BÉZOUTIANS AND $\mathbb{A}^1$-BROUWER DEGREES — MACAULAY2

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With $\mathrm{GW}(k)$ in hand, we want to implement algorithms for computing *local* and *global* $\mathbb{A}^1$-*Brouwer degrees* of endomorphisms of affine space.

Standing assumption: Fields are not of characteristic 2.

1. Background speedrun

The local *Brouwer degree* of a map $f : M \to N$ between n-manifolds at a point p encodes roughly how the function f is transforming space around the point p. This is an important topological invariant. Leveraging the machinery of motivic homotopy theory, one can analogously define an $\mathbb{A}^1$-*Brouwer degree*, valued in the Grothendieck–Witt ring $\mathrm{GW}(k)$, following work of Morel, Kass-Wickelgren, Levine, and others. This can be computed on affine charts, analogous to the classical setting. The goal of this project is to implement algorithms for computing $\mathbb{A}^1$-Brouwer degrees into Macaulay2.

2. $\mathbb{A}^1$-Brouwer degrees

Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of affine n-space, and let p be an isolated preimage of zero. Then we can compute the *local* $\mathbb{A}^1$-*Brouwer degree* of f at p, denoted

$$\deg_p^{\mathbb{A}^1}(f) \in \mathrm{GW}(k).$$

This is an invariant on the local k-algebra

$$Q_p(f) := \frac{k[x_1, \ldots, x_n]_p}{(f_1, \ldots, f_n)}.$$

Here's a procedure to do this, called a *Bézoutian bilinear form*:

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(1) Introduce some dummy variables $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$, and consider the quantities

$$\Delta_{ij} := \frac{f_i(Y_1, \dots, Y_{j-1}, X_j, \dots, X_n) - f_i(Y_1, \dots, Y_j, X_{j+1}, \dots, X_n)}{X_j - Y_j}.$$

Think about this as analogous to a partial derivative. This quantity lives in the tensor product

$$\Delta_{ij} \in Q_p(f) \otimes_k Q_p(f).$$

(2) Define the *Bézoutian* as the determinant as i and j vary:

$$\text{Béz}(f) := \det(\Delta_{ij}) \in Q_p(f) \otimes_k Q_p(f).$$

(3) For any k-vector space basis $a_1, \dots, a_r$ of $Q_p(f)$, we can express the Bézoutian as

$$\text{Béz}(f) = \sum_{i,j} b_{ij} a_i \otimes a_j,$$

for some scalars $b_{ij} \in k$.

(4) Observe that the matrix $(b_{ij})_{i,j}$ is an invertible symmetric matrix. We call this the *Bézoutian bilinear form*

Theorem 2.1. [BMP23, 1.2] There is an isomorphism in $\mathrm{GW}(k)$:

$$\deg_p^{\mathbb{A}^1}(f) \cong (b_{ij}).$$

Let's discuss the global story before doing examples.

3. Global $\mathbb{A}^1$-Brouwer degrees

Given an endomorphism $f : \mathbb{A}_k^n \to \mathbb{A}_k^n$ all of whose zeros are isolated, we can compute the *global* $\mathbb{A}^1$-Brouwer degree as the sum of the local degrees:

$$\deg^{\mathbb{A}^1}(f) := \sum_{p \in f^{-1}(0)} \deg_p^{\mathbb{A}^1}(f).$$

Solving for the fiber of zero is a computationally costly step in computing a global degree. One of the main results of [BMP23] is that we can circumvent this step of the computation!

Theorem 3.1. [BMP23, 5.4] The global $\mathbb{A}^1$-Brouwer degree is isomorphic to the Bézoutian bilinear form on the global k-algebra $Q(f) = k[x_1, \dots, x_n]/f$.

Now's a good time for an example.

Example 3.2. Let $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ be given by

$$f(x) = x(x-1)(x+1)^2 = x^4 + x^3 - x^2 - x.$$

Here our global algebra is

$$Q(f) = \frac{k[x]}{x^4 + x^3 - x^2 - x}.$$

Since there is one polynomial and one variable, the Bézoutian is equal to Δ_{11}:

$$\begin{aligned}\text{Béz}(f) = \Delta_{11} = \frac{f(X)-f(Y)}{X-Y} &= \frac{(X^4+X^3-X^2-X)-(Y^4+Y^3-Y^2-Y)}{X-Y} \\ &= X^3 + X^2Y + XY^2 + Y^3 + X^2 + XY + Y^2 - X - Y - 1.\end{aligned}$$

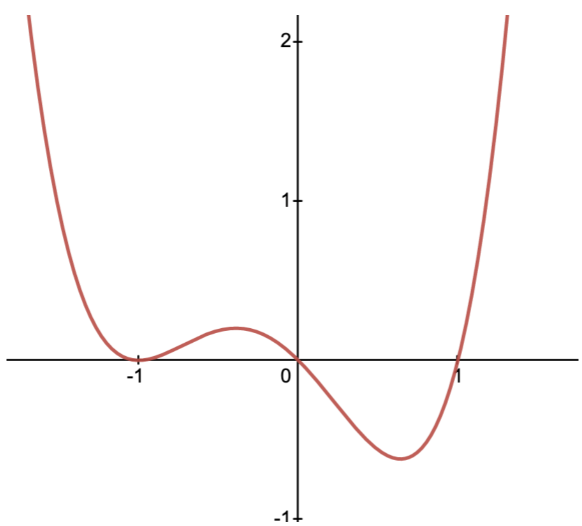
Now pick a k-vector space basis for $Q(f)$. An easy basis is $\{1, x, x^2, x^3\}$. This extends to a basis for $Q(f) \otimes_k Q(f)$:

$$Q(f) \otimes_k Q(f) = \text{span}_k \left\{X^iY^j \colon 0 \leq i, j \leq 3\right\}.$$

Now let's write out the coefficients we see on the Bézoutian. A nice way to do this is write the basis elements in the X_i's labeling the columns, and the Y_i's labeling the rows:

	1	X	X^2	X^3
1	-1	-1	1	0
Y	-1	1	0	0
Y^2	1	1	0	0
Y^3	1	0	0	0

This is isomorphic to $2\mathbb{H}$. Look at the graph of the function for intuition:



This degree is picking up the roots with multiplicity and orientation! The local degree at $x = -1$ will be $\mathbb{H}$, the local degree at the origin is $\langle -1 \rangle$, and at $x = 1$ it is $\langle 1 \rangle$. Let's verify this with a local computation.

Example 3.3. Let's compute $\deg^{\mathbb{A}^1}_{(x+1)}(f)$. We can compute the Bézoutian before localizing, and look at its image in the tensor of the local algebras:

$$\text{Béz}(f) \in \frac{k[X]_{(X+1)}}{f(X)} \otimes_k \frac{k[Y]_{(Y+1)}}{f(Y)}.$$

The problem then reduces to finding an appropriate k-vector space basis for $Q_p(f)$. Observe that

$$\frac{k[x]_{(x+1)}}{f(x)} = \frac{k[x]_{(x+1)}}{x(x-1)(x+1)^2} \cong \frac{k[x]_{(x+1)}}{(x+1)^2}.$$

So it makes sense to take $\{1, x\}$ as a basis. Observe now that $x^2 = -2x - 1$. We can rewrite the Bézoutian then as

$$\begin{aligned}
&X^3 + X^2Y + XY^2 + Y^3 + X^2 + XY + Y^2 - X - Y - 1 \\
&\equiv X(-2X-1) + (-2X-1)Y + X(-2Y-1) + Y(-2Y-1) + (-2X-1) + XY \\
&\qquad + (-2Y-1) - X - Y - 1 \\
&= -2X^2 - 3XY - 2Y^2 - 5X - 5Y - 3 \\
&\equiv -3XY - X - Y + 1.
\end{aligned}$$

This is now expressed in terms of our basis elements. We get then that

$$\deg^{\mathbb{A}^1}_{(x+1)}(f) = \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}.$$

Diagonalizing (e.g. $R_2 = R_2 + R_1$ and the same for columns) we get the hyperbolic form.

4. Normal bases

There are two computational questions that remain:

1. How do we compute a k-vector space basis for one of these sorts of local or global k-algebras?
2. How do we obtain an expression for the Bézoutian in terms of these basis elements?

The first question is answered by classical commutative algebra! If any of the following is unfamiliar to you, I wrote a blog post about it here.

Theorem 4.1. (Macaulay) The standard monomials give a k-vector space basis for $k[x_1, \ldots, x_n]/I$ for $I \trianglelefteq k[x_1, \ldots, x_n]$ any ideal.

The local algebra might look more daunting, but we can use a classical trick to turn a local algebra into a quotient!

Proposition 4.2. If p is an isolated zero of the ideal $I \trianglelefteq k[x_1, \ldots, x_n]$, then we have an isomorphism of k-algebras

$$\frac{k[x_1, \ldots, x_n]_p}{I_p} \cong \frac{k[x_1, \ldots, x_n]}{(I : (I : p^\infty))}.$$

So we can saturate at the prime, and then take the ideal quotient by the saturation. Computing a k-vector space basis for $k[x_1, \ldots, x_n]_p/I$ then reduces to computing a normal basis for $k[x_1, \ldots, x_n]/(I : (I : p^\infty))$.

The second question should(?) be an ordering thing. As the normal basis consists of monomials, they can be expressed in some appropriate ordering on monomials (e.g. Macaulay2 defaults to grevlex). Asking for an expression of the Bézoutian in a quotient ring should by default give a reduced form of it that is expressible in the standard monomials. We might want to draft a quick proof of why this works for the accompanying documentation.

4.1. **Coding goals.**

Goal 4.3. Implement local and global $\mathbb{A}^1$-degree algorithms into Macaulay2.

(1) Given an endomorphism of affine space with isolated zeros, compute its global $\mathbb{A}^1$-degree.
(2) Given an endomorphism of affine space and an isolated zero, compute its local $\mathbb{A}^1$-degree.
(3) (Optional but would be amazing) Given an endomorphism of affine space, *compute its zeros* and all the accompanying local degrees.

Note: This has already been implemented into Sage: https://github.com/shmckean/A1-degree/, following the procedures outlined above. Our goal is to expand and improve upon these algorithms.

Goal 4.4. Potentially, use some of the calculation rules outlined in [BMP23, §6] to speed up $\mathbb{A}^1$-degree computations.

References

[BMP23] Thomas Brazelton, Stephen McKean, and Sabrina Pauli, *Bézoutians and the $\mathbb{A}^1$-degree*, 2023. 2, 5