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Square-free Gröbner degenerations

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Abstract Let I be a homogeneous ideal of $S = K[x_1, \dots, x_n]$ and let J be an initial ideal of I with respect to a term order. We prove that if J is radical then the Hilbert functions of the local cohomology modules supported at the homogeneous maximal ideal of S/I and S/J coincide. In particular, $\operatorname{depth}(S/I) = \operatorname{depth}(S/J)$ and $\operatorname{reg}(S/I) = \operatorname{reg}(S/J)$.

1 Introduction

Let K be a field and let S be the polynomial ring $K[x_1, \dots, x_n]$ equipped with the standard graded structure. Let M be a finitely generated graded S-module. We denote by $\beta_{ij}(M)$ the (i, j)th Betti number of M, and by $h^{ij}(M)$ the dimension of the degree j component of the ith local cohomology module $H^i_{\mathfrak{m}}(M)$ supported at the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Furthermore $\operatorname{reg}(M)$ denotes the Castelnuovo–Mumford regularity of M and $\operatorname{depth}(M)$ its depth. Finally recall that a non-zero Betti number $\beta_{i,i+j}(M)$ is called extremal if $\beta_{h,h+k}(M) = 0$ for every $h \geq i$, $k \geq j$ with $(h, k) \neq (i, j)$.

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Let I be a homogeneous ideal of S. Given a term order $<$ on S we may associate to I the initial ideal $J = \text{in}(I)$. It is well-known that S/J can be realized as the special fiber of a flat family whose generic fiber is S/I and the process of replacing I with J is called a Gröbner degeneration. If the ideal I is in generic coordinates then the corresponding initial ideal is called the generic initial ideal of I with respect to the given term order and it is denoted by $\text{gin}(I)$, see [24, Chapter 15.9]. It is known that Betti numbers and local cohomology dimensions can only grow under Gröbner degeneration, i.e. $\beta_{ij}(S/I) \leq \beta_{ij}(S/J)$ and $h^{ij}(S/I) \leq h^{ij}(S/J)$ for all i, j. In particular the same conclusion holds for regularity and projective dimension. Simple examples show that the inequalities are in general strict. On the other hand, Bayer and Stillman proved in [5] that $\text{reg}(S/I) = \text{reg}(S/\text{gin}(I))$ and $\text{depth}(S/I) = \text{depth}(S/\text{gin}(I))$ with respect to the revlex order. Furthermore Bayer, Charalambus and Popescu [4] generalized Bayer and Stillman's result by proving that S/I and $S/\text{gin}(I)$ have the same extremal Betti numbers.

Algebras with straightening laws (ASL for short) were introduced by De Concini, Eisenbud and Procesi in [21,23] and, in a slightly different way, by Baclawski [3]. Actually they appear under the name of "ordinal Hodge algebras" in [21] while the terminology "ASL" is used by Eisenbud [23] and, later on, by Bruns and Vetter in [10]. This notion arose as an axiomatization of the underlying combinatorial structure observed in classical algebras appearing in invariant theory, commutative algebra and algebraic geometry. Any ASL A has a discrete counterpart A_D defined by square-free monomials of degree 2. It turns out that A_D can be realized as a Gröbner degeneration of A and indeed ASLs can be characterized via Gröbner degenerations [14, Lemma 5.5].

The question whether the main homological invariants of an ASL depend only on its discrete counterpart became popular among the experts and led to consider possible generalizations. Herzog was guided by these considerations to conjecture the following:

Conjecture 1.1 (Herzog) *Let I be a homogeneous ideal of a standard graded polynomial ring S and let J be an initial ideal of I with respect to a term order. Then the extremal Betti numbers of S/I and S/J coincide provided J is square-free.*

In other words, Herzog's intuition was that a square-free initial ideal behaves, with respect to the homological invariants, as the revlex generic initial ideal. However the stronger statement asserting that I and J have the same revlex gin if J is a square-free initial ideal of I turned out to be false, see Example 3.1. The above conjecture, in various forms, has been discussed in several occasions by Herzog and his collaborators. It appeared in print only recently in [18, Conjecture 1.7] and in the introduction of [27]. In this paper we solve Herzog's conjecture positively. Indeed we establish a stronger result:

Theorem 1.2 *Let I be a homogeneous ideal of a standard graded polynomial ring S and let J be an initial ideal of I with respect to a term order. Then*

$$h^{ij}(S/I) = h^{ij}(S/J)$$

for all i, j provided J is square-free.

Since the extremal Betti numbers of S/I can be characterized in terms of the values of $h^{ij}(S/I)$, Herzog's conjecture follows from Theorem 1.2.

It turns out that, in many respects, the relationship between I and a square-free initial ideal J (when it exists) is tighter than the relation between I and its revlex generic initial ideal. For example, Herzog and Sbarra proved in [26] that, over a field of characteristic 0, the assertion of Theorem 1.2 with J replaced by revlex $\mathrm{gin}(I)$, holds if and only if S/I is sequentially Cohen–Macaulay. Furthermore, a consequence of Theorem 1.2 is that, if J is a square-free initial ideal of I, then S/I satisfies Serre's condition (S_r) if and only if S/J does (see Corollary 2.11). The latter statement is false if J is replaced by the revlex $\mathrm{gin}(I)$.

The paper is structured as follows. Section 2 is devoted to prove Theorem 1.2, and to draw some immediate consequences. In Sect. 3 we discuss properties of ideals admitting a square-free initial ideal and some consequences of Theorem 1.2. We discuss as well three families of ideals with square-free initial ideals that appeared in the literature: ideals defining ASLs (3.1), Cartwright–Sturmfels ideals (3.2) and Knutson ideals (3.3). Finally we discuss some possible developments.

2 The main result

The goal of the section is to prove Theorem 1.2. We will use lex for the degree lexicographic term order, and revlex for the degree reverse lexicographic term order associated to the total order $x_1 > \cdots > x_n$.

The main ingredient of the proof is Proposition 2.4 that can be regarded as a characteristic free version of [31, Theorem 1.1]. One key ingredient in the proof of Proposition 2.4 is the notion of cohomologically full singularities that was introduced and studied in the recent preprint [20] by De Stefani, Dao and Ma. Let us recall the definition:

Definition 2.1 A Noetherian local ring $(A, \mathfrak{n})$ is cohomologically full if it satisfies the following condition. For every local ring $(B, \mathfrak{m})$ such that $\mathrm{char}(B) = \mathrm{char}(A)$ and $\mathrm{char}(B/\mathfrak{m}) = \mathrm{char}(A/\mathfrak{n})$ and every surjection of local rings $\phi : (B, \mathfrak{m}) \to (A, \mathfrak{n})$ such that the induced map $\overline{\phi} : B/\sqrt{(0)} \to A/\sqrt{(0)}$ is an isomorphism, one has that the induced map on local cohomology $H^i_{\mathfrak{m}}(B) \to H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{n}}(A)$ is surjective for all $i \in \mathbb{N}$.

We point out that the closely related notion of liftable local cohomology is introduced and discussed by Kollár and Kovács in the paper [32] which is a generalized and expanded version of [31]. The next proposition is inspired by [31, Proposition 5.1].

Proposition 2.2 *Let $(R, \mathfrak{t})$ be an Artinian local ring, and $(A, \mathfrak{n})$ be a Noetherian local flat R-algebra such that the special fiber $A/\mathfrak{t}A$ is cohomologically full. Let N be a finitely generated R-module, and $N = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_q \supseteq N_{q+1} = 0$ a filtration of submodules such that $N_j/N_{j+1} \cong R/\mathfrak{t}$ for all $j = 0\ldots, q$. Then, for all $i \in \mathbb{N}$ and $j = 0\ldots, q$, the following complex of A-modules is exact:*

$$0 \to H^i_{\mathfrak{n}}(N_{j+1} \otimes_R A) \to H^i_{\mathfrak{n}}(N_j \otimes_R A) \to H^i_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A) \to 0$$

Proof Notice that the surjection $A \xrightarrow{\phi} A/\mathfrak{t}A$ yields an isomorphism between $A/\sqrt{(0)}$ and $(A/\mathfrak{t}A)/\sqrt{(0)}$. Since the tensor product is right-exact, we have a surjection of A-modules

$$N_j \otimes_R A \xrightarrow{\beta} (N_j/N_{j+1}) \otimes_R A \cong A/\mathfrak{t}A.$$

Denoting by β' the composition of β with the isomorphism $(N_j/N_{j+1}) \otimes_R A \cong A/\mathfrak{t}A$, choose $x \in N_j \otimes_R A$ such that $\beta'(x) = 1$, and set $\alpha : A \to N_j \otimes_R A$ the multiplication by x. Then $\beta' \circ \alpha : A \to A/\mathfrak{t}A$ equals ϕ. Therefore, because $A/\mathfrak{t}A$ is cohomologically full, the induced map of A-modules

$$H^k_{\mathfrak{n}}(\beta' \circ \alpha) = H^k_{\mathfrak{n}}(\beta') \circ H^k_{\mathfrak{n}}(\alpha) : H^k_{\mathfrak{n}}(A) \to H^k_{\mathfrak{n}}(A/\mathfrak{t}A)$$

is surjective for all $k \in \mathbb{N}$, so that $H^k_{\mathfrak{n}}(\beta) : H^k_{\mathfrak{n}}(N_j \otimes_R A) \to H^k_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A)$ is surjective as well. Since A is a flat R-algebra, for each $j = 0, \ldots, q$ we have a short exact sequence of A-modules

$$0 \to N_{j+1} \otimes_R A \to N_j \otimes_R A \xrightarrow{\beta} (N_j/N_{j+1}) \otimes_R A \to 0.$$

Passing to the long exact sequence on local cohomology

$$\ldots \to H^{i-1}_{\mathfrak{n}}(N_j \otimes_R A) \xrightarrow{H^{i-1}_{\mathfrak{n}}(\beta)} H^{i-1}_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A) \to$$
$$H^i_{\mathfrak{n}}(N_{j+1} \otimes_R A) \to H^i_{\mathfrak{n}}(N_j \otimes_R A) \xrightarrow{H^i_{\mathfrak{n}}(\beta)} H^i_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A) \to \ldots,$$

since each $H^k_{\mathfrak{n}}(\beta)$ is surjective, we get the thesis. □

Proposition 2.3 *Let $J \subset S = K[x_1, \ldots, x_n]$ be a square-free monomial ideal. Then $(S/J)_{\mathfrak{m}}$ is cohomologically full.*

Proof If K has positive characteristic, then $(S/J)_{\mathfrak{m}}$ is F-pure by [29, Proposition 5.38]. Hence, in characteristic zero $(S/J)_{\mathfrak{m}}$ is of F-pure type (and therefore of F-injective type). Then $(S/J)_{\mathfrak{m}}$ is Du Bois by [40, Theorem 6.1]. So in each case, we conclude by [34, Lemma 3.3, Remark 3.4]. Alternatively, one can argue using [33, Thm.1(i)]. □

In the proposition below, $(R, \mathfrak{t})$ is a homomorphic image of a Gorenstein local ring, $P = R[x_1, \dots, x_n]$ is a standard graded polynomial ring over $R = P_0$ and A is a graded quotient of P. Denote by $\mathfrak{n}$ the unique homogeneous maximal ideal $\mathfrak{t}P + (x_1, \dots, x_n)$ of P.

Proposition 2.4 *With the notation above, assume furthermore that A is a flat R-algebra. If $(A/\mathfrak{t}A)_{\mathfrak{n}}$ is cohomologically full, then $\operatorname{Ext}^i_P(A, P)$ is a free R-module for all $i \in \mathbb{Z}$.*

Proof Let $X = \operatorname{Spec}(A_{\mathfrak{n}})$ and $Y = \operatorname{Spec}(R)$. Then $f : X \to Y$ is a flat morphism of local schemes, which is essentially of finite type and embeddable in a Gorenstein morphism. Furthermore, if $y \in Y$ is the closed point, the fiber X_y is the affine scheme $\operatorname{Spec}(A_{\mathfrak{n}}/\mathfrak{t}A_{\mathfrak{n}})$. In [31, Corollary 6.9] it is proved that, if in addition the schemes are essentially of finite type over $\mathbb{C}$ and X_y has Du Bois singularities, then $h^{-i}(\omega^{\bullet}_{X/Y})$ is flat over Y for any $i \in \mathbb{Z}$. Here $\omega^{\bullet}_{X/Y}$ denotes the relative dualizing complex of f, see [44, Section 45.24] for basic properties of relative dualizing complexes. We note that the proof of [31, Corollary 6.9] holds as well if one replaces the assumption that X_y has Du Bois singularities with the (weaker) assumption that X_y has cohomologically full singularities. In fact, the Du Bois assumption is used only in the proof of [31, Proposition 5.1], that, as we noticed in Proposition 2.2, holds true also under the assumption that the special fiber is cohomologically full.

So, also under our assumptions, we have that $h^{-i}(\omega^{\bullet}_{X/Y})$ is flat over Y for any $i \in \mathbb{Z}$. But $\omega^{\bullet}_{X/Y}$ is the sheafication of $R\operatorname{Hom}(A_{\mathfrak{n}}, P_{\mathfrak{n}})[n]$, so $\operatorname{Ext}^{n-i}_{P_{\mathfrak{n}}}(A_{\mathfrak{n}}, P_{\mathfrak{n}}) \cong \operatorname{Ext}^{n-i}_P(A, P)_{\mathfrak{n}}$ is a flat R-module. So $\operatorname{Ext}^{n-i}_P(A, P)$ is a flat R-module, and therefore $\operatorname{Ext}^{n-i}_P(A, P)_j$ is a flat R-module for all $j \in \mathbb{Z}$. Since $\operatorname{Ext}^{n-i}_P(A, P)$ is finitely generated as P-module, $\operatorname{Ext}^{n-i}_P(A, P)_j$ is actually a finitely generated flat, and so free, R-module for any $j \in \mathbb{Z}$. In conclusion, $\operatorname{Ext}^{n-i}_P(A, P)$ is a direct sum of free R-modules and hence it is a free R-module itself. □

We are ready to prove Theorem 1.2:

Proof of Theorem 1.2 By [43, Proposition 1.11] there exists a vector $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ such that $J = \operatorname{in}_w(I)$. Let t be a new indeterminate over K, $R = K[t]_{(t)}$ and $P = R[x_1, \dots, x_n]$. Equip P with the grading given by $\deg(x_i) = 1$ and $\deg(t) = 0$. Consider the w-homogenization $\hom_w(I) \subset P$

and set $A = P/\hom_w(I)$. It is well known that the inclusion $R \to A$ is flat, $A/(t) \cong S/J$ and $A \otimes_R K(t) \cong (S/I) \otimes_K K(t)$. In particular, by Proposition 2.3, the special fiber $(A/(t))_{\mathfrak{n}}$, where $\mathfrak{n}$ is the unique homogeneous maximal ideal of P, is cohomologically full. Hence, by Proposition 2.4, $\operatorname{Ext}^i_P(A, P)$ is a free R-module for any $i \in \mathbb{Z}$. So $\operatorname{Ext}^i_P(A, P)_j$ is a finitely generated free R-module for any $j \in \mathbb{Z}$. Say $\operatorname{Ext}^i_P(A, P)_j \cong R^{r_{i,j}}$. Since $t \in P$ is a nonzero divisor on A we have the short exact sequence

$$0 \to A \xrightarrow{\cdot t} A \to A/(t) \to 0.$$

Applying $\operatorname{Hom}_P(-, P)$ to it, for all $i \in \mathbb{Z}$ we get:

$$0 \to \operatorname{Coker}(\alpha_{i,t}) \to \operatorname{Ext}^{i+1}_P(A/(t), P) \to \operatorname{Ker}(\alpha_{i+1,t}) \to 0,$$

where $\alpha_{k,t}$ is the multiplication by t on $\operatorname{Ext}^k_P(A, P)$. Notice that by [8, Lemma 3.1.16] there are natural isomorphisms $\operatorname{Ext}^i_S(A/(t), S) \cong \operatorname{Ext}^{i+1}_P(A/(t), P)$. For all $i \in \mathbb{Z}$ we thus have the short exact sequence:

$$0 \to \operatorname{Coker}(\alpha_{i,t}) \to \operatorname{Ext}^i_S(A/(t), S) \to \operatorname{Ker}(\alpha_{i+1,t}) \to 0.$$

For any $j \in \mathbb{Z}$, the above short exact sequence induces a short exact sequence of K-vector spaces

$$0 \to \operatorname{Coker}(\alpha_{i,t})_j \to \operatorname{Ext}^i_S(A/(t), S)_j \to \operatorname{Ker}(\alpha_{i,t})_j \to 0.$$

Since $\operatorname{Ext}^k_P(A, P)_j \cong R^{r_{k,j}}$, we get

$$\operatorname{Ker}(\alpha_{i,t})_j = 0 \quad \text{and} \quad \operatorname{Coker}(\alpha_{i,t})_j \cong K^{r_{i,j}}.$$

Therefore $\operatorname{Ext}^i_S(A/(t), S)_j \cong K^{r_{i,j}}$. On the other hand

$$(\operatorname{Ext}^i_S(S/I, S)_j) \otimes_K K(t) \cong \operatorname{Ext}^i_{S \otimes_K K(t)}((S/I) \otimes_K K(t), S \otimes_K K(t))_j \cong$$
$$\operatorname{Ext}^i_{P \otimes_R K(t)}(A \otimes_R K(t), P \otimes_R K(t))_j \cong (\operatorname{Ext}^i_P(A, P)_j) \otimes_R K(t) \cong K(t)^{r_{i,j}}.$$

Therefore $\operatorname{Ext}^i_S(S/J, S)_j \cong K^{r_{i,j}} \cong \operatorname{Ext}^i_S(S/I, S)_j$ for all $i, j \in \mathbb{Z}$. This concludes the proof because, by Grothendieck graded duality, we have:

$$h^{ij}(S/I) = \dim_K \operatorname{Ext}^{n-i}_S(S/I, S)_{j-n} \quad \text{and} \quad h^{ij}(S/J) = \dim_K \operatorname{Ext}^{n-i}_S(S/J, S)_{j-n}$$

for all $i, j \in \mathbb{Z}$. $\square$

Remark 2.5 The proof of Theorem 1.2 works also for more general gradings. Assume $S = K[x_1, \dots, x_n]$ is equipped with a $\mathbb{Z}^m$-graded structure such that $\deg(x_i) \in \mathbb{N}^m \setminus \{0\}$. Let $I \subset S$ be a $\mathbb{Z}^m$-graded ideal with a is square-free initial ideal J, then

$$\dim_K H^i_{\mathfrak{m}}(S/I)_v = \dim_K H^i_{\mathfrak{m}}(S/J)_v \text{ for all } i \in \mathbb{N},\ v \in \mathbb{Z}^m.$$

Remark 2.6 Theorem 1.2 holds as well if one replaces the assumption that the initial ideal J is square-free with the assumption that S/J cohomologically full. A way of constructing new examples of cohomologically full rings form known ones is via flat extensions. For example, a sequence $a = a_1, \dots, a_n$ of positive integers induces a K-algebra map $\phi_a : S \to S$ defined by $\phi_a(x_i) = x_i^{a_i}$, which is indeed a flat extension. Since cohomological fullness is preserved under flat extensions (see [20, Lemma 3.4]) one has that $S/\phi_a(J)S$ is cohomologically full if S/J is cohomologically full. In particular, by Proposition 2.3 we have that $S/\phi_a(J)S$ is cohomologically full if J is a square-free monomial ideal.

Since the extremal Betti numbers of S/I can be described in terms of $h^{ij}(S/I)$ (cf. [13]), we have:

Corollary 2.7 *Let $I \subset S$ be a homogeneous ideal and J a square-free initial ideal of I. Then the extremal Betti numbers of S/I and of S/J coincide. In particular,* $\operatorname{depth}(S/I) = \operatorname{depth}(S/J)$ *and* $\operatorname{reg}(S/I) = \operatorname{reg}(S/J)$.

Remark 2.8 One can ask if $S/\sqrt{\operatorname{in}(I)}$ is Cohen–Macaulay whenever S/I is Cohen–Macaulay (independently from the fact that $\operatorname{in}(I)$ is square-free). In [45] it is proved that, if S/I is Cohen–Macaulay, $\operatorname{Proj}(S/\operatorname{in}(I))$ cannot be disconnected by removing a closed subset of codimension larger than 1, a necessary condition for the Cohen–Macaulayness of $S/\sqrt{\operatorname{in}(I)}$. However, we present two determinantal examples, one for lex and the other for revlex, such that S/I is Cohen–Macaulay but $S/\sqrt{\operatorname{in}(I)}$ not.

(1) Let $S = K[x_1, \dots, x_7]$ and I be the ideal of 2-minors of :

$$\begin{pmatrix} x_1 + x_2 & x_5 & x_4 \\ -x_5 + x_6 & x_3 + x_7 & x_5 \\ x_4 + x_7 & x_1 - x_3 & x_5 + x_7 \end{pmatrix}$$

It turns out that S/I is a 3-dimensional Cohen–Macaulay domain. With respect to lex, one has $\operatorname{depth}(S/\sqrt{\operatorname{in}(I)}) = 2$.

(2) Let $S = K[x_1, \dots, x_9]$ and I the ideal of 2-minors of :

$$\begin{pmatrix} x_3 + x_7 & x_6 & x_1 & x_5 \\ x_9 & x_4 + x_5 & x_7 & x_1 + x_2 \\ x_3 & x_3 & x_7 & x_7 - x_8 \end{pmatrix}$$

It turns out that S/I is a 3-dimensional Cohen–Macaulay ring. With respect to revlex one has $\operatorname{depth}(S/\sqrt{\operatorname{in}(I)}) = 2$.

As explained in the introduction square-free initial ideals and revlex generic initial ideals have important features in common. Next we show that from other perspectives a square-free initial ideal (when it exists!) behaves better than the revlex generic initial ideal. We recall first some definitions. Let $A = S/I$ where $I \subset S$ is a homogeneous ideal.

(1) A is Buchsbaum if for any homogeneous system of parameters $f_1, \dots, f_d$ of A,

$$(f_1, \dots, f_{i-1}) : f_i = (f_1, \dots, f_{i-1}) : \mathfrak{m} \ \text{ for all } \ i = 1, \dots, d.$$

(2) A is generalized Cohen–Macaulay if $H^i_{\mathfrak{m}}(A)$ has finite length for all $i < \dim(A)$.
(3) For $c \in \mathbb{N}$, A is Cohen–Macaulay in codimension c if $A_{\mathfrak{p}}$ is Cohen–Macaulay for any prime ideal $\mathfrak{p}$ of A such that $\operatorname{height}(\mathfrak{p}) \leq \dim(A) - c$.
(4) For $r \in \mathbb{N}$, A satisfies Serre's (S_r) condition if $\operatorname{depth}(A_{\mathfrak{p}}) \geq \min\{r, \operatorname{height}(\mathfrak{p})\}$ for any prime ideal $\mathfrak{p}$ of A.

Remark 2.9 If $I \subset S$ is an ideal and $\mathfrak{p} \subset S$ is a prime ideal of height h containing I, then for all $k \in \mathbb{N}$:

$$\operatorname{depth}(S_{\mathfrak{p}}/IS_{\mathfrak{p}}) \geq k \iff (\operatorname{Ext}^{h-i}_S(S/I, S))_{\mathfrak{p}} = 0 \ \text{ for all } \ i < k.$$

Recall that, by definition, S/I is pure if $\dim(S/\mathfrak{p}) = \dim(S/I)$ for all associated prime ideals $\mathfrak{p}$ of I. Hence from the equivalences above we can deduce that:

(1) S/I is pure $\iff \dim(\operatorname{Ext}^{n-i}_S(S/I, S)) < i$ for all $i < \dim(S/I)$.
(2) S/I is generalized Cohen–Macaulay $\iff \dim(\operatorname{Ext}^{n-i}_S(S/I, S)) \leq 0$ for all $i < \dim(S/I)$.
(3) S/I is Cohen–Macaulay in codimension $c \iff \dim(\operatorname{Ext}^{n-i}_S(S/I, S)) < c$ for all $i < \dim(S/I)$.
(4) S/I satisfies (S_r) for $r \geq 2 \iff \dim(\operatorname{Ext}^{n-i}_S(S/I, S)) \leq i - r$ for all $i < \dim(S/I)$.

For the latter equivalence, the assumption $r \geq 2$ is needed to guarantee the purity of S/I, see [38, Lemma (2.1)] for details.

Remark 2.10 Given a homogeneous ideal $I \subset S$, if $A = S/I$ is Buchsbaum then it is generalized Cohen–Macaulay, but the converse does not hold true in general. If $H^i_{\mathfrak{m}}(A)$ is concentrated in only one degree for all $i < \dim(A)$, however, A must be Buchsbaum by [39, Theorem 3.1]. If I is a square-free monomial ideal and A is generalized Cohen–Macaulay, then it turns out that $H^i_{\mathfrak{m}}(A) = (H^i_{\mathfrak{m}}(A))_0$ for all $i < \dim(A)$. In particular, for a square-free monomial ideal $I \subset S$ we have that S/I is Buchsbaum if and only if S/I is generalized Cohen–Macaulay.

Summing up, the remarks above and Theorem 1.2 give:

Corollary 2.11 *Let $I \subset S$ be a homogeneous ideal with a square-free initial ideal J. Then:*

(i) *The following conditions are equivalent:*
   (a) *S/I is generalized Cohen–Macaulay;*
   (b) *S/I is Buchsbaum;*
   (c) *S/J is generalized Cohen–Macaulay;*
   (d) *S/J is Buchsbaum.*

(ii) *For any $r \in \mathbb{N}$, S/I satisfies the (S_r) condition if and only if S/J satisfies the (S_r) condition;*

(iii) *For any $c \in \mathbb{N}$, S/I is Cohen–Macaulay in codimension c if and only if S/J is Cohen–Macaulay in codimension c.*

3 Examples and open problems

In this section, we present examples of ideals with square-free initial ideals and interesting further properties. We also discuss briefely three special classes of ideals with square-free initial ideals that have appeared in the literature, pointing at some consequences of Theorem 1.2 for each of them.

By [4] two ideals with the same revlex generic initial ideal have the same extremal Betti numbers. Hence a stronger version of Herzog's conjecture 1.1 would be the statement: if I is an ideal with a square-free initial ideal J then $\operatorname{gin}(I) = \operatorname{gin}(J)$ with respect to revlex. The following example shows that the this is not the case.

Example 3.1 Let $S = K[x_{ij} : 1 \le i, j, \le 4]$ and $<$ be the revlex order associated to the total order $x_{11} > x_{12} > x_{13} > x_{14} > x_{21} > \cdots > x_{44}$. Let I be the ideal of 2-minors of (x_{ij}). Then $J = \operatorname{in}(I) = (x_{ij}x_{hk} : i < h \text{ and } j > k)$ is square-free and quadratic and $\operatorname{gin}(I)$ and $\operatorname{gin}(J)$ differ already in degree 2. Indeed, the degree 2 part of $\operatorname{gin}(I)$ is $(x_{ij} : i = 1, 2 \text{ and } j = 1, 2, 3, 4)^2$ and the degree 2 part of $\operatorname{gin}(J)$ is obtained from that of $\operatorname{gin}(I)$ by replacing $x_{23}x_{24}, x_{24}^2$ with $x_{11}x_{31}, x_{12}x_{31}$.

If K has positive characteristic and J is a square-free monomial ideal then S/J is F-pure (cf. [29, Proposition 5.38]). However, S/I need not be F-pure if I has a square-free initial ideal:

Example 3.2 Let $S = K[x_1, \dots, x_5]$ where K has characteristic $p > 0$, and I the ideal generated by the 2-minors of the matrix:

$$\begin{pmatrix} x_4^2 + x_5^a & x_3 & x_2 \\ x_1 & x_4^2 & x_3^b - x_2 \end{pmatrix}.$$

Note that, if $\deg(x_4) = a, \deg(x_1) = \deg(x_3) = 1, \deg(x_2) = b$ and $\deg(x_5) = 2$, the ideal I is homogeneous. Singh proved in [41, Theorem 1.1] that, if $a - a/b > 2$ and $\mathrm{GCD}(p, a) = 1$, then S/I is not F-pure. However with respect to lex one has

$$\mathrm{in}(I) = (x_1x_3,\ x_1x_2,\ x_2x_3).$$

On the other hand we have:

Proposition 3.3 *Let $I \subset S$ be a homogeneous ideal with a square-free initial ideal. Then S/I is cohomologically full.*

Proof As in the proof of Theorem 1.2, setting $J = \mathrm{in}(I)$, take a weight $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ such that $J = \mathrm{in}_w(I)$. Let t be a new indeterminate over K, $R = K[t]_{(t)}$ and $P = R[x_1, \dots, x_n]$. By considering the w-homogenization $\mathrm{hom}_w(I) \subset P$, set $A = P/\mathrm{hom}_w(I)$. Since $A/(t) \cong S/J$ is cohomologically full and t is a non-zero divisor on A, then A is cohomologically full by [20, Theorem 3.1]. So, $A_t \cong (S/I) \otimes_K K(t)$ is cohomologically full by [20, Lemma 3.4], and therefore (again by [20, Lemma 3.4]), S/I is cohomologically full. $\square$

Corollary 3.4 *Let $I \subset S$ be a homogeneous ideal with a square-free initial ideal. If $J \subset S$ is a homogeneous ideal such that $\sqrt{J} = I$, then*

$$\mathrm{depth}(S/J) \leq \mathrm{depth}(S/I) \quad \textit{and} \quad \mathrm{reg}(S/J) \geq \mathrm{reg}(S/I).$$

Proof It follows from Proposition 3.3. $\square$

Given an ideal $I \subset S$, the cohomological dimension of I is defined as:

$$\mathrm{cd}(S, I) = \max\{i \in \mathbb{N} : H_I^i(S) \neq 0\}.$$

In general we have $\mathrm{cd}(S, I) = \mathrm{cd}(S, \sqrt{I})$. Furthermore, if I has height h and is generated by r polynomials we have $h \leq \mathrm{cd}(S, I) \leq r$. One may ask what is the relationship between the cohomological dimension of I and that of $\mathrm{in}(I)$. The next example shows that, in general, they are unrelated.

Example 3.5 (1) For any ideal $I \subset S$ of height h, the generic initial ideal $\mathrm{gin}(I)$ w.r.t. revlex has cohomological dimension h: in fact $\sqrt{\mathrm{gin}(I)} =$

$(x_1, \ldots, x_h)$. However, there are many ideals I of height h for which $\operatorname{cd}(S, I) > h$. Hence there are ideals for which

$$\operatorname{cd}(S, I) > \operatorname{cd}(S, \operatorname{in}(I)).$$

holds.

(2) In [45, Example 2.14] it has been considered the ideal

$$I = (x_1x_5 + x_2x_6 + x_4^2,\ x_1x_4 + x_3^2 - x_4x_5,\\ x_1^2 + x_1x_2 + x_2x_5) \subset S = K[x_1, \ldots, x_6].$$

It turns out that I is a height 3 complete intersection and S/I is a normal domain (notice that in [45, Example 2.14] there is a typo in the last equation). With respect to lex we have:

$$\sqrt{\operatorname{in}(I)} = (x_1, x_2, x_3) \cap (x_1, x_3, x_6) \cap (x_1, x_2, x_5) \cap (x_1, x_4, x_5).$$

However $S/\sqrt{\operatorname{in}(I)}$ has depth 2, so $\operatorname{cd}(S, \sqrt{\operatorname{in}(I)}) = 6 - 2 = 4$ by [33, Thm.1]. Therefore $\operatorname{cd}(S, \operatorname{in}(I)) = 4$, but $\operatorname{cd}(S, I) = 3$. Hence there are ideals for which

$$\operatorname{cd}(S, I) < \operatorname{cd}(S, \operatorname{in}(I)).$$

holds.

On the other hand we have:

Proposition 3.6 *If $I \subset S$ is a homogeneous ideal and* $\operatorname{in}(I)$ *is square-free, then*

$$\operatorname{cd}(S, I) \geq \operatorname{cd}(S, \operatorname{in}(I)).$$

Furthermore, if K has positive characteristic, then $\operatorname{cd}(S, I) = \operatorname{cd}(S, \operatorname{in}(I))$.

Proof Under this assumption S/I is cohomologically full by Proposition 3.3, so $\operatorname{cd}(S, I) \geq n - \operatorname{depth}(S/I)$ by [20, Proposition 2.5]. However, $\operatorname{depth}(S/I) = \operatorname{depth}(S/\operatorname{in}(I))$ by Corollary 2.7, and $\operatorname{cd}(S, \operatorname{in}(I)) = n - \operatorname{depth}(S/\operatorname{in}(I))$ by [33, Thm.1].

If K has positive characteristic, $\operatorname{cd}(S, I) \leq n - \operatorname{depth}(S/I)$ by the remark after Proposition 4.1 in [37]. □

Example 3.7 If K has characteristic 0, the inequality in Proposition 3.6 can be strict. For example if $S = K[X]$ where $X = (x_{ij})$ denotes an $r \times s$ generic matrix, the ideal $I \subset S$ generated by the size t minors of X has cohomological dimension $rs - t^2 + 1$ by a result of Bruns and Schwänzl in [9]. However, Sturmfels proved in [42] that the "diagonal" initial ideal $\operatorname{in}(I)$ square-free and $S/\operatorname{in}(I)$ is Cohen–Macaulay. In particular, by [33, Thm.1], $\operatorname{cd}(S, \operatorname{in}(I)) = \operatorname{height}(\operatorname{in}(I)) = (r - t + 1)(s - t + 1) < \operatorname{cd}(S, I)$.

3.1 ASL: Algebras with straightening laws

Let $A = \oplus_{i\in\mathbb{N}} A_i$ be a graded algebra and let $(H, \prec)$ be a finite poset. Let $H \to \cup_{i>0} A_i$ be an injective function. The elements of H will be identified with their images. Given a chain $h_1 \preceq h_2 \preceq \cdots \preceq h_s$ of elements of H the corresponding product $h_1 \cdots h_s \in A$ is called standard monomial. One says that A is an algebra with straightening laws on H (with respect to the given embedding H into $\cup_{i>0} A_i$) if three conditions are satisfied:

(1) The elements of H generate A as a A_0-algebra.
(2) The standard monomials are A_0-linearly independent.
(3) For every pair h_1, h_2 of incomparable elements of H there is a relation (called the straightening law)

$$h_1 h_2 = \sum_{j=1}^{u} \lambda_j h_{j1} \cdots h_{jv_j}$$

where $\lambda_j \in A_0 \setminus \{0\}$, the $h_{j1} \cdots h_{jv_j}$ are distinct standard monomials and, assuming that $h_{j1} \preceq \cdots \preceq h_{jv_j}$, one has $h_{j1} \prec h_1$ and $h_{j1} \prec h_2$ for all j.

It then follows from the three axioms that the standard monomials form a basis of A over A_0 and that the straightening laws are indeed the defining equations of A as a quotient of the polynomial ring $A_0[H] = A_0[h : h \in H]$. That is, the kernel I of the canonical surjective map $A_0[H] \to A$ of A_0-algebras induced by the function $H \to \cup_{i>0} A_i$ is generated by the straightening laws regarded as elements of $A_0[H]$, i.e.,

$$A = A_0[H]/I \quad \text{with} \quad I = (h_1 h_2 - \sum_{j=1}^{u} \lambda_j h_{j1} \cdots h_{jv_j} : h_1 \not\preceq h_2 \not\preceq h_1).$$

Remark 3.8 The ideal I is homogeneous if one equips the polynomial ring $A_0[H]$ with the $\mathbb{N}$-graded structure induced by assigning to h the degree of its image in $\cup_{i>0} A_i$.

The ideal $J = (h_1 h_2 : h_1 \not\preceq h_2 \not\preceq h_1)$ of $A_0[H]$ defines a quotient $A_D = A_0[H]/J$ which is an ASL as well, called the discrete ASL associated to H. In [21] it is proved that A_D is the special fiber of a flat family with general fiber A. Indeed, at least when A_0 is a field, one can obtain the same result by observing that with respect to (weighted) revlex associated to a total order on H that refines the given partial order $\prec$ one has $J = \mathrm{in}(I)$. More gnerally it has been observed in [14, Lemma 5.5] that ASLs can also be defined via Gröbner degenerations.

As a consequence of Theorem 1.2 and Remark 2.5 we have:

Corollary 3.9 *Let A be an ASL over a field K and let A_D be the corresponding discrete counterpart. Then* $\mathrm{depth}(A) = \mathrm{depth}(A_D)$. *In particular, A is Cohen–Macaulay if and only if A_D is Cohen–Macaulay.*

Remark 3.10 In [36], Miyazaki proved that if A is a Cohen-Macualay ASL and A_D is Buchsbaum then A_D is Cohen-Macualay.

Remark 3.11 Let $S = K[x_1, \dots, x_n]$ and let H be the set of square-free monomials different from 1 ordered by division: for $m_1, m_2 \in H$ one sets $m_1 \preceq m_2$ if and only if $m_2 | m_1$. Then S can be regarded as an ASL over H with straightening law:

$$m_1 m_2 = \mathrm{GCD}(m_1, m_2)\mathrm{LCM}(m_1, m_2).$$

This induces an ASL structure on every Stanley–Reisner ring $K[\Delta]$ associated with a simplicial complex Δ on n vertices. Here the underlying poset is given by the non-empty faces of Δ ordered by reverse inclusion. Hence the discrete ASL associated to $K[\Delta]$ is $K[\mathrm{sd}(\Delta)]$ where $\mathrm{sd}(\Delta)$ is the barycentric subdivision of Δ. By Theorem 1.2 and by Grothendieck duality we have

$$\dim(\mathrm{Ext}_S^{n-i}(K[\Delta], S)) = \dim(\mathrm{Ext}_{S'}^{N-i}(K[\mathrm{sd}(\Delta)], S')) \ \text{ for all } \ i \in \mathbb{Z}$$

where S' is a polynomial ring of dimension $N = 2^n - 1$ and dim stands for Krull dimension. This is a known fact. Indeed Yanagawa [47] proved that the Krull dimensions of the deficiency modules of a Stanley–Reisner ring are topological invariants.

A further application of our main result is a proof of the Eisenbud–Green–Harris conjecture for Cohen–Macaulay standard graded ASLs. This conjecture has been originally presented in [25] in various ways. Here we refer to the formulation discussed in the fourth section of [25].

Theorem 3.12 *Let A be a standard graded Cohen–Macaulay ASL on H. Then any Artinian reduction of A verifies Eisenbud–Green–Harris conjecture. In other words, the h-vector of A is equal to the f-vector of some simplicial complex on* $|H| - \dim(A)$ *vertices.*

Proof Let $|H| = n$, $S = K[x_1, \dots, x_n]$ and write $A = S/I$ where I is the ideal generated by the straightening laws. With respect to a suitable term order the ideal $\mathrm{in}(I)$ is generated by square-free quadratic monomials. Furthermore, $\mathrm{depth}(S/\mathrm{in}(I)) = \mathrm{depth}(S/I)$ by Corollary 2.7. So the conclusion follows using [11], because S/I and $S/\mathrm{in}(I)$ have the same Hilbert function: in fact [11, Theorem 2.1] implies that the h-vector of $S/\mathrm{in}(I)$ equals the h-vector of S/J where J is a homogeneous ideal containing $(x_1^2, \dots, x_{n-d}^2)$ with S/J Cohen–Macaulay, where $d = \dim(A)$. Therefore, considering an Artinian reduction S'/J' of S/J, where $S' = K[x_1, \dots, x_{n-d}]$, $J' \subset S'$ is still a homogeneous ideal containing $(x_1^2, \dots, x_{n-d}^2)$ and the Hilbert function of S'/J' equals the h-vector of A. If $\mathrm{in}(J')$ is any initial ideal of $J' \subset S'$, then the faces of the desired simplicial complex are those $\sigma \subset \{1, \dots, n-d\}$ such that $\prod_{i \in \sigma} x_i$ is not in $\mathrm{in}(J')$. $\square$

3.2 Cartwright–Sturmfels ideals

Cartwright–Sturmfels ideals were introduced and studied by Conca, De Negri, Gorla in a series of recent papers [15–18] inspired by the work [12]. We recall briefly their definition and main properties. Given positive integers $d_1, \dots, d_m$ one considers the polynomial ring $S = K[x_{ij} : 1 \le i \le m \text{ and } 1 \le j \le d_i]$ with $\mathbb{Z}^m$-graded structure induced by assignment $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. The group $G = \mathrm{GL}_{d_1}(K) \times \cdots \times \mathrm{GL}_{d_m}(K)$ acts on S as the group of multigraded K-algebra automorphisms. The Borel subgroup $B = B_{d_1}(K) \times \cdots \times B_{d_m}(K)$ of the upper triangular invertible matrices acts on S by restriction. An ideal J is Borel-fixed if $g(J) = J$ for all $g \in B$. A multigraded ideal $I \subset S$ is Cartwright–Sturmfels if its multigraded Hilbert function coincides with that of a Borel-fixed radical ideal. It turns out that every initial ideal of a Cartwright–Sturmfels ideal is square-free and that the set of Cartwright–Sturmfels ideals is closed under multigraded linear sections and multigraded projections.

Examples of Cartwright–Sturmfels ideals are:

(1) The ideal of 2-minors and the ideal of maximal minors of matrices of distinct variables with graded structure given by rows or columns, [15–17].
(2) Binomial edge ideals, [18].
(3) Ideals defining multigraded closure of linear spaces and multiview ideals, see [18] for the origin and definition of these notions.

So, multigraded linear sections and multigraded projections of the above ideals are Cartwright–Sturmfels as well. Note that Conjecture [18, 1.14] turns out to be a special case of (the multigraded version of) Theorem 1.2, see Remark 2.5.

Remark 3.13 A result of Brion [7] implies that S/I is a Cohen–Macaulay normal domain for every prime Cartwright–Sturmfels ideal I. Such property does not hold for for every prime ideal admitting a square-free initial ideal, see 3.15.

3.3 Knutson ideals

In this subsection we set $K = \mathbb{Q}$ or $K = \mathbb{Z}/p\mathbb{Z}$. Let $f \in S = K[x_1, \dots, x_n]$ be such that $\operatorname{in}(f)$ is a square-free monomial for some term order $<$.

Let $\mathcal{C}_f$ be the smallest set of ideals of S satisfying the following conditions:

(1) $(f) \in \mathcal{C}_f$;
(2) If $I \in \mathcal{C}_f$, then $I : J \in \mathcal{C}_f$ for any ideal $J \subset S$; in particular the associated primes of I are in $\mathcal{C}_f$.
(3) If $I, J \in \mathcal{C}_f$, then $I + J \in \mathcal{C}_f$ and $I \cap J \in \mathcal{C}_f$.

Example 3.14 If $f = x_1 \cdots x_n$, then $\mathcal{C}_f$ is the set of all the square-free monomial ideals of S.

The class $\mathcal{C}_f$ is introduced and studied by Knutson in [30]. The elements of $\mathcal{C}_f$ will be called Knutson ideals associated with f. Knutson proved in [30] that the initial ideal of any ideal in $\mathcal{C}_f$ is square-free and hence every ideal in $\mathcal{C}_f$ is radical. Moreover, one can prove that $\operatorname{in}(I) \neq \operatorname{in}(J)$ whenever $I, J \in \mathcal{C}_f$ are different. So $\mathcal{C}_f$ is a finite set. In positive characteristic, if $I \in \mathcal{C}_f$ then $\phi_f(I) \subset I$ where $\phi_f : S \to S$ is a special splitting (associated to the polynomial f) of the Frobenius morphism $F : S \to S$. In particular, in positive characteristic, S/I is F-pure for each $I \in \mathcal{C}_f$. Such a property does not hold for every ideal admitting a square-free initial ideal, see 3.2. In [30, Section 7], Knutson has shown that many important families of ideals are of the form $\mathcal{C}_f$ for a suitable choice of f. This apply, for example, to ideals defining matrix Schubert varieties and ideals defining Kazhdan-Lusztig varieties.

We describe below an example, originated from a discussion with Jenna Rajchgot at MSRI in 2012, of a Knutson prime ideal that is not Cohen–Macaulay that answers negatively a question we had in the preliminary version of the paper.

Example 3.15 Let $S = K[x_1, \dots, x_5]$, and $f = gh$ where $g = x_1x_4x_5 - x_2x_4^2 - x_3x_5^2$ and $h = x_2x_3 - x_4x_5$. With respect to lex one has $\operatorname{in}(f) = x_1x_2x_3x_4x_5$. We have that (g, h) is a height 2 complete intersection in $\mathcal{C}_f$. One can check that

$$\mathfrak{p} = (g, h, x_1x_3x_4 - x_3^2x_5 - x_4^3, x_1x_2x_5 - x_2^2x_4 - x_5^3)$$

is a height 2 prime ideal. Since it contains (g, h), must be associated to it, so $\mathfrak{p} \in \mathcal{C}_f$ and one can check that $S/\mathfrak{p}$ is not Cohen–Macaulay.

Next, we show how to derive the solution of a conjecture stated in [6] in the case of Knutson ideals. Given an ideal $I \subset S$, the graph with vertex set $\mathrm{Min}(I)$ and edges $\{\mathfrak{p}, \mathfrak{p}'\}$ if $\mathrm{height}(\mathfrak{p} + \mathfrak{p}') = \mathrm{height}(I) + 1$ is called the dual graph of I. The ideal I is called Hirsch if the diameter of the dual graph is bounded above from $\mathrm{height}(I)$. In [6,22] has been explained why Hirsch ideals are a natural class to consider, and there several examples of such ideals have been provided. In particular, in [6, Conjecture 1.6] has been conjectured that, if I is quadratic and S/I is Cohen–Macaulay, then I is Hirsch.

Proposition 3.16 *Let $I \subset S$ be a homogeneous Knutson ideal such that S/I satisfies (S_2), e.g. S/I is Cohen–Macaulay. If either* $\mathrm{in}(I)$ *is quadratic or* $\mathrm{height}(I) \leq 3$*, then I is Hirsch.*

Proof By the assumption $\mathrm{in}(I)$ is the Stanley–Reisner ideal I_Δ of a simplicial complex Δ on n vertices. Since S/I satisfies (S_2), S/I_Δ satisfies (S_2) as well by Corollary 2.11. In other words, Δ must be a normal simplicial complex (i.e. all links are connected in codimension one). If I_Δ is quadratic (that is Δ is flag), then I_Δ is Hirsch by [2]. If $\mathrm{height}(I_\Delta) \leq 3$, then it is simple to check that I_Δ is Hirsch (see [28, Corollary A.4] for the less trivial case in which $\mathrm{height}(I_\Delta) = 3$). In each case, we conclude because, under the assumptions of the theorem, by [22, Theorem 3.3] the diameter of the dual graph of I is bounded above from that of $\mathrm{in}(I)$. □

3.4 An open question

As shown in 3.15 not all prime ideals with a square-free initial ideal define a Cohen–Macaulay ring. On the other hand, it would be interesting to understand under what further conditions a prime ideal with a square-free initial ideal defines a Cohen–Macaulay ring. For example, we ask the following:

Question 3.17 *Let $I \subset S$ be a homogeneous prime ideal with a square-free initial ideal such that* Proj S/I *is nonsingular. Is S/I Cohen–Macaulay with a negative a-invariant?*

A similar question appeared in [46, Problem 3.6]. For ASLs, Buchweitz proved a related statement (unpublished, see [21]), that, by virtue of Theorem 1.2, can be expressed as follows:

Theorem 3.18 (Buchweitz). *Let A be an ASL domain over a field of characteristic* 0 *such that* Proj *A has rational singularities. If A is Cohen–Macaulay, then A has rational singularities.*

Special cases where Question 3.17 has a positive answer are discussed in the recent preprint [19].

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