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DEGREES OF SYMMETRIC GROTHENDIECK POLYNOMIALS AND CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. We give an explicit formula for the degree of the Grothendieck polynomial of a Grassmannian permutation and a closely related formula for the Castelnuovo-Mumford regularity of the Schubert determinantal ideal of a Grassmannian permutation. We then provide a counterexample to a conjecture of Kummini-Lakshmibai-Sastry-Seshadri on a formula for regularities of standard open patches of particular Grassmannian Schubert varieties and show that our work gives rise to an alternate explicit formula in these cases. We end with a new conjecture on the regularities of standard open patches of *arbitrary* Grassmannian Schubert varieties.

1. INTRODUCTION

Lascoux and Schützenberger [11] introduced *Grothendieck polynomials* to study the K-theory of flag varieties. Grothendieck polynomials have a recursive definition, using divided difference operators. The symmetric group S_n acts on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ by permuting indices. Let s_i be the simple transposition in S_n exchanging i and $i+1$. Then define operators on $\mathbb{Z}[x_1, x_2, \ldots, x_n]$

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}} \text{ and } \pi_i = \partial_i(1 - x_{i+1}).$$

Write $w_0 = n\, n{-}1\, \ldots\, 1$ for the **longest permutation** in S_n (in one-line notation) and take

$$\mathfrak{G}_{w_0}(x_1, x_2, \ldots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Let $w_i := w(i)$ for $i \in [n]$. Then if $w_i > w_{i+1}$, we define $\mathfrak{G}_{s_i w} = \pi_i(\mathfrak{G}_w)$. We call $\{\mathfrak{G}_w : w \in S_n\}$ the set of **Grothendieck polynomials**. Since the π_i's satisfy the same braid and commutation relations as the simple transpositions, each $\mathfrak{G}_w$ is well defined.

Grothendieck polynomials are generally inhomogeneous. The lowest degree of the terms in $\mathfrak{G}_w$ is given by the *Coxeter length* of w. The degree (i.e. highest degree of the terms) of $\mathfrak{G}_w$ can be described combinatorially in terms of pipe dreams

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(see [3, 8]), but this description is not readily computable. We seek an explicit combinatorial formula. In this paper, we give such an expression in the Grassmannian case. Our proof relies on a formula of Lenart [12].

One motivation for wanting easily-computable formulas for degrees of Grothendieck polynomials (for large classes of $w \in S_n$) comes from commutative algebra: formulas for degrees of Grothendieck polynomials give rise to closely related formulas for *Castelnuovo-Mumford regularity* of associated Schubert determinantal ideals. Recall that Castelnuovo-Mumford regularity is an invariant of a homogeneous ideal related to its minimal free resolution (see Section 4 for definitions). Formulas for regularities of Schubert determinantal ideals yield formulas for regularities of certain well-known classes of generalized determinantal ideals in commutative algebra. For example, among the Schubert determinantal ideals are ideals of $r \times r$ minors of an $n \times m$ matrix of indeterminates and one sided ladder determinantal ideals. Furthermore, many other well-known classes of generalized determinantal ideals can be viewed as defining ideals of Schubert varieties intersected with opposite Schubert cells, so degrees of specializations of *double Grothendieck polynomials* govern Castelnuovo-Mumford regularities in these cases. Thus, one purpose of this paper is to suggest a purely combinatorial approach to studying regularities of certain classes of generalized determinantal ideals.

2. Background on permutations

We start by recalling some background on the symmetric group. We follow [13] as a reference. Let S_n denote the **symmetric group** on n letters, i.e. the set of bijections from the set $[n] := \{1, 2, \ldots, n\}$ to itself. We typically represent permutations in one-line notation. The **permutation matrix** of w, also denoted by w, is the matrix which has a 1 at (i, w_i) for all $i \in [n]$, and zeros elsewhere.

The **Rothe diagram** of w is the subset of cells in the $n \times n$ grid

$$D(w) = \{(i, j) \ \mid \ 1 \leq i, j \leq n, \ w_i > j, \text{ and } w_j^{-1} > i\}.$$

Graphically, $D(w)$ is the set of cells in the grid which remain after plotting the points (i, w_i) for each $i \in [n]$ and striking out any boxes which appear weakly below or weakly to the right of these points. The **essential set** of w is the subset of the diagram

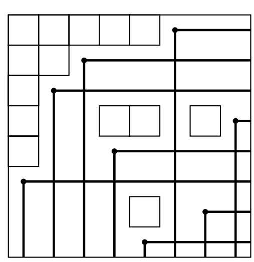
$$\mathcal{E}ss(w) = \{(i, j) \in D(w) \ \mid \ (i + 1, j), (i, j + 1) \notin D(w)\}.$$

Each permutation has an associated **rank function** defined by

$$r_w(i, j) = |\{(i', w_{i'}) \ \mid \ i' \leq i, w_{i'} \leq j\}|.$$

We write $\ell(w) := |D(w)|$ for the **Coxeter length** of w.

Example 2.1. If $w = 63284175 \in S_8$ (in one-line notation) then $D(w)$ is the following:



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Here $\mathcal{E}ss(w) = \{(1,5),(2,2),(4,5),(4,7),(5,1),(7,5)\}$.

3. Grassmannian Grothendieck polynomials

A **partition** is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. We define the **length** of λ to be $\ell(\lambda) = |\{h \in [k] \mid \lambda_h \neq 0\}|$ and the **size** of λ, denoted $|\lambda|$, to be $\sum_{i=1}^k \lambda_i$. Write $\mathcal{P}_k$ for the set of partitions of length at most k. Here, we conflate partitions with their Young diagrams, i.e. the notation $(i,j) \in \lambda$ indicates choosing the jth box in the ith row of the Young diagram of λ.

We say $w \in S_n$ has a **descent** at position k if $w_k > w_{k+1}$. A permutation $w \in S_n$ is **Grassmannian** if w has a unique descent. To each Grassmannian permutation w, we can uniquely associate a partition $\lambda \in \mathcal{P}_k$:

$$\lambda = (w_k - k, \ldots, w_1 - 1),$$

where k is the position of the descent of w.

Let w_λ denote the Grassmannian permutation associated to λ. It is easy to check that

$$|\lambda| = \ell(w_\lambda) = |D(w_\lambda)|. \tag{1}$$

Define $\mathsf{YTab}(\lambda)$ to be the set of fillings of λ with entries in $[k]$ so that

- entries weakly increase from left-to-right along rows and
- entries strictly increase from top-to-bottom along columns.

For a partition λ, the **Schur polynomial** in k variables is

$$s_\lambda(x_1, x_2, \ldots, x_k) = \sum_{T \in \mathsf{YTab}(\lambda)} \prod_{i=1}^{k} x_i^{\#i\text{'s in } T}.$$

Definition 3.1. Let $\lambda, \mu \in \mathcal{P}_k$ so that $\lambda \subseteq \mu$. Denote by $\mathsf{Tab}(\mu/\lambda)$ the set of fillings of the skew shape μ/λ with entries in $[k]$ such that

- entries strictly increase left-to-right in each row,
- entries strictly increase top-to-bottom in each column, and
- entries in row i are at most $i-1$ for each $i \in [k]$.

For ease of notation, let $\mathfrak{G}_\lambda := \mathfrak{G}_{w_\lambda}$.

Theorem 3.2 ([12, Theorem 2.2])**.** *For a Grassmannian permutation $w_\lambda \in S_n$,*

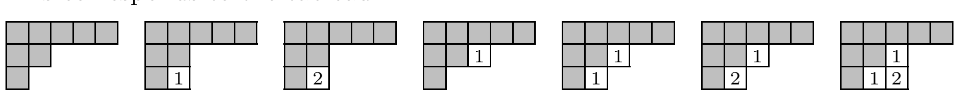
$$\mathfrak{G}_\lambda(x_1, x_2, \ldots, x_k) = \sum_{\substack{\mu \in \mathcal{P}_k \\ \lambda \subseteq \mu}} a_{\lambda\mu} s_\mu(x_1, x_2, \ldots, x_k)$$

where $(-1)^{|\mu|-|\lambda|} a_{\lambda,\mu} = |\mathsf{Tab}(\mu/\lambda)|$ and k is the unique descent of w_λ.

Example 3.3. The Grassmannian permutation $w = 24813567$ corresponds to $\lambda = (5,2,1)$. By Theorem 3.2,

$$\mathfrak{G}_{(5,2,1)}(x_1, x_2, x_3) = s_{(5,2,1)} - 2s_{(5,2,2)} - s_{(5,3,1)} + 2s_{(5,3,2)} - s_{(5,3,3)}.$$

This corresponds to the tableaux:



Definition 3.4. We say a partition μ is **maximal** for λ if $\mathsf{Tab}(\mu/\lambda) \neq \emptyset$ and $\mathsf{Tab}(\nu/\lambda) = \emptyset$ whenever $|\nu| > |\mu|$.

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The following lemma can be obtained from the proof of [12, Theorem 2.2], but we include it for completeness.

Lemma 3.5. *Fix a partition* $\lambda \in \mathcal{P}_k$. *Define* μ *by setting* $\mu_1 = \lambda_1$, *and* $\mu_i = \min\{\mu_{i-1}, \lambda_i + (i-1)\}$ *for each* $1 < i \le k$. *Then* μ *is the unique partition that is maximal for* λ.

Proof. Let ρ be any partition with $\mathsf{Tab}(\rho/\lambda) \neq \emptyset$. Since elements of $\mathsf{Tab}(\rho/\lambda)$ have strictly increasing rows, ρ/λ has at most $i-1$ boxes in row i for each i. That is, $\rho_i \le \lambda_i + (i-1)$ for each i. It follows that $\rho_i \le \mu_i$ for each i. Thus, uniqueness of μ will follow once we show that μ is maximal for λ. It suffices to produce an element $T \in \mathsf{Tab}(\mu/\lambda)$.

We will denote by $T(i,j)$ the filling by T of the box in row i and column j of μ. For each i and j with $\lambda_i < j \le \mu_i$, set

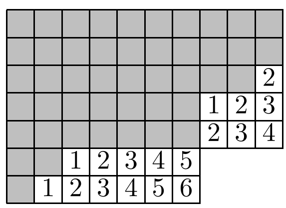
$$T(i,j) = i + j - \mu_i - 1.$$

It is easily seen that T strictly increases along rows with $T(i,j) \in [i-1]$ for each i. To see that $T \in \mathsf{Tab}(\mu/\lambda)$, it remains to note that T strictly increases down columns. Observe

$$T(i,j) - T(i-1,j) = \mu_{i-1} - \mu_i + 1 > 0.$$

$\square$

Example 3.6. If $\lambda = (10,10,9,7,7,2,1)$, the unique partition μ maximal for λ is $\mu = (10,10,10,10,10,7,7)$. Below is the tableau $T \in \mathsf{Tab}(\mu/\lambda)$ constructed in the proof of Lemma 3.5.



Definition 3.7. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, let $P(\lambda) = (P_1, P_2, \dots, P_r)$ be the set partition of $[k]$ such that $i, j \in P_h$ if and only if $\lambda_i = \lambda_j$, and $\lambda_i > \lambda_j$ whenever $i \in P_h$ and $j \in P_l$ with $h < l$.

Note that if $\lambda = (\lambda_1, \dots, \lambda_k) = (\lambda_{i_1}^{p_1}, \dots, \lambda_{i_r}^{p_r})$ in exponential notation, then $p_h = |P_h|$ for each $h \in [r]$. In the following definition, we describe a decomposition of λ into rectangles.

Definition 3.8. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition and $P(\lambda) = (P_1, P_2, \dots, P_r)$. Set $m_h = \min P_h$ for each h. Define $R(\lambda) = (R_1, R_2, \dots, R_r)$ by setting

$$R_h := \left\{ (i,j) \in \lambda \;\middle|\; i \in \bigcup_{l=1}^{h} P_l \text{ and } \lambda_{m_{h+1}} < j \le \lambda_{m_h} \right\},$$

where we take $\lambda_{m_{r+1}} := 0$.

Set $\lambda^{(h)}$ to be the partition

$$\lambda^{(h)} = \bigcup_{j=1}^{h} R_j$$

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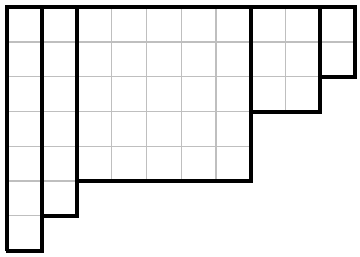
for $h \in [r]$. Equivalently, for $h \in [r-1]$, $\lambda^{(h)} = (\lambda_1 - \lambda_i, \lambda_2 - \lambda_i, \ldots, \lambda_{i-1} - \lambda_i)$ where $i = \min P_{h+1}$, and $\lambda^{(r)} = \lambda$. Set $\lambda^{(0)} := \emptyset$.

Example 3.9. For λ as in Example 3.6, one has $P_1 = \{1, 2\}$, $P_2 = \{3\}$, $P_3 = \{4, 5\}$, $P_4 = \{6\}$, and $P_5 = \{7\}$. The sets in $R(\lambda)$ are outlined below, with R_1 the rightmost rectangle and R_5 the leftmost. Considering $h = 2$, $\lambda^{(h)} = R_1 \cup R_2 = (10-7, 10-7, 9-7) = (3, 3, 2)$.



Definition 3.10. For any $n \geq 1$, let δ^n denote the **staircase shape** $\delta^n = (n, n-1, \ldots, 1)$. Given a partition μ, let

$$\mathtt{sv}(\mu) = \max\left\{k \mid \delta^k \subseteq \mu\right\}.$$

The partition $\delta^{\mathtt{sv}(\mu)}$ is called the **Sylvester triangle** of μ.

Proposition 3.11. *Suppose μ is maximal for λ and $P(\lambda) = (P_1, \ldots, P_r)$. If $i \in P_{h+1}$ for some $0 \leq h < r$, then*

$$\mu_i = \lambda_i + \mathtt{sv}(\lambda^{(h)}).$$

Proof. By Lemma 3.5, $\mu_1 = \lambda_1$ and $\mu_i = \min\{\mu_{i-1}, \lambda_i + (i-1)\}$ for $1 < i \leq k$. Clearly $P(\lambda)$ refines $P(\mu)$: if $\lambda_i = \lambda_j$, then $\mu_i = \mu_j$. Example 3.6 shows this refinement can be strict. Hence, it suffices to prove the statement when $i = \min P_{h+1}$. We work by induction on h.

When $h = 0$, $i = \min(P_1) = 1$. Since $\lambda_1 = \mu_1$, the result follows. Suppose the claim holds for some $h-1$. We show the claim holds for h. Let $i = \min P_{h+1}$. Then it suffices to show that

$$\lambda_i + \mathtt{sv}(\lambda^{(h)}) = \min\{\mu_{i-1}, \lambda_i + (i-1)\}. \tag{2}$$

Since $i = \min P_{h+1}$, it follows that $i-1 \in P_h$. By applying the inductive assumption to μ_{i-1},

$$\min\{\mu_{i-1}, \lambda_i + (i-1)\} = \min\{\lambda_{i-1} + \mathtt{sv}(\lambda^{(h-1)}), \lambda_i + (i-1)\}. \tag{3}$$

By Equations (2) and (3), the proof is complete once we show

$$\mathtt{sv}(\lambda^{(h)}) = \min\{(\lambda_{i-1} - \lambda_i) + \mathtt{sv}(\lambda^{(h-1)}), i-1\}. \tag{4}$$

Let ω, ℓ respectively denote the (horizontal) width and (vertical) length of R_h, and set $M = \mathtt{sv}(\lambda^{(h-1)})$. Equation (4) is equivalent to proving

$$\mathtt{sv}(\lambda^{(h)}) = \min\{\omega + M, \ell\}.$$

By definition, $\lambda^{(h)} = R_h \cup \lambda^{(h-1)}$, so it is straightforward to see that

$$\mathtt{sv}(\lambda^{(h)}) \leq \min\{\omega + M, \ell\}.$$

Let (M, c) be the southwest most box in the northwest most embedding of $\delta^M \subseteq \lambda^{(h-1)}$, with the indexing inherited from λ.

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Suppose first that $\ell \geq \omega+M$. Since R_h is a rectangle, $(\omega+M, c-\omega) \in \lambda^{(h)}$. Then $\delta^{\omega+M} \subseteq \lambda^{(h+1)}$ and Equation (4) follows. Otherwise, it must be that $\ell - M < \omega$. Since R_h is a rectangle, $(\ell, c-\ell+M) \in \lambda^{(h)}$. Thus, $\delta^{\ell} \subseteq \lambda^{(h+1)}$ and Equation (4) follows. □

Theorem 3.12. *Suppose $w_\lambda \in S_n$ is a Grassmannian permutation. Let $P(\lambda) = (P_1, \ldots, P_r)$. Then*

$$\deg(\mathfrak{G}_\lambda) = |\lambda| + \sum_{h\in[r-1]} |P_{h+1}| \cdot \mathtt{sv}(\lambda^{(h)}).$$

Proof. By Theorem 3.2 and Lemma 3.5, the highest nonzero homogeneous component of $\mathfrak{G}_\lambda$ is $a_{\lambda\mu}s_\mu$ where μ is maximal for λ. Since $\deg(s_\mu)$ is $|\mu|$, Proposition 3.11 implies the theorem, using the fact that $\mathtt{sv}(\lambda^{(0)}) = 0$. □

Example 3.13. Returning to λ as in Example 3.6, Theorem 3.12 states that $\deg(\mathfrak{G}_\lambda) = |\lambda| + \sum_{h=1}^{4} |P_{h+1}| \cdot \mathtt{sv}(\lambda^{(h)}) = 46 + (1\cdot 1 + 2\cdot 3 + 1\cdot 5 + 1\cdot 6) = 46 + 18 = 64$.

4. Castelnuovo-Mumford regularity of Grassmannian matrix Schubert varieties

In this section, we recall some basics of Castelnuovo-Mumford regularity and then use Theorem 3.12 to produce easily-computable formulas for the regularities of matrix Schubert varieties associated to Grassmannian permutations.

4.1. **Commutative algebra preliminaries.** Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a positively $\mathbb{Z}^d$-graded polynomial ring so that the only elements in degree zero are the constants. The **multigraded Hilbert series** of a finitely generated graded module M over S is

$$H(M; \mathbf{t}) = \sum_{\mathbf{a}\in\mathbb{Z}^d} \dim_K(M_{\mathbf{a}})\mathbf{t}^{\mathbf{a}} = \frac{\mathcal{K}(M;\mathbf{t})}{\prod_{i=1}^{n}(1-\mathbf{t}^{\mathbf{a_i}})},\ \deg(x_i) = \mathbf{a_i},$$

where if $\mathbf{a}_i = (a_i(1), \ldots, a_i(d))$, then $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_i(1)} \cdots t_d^{a_i(d)}$. The numerator $\mathcal{K}(M;\mathbf{t})$ in the expression above is a Laurent polynomial in the t_i's, called the **K-polynomial** of M. For more detail on K-polynomials, see [14, Chapter 8].

We are mostly interested in the case where S is standard graded, that is, $\deg(x_i) = 1$, and the case where $M = S/I$ where I is a homogeneous ideal with respect to the standard grading. Note that, in this case, the K-polynomial is a polynomial in a single variable t. There is a minimal free resolution

$$0 \to \bigoplus_j S(-j)^{\beta_{l,j}} \to \bigoplus_j S(-j)^{\beta_{l-1,j}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}} \to S/I \to 0$$

where $l \leq n$ and $S(-j)$ is the free S-module obtained by shifting the degrees of S by j. The **Castelnuovo-Mumford regularity** of S/I, denoted $\operatorname{reg}(S/I)$, is defined as

$$\operatorname{reg}(S/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

This invariant is a measure of the complexity of S/I and has multiple homological characterizations. For example, $\operatorname{reg}(S/I)$ is the smallest integer m for which $\operatorname{Ext}^j(S/I, S)_n = 0$, for all j and all $n \leq -m-j-1$ (see [2, Proposition 20.16]). We refer the reader to [2, Chapter 20.5] for more information on regularity.

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Let $\mathcal{K}(S/I;t)$ denote the K-polynomial of S/I with respect to the standard grading. Assume that S/I is Cohen-Macaulay and let $\mathrm{ht}_S I$ denote the height of the ideal I. Then,

$$\mathrm{reg}(S/I) = \deg \mathcal{K}(S/I;t) - \mathrm{ht}_S I. \tag{5}$$

See, for example, [1, Lemma 2.5] and surrounding explanation. In this paper, we will use this characterization of regularity.

4.2. **Regularity of Grassmannian matrix Schubert varieties.** Let X be the space of $n \times n$ matrices with entries in $\mathbb{C}$, let $\widetilde{X} = (x_{ij})$ denote an $n \times n$ generic matrix of variables, and let $S = \mathbb{C}[x_{ij}]$. Given an $n \times n$ matrix M, let $M_{[i,j]}$ denote the submatrix of M consisting of the top i rows and left j columns of M. Given a permutation matrix $w \in S_n$ we have the **matrix Schubert variety**

$$X_w := \{M \in X \mid \text{rank } M_{[i,j]} \leq \text{rank } w_{[i,j]}\},$$

which is an affine subvariety of X with defining ideal

$$I_w := \langle (r_w(i,j)+1) - \text{size minors of } \widetilde{X}_{[i,j]} \mid (i,j) \in \mathcal{E}ss(w) \rangle \subseteq S.$$

The ideal I_w, called a **Schubert determinantal ideal**, is prime [4] and is homogeneous with respect to the standard grading of S.

By [7, Theorem A], we have $\mathcal{K}(S/I_w;t) = \mathfrak{G}_w(1-t,\ldots,1-t)$, which has the same degree as $\mathfrak{G}_w(x_1,\ldots,x_n)$, since the coefficients in homogeneous components of single Grothendieck polynomials have the same sign (see, for example, [7]). Thus,

$$\mathrm{reg}(S/I_w) = \deg \mathfrak{G}_w(x_1,\ldots,x_n) - \mathrm{ht}_S I_w = \deg \mathfrak{G}_w(x_1,\ldots,x_n) - |D(w)|, \tag{6}$$

where the second equality follows because

$$\mathrm{ht}_S I_w = \mathrm{codim}_X X_w = |D(w)|$$

by [4]. We now turn our attention to the case where w is a Grassmannian permutation and retain the notation from the previous section.

Corollary 4.1. *Suppose $w_\lambda \in S_n$ is a Grassmannian permutation. Let $P(\lambda) = (P_1,\ldots,P_r)$. Then*

$$\mathrm{reg}(S/I_{w_\lambda}) = \sum_{h \in [r-1]} |P_{h+1}| \cdot \mathtt{sv}(\lambda^{(h)}).$$

Proof. This is immediate from Theorem 3.12, Equation (6), and Equation (1). □

Example 4.2. Continuing Example 3.13, Corollary 4.1 states $\mathrm{reg}(S/I_{w_\lambda}) = 18$.

Example 4.3. The ideal of $(r+1)\times(r+1)$ minors of a generic $n \times m$ matrix is the Schubert determinantal ideal of a Grassmannian permutation $w \in S_{n+m}$. Indeed, w is the permutation of minimal length in S_{n+m} such that rank $w_{[n,m]} = r$.

The corresponding partition is $\lambda = (m-r)^{(n-r)}0^r$. We have $\lambda^{(1)} = (m-r)^{(n-r)}$ and so $\mathtt{sv}(\lambda^{(1)}) = \min\{m-r, n-r\}$. Furthermore, $|P_2| = r$. Therefore,

$$\mathrm{reg}(S/I_w) = r \cdot \min\{m-r, n-r\} = r \cdot (\min\{m,n\} - r).$$

We claim no originality for the formula in Example 4.3; minimal free resolutions of ideals of $r \times r$ minors of a generic $n \times m$ matrix are well-understood (see [10] or [15, Chapter 6]).

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5. On the regularity of coordinate rings of Grassmannian Schubert varieties intersected with the opposite big cell

In this section, we discuss a conjecture of Kummini-Lakshmibai-Sastry-Seshadri [9] on Castelnuovo-Mumford regularity of coordinate rings of certain open patches of Grassmannian Schubert varieties. We provide a counterexample to the conjecture, and then we state and prove an alternate explicit formula for these regularities. We end with a conjecture on regularities of coordinate rings of standard open patches of *arbitrary* Schubert varieties in Grassmannians.

5.1. **Grassmannian Schubert varieties in the opposite big cell.** Fix $k \in [n]$ and let Y denote the space of $n \times n$ matrices of the form

$$\begin{bmatrix} M & I_k \\ I_{n-k} & 0 \end{bmatrix}, \tag{7}$$

where M is a $k\times(n-k)$ matrix with entries in $\mathbb{C}$ and I_k is a $k\times k$ identity matrix. Let $P \subseteq GL_n(\mathbb{C})$ denote the maximal parabolic of block lower triangular matrices with block rows of size $k, (n-k)$ (listed from top to bottom). Then the Grassmannian of k-planes in n-space, $Gr(k,n)$, is isomorphic to $P\backslash GL_n(\mathbb{C})$. Further, the map $\pi : GL_n(\mathbb{C}) \to Gr(k,n)$ given by taking a matrix to its coset mod P induces an isomorphism from Y onto an affine open subvariety U of $Gr(k,n)$ (often called the opposite big cell).

Let $B \subseteq GL_n(\mathbb{C})$ be the Borel subgroup of upper triangular matrices. **Schubert varieties** X_w in $P\backslash GL_n(\mathbb{C})$ are closures of orbits $P\backslash PwB$, where $w \in S_n$ is a Grassmannian permutation with descent at position k. Let Y_w denote the affine subvariety of Y defined to be $\pi|_Y^{-1}(X_w \cap U)$.

Let $\widetilde{Y}$ denote the matrix that has the form given in (7) with variables m_{ij} as the entries of M. Then, the coordinate ring of Y is $\mathbb{C}[Y] = \mathbb{C}[m_{ij} \mid i \in [k], j \in [n-k]]$, and the prime defining ideal J_w of Y_w is generated by the essential minors of $\widetilde{Y}$. That is,

$$J_w = \langle (r_w(i,j)+1) - \text{size minors of } \widetilde{Y}_{[i,j]} \mid (i,j) \in \mathcal{E}ss(w)\rangle. \tag{8}$$

5.2. **A conjecture, counterexample, and correction.** We now consider a conjecture of Kummini-Lakshmibai-Sastry-Seshadri from [9] on regularities of coordinate rings of standard open patches of certain Schubert varieties in Grassmannians. We show that this conjecture is false by providing a counterexample, and then state and prove an alternate explicit combinatorial formula for these regularities. This latter result follows immediately from our Corollary 4.1.

To state the conjecture from [9], we first translate the conventions from their paper to ours. Indeed, we use the same notation as the previous section and assume that $w \in S_n$ is a Grassmannian permutation with unique descent at position k. Suppose that $w = w_1\ w_2 \cdots w_n$ in one-line notation. Observe that w is uniquely determined from n and $(w_1, \dots, w_k)$. Suppose further that for some $r \in [k-1]$,

$$w_{k-r+i} = n-k+i \quad \text{for all } i \in [r] \tag{9}$$

and $w_1 = 1$. Let $\widetilde{w}$ be defined by $(\widetilde{w}_1, \dots, \widetilde{w}_k) = (n-w_k+1, \dots, n-w_1+1)$. Then we have

$$(\widetilde{w}_1, \dots, \widetilde{w}_k) = (k-r+1, k-r+2, \dots, k, a_{r+1}, \dots, a_{n-1}, n)$$

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for some $k < a_{r+1} < \cdots < a_{n-1} < n$. Let $a_r = k$ and $a_k = n$. For $r \leq i \leq k-1$, define $m_i = a_{i+1} - a_i$.

Conjecture 5.1 ([9, Conjecture 7.5])**.**

$$\textit{reg}(\mathbb{C}[Y]/J_w) = \sum_{i=r}^{k-1} (m_i - 1)i. \tag{10}$$

Example 5.2. We consider [9, Example 6.1]. Let J be the ideal generated by 3×3 minors of a 4×3 matrix of indeterminates. Then $J = J_w$ for $w = 1245367 \in S_7$, where $k = 4$ and $n = 7$. Then $\widetilde{w} = (3, 4, 6, 7)$. Here we see that Equation (10) yields a regularity of 2. This matches the regularity we computed in Example 4.3.

We now show that Conjecture 5.1 is not always true.

Example 5.3. Let $k = 4$, $n = 10$, $w = 145723689(10)$ so that $\widetilde{w} = (4, 6, 7, 10)$. Then $\widetilde{w}$ has the desired form. Furthermore, we have that $m_1 = 2, m_2 = 1, m_3 = 3$. Thus, by Conjecture 5.1, the regularity should be $(2-1)1 + (1-1)2 + (3-1)3 = 1 + 6 = 7$. However, a check in Macaulay2 [5] yields a regularity of 5. In fact, J_w, once induced to a larger polynomial ring, is a Schubert determinantal ideal for w, so we can use our formula from Corollary 4.1. Notice w has associated partition $\lambda = (3, 2, 2, 0)$. Then $\lambda^{(1)} = (1)$ and $\lambda^{(2)} = (3, 2, 2)$, giving $\mathrm{reg}(\mathbb{C}[Y]/J_w) = 2 \cdot \mathtt{sv}(\lambda^{(1)}) + 1 \cdot \mathtt{sv}(\lambda^{(2)}) = 2 \cdot 1 + 1 \cdot 3 = 5$.

As illustrated in Example 5.3, our formula for the regularity of a Grassmannian matrix Schubert variety given in Corollary 4.1 corrects Conjecture 5.1 whenever the ideal J_w is equal (up to inducing the ideal to a larger ring) to the Schubert determinantal ideal I_w. In fact, each Grassmannian permutation considered in [9, Conjecture 7.5] is of this form. This follows because all the essential set of such w is contained in $w_{[k,n-k]}$ by Equation (9).

Corollary 5.4. *Let $w_\lambda \in S_n$ be a Grassmannian permutation with descent at position k such that $w_1 = 1$ and for some $r \in [k-1]$, $w_{k-r+i} = n-k+i$ for $i \in [r]$. Let $P(\lambda) = (P_1, \ldots, P_r)$. Then*

$$\mathrm{reg}(\mathbb{C}[Y]/J_{w_\lambda}) = \sum_{h \in [r-1]} |P_{h+1}| \cdot \mathtt{sv}(\lambda^{(h)}).$$

5.3. **A conjecture for the general case.** We end the paper with a conjecture for the regularity of $\mathbb{C}[Y]/J_w$ where w is an *arbitrary* Grassmannian permutation with descent at position k. We begin with some preliminaries.

First note that $\mathbb{C}[Y]/J_w$ is a standard graded ring. Indeed, the torus $T \subseteq GL_n(\mathbb{C})$ of diagonal matrices acts on U and on $X_w \cap U$ by right multiplication. This action induces a $\mathbb{Z}^n$-grading on $\mathbb{C}[Y]$ such that m_{ij} has degree $\vec{e}_i - \vec{e}_j$ and J_w is homogeneous. This $\mathbb{Z}^n$-grading can be coarsened to the standard $\mathbb{Z}$-grading because the T-action contains the dilation action:[1] embed $\mathbb{C}^\times \hookrightarrow T$ by sending $z \in \mathbb{C}^\times$ to the diagonal matrix that has its (i,i)-entry equal to 1 when $1 \leq i \leq n-k$ and equal to z when $n-k+1 \leq i \leq n$.

The codimension of Y_w in Y is equal to the number of boxes in the diagram $D(w)$. So, to compute the regularity $\mathrm{reg}(\mathbb{C}[Y]/J_w)$, it remains to find the degree

[1] More generally, coordinate rings of Kazhdan-Lusztig varieties $X_w \cap X_\circ^v \subseteq B_-\backslash GL_n(\mathbb{C})$ are standard graded when v, the permutation defining the opposite Schubert cell $X_\circ^v = B_-\backslash B_- v B_-$, is 321-avoiding. See [6, pg. 25] or [16, Section 4.1] for further explanation.

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of the K-polynomial of $\mathbb{C}[Y]/J_w$. By [16, Theorem 4.5], this K-polynomial can be expressed in terms of a **double Grothendieck polynomial**, $\mathfrak{G}_w(\mathbf{x};\mathbf{y})$, which is defined as follows:

$$\mathfrak{G}_{w_0}(\mathbf{x};\mathbf{y}) = \prod_{i+j\leq n}(x_i + y_j - x_i y_j).$$

The rest are defined recursively, using the same operator π_i and recurrence defined in Section 1. Note that if $G_w(\mathbf{x};\mathbf{y})$ denotes the double Grothendieck polynomials in [7], we have $G_w(\mathbf{x};\mathbf{y}) = \mathfrak{G}_w(\mathbf{1}-\mathbf{x};\mathbf{1}-\frac{\mathbf{1}}{\mathbf{y}})$.

Let $\mathbf{c} = ((1-t),(1-t),\ldots,(1-t),0,0,\ldots,0)$ be the list consisting of k copies of $1-t$ followed by $n-k$ copies of 0, and let $\tilde{\mathbf{c}} = (0,0,\ldots,0,1-\frac{1}{t},1-\frac{1}{t},\ldots,1-\frac{1}{t})$ be the list consisting of $n-k$ copies of 0 followed by k copies of $1-\frac{1}{t}$. By [16, Theorem 4.5], the K-polynomial of S/J_w is the specialized double Grothendieck polynomial $\mathfrak{G}_w(\mathbf{c};\tilde{\mathbf{c}})$.[2] Consequently, we are reduced to computing the degree of this polynomial.

Example 5.5. Let $w = 132$ and $k = 2$. Then

$$\mathfrak{G}_w(\mathbf{x};\mathbf{y}) = (x_2+y_1-x_2y_1)+(x_1+y_2-x_1y_2)-(x_1+y_2-x_1y_2)(x_2+y_1-x_2y_1).$$

Letting $\mathbf{c} = (1-t,1-t,0)$ and $\widetilde{\mathbf{c}} = (0,1-\frac{1}{t},1-\frac{1}{t})$, one checks that $\mathfrak{G}_w(\mathbf{c};\widetilde{\mathbf{c}}) = (1-t)$ which is the K-polynomial of S/J_w with respect to the standard grading.

For the reader familiar with pipe dreams (see, e.g. [3] and [8]), we note that the degree of $\mathfrak{G}_w(\mathbf{c};\tilde{\mathbf{c}})$ is the maximum number of plus tiles in a (possibly non-reduced) pipe dream for w with all of its plus tiles supported within the northwest justified $k\times(n-k)$ subgrid of the $n\times n$ grid. This follows from [16]. However, this is not a very explicit formula for degree.

We now turn to our conjecture. It asserts that the degree of the K-polynomial of $\mathbb{C}[Y]/J_w$ for a Grassmannian permutation $w\in S_n$ with descent at position k can be computed in terms of the degree of a Grothendieck polynomial of an associated *vexillary* permutation. This will be a much more easily computable answer than a pipe dream formula because the first, third, and fifth authors will give an explicit formula for degrees of vexillary Grothendieck polynomials in the sequel.

A permutation $w\in S_n$ is **vexillary** if it contains no 2143-pattern, i.e. there are no $i<j<k<l$ such that $w_j<w_i<w_l<w_k$. For example, $w = \underline{3}25\underline{164}$ is not vexillary since it contains the underlined 2143 pattern.

Suppose $w_\lambda \in S_n$ is Grassmannian with descent k. Define $\lambda' = (\lambda_1,\ldots,\lambda_{\ell(\lambda)})$ and $\phi(\lambda) = (\phi_1,\ldots,\phi_{\ell(\lambda)})$ as follows. For $i\in[\ell(\lambda)]$,

$$\phi_i = \begin{cases} i+\min\{(n-k)-\lambda_i, k-i\} & \text{if } \lambda_i>\lambda_{i+1} \text{ or } i=\ell(\lambda),\\ \phi_{i+1} & \text{otherwise.}\end{cases}$$

A vexillary permutation v is determined by the statistics of a partition and a flag, computed using $D(v)$ (see [13, Proposition 2.2.10]). Thus, the partition λ' and flag ϕ defined above from w_λ define at most one vexillary permutation.

Conjecture 5.6. *Fix $w_\lambda\in S_n$ Grassmannian with descent k. Then $\lambda',\phi(\lambda)$ define a vexillary permutation v, and $\deg(\mathfrak{G}_{w_\lambda}(\mathbf{c};\tilde{\mathbf{c}})) = \deg(\mathfrak{G}_v(\mathbf{x}))$. In particular, reg$(\mathbb{C}[Y]/J_{w_\lambda}) = \deg(\mathfrak{G}_v(\mathbf{x}))-|\lambda|$.*

[2]The conventions used in [16] differ from ours, so the given formula is a translation of their formula to our conventions.

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Example 5.7. Let $k = 5, n = 10$ and $w_\lambda = 1489(10)23567$. Then $\lambda' = (5,5,5,2)$ and $\phi(w_\lambda) = (3,3,3,5)$, which corresponds to the vexillary permutation $v = 678142359(10)$. Thus Conjecture 5.6 states that $\deg(\mathfrak{G}_{w_\lambda}(\mathbf{c};\tilde{\mathbf{c}})) = \deg(\mathfrak{G}_v(\mathbf{x})) = 18$, so $\mathrm{reg}(\mathbb{C}[Y]/J_{w_\lambda}) = 18 - 17 = 1$.

To compute this regularity directly, take $R = \mathbb{C}[Y] = \mathbb{C}[m_{ij} \mid 1 \leq i,j \leq 5]$. Let G denote the set of 2×2 minors of $\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix}$, and let H be the set of entries in the bottom three rows of the matrix of variables $M = (m_{ij})_{1 \leq i,j \leq 5}$. Then $G \cup H$ is a minimal generating set of J_{w_λ}. The Eagon-Northcott complex is a minimal free resolution of $R/\langle G \rangle$:

$$0 \to R(-3)^2 \to R(-2)^3 \to R \to R/\langle G \rangle \to 0.$$

From this, one directly observes that the regularity of the R-module $R/\langle G \rangle$ is 1. Modding out $R/\langle G \rangle$ by the linear forms in H does not change the regularity (see, e.g. [2, Proposition 20.20]), and hence the regularity of R/J_{w_λ} is also 1.

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