SYMMETRIC BILINEAR FORMS — MACAULAY2

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The first part of this project is to implement the *Grothendieck–Witt ring* of symmetric bilinear forms over a field k.

1. Symmetric bilinear forms

Definition 1.1. Let k be a field, and V a finite-dimensional k-vector space. A *bilinear form* is a vector space homomorphism

$$\beta\colon V \times V \to k.$$

We say that β is *symmetric* if $\beta(v,w) = \beta(w,v)$ for all $v, w \in V$, and we say that β is *non-degenerate* if $\beta(v,-)\colon V \to k$ is identically zero if and only if v is the zero vector.

Note what bilinearity allows us to do — if we pick a vector space basis $e_1, \dots, e_n$ for V, then β is completely determined by the values $\beta(e_i, e_j)$ for $1 \leq i, j \leq n$. That is, we can form a matrix attached to β. This is called a *Gram matrix* associated to a bilinear form.

Gram matrices are not unique! The same bilinear form $\beta\colon V \times V \to k$ can be expressed in different bases for V, resulting in different matrices.

Exercise 1.2. Two Gram matrices are associated to the same symmetric bilinear form if and only if they are *congruent*.

Here's an easy procedural way to think about matrix congruence. We can perform an elementary row operation if we immediately do the same operation on columns. For example, add 3 times row 2 to row 1, then add 3 times column 2 to column 1. This preserves symmetry and reflects an underlying change in the basis.

Reality check 1.3. Show that the following represent the same form, called the *hyperbolic form*, and denoted $\mathbb{H}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Reality check 1.4. A bilinear form is non-degenerate if and only if the determinant of any associated Gram matrix is nonzero.

Definition 1.5. Given two bilinear forms $\beta_1 \colon V_1 \times V_1 \to k$ and $\beta_2 \colon V_2 \times V_2 \to k$, we can define their *(block) sum* and *product*:

$$\beta_1 \oplus \beta_2 \colon V_1 \oplus V_2 \times V_1 \oplus V_2 \to k$$
$$\beta_1 \otimes \beta_2 \colon V_1 \otimes V_2 \times V_1 \otimes V_2 \to k.$$

If β_1 and β_2 are both symmetric (resp. non-degenerate) then $\beta_1 \oplus \beta_2$ will both be symmetric (resp. non-degenerate).

Two symmetric bilinear forms are *isomorphic* if they differ by an automorphism on the vector space. That is, $\beta_1 \cong \beta_2$ if there is some vector space automorphism $\phi : V \xrightarrow{\sim} V$ so that $\beta_1(-,-) = \beta_2(\phi(-), \phi(-))$.

By taking isomorphism classes of non-degenerate symmetric bilinear forms, we obtain a semiring.

Definition 1.6. The *Grothendieck–Witt ring* of a field k is defined to be the group completion, with respect to sum, of the semiring of isomorphism classes of non-degenerate symmetric bilinear forms over k.

Exercise 1.7. Show that $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$ by taking rank. Hint: every symmetric matrix over $\mathbb{C}$ is congruent to an identity matrix.

Every symmetric matrix over k is congruent to a diagonal one, meaning that any symmetric bilinear form is isomorphic to a block sum of rank one forms.

Notation 1.8. For any scalar $a \in k^\times$, we denote by $\langle a \rangle$ the rank one form

$$\begin{aligned} \langle a \rangle : k \times k &\to k \\ (x, y) &\mapsto axy. \end{aligned}$$

Proposition 1.9. The Grothendieck–Witt ring is generated by the forms $\langle a \rangle$, for $a \in k^\times$, subject to the relations:

(1) $\langle a \rangle = \langle ab^2 \rangle$ for any $b \in k^\times$
(2) $\langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle$ for any $a + b \neq 0$
(3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle = \mathbb{H}$.

1.1. **Coding goals.**

Goal 1.10. Implement the Grothendieck–Witt ring $\mathrm{GW}(k)$ into Macaulay2.

(1) Given a symmetric invertible matrix over a field k, turn it into a class in $\mathrm{GW}(k)$.
(2) Given an effective[1] class in $\mathrm{GW}(k)$, produce a diagonal Gram matrix with represents it.
(3) Given two symmetric invertible matrices over k, check whether they represent the same element in $\mathrm{GW}(k)$.
(4) Implement basic invariants of symmetric bilinear forms (rank, signature, discriminant).
(5) Implement sum and tensor into $\mathrm{GW}(k)$.

Goal 1.11. (Optional, but useful for $\mathbb{A}^1$-degree stuff) Include algorithms that recognize and exploit sparsity of Gram matrices in order to more efficiently diagonalize them. (I'll include notes on this).

References

[BMP23] Thomas Brazelton, Stephen McKean, and Sabrina Pauli, *Bézoutians and the $\mathbb{A}^1$-degree*, 2023.

[Lam05] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005. MR 2104929

[1] Here I'm using the word *effective* to mean "not virtual." That is, the integral coefficients on a minimal representation of the class in generators are all positive. Equivalently, it is represented by a *literal* symmetric bilinear form, rather than a formal difference of them.