

SYMMETRIC BILINEAR FORMS — MACAULAY2

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The first part of this project is to implement the *Grothendieck–Witt ring* of symmetric bilinear forms over a field k .

1. SYMMETRIC BILINEAR FORMS

Definition 1.1. Let k be a field, and V a finite-dimensional k -vector space. A *bilinear form* is a vector space homomorphism

$$\beta: V \times V \rightarrow k.$$

We say that β is *symmetric* if $\beta(v, w) = \beta(w, v)$ for all $v, w \in V$, and we say that β is *non-degenerate* if $\beta(v, -): V \rightarrow k$ is identically zero if and only if v is the zero vector.

Note what bilinearity allows us to do — if we pick a vector space basis e_1, \dots, e_n for V , then β is completely determined by the values $\beta(e_i, e_j)$ for $1 \leq i, j \leq n$. That is, we can form a matrix attached to β . This is called a *Gram matrix* associated to a bilinear form.

Gram matrices are not unique! The same bilinear form $\beta: V \times V \rightarrow k$ can be expressed in different bases for V , resulting in different matrices.

Exercise 1.2. Two Gram matrices are associated to the same symmetric bilinear form if and only if they are *congruent*.

Here’s an easy procedural way to think about matrix congruence. We can perform an elementary row operation if we immediately do the same operation on columns. For example, add 3 times row 2 to row 1, then add 3 times column 2 to column 1. This preserves symmetry and reflects an underlying change in the basis.

Reality check 1.3. Show that the following represent the same form, called the *hyperbolic form*, and denoted \mathbb{H} :

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Reality check 1.4. A bilinear form is non-degenerate if and only if the determinant of any associated Gram matrix is nonzero.

Definition 1.5. Given two bilinear forms $\beta_1: V_1 \times V_1 \rightarrow k$ and $\beta_2: V_2 \times V_2 \rightarrow k$, we can define their *(block) sum* and *product*:

$$\begin{aligned}\beta_1 \oplus \beta_2 &: V_1 \oplus V_2 \times V_1 \oplus V_2 \rightarrow k \\ \beta_1 \otimes \beta_2 &: V_1 \otimes V_2 \times V_1 \otimes V_2 \rightarrow k.\end{aligned}$$

If β_1 and β_2 are both symmetric (resp. non-degenerate) then $\beta_1 \oplus \beta_2$ will both be symmetric (resp. non-degenerate).

Two symmetric bilinear forms are *isomorphic* if they differ by an automorphism on the vector space. That is, $\beta_1 \cong \beta_2$ if there is some vector space automorphism $\phi: V \xrightarrow{\sim} V$ so that $\beta_1(-, -) = \beta_2(\phi(-), \phi(-))$.

By taking isomorphism classes of non-degenerate symmetric bilinear forms, we obtain a semiring.

Definition 1.6. The *Grothendieck–Witt ring* of a field k is defined to be the group completion, with respect to sum, of the semiring of isomorphism classes of non-degenerate symmetric bilinear forms over k .

Exercise 1.7. Show that $\text{GW}(\mathbb{C}) \cong \mathbb{Z}$ by taking rank. Hint: every symmetric matrix over \mathbb{C} is congruent to an identity matrix.

Every symmetric matrix over k is congruent to a diagonal one, meaning that any symmetric bilinear form is isomorphic to a block sum of rank one forms.

Notation 1.8. For any scalar $a \in k^\times$, we denote by $\langle a \rangle$ the rank one form

$$\begin{aligned}\langle a \rangle &: k \times k \rightarrow k \\ (x, y) &\mapsto axy.\end{aligned}$$

Proposition 1.9. The Grothendieck–Witt ring is generated by the forms $\langle a \rangle$, for $a \in k^\times$, subject to the relations:

- (1) $\langle a \rangle = \langle ab^2 \rangle$ for any $b \in k^\times$
- (2) $\langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle$ for any $a+b \neq 0$
- (3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle = \mathbb{H}$.

1.1. Coding goals.

Goal 1.10. Implement the Grothendieck–Witt ring $\text{GW}(k)$ into Macaulay2.

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- (1) Given a symmetric invertible matrix over a field k , turn it into a class in $\text{GW}(k)$.
 - (2) Given an effective¹ class in $\text{GW}(k)$, produce a diagonal Gram matrix with represents it.
 - (3) Given two symmetric invertible matrices over k , check whether they represent the same element in $\text{GW}(k)$.
 - (4) Implement basic invariants of symmetric bilinear forms (rank, signature, discriminant).
 - (5) Implement sum and tensor into $\text{GW}(k)$.

REFERENCES

- [Lam05] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005. MR 2104929

¹Here I'm using the word *effective* to mean “not virtual.” That is, the integral coefficients on a minimal representation of the class in generators are all positive. Equivalently, it is represented by a *literal* symmetric bilinear form, rather than a formal difference of them.