WHEN IS k A DIRECT SUMMAND OF IT'S SYZYGY?

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Abstract. These are WORKING NOTES.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring or standard graded algebra over k. We say that R is (kS_i) if the syzygy module $\Omega^i_R k$ of k contains k as a direct summand. Trivially, every ring is (kS_0). Clearly if R is (kS_i) for some $i > 0$, then $\operatorname{depth} R = 0$. We write

$$\mathrm{kSS}(R) := \{i > 0 \mid R \text{ is } (\mathrm{kS}_i)\}$$

for the sequence of values for which R is (kS_i). We write $\langle a, b, c..\rangle$ for the semigroup generated by $a, b, c..$

We are interested in the property since it displays surprisingly subtle behavior. Also, it is relevant to deformation theory: if R deforms to R' which is (kS_i), then R is (kS_i) or (kS_{i+1}). See Proposition 1.1, part 6. At the same time, it is concrete enough to test using computer programs like Macaulay 2, and indeed by using Macaulay 2 that we were able to identify what seem to be patterns.

1.1. Known results.

Proposition 1.1. *Let $S = k[x_1, \dots, x_n]$ and $R = S/I$.*

(1) $\mathrm{kSS}(R)$ *is a semigroup: if R is (kS_i) and (kS_j), then R is (kS_{i+j}).*
(2) If R is (kS_2), then $\mathrm{kSS}(R)$ is $2, 3, \dots$. This happens if and only if R is a Burch ring in the sense of [1]
(3) If R is Gorenstein and i is an odd integer, then R is (kS_i) if and only if R is a field.
(4) If R is Gorenstein and i is an even integer, then R is (kS_i) if and only if R is an Artinian hypersurface.
(5) Let $(S, \mathfrak{m})$ be depth 1 and $x, y \in \mathfrak{m} - \mathfrak{m}^2$ be nonzero divisors. If $S/(x)$ is (kS_i), then $S/(y)$ is (kS_i) or (kS_{i+1}).
(6) When $n = 2$, there are only 4 patterns: $\mathrm{kSS}(R) = (0), \langle 1\rangle, \langle 2, 3\rangle, \langle 3, 5, 7\rangle$, with respective examples $I = (a^2, b^2), (a^2, ab, b^2), (a^3, a^2b, b^3), (a^4, a^2b^2, b^4)$.
(7) If R is an artinian local Golod ring of embedding dimension n, then $n + 1 \in \mathrm{kSS}(R)$.

Proof. (7) Suppose that the embedding dimension of R is n. The top homology group of the Koszul complex of the maximal ideal in R is isomorphic (up to shift, in the graded case) to the socle of R. Eagon's construction of the resolution of the residue field of a local ring (explained in [3]) involves a surjection from a free summand of the $n + 2$-nd term in the resolution onto this homology group. Thus when the Golod resolution is minimal, the whole socle of R is a free summand of the $n + 1$-st syzygy of the residue field. ■

Motivated by the above Proposition, we define $\mathrm{kSI}(R) := \inf\{i > 0 \mid R \text{ is } (\mathrm{kS}_i)\}$, the (kS)-index of R.

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Remark 1.2. Probably the analog of Proposition 5 also holds for depth $S = t$; that is, for two linear regular sequences of length t $\underline{x}, \underline{y}$, if $S/(\underline{x})$ is (kS_i) then $S/(\underline{y})$ is (kS_j) for some j such that $i \leq j \leq i + t$.

1.2. **Numerical evidence.** Macaulay 2 computation with graded rings in 2,3,4 variables yields some interesting conjectures, at least for the graded, Artinian case. Let $R = S/I$ where $S = k[x_1, \dots, x_n]$ and I be homogenous ideal.

Conjecture 1.3.
1. For all local artinian rings R with $\operatorname{edim} R = n$, the semigroup $\mathrm{kSS}(R)$ consists of the set of all consecutive integers $m, m+1, m+2, \dots$ starting with some $1 \leq m \leq 1 + \operatorname{edim} R$ *except* for the sequence $n+1, n+3, n+4 \dots$. Each of these possibilities does occur. In particular, if $n+1$ is not in $\mathrm{kSS}(R)$ then $\mathrm{kSS}(R)$ is empty.
2. If the map $t_i(I) : \mathrm{Tor}_i^S(\mathfrak{m}I, k) \to \mathrm{Tor}_i^S(I, k)$ is surjective, then R is not (kS_{n+1-i}).
3. All possible patterns can be realized by monomial ideals

Remark 1.4. For the ideals

$$\begin{aligned}
I_1 &= (a^4, b^4, c^4, ab^3), \\
I_2 &= (a^4, b^4, c^4, ab^3, cb^3), \\
I_3 &= (a^4, b^4, c^4, ab^3, bc^3), \\
I_4 &= (a^4, b^4, c^4, abc), \\
I_5 &= (a^4, b^4, c^4, ab^3, b^2c^2),
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{kSS}(S/I_1) &= (0), \\
\mathrm{kSS}(S/I_2) &= (2, 3, 4, 5, ...), \\
\mathrm{kSS}(S/I_3) &= (3, 4, 5, 6, ...), \\
\mathrm{kSS}(S/I_4) &= (4, 5, 6, 7, ...) \\
\mathrm{kSS}(S/I_5) &= (4, 6, 7, 8, ...),
\end{aligned}$$

displaying all the (conjecturally) possible patterns. The examples I_1, I_4 also show that the converse to Conjecture 1.3 (2) need not hold.

Remark 1.5. A result that appeared in [4, Proposition 1.1] asserts that if the matrix $\delta_i : F_i \to F_{i-1}$ in a minimal resolution of I has only entries in $\mathfrak{m}^2$, then the map $t_i(I) : \mathrm{Tor}_i^S(\mathfrak{m}I, k) \to \mathrm{Tor}_i^S(I, k)$ is surjective. So for instance if we start with a monomial ideal I and replace each x_i with x_i^2, we have by Conjecture 1.3 (1) and (2) that $\mathrm{kSI}(R) \geq n+1$, and it can only be infinite or $n+1$. For example, if we take $I = (x_1^4, ..., x_n^4, x_1^2x_2^2, ..., x_1^2x_n^2)$, the conjectures would imply that $\mathrm{kSS}(R) = (n+1, n+3, n+4, \dots)$. We have tested this up to $n = 5$ in Macaulay 2.

More generally, a small number of computational experiments suggested the following. Let G be a graph with edge ideal I_G. Let J_G be the ideal obtained from I_G by squaring all the monomials corresponding to the edges of G and adding the ideal generated by all the 4th powers of variable. For instance, if G is the triangle $J_G = (a^4, b^4, c^4, a^2b^2, b^2c^2, a^2c^2)$. We conjecture $\mathrm{kSS}(S/J_G)$ is empty unless G contains $\{x_a x_i\}$ for fixed a and all $i \neq a$.

Remark 1.6. Points in projective space: If $I \subset S = k[x_0, \dots, x_d]$ defines n general points in $\mathbb{P}^d$ then $R = S/(I, y)$ tends to be Burch for a linear nonzero divisor y (we will say that those points are Burch). For $\mathbb{P}^2$, Burch is equivalent to the Hilbert-Burch matrix contains some linear forms. Then it follows from the results in [2, Chapter 3] that the n general points are *not* Burch when $n = \binom{p}{2} + \left\lfloor \frac{p}{2} \right\rfloor$ for some p.

Exceptions in $\mathbb{P}^3$ so far are when $n = 6, 12(???)$.

2. A criterion for the condition in Conjecture 1.3 (2)

Let $\mathrm{gr}(X)$ be the associated graded object using the maximal ideal $\mathfrak{m}$.

Proposition 2.1. *If $\mathrm{Tor}_i^{\mathrm{gr}(S)}(\mathrm{gr}(I), k)_i = 0$, then the map $t_i(I) : \mathrm{Tor}_i^S(\mathfrak{m}I, k) \to \mathrm{Tor}_i^S(I, k)$ is surjective. Consequently, if $\mathrm{Tor}_i^{\mathrm{gr}(S)}(\mathrm{gr}(I), k)_i = 0$, then $t_j(I)$ is surjective for all $j \geq i$.*

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