weakly homotopic to the embedding $S_n(\mathscr{F}) \subset \Phi_n^+$. This gives the injectivity of the mapping of homotopy classes $[\operatorname{Nerv} J, \operatorname{Nerv} S_n\mathscr{F}] \to [\operatorname{Nerv} J, \operatorname{Nerv} I_n]$ for a finite category J. It surjectivity is established using the quasi-isomorphism $F(\tau_{\leq\nu} G(W)) \to W$ for large ν together with the truncation $\tau_{\leq\nu} G(W)$ for $W \in I_n$ with respect to the grading of modules. This proves the theorem.

2.4. COROLLARY. *In the homotopy category there is a fibration*

$$\prod_{i\in\mathbf{Z}} \mathscr{K}(k) \xrightarrow{\sigma} \mathscr{K}(\mathscr{M}(A)) \to \mathscr{K}(\operatorname{Sh} X),$$

where $\mathscr{K}(k) = \mathscr{K}(\mathrm{Vect}(k))$, *the mapping* σ *multiplies the ith copy of* $\mathscr{K}(k)$ *by the graded A-module* $A[i]$.

2.5. LEMMA. *The boundary operators in the localization exact sequence for* $\mathscr{F} \subset \mathscr{M}^*(B) \to \mathrm{Sh}(X)^{\mathrm{op}}$ *are trivial.*

PROOF. We consider the morphism of fibrations

$$\begin{array}{ccccc}
\mathscr{K}(\mathscr{F}) & \longrightarrow & \mathscr{K}(\mathscr{M}^*(B)) & \longrightarrow & \mathscr{K}((\mathrm{Sh}\,X)^{\mathrm{op}}) \\
{\scriptstyle j}\big\downarrow{\scriptstyle S} & & \big\downarrow & & \big\downarrow \\
\mathscr{K}(\mathscr{F}_S) & \longrightarrow & \mathscr{K}(\mathscr{M}^*(S)) & \longrightarrow & \mathscr{K}((\mathrm{Sh}\,P(E))^{\mathrm{op}})
\end{array}$$

where $\mathscr{F}_S$ is the category of finite-dimensional graded $S^{\cdot}(E^*)$-modules induced by the direct image functor. This gives that the composition

$$K_i(\mathrm{Sh}\,X) \to K_i(P(E)) \xrightarrow{\partial_{P(E)}} K_{i-1}(\mathscr{F}_S)$$

coincides with $j_* \circ \partial_X$. And for $\partial_{P(E)}$ triviality it is known. □

Since the exact homotopy sequence of the fibration of Corollary 2.4 is the localization sequence under consideration, we will obtain exact sequences

$$0 \to K_i(k) \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}] \to K_i(\mathscr{M}(A)) \to K_i(\mathrm{Sh}\,X) \to 0.$$

§3. Dévissage of the category of A-modules

3.1. Notations and assumptions. $S^2E^{\cdot} = \Delta_0 \supset \Delta_1 \supset \cdots$ is the space of quadratic forms on E with the filtration by the dimension of the kernel.

$\tilde{\mathscr{L}} \simeq k^m \subset S^2E^{\cdot}$ is the space of quadratic forms vanishing on X. In this section we shall assume that $\tilde{\mathscr{L}}$ intersects all the $\Delta_i - \Delta_{i+1}$ transversally.

$\mathscr{L} = P(\tilde{\mathscr{L}}) \simeq P^{m-1}$; $\mathscr{L}_i$ are the images of $\tilde{\mathscr{L}} \cap \Delta_i - \{0\}$ in $\mathscr{L}$; $V_i = \mathscr{L}_i - \mathscr{L}_{i+1}$.

$\tilde{V}_i = V_i$ if $i \not\equiv N \pmod 2$, and $\tilde{V}_i =$ the canonical two-sheeted unramified "discriminant" covering of V_i, if $i \equiv N \pmod 2$.

$\mathscr{M} = \mathscr{M}(A)$ is the category of finitely generated $\mathbf{Z}$-graded left A-modules.

$A^{(i)} = A \otimes_{k[h_1, \dots, h_m]} k[\Delta_i]$.

The indices $\overline{0}$ and $\overline{1}$ denote a $\mathbf{Z}/2$-grading, including the one induced by the $\mathbf{Z}$-grading. For example, $A_{\overline{0}} = \bigoplus A_{2i}$.

$\mathscr{M}_i \subset \mathscr{M}$ is the Serre subcategory, consisting of the modules which, when considered as $k[h_1, \dots, h_m]$-modules, have support contained in Δ_i.

$\mathscr{M}_\infty \subset \mathscr{M}$ is the category of finite-dimensional modules (which coincides with $\mathscr{M}_i$ for large i).

$\mathscr{M}_i^0$ is the category of $2\mathbf{Z}$-graded $A_{\overline{0}}^{(i)}$-modules.

$\underline{A}^{(i)}$ is the $\mathbf{Z}$-graded sheaf of algebras over $\mathscr{L}_i \subset \mathscr{L} \simeq P^{m-1}$, whose stalk over a (not necessarily k-rational) point $l \in \mathscr{L}_i$ is A/J_lA, where $J_l \subset k[h_1, \dots, h_m]$ is the graded ideal corresponding to l.

$\underline{\mathrm{Rad}}^{(i)} \subset \underline{A}^{(i)}$ is the sheaf of graded ideals whose stalk over a point $l \in \mathscr{L}_i$ is $(\mathrm{Ker}\, q) \cdot (A/J_lA)$, where $q \in S^2(E \otimes k(l))^*$ is the quadratic form corresponding to l.

$\mathrm{Rad}^{(i)} = H^0(\mathscr{L}_i, \underline{\mathrm{Rad}}^{(i)})$ is the corresponding ideal in $A^{(i)}$.

$\mathscr{M}_i'$ is the category of $\mathbf{Z}$-graded $A^{(i)}$-modules killed by the ideal $\mathrm{Rad}^{(i)}$.

$\mathscr{M}_i'^0$ is the category of $2\mathbf{Z}$-graded $A_{\overline{0}}^{(i)}$-modules killed by $\mathrm{Rad}_{\overline{0}}^{(i)}$.

$\mathscr{M}'_{i,i+1}$, $M^0_{i,i+1}$, and $M'^0_{i,i+1}$ are the Serre subcategories in $\mathscr{M}'_1$, $\mathscr{M}^0_i$, and $\mathscr{M}'^0_i$, respectively, consisting of the modules whose supports as $k[h_1, \dots, h_m]$-modules are contained in Δ_{i+1}.

$\underline{R} = \bigoplus_j (\Lambda^{2j}(E) \otimes \mathscr{O}_{\mathscr{L}}(-j))$ is the sheaf of algebras over $\mathscr{L} \simeq P^{m-1}$ corresponding to the graded $k[h_1, \dots, h_m]$-module $A_{\bar{0}}$.

$\underline{R}_i$ is the sheaf of simple algebras on $\tilde{V}_i$ whose descent is the semisimplification of the sheaf of algebras $\underline{R}|_{V_i}$.

$\underline{R}$-mod is the category of coherent sheaves of left $\underline{R}$-modules on $\tilde{V}_i$.

3.2. LEMMA. *The functor $\varphi\colon \mathscr{M}'_i/\mathscr{M}'_{i,i+1} \to \mathscr{M}'^0_i/\mathscr{M}'^0_{i,i+1}$, taking W to $W_{\bar{0}}$, is an equivalence of categories.*

PROOF. For the usual Clifford algebra $C = \mathrm{Cliff}(E, q)$ of a nondegenerate quadratic form q the analogous functor $\overline{\varphi}$: $\{\mathbf{Z}/2\text{-graded } C\text{-modules}\} \to \{C_{\bar{0}}\text{-modules}\}$ gives an equivalence and has the inverse $\overline{\psi}\colon L \mapsto L \otimes_{C_{\bar{0}}} C$. Let $\psi\colon \mathscr{M}'^0_i \to \mathscr{M}'_i$ be the functor given by the analogous formula. The natural transformations $\mathrm{id} \to \varphi\psi$ and $\psi\varphi \to \mathrm{id}$ give isomorphisms over each point of the variety $\mathscr{L}_i \to \mathscr{L}_{i+1}$.

3.3. PROPOSITION. *There are natural isomorphisms*

$$K_\bullet(\mathscr{M}_i/\mathscr{M}_{i+1}) \simeq K_\bullet(\underline{R}_i\text{-mod}).$$

PROOF. We consider the functors

$$\mathscr{M}_i/\mathscr{M}_{i+1} \leftarrow \mathscr{M}'_i/\mathscr{M}'_{i,i+1} \xrightarrow{\varphi} \mathscr{M}'^0_i/\mathscr{M}'^0_{i,i+1} \to \underline{R}_i\text{-mod},$$

the left one of which is induced by the embedding and gives an isomorphism on $K_\bullet$ by Quillen's dévissage theorem, and the right one is the equivalence given by Serre's theorem.

3.4. COROLLARY. *The functor of the even part gives isomorphisms*

$$K_\bullet(\mathscr{M}/\mathscr{M}_\infty) \xrightarrow{\sim} K_\bullet(\underline{R}\text{-mod}).$$

3.5. LEMMA. *Let $\underline{L}$ be an Azumaya algebra of rank d on a variety Y. The forgetful functor f and the functor $? \otimes_{\mathscr{O}_Y} \underline{L}$ induce isomorphisms*

$$K_\bullet(\underline{L}\text{-mod}) \otimes \mathbf{Z}[1/d] \rightleftarrows K_\bullet(Y) \otimes \mathbf{Z}[1/d].$$

PROOF. For $\mathscr{G} \in \mathrm{Sh}(Y)$, $f(\mathscr{G} \otimes_{\mathscr{O}_Y} \underline{L}) = \mathscr{G} \otimes f(\underline{L})$. Since $f(\underline{L})$ is invertible in $K_0(\mathrm{Sh}\, Y) \otimes \mathbf{Z}[1/d]$, the assertion follows from the following lemma.

3.6. LEMMA. *For $M \in \underline{L}$-mod there is a natural isomorphism of L-modules: $f(M) \otimes_{\mathscr{O}_Y} \underline{L} \to M \otimes_{\mathscr{O}_Y} f(\underline{L})$.*

PROOF. Let $Z \xrightarrow{p} Y$ be the Brauer-Severi variety corresponding to $\underline{L}$; J the tautological sheaf of "ideals" on Z; and $\mathrm{Sh}_{-1}(Z)$ the category of coherent sheaves on Z which on each geometric fiber have the form $\mathscr{O}(-1)^r$ (or, equivalently, the $\mathscr{E} \in \mathrm{Sh}(Z)$ such that $\mathscr{E} \otimes_{\mathscr{O}_Z} J^*$ lift from Y).

The functors

$$\mathrm{Sh}_{-1}(Z) \underset{\beta}{\overset{\alpha}{\leftrightarrows}} \underline{L}\text{-mod}$$

defined by the formulas $\alpha(\mathscr{E}) = p_*(\mathscr{E} \otimes J^*)$ (the action of L is induced by the action of $p^*\underline{L} = J^* \otimes J$ on J^*) and $\beta(M) = p^*M \otimes_{p^*L} J$ give equivalences of categories that are inverse to each other. Relative to the identifications given by them, the functors

$f(?) \otimes_{\mathscr{O}_Y} \underline{L}$ and $? \otimes_{\mathscr{O}_Y} f(\underline{L})$, mapping the category $\underline{L}$-mod into itself, are taken to the functors

$$\mathscr{E} \mapsto p^*(p_*(\mathscr{E} \otimes_{\mathscr{O}_Z} J^*)) \otimes_{\mathscr{O}_Z} J \quad \text{and} \quad \mathscr{E} \mapsto \mathscr{E} \otimes_{\mathscr{O}_Z} p^*\underline{L}$$

respectively. Since $\mathscr{E} \otimes_{\mathscr{O}_Z} J^*$ is trivial on the fibers of p, $p^*(p_*(\mathscr{E} \otimes J^*)) \simeq \mathscr{E} \otimes J^*$, and the first functor is isomorphic to $\mathscr{E} \otimes J^* \otimes J \simeq \mathscr{E} \otimes p^*\underline{L}$.

3.7. LEMMA. *The boundary operators in the localization sequence for $\mathscr{M}_\infty(A) \subset \mathscr{M}(A) \to \mathscr{M}(A)/\mathscr{M}_\infty(A)$ are trivial.*

PROOF. Let $\mathrm{SI}.(B) \subset S.(\mathscr{M}^*(B))$ be the simplicial subcategory defined using finite complexes of graded B-modules dual to free modules. Arguments analogous to the proof of Proposition 1.8 and part b) of Theorem 2.3 show that the fibration

$$S.(\mathscr{M}_\infty(A)) \to S.(\mathscr{M}(A)) \to S.(\mathscr{M}(A)/\mathscr{M}_\infty(A)),$$

defining the localization sequence, is isomorphic to

$$\mathrm{SI}.(B) \to S.(\mathscr{M}^*(B)) \to S.^-(\mathscr{M}(A)/\mathscr{M}_\infty(A))$$

is the homotopy category. Here the second arrow is the composition

$$S.(\mathscr{M}^*(B)) \to S.^-(\mathscr{M}(A)) \to S.^-(\mathscr{M}(A)/\mathscr{M}_\infty(A)).$$

We consider the morphism of fibrations:

$$\begin{array}{ccccc}
\mathrm{SI}.(B) & \longrightarrow & S.(\mathscr{M}^*(B)) & \longrightarrow & S.^-(\mathscr{M}(A)/\mathscr{M}_\infty(A)) \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow \\
S.\mathscr{M}^*(S(E^*)) & \longrightarrow & S.\mathscr{M}^*(S(E^*)) & \longrightarrow & \{\cdot\}
\end{array}$$

giving that the composition

$$\pi_i|S.(\mathscr{M}(A)/\mathscr{M}_\infty(A))| \xrightarrow{\partial} \pi_{i-1}|\mathrm{SI}.(B)| \xrightarrow{\alpha_*} \pi_{i-1}|S.\mathscr{M}^*(S(E^*))|$$

is equal to zero for all i. We now observe that the morphisms α_* is injective, since after the identification of $\pi_{i-1}|\mathrm{SI}.(B)|$ and $\pi_{i-1}|S.\mathscr{M}^*(S(E^*))|$ with $K_{i-2}(k)\otimes_{\mathbf{Z}}\mathbf{Z}[t, t^{-1}]$ it takes $a \cdot t^j$ to $a \cdot \sum_{\nu=0}^m (-1)^\nu \binom{m}{\nu} t^{j+\nu}$.

3.8. EXAMPLE. Let $X \subset \mathrm{P}(E)$ be the smooth intersection of two or three even-dimensional quadrics, and let $Y \to \mathscr{L}$, $\mathscr{L} \cong P^1$ or P^2, be the two-sheeted covering, ramified along the discriminant divisor $\mathscr{L}_1 \subset \mathscr{L}$. The sheaf of algebras R on $\mathscr{L}$ is the descent of a uniquely determined sheaf of simple algebras R' on Y (Y is the spectrum of the sheaf of the centers of the algebras of $\underline{R}$); the $2\mathbf{Z}$-graded $A_{\bar{0}}$-module $A_{\bar{1}}[1]$ corresponds to a sheaf $\underline{S}$ of $\underline{R}$-modules on $\mathscr{L}$ and, hence, to a sheaf $\underline{S}'$ of $\underline{R}'$-modules on Y.

We denote by $U_j \subset K_j(X) \otimes \mathbf{Z}[1/2]$ the subgroup generated by the images of $\mathscr{O}_X(i) \otimes K_j(k) \otimes \mathbf{Z}[1/2]$ for all $i \in \mathbf{Z}$. We denote by $W_j \subset K_j(Y) \otimes \mathbf{Z}[1/2]$ the subgroup generated by the images of $\underline{R}'(i) \otimes K_j(k) \otimes \mathbf{Z}[1/2]$ and $\underline{S}'(i) \otimes K_j(k) \otimes \mathbf{Z}[1/2]$, $i \in \mathbf{Z}$, where $\mathscr{O}_Y(1)$ is lifted from $\mathscr{L}$. The arguments of this section combined with Lemma 2.6 give isomorphisms between $K_j(X) \otimes \mathbf{Z}[1/2]/U_j$ and $K_j(Y) \otimes \mathbf{Z}[1/2]/W_j$. In the case when Y is a curve and the field k is algebraically closed, we can manage without multiplying by $\mathbf{Z}[1/2]$, since $\underline{R}' \simeq \operatorname{End}\Sigma$ for some Σ; for this we need to replace $\underline{R}'$ by Σ and $\underline{S}'$ by $\alpha^*\Sigma$ in the definition of W_j, where $\alpha\colon Y \to Y$ is the involution.

These isomorphisms can be given more explicitly. In fact, on $X \times Y$ there is a complex of sheaves

$$\cdots \to \mathscr{O}(-2) \boxtimes \underline{R}'(-1) \to \mathscr{O}(-1) \boxtimes \underline{S}' \to \mathscr{O} \boxtimes \underline{R}'. \tag{3.1}$$

This complex is the resolution of some bundle L. Let p_X and p_Y be the projections of $X \times Y$ onto the factors. If $\mathscr{G}$ is a sufficiently "negative" sheaf on X, then $Rp_{Y*}(p_X^*(\mathscr{G}) \otimes L)$ is isomorphic to the complex

$$\{\cdots \to H^{\dim X}(X, \mathscr{G}(-2)) \otimes \underline{R}'(-1) \to H^{\dim X}(X, \mathscr{G}(-1)) \otimes \underline{S}' \to H^{\dim X}(X, \mathscr{G}) \otimes \underline{R}'\}.$$

Therefore the composition of mappings

$$K_j(X) \xrightarrow{H_*} K_j(\mathscr{M}^*(B)/\mathscr{F}) \xrightarrow{\overline{F}_*} K_j(\mathscr{M}(A))/\left(\bigoplus_{i\in\mathbf{Z}} K_j(k)\right)$$
$$\to K_j(R'\text{-mod})/K_j(k) \otimes (\text{the classes of } R'(i), S'(i), i \in \mathbf{Z}) \xrightarrow{f_*} K_j(Y)/W_j,$$

where H takes $\mathscr{G}$ to $\bigoplus_{i\leq 0} H^{\dim X}(X, \mathscr{G}(i))$ and f is the forgetful functor, is $p_{Y*}(p_X^*(?) \otimes L)$ (cf. [7]). In the case of the intersection of two quadrics and $k = \bar{k}$ we can replace $\underline{R}'$ by Σ and $\underline{S}'$ by $\alpha^*\Sigma$ in the complex (3.1).

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