A1 Brouwer: diagonilzation and Witt chains

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Diagonilizing a bilinear form

Option 1

Given form β with non-singular Gram matrix A, find a vector v such that $v^t A v \neq 0$ (one of $e_1, e_2, \ldots, e_n, e_1 + \cdots + e_n$ will work). Then let $d = v^t A v$ and find a basis for $\ker(v^t A)$. It turns out that

$$\beta = \langle d \rangle + \beta|_{\ker(v^t A)}$$

(Lam I.2.3). Then compute the $(n-1) \times (n-1)$ Gram matrix for $\beta|_{\ker(v^t A)}$ and proceed by recursion.

Option 2

Apply column operations $E_1, \ldots, E_k$ to put A into upper triangular form, $AE_1 \cdots E_k$. Then $E_k^t \cdots E_1^t A E_1 \cdots E_k$ is a diagonilization of A.

Witt Decomposition

Every non-degenerate bilinear form β splits up as a hyperbolic and anisotropic part, $\beta = \beta_h + \beta_a$. Where $\beta_h = \mathbb{H} + \mathbb{H} + \cdots + \mathbb{H}$ is just a sum of $\mathbb{H}$ and β_a contains no copies of $\mathbb{H}$. This could be a start for putting a diagonal Gram matrix into a more canonical form.

An algorithm to get this decomposition:

- Find x such that $x^t A x \neq 0$
- Pick $y \notin \ker(x^t A)$
- Let $V' = \ker(x^t A) \cap \ker(y^t A)$
- Then $\beta = \mathbb{H} + \beta|_{V'}$ (Lam I.3.4). Repeat recursively on $\beta|_{V'}$

Witt chains

This is a potential way to check if two diagonal Gram matrices $A = \mathrm{diag}(a_1, \ldots, a_n)$ and $B = \mathrm{diag}(b_1, \ldots, b_n)$ are congruent.

Proposition (Lam I.5.1): In the case that A, B are rank 2 ($n = 2$), $A \cong B$ if and only if $a_1 a_2 = b_1 b_2 r^2$ for some square r^2 ($r \in k^\times$) and there is an element $l \in k^\times$ represented by both forms (i.e. $l = a_1 x^2 + a_2 y^2 = b_1 x'^2 + b_2 y'^2$ for some $x, y, x', y' \in k$).

A **Witt move** on $A = \mathrm{diag}(a_1, \ldots, a_n)$ replaces a pair of values a_i, a_j on the diagonal with an equivilant pair (as given by the conditions above), a_i', a_j'. The Chain equivilance theorem (I.5.2) states that for any two congruent diagonal matrices A, B, there is a chain of Witt moves that gets you from A to B.

Example: The form $\langle 2, 3, 6 \rangle$ should be $\langle 1, 1, 1 \rangle$, so we can try out the algorithm on this as a test case.