PRISM TABLEAUX FOR ALTERNATING SIGN MATRIX VARIETIES

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ABSTRACT. A prism tableau is a set of reverse semistandard tableaux, each positioned within an ambient grid. Prism tableaux were introduced to provide a formula for the Schubert polynomials of A. Lascoux and M.P. Schützenberger. This formula directly generalizes the well known expression for Schur polynomials as a sum over semistandard tableaux. Alternating sign matrix varieties generalize the matrix Schubert varieties of W. Fulton. We use prism tableaux to give a formula for the multidegree of an alternating sign matrix variety.

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Date: October 16, 2018.

1. Introduction

An **alternating sign matrix** (ASM) is a square matrix with entries in $\{-1, 0, 1\}$ so that

(A1) the nonzero entries in each row and column alternate in sign and
(A2) each row and column sums to 1.

Let $\mathsf{ASM}(n)$ be the set of all $n \times n$ ASMs. The enumeration of ASMs has drawn much interest, the sequence for $n \geq 1$ being

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \ldots.$$

There is a closed form expression for this sequence; the celebrated *alternating sign matrix conjecture* of W. H. Mills–D. P. Robbins–H. Rumsey [MRR83] asserts that

$$|\mathsf{ASM}(n)| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

The original proof was given by D. Zeilberger [Zei96]. A second proof was given by G. Kuperberg [Kup96] using the six-vertex model of statistical mechanics. See *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*, by D. Bressoud, for the link between ASMs and hypergeometric series, plane partitions, and lattice paths [Bre99].

Each $A = (a_{ij})_{i,j=1}^n \in \mathsf{ASM}(n)$ has an associated **corner sum function**

$$r_A(i, j) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} a_{k\ell}. \tag{1}$$

Corner sum functions define a lattice structure on $\mathsf{ASM}(n)$; say

(2) $A \leq B$ if and only if $r_A(i, j) \geq r_B(i, j)$ for all $1 \leq i, j \leq n$.

Restricted to permutation matrices, (2) is the **Bruhat order** on the symmetric group $\mathcal{S}_n$. A. Lascoux and M.P. Schützenberger showed that $\mathsf{ASM}(n)$ is the smallest lattice which contains $\mathcal{S}_n$ as an order embedding [LS96].

A **partition** is a weakly decreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_h).$$

The **length** of λ is $\ell(\lambda) := |\{i : \lambda_i \neq 0\}|$. Fix tuples of partitions and positive integers

(3) $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ and $\mathbf{d} = (d_1, \ldots, d_k)$ so that $d_i \geq \ell(\lambda^{(i)})$ for all i.

We associate to each $(\boldsymbol{\lambda}, \mathbf{d})$ an ASM, denoted $A_{\boldsymbol{\lambda},\mathbf{d}}$, which is the least upper bound of a list of *Grassmannian* permutations. Conversely, for any ASM, there exists some $(\boldsymbol{\lambda}, \mathbf{d})$ so that $A = A_{\boldsymbol{\lambda},\mathbf{d}}$.

Prism tableaux were first defined in [WY15]. We give a more general definition here. A *prism tableau* for $(\boldsymbol{\lambda}, \mathbf{d})$ is a k-tuple of *reverse semistandard tableaux*, with shapes and labels determined by the pair $(\boldsymbol{\lambda}, \mathbf{d})$. We write $\mathtt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ for the set of *minimal prism tableaux* for $(\boldsymbol{\lambda}, \mathbf{d})$ which have no *unstable triples*. These terms are defined in Section 2.3. Each prism tableau has an associated weight monomial $\mathtt{wt}(\mathcal{T})$. Let

$$\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \sum_{\mathcal{T} \in \mathtt{Prism}(\boldsymbol{\lambda},\mathbf{d})} \mathtt{wt}(\mathcal{T}). \tag{4}$$

Call $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ an **ASM polynomial**.

If $\boldsymbol{\lambda} = (\lambda)$ and $\mathbf{d} = (d)$, the polynomial $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ is the **Schur polynomial** $s_\lambda(x_1, \ldots, x_d)$. This follows immediately from the usual definition of s_λ as a weighed sum over *semistandard tableaux*. The *Schubert polynomials* $\{\mathfrak{S}_w : w \in \mathcal{S}_\infty\}$ of A. Lascoux and M.P. Schützenberger [LS82] generalize Schur polynomials. The purpose of [WY15] was to provide a prism formula for Schubert polynomials. We prove the following generalization.

Theorem 1.1. $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \displaystyle\sum_{w \in \texttt{MinPerm}(A_{\boldsymbol{\lambda},\mathbf{d}})} \mathfrak{S}_w.$

Here, $\texttt{MinPerm}(A)$ denotes the set permutations above A in $\mathsf{ASM}(n)$ which have the minimum possible length. Our proof of Theorem 1.1 is purely combinatorial; we give a bijection between $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ and the set of facets of the *subword complexes* ([KM04]) for each $w \in \texttt{MinPerm}(A)$. The Schubert polynomial is a weighted sum over the facets of its corresponding subword complex [FK96, BB93, KM05].

In Section 4.3, we define a map from the set of all prism tableaux to a simplicial complex $\Delta(Q_{n\times n}, A)$, which is itself a union subword complexes. Restricted to $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$, this map is a bijection onto the set of maximal dimensional facets in $\Delta(Q_{n\times n}, A)$ (see Theorem 4.16).

$\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ also has a geometric interpretation; it is the *multidegree* of an *alternating sign matrix variety*. Write $\mathsf{Mat}(n)$ for the space of $n \times n$ matrices over an algebraically closed field $\Bbbk$. Given $M \in \mathsf{Mat}(n)$, let $M_{[i],[j]}$ be the submatrix of M which consists of the first i rows and j columns of M. We define the **alternating sign matrix variety**

$$X_A := \{M \in \mathsf{Mat}(n) : \operatorname{rank}(M_{[i],[j]}) \leq r_A(i,j) \text{ for all } 1 \leq i,j \leq n\}. \tag{5}$$

If $w \in \mathcal{S}_n$, then X_w is a **matrix Schubert variety** as defined in [Ful92].

ASM varieties are stable under multiplication by the group of invertible, diagonal matrices $\mathsf{T} \subset \mathsf{GL(n)}$. There is a corresponding $\mathbb{Z}^n$ grading and multidegree

$$\mathcal{C}(X_A; \mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_n].$$

Whenever $w \in \mathcal{S}_n$, we have $\mathfrak{S}_w = \mathcal{C}(X_w; \mathbf{x})$. This was shown in [KM05] and is equivalent to earlier statements in the language of equivariant cohomology [FR03] and degeneracy loci [Ful92]. We show $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ is the multidegree of the ASM variety $X_{A_{\boldsymbol{\lambda},\mathbf{d}}}$.

Theorem 1.2. *Fix $\boldsymbol{\lambda}$ and $\mathbf{d}$ as in (3). Then*

$$\mathcal{C}(X_{A_{\boldsymbol{\lambda},\mathbf{d}}}; \mathbf{x}) = \mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}.$$

The irreducible components of X_A are always matrix Schubert varieties. Theorem 1.2 follows from Theorem 1.1 and the additivity of multidegrees.

We also discuss the explicit connection of prism tableaux to Gröbner geometry of X_A. Let $Z = (z_{ij})_{i,j=1}^n$ be the generic $n \times n$ matrix. Define the **ASM ideal** by

$$I_A := \langle \text{ minors of size } r_A(i,j) + 1 \text{ in } Z_{[i],[j]} \rangle. \tag{6}$$

It is immediate that I_A provides set-theoretic equations for X_A. For any $A \in \mathsf{ASM}(n)$, we I_A is radical. This follows from the Frobenius splitting argument given in [Knu09, Section 7.2]. We make the connection to ASM varieties explicit.

Proposition 1.3 ([Knu09]). *Fix any antidiagonal term order $\prec$ on $\Bbbk[Z]$.*

(1) The essential (and hence defining) generators of I_A form a Gröbner basis under $\prec$.

(2) I_A is radical and its initial ideal is a square-free monomial ideal.
(3) The Stanley-Reisner complex of $\texttt{init}(X_A)$ *is* $\Delta(Q_{n\times n}, A)$.

Since $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ is in weight preserving bijection with the facets of maximum dimension in $\Delta(Q_{n\times n}, A)$, this yields a second proof of Theorem 1.2.

2. Prism tableaux and ASMs

2.1. **Rothe diagrams for ASMs.** We start by presenting a generalization of Rothe diagrams to ASMs. Following [MRR83], say $A = (a_{ij})_{i,j=1}^n \in \mathsf{ASM}(n)$ has an **inversion** in position (i, j) if

$$\sum_{(k,l):i<k \text{ and } j<l} a_{il}a_{kj} = 1. \tag{7}$$

Write $[n] := \{1, \dots, n\}$. Then

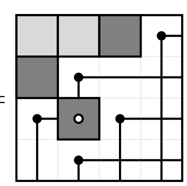
$$D(A) := \{(i, j) : (i, j) \text{ is an inversion of } A\} \subset n \times n \tag{8}$$

is the **Rothe diagram** of A. We represent $D(A)$ graphically. Our convention is to visually indicate the ASM by placing a black dot for each 1 in A and a white dot for each -1. The **essential set** $\mathcal{E}ss(A)$ consists of the southeast most corners of each connected component of $D(A)$,

$$\mathcal{E}ss(A) := \{(i, j) \in D(A) : (i + 1, j), (i, j + 1) \notin D(A)\}.$$

Example 2.1.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad D(A) =$$



The boxes of the diagram of A are shaded gray. The essential boxes are dark gray. □

The sum in (7) factorizes

$$\sum_{(k,l):i<k \text{ and } j<l} a_{il}a_{kj} = \left(\sum_{k=i+1}^{n} a_{kj}\right)\left(\sum_{l=j+1}^{n} a_{il}\right) = \left(1 - \sum_{k=1}^{i} a_{kj}\right)\left(1 - \sum_{l=1}^{j} a_{il}\right). \tag{9}$$

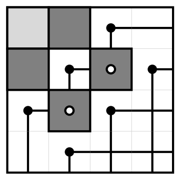
See [BMH95]. By conditions (A1) and (A2) the factors in RHS of (9) product are always 0 or 1. In order for (i, j) to be an inversion, both must be 1. Visually, this amounts to striking out hooks to the right and below each black dot which stop just before they encounter a box which contains a white dot. The boxes which remain are the elements of $D(A)$.

Notice that $D(A)$ is similar to the ASM diagram defined by A. Lascoux [Las08]. However, our conventions on inversions differ; we include the set of *negative inversions* in our diagram. If w is a permutation matrix, $D(w)$ and $\mathcal{E}ss(w)$ coincide with the usual Rothe diagram and essential set, as defined in [Ful92]. Any permutation is uniquely determined by the restriction of the corner sum function to its essential set [Ful92, Lemma 3.10]. The same statement holds more generally for ASMs, see Proposition 3.11.

Given $w \in \mathcal{S}_n$, the permutation matrix of w is an $n \times n$ matrix with a one in each of the $(i, w(i))$ positions and zeros elsewhere. This defines an embedding of $\mathcal{S}_n \hookrightarrow \mathsf{ASM}(n)$. We freely identify each permutation with its permutation matrix. The length $\ell(w)$ of $w \in \mathcal{S}_n$ is the number of inversions, or equivalently $\ell(w) = |D(w)|$. Say

$$\texttt{deg}(A) = \min\{\ell(w) : w \in \mathcal{S}_n \text{ and } w \geq A\}. \tag{10}$$

Example 2.2. In general $\texttt{deg}(A) \neq |D(A)|$. For example, suppose A is the ASM whose diagram is pictured below.

$D(A) =$ 

$$r_A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad r_{3412} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Since $r_{3412} < r_A$ we have $3412 \geq A$. Therefore

$$\texttt{deg}(A) \leq \ell(3412) = 4 < |D(A)|.$$

By checking all $w \geq A$, the reader may verify $\texttt{deg}(A) = 4$. □

2.2. **Grassmannian and biGrassmannian permutations.** We recall standard facts on permutations. See [Man01] as a reference. A **descent** of $w \in \mathcal{S}_n$ is a position i so that $w(i) > w(i+1)$. A permutation is **Grassmannian** if it has a unique descent. Let $\mathcal{G}_n$ denote the set of Grassmannian permutations in $\mathcal{S}_n$. If $u \in \mathcal{G}_n$, write $\texttt{des}(u)$ for the position of its descent.

Let $u \in \mathcal{G}_n$. Let

$$\lambda_i^{(u)} = u(\texttt{des}(u) - i + 1) - (\texttt{des}(u) - i + 1).$$

Equivalently, $\lambda_i^{(u)}$ is the number of boxes in row $\texttt{des}(u) - i + 1$ of $D(u)$. Since u has a unique descent at $\texttt{des}(u)$, we have $\lambda_i^{(u)} \geq \lambda_{i+1}^{(u)}$ for all $i = 1, \ldots, \texttt{des}(u) - 1$. Then

$$\lambda^{(u)} = (\lambda_1^{(u)}, \lambda_2^{(u)}, \ldots, \lambda_{\texttt{des}(u)}^{(u)})$$

is a partition with $\ell(\lambda^{(u)}) \leq \texttt{des}(u)$ and $\lambda_1^{(u)} \leq n - \texttt{des}(u)$.

Write $a \times b$ for the partition whose Young diagram has a rows of length b.

Lemma 2.3. *The map $u \mapsto (\lambda^{(u)}, \texttt{des}(u))$ defines a bijection between $\mathcal{G}_n$ and pairs (λ, d) with*

$$\lambda \subseteq d \times (n - d)$$

i.e. partitions with $\lambda_1 \leq n - d$ and $\ell(\lambda) \leq d$.

Let $[\lambda, d]_g$ be the Grassmannian with $\lambda^{([\lambda,d]_g)} = \lambda$ and $\texttt{des}([\lambda, d]_g) = d$. If $\lambda = (0)$ is the empty partition, then let $[\lambda, d]_g = \mathrm{id}$. Let $(\boldsymbol{\lambda}, \mathbf{d})$ be as in (3). We can always choose n large enough so that $\lambda^{(i)} \subseteq d_i \times (n - d_i)$ for all $i = 1, \ldots, k$. Then let

$$\mathbf{u}_{\boldsymbol{\lambda},\mathbf{d}} := ([\lambda^{(1)}, d_1]_g, \ldots, [\lambda^{(k)}, d_k]_g) \tag{11}$$

and

$$A_{\boldsymbol{\lambda},\mathbf{d}} := \vee \mathbf{u}_{\boldsymbol{\lambda},\mathbf{d}} \in \mathsf{ASM}(n). \tag{12}$$

A permutation is **biGrassmannian** if both it, and its inverse, are Grassmannian. Write $\mathcal{B}_n$ for the set of biGrassmannian permutations in $\mathcal{S}_n$. A permutation is biGrassmannian if and only if its diagram is a rectangle. Elements of $\mathcal{B}_n$ are naturally labeled by triples of integers (i, j, r) which satisfy the following conditions:

(B1) $1 \leq i, j$
(B2) $0 \leq r < \min(i, j)$
(B3) $i + j - r \leq n$.

Let I_k denote the $k \times k$ identity matrix. Then we write

$$[i, j, r]_b := \left(\begin{array}{c|c|c|c} I_r & & & \\ \hline & & I_{i-r} & \\ \hline & I_{j-r} & & \\ \hline & & & I_{n-i-j+r} \end{array}\right) \tag{13}$$

for the (unique) biGrassmannian encoded by this triple. In the case $r = \min(i, j)$, let $[i, j, r]_b$ be the identity permutation.

There are multiple labeling conventions for biGrassmannians in the literature (see e.g. [LS96], [Rea02], [Kob13]). We have chosen ours so the following properties hold.

Lemma 2.4. *Let $u = [i, j, r]_b \in \mathcal{B}_n$. Then*

(1) $\mathcal{E}ss(u) = \{(i, j)\}$
(2) $r_u(i, j) = r$.
(3) $\mathtt{des}(u) = i$
(4) $\lambda^{(u)} = (i - r_u(i, j)) \times (j - r_u(i, j))$.

Lemma 2.4 is immediate from (13).

2.3. **Prism tableaux.** Each partition λ has an associated **Young diagram** which consists of left justified boxes with λ_1 boxes in the bottom row, λ_2 in the next, and so on. We will freely identify λ with its Young diagram. A **reverse semistandard tableau** is a filling of λ with positive integers so that labels

(T1) weakly decrease within rows (from left to right) and
(T2) strictly decrease (from bottom to top) within columns.

We write $\mathtt{RSSYT}(\lambda, d)$ for the set of reverse semistandard fillings of λ which use labels from the set $[d]$.

Example 2.5. Let $\lambda = (4, 4, 2, 1)$ and $d = 7$.

1			
3	1		
4	4	2	2
6	6	4	3

The tableau pictured above is an element of $\mathtt{RSSYT}(\lambda, 7)$. □

We define

$$\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) = \mathtt{RSSYT}(\lambda^{(1)}, d_1) \times \ldots \times \mathtt{RSSYT}(\lambda^{(k)}, d_k).$$

An element of $\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is called a **prism tableau**.

For the discussion which follows, it is not enough to merely think of a prism tableau as a tuple of reverse semistandard tableaux. Rather, we think of each of the component tableaux as having a position in the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ grid. We use matrix coordinates to refer boxes in the grid; (i,j) indicates the box in the ith row (from the top) and jth column (from the left) of the grid. An **antidiagonal** of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ consists of the boxes

$$\{(i,1),(i-1,2),\ldots,(1,i)\}.$$

We identify the shape of each $\lambda^{(i)}$ with

$$\lambda^{(i)} = \{(a,b) : b \leq \lambda^{(i)}_{d_i-a+1}\} \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}. \tag{14}$$

The **prism shape** for $(\boldsymbol{\lambda}, \mathbf{d})$ is obtained by overlaying the $\lambda^{(i)}$'s:

$$\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d}) := \bigcup_{i=1}^{k} \{(a,b) : b \leq \lambda^{(i)}_{d_i-a+1}\}. \tag{15}$$

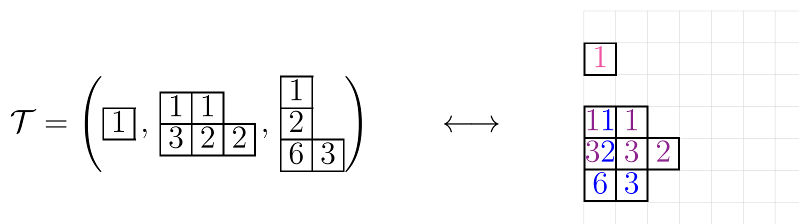
From this perspective, a prism tableau for $(\boldsymbol{\lambda}, \mathbf{d})$ is a filling of $\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d})$ which assigns a label of color i from the set $\{1,2,\ldots,d_i\}$ to each $(a,b) \in \lambda^{(i)}$ so that labels of color i weakly decrease along rows from left to right and strictly decrease along columns from bottom to top. Such fillings are in immediate bijection with $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$. As such, we freely identify these two representations of a prism tableau.

Weight $\mathcal{T}$ as follows:

$$\texttt{wt}(\mathcal{T}) = \prod_{i=1}^{\infty} x_i^{n_i}$$

where n_i is the number of antidiagonals which contain the label i (in any color).

Example 2.6. Let $\boldsymbol{\lambda} = ((1),(3,2),(2,1,1))$ and $\mathbf{d} = (2,5,6)$. Below, we give an example of $\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$.



The corresponding weight monomial is $\texttt{wt}(\mathcal{T}) = x_1^3 x_2^2 x_3^3 x_6$. $\square$

Let

$$\texttt{deg}(\boldsymbol{\lambda}, \mathbf{d}) = \min\{\texttt{deg}(\texttt{wt}(\mathcal{T})) : \mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})\}. \tag{16}$$

$\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is **minimal** if $\texttt{deg}(\texttt{wt}(\mathcal{T})) = \texttt{deg}(\boldsymbol{\lambda}, \mathbf{d})$. Let ℓ_c be a label ℓ of color c. Labels $\{\ell_c, \ell_d, \ell'_e\}$ in the same antidiagonal form an **unstable triple** if $\ell < \ell'$ and replacing the ℓ_c with ℓ'_c gives a prism tableau. Write

$$\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d}) = \{\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) : \mathcal{T} \text{ is minimal and has no unstable triples}\}. \tag{17}$$

We now describe two ways of taking an ASM as a input and producing a pair $(\boldsymbol{\lambda}, \mathbf{d})$ so that $A = A_{\boldsymbol{\lambda},\mathbf{d}}$. Both procedures are entirely combinatorial. We start with BiGrassmannian prism tableaux, which were defined in [WY15].

Definition 2.7 (BiGrassmannian Prism Tableaux). Suppose

$$\mathcal{E}ss(A) = \{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\}.$$

Let

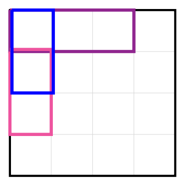
$$\beta^{(\ell)} = (i_\ell - r_A(i_\ell, j_\ell)) \times (j_\ell - r_A(i_\ell, j_\ell)). \tag{18}$$

Define $\boldsymbol{\beta}_A = (\beta^{(1)}, \ldots, \beta^{(k)})$ and $\mathbf{b}_A = \{i_1, \ldots, i_k\}$. The **biGrassmannian prism shape** is $\mathbb{S}_B(A) := \mathbb{S}(\boldsymbol{\beta}_A, \mathbf{b}_A)$. Write $\texttt{Prism}_B(A) := \texttt{Prism}(\boldsymbol{\beta}_A, \mathbf{b}_A)$.

Example 2.8. Let A be as in Example 2.1. Then $\mathcal{E}ss(A) = \{(1,3), (2,1), (3,2)\}$.

(i_ℓ, j_ℓ)	$r_A(i_\ell, j_\ell)$	$\beta^{(\ell)}$
$(1,3)$	0	1×3
$(2,1)$	0	2×1
$(3,2)$	1	2×1

Using the table above, we construct the shape $\mathbb{S}_B(A)$.



There are only three prism fillings of $\mathbb{S}_B(A)$.

$$\mathcal{T}_1 = \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 22 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \mathcal{T}_2 = \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 21 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \mathcal{T}_3 = \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 21 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

The corresponding weight monomials are $\texttt{wt}(\mathcal{T}_1) = x_1^3 x_2 x_3$, $\texttt{wt}(\mathcal{T}_2) = x_1^3 x_2 x_3$, and $\texttt{wt}(\mathcal{T}_3) = x_1^3 x_2^2$. These all have the same degree, and so each tableaux is minimal. $\mathcal{T}_1$ can be obtained from $\mathcal{T}_2$ by replacing the pink 1 with a 2. So $\mathcal{T}_2$ has an unstable triple. So we conclude

$$\texttt{Prism}_B(A) = \{T_1, T_3\}.$$

Then $\mathfrak{A}_{\boldsymbol{\beta}_A, \mathbf{b}_A} = x_1^3 x_2 x_3 + x_1^3 x_2^2$. □

We now introduce the parabolic prism model. Our definition uses the *monotone triangles* of W. H. Mills, D. P. Robbins, and H. Rumsey [MRR83]. Given

$$A = (a_{ij})_{i,j=1}^n \in \mathsf{ASM}(n)$$

let C_A be the matrix of partial column sums, i.e. $C_A(i,j) = \sum_{\ell=1}^{i} a_{\ell j}$. The ith row of m_A records (in increasing order) the positions of the 1s in the ith row of C_A. The array m_A is called a **monotone triangle**.

Example 2.9.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad C_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad m_A = \begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

□

There is explicit dictionary between monotone triangles and corner sum matrices. Entry (i,j) of m_A indicates the position of the jth ascent in row i of r_A, i.e,

(19) $$m_A(i,j) = a \text{ if and only if } r_A(i,a-1) = j-1 \text{ and } r_A(i,a) = j.$$

Given A and $1 \leq \ell \leq n$, we define

(20) $$\lambda^{(A,\ell)} = (m_A(\ell,\ell) - \ell, m_A(\ell,\ell-1) - (\ell-1), \ldots, m_A(\ell,1) - 1).$$

Since m_A strictly increases along rows, $\lambda^{(A,\ell)}$ is a partition. By construction,

$$\lambda^{(A,\ell)} \subseteq \ell \times (n-\ell).$$

Notice if $u \in \mathcal{G}_n$, then $\lambda^{(u,\mathtt{des}(u))} = \lambda^{(u)}$. If $w \in \mathcal{S}_n$, then $[\lambda^{(A,\ell)}, \ell]_g$ is the minimal length coset representative for w in the maximal parabolic subgroup generated by removing $(\ell, \ell+1)$ from the list of simple transpositions in $\mathcal{S}_n$.

Definition 2.10 (Parabolic Prism Tableaux). Write

$$\{i : (i,j) \in \mathcal{E}ss(A)\} = \{i_1, \ldots, i_k\}$$

for the indices of essential rows of A. Let

$$\boldsymbol{\rho}_A = (\lambda^{(A,i_1)}, \lambda^{(A,i_2)}, \ldots, \lambda^{(A,i_k)}) \text{ and } \mathbf{p}_A = (i_1, \ldots, i_k).$$

Then define the **parabolic prism shape**

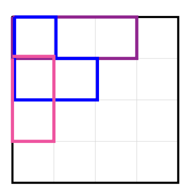
$$\mathbb{S}_P(A) = \mathbb{S}(\boldsymbol{\rho}_A, \mathbf{p}_A).$$

We abbreviate $\mathtt{Prism}_P(A) := \mathtt{Prism}(\boldsymbol{\rho}_A, \mathbf{p}_A)$.

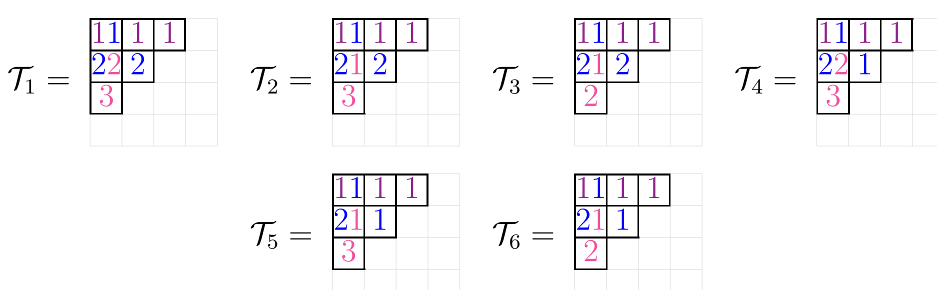
Example 2.11. Let A be as in Example 2.1. Then

$$m_A = \begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}.$$

The essential rows are $\mathbf{p}_A = (1,2,3)$ and $\boldsymbol{\rho}_A = ((3),(2,1),(1,1))$.



We list the possible prism fillings of $(\boldsymbol{\rho}_A, \mathbf{p}_A)$.



i	1	2	3	4	5	6
$\mathtt{wt}(\mathcal{T}_i)$	$x_1^3x_2^2x_3$	$x_1^3x_2^2x_3$	$x_1^3x_2^2$	$x_1^3x_2x_3$	$x_1^3x_2x_3$	$x_1^3x_2^2$
minimal	no	no	yes	yes	yes	yes

Among the minimal tableaux, $\mathcal{T}_4$ is obtained by replacing the unstable triple in $\mathcal{T}_5$. Likewise, replacing the unstable triple $\mathcal{T}_6$ produces $\mathcal{T}_3$. So $\mathtt{Prism}_P(A) = \{\mathcal{T}_3, \mathcal{T}_4\}$. Then

$$\mathfrak{A}_{\boldsymbol{\rho}_A, \mathbf{p}_A} = x_1^3 x_2^2 + x_1^3 x_2 x_3.$$

Notice that $\mathfrak{A}_{\boldsymbol{\beta}_A, \mathbf{b}_A} = \mathfrak{A}_{\boldsymbol{\rho}_A, \mathbf{p}_A}$. This holds in general as a consequence of Theorem 1.2 and the next proposition. $\square$

Proposition 2.12. *(1)* $A = A_{\boldsymbol{\beta}_A, \mathbf{b}_A}$.
(2) $A = A_{\boldsymbol{\rho}_A, \mathbf{p}_A}$.

We will postpone the proof to Section 3.3.

3. The Lattice of ASMs

3.1. **Preliminaries on posets and lattices.** We follow [LS96] and [Rea02] as references. A **partially ordered set** (or poset) is a set $\mathcal{P}$ equipped with a binary relation $\leq$ which satisfies the axioms of *reflexivity*, *antisymmetry*, and *transitivity*. If $a \leq b$ and $a \neq b$ we write $a < b$. Given $a, b \in \mathcal{P}$ we say b **covers** a if $a < b$ and whenever $a \leq c \leq b$, we have $c = a$ or $c = b$.

An element $a \in \mathcal{P}$ is **minimal** in $\mathcal{P}$ if whenever $b \in \mathcal{P}$ so that $b \leq a$ we have $a = b$. Similarly, $a \in \mathcal{P}$ is **maximal** in $\mathcal{P}$ if whenever $b \in \mathcal{P}$ so that $b \geq a$ we have $a = b$. Write $\mathtt{MIN}(\mathcal{P})$ for the set of minimal elements in $\mathcal{P}$ and $\mathtt{MAX}(\mathcal{P})$ for the maximal elements.

The **join** of $\mathcal{S} \subseteq \mathcal{P}$ (when it exists) is the least upper bound of $\mathcal{S}$. Similarly, the **meet** is the greatest lower bound. The join and meet are denoted $\vee$ and $\wedge$ respectively.

An element $a \in \mathcal{P}$ is **basic** if $a \neq \vee\mathcal{S}$ whenever $a \notin \mathcal{S}$. The set of basic elements in $\mathcal{P}$ is called the **base** of $\mathcal{P}$. Let $\mathbb{P}(\mathcal{S})$ denote the **power set** of $\mathcal{S}$, that is the set of all subsets of $\mathcal{S}$. $\mathbb{P}(\mathcal{S})$ has the natural structure of a poset by inclusion of sets. Given any subset $\mathcal{C} \subseteq \mathcal{P}$, define $\pi_{\mathcal{C}} : \mathcal{P} \to \mathbb{P}(\mathcal{C})$ by $\pi_{\mathcal{C}}(a) = \{c \in \mathcal{C} : c \leq a\}$. The base is characterized by the following property.

Proposition 3.1 ([LS96, Proposition 2.4])**.** *Let $\mathcal{B}$ be the base of a finite poset $\mathcal{P}$. The projection $\pi_{\mathcal{B}}$ is an order isomorphism onto its image. Furthermore, if any $\mathcal{C} \subseteq \mathcal{P}$ has this property, then $\mathcal{B} \subseteq \mathcal{C}$.*

As a consequence, any element $a \in \mathcal{P}$ is uniquely encoded by the set $\pi_{\mathcal{B}}(a)$. Furthermore, $a = \vee\pi_{\mathcal{B}}(a)$ (see [Rea02, Proposition 9]). In particular, $a = \vee\mathtt{MAX}(\pi_{\mathcal{B}}(a))$.

A **lattice** $\mathcal{L}$ is a poset in which every pair of elements has a join and a meet. Basic elements in a lattice are also known as **join-irreducibles** and have the characterization that they cover a unique element. A **sublattice** of $\mathcal{L}$ is a subset $\mathcal{L}' \subseteq \mathcal{L}$ which is itself a lattice and has the *same* operations of join and meet as $\mathcal{L}$.

Assume $\mathcal{S}$ is a totally ordered set and I some indexing set. Let

$$\mathcal{S}^I := \{(a_i)_{i \in I} : a_i \in \mathcal{S} \text{ for all } i \in I\}.$$

There is a natural partial order on $\mathcal{S}^I$ by entrywise comparison. Explicitly, if $\mathbf{a} = (a_i)_{i \in I}$ and $\mathbf{b} = (b_i)_{i \in I}$ in $\mathcal{S}^I$, then

$$\mathbf{a} \leq \mathbf{b} \text{ if and only if } a_i \leq b_i \text{ for all } i \in I.$$

Write

$$\mathbf{max}(\mathbf{a}, \mathbf{b}) := (\max(a_i, b_i))_{i \in I} \quad \text{and} \quad \mathbf{min}(\mathbf{a}, \mathbf{b}) := (\min(a_i, b_i))_{i \in I}.$$

Lemma 3.2. *(1) $\mathcal{S}^I$ is a lattice with $\mathbf{a} \vee \mathbf{b} = \mathbf{max}(\mathbf{a}, \mathbf{b})$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{min}(\mathbf{a}, \mathbf{b})$.*
(2) If a subset of $\mathcal{S}^I$ is closed under joins and meets, then it is a sublattice of $\mathcal{S}^I$ (and hence is itself a lattice).

Since $\mathcal{S}^I$ is a cartesian product of lattices, (1) is immediate. Likewise, (2) follows from the definition of a sublattice.

A lattice is **complete** if every subset has a join and meet. Any finite lattice is automatically complete. The **Dedekind-MacNeille** completion of $\mathcal{P}$ is the smallest complete lattice which contains $\mathcal{P}$ as an order embedding. Any finite poset has the same base as its Dedekind-MacNeille completion [Rea02, Proposition 28].

3.2. **The Dedekind-MacNeille completion of the symmetric group.** Write

$$\mathsf{R}(n) := \{r_A : A \in \mathsf{ASM}(n)\}.$$

For convenience, define $r_A(i,j) = 0$ whenever $i = 0$ or $j = 0$. Then

$$a_{ij} = r_A(i,j) - r_A(i,j-1) - r_A(i-1,j) + r_A(i-1,j-1) \tag{21}$$

recovers the (i,j) entry of A [RR86]. As such, the map $A \mapsto r_A$ defines a bijection between $\mathsf{ASM}(n)$ and $\mathsf{R}(n)$. The following lemma characterizes corner sums of ASMs.

Lemma 3.3 ([RR86, Lemma 1])**.** *Let A be an $n \times n$ matrix. Then $A \in \mathsf{ASM}(n)$ if and only if*

(R1) $r_A(i,n) = r_A(n,i) = i$ for all $i = 1, \ldots, n$.
(R2) $r_A(i,j) - r_A(i-1,j)$ and $r_A(i,j) - r_A(i,j-1) \in \{0,1\}$ for all $1 \leq i, j \leq n$.

Lemma 3.4. *$\mathsf{R}(n)$ is a distributive lattice with join and meet given by $r_A \vee r_B = \mathbf{max}(r_A, r_B)$ and $r_A \wedge r_B = \mathbf{min}(r_A, r_B)$, respectively.*

Lemma 3.4 follows from Lemma 3.2 by verifying that (R1) and (R2) are preserved under taking minimums and maximums. The lattice of ASMs was initially studied by N. Elkies, G. Kuperberg, M. Larsen, and J. Propp [EKLP92]. The definition in *ibid* is in terms of height functions, which are in obvious order reversing bijection with corner sum matrices. The order on $\mathsf{ASM}(n)$ can also be defined using monotone triangles; this perspective was used in [LS96].

Lemma 3.5 ([LS96, Lemma 5.4])**.** *The Dedekind-MacNeille completion of $\mathcal{S}_n$ is isomorphic to $\mathsf{ASM}(n)$. The base of the $\mathcal{S}_n$, and hence $\mathsf{ASM}(n)$, is $\mathcal{B}_n$.*

In [LS96], A. Lascoux and M. P. Schützenberger also give the base for type B Coxeter groups. M. Geck and S. Kim determined the base for all finite Coxeter groups [GK97].

Let

$$\texttt{biGr}(A) = \texttt{MAX}(\pi_{\mathcal{B}_n}(A)) \tag{22}$$

be the maximal biGrassmannians in $\pi_{\mathcal{B}_n}(A)$. Then as a consequence of Lemma 3.5 and Proposition 3.1

$$A = \vee \pi_{\mathcal{B}_n}(A) = \vee \texttt{biGr}(A). \tag{23}$$

Notice that (23) determines a pair $(\boldsymbol{\lambda}, \mathbf{d})$ so that $A = A_{\boldsymbol{\lambda},\mathbf{d}}$. If $\texttt{biGr}(A) = \{u_1, \ldots, u_k\}$ then setting $\lambda^{(i)} = \lambda^{(u_i)}$ and $d_i = \texttt{des}(u_i)$ produces the desired prism shape.

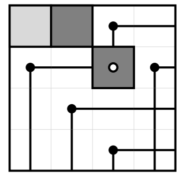
We also define

$$\texttt{Perm}(A) := \texttt{MIN}(\{w \in \mathcal{S}_n : w \geq A\}) \tag{24}$$

and

$$\texttt{MinPerm}(A) := \{w \in \texttt{Perm}(A) : \ell(w) = \texttt{deg}(A)\}. \tag{25}$$

Example 3.6. Let A be the ASM whose diagram is pictured below.



By direct verification, we may compute $\texttt{Perm}(A) = \{3412, 4123\}$. Since $\ell(3412) = 4$ and $\ell(4123) = 3$, we have $\texttt{MinPerm}(A) = \{4123\}$. □

3.3. **Corner sums and biGrassmannians.** In this section, we discuss the specific connection of $\texttt{biGr}(A)$ to $\mathcal{E}ss(A)$. Furthermore, we review known facts about biGrassmannian permutations and the Bruhat order. We then use this to prove Proposition 2.12.

The definition of $\mathcal{E}ss(A)$ generalizes W. Fulton's definition of the essential set of a permutation matrix. However, there is another characterization in terms of corner sum matrices. This is taken as the definition elsewhere in the literature, for example see [For08] or [Kob13]. We prove these definitions are equivalent.

Lemma 3.7. $\mathcal{E}ss(A) = \{(i,j) : r_A(i,j) = r_A(i-1,j) = r_A(i,j-1)$ *and* $r_A(i,j)+1 = r_A(i+1,j) = r_A(i,j+1)\}$.

Proof. By (9), if $(i,j) \in D(A)$ if and only if $\sum_{k=1}^{i} a_{kj} = 0$ and $\sum_{l=1}^{j} a_{il} = 0$. Since

$$r_A(i,j) - r_A(i-1,j) = \sum_{l=1}^{j} a_{il} \quad \text{and} \quad r_A(i,j) - r_A(i,j-1) = \sum_{k=1}^{i} a_{kj}$$

we have

$$(i,j) \in D(A) \quad \text{if and only if} \quad r_A(i,j) = r_A(i-1,j) = r_A(i,j-1). \tag{26}$$

($\subseteq$) Assume $(i,j) \in \mathcal{E}ss(A)$. By definition, $(i+1,j), (i,j+1) \notin D(A)$. Since $(i,j) \in \mathcal{D}(A)$, applying (26) and (R2), we have

$$r_A(i,j) = r_A(i,j-1) \leq r_A(i+1,j-1) \leq r_A(i+1,j).$$

If $r_A(i+1,j) = r_A(i,j)$ then $r_A(i+1,j) = r_A(i+1,j-1)$. Then by (26), we have $(i+1,j-1) \in \mathcal{D}(A)$, contradicting $(i,j) \in \mathcal{E}ss(A)$. So $r_A(i,j)+1 = r_A(i+1,j)$. The argument for $r_A(i,j)+1 = r_A(i,j+1)$ is entirely analogous.

($\supseteq$) By assumption, $r_A(i,j) = r_A(i-1,j) = r_A(i,j-1)$. Then applying (26), $(i,j) \in \mathcal{D}(A)$. Since $r_A(i,j) \neq r_A(i+1,j)$ and $r_A(i,j) \neq r_A(i,j+1)$, we conclude

$$(i+1,j), (i,j+1) \notin \mathcal{D}(A).$$

So $(i,j) \in \mathcal{E}ss(A)$. □

Lemma 3.8. $[i,j,r]_b = \wedge\{A \in \mathsf{ASM}(n) : r_A(i,j) \leq r\}$.

See [BS17, Theorem 30] for a proof. An analogous statement in terms of monotone triangles appears in [LS96]. Note in particular,

$$\text{if } r_A(i,j) \leq r, \text{ we have } [i,j,r]_b \leq A. \tag{27}$$

This is a special case of the generalized essential criterion given in [Kob13].

Lemma 3.9. *Fix $A \in \mathsf{ASM}(n)$.*

(1) For all $1 \leq i, j \leq n$, $[i, j, r_A(i, j)]_b \in \mathcal{B}_n$ or $[i, j, r_A(i, j)]_b = \mathrm{id}$.
(2) $A = \vee\{[i, j, r_A(i, j)]_b : 1 \leq i, j \leq n\}$.

Proof. (1) From (R2) we must have $r_A(i, j) \leq \min\{i, j\}$. If this is an equality, we have $[i, j, r_A(i, j)]_b = \mathrm{id}$ and we are done. So assume not. By (R1), $r_A(i, n) = i$. As a consequence of (R2), $n - j \geq i - r_A(i, j)$. Then $i + j - r_A(i, j) \leq n$. So the conditions (B1)-(B3) are satisfied.

(2) Let $A' = \vee\{[i, j, r_A(i, j)]_b : 1 \leq i, j \leq n]\}$. By Lemma 3.8, A is an upper bound to each $[i, j, r_A(i, j)]_b$. So $A \geq A'$ and $r_A \leq r_{A'}$. Since $r_{A'}$ is entrywise the minimum of the corner sum matrices of the $[i, j, r_A(i, j)]_b$'s, in particular, $r_{A'}(i, j) \leq r_{[i,j,r_A(i,j)]_b}(i, j) = r_A(i, j)$. Then $r_{A'} \leq r_A$. As such, $r_{A'} = r_A$ and so $A' = A$. $\square$

Lemma 3.10. *Assume $A \neq I_n$. If $(i, j) \notin \mathcal{E}ss(A)$, then there is some (i', j') so that*

$$[i, j, r_A(i, j)]_b < [i', j', r_A(i', j')]_b.$$

Proof. Let $u = [i, j, r_A(i, j)]_b$. If $r_A(i, j) = \min\{i, j\}$ then u is the identity and hence smaller than any biGrassmannian. So assume $r_A(i, j) < \min\{i, j\}$. Since $A \neq I_n$, there is some (i', j') exists for which $[i', j', r_A(i', j')]_b \in \mathcal{B}_n$ (e.g. some $(i', j') \in \mathcal{E}ss(A)$).

Applying Lemma 3.7 and (R2), there are four potential ways for (i, j) to fail to be in $\mathcal{E}ss(A)$.

Case 1: $r_A(i, j) = r_A(i - 1, j) + 1$.

Since we have assumed $r_A(i, j) < \min\{i, j\}$ and $r_A(0, j) = 0$, we must have $i > 1$. So let $u' = [i - 1, j, r_A(i - 1, j)]_b$.

Then $r_{u'}(i, j) = r_{u'}(i - 1, j) + 1 = r_A(i - 1, j) + 1 = r_A(i, j) = r_u(i, j)$ so by Lemma 3.8, $u \leq u'$.

Case 2: $r_A(i, j) = r_A(i, j - 1) + 1$.

The argument is entirely analogous to Case 1.

Case 3: $r_A(i, j) = r_A(i + 1, j)$.

Now let $u' = [i + 1, j, r_A(i + 1, j)]_b$. Then

$$r_{u'}(i, j) = r_{u'}(i + 1, j) = r_A(i + 1, j) = r_A(i, j) = r_u(i, j).$$

Applying Lemma 3.8, we have $u < u'$.

Case 4: $r_A(i, j) = r_A(i, j + 1)$.

This is essentially the same as Case 3. $\square$

The following proposition shows how to recover $\texttt{biGr}(A)$ from $\mathcal{E}ss(A)$.

Proposition 3.11. $\texttt{biGr}(A) = \{[i, j, r_A(i, j)]_b : (i, j) \in \mathcal{E}ss(A)\}$.

Proposition 3.11 is discussed in [LS96, Section 5], using essential points of monotone triangles. It can be found in a slightly more general context in [For08, Theorem 5.1]. As an immediate consequence, A is determined by the restriction of r_A to $\mathcal{E}ss(A)$. This generalizes [Ful92, Lemma 3.10].

Proof of Proposition 3.11. First note that

$$\texttt{biGr}(A) \subseteq \{[i,j,r_A(i,j)]_b : (i,j) \in \mathcal{E}ss(A)\}. \tag{28}$$

If $A = I_n$ then $\texttt{biGr}(A) = \{\} = \mathcal{E}ss(A)$. So assume not.

By Lemma 3.10, whenever $(i,j) \notin \mathcal{E}ss(A)$, there is some (i',j') so that

$$[i,j,r_A(i,j)]_b < [i',j',r_A(i',j')]_b.$$

We may iteratively apply the Lemma 3.10 to construct a chain of inequalities

$$[i,j,r_A(i,j)]_b < [i',j',r_A(i',j')]_b < \ldots < [i'',j'',r_A(i'',j'')]_b$$

with $(i'',j'') \in \mathcal{E}ss(A)$. Therefore

$$\begin{aligned} A &= \vee\{[i,j,r_A(i,j)]_b : 1 \leq i,j \leq n]\} && \text{(by Lemma 3.9)} \\ &= \vee\{[i,j,r_A(i,j)]_b : (i,j) \in \mathcal{E}ss(A)]\}. \end{aligned}$$

In particular, by (28) any biGrassmannian below A has an upper bound in

$$\{[i,j,r_A(i,j)]_b : (i,j) \in \mathcal{E}ss(A)\}.$$

Claim 3.12. *$\{[i,j,r_A(i,j)]_b : (i,j) \in \mathcal{E}ss(A)\}$ is an antichain, i.e. its elements are all incomparable.*

Proof. Take $(i,j),(i',j') \in \mathcal{E}ss(A)$. Write $u = [i,j,r_A(i,j)]_b$ and $u' = [i',j',r_A(i',j')]_b$.

Case 1: $r_{u'}(i,j) \leq r$.

Since $u' \leq A$, we have $r_{u'} \geq r_A$. In particular, $r_{u'}(i,j) \geq r$. So $r_{u'}(i,j) = r$. By condition (R2), $r_{u'}(i-1,j), r_{u'}(i,j-1) \in \{r-1,r\}$ and $r_{u'}(i+1,j), r_{u'}(i,j+1) \in \{r,r+1\}$. But since $(i,j) \in \mathcal{E}ss(A)$ and $r_{u'} \geq r_A$, applying Lemma 3.7 we are forced to have

$$r_{u'}(i-1,j) = r_{u'}(i,j-1) = r = r_A(i-1,j) = r_A(i,j-1)$$

and

$$r_{u'}(i+1,j) = r_{u'}(i,j+1) = r+1 = r_A(i+1,j) = r_A(i,j+1).$$

Then $(i,j) \in \mathcal{E}ss(u')\{(i',j')\}$. As such, $u' = u$.

Case 2: $r_{u'}(i,j) > r$.

Then $r_{u'}(i,j) > r_u(i,j)$. So immediately, we conclude $u \ngeq u'$.

We may reverse the roles of u and u' in the above argument. Therefore either u and u' are incomparable or $u = u'$. □

As a consequence of Claim 3.12, we have shown that $\{[i,j,r_A(i,j)]_b : (i,j) \in \mathcal{E}ss(A)\}$ is an antichain of biGrassmannian permutations whose least upper bound is A. Therefore, $\texttt{biGr}(A) = \{[i,j,r_A(i,j)]_b : (i,j) \in \mathcal{E}ss(A)\}$. □

Lemma 3.13. *Suppose $A \in \mathsf{ASM}(n)$ and $u \in \mathcal{G}_n$. If $r_A(\texttt{des}(u),j) \leq r_u(\texttt{des}(u),j)$ for all $j = 1,\ldots,n$, then $u \leq A$.*

Proof. Let $i = \texttt{des}(u)$. Since $r_A(i,j) \leq r_u(i,j)$, we have $A \geq [i,j,r_u(i,j)]_b$. Since u is Grassmannian, all of its essential boxes occur in row i. Then by Lemma 3.8 we have $u' \leq A$ for all $u' \in \mathcal{E}ss(u)$. So A is an upper bound to $\mathcal{E}ss(u)$. Then $A \geq u = \vee\mathcal{E}ss(u)$. □

With the above lemmas, we are now ready to prove Proposition 2.12.

Proof of Proposition 2.12. (1) Let $\mathcal{E}ss(A) = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ and

$$\beta^{(\ell)} = (i_\ell - r_A(i_\ell, j_\ell)) \times (j_\ell - r_A(i_\ell, j_\ell))$$

as in (18). By construction,

$$[\beta^{(\ell)}, i_\ell]_b = [i_\ell, j_\ell, r_A(i_\ell, j_\ell)]_b. \tag{29}$$

Therefore,

$$\begin{aligned} A &= \vee \mathtt{biGr}(A) && \text{(by (23))} \\ &= \vee\{[i, j, r_A(i, j)]_b : (i, j) \in \mathcal{E}ss(A)\} && \text{(by Proposition 3.11)} \\ &= \vee\{[\beta^{(1)}, i_1]_g, \ldots, [\beta^{(k)}, i_k]_g\} && \text{(by (29))} \\ &= A_{\boldsymbol{\beta}_A, \mathbf{b}_A} && \text{(by (12)).} \end{aligned}$$

(2) Let $u = [\lambda^{(A,i)}, i]_g$. Since $\lambda^{(A,i)} = \lambda^{(u,i)}$, we must have

$$m_A(i, j) = m_u(i, j) \quad \text{for all} \quad j = 1, \ldots, i.$$

Applying (19), we have

$$r_A(i, j) = r_u(i, j) \text{ for all } j = 1, \ldots, n. \tag{30}$$

Let $\mathbf{p}_A = (i_1, \ldots, i_k)$ be the essential rows of A and let $\boldsymbol{\rho}_A = (\lambda^{(A,i_1)}, \ldots, \lambda^{(A,i_k)})$ Then $\mathbf{u}_{\boldsymbol{\rho}_A, \mathbf{p}_A} = ([\lambda^{(A,i_1)}, i_1]_g, \ldots, [\lambda^{(A,i_k)}, i_k]_g)$.

By (30) and Lemma 3.13, $[\lambda^{(A,i_\ell)}, i_\ell]_g \leq A$ for all $\ell = 1, \ldots, k$. As such, A is an upper bound to $\mathbf{u}_{\boldsymbol{\rho}_A, \mathbf{p}_A}$ and hence

$$A_{\boldsymbol{\rho}_A, \mathbf{p}_A} = \vee \mathbf{u}_{\boldsymbol{\rho}_A, \mathbf{p}_A} \leq A. \tag{31}$$

On the other hand, by (30), $r_{[\lambda^{(A,i)}, i]_g}(i, j) = r_A(i, j)$. Then by Lemma 3.8,

$$[i, j, r_A(i, j)]_b \leq [\lambda^{(A,i)}, i]_g \quad \text{for all} \quad 1 \leq i, j \leq n.$$

In particular, if $u \in \mathtt{biGr}(A)$, then there is some i_ℓ in the list $\mathbf{p}_A$ so that

$$u \leq [\lambda^{(A_{i_\ell})}, i_\ell]_g \leq A_{\boldsymbol{\rho}_A, \mathbf{p}_A}.$$

So $A_{\boldsymbol{\rho}_A, \mathbf{p}_A}$ is an upper bound to $\mathtt{biGr}(A)$ and hence

$$A = \vee \mathtt{biGr}(A) \leq A_{\boldsymbol{\rho}_A, \mathbf{p}_A}. \tag{32}$$

Therefore, by (31) and (32), $A = A_{\boldsymbol{\rho}_A, \mathbf{p}_A}$. □

We note that the parabolic model could also have been defined using a partition shape for *every* row of A. This has the drawback of having more redundant labels in each tableau. However, the prism shapes have a direct connection to the poset of ASMs

$$A \leq B \quad \text{if and only if} \quad \lambda^{(A,i)} \subseteq \lambda^{(B,i)} \quad \text{for all} \quad i = 1, \ldots, n.$$

This generalizes the following description of the poset of Grassmannian permutations with a fixed descent. Take $u, v \in \mathcal{G}_n$ with $\mathtt{des}(u) = \mathtt{des}(v)$. Then

$$u \leq v \quad \text{if and only if} \quad \lambda^{(u)} \subseteq \lambda^{(v)}.$$

3.4. **Inclusions of ASMs.** There is a natural inclusion $\iota : \mathsf{ASM}(n) \to \mathsf{ASM}(n+1)$ defined by

$$A \mapsto \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right).$$

We write

$$\mathsf{ASM}(\infty) := \bigcup_{n=1}^{\infty} \mathsf{ASM}(n)/\sim$$

where $\sim$ is the equivalence relation generated by $A \sim \iota(A)$. Let

$$\mathcal{S}_\infty = \bigcup_{n=1}^{\infty} \mathcal{S}_n/\sim.$$

When context is clear, we will freely identify an equivalence class its representatives. We write $A \in \mathsf{ASM}(n)$ to indicate that A has a representative which is an element of $\mathsf{ASM}(n)$.

Observe that

$$A \leq B \text{ if and only if } \iota(A) \leq \iota(B). \tag{33}$$

To see this, notice that $r_{\iota(A)}(i, n+1) = r_{\iota(A)}(n+1, i) = i$ for any $A \in \mathsf{ASM}(n)$. Thus $\mathsf{ASM}(\infty)$ inherits the structure of a poset from the finite case. In particular, for any n, there is an order embedding

$$\mathsf{ASM}(n) \hookrightarrow \mathsf{ASM}(\infty).$$

To compare two classes in $\mathsf{ASM}(\infty)$, we may take N large enough so that there are representatives in $A, B \in \mathsf{ASM}(N)$. Due to (33) the resulting order does not depend on the choice of N. Pairwise, joins and meets still exist so $\mathsf{ASM}(\infty)$ is a lattice. However it is *not* complete; in particular, the entire lattice $\mathsf{ASM}(\infty)$ has no upper bound.

Note that if $u \in \mathcal{G}_n$, we have

$$(\lambda^{(u)}, \mathtt{des}(u)) = (\lambda^{(\iota(u))}, \mathtt{des}(\iota(u))).$$

So the bijection in Lemma 2.3 is stable under inclusion. Write $\mathcal{G}_\infty$ and $\mathcal{B}_\infty$ for the sets of Grassmannian and biGrassmannian permutations in $\mathcal{S}_\infty$. Diagrams are also stable under inclusion, i.e. $D(A) = D(\iota(A))$. Therefore

$$\mathtt{biGr}(\iota(A)) = \{\iota(u) : u \in \mathtt{biGr}(A)\}. \tag{34}$$

Therefore, elements of $\mathsf{ASM}(\infty)$ are encoded by (finite) antichains in $\mathcal{B}_\infty$.

3.5. **Partial ASMs.** We now discuss another poset, which is closely related to $\mathsf{ASM}(n)$. A **partial alternating sign matrix** is a matrix with entries in $\{-1, 0, 1\}$ so that

(1) the nonzero entries in each row and column alternate in sign,
(2) each row and column sums to 0 or 1, and
(3) the first nonzero entry of any row or column is 1.

A **partial permutation** is a partial ASM with entries in $\{0, 1\}$. Write $\mathsf{PA}(n)$ for the set of $n \times n$ partial ASMs and $\mathsf{P}(n)$ for the set of $n \times n$ partial permutation matrices. We sometimes say A (or w) is an **honest** ASM (or **honest** permutation) to emphasize that $A \in \mathsf{ASM}(n)$ (or $w \in \mathcal{S}_n$).

As in the case of ASMs, we may endow $\mathsf{PA}(n)$ with the structure of a poset by comparison of corner sum functions. M. Fortin studied $\mathsf{PA}(n)$, showing that it is the Dedekind-MacNeille completion of $\mathsf{P}(n)$ [For08, Section 6]. Here, partial permutation matrices are identified with *partial injective functions*. The poset structure defined by corner sum matrices agrees with the extended Bruhat order defined by L. E. Renner in [Ren05].

Lemma 3.14. *Every $A \in \mathsf{PA}(n)$ has a canonical completion to $\widetilde{A} \in \mathsf{ASM}(N)$, with $n \leq N \leq 2n$.*

Proof. The construction is similar to the one in for partial permutations found in [MS04, Proposition 15.8]. Starting from the top row of A, if sum of row i is zero, append a new column to A with a 1 in the ith row. Continue in this way from top to bottom. Then starting from the leftmost column, if column j sums to zero, add a new row with a 1 in position j. Let $\widetilde{A}$ be the matrix obtained by this procedure.

By construction, $\widetilde{A}$ satisfies (A1); nonzero entries alternate in sign along rows and columns. Also, the entries of within each row and column of $\widetilde{A}$ sum to 1, so (A2) holds. As such, the sum of all entries in A counts the total number of rows, as well as the number of columns. So $\widetilde{A}$ is square. At most n columns and rows were added. So $\tilde{A} \in \mathsf{ASM}(N)$ for some $n \leq N \leq 2n$. □

Example 3.15. If $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$ then $\widetilde{A} = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$. Since the sum of the entries of A is 1, $N = 2n - 1 = 5$. □

For $w \in \mathsf{P}(n)$ we define the **length** of w to be $\ell(w) := \ell(\widetilde{w})$. Similarly, we define the diagram $D(A) := D(\widetilde{A})$. By construction, $D(A)$ is contained in the $n \times n$ grid.

Lemma 3.16. $r_A \geq r_B$ *if and only if* $r_{\widetilde{A}} \geq r_{\widetilde{B}}$.

Proof. If $r_{\widetilde{A}} \geq r_{\widetilde{B}}$ it is immediate that $r_A \geq r_B$.

Now assume that $r_A \geq r_B$. By construction, the essential set of both $\widetilde{A}$ and $\widetilde{B}$ is contained in the first n rows and columns. So for any $(i,j) \in \mathcal{E}ss(\widetilde{A})$, we have $r_{\widetilde{B}}(i,j) \leq r_{\widetilde{A}}(i,j)$ and therefore $[i,j,r_{\widetilde{A}}(i,j)]_b \leq B$. Then $\widetilde{B}$ is an upper bound to $\mathtt{biGr}(A)$ and so $\widetilde{A} \leq B$ which implies $r_{\widetilde{A}} \geq r_{\widetilde{B}}$. □

Taking the inclusion of $\widetilde{A}$ into $\mathsf{ASM}(2n)$ is an order embedding $\mathsf{PA}(n) \hookrightarrow \mathsf{ASM}(2n)$. As such, we may study the order on $\mathsf{PA}(n)$ by identifying each partial ASM with its image under the above inclusion.

A **partial biGrassmannian** is an element $b \in \mathsf{P}(n)$ so that $\widetilde{b} \in \mathcal{S}_{2n}$ is biGrassmannian. Again, these are indexed by triples (i,j,r) but we omit condition (B3). Write $[i,j,r_{ij}]_b$ for the partial biGrassmannian in $\mathsf{P}(n)$. By [For08], these are the basic elements of $\mathsf{PA}(n)$.

Notice, that restrictions of honest ASMs to northwest submatrices produce partial ASMs. Take $A \in \mathsf{ASM}(N)$. Then if $n \leq N$, we have $A_{[n],[n]} \in \mathsf{PA}(n)$. Notice $A \leq B$ implies $A_{[n],[n]} \leq B_{[n],[n]}$. However, the converse certainly does not hold. However, in the case $A = \widetilde{A}_{[n],[n]}$, we do have $A \leq B$ whenever $A_{[n],[n]} = B_{[n],[n]}$. This follows since $\mathcal{E}ss(A) \subseteq n \times n$ and so $u \leq B$ for all $u \in \mathtt{biGr}(A)$.

4.1. **Simplicial complexes.** Recall that $\mathbb{P}(S)$ denotes the power set of S. A **simplicial complex** Δ is a subset of $\mathbb{P}([N])$ so that whenever $f \in \Delta$ and $f' \subseteq f$, we have $f' \in \Delta$. An element $f \in \Delta$ is called a **face**. The **dimension** of f is $\dim(f) = |f| - 1$. Write

$$\dim(\Delta) = \max\{\dim(f) : f \in \Delta\}.$$

If $f \in \Delta$, the **codimension** of f is $\operatorname{codim}(f) = \dim(\Delta) - \dim(f)$. The set of faces of Δ ordered by inclusion form a poset. Let

$$F(\Delta) = \mathtt{MAX}(\Delta) \tag{35}$$

denote the set of **facets** of Δ, i.e. the maximal faces. Then define

$$F_{\mathtt{max}}(\Delta) = \{f \in \Delta : \operatorname{codim}(f) = 0\}. \tag{36}$$

Necessarily, $F_{\mathtt{max}}(\Delta) \subseteq F(\Delta)$. When this containment is an equality, Δ is called **pure**.

Given two simplicial complexes $\Delta_1, \Delta_2 \subseteq \mathbb{P}([N])$, we may refer without ambiguity to the intersection (or union) of Δ_1 and Δ_2; it is precisely their intersection (or union) as sets. A straightforward verification shows that $\Delta_1 \cap \Delta_2$ and $\Delta_1 \cup \Delta_2$ are themselves simplicial complexes.

Lemma 4.1. *Fix simplicial complexes $\Delta_1, \ldots, \Delta_k \subseteq \mathbb{P}([N])$. Let $\Delta = \Delta_1 \cap \ldots \cap \Delta_k$. Then*

$$F(\Delta) \subseteq \{f_1 \cap \ldots \cap f_k : f_i \in F(\Delta_i)\}.$$

Proof. Fix $f \in F(\Delta) \subseteq \Delta$. Then $f \in \Delta_i$ for all i. For each i, there exists some $f_i \in F(\Delta_i)$ such that $f \subseteq f_i$. Therefore,

$$f \subseteq f_1 \cap \ldots \cap f_k \subseteq f_i \text{ for all } i = 1, \ldots, k. \tag{37}$$

Then $f_1 \cap \ldots \cap f_k \in \Delta_i$ for all i. As such,

$$f \subseteq f_1 \cap \ldots \cap f_k \in \Delta_1 \cap \ldots \cap \Delta_k = \Delta. \tag{38}$$

Since $f \in F(\Delta)$, the containment in (38) is actually an equality. $\square$

Let $\Bbbk[\mathbf{z}] = \Bbbk[z_1, \ldots, z_N]$. Given $\mathbf{v} = (v_1, \ldots, v_N) \in \mathbb{N}^N$, write $\mathbf{z}^{\mathbf{v}} := \prod_{i=1}^N z_i^{v_i}$. If $\mathbf{v} \in \{0,1\}^N$, then $\mathbf{z}^{\mathbf{v}}$ is a **square-free monomial**. An ideal is called a **square-free monomial ideal** if it has a generating set of square-free monomials. Stanley-Reisner theory describes the correspondence between square-free monomial ideals in $\Bbbk[\mathbf{z}]$ and simplicial complexes $\Delta \subseteq \mathbb{P}([N])$. We give a brief overview. For more background, see [MS04, Chapter 1].

Notice square-free monomials in $\Bbbk[F]$ correspond to faces in $\mathbb{P}([N])$. Given $f \in \mathbb{P}([N])$, write $\mathbf{z}^f = \displaystyle\prod_{i \in f} z_i$.

Definition 4.2. The **Stanley-Reisner ideal** of Δ is

$$I_\Delta = \langle \mathbf{z}^f : f \notin \Delta \rangle.$$

The quotient $\Bbbk[\mathbf{z}]/I_\Delta$ is called the **Stanley-Reisner ring** of Δ.

Write $\mathfrak{m}_f = \langle z_i : i \in f \rangle$ and let $\overline{f} = [N] - f$.

Theorem 4.3. [MS04, Theorem 1.7] *The map $\Delta \mapsto I_\Delta$ is a bijection between square-free monomial ideals in $\Bbbk[\mathbf{z}]$ and simplicial complexes $\Delta \subseteq \mathbb{P}([N])$. I_Δ can be expressed as an intersection of monomial prime ideals*

$$I_\Delta = \bigcap_{f \in \Delta} \mathfrak{m}_{\overline{f}}. \tag{39}$$

Explicitly, the inverse map takes a square-free monomial ideal I to

$$\Delta(I) := \{f \subseteq [N] : \mathbf{z}^f \notin I\}.$$

Given a square-free monomial ideal I, we say $\Delta(I)$ is the **Stanley-Reisner complex** associated to I.

The following lemma is straightforward from Definition 4.2, but we give the details.

Lemma 4.4. *Let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be a set square-free monomial ideals with $I_\alpha \subseteq \Bbbk[z_1, \ldots, z_N]$. Then $\Delta(\sum_{\alpha \in \mathcal{A}} I_\alpha) = \bigcap_{\alpha \in \mathcal{A}} \Delta(I_\alpha)$.*

Proof. A generating set for $\sum_{\alpha \in \mathcal{A}} I_\alpha$ can be obtained by concatenation of the generating sets for the I_α's. So it is a square-free monomial ideal. Notice a monomial $m \in \sum_{\alpha \in \mathcal{A}} I_\alpha$ if and only if $m \in I_\alpha$ for some $\alpha \in \mathcal{A}$.

Assume $f \subseteq [N]$. Then,

$$\begin{aligned} f \in \Delta(\sum_{\alpha \in \mathcal{A}} I_\alpha) &\iff \mathbf{z}^f \notin \sum_{\alpha \in \mathcal{A}} I_\alpha \\ &\iff \mathbf{z}^f \notin I_\alpha \text{ for all } \alpha \in \mathcal{A} \\ &\iff f \in \Delta(I_\alpha) \text{ for all } \alpha \in \mathcal{A} \\ &\iff f \in \bigcap_{\alpha \in \mathcal{A}} \Delta(I_\alpha). \end{aligned}$$

□

4.2. **Subword complexes.** We now recall the definition of a subword complex, following [KM04]. Let Π be a Coxeter group minimally generated by simple reflections Σ. A **word** is an ordered list $Q = (s_1, \ldots, s_m)$ of simple reflections in Σ. A **subword** of Q is an ordered subsequence $P = (s_{i_i}, \ldots, s_{i_k})$. Subwords of Q are naturally identified with faces of the simplicial complex $\mathbb{P}([m])$.

A word P **represents** $w \in \Pi$ if $w = s_1 \cdots s_m$ and $\ell(w) = m$, i.e. the ordered product is a reduced expression for w. We say Q **contains** w if Q has a subword which represents w. Then define the **subword complex**

$$\Delta(Q, w) = \{Q - P : P \text{ contains } w\}.$$

We will abbreviate $\mathcal{F}_P := Q - P$. Immediately by definition,

$$\mathcal{F}_P \subseteq \mathcal{F}_{P'} \quad \text{if and only if} \quad P \supseteq P'. \tag{40}$$

A well known characterization of the Bruhat order on $\mathcal{S}_n$ is via subwords:

$$w \geq v \text{ if and only if some (and hence every) reduced word for } w \text{ contains } v. \tag{41}$$

See [Hum92, Section 5.10]. This is equivalent to the order on $\mathcal{S}_n$ as defined in (2). See [BB06, Theorem 2.1.5] for a proof.

The **Demazure algebra** of (Π, Σ) over a ring R is freely generated by $\{e_w : w \in \Pi\}$ with multiplication given by

$$e_w e_s = \begin{cases} e_{ws} & \text{if } \ell(ws) > \ell(w) \\ e_w & \text{if } \ell(ws) < \ell(w). \end{cases}$$

If $Q = s_1 \dots s_k$, the **Demazure product** $\delta(Q)$ is defined by the product $e_{s_1} \cdots e_{s_m} = e_{\delta(Q)}$. The faces of $\Delta(Q, w)$ have a natural description in terms of the Demazure product.

Lemma 4.5 **([KM04, Lemma 3.4]).** *$\delta(P) \geq w$ if an only if P contains w.*

Then $\Delta(Q, w) = \{\mathcal{F}_P : \delta(P) \geq w\}$. This motivates the following definition. Let $\Pi = \mathcal{S}_n$ and $\Sigma = \{(i, i{+}1) : i = 1, \dots, n{-}1\}$ be the set of simple transpositions. Given $A \in \mathsf{ASM}(n)$, define

$$\Delta(Q, A) = \{\mathcal{F}_P : \delta(P) \geq A\}. \tag{42}$$

This is itself a simplicial complex, but need not be a subword complex. Immediately from the definition,

$$\text{if } A \geq B \text{ then } \Delta(Q, A) \subseteq \Delta(Q, B). \tag{43}$$

We will show that $\Delta(Q, A)$ is a union of subword complexes. In particular, if $A \in \mathsf{ASM}(m)$ with $m \leq n$ each of these subword complexes correspond to permutations in $\mathcal{S}_m$.

Lemma 4.6. *Suppose $w \in \mathcal{S}_\infty$ is an upper bound to $\{w_1, \dots, w_k\} \subseteq \mathcal{S}_m$. Then there exists $w' \in S_m$ so that $\vee\{w_1, \dots, w_k\} \leq w' \leq w$.*

Proof. Let P be a reduced word for w. By (41), P contains a subword, P_i which represents w_i for each i. Let $P' = \bigcup_{i=1}^k P_i \subseteq P$. By Lemma 4.5, since P' contains each of the w_i's, we have $\delta(P') \geq w_i$ for all i. So $\delta(P')$ is an upper bound to $\{w_1, \dots, w_k\}$. Again, by Lemma 4.5, P' contains $\delta(P')$ and hence P contains $\delta(P')$. So

$$w = \delta(P) \geq \delta(P').$$

Finally, the word $\mathcal{P}'$ uses only simple transpositions from S_m, so $\delta(P') \in \mathcal{S}_m$. ☐

As a corollary, we obtain the following.

Corollary 4.7. *(1) $\mathtt{Perm}(A) = \mathtt{MIN}(\{w \in \mathcal{S}_\infty : w \geq A\})$.*
(2) $\mathtt{Perm}(A) = \mathtt{MIN}(\{w \in \mathsf{P}(n) : w \geq A\})$.

Proof. (1) This is immediate from Lemma 4.6.

(2) Fix $w \in \mathsf{P}(n)$. Consider the inclusions $\widetilde{A}, \widetilde{w} \in \mathsf{ASM}(2n)$. Then $\widetilde{w} \geq A$ is an upper bound to $\mathtt{biGr}(\widetilde{A}) = \mathtt{biGr}(A)$. Applying Lemma 4.6, we obtain $w' \in \mathcal{S}_n$ with $\widetilde{A} \leq w' \leq \widetilde{w}$. Since $w' \in \mathcal{S}_n$,we may take its representative $\widetilde{w'} \in \mathsf{ASM}(2n)$. So $\widetilde{A} \leq \widetilde{w'} \leq \widetilde{w}$. Applying Lemma 3.16, we see that $A \leq w' \leq w$. So the statement follows. ☐

Proposition 4.8. *Fix a word Q and $A \in \mathsf{ASM}(n)$.*

(1) $\Delta(Q, A) = \displaystyle\bigcup_{w \in \mathtt{Perm}(A)} \Delta(Q, w)$.

(2) If $A = \vee\{A_1, \dots, A_k\}$ then

$$\Delta(Q, A) = \bigcap_{i=1}^{k} \Delta(Q, A_i).$$

(3) $F(\Delta(Q,A)) = \{\mathcal{F}_P : P \text{ is a reduced expression for some } w \in \texttt{Perm}(A)\}$.

Proof. (1) Since $w \geq A$, applying (43) we have

$$\Delta(Q,A) \supseteq \bigcup_{w \in \texttt{Perm}(A)} \Delta(Q,w).$$

If $\mathcal{F}_P \in \Delta(Q,A)$ then $\delta(P) \geq A$. By (24) and Corollary 4.7, there exists $w \in \texttt{Perm}(A)$ so that $\delta(P) \geq w \geq A$. Then $\mathcal{F}_P \in \Delta(Q,w)$. Therefore, $\Delta(Q,A) \subseteq \bigcup\limits_{w \in \texttt{Perm}(A)} \Delta(Q,w)$.

(2) Since $A \geq A_i$, applying (43), we have that $\Delta(Q,A) \subseteq \Delta(Q,A_i)$ for each $i = 1,\ldots,k$. So

$$\Delta(Q,A) \subseteq \bigcap_{i=1}^{k} \Delta(Q,A_i).$$

If $\mathcal{F}_P \in \Delta(Q,A_i)$ for all i, then $\delta(P) \geq A_i$ for all i. Since $A = \vee\{A_1,\ldots,A_k\}$ we must have $\delta(P) \geq A$. So $\mathcal{F}_P \in \Delta(Q,A)$.

(3) Suppose P is a reduced expression for some $w \in \texttt{Perm}(A)$. If P contains P' and $\mathcal{F}_{P'} \in \Delta(Q,A)$ then $w = \delta(P) \geq \delta(P') \geq A$. By (24), $\delta(P') = w$. Since P is a reduced expression for w, we have $P = P'$. Therefore $\mathcal{F}_P \in \Delta(Q,w)$. $\square$

For the rest of this section, we focus on a fixed ambient word Q. Write s_i for the simple transposition $(i,i+1) \in \mathcal{S}_{2n}$. Define the **square word**

$$Q_{n\times n} = s_n\, s_{n-1}\, \ldots\, s_1 \quad s_{n+1}\, s_n\, \ldots\, s_2 \quad \ldots \quad s_{2n-1}\, s_{2n-2}\, \ldots s_n.$$

Order the boxes of the $n \times n$ grid by reading along rows from right to left, starting with the top row and working down to the bottom. This ordering identifies each letter of $Q_{n\times n}$ with a cell in the $n \times n$ grid.

A **plus diagram** is a subset of the $n \times n$ grid. We indicate (i,j) is in the plus diagram by marking its position in the grid with a $+$. The identification of the letters in $Q_{n\times n}$ with the grid defines a natural bijection between subwords of $Q_{n\times n}$ and plus diagrams. As such, we freely identify each word with its plus diagram.

Example 4.9. When $n = 3$, we have

$$Q_{n\times n} = s_3 s_2 s_1 s_4 s_3 s_2 s_5 s_4 s_3.$$

Below, we label the entries of the 3×3 grid with their corresponding simple transpositions. We also give a subword of $Q_{3\times 3}$ and its corresponding plus diagram.

$$\begin{matrix} s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \\ s_3 & s_4 & s_5 \end{matrix} \qquad\qquad s_3 - - - s_3\, s_2 - - - \qquad\qquad \begin{matrix} \cdot & \cdot & + \\ + & + & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

Notice that P is not a reduced expression, $s_3 s_3 s_2 = s_2$. Therefore, it is not a facet of $\Delta(Q_{n\times n}, A)$ for any $A \in \mathsf{ASM}(n)$. $\square$

For brevity, write $\Delta_A := \Delta(Q_{n\times n}, A)$. Assign $\mathcal{F}_P$ the weight

$$\texttt{wt}(\mathcal{F}_P) = \prod_{i=1}^{n} x_i^{n_i} \quad \text{where} \quad n_i = |\{j : (i,j) \in P\}|.$$

When $w \in \mathcal{S}_n$, the complex Δ_w is a pure simplicial complex. Its facets are in immediate bijection with pipe dreams (also known as RC-graphs).

Theorem 4.10 ([FK96, BB93, KM05])**.**

$$\mathfrak{S}_w = \sum_{\mathcal{F}_P \in F(\Delta_w)} \mathtt{wt}(\mathcal{F}_P). \tag{44}$$

For permutations, Δ_w is the Stanley-Reisner complex of a degeneration of the Schubert determinantal ideal I_w [KM05, Theorem B]. The same holds for I_A and Δ_A, see Section 5.4 for details.

As a consequence of Theorem 4.10, we have the following corollary.

Corollary 4.11. *(1)* $\displaystyle\sum_{w \in \mathtt{Perm}(A)} \mathfrak{S}_w = \sum_{\mathcal{F}_P \in F(\Delta_A)} \mathtt{wt}(\mathcal{F}_P).$

(2) $\displaystyle\sum_{w \in \mathtt{MinPerm}(A)} \mathfrak{S}_w = \sum_{\mathcal{F}_P \in F_{\mathtt{max}}(\Delta_A)} \mathtt{wt}(\mathcal{F}_P).$

Proof. (1) By Proposition 4.8,

$$F(\Delta_A) = \bigcup_{w \in \mathtt{Perm}(A)} F(\Delta_w). \tag{45}$$

$\mathcal{F}_P$ is a facet of Δ_w if and only if P represents w. A subword can represent at most one permutation, so the union in (45) is disjoint. Therefore, applying (44), we have

$$\sum_{w \in \mathtt{Perm}(A)} \mathfrak{S}_w = \sum_{w \in \mathtt{Perm}(A)} \sum_{\mathcal{F}_P \in F(\Delta_w)} \mathtt{wt}(\mathcal{F}_P) = \sum_{\mathcal{F}_P \in F(\Delta_A)} \mathtt{wt}(\mathcal{F}_P).$$

(2) Observe that

$$F_{\mathtt{max}}(\Delta_A) = \bigcup_{w \in \mathtt{MinPerm}(A)} F(\Delta_w).$$

Again this union is disjoint. So the result follows. □

4.3. **Proof of Theorem 1.1.** Take $T \in \mathtt{RSSYT}(\lambda, d)$ and write T_{ij} for the entry of T which is in the ith row and jth column in the ambient $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ grid (as in (14)). Define a plus diagram P_T by placing a plus in position $(T_{ij}, i + j - T_{ij})$ for each label in T. This in turn defines a map $T \mapsto \mathcal{F}_{P_T}$. Define $\Phi_{\lambda,d}(T) = \mathcal{F}_{P_T}$.

Proposition 4.12. $\Phi_{\lambda,d} : \mathtt{RSSYT}(\lambda, d) \to F(\Delta_{u_{\lambda,d}})$ *is a bijection.*

Proposition 4.12 is well known. For a proof, see e.g. [KMY09, Proposition 5.3]. Define

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}} : \mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) \to \Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}} \tag{46}$$

where

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(T^{(1)}, \ldots, T^{(k)}) = \Phi_{\lambda^{(1)},d_1}(T^{(1)}) \cap \ldots \cap \Phi_{\lambda^{(k)},d_k}(T^{(k)}). \tag{47}$$

Equivalently, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(T^{(1)}, \ldots, T^{(k)}) = \mathcal{F}_{P_{\mathcal{T}}}$ where $P_{\mathcal{T}} = \bigcup_{i=1}^{k} P_{T^{(i)}}$. By part (2) of Proposition 4.8, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}$ is well defined.

Example 4.13. Continuing Example 2.6, we have

$$\mathcal{T} = \begin{array}{|c|c|c|} \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \multicolumn{3}{c}{} \\ \cline{1-2} 11 & 1 & \multicolumn{1}{c}{} \\ \hline 32 & 3 & 2 \\ \hline 6 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \mapsto \qquad P_{\mathcal{T}} = \begin{array}{ccccccc} \cdot & + & \cdot & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & + & \cdot & + & \cdot \\ \cdot & \cdot & + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

□

Lemma 4.14. *$\Phi_{\boldsymbol{\lambda},\mathbf{d}}$ is weight preserving.*

Proof. The plus diagram P_T has a plus in position (i, j) if and only if there is some a so that $T_{a,i+j-a} = i$.

Let $\mathcal{F}_P = \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T})$. Then $P = P_{T^{(1)}} \cup \ldots \cup P_{T^{(k)}}$. Therefore, $(i, j) \in P$ if and only if $(i, j) \in P_{T^{(\ell)}}$ for some $\ell \in [k]$. As such, $(i, j) \in P$ if and only if the label i appears in the $(i + j)$th antidiagonal of $\mathcal{T}$. Therefore,

$$\mathtt{wt}(\mathcal{F}_P) = \prod_{(i,j)\in P} x_i = \prod_i x_i^{n_i} = \mathtt{wt}(\mathcal{T}).$$ □

Notice by Lemma 4.1 and Proposition 4.12,

$$F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}}) \subseteq \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})). \tag{48}$$

If $\mathcal{T}$ is minimal, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) \in F_{\mathtt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$. This implies

$$\mathtt{deg}(\boldsymbol{\lambda}, \mathbf{d}) = \mathtt{deg}(A_{\boldsymbol{\lambda},\mathbf{d}}). \tag{49}$$

For permutation matrices, $\mathtt{Perm}(w) = w$ and so $\mathtt{deg}(w) = \ell(w)$. This shows the original definition for a minimal prism tableau given in [WY15] agrees with the definition stated here.

Call $\mathcal{T}$ **facet** if $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) \in F(\Delta_A)$. Let $\mathtt{Facet}(\boldsymbol{\lambda}, \mathbf{d}) \subseteq \mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ denote the set of facet prism tableaux. Write $\mathtt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d}) \subseteq \mathtt{Facet}(\boldsymbol{\lambda}, \mathbf{d})$ for the set of facet tableaux which have no unstable triples. By (48),

$$\mathtt{Perm}(A_{\boldsymbol{\lambda},\mathbf{d}}) = \{w : \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) \in \Delta_w \text{ for some } \mathcal{T} \in \mathtt{Facet}(\boldsymbol{\lambda}, \mathbf{d})\}. \tag{50}$$

Example 4.15. Let A be as in Example 3.6. Set $\boldsymbol{\lambda} = ((2), (2))$ and $\mathbf{d} = (1, 2)$. Notice that $\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d})$ is *both* the parabolic and the biGrassmannian prism shape for A. So $A_{\boldsymbol{\lambda},\mathbf{d}} = A$. There are three prism fillings of $\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d})$, listed below.

$$\mathcal{T}_1 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \qquad \mathcal{T}_2 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 1 \\ \hline \end{array} \qquad \mathcal{T}_3 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

Only $\mathcal{T}_1$ is minimal, so $\mathtt{Prism}(\boldsymbol{\lambda}, \mathbf{d}) = \{\mathcal{T}_1\}$. Therefore $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \mathtt{wt}(\mathcal{T}_1) = x_1^3$. The above prism tableaux correspond to the following plus diagrams.

$$P_1 = \begin{array}{cccc} + & + & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad P_2 = \begin{array}{cccc} + & + & + & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad P_3 = \begin{array}{cccc} + & + & \cdot & \cdot \\ + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

Since $P_2 \supsetneq P_1$, we have $\mathcal{F}_{P_2} \subsetneq \mathcal{F}_{P_1}$. Therefore $\mathcal{T}_2$ is not a facet prism tableau. There are no plus diagrams in the image of $\Phi_{\boldsymbol{\lambda},\mathbf{d}}$ which are strictly contained in P_1 or P_3, so by (48), $\mathcal{F}_{P_1}, \mathcal{F}_{P_3} \in F(\Delta_A)$. So $\mathcal{T}_1, \mathcal{T}_3 \in \mathtt{Facet}(\boldsymbol{\lambda}, \mathbf{d})$.

The word corresponding to P_1 is $s_3 s_2 s_1 = 4123$ and the word for P_3 is $s_2 s_1 s_3 s_2 = 3412$. Therefore $\mathtt{Perm}(A) = \{4123, 3124\}$ and $\mathtt{MinPerm}(A) = \{4123\}$. $\square$

Theorem 4.16. *(1) $F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$ is in weight preserving bijection with $\mathtt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d})$.*
(2) The bijection in (1) restricts to a bijection between $F_{\mathtt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$ and $\mathtt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$.

Theorem 1.1 follows as an immediate consequence of Theorem 4.16 and Corollary 4.11.

(51) $$\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \sum_{T \in \mathtt{Prism}(\boldsymbol{\lambda},\mathbf{d})} \mathtt{wt}(T) = \sum_{w \in \mathtt{MinPerm}(A_{\boldsymbol{\lambda},\mathbf{d}})} \mathfrak{S}_w.$$

Similarly, we have

(52) $$\sum_{\mathcal{T} \in \mathtt{StableFacet}(\boldsymbol{\lambda},\mathbf{d})} \mathtt{wt}(\mathcal{T}) = \sum_{w \in \mathtt{Perm}(A_{\boldsymbol{\lambda},\mathbf{d}})} \mathfrak{S}_w.$$

For our proof of Theorem 4.16, we analyze the fibers of $\Phi_{\boldsymbol{\lambda},\mathbf{d}}$

(53) $$\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P) = \{\mathcal{T} \in \mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) : \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) = \mathcal{F}_P\}.$$

For an arbitrary face of $\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}}$, this fiber may be empty. However, by (48), facets have nonempty fibers. In Proposition 4.19 we show that the fiber of any facet has the structure of a lattice. Furthermore, the maximum element of $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is the only tableau in the fiber with no unstable triples.

Order $\mathtt{RSSYT}(\lambda, d)$ and $\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{u})$ by entrywise comparison.

Proposition 4.17. *(1) $\mathtt{RSSYT}(\lambda, d)$ is a lattice.*
(2) $\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{u})$ is a lattice.

Proof. (1) Given $T, U \in \mathtt{RSSYT}(\lambda, d)$, we claim $T \wedge U = \mathbf{min}(T, U)$ and $T \vee U = \mathbf{max}(T, U)$. Note that

(54) $$\text{if } a_1 \le a_2 \text{ and } b_1 \le b_2 \text{ then } \min\{a_1, b_1\} \le \min\{a_2, b_2\}.$$

Similarly,

(55) $$\text{if } a_1 < a_2 \text{ and } b_1 < b_2 \text{ then } \min\{a_1, b_1\} < \min\{a_2, b_2\}.$$

The same statements hold when replacing $\min$ with $\max$. Therefore, conditions (T1) and (T2) are preserved under taking entrywise minimums and maximums. Furthermore, $\mathbf{min}(T, U)$ and $\mathbf{max}(T, U)$ use only labels from $[d]$. Then

$$\mathbf{min}(T, U), \mathbf{max}(T, U) \in \mathtt{RSSYT}(\lambda, d).$$

By applying Lemma 3.2, we see that $\mathtt{RSSYT}(\lambda, d)$ is a lattice.

(2) By (1), $\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is a product of lattices. So $\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is itself a lattice. Again, $\mathcal{T} \wedge \mathcal{U} = \mathbf{min}(\mathcal{T}, \mathcal{U})$ and $\mathcal{T} \vee \mathcal{U} = \mathbf{max}(\mathcal{T}, \mathcal{U})$. $\square$

Write

(56) $$\mathtt{RSSYT}_P(\lambda, d) := \{T \in \mathtt{RSSYT}(\lambda, d) : \Phi_{\lambda,d}(T) \supseteq \mathcal{F}_P\}.$$

Lemma 4.18. *(1) Suppose $T, U \in \texttt{RSSYT}_P(\lambda, d)$. Then*

$$\Phi_{\lambda,d}(T \vee U) \supseteq \mathcal{F}_P \quad \text{and} \quad \Phi_{\lambda,d}(T \wedge U) \supseteq \mathcal{F}_P.$$

As such, $\texttt{RSSYT}_P(\lambda, d)$ is a lattice.

(2) Suppose $T, U \in \texttt{RSSYT}_P(\lambda, d)$ with $T < U$. Then there exists $V \in \texttt{RSSYT}_P(\lambda, d)$ so that $T < V \leq U$ and V differs from T by increasing the value of a single entry.

(3) Take $\mathcal{T}, \mathcal{U} \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$. Then

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \vee \mathcal{U}) \supseteq \mathcal{F}_P \quad \text{and} \quad \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) \supseteq \mathcal{F}_P.$$

Proof. (1) Let $T, U \in \texttt{RSSYT}_P(\lambda, d)$. Fix an antidiagonal D of λ. Let $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_m\}$ be the ordered lists of labels which appear in antidiagonal D of T and U respectively. Then the entries in antidiagonal D of $T \vee U$ are

$$\{\max(a_1, b_1), \ldots, \max(a_m, b_m)\} \subseteq \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}.$$

Since this holds for every antidiagonal,

$$P_{T \vee U} \subseteq P_T \cup P_U.$$

Therefore,

$$\Phi_{\lambda,d}(T \vee U) \supseteq \Phi_{\lambda,d}(T) \cap \Phi_{\lambda,d}(U) \supseteq \mathcal{F}_P.$$

The argument for $T \wedge U$ is the same.

(2) Suppose $T, U \in \texttt{RSSYT}_P(\lambda, d)$ and $T < U$. Since $T < U$ all entries of T are (weakly) less than the entries of U. Let $S = \{(i,j) : T_{ij} < U_{ij}\}$. Since $T \neq U$, there is some entry of T that is strictly less than the corresponding entry in U, so $S \neq \emptyset$.

Since S is finite and nonempty, there is some $(i,j) \in S$ so that $(i+1, j), (i, j-1) \notin S$. Then replace the (i,j) entry of T with U_{ij} and call this V. Then $V \in \texttt{RSSYT}(\lambda, d)$. Furthermore, the entries in each antidiagonal of V form a subset of the union of the antidiagonal entries of T and U so $\Phi_{\lambda,d}(V) \supseteq \Phi_{\lambda,d}(T) \cap \Phi_{\lambda,d}(U) \supseteq \mathcal{F}_P$. Then we have produced $V \in \texttt{RSSYT}_P(\lambda, d)$ so that $T < V < U$.

(3) By definition, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) = \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{U}) = \mathcal{F}_P$. Then $\Phi_{\lambda^{(i)},d_i}(T^{(i)}) \supseteq \mathcal{F}_P$ and $\Phi_{\lambda^{(i)},d_i}(U^{(i)}) \supseteq \mathcal{F}_P$ for all $i = 1, \ldots, k$. Applying (1), we have

$$\Phi_{\lambda^{(i)},d_i}(T^{(i)} \vee U^{(i)}) \supseteq \mathcal{F}_P \text{ and } \Phi_{\lambda^{(i)},d_i}(T^{(i)} \wedge U^{(i)}) \supseteq \mathcal{F}_P.$$

Therefore,

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \vee \mathcal{U}) = \Phi_{\lambda^{(1)},d_1}(T^{(1)} \vee U^{(1)}) \cap \ldots \cap \Phi_{\lambda^{(k)},d_k}(T^{(k)} \vee U^{(k)}) \supseteq \mathcal{F}_P$$

and

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) = \Phi_{\lambda^{(1)},d_1}(T^{(1)} \wedge U^{(1)}) \cap \ldots \cap \Phi_{\lambda^{(k)},d_k}(T^{(k)} \wedge U^{(k)}) \supseteq \mathcal{F}_P. \qquad \square$$

Proposition 4.19. *Fix $\mathcal{F}_P \in F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$.*

(1) $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is a lattice.

(2) Suppose $\mathcal{T}, \mathcal{U} \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ with $\mathcal{T} < \mathcal{U}$. Then $\mathcal{T}$ has an unstable triple.

(3) $|\texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)| = 1.$

Proof. (1) $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is a subposet of $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$. So it is enough to show that $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is closed under taking joins and meets.

By Lemma 4.18, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) \supseteq \mathcal{F}_P$. Since $\mathcal{F}_P \in F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$, this containment is actually an equality. Therefore, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$, i.e., $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is closed under joins. The argument for meets is the same. So we conclude $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is a lattice.

(2) Suppose $\mathcal{U} > \mathcal{T}$. In particular, for some i, we have $U^{(i)} > T^{(i)}$. By the part 2 of Lemma 4.18, there is $V \in \texttt{RSSYT}_P(\lambda^{(i)}, d_i)$ with $U^{(i)} \geq V > T^{(i)}$ so that V differs from $T^{(i)}$ by increasing the value a single entry.

Since $\Phi_{\lambda^{(\ell)},d_\ell}(T^{(\ell)}) \supseteq \mathcal{F}_P$ for all $\ell = 1, \ldots, k$ and $\Phi_{\lambda^{(i)},d_i}(V) \supseteq \mathcal{F}_P$, we have

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(T^{(1)}, \ldots, T^{(i-1)}, V, T^{(i+1)}, \ldots T^{(k)}) \supseteq \mathcal{F}_P. \tag{57}$$

Since $\mathcal{F}_P$ is a facet, (57) is an equality. Then

$$(T^{(1)}, \ldots, T^{(i-1)}, V, T^{(i+1)}, \ldots T^{(k)}) \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P).$$

So $\mathcal{T}$ has an unstable triple.

(3) By (1), $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is a lattice. In particular, it is finite and nonempty so has a unique maximum element.

By part (2), if $\mathcal{T}$ is not the maximum of $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$, then it has an unstable triple. Conversely, if $\mathcal{T}$ has an unstable triple, then by definition, there is $\mathcal{T}' \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ with $\mathcal{T} < \mathcal{T}'$. So $\mathcal{T}$ is not the maximum. Therefore, $\texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$ is the maximum of $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$. So

$$|\texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)| = 1. \qquad \square$$

Proof of Theorem 4.16. (1) Define $\Psi : F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}}) \to \texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d})$ by mapping $\mathcal{F}_P$ to the unique element in $\texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P)$. By Proposition 4.19 part (3), this is well defined. Injectivity follows since

$$\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_P) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{P'}) = \emptyset$$

whenever $P \neq P'$.

Given $\mathcal{T} \in \texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d})$, let $\mathcal{F}_P = \Phi_{\boldsymbol{\lambda},\mathbf{u}}$. By the definition of $\texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d})$, we have $\mathcal{F}_P \in F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$. Then $\mathcal{T} = \Psi(\mathcal{F}_P)$. As such, Ψ is surjective.

Since $\Phi_{\boldsymbol{\lambda},\mathbf{u}}$ is weight preserving, $\Phi(F_{\texttt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})) = \texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$. $\square$

5. Multidegrees and ASM varieties

5.1. **Multidegrees.** In this section, we review multidegrees. See [MS04, Chapter 8] for an introduction. We say $\Bbbk[\mathbf{z}]$ is **multigraded** by $\mathbb{Z}^n$ if there is a semigroup homomorphism $\mathtt{d} : \mathbb{N}^N \to \mathbb{Z}^n$. We may interpret $\mathtt{d}$ as a map from monomials in $\Bbbk[\mathbf{z}]$ to elements of $\mathbb{Z}^n$. As such, we write $\mathtt{d}(\mathbf{z}^{\mathbf{v}}) := \mathtt{d}(\mathbf{v})$.

Write $\Bbbk[\mathbf{z}]_{\mathbf{a}}$ for the $\Bbbk$ vector space which has as a basis the monomials of degree $\mathbf{a}$,

$$\{\mathbf{z}^{\mathbf{v}} : \mathtt{d}(\mathbf{z}^{\mathbf{v}}) = \mathbf{a}\}.$$

As a vector space, $\Bbbk[\mathbf{z}] = \bigoplus_{\mathbf{a} \in \mathcal{A}} \Bbbk[\mathbf{z}]_{\mathbf{a}}$. A $\Bbbk[\mathbf{z}]$-module M is multigraded by $\Bbbk[\mathbf{z}]$ if it has a direct sum decomposition $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ which satisfies

$$\Bbbk[\mathbf{z}]_{\mathbf{a}} \cdot M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$. We will assume that the multigrading is **positive**, that is each of the graded pieces of $\Bbbk[\mathbf{z}]$ are finite dimensional as $\Bbbk$ vector spaces.

Let $\mathcal{C}$ be a function from finitely generated, graded S modules to $\mathbb{Z}[x_1, \ldots, x_n]$. We say $\mathcal{C}$ is **additive** if for each M

$$\mathcal{C}(M; \mathbf{x}) = \sum_{i=1}^{k} \mathtt{mult}(M, \mathfrak{p}_i)\mathcal{C}(\Bbbk[\mathbf{z}]/\mathfrak{p}_i; \mathbf{x}). \tag{58}$$

Here, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ is the set of maximal dimensional associated primes of M and $\mathtt{mult}(M, \mathfrak{p})$ is the *multiplicity* of M at $\mathfrak{p}$. See [Eis95, Section 3.6].

Fix a monomial term order on $\Bbbk[\mathbf{z}]$. If $f \in \Bbbk[\mathbf{z}]$, write $\mathtt{init}(f)$ for its lead term. The **initial ideal** of I is

$$\mathtt{init}(I) := \{\mathtt{init}(f) : f \in I\}.$$

$\mathcal{C}$ is **degenerative** if given a graded free presentation F/K, we have

$$\mathcal{C}(F/K; \mathbf{x}) = \mathcal{C}(F/\mathtt{init}(K); \mathbf{x}). \tag{59}$$

Write $\langle \mathbf{a}, \mathbf{x} \rangle := a_1x_1 + a_2x_2 + \ldots + a_nx_n$.

Theorem 5.1. [MS04, Theorem 8.44] *There is a unique function $\mathcal{C}$ which is additive and degenerative so that*

$$\mathcal{C}(\Bbbk[\mathbf{z}]/\langle z_{i_1}, \ldots, z_{i_k}\rangle; \mathbf{x}) = \prod_{\ell=1}^{k} \langle \mathtt{d}(z_{i_\ell}), \mathbf{x}\rangle. \tag{60}$$

$\mathcal{C}(M; \mathbf{x})$ is called the **multidegree** of M.

Lemma 5.2. *Suppose I is a square-free monomial ideal in $\Bbbk[\mathbf{z}]$. Then*

$$\mathcal{C}(\Bbbk[\mathbf{z}]/I; \mathbf{x}) = \sum_{f \in F_{\mathtt{max}}(\Delta(I))} C(\Bbbk[\mathbf{z}]/\mathfrak{m}_{\overline{f}}; \mathbf{x}).$$

Proof. Since I is square-free, it has the prime decomposition

$$I = \bigcap_{f \in F(\Delta(I))} \mathfrak{m}_{\overline{f}}.$$

Squarefree monomial ideals are radical, and so

$$\mathtt{mult}(\Bbbk[\mathbf{z}]/I, \Bbbk[\mathbf{z}]/\mathfrak{m}_{\overline{f}}) = 1$$

whenever $f \in F(\Delta(I))$. The maximal dimensional associated primes of $\Bbbk[\mathbf{z}]$ are

$$\{\mathfrak{m}_{\overline{f}} : f \in F_{\mathtt{max}}(\Delta(I)).$$

Applying additivity,

$$\mathcal{C}(\Bbbk[\mathbf{z}]/I; \mathbf{x}) = \sum_{f \in F_{\mathtt{max}}(\Delta(I))} \mathcal{C}(\Bbbk[z])/\mathfrak{m}_{\overline{f}}; \mathbf{x}). \qquad \square$$

5.2. **ASM varieties.** Recall $\mathsf{Mat}(n)$ is the space of $n \times n$ matrices over an algebraically closed field $\Bbbk$. Write $\mathsf{GL}(n)$ for the invertible matrices in $\mathsf{Mat}(n)$ and T for the torus of diagonal matrices in $\mathsf{GL}(n)$. There is a natural action of $\mathsf{GL}(n)$, and hence T, on $\mathsf{Mat}(n)$ by left multiplication. A variety $X \subseteq \mathsf{Mat}(n)$ is T stable if $T \cdot X \subseteq X$.

Let $Z = (z_{ij})_{i,j=1}^n$ be a matrix of generic variables and write $\Bbbk[Z] = \Bbbk[z_{11}, z_{12}, \dots, z_{nn}]$ for the coordinate ring of $\mathsf{Mat}(n)$. Let $\mathtt{d}$ be the degree map defined by $\mathtt{d}(z_{ij}) = i$. The multigrading defined by the degree map corresponds to the action of T on $\mathsf{Mat}(n)$. In particular, T stable subvarieties of $\mathsf{Mat}(n)$ have coordinate rings that are $\Bbbk[Z]$-graded modules. When $\Bbbk[Z]/I$ is the coordinate ring of $X \subseteq \mathsf{Mat}(n)$, write $\mathcal{C}(X; \mathbf{x}) := \mathcal{C}(\Bbbk[Z]/I; \mathbf{x})$. In this situation, (58) becomes

$$\mathcal{C}(X; \mathbf{x}) = \sum_{i=1}^{k} \mathcal{C}(X_i; \mathbf{x}) \tag{61}$$

where $\{X_1, \dots, X_k\}$ are the maximal dimensional irreducible components of X. Since I is radical, (61) is a multiplicity free sum.

Given an $n \times n$ matrix M, write $M_{[i],[j]}$ for the submatrix of M which consists of the first i rows and j columns of M. Fix $w \in \mathsf{P}(n)$. The **matrix Schubert variety** is

$$X_w := \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \leq r_w(i,j) \text{ for all } 1 \leq i, j \leq n\}. \tag{62}$$

Matrix Schubert varieties generalize classical determinantal varieties. They were studied by W. Fulton, who showed that they are irreducible [Ful92].

Given $A \in \mathsf{ASM}(n)$, we define the **alternating sign matrix variety**

$$X_A := \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \leq r_A(i,j) \text{ for all } 1 \leq i, j \leq n\}.$$

Immediately by definition,

$$\text{if} \quad A \leq B \quad \text{then} \quad X_A \supseteq X_B. \tag{63}$$

Let $\mathsf{B}_-, \mathsf{B} \subset \mathsf{GL}_n$ be the Borel subgroups of lower triangular and upper triangular matrices respectively. There is a left action of $\mathsf{B}_- \times \mathsf{B}$ on $\mathsf{Mat}(n)$ given by

$$(b_1, b_2) \cdot M := b_1 M b_2^{-1}. \tag{64}$$

Write $\Omega_w = \mathsf{B}_- \times \mathsf{B} \cdot w$ for the orbit through the partial permutation w. We recall some facts about $\mathsf{B}_- \times \mathsf{B}$ orbits. See [MS04, Chapter 15].

Proposition 5.3. *(1) If $M \in \Omega_w$, then $\operatorname{rank}(M_{[i],[j]}) = r_w(i,j)$ for all $1 \leq i, j \leq n$.*
(2) There is a unique $w \in \mathsf{P}(n)$ in each $\mathsf{B}_- \times \mathsf{B}$ orbit.
(3) $X_w = \overline{\Omega_w}$. Furthermore, X_w is irreducible and has dimension $n^2 - \ell(w)$.

Proof. (1) The action of $\mathsf{B}_- \times \mathsf{B}$ on $M \in \mathsf{Mat}(n)$ is by row operations which sweep downwards and column operations which sweep to the right. Restricted to $M_{[i],[j]}$ this action is just row and column operations within $M_{[i],[j]}$. So $\operatorname{rank}(M_{[i],[j]})$ is stable under this action for all $1 \leq i, j \leq n$. In particular, $\operatorname{rank}(w_{[i],[j]}) = r_w(i,j)$, so the result follows.

(2) This is Proposition 15.27 in [MS04].

(3) See Theorem 15.31 [MS04]. □

X_A has the following set theoretic descriptions as unions and intersections of other ASM varieties.

Proposition 5.4. *(1)* $X_A = \bigcup_{w\in\mathtt{Perm}(A)} X_w.$

(2) If $A = \vee\{A_1,\dots,A_k\}$*, then* $X_A = \bigcap_{i=1}^{k} A_i$.

Proof. (1) ($\subseteq$) Fix $M \in X_A$. Then $M \in \Omega_w$ for some $w \in \mathsf{P}(n)$ and $r_w \le r_A$. By Corollary 4.7, there exists $w' \in \mathtt{Perm}(A)$ so that $w \ge w'$. So $M \in X_w \subseteq X_{w'}$. Hence $M \in \bigcup_{w\in\mathtt{Perm}(A)} X_w$.

($\supseteq$) If $w \in \mathtt{Perm}(A)$ then $w \ge A$. So by (63), $X_A \supseteq X_w$. Therefore,

$$X_A \supseteq \bigcup_{w\in\mathtt{Perm}(A)} X_w.$$

(2) ($\subseteq$) $A \ge A_i$ for all i. So by (63), $X_A \subseteq \bigcap_{i=1}^k X_{A_i}$.

($\supseteq$) Take $M \in \bigcap_{i=1}^k X_{A_i}$. Then $M \in \Omega_w$ for some $w \in \mathsf{P}(n)$. Since $M \in X_{A_i}$ for all i, we have $w \ge A_i$ for all i. So $w \ge A = \vee\{A_1,\dots,A_k\}$. Therefore $M \in \Omega_w \subseteq X_A$. □

W. Fulton showed that each X_w is defined by a smaller set of **essential** conditions,

$$X_w = \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \le r_w(i,j) \text{ for all } (i,j) \in \mathcal{E}ss(w)\}. \tag{65}$$

By Proposition 5.4, $X_A = \bigcap_{u\in\mathtt{biGr}(A)} X_u$. Therefore, ASM varieties are also defined by essential conditions.

$$X_A = \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \le r_A(i,j) \text{ for all } (i,j) \in \mathcal{E}ss(A)\}. \tag{66}$$

The rank of *any* submatrix is preserved under the action of T, so X_w is T stable. By [KM05, Theorem A], when $w \in \mathcal{S}_n$,

$$\mathcal{C}(X_w;\mathbf{x}) = \mathfrak{S}_w. \tag{67}$$

Proposition 5.5. $\mathcal{C}(X_A;\mathbf{x}) = \sum_{w\in\mathtt{MinPerm}(A)} \mathfrak{S}_w.$

Proof. As a consequence of Proposition 5.4, the top dimensional irreducible components of X_A are $\{X_w : w \in \mathtt{MinPerm}(A)\}$. Then using the additivity property of multidegrees and (67), we have

$$\mathcal{C}(X_A;\mathbf{x}) = \sum_{w\in\mathtt{MinPerm}(A)} \mathcal{C}(X_w;\mathbf{x}) = \sum_{w\in\mathtt{MinPerm}(A)} \mathfrak{S}_w.$$ □

Theorem 1.2 follows as an immediate consequence of Proposition 5.5 and Theorem 1.1.

5.3. **Northwest rank conditions.** It is possible to consider more general rank conditions than those defined by corner sums of ASMs. Let $\mathbf{r} = (r_{ij})_{i,j=1}^n$ with $r_{ij} \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. The **northwest rank variety** is

$$X_{\mathbf{r}} := \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \le r_{ij} \text{ for all } 1 \le i,j \le n\}. \tag{68}$$

W. Fulton showed that $X_{\mathbf{r}}$ is irreducible if and only if $\mathbf{r} = r_w$ for some $w \in \mathsf{P}(n)$. It is stable under the $\mathsf{B}_- \times \mathsf{B}$ orbit, so decomposes as a union of (partial) matrix Schubert varieties. Z. Xu-an and G. Hongzhu classified northwest rank varieties and gave an algorithm to decompose them into their irreducible components [ZG08].

We give an alternative discussion using the order theoretic properties of partial ASMs. *A priori*, $X_{\mathbf{r}}$ appears to be a more general object than an ASM variety. We will show, up to an affine factor, $X_{\mathbf{r}}$ is isomorphic to some ASM variety. Furthermore, $X_{\mathbf{r}} = X_{r_A}$ for some $A \in \mathsf{PA}(n)$. So northwest rank varieties are indexed by partial ASMs.

Lemma 5.6. *Let $A \in \mathsf{PA}(n)$ and $\widetilde{A} \in \mathsf{ASM}(N)$ its completion to an honest ASM. Then*

$$X_A \times \Bbbk^{N^2-n^2} \cong X_{\widetilde{A}}.$$

Proof. By construction, $r_{\widetilde{A}}(i,j) = r_A(i,j)$ for all $1 \le i,j \le n$. Therefore, if $M \in X_{\widetilde{A}}$ then $M_{[n],[n]} \in X_A$. Conversely, fix $L \in X_A$. We have $L \in \Omega_w$ for some $w \in \mathsf{P}(n)$, with $w \ge A$. Let L' be any matrix in $\mathsf{Mat}(N)$ so that $L'_{[n],[n]} = L$. Then $L' \in \Omega_v$ for some $v \in \mathsf{P}(N)$. Consider the completions $\widetilde{w}, \widetilde{v} \in \mathsf{ASM}(\infty)$. Since $A \le w$, we have $\widetilde{A} \le \widetilde{w} \in \mathsf{ASM}(\infty)$.

By construction, $\widetilde{w}$ is the minimum among elements of $\mathsf{ASM}(\infty)$ which restrict to w in $\mathsf{P}(n)$. Since $v_{[n],[n]} = w$, we have $\widetilde{A} \le \widetilde{w} \le \widetilde{v} \in \mathsf{ASM}(\infty)$. Then $\widetilde{A} \le \widetilde{v}_{[N],[N]} = v \in \mathsf{PA}(N)$. Therefore, $L' \in \Omega_v \subseteq X_{\widetilde{A}}$. As such, $X_{\widetilde{A}} \cong \{L \times \Bbbk^{N^2-n^2} : L \in X_A\}$. $\square$

Fix a rank function $\mathbf{r} = (r_{ij})_{i,j=1}^n$. Let

$$A_{\mathbf{r}} = \vee\{[i,j,r_{ij}]_b : r_{ij} < n\} \in \mathsf{PA}(n). \tag{69}$$

Proposition 5.7. $X_{A_{\mathbf{r}}} = X_{\mathbf{r}}$.

Proof. If $r_{ij} \ge n$, it is a vacuous rank condition on matrices in $\mathsf{Mat}(n)$. So we ignore these entries of $\mathbf{r}$. By definition, $X_{A_{\mathbf{r}}} = \bigcap X_{[i,j,r_{ij}]_b}$ with the intersection taken over (i,j) indexing nonvacuous rank conditions.

If $M \in X_{\mathbf{r}}$ we have $M \in X_{[i,j,r_{ij}]_b}$ for all $1 \le i,j \le n$. So $M \in X_{A_{\mathbf{r}}}$. Conversely, if $M \in X_{A_{\mathbf{r}}}$, then $\operatorname{rank}(M_{[i,j]}) \le r_A(i,j) \le r_{ij}$ whenever r_{ij} is a nonvacuous rank condition. So $M \in X_{\mathbf{r}}$. $\square$

Notice that unions of matrix Schubert varieties need not be northwest rank varieties.

Example 5.8. Let $X = X_{132} \cup X_{213}$. If $X = X_{\mathbf{r}}$, then $r_{132}, r_{213} \le \mathbf{r}$. So $r_{132} \wedge r_{213} \le \mathbf{r}$. But $r_{132} \wedge r_{213} = r_{123}$. Since $\mathtt{dim}(X_{123}) > \mathtt{dim}(X)$, it follows that X can not be defined by a list of northwest rank conditions. $\square$

5.4. **ASM determinantal ideals.** We now turn our discussion to defining ideals for ASM varieties. Define the **ASM ideal** by

$$I_A := \langle \text{ minors of size } r_A(i,j)+1 \text{ in } Z_{[i],[j]}\rangle. \tag{70}$$

A matrix has rank at most r if and only if all of its minors of size $r+1$ vanish. As such, I_A set-theoretically cuts out X_A. Furthermore, I_A has generators which are homogeneous for the $\mathbb{Z}^n$ grading on $\Bbbk[Z]$.

Lemma 5.9. *(1) If $r_A \le r_B$ then $I_A \supseteq I_B$.*
(2) $I_A = \displaystyle\sum_{u \in \mathtt{biGr}(A)} I_u = \langle$ minors of size $r_A(i,j)+1$ in $Z_{[i],[j]} : (i,j) \in \mathcal{E}ss(A)\rangle$.

Proof. (1) Define

$$I^r_{i,j} = \langle \text{ minors of size } r+1 \text{ in } Z_{[i],[j]}\rangle. \tag{71}$$

We may compute each minor by iteratively doing row expansions. As such,

$$\text{if } r \le r' \text{ then } I_{i,j}^{r} \supseteq I_{i,j}^{r'}. \tag{72}$$

So suppose $r_A \le r_B$. Then

$$I_A = \sum_j \sum_i I_{i,j}^{r_A(i,j)} \supseteq \sum_j \sum_i I_{i,j}^{r_B(i,j)} = I_B.$$

(2) For each (i,j) there is $u \in \texttt{biGr}(A)$ so that $r_A(i,j) = r_u(i,j)$. As such, $I_{i,j}^{r_A(i,j)} \subseteq I_u$ for some u in $\texttt{biGr}(A)$. By part (1), $I_u \subseteq I_A$, for all $u \in \texttt{biGr}(A)$. Therefore

$$I_A = \sum_j \sum_i I_{i,j}^{r_A(i,j)} \subseteq \sum_{u \in \texttt{biGr}(A)} I_u \subseteq I_A.$$

□

To distinguish between the two generating sets of I_A, we refer to

$$\texttt{Gen}(A) = \{ \text{ minors of size } r_A(i,j)+1 \text{ in } Z_{[i],[j]} \}$$

as the **defining generators** of I_A. Call

$$\texttt{EssGen}(A) = \{ \text{ minors of size } r_A(i,j)+1 \text{ in } Z_{[i],[j]} : (i,j) \in \mathcal{E}ss(A) \}$$

the **essential generators** of I_A.

Example 5.10. Let A be as in Example 3.6. We have $\mathcal{E}ss(A) = \{(1,2),(2,3)\}$. Furthermore, $r_A(1,2) = 0$ and $r_A(2,3) = 1$. Applying Lemma 5.9 yields

$$\begin{aligned} I_A &= \langle z_{11}, z_{12}, \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \rangle \\ &= \langle z_{11}, z_{12,}, z_{13}z_{21}, z_{13}z_{22} \rangle \\ &= \langle z_{11}, z_{12}, z_{21}, z_{22} \rangle \cap \langle z_{11}, z_{12}, z_{13} \rangle \\ &= I_{3412} \cap I_{4123}. \end{aligned}$$

This agrees with the irreducible decomposition $X_A = X_{3412} \cup X_{4123}$. Notice by additivity, $\mathcal{C}(\Bbbk[Z]/I_A; \mathbf{x}) = \mathcal{C}(\Bbbk[Z]/I_{4123}) = x_1^3 = \mathfrak{S}_{4123}$. □

An **antidiagonal** term order on $\Bbbk[Z]$ is a term order for which the lead term of any minor in Z is the product of its antidiagonal terms. From now on, fix an antidiagonal term order $\prec$ on $\Bbbk[Z]$. A **Gröbner basis** for I is a set $\{g_1, \ldots, g_k : g_i \in \Bbbk[Z]\}$ so that

(1) $I = \langle g_1, \ldots, g_k \rangle$, and
(2) $\texttt{init}(I) = \langle \texttt{init}(g_1), \ldots, \texttt{init}(g_k) \rangle$.

Proof of Proposition 1.3. (1) If $w \in \mathcal{S}_n$, by Section 7.2 of [Knu09], there is a Frobenius splitting for which X_w is compatibly split. Since $X_A = \bigcap_{u \in \texttt{biGr}(A)} X_u$, it is also compatibly split.

By the argument in [Stu90], $\texttt{EssGen}(u)$ is a Gröbner basis for I_u. Since $\mathcal{B}_n$ is the base of $\mathsf{ASM}(n)$, we may apply part (2) of [KM05, Theorem 6], since

$$\texttt{EssGen}(A) = \bigcup_{u \in \texttt{biGr}(A)} \texttt{EssGen}(u),$$

it is a Gröbner basis for I_A. Since $\texttt{Gen}(A) \supseteq \texttt{EssGen}(A)$, we have that $\texttt{Gen}(A)$ is also a Gröbner basis for I_A.

(2) The lead terms of $\texttt{EssGen}(A)$ are are square-free, hence $\texttt{init}(I_A)$ is radical. Since I_A degenerates to a radical ideal, it is itself radical.

(3) By [KM05, Theorem B], if $w \in S_n$,

$$\Delta(\texttt{init}(I_w)) = \Delta(Q_{n\times n}, w). \tag{73}$$

From part (1),

$$\texttt{init}(I_A) = \sum_{u \in \texttt{biGr}(A)} \texttt{init}(I_u).$$

Therefore,

$$\begin{aligned}
\Delta(\texttt{init}(I_A)) &= \bigcap_{u \in \texttt{biGr}(A)} \Delta(\texttt{init}(I_u)) && \text{(by Lemma 4.4)}\\
&= \bigcap_{u \in \texttt{biGr}(A)} \Delta(Q_{n\times n}, u) && \text{(by (73))}\\
&= \Delta(Q_{n\times n}, A) && \text{(by part (2) of Proposition 4.8.)}
\end{aligned}$$

□

The discussion in [Knu09] assumes $\Bbbk = \mathbb{Q}$. However, since the defining generators of I_A have coefficients in $\{\pm 1\}$, the generators are actually Gröbner over $\mathbb{Z}$, and so the statement holds more generally. Applying Lemma 5.2, we can also compute $\mathcal{C}(X_A; \mathbf{x})$ as the weighted sum over

$$F_{\texttt{max}}(\Delta(\texttt{init}(I_A))) = F_{\texttt{max}}(\Delta_A).$$

Theorem 4.16 gives a weight preserving bijection between $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ and $F_{\texttt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$. This produces a specific connection between the Gröbner geometry of $X_{A_{\boldsymbol{\lambda},\mathbf{d}}}$ and prism tableaux.

Acknowledgements

I thank my advisor, Alexander Yong, for his guidance throughout this project. I also thank Allen Knutson for suggesting this direction of research and Jessica Striker for helpful conversations about alternating sign matrices. I was supported by a UIUC Campus Research Board and by an NSF Grant. This work was partially completed while participating in the trimester "Combinatorics and Interactions" at the Institut Henri Poincaré. My travel support was provided by NSF Conference Grant 1643027. I was funded by the Ruth V. Shaff and Genevie I. Andrews Fellowship. I used Sage and Macaulay2 during the course of my research.

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