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Homological properties of the homology algebra of the Koszul complex of a local ring: Examples and questions

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A R T I C L E   I N F O

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A B S T R A C T

Let R be a local commutative noetherian ring and HKR the homology ring of the corresponding Koszul complex. We study the homological properties of HKR in particular the Avramov spectral sequence. When the embedding dimension of R is four and when R can be presented with quadratic relations we have found 101 cases where this spectral sequence degenerates and only three cases where it does not degenerate. We also determine completely the Hilbert series of the bigraded Tor of these HKR in Tables A–D at the end of the paper. We also study some higher embedding dimensions. Among the methods used are the programme `BERGMAN` by Jörgen Backelin et al., the `Macaulay2`-package `DGAlgebras` by Frank Moore, combined with results by Govorov, Clas Löfwall, Victor Ufnarovski and others.

0. Introduction and Main Theorem

Let (R, m, k) be a local commutative noetherian ring with maximal ideal m and residue field $k = R/m$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a minimal set of generators of the

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maximal ideal m. The Koszul complex of R is the exterior algebra $\oplus_{i=0}^{n} \Lambda^i R^n$ of a free R-module of rank n equipped with the differential:

$$d(T_{j_1} \wedge \ldots \wedge T_{j_i}) = \sum_{l=1}^{i} (-1)^{l+1} x_{j_l} T_{j_1} \wedge \ldots \wedge \widehat{T}_{j_l} \wedge \ldots \wedge T_{j_i}$$

It will be denoted by $K(\mathbf{x}, R)$ or KR since it is essentially independent of $\mathbf{x}$. It is a differential graded algebra and its homology algebra HKR is a skew-commutative graded algebra over k. In the case that R can be represented as a quotient of a regular local ring $(\tilde{R}, \tilde{m})$ as $R = \tilde{R}/a$, where $a \subset \tilde{m}^2$ (passing to a completion of R we can always assume that this is the case for the problems we are studying), we have that HKR is isomorphic to a Tor-algebra:

$$HKR \simeq \mathrm{Tor}_*^{\tilde{R}}(\tilde{R}/a, k)$$

This algebra HKR has been studied in various special cases by many authors: it is an exterior algebra if and only if R is a local complete intersection; it is a Poincaré duality algebra if and only if R is a Gorenstein local ring [2] and if R is a Golod ring then the square of the augmentation ideal (*i.e.* the ideal generated by the generators of positive degree) of HKR is 0 (the converse is however not true, even for rings with monomial relations [9]). But the general structure of HKR (and in particular its homological properties) can be very complicated, even if R is a Koszul ring as we will see below. The aim of the present paper is to combine different methods towards studying HKR. We will illustrate our methods on very precise examples, thereby discovering some new unexpected phenomena. We note that HKR is graded, and one of our aims is to determine the double series

$$\Phi_R(x, y) = \sum_{p \geq 0, q \geq 0} |\mathrm{Tor}_{p,q}^{HKR}(k, k)| x^p y^q \tag{1}$$

(where $|V|$ denotes the dimension of a k-vector space V) for most (probably essentially all) quadratic rings R of embedding dimension 4: Let $P_R(z) = \sum_{n \geq 0} |\mathrm{Tor}_n^R(k, k)| z^n$. We will see that with the exception of three explicit cases we have $\Phi_R(z, z) = P_R(z)/(1+z)^4$. This last equality can be expressed by saying that the Avramov spectral sequence *degenerates i.e.* degenerates from the second page. Recall that this spectral sequence was introduced in [3] and further studied in [4, formula (6.2.1)]. It is as follows for any local ring R

$$E_n^2 = \bigoplus_{p+q=n} E_{p,q}^2 = \bigoplus_{p+q=n} \mathrm{Tor}_{p,q}^{HKR}(k, k) \Longrightarrow \mathrm{Tor}^R(k, k) \otimes_{KR \otimes k} k$$

In particular there is a coefficientwise inequality $\ll$ of formal power series

$$\frac{P_R(z)}{(1+z)^m} \ll \Phi_R(z,z)$$

where m is the embedding dimension of R. Of the three exceptional cases in embedding dimension 4, two are Koszul rings with Hilbert series $1+4z+3z^2$ and $(1+3t-2t^3)/(1-t)$ respectively. In these two cases *HKR* is far from being a Koszul ring, indeed for the first case the bigraded Ext-algebra $\mathrm{Ext}^*_{HKR}(k,k)$ has also generators in bidegrees $(3,8)$ and $(3,9)$. Since there are other Koszul rings having the same series $1+4z+3z^2$ without this strange behavior, we have here isolated new homological invariants even for Koszul rings. Let us recall that the Avramov spectral sequence always degenerates in embedding dimension ≤ 3 and for higher embedding dimensions a consequence of it can be expressed by saying that *HKR* and its Massey products determine $P_R(z)$. The converse is not true and the present paper shows that by studying the homology of *HKR* directly we sometimes obtain interesting invariants for R. Our methods also show e.g. that for the ring $k[x,y,z,u]/(x^2,xy,yz,zu,u^2)$ the Avramov spectral sequence *does* degenerate, despite the fact that there is no differential algebra structure on the minimal resolution of the ideal (x^2,xy,yz,zu,u^2) over $k[x,y,z,u]$, thereby answering a question by Avramov in [4] in the negative. We will use a very useful package `DGAlgebras` (written by Frank Moore [15]) which runs under Macaulay2. This package gives in particular an explicit presentation with generators and relations for *HKR*, when you read in the ring R. This presentation will then be analyzed, using results by Clas Löfwall [11] (the cube of the augmentation ideal of *HKR* is often 0), Jörgen Backelin et al. [6] (the programme `BERGMAN`), Victor Ufnarovski (how to use `BERGMAN` for determining Hilbert series of noncommutative graded algebras, where some of the generators have degrees > 1) and others. In the tables of Appendix A we have described 83 different cases of the double series

$$P_R(x,y) = \sum_{p\geq 0, q\geq 0} |\mathrm{Tor}^R_{p,q}(k,k)| x^p y^q \tag{2}$$

when R is a quotient of a polynomial ring $k[x,y,z,u]$ with an ideal generated by quadratic forms in (x,y,z,u). (Note that the previous $P_R(z) = P_R(z,1)$.) We have probably obtained all different such cases, but inside each case there are sometimes variations due to the behavior of *HKR*, described in Tables A, B, C, D at the end of this paper (this is like the periodic table of the elements, where we also have isotopes). Examples of this are the cases **46**, **63** and **71**. More precisely here is our

Main Theorem. *Let $R = k[x,y,z,u]/(f_1,f_2,\ldots,f_t)$, where k is a field of characteristic 0 and the f_i are quadratic forms in the variables x,y,z,u and $P_R(x,y)$ is given by (2) above. Then there are 83 different cases known for the $P_R(x,y)$ (they are given in Appendix A). Inside each of these cases $P_R(x,y)$ there are sometimes "isotopes" having different homological properties of the homology algebra HKR of the Koszul complex. In total there are 21 such extra "isotopes" known, the most extreme case of $P_R(x,y)$ is case*

71 *which is a Koszul algebra, but which has 4 extra isotopes. With the exception of three of these* $104 = 83 + 21$ *cases we have that* $\Phi_R(z,z) = P_R(z)/(1+z)^4$ *(i.e. the Avramov spectral sequence degenerates). The three exceptional cases being*

a) *Case 63ne (an isotope of Case 63) given by the ideal* $(x^2, xy, xz+u^2, xu, y^2+z^2, zu)$ *and where*

$$(1+z)^4/P_{R_{63ne}}(z) - 1/\Phi_{R_{63ne}}(z,z) = z^7(1+z)/\left(1-z+z^2\right)$$

b) *Case 71v16 (an isotope of Case 71) given by the ideal* $(xz+u^2, xy, xu, x^2, y^2+z^2, zu, yz)$ *and where*

$$(1+z)^4/P_{R_{71v16}}(z) - 1/\Phi_{R_{71v16}}(z,z) = z^7(1+z)/\left(1-z+z^2\right)$$

c) *Case 60 given by the ideal* $(x^2+yz+u^2, xy, zu, z^2, xz+yu, xu)$ *and where*

$$(1+z)^4/P_{R_{60}}(z) - 1/\Phi_{R_{60}}(z,z) = z^7(1+z)\left(1-3z^2-z^3\right)/\left(1-z+z^2\right)$$

Note that a) and b) are Koszul rings. We have also calculated the two-variable $\Phi_R(x,y)$ for the 104 cases. The results are given in Tables A–D at the end of the paper and the methods used for this are mentioned in Section 5. In Section 1 we start with case a) of the Main Theorem as a typical example of the calculation of *HKR* and its homological properties. In Sections 2 and 3 we treat the cases c) and b) of our Main Theorem. Section 4 solves a question of Avramov and at the end of the paper we give the tables (Tables A–D just mentioned) where hopefully all cases of Φ_R for the embedding dimension 4 cases (quadratic relations) have been calculated. We assume the characteristic of k to be 0 throughout except in Section 6 where we present conjectures and some results for the higher embedding dimension cases.

We finally wish to thank Clas Löfwall, Jörgen Backelin and Victor Ufnarovski for stimulating discussions. I also thank Luchezar Avramov and the referee for useful comments and suggestions.

1. A Koszul ring in 4 variables with 6 relations which is exceptional

Recall first that if *HKR* is the homology of the Koszul complex of a ring R of embedding dimension m as in the introduction, then there is a spectral sequence (the Avramov spectral sequence [4])

$$E_n^2 = \oplus_{p+q=n} \mathrm{Tor}_{p,q}^{HKR}(k,k) => \mathrm{Tor}^R(k,k) \otimes_{KR \otimes k} k$$

leading to the inequality

$$P_R(z)/(1+z)^m \le \sum_{n\ge 0}\sum_{p+q=n} |\mathrm{Tor}^{HKR}_{p,q}(k,k)| z^n$$

where we have equality if and only if the spectral sequence degenerates.

Before we analyze this spectral sequence, in particular E^2_n we first recall some results about quadratic algebras and their Koszul duals. A quadratic algebra over a field k is a quotient $A = T(V)/(W)$ where V is a vector space over k, $T(V)$ is the tensor algebra over V, W is a vector subspace of $V \otimes_k V$ and (W) is the two-sided ideal in $T(V)$ generated by W. This algebra is graded if we give the generators of V the grade 1 (the generators of W correspond to quadratic relations). Let us now assume that V has a finite basis. The quadratic (Koszul) dual $A^!$ of A is defined as follows: the inclusion map $\phi : W \to V \otimes V$ defines by duality an exact sequence

$$0 \longrightarrow \phi^\perp \longrightarrow V^* \otimes V^* \longrightarrow W^* \longrightarrow 0$$

where the dual of a k-vector space U is denoted by $U^* = \mathrm{Hom}_k(U,k)$. (Note that $(V \otimes_k V)^*$ can now be identified with $V^* \otimes_k V^*$.) Here $\phi^\perp$ is the subspace of $V^* \otimes V^*$ consisting of those maps $f : V \otimes V \to k$ such that f restricted to W is zero. Then we define $A^! = T(V^*)/(\phi^\perp)$. Clearly $A^!$ is also a quadratic algebra and we have that $(A^!)^!$ is isomorphic to A. The commutative polynomial ring $k[x,y,z,u]$ can e.g. be written as a quadratic algebra $A = k\langle x,y,z,u\rangle/(xy-yx, xz-zx, xu-ux, yz-zy, yu-uy, zu-uz)$ where $k\langle x,y,z,u\rangle$ is the free associative algebra on generators corresponding to the x,y,z,u and in this case

$$\begin{aligned} A^! = {} & k\langle X,Y,Z,U\rangle/ \\ & \big(X^2, Y^2, Z^2, U^2, XY+YX, XZ+ZX, XU+UX, YZ+ZY, YU+UY, ZU+UZ\big) \end{aligned}$$

where the variables X,Y,Z,U are dual variables to the x,y,z,u so that $A^!$ is the exterior algebra on the X,Y,Z,U, and $(A^!)^!$ gives back $k[x,y,z,u]$. In general, if A is any quadratic algebra, it can be shown that $A^!$ can be identified with the subalgebra of the Yoneda Ext-algebra $\mathrm{Ext}^*_A(k,k)$, generated by $\mathrm{Ext}^1_A(k,k)$. These definitions also work if A is homogeneous but also has relations in higher degrees as happens sometimes for our examples. Let us now consider the ring

$$R = k[x,y,z,u]/\big(x^2, xy, zu, xu, y^2+z^2, xz+u^2\big)$$

This is the case a) of the main theorem above. The Koszul dual of this ring is

$$R^! = k\langle X,Y,Z,U\rangle/(YU+UY, YZ+ZY, YY-ZZ, XZ+ZX-UU)$$

and the Gröbner basis of the ideal in $R^!$ is *quadratic* if we use the ordering of the variables Y,U,Z,X. Thus $R^!$ and R are Koszul rings and their Hilbert series are $(1+z)/(1-3z+2z^3)$ and $(1+3t-2t^3)/(1-t)$ respectively. In general if A is (skew)commutative, the Hilbert series of $A^!$ can be very complicated (in the examples I and III of Section 6 these series are transcendental functions). In this case a) of the main theorem $R^!$ has a nice series. However, the homology of the Koszul complex of R, namely HKR and its homology as an algebra are rather complicated: We now use the Macaulay2 package `DGAlgebras` with the input file:

```
loadPackage(''DGAlgebras'')
R=QQ[x,y,z,u]/ideal(x*z+u^2,x*y,x*u,x^2,y^2+z^2,z*u)
res(ideal R); betti oo
res(coker vars R,LengthLimit => 7); betti oo
HKR=HH koszulComplexDGA(R)
generators HKR
for n from 1 to length(generators HKR) list degree X_n
ideal HKR
I=ideal(vars HKR)
I^2; trim(oo)
I^3; trim(oo)
res(coker vars HKR,LengthLimit => 4);
betti oo
```

From the output file of this we get first the important fact that the algebra HKR has the cube of its augmentation ideal $I = \overline{HKR}$ equal to 0. This first result is useful for determining the homological properties of HKR, in particular for determining the series

$$\Phi_R(x,y) = \sum_{p,q\geq 0} |\mathrm{Tor}_{p,q}^{HKR}(k,k)| x^p y^q.$$

Indeed, we can now use a general result by Clas Löfwall, which essentially says that many homological properties of a graded algebra, whose augmentation ideal has cube 0 are determined by its Hilbert series and the subalgebra of the whole Ext-algebra generated (under Yoneda product) of the $\mathrm{Ext}^1(k,k)$ just introduced. His first results from 1976 in this direction are in [12]. More precisely, Theorem 1.3 on page 310 of [11] (note that the citation page 19 in this Theorem should be page 309 of [11]) says that using the notations of [11] our $\Phi_R(x,y)$ of (1) is equal to Löfwall's $P_{HKR}(x,1,y,1)$ and the Löfwall formula gives

$$\Phi_R(x,y) = xH_A(x,y)/\big(1 + x - H_A(x,y)\big(1 - H_{I/I^2}(y)x + H_{I^2}(y)x^2\big)\big) \tag{3}$$

where $H_A(x,y)$ is the Hilbert series in two variables of the algebra A which is the subalgebra generated by $\mathrm{Ext}^1_{HKR}(k,k)$ of the Yoneda Ext-algebra $\mathrm{Ext}^*_{HKR}(k,k)$. This algebra is equal also to the Koszul dual of HKR, which in his case can be presented as

$$T\big((I/I^2)^*\big)/\big(\big(im\big(I/I^2 \otimes I/I^2 \to I^2\big)^*\big)$$

where T is the tensor algebra on $(I/I^2)^*$ which is the k-vector space dual of I/I^2 and $(im(I/I^2 \otimes I/I^2 \to I^2)^*)$ is the twosided ideal in T generated by the dual of the image of the natural multiplication map of HKR. It is thus natural to write $HKR^!$ instead of A and we will often write the preceding formula as

$$1/\Phi_R(x,y) = (1+1/x)/HKR^!(x,y) - HKR(-x,y)/x \tag{4}$$

where $HKR(x,y)$ is the Hilbert series in two variables of HKR. We now use this result and continue the analysis of the output file. Formula (3) shows that to get $\Phi_R(x,y)$ we need the Hilbert series $A(x,y) = HKR^!(x,y)$ of A and the two Hilbert series $H_{I/I^2}(y)$ and $H_{I^2}(y)$, the most difficult part being the determination of $A(x,y)$ and we start with that. We see that we can determine a presentation of HKR in our special (and typical) case as having 6 generators $X_1,\dots,X_6$ of degree $(1,2)$, 6 generators $X_7,\dots,X_{12}$ of degree $(2,3)$ and one generator X_{13} of degree $(3,4)$ (here the second element of these pairs of degrees comes from the grading of R and will be ignored), satisfying the following *quadratic* relations:

$$X_5X_6, X_4X_6, X_2X_6, X_1X_6, X_4X_5 - X_3X_6, X_2X_5, X_1X_5, X_2X_4, X_1X_4, X_1X_3, X_1X_2,$$
$$X_6X_{12}, X_5X_{12}, X_2X_{12}, X_1X_{12}, X_6X_{11} - X_4X_{12}, X_5X_{11} - X_3X_{12}, X_1X_{11}, X_6X_{10},$$
$$X_5X_{10}, X_4X_{10}, X_2X_{10}, X_1X_{10}, X_6X_9, X_5X_9, X_4X_9, X_3X_9 + X_4X_{12}, X_2X_9,$$
$$X_1X_9, X_6X_8, X_5X_8 - X_4X_{12}, X_4X_8, X_3X_8 + X_4X_{11}, X_2X_8, X_1X_8, X_6X_7, X_5X_7,$$
$$X_4X_7, X_3X_7 + X_2X_{11}, X_2X_7, X_1X_7, X_6X_{13}, X_5X_{13}, X_4X_{13}, X_2X_{13}, X_1X_{13}, X_{12}X_{12},$$
$$X_{11}X_{12}, X_{10}X_{12}, X_9X_{12}, X_8X_{12}, X_7X_{12}, X_{10}X_{11} + X_3X_{13}, X_9X_{11},$$
$$X_8X_{11}, X_7X_{11}, X_{10}X_{10}, X_9X_{10}, X_8X_{10}, X_7X_{10}, X_9X_9,$$
$$X_8X_9, X_7X_9, X_8X_8, X_7X_8, X_7X_7, X_{12}X_{13}, X_{11}X_{13}, X_{10}X_{13}, X_9X_{13}, X_8X_{13}, X_7X_{13}.$$

Now, using these relations we get that the Koszul dual A of HKR is the quotient of the free associative algebra $k\langle Y_1,\dots,Y_{13}\rangle$ on the dual generators Y_i of the X_i with the ideal generated by

$$[Y_2,Y_3], [Y_3,Y_4], [Y_3,Y_5], [Y_4,Y_5] + [Y_3,Y_6], [Y_3,Y_{10}], [Y_3,Y_{11}], Y_{11}Y_{11}, [Y_3,Y_7] - [Y_2,Y_{11}],$$
$$[Y_5,Y_{11}] + [Y_3,Y_{12}], [Y_{10},Y_{11}] - [Y_3,Y_{13}], [Y_3,Y_8] - [Y_4,Y_{11}],$$
$$[Y_4,Y_{12}] + [Y_6,Y_{11}] + [Y_5,Y_8] - [Y_3,Y_9].$$

Here we have written e.g. $[Y_2,Y_3] = Y_2Y_3 - Y_3Y_2$ and e.g. $[Y_3,Y_{13}] = Y_3Y_{13} - Y_{13}Y_3$ for the "odd" generators $Y_i, i = 1\dots6$ and Y_{13}. Furthermore we have written e.g. $[Y_{10},Y_{11}] = Y_{10}Y_{11} + Y_{10}Y_{11}$ for the "even" generators and e.g. $[Y_3,Y_{10}] = Y_3Y_{10} + Y_{10}Y_3$ when the even and odd generators are mixed. Now to use the formula of Löfwall, we need to calculate the Hilbert series $A(x,y)$ of A with `BERGMAN` [6] using the presentation we have just determined, where the bigrading of the preceding generators should now be 1 for the first degree (corresponding to x) and for the second degree (corresponding to y) we should have 1 for $Y_i(i = 1,\dots6)$, 2 for $Y_i(i = 7,\dots12)$ and 3 for Y_{13}. But `BERGMAN` can for the moment only directly handle Hilbert series of algebras where the generators have only degree 1. But if for the moment we are only interested in the generating series of the total terms

$$\Phi_R(x,x) = \sum_{n=0}^{\infty} \sum_{p+q=n} |\mathrm{Tor}^{HKR}_{p,q}(k,k)| x^n$$

which occur in the Avramov spectral sequence it is sufficient to determine the Hilbert series $H_A(x,x)$ of A, i.e. to give the variables $Y_i (i = 1, \dots 6)$ degree 2, the variables $Y_i (i = 7, \dots 12)$ degree 3 and the variable Y_{13} degree 4, keep the relations as above and use the following Lemma that Victor Ufnarovski has been kind to communicate to me:

Lemma *(Ufnarovski). Let A be a graded algebra which is quotient of a free algebra with n generators Y_i of degrees $d_i \geq 1$ by an ideal $I = (f_1, \dots, f_m)$ which is homogeneous with respect to the given grading. Let now B be a new graded algebra obtained from A as follows: Introduce a new variable t of degree 1 and new variables Z_i of degrees 1, and let B be the quotient of the free associative algebra on the variables t, Z_i with the "same" ideal as in A, but where we have replaced each Y_i by $t^{d_i - 1} Z_i$. Then we have the following relation between the Hilbert series of A and B:*

$$1/A(z) = 1/B(z) + (n+1)z - \sum_{i=1}^{n} z^{d_i}$$

i.e. we can reduce the calculation of $A(z)$ to the calculation of $B(z)$ which can be done directly in `BERGMAN` since all generators of B have degree 1.

The proof of the Lemma will be given at the end of this section.

Remark. There is an analogous result in Ufnarovski's book [19], Lemma on page 141.

We now continue with the proof of case a) of our main theorem:

We follow the recipe of the Ufnarovski Lemma. Recall that the variables Y_i of degree $(1,2)$ should now be considered to have degree 2 etc.

We introduce a new variable t of degree 1 and replace the variables Y_i above as follows:

$$Y_i = tZ_i,\ 1 \leq i \leq 6;\ Y_i = t^2 Z_i,\ 7 \leq i \leq 12,\ Y_{13} = t^3 Z_{13}$$

where the new Z_i:s are given degree 1. Now the algebra A becomes the quotient of the free associative algebra in 14 generators of degree 1: $k\langle Z_1, \dots, Z_6, t, Z_7 \dots, Z_{13}\rangle$ with the ideal above where e.g. $[Y_2, Y_3]$ is replaced by $tZ_2tZ_3 - tZ_3tZ_2$ and $[Y_{10}, Y_{11}] - [Y_3, Y_{13}]$ is replaced by

$$t^2 Z_{10} t^2 Z_{11} + t^2 Z_{11} t^2 Z_{10} - tZ_3 t^3 Z_{13} + t^3 Z_{13} t Z_3$$

etc. The new algebra is denoted by B and the old one by A and the Ufnarovski Lemma gives that we have the following relation between their Hilbert series $H_A(z)$ and $H_B(z)$:

$$1/H_A(z) = 1/H_B(z) + 14z - 6z^2 - 6z^3 - z^4 \tag{5}$$

It remains to calculate $H_B(z)$. For this we can now directly use BERGMAN on the following input file which we call inbcaseB (we have written xi instead of Z_i):

```
(noncommify)
(algforminput)
vars x1,x2,x3,x4,x5,x6,t,x7,x8,x9,x10,x11,x12,x13;
t*x2*t*x3-t*x3*t*x2,t*x3*t*x4-t*x4*t*x3,t*x3*t*x5-t*x5*t*x3,
t*x4*t*x5-t*x5*t*x4+t*x3*t*x6-t*x6*t*x3,t*x3*t^2*x10+t^2*x10*t*x3,
t*x3*t^2*x11+t^2*x11*t*x3,
t^2*x11*t^2*x11,t*x3*t^2*x7+t^2*x7*t*x3-t*x2*t^2*x11-t^2*x11*t*x2,
t*x5*t^2*x11+t^2*x11*t*x5+t*x3*t^2*x12+t^2*x12*t*x3,
t^2*x10*t^2*x11+t^2*x11*t^2*x10-t*x3*t^3*x13+t^3*x13*t*x3,
t*x3*t^2*x8+t^2*x8*t*x3-t*x4*t^2*x11-t^2*x11*t*x4,
t*x4*t^2*x12+t^2*x12*t*x4+t*x6*t^2*x11+t^2*x11*t*x6+t*x5*t^2*x8+
t^2*x8*t*x5-t*x3*t^2*x9-t^2*x9*t*x3;
```

The following command in BERGMAN:

```
(ncpbhgroebner ''inbcaseB'' ''t1'' ''t2'' ''t3'')
```

gives after a few seconds t3, i.e. the Hilbert series for B in degrees ≤ 20 and ≥ 2, so that

$$1/H_B(z) = 1 - 14z + 4z^4 + 6z^5 + 2z^6 - z^8 - z^9 + z^{11} + z^{12} - z^{14} - z^{15} + z^{17} + z^{18} - z^{20} + O\big(z^{21}\big) \tag{6}$$

leading to the guess (alternatively using the maple command convert('',ratpoly) on the previous formula (6)) that

$$1/H_B(z) = \frac{1 - 15z + 15z^2 - 14z^3 + 4z^4 + 2z^5 + 4z^7 + z^8}{1 - z + z^2} \tag{7}$$

But this is not a proof. To get a proof we will analyze the file t1, i.e. the Gröbner basis of the ideal in B. For the ordering of the variables we have given in inbcaseB this Gröbner basis is infinite. But, at my request some time ago Jörgen Backelin wrote an addition permutebreak.sl to the programme BERGMAN which checks the Gröbner bases up to a certain given degree of the different permutations of the variables of a given input to the programme BERGMAN. In the present case this seems rather discouraging in view of the fact that $14! = 87178291200$ but rather immediately we see that switching the variables x2,x3 to x3,x2 in the vars line of inbcaseB gives a finite Gröbner bases (in degrees ≤ 8). Even better: after a short time with breaking degree 12, checking about 2700 cases, we obtain that the ordering x1,x2,x4,x5,x6,t,x3,x7,x8,x9,x10,x11,x12,x13 gives a finite Gröbner bases with 4 elements of degree 2, 6 elements of degree 3 and 2 elements of degree 4. We therefore get 12 leading monomials in the free algebra and the general structure of the Hilbert series follows from Govorov [8, Theorem 2] and formula (7) follows. Now formula (5) gives that

$$1/A(z) = \frac{1 - 15z + 15z^2 - 14z^3 + 4z^4 + 2z^5 + 4z^7 + z^8}{1 - z + z^2} + 14z - 6z^2 - 6z^3 - z^4$$

$$= \frac{1 - z - 5z^2 + 3z^4 - 3z^5 - z^6 + 4z^7 + z^8}{1 - z + z^2}$$

and now to use the Löfwall formula we only need to observe that I^2 is equal to

$$X_3X_6, X_3X_5, X_3X_4, X_2X_3, X_4X_{12}, X_3X_{12}, X_4X_{11}, X_3X_{11}, X_2X_{11}, X_3X_{10}, X_3X_{13}, X_{11}X_{11}$$

and $I^3 = 0$ so that

$$HKR(x, y) = 1 + 6xy + 6xy^2 + xy^3 + 4x^2y^2 + 6x^2y^3 + 2x^2y^4$$

and

$$\begin{aligned} 1/\Phi_R(z, z) &= (1 + 1/z)/A(z) - \bigl(1 - 6z^2 - 6z^3 + 3z^4 + 6z^5 + 2z^6\bigr)/z \\ &= \frac{(1 + z)(1 - 2z - 3z^2 + 3z^3 - 3z^5 + 2z^6 + z^7)}{1 - z + z^2} \end{aligned}$$

Now, since R is a Koszul ring with Hilbert series $P_R(z) = \frac{1+z}{1-3z+2z^3}$ given above, it follows in particular that *if* the Avramov spectral sequence degenerated, we would have had $P_R(z)/(1 + z)^4 = \Phi_R(z, z)$. But

$$(1 + z)^4/P_R(z) - 1/\Phi_R(z, z) = \frac{z^7(1 + z)}{1 - z + z^2}$$

shows that this is not the case (there is a non-zero Massey product in KR of degree 7).

Remark. In Section 5 we will see how to calculate the series of $\Phi_R(x, y)$ using the extra degree in HKR coming from the Koszul homology. There is also a third degree, coming from the grading of R, that we will not use.

Proof of the Ufnarovski Lemma (Ufnarovski). We need the theory of n-chains [19] of an algebra such as $A = k\langle Y_1, Y_2, \ldots\rangle/(f_1, \ldots f_m)$. We use deglex order. It is known (cf. [19], p. 51) that

$$1/A(z) = 1 - H_X(z) + H_{C_1}(z) - H_{C_2}(z) + H_{C_3}(z) - \ldots$$

where $X = Y_1, Y_2, \ldots$ are the generators of A, and $H_X(z) = \sum z^{d_i}$ is the Hilbert series (i.e. the generating series) of X, and where C_1 = the leading terms of the relations f_i (we use the deglex order) and $H_{C_1}(z)$ is the generating series of C_1, similarly for the n-chains C_n for $n \geq 2$. We now introduce an auxiliary algebra D which I write

$$D = k\langle Y_1, Y_2, \ldots, t, Z_1, Z_2, \ldots\rangle/\bigl(f_1, \ldots, f_m, Y_i - t^{d_i - 1}Z_i \ldots\bigr)$$

Now if the order is deglex such that

$$Y_1 > Y_2 > \ldots > t > Z_1 > Z_2 > \ldots Z_n$$

For this algebra D we have the new $X' = X \bigcup \{t, Z_1, \ldots, Z_n\}$ and the new

$$C_1' = C_1 \bigcup \{t^{d_i - 1} Z_i\}$$

but the higher C_i for $i \geq 2$ are the same. It follows that

$$H_D(z) = H_A(z) - (n+1)z + \sum_{i=1}^{n} z^{d_i}$$

On the other hand, consider the order

$$Y_1 > Y_2 > \ldots Z_1 > Z_2 > \ldots > t$$

We have that Y_i can be replaced by $t^{d_i-1}Z_i$ and we get the algebra

$$B = k\langle Z_1, Z_2, \ldots Z_n, t\rangle / (g_1, g_2, \ldots, g_m)$$

where the relations g_i are obtained from the f_i by replacing all Y_i by $t^{d_i-1}Z_i$. Clearly $H_B(z) = H_D(z)$ and the lemma is proved. $\square$

2. One more exceptional case

This case will be treated as in Section 1 so we can be more brief here. This is case c) in the main Theorem of Section 0 i.e. the ring $R_{60} = k[x, y, z, u]/(x^2 + yz + u^2, xz + yu, xu, xy, z^2, xu)$. The resolution of the ideal of R_{60} over $k[x, y, z, u]$ has Betti numbers:

$$\begin{pmatrix} 1 & . & . & . & . \\ . & 6 & 5 & . & . \\ . & . & 6 & 9 & 3 \end{pmatrix}$$

and an application of the programme `DGAlgebras` gives again that the algebra HKR_{60} has the cube of the augmentation ideal $\overline{HKR_{60}}$ equal to 0. Furthermore we have 6 generators $X_1, \ldots, X_6$ of degree $(1, 2)$ and 5 generators $X_7, \ldots, X_{11}$ of degree $(2, 3)$ satisfying the following *quadratic* relations:

$$X_5X_6, X_2X_6, X_1X_6, X_4X_5 - X_3X_6, X_3X_5, X_2X_5, X_1X_5, X_2X_4, X_2X_3,$$
$$X_6X_{11}, X_5X_{11}, X_2X_{11}, X_1X_{11}, X_6X_{10}, X_5X_{10}, X_4X_{10} - X_3X_{11}, X_3X_{10},$$
$$X_2X_{10}, X_6X_9 - X_4X_{11}, X_5X_9 - X_3X_{11}, X_2X_9, X_6X_8, X_5X_8, X_2X_8 + X_1X_{10},$$
$$X_1X_8, X_6X_7 - X_3X_{11}, X_5X_7, X_4X_7 + X_3X_9, X_3X_7, X_2X_7, X_{11}X_{11}, X_{10}X_{11}, X_9X_{11},$$
$$X_8X_{11}, X_7X_{11}, X_{10}X_{10}, X_9X_{10}, X_8X_{10}, X_7X_{10}, X_7X_9, X_8X_8, X_7X_7.$$

Now, using these relations we get that the Koszul dual of *HKR* is the quotient of the free algebra $k\langle Y_1, \ldots, Y_{11}\rangle$ on the dual generators Y_i of the X_i with the ideal generated by

$$\begin{gathered}[Y_1,Y_2],[Y_1,Y_3],[Y_1,Y_4],[Y_3,Y_4],[Y_3,Y_6]+[Y_4,Y_5],[Y_4,Y_6],[Y_1,Y_7],[Y_1,Y_9],[Y_3,Y_8],\\ [Y_4,Y_8],[Y_4,Y_9],[Y_7,Y_8],[Y_8,Y_9],Y_9Y_9,[Y_2,Y_8]-[Y_1,Y_{10}],[Y_4,Y_7]-[Y_3,Y_9],\\ [Y_6,Y_9]+[Y_4,Y_{11}],[Y_3,Y_{11}]+[Y_5,Y_9]+[Y_4,Y_{10}]+[Y_6,Y_7] \qquad (8)\end{gathered}$$

Here we have again written e.g. $[Y_1, Y_2] = Y_1Y_2 - Y_2Y_1$ for the "odd" generators $Y_i, i = 1 \ldots 6$, and e.g. $[Y_8, Y_9] = Y_8Y_9 + Y_9Y_8$ for the "even" generators $Y_i, i = 7 \ldots 11$, and e.g. $[Y_2, Y_8] = Y_2Y_8 + Y_8Y_2$ when even and odd generators are mixed. We want to calculate the Hilbert series of this quotient, when the generators Y_i are given the degree 2 for $i = 1, \ldots, 6$ and degree 3 for $i = 7, \ldots, 11$. For this we use again the programme BERGMAN and the idea of Victor Ufnarovski to handle the gradings, i.e. we introduce a new variable t of degree 1 and replace the variables Y_i in (8)

$$Y_i = tZ_i, 1 \le i \le 6; Y_i = t^2Z_i, 7 \le i \le 11$$

where the Z_i:s also have degree 1. Now the algebra A becomes the quotient of the free algebra in 12 generators of degree 1: $k\langle Z_1, \ldots, Z_6, t, Z_7, \ldots, Z_{11}\rangle$ with the ideal (8) above where e.g. $[Y_1, Y_2]$ is replaced by $tZ_1tZ_2 - tZ_2tZ_1$ and e.g. Y_9Y_9 is replaced by $t^2Z_9t^2Z_9$ and $[Y_6, Y_9] + [Y_4, Y_{11}]$ is replaced by

$$tZ_6t^2Z_9 + t^2Z_9tZ_6 + tZ_4t^2Y_{11} + t^2Y_{11}tZ_4$$

etc. If the new algebra is denoted by B and the old one by A, then by the theory of Ufnarovski we have the following relation between their Hilbert series:

$$1/H_A(z) = 1/H_B(z) + 12z - 6z^2 - 5z^3$$

It remains to calculate $H_B(z)$.

After these preparations we are now ready again to use the programme BERGMAN and the command in BERGMAN

(ncpbhgroebner "inbcaseB" "t1" "t2" "t3")

After a few seconds we get t3 (i.e. the Hilbert series for B) in degrees ≤ 20 and ≥ 2 and we obtain with maple

$$\begin{aligned}S := \mathtt{series}(1/(1+12z+\mathtt{t3}, z, 21) = 1 - 12z + 6z^4 + 9z^5 + 2z^6 - 3z^7 - 4z^8 - 2z^9\\ + z^{10} + 3z^{11} + 2z^{12} - z^{13} - 3z^{14} - 2z^{15} + z^{16} + 3z^{17} + 2z^{18} - z^{19} - 3z^{20} + O(z^{21})\end{aligned}$$

and the maple command `convert(S, ratpoly)` on the previous series gives the rational function:

$$\frac{(1-z)(z^9+2z^8+z^7-3z^6-2z^5-5z^4-11z^3+z^2-12z+1)}{(1-z+z^2)} \tag{9}$$

Here is again how one should proceed to get a proof of this formula. If we can find a permutation of the 12 variables in `inbcaseB` so that the Gröbner basis `t1` is finite, then the Hilbert series of B is equal to the Hilbert series of the free algebra on the 12 variables divided by the ideal generated by the monomial leading terms of this finite Gröbner basis, and this last Hilbert series is rational of a special form according to Govorov. Again we use Jörgen Backelin's addition `permutebreak.sl` to the programme `BERGMAN` and we obtain that the order of the variables $x3, t, x1, x2, x5, x6, x4, y8, y7, y9, y10, y11$ gives a finite Gröbner basis with 6 elements of degree 4, 9 elements of degree 5 and 3 elements of degree 6. We therefore get 18 leading monomials in the free algebra and the general structure of the Hilbert series follows from Govorov ([8, Theorem 2]) and the formula (9) follows. We have now all we need to prove the following

Theorem. *Let* $R_{60} = k[x,y,z,u]/(x^2+yz+u^2, xz+yu, zu, xy, z^2, xu)$. *Then*

i)

$$\begin{aligned}\Phi_{R_{60}}(z,z) &= \sum_{n=0}^{\infty}\sum_{p+q=n} |\mathrm{Tor}_{p,q}^{HKR}(k,k)| z^n \\ &= \frac{(1-z+z^2)}{(1+z)(z^9-z^8+z^7+z^6-4z^5+z^4+4z^3-3z^2-2z+1)}\end{aligned}$$

ii) $P_{R_{60}}(z) = (1+z)/(-z^6-z^5+3z^3-3z+1)$

iii) $1/\Phi_{R_{60}}(z,z) - (1+z)^4/P_{R_{60}}(z) = z^7(1+z)(1-3z^2-z^3)/(1-z-z^2)$.

It follows that the Avramov spectral sequence does not degenerate (the two series differ starting in degree 7).

Proof. For the algebra HKR we have that the cube of the augmentation ideal $\overline{HKR}$ is zero. Furthermore the rational function $1/HKR^!(z)$ is given by the formula (9) above and the two-variable form of the Hilbert series of HKR is given by the formula

$$HKR(x,y) = 1 + 6xy + 5xy^2 + 6x^2y^2 + 9x^2y^3 + 3x^2y^4$$

This gives the one-variable Hilbert series $HKR(-z,z)$ and the formula

$$1/P_{HKR}(z) = (1+1/z)/HKR^!(z) - HKR(-z,z)/z$$

gives the result i).

To prove ii) we first observe that the $R^! = k\langle X, Y, Z, U\rangle/(Y^2, X^2 - YZ - ZY, XZ + ZX - YU - UY, YZ + ZY - U^2)$, where the X, Y, Z, U are the dual variables to x, y, z, u. There are only 24 permutations of the variables X, Y, Z, U and one sees that the order X, Y, U, Z gives in BERGMAN a finite Gröbner basis, and that $1/R^!(z) = (1-z)^2(1 - 2z - z^2 - z^3)$. On the other hand the Hilbert series $R(z) = (1 - 3z + 3z^3)/(1 - z) = 1 + 4z + 4z^2 + z^3 + z^4 + z^5 + \cdots$ and according to Theorem B.9 by Clas Löfwall (cf. page 310 of [18]) the ring R satisfies a condition $\mathcal{M}_3$ so that we still have the formula $1/P_R(z) = (1+1/z)/R^!(z) - R(-z)/z$ and this gives ii) and the assertion iii) follows. □

3. One final exceptional case for a local ring that is artinian and a Koszul algebra

In Section 4 we will treat a Koszul local ring (case 46) whose Koszul algebra has properties that show that the Avramov spectral sequence degenerates (solving a problem of Avramov). In Section 2 we found a quadratic local ring (case 60) whose Avramov spectral sequence does not degenerate. But case 60 is not a Koszul algebra. Now we pass to a case of a Koszul local ring (case 71) which has further rather unexpected properties and which is case b) of the Main Theorem.

Theorem. *There are at least five local commutative rings having Hilbert series* $(1+z)(1+3z)$ *which are Koszul algebras and which have different homological properties. Furthermore for one of these five cases the Avramov spectral sequence does not degenerate.*

Proof. For the ring of case 71 in Appendix A we have the presentation

$$R_{71} = k[x, y, z, u]/\big(x^2, y^2, z^2, u^2, xy, zu, yz + xu\big).$$

But we also have three other rings which are Koszul algebras and have the same Hilbert series:

$$R_{71v4} = \frac{k[x, y, z, u]}{(x^2 + u^2, xy, xu, y^2, yz, z^2, zu)} \quad R_{71v7} = \frac{k[x, y, z, u]}{(x^2, y^2, z^2, xz + u^2, xu, yz, zu)}$$
$$R_{71v5} = \frac{k[x, y, z, u]}{(x^2 + xy, x^2 + yz, xy + y^2, z^2, xu, zu, xz + u^2)}$$

but the Betti numbers of the resolution of the preceding *four* ideals are:

$$\begin{pmatrix} 1 & . & . & . & . \\ . & 7 & 8 & 2 & . \\ . & . & 5 & 8 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & . & . & . & . \\ . & 7 & 8 & 3 & . \\ . & . & 6 & 8 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & . & . & . & . \\ . & 7 & 8 & 1 & . \\ . & . & 4 & 8 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & . & . & . & . \\ . & 7 & 8 & . & . \\ . & . & 3 & 8 & 3 \end{pmatrix}$$

so that the rings can not be isomorphic. Their *HKR* are however rather easy, in one of these cases *HKR* has only monomial relations, but strangely enough the case of a R_{71} with only monomial relations gives a more complicated *HKR*.

There remains one case R_{71v16} which is *very* different from the preceding and leads to a new phenomenon. We now treat it in detail: The ring

$$R_{71v16} = k[x,y,z,u]/\big(xz+u^2, xy, xu, x^2, y^2+z^2, zu, yz\big)$$

is also a Koszul algebra and it has the same matrix of Betti numbers as R_{71} but the algebraic behavior of its *HKR* is very different. It still has the cube of the augmentation ideal of *HKR* equal to 0, it has seven generators $X_1,\dots,X_7$ of degree $(1,2)$, eight generators $X_8,\dots,X_{15}$ of degree $(2,3)$ and two generators X_{16}, X_{17} of degree $(3,4)$ and an applications of the programme `DGAlgebras` gives that these generators satisfy 128 *quadratic relations* of which only the following 16 are non-monomial:

$$\begin{gathered} X_5X_6 - X_3X_7, X_2X_3 - X_4X_5, X_7X_{13} - X_5X_{15}, X_6X_{13} - X_3X_{15}, X_3X_{12} - X_5X_{14}, \\ X_3X_{11} + X_5X_{15}, X_6X_{10} - X_5X_{15}, X_4X_{10} - X_2X_{13}, X_3X_{10} + X_5X_{13}, X_5X_9 + X_2X_{13}, \\ X_3X_9 + X_4X_{13}, X_3X_8 + X_2X_{13}, X_3X_{16} - X_5X_{17}, X_{13}X_{14} + X_3X_{17}, X_{10}X_{14} + X_5X_{17}, \\ X_{12}X_{13} + X_5X_{17} \end{gathered}$$

and since the following 6 quadratic monomials do *not* occur among the 128 relations:

$$X_3X_4,\ X_3X_5,\ X_3X_6,\ X_3X_{13},\ X_3X_{14},\ X_{13}X_{13}$$

it follows that the Koszul dual $HKR^!$ is equal to the quotient of the free associative algebra on the 17 variables $Y_i, 1 \le i \le 17$ (which are dual to the X_i) with respect to the ideal generated by the 16 quadratic relations:

$$\begin{gathered} [Y_3,Y_4], [Y_3,Y_5], [Y_3,Y_6], [Y_3,Y_{13}], [Y_3,Y_{14}], Y_{13}Y_{13}, [Y_2,Y_3] + [Y_4,Y_5], [Y_5,Y_6] + [Y_3,Y_7] \\ [Y_3,Y_{12}] + [Y_5,Y_{14}], [Y_3,Y_{10}] - [Y_5,Y_{13}], [Y_3,Y_{15}] + [Y_6,Y_{13}], [Y_3,Y_9] - [Y_4,Y_{13}], \\ [Y_3,Y_{17}] - [Y_{13},Y_{14}], [Y_2,Y_{13}] - [Y_3,Y_8] + [Y_4,Y_{10}], [Y_5,Y_{15}] \\ + [Y_7,Y_{13}] - [Y_6,Y_{10}] - [Y_3,Y_{11}], \\ [Y_5,Y_{17}] - [Y_{12},Y_{13}] - [Y_{10},Y_{14}] + [Y_3,Y_{16}] \end{gathered}$$

Here $[,]$ means as before the graded commutator so that $[Y_i,Y_j] = Y_iY_j - Y_jY_i$ if $1 \le i \le 7$ or $i = 16$ or $i = 17$ and $1 \le j \le 7$ or $j = 16$ or $j = 17$. Furthermore we have $[Y_s,Y_t] = Y_sY_t + Y_tY_s$ otherwise. We now need to calculate the Hilbert series of $HKR^!$ where we have given the variables $Y_1,\dots,Y_7$ the degrees 2, the variables $Y_8,\dots,Y_{15}$ the degrees 3 and the variables Y_{16}, Y_{17} the degrees 4. We use the Ufnarovski trick and introduce a new variable T of degree 1 and replace everywhere in $HKR^!$ the variables Y_i according to the formulae $Y_i = TZ_i$ for $1 \le i \le 7$, $Y_i = T^2Z_i$ for $8 \le i \le 15$ and $Y_i = T^3Z_i$ for $i = 16, 17$. The modified algebra A has generators of degree 1 and can be put into `BERGMAN` and its Hilbert series $A(z)$ is related to $HKR^!(z)$ by the Ufnarovski formula:

$$\frac{1}{HKR^!(z)} = \frac{1}{A(z)} + 18z - 7z^2 - 8z^3 - 2z^4$$

The calculations in BERGMAN give after a few minutes $A(z)$ up to degree 22, giving that

$$1/A(z) = 1 - 18z + 5z^4 + 8z^5 + 3z^6 - z^8 - z^8 + z^{11} + z^{12} - z^{14} - z^{15} + z^{17} + z^{18} - z^{20} - z^{21} + O\big(z^{23}\big)$$

indicating (using the maple command convert(-,ratpoly) as earlier) that

$$\frac{1}{A(z)} = \frac{1 - 19z + 19z^2 - 18z^3 + 5z^4 + 3z^5 + 5z^7 + 2z^8}{1 - z + z^2}$$

To prove this we use again the permutebergman addition by Jörgen Backelin to BERGMAN and we try to find a permutation of the variables $T, Z_i, i = 1 \dots 17$ that gives a finite Gröbner basis when we calculate the Hilbert series of the algebra A. Since the number of permutations of 18 variables is $18! = 6402373705728000$ this seems to be a hopeless task, but after letting the programme run through only the 24000 first permutations (this took about an hour) we finally got the permutation

$$Z1, Z3, Z4, Z6, Z7, T, Z2, Z5, Z8, Z9, Z10, Z11, Z12, Z13, Z14, Z15, Z16, Z17$$

for which the Gröbner basis is finite: it has 5 elements of degree 4, 8 elements of degree 5 and 3 elements of degree 6. Therefore, according to the result of Govorov the Hilbert series $A(z)$ is rational as indicated. We have now all we need to prove the following

Theorem. *Let $R = k[x, y, z, u]/(xz + u^2, xy, xu, x^2, y^2 + z^2, zu, yz)$. Then R is a Koszul algebra with Hilbert series $R(z) = 1 + 4z + 3z^2$ so that $P_R(z) = 1/(1 - 4z + 3z^2)$. But HKR is far from being a Koszul algebra. Indeed:*

a)

$$\begin{aligned}\Phi_R(z,z) &= \sum_{n=0}^{\infty} \sum_{p+q=n} \left|\operatorname{Tor}_{p,q}^{HKR}(k,k)\right| z^n \\ &= \frac{(1 - z + z^2)}{(1+z)(2z^7 + 2z^6 - 4z^5 + z^4 + 3z^3 - 4z^2 - 2z + 1)}\end{aligned}$$

b) *Furthermore $P_R(z) = 1/(1 - 4z + 3z^2)$ so that*

$$(1+z)^4/P_R(z) - 1/\Phi_R(z,z) = z^7(1+z)/\big(1 - z + z^2\big)$$

It follows again that the Avramov spectral sequence does not degenerate.

Proof. We have just determined $1/A(z)$ and it follows that

$$\frac{1}{HKR^!(z,z)} = \frac{1 - z - 6z^2 - z^3 + 4z^4 - 3z^5 - 2z^6 + 5z^7 + 2z^8}{1 - z + z^2}$$

But the two-variable version of the Hilbert series of HKR is

$$HKR(x,y) = 1 + 7xy + 8xy^2 + 2xy^3 + 5x^2y^2 + 8x^2y^3 + 3x^2y^4$$

This gives again a one-variable $HKR(-z,z) = 1-7z^2-8z^3+3z^4+8z^5+3z^6$. The formula by Clas Löfwall $1/P_{HKR}(z,z) = (1+1/z)/HKR^!(z,z) - HKR(-z,z)/z$ now gives

$$1/P_{HKR}(z,z) = \frac{(1+z)(1 - 2z - 4z^2 + 3z^3 + z^4 - 4z^5 + 2z^6 + 2z^7)}{1 - z + z^2}$$

and the Theorem is proved. $\square$

4. The other embedding dimension four cases

We have just treated the three interesting cases of the Main Theorem in Section 0. For the other cases we have that the Avramov spectral sequence *does* degenerate and they are treated in an analogous way. Here we will only treat case 46 in detail because it solves a problem that was left open in Avramov [4]. The case 46 in the ring $k[x,y,z,u]/(x^2, xy, yz, zu, u^2)$, and it is also studied in [5, page 21].

Application as before of the programme `DGAlgebras` gives that the algebra HKR has the cube of the augmentation ideal equal to 0 and that HKR has 5 generators $X_1, \dots, X_5$ of degree $(1,2)$ and 4 generators X_6, X_7, X_8, X_9 of degree $(2,3)$. These generators have only quadratic relations and they are:

$$X_4X_5, X_3X_5, X_2X_5, X_3X_4, X_2X_3, X_1X_3, X_1X_2, X_5X_9, X_4X_9, X_3X_9, X_2X_9,$$
$$X_5X_8, X_4X_8, X_3X_8, X_5X_7, X_4X_7 - X_2X_8, X_3X_7, X_2X_7, X_1X_7, X_5X_6 - X_1X_9,$$
$$X_3X_6, X_2X_6, X_1X_6, X_9X_9, X_8X_9, X_7X_9, X_6X_9, X_8X_8, X_7X_8, X_7X_7, X_6X_7, X_6X_6$$

so that the Koszul dual $HKR^!$ is the quotient of the free algebra $k\langle Y_1, \dots, Y_9\rangle$ in the dual generators Y_i of the X_i with the ideal generated by

$$[Y_1,Y_4], [Y_1,Y_5], [Y_2,Y_4], [Y_1,Y_8], [Y_4,Y_6], [Y_6,Y_8], [Y_4,Y_7] + [Y_2,Y_8], [Y_5,Y_6] + [Y_1,Y_9]$$

Here again we have written e.g. $[Y_1,Y_4] = Y_1Y_4 - Y_4Y_1$ for the "odd" generators and e.g. $[Y_6,Y_8] = Y_2Y_8 + Y_8Y_2$ for the "even" generators and e.g. $[Y_4,Y_7] = Y_4Y_7 + Y_7Y_4$ when the even and odd generators are mixed. We use the previous methods to calculate the Hilbert series of our quotient when the generators $Y_1, \dots, Y_5$ are given the degrees 2 and the generators $Y_6, \dots, Y_9$ are given the degree 3, i.e. we introduce a new variable

t of degree 1 and replace the variables Y_i by tZ_i for $1 \leq i \leq 5$ and $t^2 Z_i$ for $6 \leq i \leq 9$. where the Z_i now have degree 1. Our algebra $HKR^!$ is now replaced by the quotient of $k\langle Z_1, \ldots, Z_5, t, Z_6, \ldots, Z_9\rangle$ by the ideal of $HKR^!$ where e.g. $Y_1Y_4 - Y_4Y_1$ is replaced by $tZ_1tZ_4 - tZ_4tZ_1$ etc. If this new algebra is denoted by B then again by the theory of Ufnarovski we have in this special case the following relation between the their Hilbert series $H_A(z)$ and $H_B(z)$

$$1/H_A(z) = 1/H_B(z) + 10z - 5z^2 - 4z^3$$

and $H_B(z)$ is calculated by `BERGMAN` as before. We get:

$$1/H_B(z) = 1 - 10z + 3z^4 + 4z^5 + z^6$$

since the permutation $Z_1, Z_2, Z_3, t, Z_4, \ldots, Z_9$ of the variables in B gives a finite Gröbner basis (3 elements of degree 4, 4 elements of degree 5 and one element of degree 6).

We are now ready to calculate $\Phi_R(x, x)$ of our *HKR*. We have the formula of Löfwall

$$1/\Phi_R(z, z) = (1 + 1/z)/A(z) - HKR(-z, z)/z$$

where we have just found that

$$\begin{aligned} 1/A(z) &= \left(1 - 10z + 3z^4 + 4z^5 + z^6\right) + 10z - 5z^2 - 4z^3 \\ &= 1 - 5z^2 - 4z^3 + 3z^4 + 4z^5 + z^6 \end{aligned}$$

For the Hilbert series in two variables of *HKR* we observe that since

$$\overline{HKR}^2 = X_1X_5, X_2X_4, X_1X_4, X_1X_9, X_2X_8, X_1X_8, X_4X_6, X_6X_8$$

that

$$HKR(x, y) = 1 + 5xy + 4xy^2 + 3x^2y^2 + 4x^2y^3 + x^2y^4$$

so that $HKR(-z, z) = 1 - 5z^2 - 4z^3 + 3z^4 + 4z^5 + z^6$ so that $1/\Phi_R(z, z) = 1 - 5z^2 - 4z^3 + 3z^4 + 4z^5 + z^6$ too. But since R is a Koszul ring (quadratic monomial relations) with Hilbert series $H_R(z) = (1 + 3z + z^2 - z^3)/(1 - z)$ we have that (Fröberg) $P_R(z) = 1/H_R(-z)$ so that $(1 + z)^4/P_R(z) = 1 - 5z^2 - 4z^3 + 3z^4 + 4z^5 + z^6$ too and therefore the Avramov spectral sequence degenerates.

5. More precise $\Phi_R(x, y)$

In the preceding sections we have shown how to calculate $\Phi_R(x, x)$ for some examples R. In this section we will give an indication about how to generalize this to the calculation of the double series $\Phi_R(x, y)$. The complete calculation results for the quadratic embedding dimension ≤ 4 case are given in Tables A–D at the end of the paper. Let us illustrate these more precise calculations with the simple example of case 46 which we have just treated for the calculation of $\Phi_R(x,x)$. Note first that the output file for the case 46 gives for the dimensions of the $\mathrm{Tor}^{HKR}_{p,q}(k,k)$ the integers:

0 :	1	5	22	95	409	1760	7573	32585
1 :	.	4	36	236	1364	7368	38152	191908
2 :	.	.	15	198	1723	12438	80628	487202
3 :	.	.	.	56	976	10576	91448	690904
4 :	.	.	.	.	209	4527	58685	590894
5 :	.	.	.	.	.	780	20196	304696
6 :	.	.	.	.	.	.	2911	87692
7 :	.	.	.	.	.	.	.	10864

showing that $\Phi_R(x,y)$ should start as

$$\Phi_R(x,y) = 1 + xy(5+4y) + x^2y^2\big(22+36y+15y^2\big) + x^3y^3\big(95+236y+198y^2+56y^3\big) + \cdots$$

But note that the calculation of $\Phi_R(z,z)$ in the preceding section gives

$$\Phi_R(z,z) = 1 + 5z^2 + 4z^3 + 22z^4 + 36z^5 + 110z^6 + 236z^7 + 607z^8 + 1420z^9 + 3483z^{10} + \cdots$$

and as you see this is not sufficient to get the preceding columns e.g. to find the decomposition $110 = 15 + 95 = |\mathrm{Tor}^{HKR}_{2,4}(k,k)| + |\mathrm{Tor}^{HKR}_{3,3}(k,k)|$ etc. But now we use the Löfwall formula (4) where we replace x by z^u for any integer $u \geq 1$ and y by z:

$$1/\Phi_R\big(z^u,z\big) = \big(1+1/z^u\big)/H_A\big(z^u,z\big) - \big(1 - H_{I/I^2}(z)z^u + H_{I^2}(z)z^{2u}\big)z^u \qquad (10)$$

gives that $H_A(z^u,z)$ is the Hilbert series of a new variant of the algebra A where all previous generators of degree s have been replaced by generators of degree $s+u-1$. Now the Hilbert series of A with generators of these degrees can still be calculated with `BERGMAN`. We illustrate this with the case 46 just treated when $u = 31$. The new input file for `BERGMAN` will therefore be:

```
(noncommify)
(algforminput)
vars x1,x2,x3,x4,x5,t,x6,x7,x8,x9;
t^31*x1*t^31*x4-t^31*x4*t^31*x1,t^31*x2*t^31*x4-t^31*x4*t^31*x2,
t^31*x1*t^31*x5-t^31*x5*t^31*x1,t^31*x1*t^32*x8+t^32*x8*t^31*x1,
t^31*x4*t^32*x6+t^32*x6*t^31*x4,t^32*x6*t^32*x8+t^32*x8*t^32*x6,
t^31*x4*t^32*x7+t^32*x7*t^31*x4+t^31*x2*t^32*x8+t^32*x8*t^31*x2,
t^31*x5*t^32*x6+t^32*x6*t^31*x5+t^31*x1*t^32*x9+t^32*x9*t^31*x1;
```

Note that here the generators still all have degree 1 and `BERGMAN` now gives for this infile the inverse of its Hilbert series:

$$1 - 10z + 3z^{64} + 4z^{65} + z^{66}$$

(the permutation $x1, x2, x3, t, x4, x5, x6, x7, x8, x9$ still gives a finite Gröbner basis), and the Ufnarovski method now still gives that $1/A(z^{31}, z) = 1+3z^{64}+4z^{65}+z^{66}-5z^{32}-4z^{33}$. On the other hand the Hilbert series of $HKR(x, y) = 1+5xy+4xy^2+3x^2y^2+4x^2y^3+x^2y^4$ so that $HKR(-z^{31}, z) = 1 - 5z^{32} - 4z^{33} + 3z^{64} + 4z^{65} + z^{66}$ and the formula (10) now also gives

$$\begin{aligned} 1/\Phi_R\big(z^{31}, z\big) &= \big(1 + 1/z^{31}\big)/A\big(z^{31}, z\big) - HKR\big(-z^{31}, z\big)/z^{31} \\ &= 1 + 3z^{64} + 4z^{65} + z^{66} - 5z^{32} - 4z^{33} \end{aligned}$$

But the Taylor series of $1/(1 + 3z^{64} + 4z^{65} + z^{66} - 5z^{32} - 4z^{33})$ is

$$\begin{aligned} &1 + z^{32}(5 + 4z) + z^{64}\big(22 + 36z + 15z^2\big) + z^{96}\big(95 + 236z + 198z^2 + 56z^3\big) \\ &\quad + z^{128}\big(409 + 1364z + 1723z^2 + 976z^3 + 109z^4\big) \\ &\quad + z^{160}\big(1760 + 7368z + 12438z^2 + \cdots\big) + \cdots \end{aligned}$$

which "explains" the columns above at the beginning of this section and gives an indication that

$$1/\Phi_R(x, y) = 1 - \big(5 + 4y + y^2\big)xy + \big(3 + 4y + y^2\big)x^2y^2$$

In a similar way one treats all the quadratic embedding dimension 4 cases and the results are given in Tables A–D that follow at the end of the paper.

6. Higher embedding dimensions. Four examples

Many new phenomena come up in connection with higher embedding dimensions of local rings. Here we will just mention four very different cases of $R = k[x, y, z, u, v]/J$ where the J is an ideal generated by quadratic forms in x, y, z, u, v

Case I $R_I = k[x, y, z, u, v]/(y^2, u^2, yz + xu, zu + yv, z^2 - yu - xv)$

Case II $R_{II} = k[x, y, z, u, v]/(xu - xv, xv - zu, yv, u^2, v^2)$

Case III $R_{III} = k[x, y, z, u, v]/(xz + yu, yv, zu + uv, z^2, v^2)$

Case IV $R_{IV} = k[x, y, z, u, v]/(x^2, xy, yz, zu, uv, v^2, (x + y)(u + v))$

In **Case I** and in **Case III** the resolution of J over $k[x, y, z, u, v]$ has the same diagram of Betti numbers:

$$\begin{pmatrix} 1 & . & . & . & . & . \\ . & 5 & 3 & . & . & . \\ . & . & 7 & 10 & 5 & 1 \end{pmatrix}$$

but in **Case II** and **Case IV** the diagrammes of Betti numbers are:

$$\begin{pmatrix} 1 & . & . & . & . & . \\ . & 5 & 3 & . & . & . \\ . & . & 8 & 12 & 6 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & . & . & . & . & . \\ . & 7 & 7 & . & . & . \\ . & . & 7 & 14 & 7 & 1 \end{pmatrix}$$

respectively. The homological properties of HKR and R are *very* different in the four cases:

Case I This is the case that Löfwall and I studied in [14], which also solved an old problem (the "Opera case"). In [14] we proved that the Hilbert series of $R^!$ is transcendental and different for all values of the characteristic of the field k. Similar results are true for $HKR^!$ and the Avramov spectral sequence does not degenerate. We will just give a few indications below.

Case II In [16] I found, inspired by Lemaire [10], a case of a local ring (R, m) where the Yoneda Ext-algebra $\mathrm{Ext}_R^*(k,k)$ was not finitely generated. Furthermore that $R^!$ had global dimension 3. This is a variant of that example which can be treated as in Remark 3 of page 220 of [17] and similar assertions are true for HKR. Furthermore the Avramov spectral sequence degenerates.

Case III This is a variant of the first known example due to Anick [1] (cf. also [13]) where $P_R(z)$ is transcendental. Here HKR seems to have the same properties and the Avramov spectral sequence degenerates at least up to degrees 20.

Case IV This is an example related to an edge ideal where we are very close to degeneration of the Avramov spectral sequence.

More details: **Case I**.

We apply as before `DGAlgebras` and obtain that HKR has the cube of the augmentation ideal equal to 0. Furthermore HKR has 9 generators: five X_1, X_2, X_3, X_4, X_5 of degree $(1, 2)$, three generators X_6, X_7, X_8 of degree $(2, 3)$ and one X_9 of degree $(5, 7)$. Furthermore these generators only satisfy 17 quadratic relations of which the following 9 are non-monomial relations:

$$X_3X_4 - X_2X_5, X_1X_4 - X_3X_5, X_2X_3 + X_1X_5, X_4X_7 - X_5X_8, X_5X_6 + X_3X_8,$$
$$X_4X_6 - X_5X_7 + X_2X_8, X_3X_6 + X_1X_7, X_2X_6 + X_3X_7 + X_1X_8, X_7X_7 + 2X_6X_8$$

and since the following 11 quadratic monomials do *not* occur among the 17 quadratic relations

$$X_1X_2, X_1X_3, X_2X_4, X_4X_5, X_1X_6, X_2X_7, X_4X_8, X_6X_6, X_6X_7, X_7X_8, X_8X_8$$

it follows that $HKR^!$ is the quotient of the free algebra on the 9 generators Y_i (which are dual to the X_i) by the two-sided ideal generated by the 21 quadratic relations:

$$[Y_1, Y_2], [Y_1, Y_3], [Y_2, Y_4], [Y_4, Y_5], [Y_1, Y_6], [Y_2, Y_7], [Y_4, Y_8], Y_6Y_6, [Y_6, Y_7], [Y_7, Y_8], Y_8Y_8,$$
$$[Y_1, Y_4] + [Y_3, Y_5], [Y_2, Y_3] - [Y_1, Y_5], [Y_3, Y_4] + [Y_2, Y_5], [Y_2, Y_6] - [Y_3, Y_7],$$

$$[Y_3,Y_7]-[Y_1,Y_8],[Y_3,Y_6]-[Y_1,Y_7],[Y_4,Y_6]+[Y_5,Y_7],[Y_5,Y_7]+[Y_2,Y_8],$$
$$[Y_5,Y_6]-[Y_3,Y_8],[Y_4,Y_7]+[Y_5,Y_8],[Y_7,Y_7]-[Y_6,Y_8]$$

Here the graded commutators are as before and adding as before a new variable T giving a new algebra A generated in degree 1 such that

$$\frac{1}{HKR^!(z)}=\frac{1}{A(z)}+10z-5z^2-3z^3-z^6$$

We find using BERGMAN that

$$\begin{aligned}\frac{1}{A(z)}&=1-10z+7z^4+10z^5+5z^6-10z^8-27z^9-25z^{10}+3z^{11}+19z^{12}+27z^{13}\\&+56z^{14}+57z^{15}+z^{16}-42z^{17}-58z^{18}-101z^{19}-124z^{20}-51z^{21}+48z^{22}\\&+97z^{23}+154z^{24}+204z^{25}+\dots\end{aligned}$$

This gives $\frac{1}{HKR^!(z)}$ and since the two-variable Hilbert series of HKR is $1+5xy+3xy^2+xy^5+7x^2y^2+10x^2y^3+5x^2y^4$ and therefore the one variable series with a minus sign in the first variable is $h=1-5z^2-3z^3+7z^4+10z^5+4z^6$, the formula

$$\frac{1}{P_{HKR}(z)}=(1+1/z)/HKR^!(z)-h/z$$

gives

$$\begin{aligned}P_{HKR}(z)&=1+5z^2+3z^3+18z^4+20z^5+60z^6+93z^7+221z^8+415z^9\\&+929z^{10}+1936z^{11}+\dots\end{aligned}$$

whereas

$$\begin{aligned}\frac{P_R(z)}{(1+z)^5}&=1+5z^2+3z^3+18z^4+20z^5+59z^6+92z^7+216z^8+407z^9\\&+907z^{10}+1897z^{11}+\dots\end{aligned}$$

and they start differing in degree 6 so that the Avramov spectral sequence does not degenerate. The preceding calculations are in characteristic 0. But e.g. in characteristic 7 we have

$$\begin{aligned}\frac{P_R(z)}{(1+z)^5}&=1+5z^2+3z^3+18z^4+20z^5+60z^6+93z^7+221z^8+415z^9+928z^{10}\\&+1935z^{11}+\dots\end{aligned}$$

In characteristic 7 we also have

$$P_{HKR}(z) = 1 + 5z^2 + 3z^3 + 18z^4 + 20z^5 + 60z^6 + 93z^7 + 221z^8 + 415z^9 + 929z^{10} + 1936z^{11} + \dots$$

and these two series differ in degree 10, so that the Avramov spectral sequence does not degenerate.

Similar results are true in characteristic 2, 3, 5, 11, 13, etc. But characteristic 2 is slightly different.

Important observation: The relations between the first 5 variables X_i in *HKR* are the same in all characteristic:

$$X_3X_4 - X_2X_5, X_1X_4 - X_3X_5, X_2X_3 + X_1X_5$$

But the quotient of the exterior algebra in five variables with these 3 relations has a homology with transcendental Hilbert series which is different for all characteristic (it is the "skew" counterpart of the present **Case I** for ordinary polynomial rings). It occurs among the relations that Eisenbud and Koh have classified in [7].

More details: **Case II**. Now the ring is $R = k[x, y, z, u, v]/(xu - xv, xv - zu, yv, u^2, v^2)$. We have that the Hilbert series $R^!(z) = \frac{(1+z)^3(1-z)}{(1-z-z^2)^3}$ and $R(z) = \frac{1+2z-2z^2-3z^3+4z^4-z^5}{(1-z)^3}$ and

$$1/P_R(z) = (1 + 1/z)/R^!(z) - R(-z)/z = \frac{(1 - 2z - z^2)(1 - z - z^2 + z^3 - z^4)}{(1 + z)^3(1 - z)}$$

We now apply `DGAlgebras` and we obtain that *HKR* has five generators X_1, X_2, X_3, X_4, X_5 of degree $(1, 2)$, three generators X_6, X_7, X_8 of degree $(2, 3)$, one generator X_9 of degree $(2, 4)$, three generators X_{10}, X_{11}, X_{12} of degree $(3, 5)$, three generators X_{13}, X_{14}, X_{15} of degree $(4, 6)$ and one generator X_{16} of degree $(5, 7)$. Furthermore these generators satisfy 107 quadratic relations of which only the following four are non-monomial relations:

$$X_1X_3 + X_3X_4, X_1X_2 + X_3X_4, X_1X_8 - X_4X_8, X_2X_6 - X_3X_6$$

and since the following 16 quadratic monomials do *not* occur among the 107 quadratic relations:

$$X_1X_4, X_1X_5, X_2X_3, X_2X_4, X_2X_5, X_3X_5, X_1X_7, X_2X_7, X_2X_8, X_3X_7, X_4X_6, X_5X_6, X_5X_8, X_6X_7, X_6X_8, X_7X_8$$

it follows that $HKR^!$ is the quotient of the free algebra on the 16 generators Y_i (which are dual to the X_i) by the two-sided ideal generated by the 19 quadratic relations

$$[Y_1, Y_4], [Y_1, Y_5], [Y_2, Y_3], [Y_2, Y_4], [Y_2, Y_5], [Y_3, Y_5], [Y_1, Y_7], [Y_2, Y_7], [Y_2, Y_8], [Y_3, Y_7], [Y_4, Y_6], [Y_5, Y_6], [Y_5, Y_8], [Y_6, Y_7], [Y_6, Y_8], [Y_7, Y_8], [Y_3, Y_4] - [Y_1, Y_3] - [Y_1, Y_2], \quad (xxx)$$

$$[Y_1, Y_8] + [Y_4, Y_8], [Y_2, Y_6] + [Y_3, Y_6]$$

Here $[,]$ means the graded commutator so that $[Y_i, Y_j] = Y_iY_j - Y_jY_i$ if $1 \le i \le 5$ or $i = 10, 11, 12, 16$ *and* $1 \le j \le 5$ or $j = 10, 11, 12, 16$. Furthermore we have $[Y_s, Y_t] = Y_sY_t + Y_tY_s$ otherwise. To calculate $HKR^!(z)$ we use again the Ufnarovski trick and replace the Y_i with $Z_iT^{degY_i - 1}$ everywhere in order to get a new algebra A with generators of degree 1. Calculations in `BERGMAN` give after a few minutes

$$\frac{1}{A(z)} = 1 - 17z + 7z^4 + 9z^5 + 2z^6 - 3z^7 - 4z^8 - 4z^9 - 4z^{10} - 4z^{11} - 4z^{12} - 4z^{13} - 4z^{14} + O\bigl(z^{15}\bigr)$$

leading as before to

$$\begin{aligned}\frac{1}{HKR^!(z)} &= \frac{1}{A(z)} + 17z - 5z^2 - 4z^3 - 3z^4 - 3z^5 - z^6 \\ &= \frac{(1+z)^2(1 - 3z + 4z^3 - 2z^5 - z^6)}{1 - z}\end{aligned}$$

Furthermore the cube of the augmentation ideal of HKR is equal to zero, and the square of that ideal is generated by

$$X_3X_5, X_2X_5, X_1X_5, X_3X_4, X_2X_4, X_1X_4, X_2X_3, X_5X_8, X_4X_8, X_2X_8, X_3X_7,$$
$$X_2X_7, X_1X_4, X_5X_6, X_4X_6, X_3X_6, X_7X_8, X_6X_8, X_6X_7$$

leading to the bigraded Hilbert series

$$HKR(x, y) = 1 + 5xy + 4xy^2 + 3xy^3 + 3xy^4 + xy^5 + 7x^2y^2 + 9x^2y^3 + 3x^2y^4$$

and thus $H(-z, z) = 1 - 5z^2 - 4z^3 + 4z^4 + 6z^5 + 2z^6$ so that

$$\frac{1}{P_{HKR}(z)} = (1 + 1/z)/HKR^!(z) - \bigl(1 - 5z^2 - 4z^3 + 4z^4 + 6z^5 + 2z^6\bigr)/z$$

so that

$$P_{HKR}(z) = \frac{1 - z}{(1+z)^2(1 - 2z - z^2)(1 - z - z^2 + z^3 + z^4)}$$

But

$$P_R(z) = \frac{(1+z)^3(1 - z)}{(1 - 2z - z^2)(1 - z - z^2 + z^3 + z^4)}$$

so the Avramov spectral sequence degenerates. Furthermore $\mathrm{Ext}^*_{HKR}(k, k)$ is not finitely generated.

More details: **Case III** Now the ring is $R = k[x, y, z, u, v]/(xz + yu, zu + uv, yv, z^2, v^2)$. We have that

$$R^!(z) = \left(\frac{1+z}{1-z-z^2}\right)^2 \prod_{n=1}^{\infty} \frac{1+z^{2n-1}}{1-z^{2n}}$$

and

$$R(z) = \frac{1+3z+z^2-2z^3+z^4}{(1-z)^2}$$

and the formula (it is still valid)

$$\frac{1}{P_R(z)} = (1+1/z)/R^!(z) - R(-z)/z$$

gives $P_R(z)$.

On the other hand `DGAlgebras` gives that HKR has five generators X_1, X_2, X_3, X_4, X_5 of degree $(1, 2)$, three generators X_5, X_7, X_8 of degree $(2, 3)$, one generator X_9 of degree (4,6) and one generator X_{10} of degree $(5, 7)$. Furthermore, these generators satisfy 29 quadratic relations of which only the following are non-monomial relations:

$$X_1X_3 + X_3X_4, X_1X_2 + X_3X_5, X_1X_8 - X_4X_8, X_1X_6 + X_3X_7 + X_5X_8$$

Since $IHKR^3 = 0$ and

$$IHKR^2 = (X_3X_5, X_2X_5, X_1X_5, X_3X_4, X_2X_4, X_1X_4, X_2X_3, X_5X_8, X_4X_8, X_2X_8, X_3X_7,$$
$$X_2X_7, X_1X_7, X_5X_6, X_4X_6, X_3X_6, X_2X_6, X_7X_8, X_6X_8, X_6X_7, X_6X_6)$$

It follows that $HKR^!$ is the quotient of the free algebra on the ten generators Y_i (which are dual to the X_i) by the twosided ideal generated by the 21 quadratic relations

$$[Y_1, Y_4], [Y_1, Y_5], [Y_2, Y_3], [Y_2, Y_4], [Y_2, Y_5], [Y_1, Y_2] - [Y_3, Y_5], [Y_1, Y_3] - [Y_3, Y_4],$$
$$[Y_1, Y_7], [Y_2, Y_6], [Y_2, Y_7], [Y_2, Y_8], [Y_3, Y_6], [Y_4, Y_6], [Y_5, Y_6], [Y_1, Y_6] - [Y_3, Y_7],$$
$$[Y_3, Y_7] - [Y_5, Y_8], [Y_1, Y_8] - [Y_5, Y_8], Y_6Y_6, [Y_6, Y_7], [Y_6, Y_8], [Y_7, Y_8]$$

where $[,]$ again means the graded commutator so that $[Y_i, Y_j] = Y_iY_j - Y_jY_i$ if $1 \le i \le 5$ and $1 \le j \le 5$. Furthermore $[Y_i, Y_j] = Y_iY_j + Y_jY_i$ otherwise. Note that here the generators should have the degree 2,3,4,5 and we again apply the Ufnarovski trick introducing a new extra variable T and replacing everywhere Y_i by $T^{degY_i - 1}Z_i$ gives a new algebra A with generators of degree 1, such that

$$1/HKR^!(z) = 1/A(z) + 11z - 5z^2 - 3z^3 - x^5 - z^6$$

Again BERGMAN gives $A(z)$ easily up to degree 20 and we get:

$$1/HKR^!(z) = 1 - 5z^2 - 3z^3 + 7z^4 + 9z^5 + 3z^6 - 5z^7 - 13z^8 - 11z^9 + 2z^{10} + 9z^{11} + z^{12} - 5z^{13} + 2z^{14} + 7z^{15} + 4z^{16} + 5z^{17} + 6z^{18} - 2z^{19} - 8z^{20} \dots$$

and since $HKR(x, y) = 1 + 5xy + 3xy^2 + xy^4 + xy^5 + 7x^2y^2 + 10x^2y^3 + 4x^2y^4$ we get $HKR(-z, z) = 1 - 5z^2 - 3z^3 + 7z^4 + 9z^5 + 3z^5$ and the Löfwall formula

$$1/\Phi_R(z, z) = 1/P_{HKR}(z, z) = (1 + 1/z)/HKR^!(z) - HKR(-z, z)/z$$

gives

$$\Phi_R(z, z) = 1 + 5z^2 + 3z^3 + 18z^4 + 21z^5 + 66z^6 + 111z^7 + 274z^8 + 549z^9 + 1251z^{10} \dots$$

and the Avramov spectral sequence degenerates up to degree 20 (and probably up to degree ∞).

More details **Case IV**:

Finally the ring is $R = k[x, y, z, u, v]/(x^2, xy, yz, zu, uv, v^2, (x + y)(u + v))$. This is a Koszul ring (proof below) with Hilbert series $(1 + 4t + 3t^2 - t^3)/(1 - t)$ so that the right hand side of the Avramov spectral sequence is

$$P_R(z)/(1 + z)^5 = 1 + 7z^2 + 7z^3 + 42z^4 + 84z^5 + 287z^6 + 734z^7 + 2156z^8 + \dots$$

On the other hand the dimensions of the $\mathrm{Tor}^{HKR}_{p,q}(k, k)$ are given by the table (using DGAlgebras):

0 :	1	7	42	245
1 :	.	7	84	735
2 :	.	.	42	735
3 :	.	.	.	245
4 :	.	1	14	133
5 :	.	.	14	266
6 :	.	.	.	133
7 :	.	.	.	.
8 :	.	.	1	21
9 :	.	.	.	21

From this we see that

$$|\mathrm{Tor}^{HKR}_{1,5}(k, k)| + |\mathrm{Tor}^{HKR}_{2,4}(k, k)| + |\mathrm{Tor}^{HKR}_{3,3}(k, k)| = 1 + 42 + 245 = 288$$

which is one unit higher than 287, so the Avramov spectral sequence does not degenerate. Now the differential in the spectral sequence $E^2_{3,4} \to E^2_{1,5}$ must be *onto* the onedimensional vector space $E^2_{1,5}$, since if the image were zero, we would have from the exact sequence

$$0 \to E^3_{3,4} \to E^2_{3,4} \to E^2_{1,5}$$

that the dimension of $E^3_{3,4}$ would be the same as the dimension of $E^2_{3,4}$ which is 735. But $E^3_{3,4} = E^\infty_{3,4}$ can not be bigger than 734. Thus the edge homomorphism $E^2_{1,5} \to E^\infty_{1,5}$ is zero. Using the same methods we used earlier in the present paper we obtain the result

$$(1+z)^5/P_R(z) - 1/\Phi_R(z,z) = z^6(1+z).$$

Finally, here is the proof that R is a Koszul ring: the edge ring C_7 is

$$k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]/(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_1)$$

and it is a Koszul ring (Fröberg). Now divide out by the R-sequence $x_1 + x_2 + x_3$ and $x_5 + x_6 + x_7$, *i.e.* replace x_1 by $-x_2 - x_3$ and x_7 by $-x_5 - x_6$ and change the notation slightly to get our R.

Appendix A. A Description of the Homological Behaviour of families of quadratic forms in four variables

This is appendix to our present paper (2015) about the homological properties of the homology of the Koszul algebra of *codim* $\geq$ 4 local rings. Part of this appendix was originally included as pages 86–95 in the volume "Syzygies and Geometry", October 7–8, 1995, AMS Special Session and International Conference, Northeastern University, Boston, 1995, edited by Antony Iarrobino, Alex Martsinkovsky and Jerzy Weyman. But that volume had limited circulation, and here we present an update where furthermore hopefully all misprints in the tables have been corrected (I thank Aldo Conca for help with this).

Let k be a field (for simplicity we assume here that it is of characteristic 0), $k[X_1, \ldots, X_n]$ the commutative polynomial ring in n variables over k and let $f_1, \ldots, f_t$ be quadratic forms in the X_i:s. Consider the quotient ring

$$R = k[X_1, \ldots, X_n]/(f_1, \ldots f_r) \tag{1}$$

which is a graded vector space over k. If V is a vector space over k, we denote by $|V|$ the dimension $\dim_k(V)$ of V over k. Let $R(t) = \sum_{i=0}^{\infty} |R^i| t^i$ be the Hilbert series of R. In Tables 1–2 below you find a description of the 60 possible series $R(t)$ when $n = 4$ (the "example" column in Tables 1–2 refers to the examples in Tables 3–7 below).

Let $\mathrm{Ext}^*_R(k,k) = \mathrm{Ext}^{*,*}_R(k,k)$ be the Yoneda Ext-algebra of R. This space is dual to $\mathrm{Tor}^R_{*,*}(k,k)$ and the second grading comes from the grading of R. In Tables 3–7 you find the possible 83 series

Table 1
The first 30 possible Hilbert series $H_1, \ldots, H_{30}$.

	Rational form	Taylor series (Hilbert function)							Example
H_1	$1+4t$	1	4	0	0	0	0	0	83
H_2	$1+4t+t^2$	1	4	1	0	0	0	0	81
H_3	$(1+3t-3t^2)/(1-t)$	1	4	1	1	1	1	1	82
H_4	$1+4t+2t^2$	1	4	2	0	0	0	0	78
H_5	$(1+3t-2t^2-t^3)/(1-t)$	1	4	2	1	1	1	1	79
H_6	$(1+3t-2t^2)/(1-t)$	1	4	2	2	2	2	2	80
H_7	$1+4t+3t^2=(1+t)(1+3t)$	1	4	3	0	0	0	0	71
H_8	$1+4t+3t^2+t^3$	1	4	3	1	0	0	0	73
H_9	$(1+3t-t^2-2t^3)/(1-t)$	1	4	3	1	1	1	1	72, 74
H_{10}	$(1+3t-t^2-t^3)/(1-t)$	1	4	3	2	2	2	2	75
H_{11}	$(1+3t-t^2)/(1-t)$	1	4	3	3	3	3	3	76
H_{12}	$(1+2t-4t^2+2t^3)/(1-t)^2$	1	4	3	4	5	6	7	77
H_{13}	$1+4t+4t^2=(1+2t)^2$	1	4	4	0	0	0	0	54, 55, 56, 57
H_{14}	$1+4t+4t^2+t^3=(1+t)(1+3t+t^2)$	1	4	4	1	0	0	0	61
H_{15}	$(1+3t-3t^3)/(1-t)$	1	4	4	1	1	1	1	58, 59, 60, 62
H_{16}	$(1+3t-2t^3-t^4)/(1-t)$	1	4	4	2	1	1	1	64
H_{17}	$(1+2t-2t^2)(1+t)/(1-t)$	1	4	4	2	2	2	2	63, 65
H_{18}	$(1+3t-t^3)/(1-t)$	1	4	4	3	3	3	3	66
H_{19}	$(1+3t)/(1-t)$	1	4	4	4	4	4	4	68
H_{20}	$(1+2t-3t^2+t^4)/(1-t)^2$	1	4	4	4	5	6	7	67, 69
H_{21}	$(1+2t-3t^2+t^3)/(1-t)^2$	1	4	4	5	6	7	8	70
H_{22}	$1+4t+5t^2$	1	4	5	0	0	0	0	29
H_{23}	$1+4t+5t^2+t^3$	1	4	5	1	0	0	0	30, 32
H_{24}	$(1+3t+t^2-4t^3)/(1-t)$	1	4	5	1	1	1	1	31, 33
H_{25}	$1+4t+5t^2+2t^3=(1+t)^2(1+2t)$	1	4	5	2	0	0	0	38
H_{26}	$(1+3t+t^2-3t^3-t^4)/(1-t)$	1	4	5	2	1	1	1	39
H_{27}	$(1+3t+t^2-3t^3)/(1-t)$	1	4	5	2	2	2	2	34, 35, 36, 37, 40
H_{28}	$(1+2t-t^2-t^3)(1+t)/(1-t)$	1	4	5	3	2	2	2	43
H_{29}	$(1+2t)(1+t-t^2)/(1-t)$	1	4	5	3	3	3	3	41, 42, 44
H_{30}	$(1+2t-t^2)(1+t)/(1-t)$	1	4	5	4	4	4	4	46

In Table 2 you can see the last 30 possible Hilbert series.

Table 2
The last 30 possible Hilbert series $H_{31}, \ldots, H_{60}$.

	Rational factored form	Taylor series (Hilbert function)							Example
H_{31}	$(1+2t-2t^2-2t^3+2t^4)/(1-t)^2$	1	4	5	4	5	6	7	45, 47, 48
H_{32}	$(1+3t+t^2)/(1-t)$	1	4	5	5	5	5	5	50
H_{33}	$(1+2t-2t^2-t^3+t^4)/(1-t)^2$	1	4	5	5	6	7	8	49, 51
H_{34}	$(1+2t-2t^2)/(1-t)^2$	1	4	5	6	7	8	9	52
H_{35}	$(1+2t-2t^2+t^3)/(1-t)^2$	1	4	5	7	9	11	13	53
H_{36}	$1+4t+6t^2+4t^3+t^4$	1	4	6	4	1	0	0	11
H_{37}	$(1+3t+2t^2-2t^3-3t^4)/(1-t)$	1	4	6	4	1	1	1	12
H_{38}	$(1+3t+2t^2-2t^3-2t^4)/(1-t)$	1	4	6	4	2	2	2	13
H_{39}	$(1+3t+2t^2-2t^3-t^4)/(1-t)$	1	4	6	4	3	3	3	14
H_{40}	$(1+3t+2t^2-2t^3)/(1-t)$	1	4	6	4	4	4	4	15
H_{41}	$(1+2t-t^2-4t^3+3t^4)/(1-t)^2$	1	4	6	4	5	6	7	16
H_{42}	$(1+t-t^2)(1+t)^2/(1-t)$	1	4	6	5	4	4	4	18
H_{43}	$(1+3t+2t^2-t^3)/(1-t)$	1	4	6	5	5	5	5	17, 19
H_{44}	$(1+2t-t^2-3t^3+2t^4)/(1-t)^2$	1	4	6	5	6	7	8	20
H_{45}	$(1+t)(1+2t)/(1-t)$	1	4	6	6	6	6	6	23
H_{46}	$(1+t-t^2)^2/(1-t)^2$	1	4	6	6	7	8	9	21, 22, 24
H_{47}	$(1+2t-t^2-t^3)/(1-t)^2$	1	4	6	7	8	9	10	26
H_{48}	$(1+t-2t^2+t^3)(1+t)/(1-t)^2$	1	4	6	7	9	11	13	25
H_{49}	$(1+2t-t^2)/(1-t)^2$	1	4	6	8	10	12	14	27
H_{50}	$(1+t-3t^2+3t^3-t^4)/(1-t)^3$	1	4	6	10	15	21	28	28

Table 2 (*continued*)

	Rational factored form	Taylor series (Hilbert function)							Example
H_{51}	$(1+t)^3/(1-t)$	1	4	7	8	8	8	8	5
H_{52}	$(1+2\,t-2\,t^3)/(1-t)^2$	1	4	7	8	9	10	11	6
H_{53}	$(1+2\,t-2\,t^3+t^4)/(1-t)^2$	1	4	7	8	10	12	14	7
H_{54}	$(1+t-t^2)(1+t)/(1-t)^2$	1	4	7	9	11	13	15	8
H_{55}	$(1+2\,t)/(1-t)^2$	1	4	7	10	13	16	19	9
H_{56}	$(1+t-2\,t^2+t^3)/(1-t)^3$	1	4	7	11	16	22	29	10
H_{57}	$(1+t)^2/(1-t)^2$	1	4	8	12	16	20	24	3
H_{58}	$(1+t-t^2)/(1-t)^3$	1	4	8	13	19	26	34	4
H_{59}	$(1+t)/(1-t)^3$	1	4	9	16	25	36	49	2
H_{60}	$1/(1-t)^4$	1	4	10	20	35	56	84	1

Table 3
The first 19 cases (case numbers in **boldface** indicate depth 0 rings).

Case	The graded Betti numbers								Ideal gen:s (ex.)	$R(t)$	$R^!(t)$
1	1	4	6	4	1	0	0	0	0	H_{60}	A_{60}
2	1	4	7	8	8	8	8	8	x^2	H_{59}	A_{59}
3	1	4	8	12	16	20	24	28	x^2, y^2	H_{57}	A_{57}
4	1	4	8	13	21	34	55	89	x^2, xy	H_{58}	A_{58}
5	1	4	9	16	25	36	49	64	x^2, y^2, z^2	H_{51}	A_{51}
6	1	4	9	16	25	36	49	64	x^2, y^2+xz, yz	H_{52}	A_{51}
	–	–	–	1	6	21	56	126			
	–	–	–	–	–	–	1	8			
7	1	4	9	16	25	36	49	64	$x^2+y^2, z^2+u^2,$	H_{53}	A_{51}
	–	–	–	2	12	42	112	252	$xz+yu$		
	–	–	–	–	–	–	4	32			
8	1	4	9	17	30	51	85	140	x^2, y^2, xz	H_{54}	A_{54}
9	1	4	9	18	36	72	144	288	x^2, xy, y^2	H_{55}	A_{55}
10	1	4	9	19	41	88	189	406	x^2, xy, xz	H_{56}	A_{56}
11	1	4	10	20	35	56	84	120	x^2, y^2, z^2, u^2	H_{36}	A_{36}
12	1	4	10	20	35	56	84	120	$x^2+xy, y^2+xu,$	H_{37}	A_{36}
	–	–	–	–	–	–	–	–	z^2+xu, u^2+zu		
	–	–	–	1	7	29	91	239			
	–	–	–	–	–	–	–	–			
	–	–	–	–	–	–	1	10			
13	1	4	10	20	35	56	84	120	$x^2+z^2+u^2,$	H_{38}	A_{36}
	–	–	–	1	6	22	62	148	$y^2, xz, yu+zu$		
	–	–	–	–	–	–	1	8			
14	1	4	10	20	35	56	84	120	$xz, y^2, z^2+u^2,$	H_{39}	A_{36}
		–	–	2	13	51	153	387	$yu+zu$		
		–	–	–	–	–	4	36			
15	1	4	10	20	35	56	84	120	$xz, y^2, yz+u^2,$	H_{40}	A_{36}
		–	–	3	20	80	244	626	$yu+zu$		
		–	–	–	–	–	9	84			
16	1	4	10	20	35	56	84	120	$xy+z^2+yu, y^2,$	H_{41}	A_{36}
		–	–	4	26	103	312	797	$yu+zu, xz$		
		–	–	–	–	–	16	144			
17	1	4	10	21	39	66	104	155	$xz, yz+xu,$	H_{43}	$A_{43.1}$
		–	–	–	1	8	36	121	$y^2, yu+zu$		
18	1	4	10	21	40	72	125	212	x^2, y^2, z^2, yu	H_{42}	A_{42}
19	1	4	10	21	40	72	125	212	$xz, y^2,$	H_{43}	A_{42}
		–	–	1	7	29	93	255	$yu+zu, u^2$		
		–	–	–	–	–	1	10			

Table 4
Cases 20–37.

Case	The graded Betti numbers								Ideal gen:s (ex.)	$R(t)$	$R^!(t)$
20	1	4	10	21	40	72	125	212	$xz, y^2,$	H_{44}	A_{42}
		–	–	2	13	52	163	440	$yu + z^2,$		
		–	–	–	–	–	4	36	$yu + zu$		
21	1	4	10	22	45	88	167	310	$xz, y^2, z^2, yu + zu$	H_{46}	A_{46}
22	1	4	10	22	45	88	167	311	$x^2 + xy, xu,$	H_{46}	$A_{46.1}$
	–	–	–	–	–	–	1	7	$xz + yu, y^2$		
23	1	4	10	22	46	94	190	382	xz, xu, y^2, z^2	H_{45}	A_{45}
24	1	4	10	22	46	94	190	382	$xz, y^2,$	H_{46}	A_{45}
		–	–	1	6	23	72	201	$yz + z^2,$		
		–	–	–	–	–	1	8	$yu + zu$		
25	1	4	10	23	51	111	240	517	x^2, xy, xz, u^2	H_{48}	A_{48}
26	1	4	10	23	52	117	263	591	xz, y^2, yu, zu	H_{47}	A_{47}
27	1	4	10	24	58	140	338	816	xy, xz, y^2, yz	H_{49}	A_{49}
28	1	4	10	26	69	181	476	1252	x^2, xy, xz, xu	H_{50}	A_{50}
29	1	4	11	24	46	80	130	200	$x^2 + xy, y^2 + xu,$	H_{22}	$A_{22.1}$
	–	–	–	5	36	159	536	1519	$z^2 + xu, zu + u^2,$		
	–	–	–	–	–	–	25	260	yz		
30	1	4	11	25	50	91	154	246	$xy + u^2, xz,$	H_{23}	$A_{23.1}$
	–	–	–	1	9	46	175	550	$x^2 + z^2 + u^2$		
	–	–	–	–	–	–	1	14	$y^2, yu + zu$		
31	1	4	11	25	50	91	154	246	$x^2 - y^2, y^2 - z^2,$	H_{24}	$A_{23.1}$
	–	–	–	2	16	77	282	864	$z^2 - u^2, xz + yu,$		
	–	–	–	–	–	–	4	48	$-x^2 + xy - yz + xu$		
32	1	4	11	25	51	97	176	309	$x^2 + z^2, xz,$	H_{23}	$A_{23.2}$
	–	–	–	2	15	68	238	708	$y^2, yu + zu,$		
	–	–	–	–	–	–	4	44	u^2		
33	1	4	11	25	51	97	176	309	$x^2 + xy, y^2 + yz,$	H_{24}	$A_{23.2}$
	–	–	–	3	22	99	345	1024	$y^2 + xu, z^2 + xu,$		
	–	–	–	–	–	–	9	96	$zu + u^2$		
34	1	4	11	26	55	106	190	322	$x^2 + xy + yu + u^2, y^2, xz,$	H_{27}	$A_{27.1}$
	–	–	–	–	–	5	38	172	$x^2 + z^2 + u^2, yu + zu$		
35	1	4	11	26	55	108	201	360	$x^2 + z^2 + u^2, y^2, xz,$	H_{27}	$A_{27.2}$
	–	–	–	–	2	16	76	278	$xy + yz + yu, yu + zu$		
36	1	4	11	26	56	114	223	424	$x^2 + y^2, z^2, u^2,$	H_{27}	$A_{27.3}$
	–	–	–	1	8	38	140	441	$yz - yu,$		
	–	–	–	–	–	–	1	12	$xz + zu$		
37	1	4	11	26	56	114	223	425	$x^2, y^2, xy - zu,$	H_{27}	$A_{27.4}$
	–	–	–	1	8	38	141	448	$yz - xu,$		
	–	–	–	–	–	–	1	12	$(x - y)(z - u)$		

Table 5
Cases 38–57.

Case	The graded Betti numbers								ideal gen:s (ex.)	$R(t)$	$R^!(t)$
38	1	4	11	26	57	120	247	502	x^2, y^2, z^2, zu, u^2	H_{25}	A_{25}
39	1	4	11	26	57	120	247	502	$x^2 + yz + u^2,$	H_{26}	A_{25}
	–	–	–	1	7	31	109	334	$xz + z^2 + yu,$		
	–	–	–	–	–	–	1	10	xy, xu, zu		
40	1	4	11	26	57	120	247	502	$x^2 - xu, xu - y^2,$	H_{27}	A_{25}
	–	–	–	2	14	62	218	668	$y^2 - z^2, z^2 - u^2,$		
	–	–	–	–	–	–	4	40	$xz + yu$		
41	1	4	11	27	62	137	295	624	xy, y^2, z^2, zu, u^2	H_{29}	A_{29}
42	1	4	11	27	62	137	296	632	$x^2 + xy, zu, y^2,$	H_{29}	$A_{29.1}$
	–	–	–	–	–	1	8	30	$xu, xz + yu$		
43	1	4	11	27	63	144	326	735	x^2, y^2, yz, zu, u^2	H_{28}	A_{28}
44	1	4	11	27	63	144	326	735	$xz, yz, y^2,$	H_{29}	A_{28}
	–	–	–	1	7	31	111	352	$yu + zu,$		
	–	–	–	–	–	–	1	10	$z^2 + u^2$		

Table 5 (*continued*)

Case	The graded Betti numbers								ideal gen:s (ex.)	$R(t)$	$R^!(t)$
45	1	4	11	28	68	162	382	896	$xy+yz, xy+z^2+yu,$	H_{31}	A_{31}
									$yu+zu, y^2, xz$		
46	1	4	11	28	69	168	407	984	x^2, xy, yz, zu, u^2	H_{30}	A_{30}
47	1	4	11	28	69	168	407	984	$x^2+xy, y^2,$	H_{31}	A_{30}
	–	–	–	1	6	25	88	282	$xu, xz+yu,$		
	–	–	–	–	–	–	1	8	$-x^2+xz-yz$		
48	1	4	11	28	69	169	413	1009	$xy, z^2+yu,$	H_{31}	$A_{31.1}$
	–	–	–	1	7	31	113	370	$yu+zu, y^2,$		
	–	–	–	–	–	–	1	10	xz		
49	1	4	11	29	75	193	496	1274	xz, y^2, z^2, yu, zu	H_{33}	A_{33}
50	1	4	11	29	76	199	521	1364	x^2, xy, xz, y^2, z^2	H_{32}	A_{32}
51	1	4	11	29	76	199	521	1364	$xy, xz, yz+xu,$	H_{33}	A_{32}
	–	–	–	1	6	25	90	300	z^2, zu		
	–	–	–	–	–	–	1	8			
52	1	4	11	30	82	224	612	1672	x^2, xy, xz, y^2, yz	H_{34}	A_{34}
53	1	4	11	31	88	249	705	1996	y^2-u^2, xz, yz, z^2, zu	H_{35}	A_{35}
54	1	4	12	32	80	192	448	1024	$x^2, xz, y^2, z^2, yu+zu, u^2$	H_{13}	A_{13}
55	1	4	12	32	80	192	449	1034	$x^2+xy, xz+yu,$	H_{13}	$A_{13.1}$
	–	–	–	–	–	1	10	57	$xu, y^2, z^2, zu+u^2$		
56	1	4	12	32	80	193	457	1072	$x^2+xz+u^2, xy,$	H_{13}	$A_{13.2}$
	–	–	–	–	1	9	48	199	xu, x^2-y^2, z^2, zu		
57	1	4	12	32	81	200	488	1184	$x^2+yz+u^2, xu,$	H_{13}	$A_{13.3}$
	–	–	–	1	8	40	160	562	$x^2+xy, xz+yu,$		
	–	–	–	–	–	–	1	12	$zu+u^2, y^2+z^2$		

Table 6
Cases 58–74.

Case	The graded Betti numbers								Ideal gen:s,ex.	$R(t)$	$R^!(t)$
58	1	4	12	33	87	225	576	1467	$x^2+xy, x^2+zu, y^2,$	H_{15}	A_{15}
									$z^2, xz+yu, xu$		
59	1	4	12	33	87	225	576	1468	$x^2-y^2, xy, xu,$	H_{15}	$A_{15.1}$
	–	–	–	–	–	–	1	8	$z^2, zu, xz+yu$		
60	1	4	12	33	87	225	577	1474	$x^2+yz+u^2, xz+yu,$	H_{15}	$A_{15.2}$
	–	–	–	–	–	1	7	33	zu, xy, z^2, xu		
61	1	4	12	33	88	232	609	1596	$x^2-y^2, xy, z^2,$	H_{14}	A_{14}
									xu, zu, u^2		
62	1	4	12	33	88	232	609	1596	$x^2-y^2, xy,$	H_{15}	A_{14}
	–	–	–	1	7	33	129	455	$xu, yz+yu,$		
	–	–	–	–	–	–	1	10	z^2, zu		
63	1	4	12	34	94	258	706	1930	$x^2, xy, xu, y^2, z^2, zu$	H_{17}	A_{17}
64	1	4	12	34	95	265	739	2061	$x^2-y^2, xy,$	H_{16}	A_{16}
									z^2, xu, yu, zu		
65	1	4	12	34	95	265	739	2061	$x^2, xy, xz,$	H_{17}	A_{16}
	–	–	–	1	7	33	131	475	$y^2, yu+z^2,$		
	–	–	–	–	–	–	1	10	$yu+zu$		
66	1	4	12	35	101	291	838	2413	$xz, y^2, yu, z^2, zu, u^2$	H_{18}	A_{18}
67	1	4	12	36	107	318	945	2808	xy, xz, y^2, yu, z^2, zu	H_{20}	A_{20}
68	1	4	12	36	108	324	972	2916	$x^2, xy, xz, y^2, yz, z^2$	H_{19}	A_{19}
69	1	4	12	36	108	324	972	2916	$x^2, xz, xu,$	H_{20}	A_{19}
	–	–	–	1	6	27	108	405	$xy-zu,$		
	–	–	–	–	–	–	1	8	yz, z^2		
70	1	4	12	37	114	351	1081	3329	x^2, xy, xz, xu, y^2, yz	H_{21}	A_{21}
71	1	4	13	40	121	364	1093	3280	$x^2, y^2, z^2, u^2,$	H_7	A_7
									$xy, zu, yz+xu$		

(*continued on next page*)

Table 6 (*continued*)

Case	The graded Betti numbers								Ideal gen:s,ex.	$R(t)$	$R^!(t)$
72	1	4	13	41	128	399	1243	3872	$x^2-y^2, xy, yz,$		
									$zu, z^2, xz+yu, xu$	H_9	A_9
73	1	4	13	41	129	406	1278	4023	$x^2, y^2, z^2, u^2,$	H_8	A_8
									zu, yu, xu		
74	1	4	13	41	129	406	1278	4023	$x^2, xy+z^2,$	H_9	A_8
	–	–	–	1	7	35	151	604	$yz, xu, yu,$		
	–	–	–	–	–	–	1	10	zu, u^2		

Table 7
Cases 75–83.

Case	The graded Betti numbers								Ideal gen:s	$R(t)$	$R^!(t)$
75	1	4	13	42	135	434	1395	4484	$x^2, xy, xz, xu, y^2, yz, u^2$	H_{10}	A_{10}
76	1	4	13	43	142	469	1549	5116	$x^2, xy, xz, xu, z^2, zu, yu$	H_{11}	A_{11}
77	1	4	13	44	148	498	1676	5640	$x^2, xy, xz, xu, y^2, yz, yu$	H_{12}	A_{12}
78	1	4	14	48	164	560	1912	6528	$x^2, xy, y^2, z^2, zu, u^2,$	H_4	A_4
									$xz+yu, yz-xu$		
79	1	4	14	49	171	597	2084	7275	$x^2, xy, xz, xu, y^2, yu, z^2, zu$	H_5	A_5
80	1	4	14	50	178	634	2258	8042	$x^2, xy, xz, y^2, yz, yu, z^2, zu$	H_6	A_6
81	1	4	15	56	209	780	2911	10 864	$x^2, y^2, z^2, u^2, xy, xz,$	H_2	A_2
									$yz-xu, yu, zu$		
82	1	4	15	57	216	819	3105	11 772	$x^2, xy, xz, xu, y^2,$	H_3	A_3
									zu, u^2, yz, yu		
83	1	4	16	64	256	1024	4096	16 384	$x^2, y^2, z^2, u^2, xy, xz,$	H_1	A_1
									xu, yz, yu, zu		

Table 8
The 16 series $R^!(t)$ that do not correspond to Koszul algebras.

Rings	$R^!(t)$	Numerator of $R^!(t)$	Denominator of $R^!(t)$
17	$A_{43.1}$	$(1+t)^4(1+t^3)$	$(1-t^2)^4$
22	$A_{46.1}$	$(1+t)^2(1-t+t^2)$	$(1-t)^2(1-t-t^2-t^4-t^5)$
29	$A_{22.1}$	$(1+t)^4$	$(1-t^2)^5$
30, 31	$A_{23.1}$	$(1+t)^4(1+t^3)$	$(1-t^2)^5$
32, 33	$A_{23.2}$	$(1+t)^3$	$(1-t^2)^3(1-t-t^2)$
34	$A_{27.1}$	$(1+t)^4(1+t^3)^2$	$(1-t^2)^5(1-t^4)$
35	$A_{27.2}$	$(1+t)^3(1+t^3)$	$(1-t^2)^3(1-t-t^2)$
36	$A_{27.3}$	$(1+t)^2$	$(1-t^2)(1-t-t^2)^2$
37	$A_{27.4}$	$(1+t)^3(1+t^3)$	$(1-t^2)^3(1-t-t^2-t^4-t^5)$
42	$A_{29.1}$	$(1+t)^2(1+t^3)$	$(1-t^2)^2(1-2t-t^4)$
48	$A_{31.1}$	$(1+t)^2$	$1-2t-2t^2+2t^3+t^4-t^5$
55	$A_{13.1}$	$(1-t+t^2)^2$	$(1-t)^3(1-3t+3t^2-3t^3)$
56	$A_{13.2}$	$(1-t+t^2)$	$(1-t)^2(1-3t+2t^2-t^3)$
57	$A_{13.3}$	1	$(1-t)^2(1-2t-t^2)$
59	$A_{15.1}$	$(1+t^3)$	$(1-t)^2(1-2t-t^2-2t^4-t^5)$
60	$A_{15.2}$	1	$(1-t)^2(1-2t-t^2-t^3)$

For e.g. the ring 34 the nilpotency degree of η is 5.

$$P_R(x,y) = \sum_{i,j\geq 0} |\mathrm{Tor}^R_{i,j}(k,k)| x^i y^j \tag{2}$$

for $n=4$. Indeed, these Tables 3–7 should be read in conjunction with Tables 1–2 and Table 8 which describes the relevant non-Koszul cases. Let me be more precise: The sub-algebra of $\mathrm{Ext}^*_R(k,k)$ generated by $\mathrm{Ext}^1_R(k,k)$ is denoted by $R^!$ and its Hilbert series

Table A
Double Tor of HKR.

R	ideal gen:s (ex.)	1/Double Tor of HKR
1	0	1
2	x^2	$1 - xy$
3	x^2, y^2	$1 - 2xy + x^2y^2$
4	x^2, xy	$1 - (2 + y)xy$
5	x^2, y^2, z^2	$1 - 3xy + 3x^2y^2 - x^3y^3$
6	$x^2, y^2 + xz, yz$	$1 - (3 + y + 2y^2)xy + (3 - y)x^2y^2 - x^3y^3$
7	$x^2 + y^2, z^2 + u^2, xz + yu$	$1 - (3 + 2y + 4y^2 + y^3)xy + (3 - y)x^2y^2 - x^3y^3$
8	x^2, y^2, xz	$1 - (3 + y)xy + (2 + y)x^2y^2$
9	x^2, xy, y^2	$1 - (3 + 2y)xy$
10	x^2, xy, xz	$1 - (3 + 3y + y^2)xy$
11	x^2, y^2, z^2, u^2	$1 - 4xy + 6x^2y^2 - 4x^3y^3 + x^4y^4$
12	$x^2 + xy, y^2 + yu,$ $x^2 + xu, u^2 + zu$	$1 - (4 + y + 3y^2 + 3y^3)xy + (6 - y^2)x^2y^2 - 4x^3y^3 + x^4y^4$
13	$x^2 + z^2 + u^2,$ $y^2, xz, yu + zu$	$1 - (4 + y + 2y^2)xy + (6 + 2y^2)x^2y^2 - (4 - y)x^3y^3 + x^4y^4$
14	$xz, y^2, z^2 + u^2, yu + zu$	$1 - (4 + 2y + 5y^2)xy + (6 - 3y + 2y^2)x^2y^2 - (4 - y)x^3y^3 + x^4y^4$
15	$xz, y^2, yz + u^2, yu + zu$	$1 - (4 + 3y + 8y^2 + 2y^3)xy + (6 - 4y)x^2y^2 - (4 - y)x^3y^3 + x^4y^4$
16	$xy + z^2 + yu, y^2, yu + zu, xz$	$1 - (4 + 4y + 10y^2 + 3y^3)xy + (6 - 4y)x^2y^2 - (4 - y)x^3y^3 + x^4y^4$
17	$xz, yz + xu, y^2, yu + zu$	$(1 - (4 + y^2)xy + (6 - 4y - 6y^2 - 5y^3 - y^4)x^2y^2$ $- (4 - y)x^3y^3 + x^4y^4)/(1 + xy^2)$
18	x^2, y^2, z^2, yu	$1 - (4 + y)xy + (5 + 2y)x^2y^2 - (2 + y)x^3y^3$
19	$xz, y^2, yu + zu, u^2$	$1 - (4 + 2y + 3y^2 + y^3)xy + (5 - y^2)x^2y^2 - (2 + y)x^3y^3$
20	$xz, y^2, yu + z^2, yu + zu$	$1 - (4 + 3y + 5y^2 + 2y^3)xy + (5 + 2y)x^2y^2 - (2 + y)x^3y^3$
21	$xz, y^2, z^2, yu + zu$	$1 - (4 + 2y)xy + (4 + 4y + y^2)x^2y^2$
22	$x^2 + xy, xu, xz + yu, y^2$	$1 - (4 + y)xy + (4 + 4y - 2y^3 - y^4)x^2y^2 - (1 + 4y + y^2)x^3y^4$ $+ (-1 - 2y + y^2)x^4y^4$
23	xz, xu, y^2, z^2	$1 - (4 + 2y)xy + (3 + 2y)x^2y^2$
24	$xz, y^2, yz + z^2, yu + zu$	$1 - (4 + y)xy + (5 + 2y)x^2y^2 - (2 + y)x^3y^3$
25	x^2, xy, xz, u^2	$1 - (4 + 3y + y^2)xy + (3 + 3y + y^2)x^2y^2$
26	xz, y^2, yu, zu	$1 - (4 + 3y)xy + (1 + y)x^2y^2$
27	xy, xz, y^2, yz	$1 - (4 + 4y + y^2)xy$
28	x^2, xy, xz, xu	$1 - (4 + y)xy + (5 + 2y)x^2y^2 - (2 + y)x^3y^3$
29	$x^2 + xy, y^2 + xu,$ $z^2 + xu, zu + u^2$	$1 - (5 + 5y + 16y^2 + 5y^3)xy + (10 - 10y)x^2y^2$ $- (10 - 5y)x^3y^3 + (5 - y)x^4y^4 - x^5y^5$
30	$xy + u^2, xz, x^2 + z^2 + u^2,$ $y^2, yu + zu$	$\frac{1 - (5 + y + 5y^2)xy + (10 - 10y - 10y^2 - 10y^3 - y^4)x^2y^2 - (10 - 5y - y^3)x^3y^3 + (5 - y)x^4y^4 - x^5y^5}{1 + xy^2}$
31	$x^2 - y^2, y^2 - z^2, z^2 - u^2,$ $xz + yu, -x^2 + xy - yz + xu$	$\frac{1 - (5 + 2y + 8y^2 + 3y^3)xy + (10 - 10y - 12y^2 - 13y^3 - 4y^4)x^2y^2 - (10 - 5y)x^3y^3 + (5 - y)x^4y^4 - x^5y^5}{1 + xy^2}$
32	$x^2 + z^2, xz, y^2, yu + zu, u^2$	$1 - (5 + 3y + 7y^2 + 2y^3)xy + (9 - 3y - 2y^2)x^2y^2$ $- (7 + y - y^2)x^3y^3 + (2 + y)x^4y^4$

Table B
Double Tor of HKR.

R	ideal gen:s (ex.)	1/Double Tor of HKR
33	$x^2 + xy, y^2 + yz,$ $y^2 + xu, z^2 + xu, zu + u^2$	$1 - (5 + 4y + 10y^2 + 4y^3)xy$ $+ (9 - 4y - 3y^2)x^2y^2 - (7 + y - y^2)x^3y^3$ $+ (2 + y)x^4y^4$
34	$x^2 + xy + yu + u^2, y^2, xz$ $x^2 + z^2 + u^2, yu + zu$	$(1 - 5xy + (9 - 5y - 18y^2 - 15y^3 - 4y^4)x^2y^2$ $- (5 + 5y + 9y^3 + 10y^4 + 3y^5)x^3y^3$ $+ (-5 + 9y)x^4y^4 + (9 - 5y)x^5y^5$ $- (5 - y)x^6y^6 + x^7y^7)/(1 + xy^2)^2$
35	$x^2 + z^2 + u^2, y^2, xz$ $xy + yz + yu, yu + zu$	$(1 - (5 + y + 2y^2 + y^3)xy + (9 - 4y - 12y^2$ $- 10y^3 - 3y^4)x^2y^2 - (7 + y - y^2)x^3y^3$ $+ (2 + y)x^4y^4)/(1 + xy^2)$
36	$x^2 + y^2, z^2, u^2$ $yz - yu, xz + zu$	$1 - (5 + 3y + 4y^2 + 2y^3)xy + (8 + 2y$ $- 3y^2 - y^3)x^2y^2 - (4 + 4y + y^2)x^3y^3$

(*continued on next page*)

Table B (*continued*)

R	ideal gen:s (ex.)	1/Double Tor of *HKR*
37	$x^2, y^2, xy - zu$ $yz - xu, (x-y)(z-u)$	$(1-(5+2y+4y^2+2y^3)xy+(8-3y-7y^2 - 7y^3-3y^4)x^2y^2-(4-2y + 2y^2+3y^3)x^3y^3+(-1-4y - y^2+y^3)x^4y^4+(1+2y+y^2)x^5y^5)/(1+xy^2)$
38	x^2, y^2, z^2, zu, u^2	$1-(5+2y)xy+(7+4y)x^2y^2-(3+2y)x^3y^3$
39	$x^2+yz+u^2, xz+z^2+yu,$ xy, xu, zu	$1-(5+3y+3y^2+y^3)xy+(7+2y-y^2)x^2y^2 - (3+2y)x^3y^3$
40	$x^2-xu, xu-y^2, y^2-z^2$ $z^2-u^2, xz-yu$	$1-(5+4y+6y^2+3y^3)xy+(7+y-2y^2)x^2y^2 - (3+2y)x^3y^3$
41	xy, y^2, z^2, zu, u^2	$1-(5+3y)xy+(6+7y+2y^2)x^2y^2$
42	$x^2+xy, zu, y^2,$ $xu, xz+yu$	$(1-(5+2y)xy+(6+y-5y^2-5y^3-2y^4)x^2y^2 + (-1+y-y^3)x^3y^3+(-1-2y-y^2)x^4y^4)/ (1+xy^2)$
43	x^2, y^2, yz, zu, u^2	$1-(5+3y)xy+(5+4y)x^2y^2-(1+y)x^3y^3$
44	xz, yz, y^2 $yu+zu, z^2+u^2$	$1-(5+4y+3y^2+2y^3)xy+(5+3y-y^2)x^2y^2 - (1+y)x^3y^3$
45	$xy+yz, xy+z^2+yu,$ $yu+zu, y^2, xz$	$1-(5+4y)xy+(4+6y+2y^2)x^2y^2$
46	x^2, xy, yz, zu, u^2	$1-(5+4y)xy+(3+4y+y^2)x^2y^2$
46 va	$xz+u^2, xy, xu, x^2,$ $zu+y^2+z^2$	$1-(5+4y+y^2)xy+(4+4y+y^2)x^2y^2$
47	$x^2+xy, y^2, xu,$ $xz+yu, -x^2+xz-yz$	$1-(5+5y+2y^2+y^3)xy+(3+4y+y^2)x^2y^2$
48	$xy, z^2+yu, yu+zu, y^2, xz$	$1-(5+5y+3y^2+2y^3)xy+(3+2y-y^2)x^2y^2 - (1+y)x^3y^3$
49	xz, y^2, z^2, yu, zu	$1-(5+5y+y^2)xy+(2+3y+y^2)x^2y^2$
50	x^2, xy, xz, y^2, z^2	$1-(5+5y+y^2)xy+(1+y)x^2y^2$
51	$xy, xz, yz+xu, z^2, zu$	$1-(5+6y+3y^2+y^3)xy+(1+y)x^2y^2$
52	x^2, xy, xz, y^2, yz	$1-(5+6y+2y^2)xy$
53	y^2-u^2, xz, yz, z^2, zu	$1-(5+7y+4y^2+y^3)xy$
54	$x^2, xz, y^2, z^2, yu+zu, u^2$	$1-(6+4y)xy+(9+12y+4y^2)x^2y^2$

by $R^!(t)$. For so-called Koszul algebras R we have $P_R(x,y) = R^!(xy)$ which implies in particular (put $x=-1, y=t$) that

$$R(t)R^!(-t) = 1. \tag{3}$$

In all cases for $n \leq 4$, *except one case*, we have the formula

$$P_R(x,y)^{-1} = (1+1/x)/R^!(xy) - R(-xy)/x, \tag{4}$$

and in the only exceptional case (case **12** of Table 3) we have

$$P_R(x,y)^{-1} = \left(1-1/x^2\right)/R^!(xy) + R(-xy)/x^2. \tag{5}$$

Tables 3–7 give the Betti numbers as the programme MACAULAY presents them; thus the first horizontal line for each case gives the $|\mathrm{Tor}^R_{i,i}(k,k)|$, the second line gives the $|\mathrm{Tor}^R_{i,i+1}(k,k)|$, etc. If the ring is a Koszul algebra there is only one horizontal line and if $R(t)$ is given as e.g. H_2 (which happens in case **81**), then the corresponding $R^!(t)$ (which is determined by (3)) is denoted by e.g. A_2. In the non-Koszul cases we have special

Table C
Double Tor of HKR.

R	ideal gen:s (ex.)	1/Double Tor of HKR
55	$x^2+xy, xz+yu, xu,$ $y^2, z^2, zu+u^2$	$(1-(6+2y)xy+(9-y-9y^2-9y^3-4y^4)x^2y^2$ $-(1-6y+7y^2+18y^3+12y^4+4y^5)x^3y^3$ $+(-6-13y-3y^2+3y^3)x^4y^4+(3+6y+3y^2)x^5y^5)/$ $(1+xy^2)^2$
56	$x^2+xz+u^2, xy, xu,$ x^2-y^2, z^2, zu	$(1-(6+3y+y^2+y^3)xy+(9+y-11y^2$ $-10y^3-4y^4)x^2y^2+(-4-y+y^2-y^3)x^3y^3$ $-(1+y)x^4y^5))/(1+xy^2)$
57	$x^2+yz+u^2, xy, xu, zu,$ $y^2+z^2, xz+yu$	$1-(6+5y+4y^2+3y^3)xy+(8+5y-3y^2-y^3)x^2y^2$ $-(3+4y+y^2)z^3y^3$
57 v2	$x^2+y^2+z^2, xy,$ $xu, yz, zu, xz+u^2$	$1-(6+5y+5y^2+2y^3)xy+(9+4y-2y^2-y^3)x^2y^2$ $-(4+4y+y^2)z^3y^3$
58	$xy, x^2+zu, y^2, z^2, yu+xz, xu$	$1-(6+5y)xy+(6+9y+3y^2)x^2y^2$
59	$x^2-y^2, xy, xu, z^2, zu, xz+yu$	$(1-(6+4y)xy+(6+3y-3y^2-3y^3-2y^4)x^2y^2$ $+(4+2y-3y^2-y^3)x^3y^4-(1+4y+$ $4y^2+y^3)x^4y^4)/(1+xy^2)$
59 va	$x^2-y^2, xy, yz, zu, xz+u^2, xu$	$(1-(6+4y+y^2)xy+(7+3y-3y^2-3y^3-y^4)x^2y^2$ $+(4+2y-2y^2-y^3)x^3y^4-(2+5y$ $+4y^2+y^3)x^4y^4)/(1+xy^2)$
60	$x^2+yz+u^2, xy, zu,$ $z^2, xz+yu, xu$	$(1-(6+4y)xy+(6+2y-5y^2-4y^3-2y^4)x^2y^2$ $+(-1+3y+4y^2-y^3-y^4)x^3y^3$ $-(1+2y+y^2)x^4y^5)/(1+xy^2)$
61	$x^2-y^2, xy, z^2, xu, zu, u^2$	$1-(6+5y+y^2)xy+(6+6y+y^2)x^2y^2-(y+1)x^3y^3$
62	$x^2-y^2, xy, xu, yz+yu, z^2, zu$	$1-(6+6y+3y^2+3y^3)xy+(5+6y-y^2)x^2y^2-x^3y^4$
62 va	$x^2+yz+u^2, yu,$ zu, xy, z^2, xu	$1-(6+6y+4y^2+2y^3)xy+(6+5y)x^2y^2-(1+y)x^3y^3$
63	$x^2, xy, xu, y^2, z^2, zu$	$1-(6+6y+y^2)xy+(4+6y+2y^2)x^2y^2$
63 v4	$y^2, xz+yu, zu, xy, z^2, xu$	$1-(6+6y)xy+(3+6y+2y^2)x^2y^2$
63 v8	$x^2, xy, xu, yu, z^2,$ $xz+u^2, y^2+z^2+zu$	$1-(6+6y+2y^2)xy+(5+6y+2y^2)x^2y^2$
63 ne	$x^2, xy, xz+u^2, xu, y^2+z^2, zu$	$(1-(6+5y+y^2)xy+(4-4y^2-2y^3-y^4)x^2y^2$ $+(4+5y+y^2)x^3y^4)/(1+xy^2)$
64	$x^2-y^2, xy, z^2, xu, yu, zu$	$1-(6+6y+y^2)xy+(3+3y)x^2y^2-(1+y)x^3y^3$
65	$x^2, xy, xz, y^2, yu+z^2, yu+zu$ $x^2, xy, xz, y^2, yu+z^2, yu+zu$	$1-(6+7y+4y^2+2y^3)xy+(3+2y-y^2)x^2y^2$ $-(1+y)x^3y^3$
66	$xz, y^2, yu, z^2, zu, u^2$	$1-(6+7y+2y^2)xy+(2+3y+y^2)x^2y^2$
66 v5	$xy, xz+u^2, xu, yu, zu, z^2$	$1-(6+7y+y^2)xy+(1+3y+y^2)x^2y^2$
67	xy, xz, y^2, yu, z^2, zu	$1-(6+8y+3y^2)xy+(1+2y+y^2)x^2y^2$
68	$x^2, xy, xz, y^2, yz, z^2$	$1-(6+8y+3y^2)xy$
68v	$x^2, xy, xz, xu, u^2, y^2+z^2+zu$	$1-(6+8y+4y^2+y^3)xy+(1+y)x^2y^2$
69	$x^2, xz, xu, xy-zu, yz, z^2$	$1-(6+9y+5y^2+y^3)xy$
70	x^2, xy, xz, xu, y^2, yz	$1-(6+9y+5y^2+y^3)xy$
71	$x^2, y^2, z^2, u^2, xy, zu, yz+xu$	$1-(7+8y+2y^2)xy+(5+8y+3y^2)x^2y^2$

Table D
Double Tor of HKR.

R	ideal gen:s (ex.)	1/Double Tor of HKR
71 v16	$x^2, y^2+z^2, xy, yz, zu, xz+u^2, xu$	$(1-(7+7y+2y^2)xy+(5+y-3y^3-y^4)x^2y^2$ $+(5+7y+2y^2)x^3y^4)/(1+xy^2)$
71 v4	$x^2+u^2, xy, xu, y^2, yz, z^2, zu$	$1-(7+8y+3y^2)xy+(6+8y+3y^3)x^2y^2$
71 v7	$x^2, y^2, z^2, xz+u^2, xu, yz, zu$	$1-(7+8y+y^2)xy+(4+8y+3y^3)x^2y^2$
71 v5	$x^2+xy, x^2+yz, xy+y^2, z^2,$ $z^2, xu, zu, xz+u^2$	$1-(7+8y)xy+(3+8y+3y^3)x^2y^2$
72	$x^2-y^2, z^2, xy, yz, zu, xu$	$1-(7+9y+2y^2)xy+(2+5y+2y^3)x^2y^2$
72 v1	$xu+u^2, x^2+xy, y^2+xu, y^2+yz,$ $y^2+yz, yu+zu, z^2+xu, zu+u^2$	$1-(7+9y+3y^2)xy+(3+5y+2y^3)x^2y^2$
72 v2e	$yz, x^2+xy, xz+yu, xu, z^2, zu, x^2+u^2$	$1-(7+9y+1y^2)xy+(1+5y+2y^3)x^2y^2$

(continued on next page)

Table D (*continued*)

R	ideal gen:s (ex.)	1/Double Tor of HKR
73	$x^2, y^2, z^2, u^2, xu, yu, zu$	$1-(7+9y+4y^2+y^3)xy+(3+3y)x^2y^2$ $-(1+y)x^3y^3$
74	$x^2, xy+z^2, yz, xu, yu, zu, u^2$	$1-(7+10y+7y^2+3y^3)xy+(3+2y-y^2)x^2y^2$ $-(1+y)x^3y^3$
75	$x^2, xy, xz, y^2, yz, yu, u^2$	$1-(7+10y+5y^2+y^3)xy+(2+3y+y^2)x^2y^2$
75 v1	$y^2, xz, yz+xu, z^2, yu, zu, u^2$	$1-(7+10y+4y^2)xy+(1+2y+y^2)x^2y^2$
75 v2	$xy, xz+u^2, xu, yz+u^2, yu, zu, z^2$	$1-(7+10y+3y^2)xy+(2y+y^2)x^2y^2$
76	$x^2, xy, xz, xu, z^2, zu, yu$	$1-(7+11y+6y^2+y^3)xy$
77	$x^2, xy, xz, xu, y^2, yz, yu$	$1-(7+12y+8y^2+2y^3)xy$
78	$x^2, xy, y^2, z^2, zu, u^2, xz+yu, yz-xu$	$1-(8+12y+5y^2)xy+(2+4y+y^2)x^2y^2$
78 v1	$x^2, y^2, z^2, u^2, xy, xz, xu, yu$	$1-(8+12y+6y^2+y^3)xy+(3+5y+2y^2)x^2y^2$
78 v2e	$xu, yu+xz, yz, x^2, y^2, z^2+xy, u^2, zu$	$1-(8+12y+3y^2)xy+(4y+2y^2)x^2y^2$
78 v3v	$xu, yu+xz, yz, x^2, y^2, z^2+xz, u^2, zu$	$1-(8+12y+4y^2)xy+(1+4y+2y^2)x^2y^2$
79	$x^2, xy, xz, xu, y^2, yu, z^2, zu$	$1-(8+13y+7y^2+y^3)xy+(1+2y+y^2)x^2y^2$
80	$x^2, xy, xz, yz, y^2, yu, z^2, zu$	$1-(8+14y+9y^2+2y^3)xy$
81	$x^2, y^2, z^2, u^2, xy, xz, yz-xu, yu, zu$	$1-(9+16y+9y^2)xy+x^2y^4$
81 va	$x^2, y^2, z^2, u^2, xy, xz, xu, yu, zu$	$1-(9+16y+10y^2+2y^3)xy+(1+2y+y^2)x^2y^2$
82	$x^2, xy, xz, xu, y^2, zu, u^2, yz, yu$	$1-(9+17y+12y^2+3y^3)xy$
83	$x^2, y^2, z^2, u^2, xy, xz, xu, yz, yu, zu$	$1-(10+20y+15y^2+4y^3)xy$

Case 81 is Gorenstein.

notations for the $R^!(t)$ which are given explicitly in Table 8, and we use the formula (4) and (in case **12**) the formula (5) to get $P_R(x,y)$.

Among these 83 cases there are 68 depth 0 cases which are new (they are given in **boldface** in Tables 2–7) and 15 cases of positive depth which correspond to $n \leq 3$ and which were known before [a1].

Furthermore $\mathrm{Ext}_R^{*,*}(k,k)$ and its subalgebra $R^!$ are not only Hopf algebras, but also enveloping algebras of graded Lie algebras $g^{*,*}$ and η^*. In [a3,a4] the relations between the $g^{*,*}$ and η^* are explained. The Lie algebras η^* that correspond to Table 8 are either nilpotent (maximal degree of nilpotency is 4, which means that $\eta^5 = 0$ but $\eta^4 \neq 0$) or nice extensions of free Lie algebras (or product of two free Lie algebras) by nilpotent Lie algebras.

Remarks about the proofs: In all the 83 cases the corresponding dual algebra $R^!$ can be written down and it is a quotient of a free algebra in 4 variables with an ideal generated by quadratic elements. We now want to determine the Hilbert series of that quotient, i.e. $R^!(t)$. For this we use the programme BERGMAN which calculates a Gröbner basis of that ideal. It turns out that in all the 83 cases we can find a suitable permutation of the variables for which this ideal has a *finite* Gröbner basis. Then it follows that the quotient has a rational series (Govorov and Backelin). Let me illustrate this explicitly for the most complicated case in Table 8, namely case 37. Here $R_{37} = k[x,y,z,u]/(x^2, y^2, xy-zu, yz-xu, (x-y)(z-u))$. It follows that the Koszul dual $R^!_{37}$ is given by the formula:

$$R^!_{37} = \frac{k\langle X,Y,Z,U\rangle}{(Z^2, U^2, XY+YX+ZU+UZ, XZ+ZX-YU-UY, 2XZ+2ZX+YZ+ZY+XU+UX)}$$

where $k\langle X,Y,Z,U\rangle$ is the free associative algebra in the dual variables X,Y,Z,U. When we plug this into BERGMAN we see that the corresponding ideal has an *infinite* Gröbner basis. However, using the `permutebergman` addition by Backelin to BERGMAN as described in [a5] we see that the order of the variables (X, U, Z, Y) gives a *finite* Gröbner basis with 5 elements in degree 2, 2 elements in degree 3, 2 in degree 4, 1 in degree 5 and 1 in degree 6. Therefore the Hilbert series is rational and can be explicitly determined. Furthermore it is easy to see that the conditions L_3 are satisfied in all cases, except case 12 of Table 3, where we have L_4. This explains the formulae (4) and (5). Our results are also related to the rational homotopy Lie algebras of CW-complexes [a2,a6].

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