SOCLE SUMMANDS

DAVID EISENBUD AND HAI-LONG DAO

ABSTRACT.

INTRODUCTION

Throughout this paper $(S, \mathfrak{m}_S, k)$ denotes a regular local ring of dimension n and $(R = S/I, \mathfrak{m}_R, k)$ denotes a factor ring with $I \subset \mathfrak{m}_S^2$. Let $(K = \wedge R^n, \partial)$ be the Koszul complex of the maximal ideal of R. Let $B_i \subset C_i \subset K_i$ be the modules of boundaries and cycles, and let $H_i = C_i/B_i$ be the homology.

Because the x_i annihilate the socle, we have $\operatorname{soc} K_i \subset C_i$ and thus $\operatorname{soc} K_i = \operatorname{soc} C_i$ for all i. Let $H_i' \subset H_i$ be the image of $\operatorname{soc} K_i$ in H_i.

1. THE PERSISTENCE OF SOCLE SUMMANDS IN KOSZUL CYCLES

Proposition 1.1. *An element $s \in \operatorname{soc} K_i$ generates a summand of C_i if and only if $s \notin B_i \cap \mathfrak{m}^2 K_i$. Thus if $I \subset \mathfrak{m}_S^3$, so that $\operatorname{soc} K_i = (\operatorname{soc} R)K_i \subset \mathfrak{m}_R^2 K_i$, then $H_i' \neq 0$ is isomorphic to a maximal socle summand of C_i.*

Proof. Suppose that $s \in \operatorname{soc} K_i$. If $s \notin \mathfrak{m}_R^2 K_i$ then since $C_i \subset \mathfrak{m}_R K_i$ we see that $s \notin \mathfrak{m}_R C_i$, so s generates a summand of C_i. On the other hand, if $s \notin B_i$ then since H_i is annihilated by $\mathfrak{m}_R$ we have $\mathfrak{m}_R C_i \subset B_i$, so $s \notin \mathfrak{m}_R C_i$ and again s generates a summand of C_i.

Conversely, suppose that $s \in \mathfrak{m}_R^2 \cap B_i$. The map ∂ induces the map diagonal map

$$\Delta : \wedge^{i+1} k^n \cong K_{i+1}/\mathfrak{m}_R K_{i+1} \to \mathfrak{m}_R K_i/\mathfrak{m}_R^2 K_i \cong k^n \otimes \wedge^i k^n,$$

of the exterior algebra $\wedge k^n$, which is a monomorphism. Thus $s = \partial t$ with $t \in \mathfrak{m}_R K_{i+1}$. But then $\partial t \in \mathfrak{m}_R B_i \subset \mathfrak{m}_R C_i$, so s does not generate a summand of C_i. ∎

Theorem 1.2. *If C_i has a socle summand and $i < n$, then C_{i+1} also has a socle summand.*

Proof. Suppose that $s \in \operatorname{soc} C_i$ generates a summand. By Proposition 1.1, $s \notin B_i$ or $s \notin \mathfrak{m}_R^2 K_i$.

Suppose first that $s \notin \mathfrak{m}_R^2 K_i$. Since $\operatorname{soc} C_i = \operatorname{soc} K_i = (\operatorname{soc} R)K_i$ we see that there is some $a \in \operatorname{soc} R \setminus \mathfrak{m}_R^2$. Since $C_j \subset \mathfrak{m}_R K_j$, the element a

times any generator of K_j is contained in $\operatorname{soc} K_j \setminus \mathfrak{m}_R C_j$. Thus there are socle summands in every C_j.

Let $\{e_i\}$ be a dual basis to a basis $\{f_i\}$ of $K_1 = R^n$. The elements of R^{n*} act as (graded) derivations of the underlying exterior algebra K, and we may write $\partial = \sum x_i e_i$. Note that ∂ anti-commutes with the action of each e_j.

We claim that any element r of K_i can be represented in the form $r = \sum_j e_j(t_j)$ with $t_j \in K_{i+1}$ and that if $r \in \operatorname{soc} K_i$ then the t_i may be taken to be in $\operatorname{soc} K_{i+1}$. To see this, write r as a sum of monomials in the f_i of degree $i < n$. If $m = f_{j_1} \wedge \cdots \wedge f_{j_i}$ and f_j is not among the factors f_{j_ℓ}, then $e_j(m) = 0$ so $m = e_j(f_j \wedge m)$. Since $\operatorname{soc} K_i = (\operatorname{soc} R)K_i$, the desired assertion follows.

Now suppose that $s \in C_i \cap \mathfrak{m}_R^2 K_i$ generates a socle summand, and write $s = \sum e_j(t_j)$ with $t_j \in \operatorname{soc} K_{i+1}$. If $i + 1 = n$ then t_j automatically generates a socle summand, so we may suppose that $i \leq n - 2$. If none of the t_j generates a socle summand, then by Proposition 1.1 each t_j is in B_{i+1}. Write $t_j = \partial r_j$ with $r_j \in K_{i+2}$. Since e_i anti-commutes with ∂, $s = \sum_j e_j t_j = -\partial \sum e_j r_j$, so s is a boundary, and thus by Proposition 1.1 s does not generate a summand of C_i, a contradiction. Thus one of the t_j must generate a summand of C_{i+1}. □

Corollary 1.3. *If $I \subset \mathfrak{m}^3$, then H'_* is isomorphic to the largest socle summand of K, and is generated by $H'_n \cong \operatorname{soc} R$.*

Speculation: in the minimal free algebra representation of the resolution of k, the Koszul complex appears as a tensor factor. Is that the source of *all* the socle summands? Or at least enough of them to explain the persistence in the minimal resolution?

2. CHARACTERIZATION OF BURCH RINGS

Recall that R is *Burch* in the sense of [1] if $\mathfrak{m}_S I : (I : \mathfrak{m}_S) \neq \mathfrak{m}_S$. Here's a syzygy-style proof of a slight improvement of the result, from "Burch Ideals and Burch rings", In [1] it is shown that a local ring $(R, \mathfrak{m}, k)$ is Burch iff $\operatorname{syz}_2(k)$ has a socle summand. We give a new proof of this fact.

Suppose the embedding dimension is n and the minimal number of generators of I is g. We use the fact that the beginning of the minimal R-free resolution of k has the form

$$R^g \oplus \wedge^2 R^n \to R^n \to R$$

where $R^n \to R$ is the first Koszul map, $e_i \mapsto x_i$, and $R^g \to R^n$ is the reduction mod I of any map $S^g \to S^n$ such that the composition $S^g \to S^n \to S$ is the map onto the generators of I. (This resolution is minimal because $\mathrm{Tor}_1^S(R,k) = g$.). Thus $\mathrm{syz}_2(k)$ has a socle summand iff some socle element of $K_1 = R^n$ is in the image of R^g or of $\wedge^2 R^n$.

Theorem 2.1. *I is Burch iff some minimal generator of R^g maps to a socle element of R^n iff the module C_1 of Koszul cycles has a socle summand.*

Proof. Suppose I is Burch. If a minimal generator $f \in I$ can be written as $\sum x_i s_i$, with $s_i \in I : \mathfrak{m}$, then the corresponding generator of R^gmaps to the socle.

Conversely, suppose that $\mathrm{syz}_2(k)$ has a socle summand. If $I : \mathfrak{m}$ is contained in m^2, then this can't be a Koszul boundary, so it must come from a generator of R^g, which thus maps to a minimal generator of I that is a linear combination of socle elements.

If $I : m$ contains some element $x \notin \mathfrak{m}^2$, then $x^2 \in I$ is a minimal generator of $\mathfrak{m}^2$, and thus of I, so x comes from a minimal generator of R^g. □

3. GOLOD RINGS

Throughout this section $R, \mathfrak{m}, k$ denotes a local ring of embedding dimension n.

Golod [2] showed that, in principle, if R is the homomorphic image of a regular local ring S, then an R-free resolution of k could be defined in terms of Massey operations on the S-free resolution of R, and Shamash [5] has made this explicit. More recently, Burke [] showed how one could do the same for any R-module, using A_∞ structures on the S-free resolutions of R and M. These resolutions are minimal in the case R is a *Golod ring* and M is a "Golod module", defined by the vanishing of the Massey products or minimality of A_∞ maps.

An apparently different resolution of the residue field R, that is also minimal iff the ring is Golod, was described by Jack Eagon, and seems only to have appeared in the book of Gulliksen and Levin, [3, pp. 156–161]. When it is minimal (and perhaps always) it is of course isomorphic to the resolution constructed from the Massey operations or A_∞ structures and in all cases the free modules in the resolution may be described in the same way; but Eagon's version, surprisingly, has many fewer map components.

For example, writing K for the Koszul complex of the maximal ideal over R and F for the minimal free resolution of R over S, tensored with R, the free modules at the beginning of either resolution are:

$$\cdots \longrightarrow K_3 \oplus F_2 \longrightarrow K_2 \oplus F_1 \longrightarrow K_1 \longrightarrow K_0.$$

In the A_∞ resolution, the map $F_2 \to K_2 \oplus F_1$ involves the differential $F_2 \to F_1$ of F plus a component $F_2 \to K_2$. But in the Eagon resolution the map has only one component, a lift $F_2 \to K_2$ of the map identifying $F_2/\mathfrak{m}F_2$ with $H_2(K)$.

The description given in [3] presumably follows an unpublished version by Jack Eagon. It is very clever, but a little complicated. Here is a simpler version, adapted to a slightly special case.

To describe the socle summands in the R-syzygies of k we may assume that R is complete, and thus of the form S/I, where S is a regular local ring and I is contained in the square of the maximal ideal of S.

Let K be the Koszul complex of $\mathfrak{m}$ over R, and let F be the tensor product of R and the minimal S-free resolution of R. We define a sequence of free R modules E_i and maps $dE_{i+1} : E_{i+1} \to E_i$ inductively: Let $E_0 = K_0 = R$, and

$$E_{i+1} := K_{i+1} \oplus E_0 \otimes F_i \oplus \cdots \oplus E_{i-j} \otimes F_j \oplus \cdots \oplus E_{i-1} \otimes F_1.$$

Note that for i greater than n, the embedding dimension of R, the modules K_i and F_i are zero, so each direct sum has at most n terms

We will make E_* into a filtered complex. Set $\mathcal{F}_i(E_i) = K_i$ and, for $1 \le j \le i-1$,

$$\mathcal{F}_j(E_i) := K_i \oplus (E_0 \otimes F_{i-1}) \oplus \cdots \oplus (E_{i-j-1} \otimes F_j),$$

which is the sum of terms involving a component of K or a component of F above F_j. Thus

$$E_i = \mathcal{F}_1(E_i) \supset \mathcal{F}_2(E_i) \cdots \supset \mathcal{F}_i(E_i) \supset 0.$$

Having defined maps $dE_j : E_j \to E_{j-1}$ for $1 \le j \le i$ we define $dE_{i+1} : E_{i+1} \to E_i$ to be the sum of the Koszul differential $K_{i+1} \to K_i$, a lifting

$$\beta_{0,i} : E_0 \otimes F_i = F_i \to K_i$$

of the isomorphism $R/\mathfrak{m} \otimes F_i \cong H_i(K)$, and, for $i - j > 0$, a map

$$E_{i-j} \otimes F_j \longrightarrow \mathcal{F}_j(E_i)$$

that is the sum of

$$dE_{i-j} \otimes 1 : E_{i-j} \otimes F_j \to E_{i-j-1} \otimes F_j$$

and a map

$$\beta_{i-j,j} : E_{i-j} \otimes F_j \longrightarrow \mathcal{F}_{j+1} E_i$$

chosen as in Theorem 3.1, below.

Theorem 3.1. *(Eagon [3])*

(1) *The maps $\beta_{p,q}$ may be defined inductively to make (E, dE) into a complex; that is, having defined the maps $\beta_{p,q}$ for $p+q = i-1$, there is a choice of each $\beta_{i-j,i}$ such that $(dE_{i-j} \otimes 1) + \beta_{i-j,j}$ composed with $dE_i : E_i \to E_{i-1}$ is 0.*

(2) *With any such choice of differentials, E is an R-free resolution of k, which is minimal iff R is Golod.*

(3) *If R is Golod then the map $\beta_{i-j,j}$ may be chosen to have target $\mathcal{F}_i(E_i) = K_i$.*

Remark 3.2. In [3] the complex K is allowed to be a (possibly) infinite resolution of k over a possibly non-regular ring S mapping onto R, tensored with R; the Golod condition must of course be modified in that case. Also, the maps $\beta_{i-j,i}$ are given signs in order to make a direct comparison with the Massey operations on F.

More significantly, in [3] the map $\beta_{i-j,j}$ is chosen so that $dE_{i-j}\otimes 1+\beta_{i-j,j}$ covers the homology of a certain subcomplex. However, if $\beta_{i-j,j}$ and $\beta'_{i-j,j}$ both make the composition with dE_i zero and have (in the construction in [3]) the same target, then $\beta_{i-j,j} - \beta'_{i-j,j}$ is a boundary so $dE_{i-j} \otimes 1 + \beta'_{i-j,j}$ covers the same homology module, justifying our apparently looser description.

For example, assuming that $\beta_{i-j,j}$ is chosen as in [3] with target K_i, and $\beta'_{i-j,j}$ is chosen to have the same composition with the Koszul differential, then because the map $d : K_{i+1} \oplus F_i \to K_i$ surjects on cycles, and is a component of the differential, $dE_i + \beta'_{i-j,j} = dE_i + \beta_{i-j,j} + dh$ also covers the requisite cycles.

We write $K(R)$ for the Koszul complex of the maximal ideal of R, and $z_i(R) \subset K_i(R)$ for the i-th module of Koszul cycles. Set

$$ss(R) := \{s \mid \text{the } s\text{-th } R\text{-syzygy of } k \text{ has a socle summand}\},$$

and

$$ks(R) := \{s \mid z_s(R) \text{ has a socle summand.}\},$$

For example $z_n(R) = socle(R)$ so $n \in ks(R)$ for every R.

Theorem 3.3. *If R is a Golod ring of embedding dimension n then $ss(R) = 1 + ks(R) + \mathbb{N}$. Thus the possible semigroups $ss(R)$ are:*

(1) $s + \mathbb{N}$ *with* $s \leq n$*;*

(2) $n + 1, n + 3 + \mathbb{N}$.

Remark 3.4. The following elementary remark is useful in the proof: Let A, B, C be free modules, and $\phi : A \to B \oplus C$ a map. A generator of A maps into $soc(B \oplus C)$ iff it maps into the socle of both B and C.

Lemma 3.5. *Suppose that R is Golod. With notation as above, If a summand $G'_i \subset G_i$ maps to the socle of G_{i-1}, then then the maps $\beta_{i,j} : G_i \otimes F_j \to K_{i+j}$ may be chosen to vanish on $G'_i \otimes F_j$.*

Moreover, if there is no such summand, then no summand of G_i maps to a socle element, independent of the map to the earlier summands.

Proof. According to Theorem 3.1 the map $\beta_{i,j}$ can be chosen to be any map that makes the composition

$$\begin{tikzcd}
& & G_i \otimes F_j \arrow[dll, "\beta_{i,j}"'] \arrow[d, "dE_i \otimes F_j"] \\
K_{i+j} \arrow[d, "dE_{i+j}"'] & \oplus & G_{i-1} \otimes F_j \arrow[dll, "\beta_{i-1,j}"] \arrow[d] \\
K_{i+j-1} & \oplus & G_{i-2} \otimes F_j
\end{tikzcd}$$

zero. Because the resolution is minimal and $dE_i(G_i')$ is in the socle, we have

$$(dE_{i-1} \otimes F_j)\big((dE_i \otimes F_j)(G_i' \otimes F_j)\big) = 0.$$

Thus $\beta_{i,j}$ may be chosen to be 0 on this summand. $\square$

Proof of Theorem 3.3. If the t-th cycle module zK_t has a socle summand e, then since $\operatorname{soc} K_t \subset \mathfrak{m}^2 K_t$, e must map to a generator of $H_i(K)$, and thus e is the image of a generator of $F_t = G_0 \otimes F_t$. Thus e is a socle summand of $zG_t = \operatorname{syz}_{t+1}(k)$. Note that if the embedding dimension of R is n, then $zK_n = \operatorname{soc} K_n$, and in particular there is always a socle summand in the $n+1$-st syzygy of k.

On the other hand, suppose that zG_j has no socle summand for $j < i$ and that zK_i has no socle summand. Since zK_i has no socle summand, the image of $G_0 \otimes F_i \to K_i$ has no socle summand. Further, the differential dE_{i+1}, restricted to any component $G_\ell \otimes F_{i-\ell}$ other than $G_0 \otimes F_i$ has $dE_\ell \otimes F_{i-\ell}$ as one component, and by hypothesis the image of dE_ℓ has no socle summand. Thus the image of dE_{i+1} has no socle summand. This shows that the first s such that the s-th syzygy of k has a socle summand is one more than the first t so that zK_t has a socle summand, proving part 1.

Now suppose that the s-th syzygy of k, the image of $dE_s : G_s \to G_{s-1}$ has a socle summand. This summand is the image of a nonzero direct summand $G_s' \subset G_s$. By Lemma 3.5 the maps $\beta s, j$ can be chosen to be zero on $G_s' \otimes F_j$, and thus the differential dE_{s+j+1} restricted to $G_s' \otimes F_j$ is the map $dE_s \otimes F_j$. As long as F_j is nonzero, that is, $1 \le j \le \operatorname{pd}_S(R)$, the image of $G_s' \otimes F_j$ is a socle summand.

This shows that the j-th syzygy of k has a socle summand also for $j = s+2, \dots, s+\operatorname{pd}_S(R)+1$. In particular, since we have assumed that $\operatorname{pd}_S(R) \ge 2$, the $s+3$-rd syzygy of k has a socle summand. Repeating the argument above first with the $s+2$-nd syzygy, then with the $s+4$-th syzygy, etc, we see that the j-th syzygy has a socle summand for all $j \geq s+2$, completing the proof of part 2.

Finally, suppose that $\min ss(R) = s$ but $s \notin ks(R)$, so that the image of $G_0 \otimes F_s \subset E_{s+1}$ does not contain a socle summand. The map dE_{s+1} restricted to a component $G_i \otimes F_{s-i}$ of E_{s+1} for $0 < i < s-1$ is $dE_i \otimes F_{s+1-i} + \beta_{i,s+1-i}$ and since s was the minimum of $ss(R)$, the image of dE_i does not contain a socle summand. It follows that the image of dE_{s+1} does not contain a socle summand, proving part 3. $\square$

Putting together Theorems 3.3 and 1.2, we obtain:

Corollary 3.6. *If $(R, \mathfrak{m}_r, k)$ is a local Golod ring of embedding dimension n then the possible semigroups of homological degrees in which the resolution of k has a socle summand are $i + \mathbb{N}$ for $i \leq n$ and $n+1, n+3+\mathbb{N}$.*

4. LESCOT'S RESULTS [4]

In [4, Section 3], Lescot studied local rings $(R = S/J, \mathfrak{m}_R, k)$ where S is a regular local ring, $\mathfrak{m}_R^3 = 0$ and (to avoid trivialities) having $\operatorname{soc} R = \mathfrak{m}_R^2$ and $Hilb_R \neq 1, 2, 1$.

Lescot's analysis in example 3.8 shows that if some syzygy of k has a socle summand then some generators of J like in mm_S^3. Thus$\mathfrak{m}_S J \subsetneq \mathfrak{m}_S^3$, so $mm_S J : (J : mm_S) = mm_S J : (mm_S)^2 \neq \mathfrak{m}_S$, and it follows that R is Burch. Thus in fact all syzygies of k, starting with the second, have socle summands.

It follows that if all the generators of J are outside $\mathfrak{m}^3$, then (always assuming that $\operatorname{soc} R \;= \mathfrak{m}_R^2$) the resolution of k has no socle summands. Such an example in $k[x_1, \ldots, x_n]$ has the form $(x_1^2, \ldots, x_n^2$ together with a set of monomials that contains the products of every pair of variables without containing all of $x_i(x_1 \ldots x_n)$ for any i, such as

$$(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_3x_4).$$

with $n = 4$.

With the same conditions on R, suppose that there exists an R-module M with $\mathfrak{m}_R^2 M = 0$ such that first and second syzygies of M have no socle summands. Lescot shows that the same is true for the module k. It follows that R is not Burch; so by the result above, the resolution of k has no socle summands at all.

Question: Is the converse is true (perhaps just in this $\mathfrak{m}^3 = 0$ case)? if the resolution of k has no socle summands, could another module have them?

REFERENCES

[1] Hailong Dao, Toshinori Kobayashi, and Ryo Takahashi. Burch ideals and Burch rings. *Algebra Number Theory*, 14(8):2121–2150, 2020.

[2] E. S. Golod. Homologies of some local rings. *Dokl. Akad. Nauk SSSR*, 144:479–482, 1962.

[3] Tor H. Gulliksen and Gerson Levin. *Homology of local rings*. Queen's Papers in Pure and Applied Mathematics, No. 20. Queen's University, Kingston, Ont., 1969.

[4] Jack Lescot, Asymptotic properties of betti numbers of modules over certain rings. J. Pure and Applied Algebra 38 (1985) 287-298

[5] Jack Shamash. The Poincaré series of a local ring. *J. Algebra*, 12:453–470, 1969.

REFERENCES

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY AND THE MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY, CA 94720, USA

Email address: de@msri.org

AMS Subject Classification: Primary: Secondary: 1

Keywords: K

The first author is grateful to the National Science Foundation for partial support.