

PERIODICITY OF IDEALS OF MINORS IN FREE RESOLUTIONS

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ABSTRACT. We study the asymptotic behavior of the ideals of minors in minimal free resolutions over local rings. In particular, we prove that such ideals are eventually 2-periodic over complete intersections and Golod rings. We also establish general results on the stable behavior of ideals of minors in any infinite minimal free resolution. These ideals have intimate connections to trace ideals and cohomology annihilators. Constraints on the stable values attained by the ideals of minors in many situations are obtained, and they can be explicitly computed in certain cases.

1. INTRODUCTION

Ideals of minors of differentials in free resolutions play an important role in commutative algebra and algebraic geometry. Perhaps the most fundamental example of this is Buchsbaum-Eisenbud's acyclicity criterion [BE73], which characterizes those finite free complexes that are resolutions in terms of the ranks and ideals of minors of their differentials. There has been a tremendous amount of work on understanding the differentials in a finite free resolution; such papers are too numerous to list here. For instance, a question of great interest is: how many matrices appearing in the resolution are linear (i.e. how many have entries consisting only of 0's or degree 1 forms). This so-called N_p property has deep connections to the geometry of modules and ideals involved. Fascinating work and open conjectures abound even in this rather special case of partially linear resolutions; we refer the reader to [Far17] for a relatively short survey.

Over singular rings, most resolutions are *infinite*. There, much less is known about the differential matrices in general. Perhaps the most famous instance that we understand is Eisenbud's theory of matrix factorizations ([Eis80]), which essentially captures the phenomenon that over a local hypersurface, any minimal free resolution is eventually 2-periodic. This theory has had striking ramifications far beyond commutative algebra. Another relevant active topic is the study of Koszul rings, i.e. those whose resolutions all eventually consist of linear matrices [AE92]. Interested readers may start with a couple of comprehensive surveys on infinite resolutions [Avr10, MP15].

In this paper, we study and establish strong stabilizing patterns, including periodicity, for ideals of minors in infinite minimal resolutions. There has been little indication that such behavior is expected. Apart from the hypersurface case mentioned above, our work is motivated by a recent result in [DE23], where it is proved that over a large class of depth zero rings (technically, those with Burch index at least 2), the 7th syzygy of *any* nonfree module contains the residue field as a direct summand. Thus, the ideals of minors at high enough steps of any infinite resolution over such a ring are given by the maximal ideal and its powers. In fact, the following question (and potential answers in several cases) was raised after extensive discussions and Macaulay2 [M2] experimentations by the second author and David Eisenbud:

Question 1.1. *Given a positive integer r and a finitely generated module M over a local ring R , must there be a number $n(M)$ such that the sum of the ideals of $r \times r$ minors in $n(M)$ consecutive differential matrices of the minimal resolution of M eventually becomes constant? Is there an integer only depending on R that bounds all $n(M)$ from above?*

Note that if the ideal of minors (for a given rank) eventually become periodic, then the answer to the first part of the above question is yes. A small number of previous papers have studied the topic of ideals of minors in infinite resolutions, e.g. [EG94, EH05, KL98, Wan94], but as far as we know, none of them address periodicity or the weaker version in Question 1.1.

Our first main result is that ideals of minors in minimal free resolutions over local complete intersections are eventually 2-periodic. More precisely:

Theorem 1.2 (see Corollary 3.20 and Theorem 3.23). *Let S be a regular local ring, $I \subseteq S$ an ideal generated by a regular sequence, M a finitely generated module over the complete intersection $R = S/I$, (F, d) the minimal R -free resolution of M , and $\mathbf{I}_i^r(M)$ the ideal generated by the $r \times r$ minors of $d_i : F_i \rightarrow F_{i-1}$.*

- (1) *We have $\mathbf{I}_i^r(M) = \mathbf{I}_{i+2}^r(M)$ for all $r \geq 1$ and $i \gg 0$.*
- (2) *When F is a Shamash resolution (e.g. when M is the residue field of R), we have $\mathbf{I}_i^r(M) = \mathbf{I}_{i+2}^r(M)$ for all $r \geq 1$ and $i \geq r \cdot \dim(S) + 4(r-1) + 1$.*

We review the notion of a Shamash resolution below; see §3.1. Our proof of Part (1) invokes Eisenbud-Peeva's theory of higher matrix factorizations over complete intersections [EP16, EP21]. Our argument for Part (2) relies on an interpretation of Shamash resolutions in terms of matrix factorizations, based on insights of Burke-Stevenson [BS15], Burke-Walker [BW15], and Martin [Mar21]. Taking $r = 1$ in Part (2), one sees that the ideals of 1×1 minors in F become 2-periodic after $\dim(S) + 1$ steps, which is precisely Eisenbud's lower bound for the 2-periodicity of minimal free resolutions over hypersurfaces [Eis80, Theorem 6.1(i)]. Since every minimal free resolution over a hypersurface is a Shamash resolution, one may therefore view Theorem 3.23(2) as an extension of Eisenbud's periodicity theorem for minimal free resolutions over hypersurfaces.

The dependence of the lower bound in Theorem 3.23(2) on the rank r of the minors is necessary (Remark 3.19). Moreover, while the bound in Theorem 3.23(2) for Shamash resolutions is independent of the module M , no such global bound exists for general minimal free resolutions over complete intersections (Example 3.26).

We now turn to Golod rings. There is a sense in which the homological behavior of complete intersections and Golod rings could not be less alike: Betti numbers over complete intersections grow as slowly as possible, while Betti numbers over Golod rings grow as fast as possible. And yet, minimal free resolutions over Golod rings also have eventually 2-periodic ideals of minors and hence exhibit a finiteness property analogous to the complete intersection case:

Theorem 1.3 (see Theorem 4.9). *Let S be a regular local ring, $I \subseteq S$ an ideal such that $R = S/I$ is Golod, M a finitely generated R -module, (F, d) the minimal R -free resolution of M , and $\mathbf{I}_i^r(M)$ the ideal generated by the $r \times r$ minors of $d_i : F_i \rightarrow F_{i-1}$.*

- (1) *We have $\mathbf{I}_i^r(M) = \mathbf{I}_{i+2}^r(M)$ for all $r \geq 1$ and $i \gg 0$.*
- (2) *If $\mathrm{pd}_S(R) \geq 2$, then $\mathbf{I}_i^r(M) = \mathbf{I}_{i+1}^r(M)$ for all $r \geq 1$ and $i \gg 0$.*

Our proof of Theorem 1.3 uses Burke's structure theorem for minimal free resolutions over Golod rings in terms of A_∞ -operations and the bar construction [Bur15] (see also

[Iye97]). The proofs of Theorems 1.2 and 1.3 both exploit multiplicative structures on free resolutions: we leverage the Eisenbud operators on the Shamash construction in the complete intersection case and A_∞ -operations in the Golod case.

Golod rings are quite ubiquitous in commutative algebra. For instance, by a result of Herzog-Welker-Yassemi [HWY16, Theorem 4.1], if R is regular local and I is an ideal in R , then R/I^k is Golod for $k \gg 0$. Additionally, by work of Herzog-Huneke [HH13, Theorem 2.1(d)], if R is an \mathbb{N} -graded polynomial ring over a characteristic 0 field, and I is an ideal in R , then R/I^k and $R/I^{(k)}$ are Golod for $k \geq 2$; here, $I^{(k)}$ denotes the k^{th} symbolic power of I . Combining Theorems 1.2 and 1.3 therefore gives a large family of local rings whose minimal free resolutions have eventually 2-periodic ideals of minors.

For instance, given a local ring R with embedding dimension e and depth d , it is well-known that, when the codepth $e - d$ of R is at most 2, R is either Golod or a complete intersection. Theorems 1.2 and 1.3 therefore immediately imply:

Corollary 1.4. *If R is a local ring with codepth at most 2, then the ideals of minors in the minimal free resolution of any finitely generated R -module are eventually 2-periodic.*

It is not the case that every minimal free resolution has eventually 2-periodic ideals of minors. Indeed, there exists an Artinian Gorenstein local ring R of codepth 5 and a minimal R -free resolution whose ideals of 1×1 minors are not n -periodic for any n : see Proposition 5.1, which is essentially due to Gasharov-Peeva [GP90, Proposition 3.1]. We pose the following questions on this theme:

Question 1.5. *Does there exist a local ring R of codepth 3 or 4 and a finitely generated R -module M whose minimal free resolution does not have eventually n -periodic ideals of minors, for any $n \geq 1$?*

Question 1.6. *Does there exist a local ring R and $n > 2$ such that the ideals of minors of minimal free resolutions of finitely generated R -modules are always eventually n -periodic, but not 2-periodic?*

As an application of Theorems 1.2 and 1.3, we obtain the following periodicity result for top exterior powers of syzygies over complete intersections and Golod rings:

Corollary 1.7 (see Corollary 6.1). *Let S be a regular local ring, I an ideal of S , M a finitely generated $R = S/I$ -module, and $\text{Syz}_i(M)$ (resp. β_i) the i^{th} syzygy (resp. Betti number) of M over R .*

- (1) *If R is a complete intersection, then $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong \bigwedge^{\beta_{i+2}} \text{Syz}_{i+2}(M)$ for $i \gg 0$. If, in addition, the minimal free resolution of M is a Shamash resolution, then this isomorphism holds for $i \geq \dim(S) + 1$.*
- (2) *If R is Golod, then $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong \bigwedge^{\beta_{i+2}} \text{Syz}_{i+2}(M)$ for $i \gg 0$. If, in addition, we have $\text{pd}_S(R) \geq 2$, then $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong \bigwedge^{\beta_{i+1}} \text{Syz}_{i+1}(M)$ for $i \gg 0$.*
- (3) *Suppose $\text{depth}(R) = 0$ and that the Burch index of R (see [DE23, Definition 1.2]) is at least two. For all $i \geq 5$, we have $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong k$.*

When R is a hypersurface, Corollary 1.7(1) follows from [Eis80, Theorem 6.1(i)]. We refer the reader to [DE23] for various concrete examples of rings of depth zero and Burch index at least two.

The rest of the main results of this paper concern *stable ideals of minors* (see Definition 2.1). Given a module M with minimal free resolution F , the stable ideal of $r \times r$

minors of M is the limit as $n \rightarrow \infty$ of the sum of all the ideals of $r \times r$ minors in the smart truncation $F_{\geq n}$. Our key results in this direction are summarized as follows:

Theorem 1.8 (see Theorems 2.8, 2.13, 2.15, and 2.25). *Let (R, \mathfrak{m}, k) be a local ring and M a finitely generated R -module. Denote by $\mathbf{I}^{\text{stab}}(M, r)$ the stable ideal of $r \times r$ -minors in the minimal free resolution of M (see Definition 2.1). We have the following:*

- (1) $\mathbf{I}^{\text{stab}}(M, 1) \neq 0$.
- (2) *Let $\text{Jac}(R)$ be the Jacobian ideal of R (see [IT21, §3] for the definition of $\text{Jac}(R)$). There exists $t \geq 1$, which is independent of M , such that $\text{Jac}(R)^t \subseteq \mathbf{I}^{\text{stab}}(M, 1)$.*
- (3) *Let K be the Koszul complex on a minimal generating set of \mathfrak{m} . If $\dim_k H_1(K) \geq 2$, then $\mathbf{I}^{\text{stab}}(k, r) = \mathfrak{m}^r$ for all $r \geq 1$.*
- (4) *Assume R is a 1-dimensional analytically unramified local ring, and let \mathfrak{c}_R denote the conductor ideal of R (see Definition 2.18).*
 - (a) *We have $\mathfrak{c}_R \subseteq \mathbf{I}^{\text{stab}}(M, 1)$.*
 - (b) *Assume R is Gorenstein with infinite residue field. There exists a finitely generated R -module N such that $\mathfrak{c}_R = \mathbf{I}^{\text{stab}}(N, 1)$ if and only if R is regular or a hypersurface of multiplicity 2.*

Concerning Theorem 1.8(2), we note that Iyengar-Takahashi's definition of the Jacobian ideal of a local ring generalizes the usual notion for affine algebras over a field. Theorem 1.8(2) is an instance of the philosophy that the singularities of the ring R govern the asymptotic behavior of minimal R -free resolutions [Avr10, pp. 4]. Parts (2) and (3) of Theorem 1.8 build on closely related results of Wang [Wan94, Theorems 5.1 and 5.2] and Eisenbud-Green [EG94, Theorem 1.1], respectively. See §2.1 for a more detailed recollection of the history of results on ideals of minors in minimal free resolutions that influenced this work.

We now give a brief summary of the paper. In §2, we prove Theorem 1.8, among many other related results on stable ideals of minors. In §3 and §4, we prove Theorems 1.2 and 1.3, respectively. We discuss in §5 the aforementioned counterexample to periodicity of ideals of minors over general local rings. Finally, in §6, we explore some applications of Theorems 1.2 and 1.3, e.g. Corollary 1.7.

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NOTATION AND CONVENTIONS

- (1) All rings are assumed to be commutative, and *local* means *Noetherian local*.
- (2) We use homological indexing throughout; the shift of a complex C is given by $C[1]_j = C_{j+1}$ and $d_{C[1]} = -d_C$.
- (3) For a complex C , we let $\inf(C) = \min\{i \mid C_i \neq 0\}$, and similarly for $\sup(C)$.
- (4) Given a ring R and an R -linear map $g : F \rightarrow F'$ of free R -modules of finite rank, we denote by $\mathbf{I}^r(g)$ the ideal in R generated by the $r \times r$ minors of g . When r is

greater than the rank of F or F' , we set $\mathbf{I}^r(g) = 0$; and when either F or F' is 0, we set $\mathbf{I}^r(g) = R$.

- (5) Given a complex (C, d) of finitely generated free R -modules, let $\mathbf{I}_i^r(C)$ denote the ideal $\mathbf{I}^r(d_i : C_i \rightarrow C_{i-1})$. When R is local and M is a finitely generated R -module with minimal free resolution F , we set $\mathbf{I}_i^r(M) := \mathbf{I}_i^r(F)$; these ideals are independent of the choice of F .
- (6) Given a local ring R and a finitely generated R -module M , we denote by $\text{Syz}_n(M)$ the n^{th} syzygy in a minimal free resolution of M .
- (7) Although we work over a local ring throughout most of the paper, all of our results that are stated over local rings hold over an \mathbb{N} -graded ring R such that R_0 is local and R is a finitely generated R_0 -algebra.

2. STABLE IDEALS OF MINORS

2.1. Past Results. Perhaps the first systematic study of ideals of minors in free resolutions was undertaken by Eisenbud-Green in [EG94]. Among other results, Eisenbud-Green prove in [EG94] a conjecture of Huneke regarding the relationship between ideals of minors in a free resolution of a module M and the annihilator of M . Building on [EG94] and also results of Popescu-Roczen [PR90], Wang studies in [Wan94] the connection between the aforementioned conjecture of Huneke and the annihilator ideal of the functor $\text{Ext}_R^{d+1}(-, -)$, where R is a local ring of Krull dimension d . The main results of [Wan94] include versions of Eisenbud-Green's main result [EG94, Theorem 1.1] with the annihilator of the module replaced by the Jacobian ideal [Wan94, Theorems 5.1 and 5.2]. Some results of Wang in [Wan94] are later built upon by Iyengar-Takahashi in their study of cohomology annihilators [IT15]. Koh-Lee also extend some results of [EG94] and [Wan94] in their paper [KL98]; we revisit Koh-Lee's results in detail in §2.3.

Recent work with a specific focus on the behavior of the entries of matrices in minimal free resolutions includes the following. Motivated by a question of Eisenbud-Reeves-Totaro in [ERT94] about bounding the degrees of the entries in minimal graded free resolutions, Eisenbud-Huneke establish in [EH05] a uniform Artin-Rees-type property for the maps in the minimal free resolution of a finitely generated module over a local ring with an isolated singularity. Additionally, motivated by work of the second author and Kobayashi-Takahashi [DKT20], the second author and Eisenbud obtain in [DE23] constraints on the ideals of 1×1 minors in minimal free resolutions over local rings via their notion of the Burch index.

Our main goal in this section is to prove each part of Theorem 1.8, building on much of the above work. We provide detailed commentary throughout this section on the relationship between the present work and the results discussed above.

2.2. Definitions. While our ultimate interest is in minimal free resolutions of modules, many of our results easily extend to minimal free resolutions of (bounded below) complexes, and so we work at this level of generality throughout this section.

Definition 2.1. Let R be a local ring and $\mathbf{C}^+(\text{mod } R)$ the category of bounded below complexes of finitely generated R -modules. Suppose $M \in \mathbf{C}^+(\text{mod } R)$, and let $F \xrightarrow{\sim} M$ be its minimal free resolution (such a resolution exists and is unique; see e.g. [Rob98, Proposition 4.4.2]). For any positive integer r , define

- (1) $\mathbf{I}^{\text{tot}}(M, r) = \sum_{n \geq \inf(F)} \mathbf{I}_n^r(F)$, the *total ideal of $r \times r$ minors of M* .

- (2) $\mathbf{I}^{\text{stab}}(M, r) = \bigcap_{n \geq \inf(F)} \mathbf{I}^{\text{tot}}(F_{\geq n}, r)$, the *stable ideal* of $r \times r$ minors of M . Here, $F_{\geq n}$ denotes the smart truncation.
- (3) $\mathbf{I}^{\text{stab}}(C^+(\text{mod } R), r) = \bigcap_{M \in C^+(\text{mod } R)} \mathbf{I}^{\text{tot}}(M, r)$.

Let $C^b(\text{mod } R)$ and $C^{+,f}(\text{mod } R)$ be the full subcategories of $C^+(\text{mod } R)$ given by bounded complexes and complexes with bounded homology, respectively. We define the ideals $\mathbf{I}^{\text{stab}}(C^b(\text{mod } R), r)$, $\mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), r)$, and $\mathbf{I}^{\text{stab}}(\text{mod}(R), r)$ analogously.

Let $M \in C^+(\text{mod } R)$ with minimal free resolution F . The *projective dimension* of M is defined to be $\sup(F)$. The complex M is called *perfect* if it has finite projective dimension. The following results are immediate consequences of the definitions:

Proposition 2.2. *Let R be a local ring and $M \in C^+(\text{mod } R)$.*

- (1) *We have $\mathbf{I}^{\text{tot}}(M, 1) \neq 0$.*
- (2) *The complex M is perfect if and only if $\mathbf{I}^{\text{stab}}(M, r) = R$ for some $r > 0$.*
- (3) *$\mathbf{I}^{\text{stab}}(C^+(\text{mod } R), r) = \bigcap_{M \in C^+(\text{mod } R)} \mathbf{I}^{\text{stab}}(M, r)$.*
- (4) *R has finite global dimension if and only if $\mathbf{I}^{\text{stab}}(C^b(\text{mod } R), r) = R$.*

Remark 2.3. It is possible that $\mathbf{I}^{\text{tot}}(M, r) = 0$ for $r > 1$. For instance, take R to be Artinian and r greater than the Loewy length of R .

2.3. Extending some results of Koh-Lee. Our main results on stable ideals of minors rely on work of Koh-Lee [KL98] on ideals of minors in minimal free resolutions of modules. Since we work with minimal free resolutions of complexes rather than modules, we must slightly extend these results. The arguments we give here are minor variations of those appearing in [KL98].

The socle of a complex M is defined to be the complex $\underline{\text{Hom}}_R(k, M)$, where k is concentrated in degree zero. Given a complex (M, d^M) , set

$$s(M) := \inf\{t \geq 1 \mid \text{soc}(M) \not\subseteq \mathfrak{m}^t M + \text{im}(d^M)\};$$

this is a generalization of [KL98, Definition 1.1].

Proposition 2.4 (cf. [KL98] Proposition 1.2(i)). *Let (R, \mathfrak{m}) be a local ring, $(M, d^M) \in C^+(\text{mod } R)$, and $(N, d^N) \in C^b(\text{mod } R)$. Assume N is minimal and $\text{soc}(N) \not\subseteq \text{im}(d^N)$. Let $F \xrightarrow{\sim} M$ be the minimal free resolution of M . If an integer i satisfies*

- (a) $\inf(F) + \sup(N) \leq i \leq \sup(F) + \inf(N)$, and
- (b) $\text{Tor}_i^R(M, N) = 0$;

then $\mathbf{I}_{i+1}^1(F) \not\subseteq \mathfrak{m}^{s(N)}$.

Remark 2.5.

- (1) If $\text{soc}(M) \subseteq \text{im}(d^M)$, then $s(M) = \infty$. If $M \in C^b(\text{mod } R)$, and $\text{soc}(M) \not\subseteq \text{im}(d^M)$, then $s(M) < \infty$. In particular, the complex N in the statement of Proposition 2.4 satisfies $s(N) < \infty$.
- (2) When the complex N in Proposition 2.4 satisfies $\inf(N) \geq -1$, the assumption $i \leq \sup(F) + \inf(N)$ is superfluous, as $\mathbf{I}_{i+1}^1(F) = R$ for all $i \geq \sup(F)$.

Proof of Proposition 2.4. Set $G := F \otimes_R N$. Since we have $\text{Tor}_i^R(M, N) = 0$, the sequence $G_{i-1} \leftarrow G_i \leftarrow G_{i+1}$ is exact in the middle. Suppose j is an integer such that $N_j \neq 0$. The inequalities $\inf(N) \leq j \leq \sup(N)$ imply $i - \sup(N) \leq i - j \leq i - \inf(N)$. Applying

our assumption (a), we have $\inf(F) \leq i - j \leq \sup(F)$, i.e. $F_{i-j} \neq 0$. Thus, setting $r_k = \text{rank } F_k$, we conclude that each term of the sum

$$G_i = \bigoplus_{j=\inf(N)}^{\sup(N)} N_j^{r_{i-j}}$$

is nonzero. Since F and N are minimal, we have

$$\text{soc}(N_j)^{r_{i-j}} \subseteq \ker(d_i^G) = \text{im}(d_{i+1}^G) \subseteq \mathbf{I}_{i+1}^1(F)N_j^{r_{i-j}} + \text{im}(d_{j+1}^N)^{r_{i-j}}$$

for all $\inf(N) \leq j \leq \sup(N)$. It follows that $\text{soc}(N) \subseteq \mathbf{I}_{i+1}^1(F)N + \text{im}(d^N)$. By the definition of the integer $s(N)$, we therefore have $\mathbf{I}_{i+1}^1(F) \not\subseteq \mathfrak{m}^{s(N)}$. ■

Proposition 2.6 (cf. [KL98] Theorem 1.7(i)). *Let (R, \mathfrak{m}) be a local ring. There exists an integer $k > 0$ such that, given any $M \in \text{C}^b(\text{mod } R)$ with minimal free resolution $F \xrightarrow{\sim} M$, we have $\mathbf{I}_{i+1}^1(F) \not\subseteq \mathfrak{m}^k$ for all $i > \sup(M) + \text{depth}(R)$.*

Proof. Let $(\underline{x}) \subseteq R$ be a maximal regular sequence, and set $N := R/(\underline{x})$; if $\text{depth } R = 0$, take $N = R$. Fix $i > \sup(M) + \text{depth}(R)$. Since $\text{pd}_R(N) = \text{depth}(R)$, we have $\text{Tor}_i^R(M, N) = 0$. Certainly $\inf(F) \leq \sup(M) < i$; it therefore follows from Proposition 2.4 and Remark 2.5(2) that $\mathbf{I}_{i+1}^1(F) \not\subseteq \mathfrak{m}^k$ for $k = s(N)$. ■

Remark 2.7. The conclusion of Proposition 2.6 cannot hold in general for a smaller homological index. Indeed, consider Koszul complexes on maximal regular sequences of elements contained in arbitrarily high powers of the maximal ideal.

2.4. Constraints on stable ideals of minors.

Theorem 2.8. *Let (R, \mathfrak{m}) be a local ring. If M is a bounded complex of finitely generated R -modules, then $\mathbf{I}^{\text{stab}}(M, 1) \neq 0$.*

Proof. Let F be the minimal free resolution of M , and set $J_j := \mathbf{I}^{\text{tot}}(F_{\geq j}, 1)$. Recall that $\mathbf{I}^{\text{stab}}(M, 1) := \bigcap_{j=\inf(F)}^{\infty} J_j$. We first assume that R is \mathfrak{m} -adically complete. Suppose $\mathbf{I}^{\text{stab}}(M, 1) = 0$. By [Che43, Lemma 7], for every integer $t \geq 1$, there exists an integer $n(t)$ such that $J_{n(t)} \subseteq \mathfrak{m}^t$. By Proposition 2.6, there exists an integer k such that, for all $i > \sup(M) + \text{depth}(R) + 1$, we have $\mathbf{I}_i^1(F) \not\subseteq \mathfrak{m}^k$. In particular, $J_j \not\subseteq \mathfrak{m}^k$ for $j > \sup(M) + \text{depth}(R) + 1$. Choosing $j > \max\{\sup(M) + \text{depth}(A) + 1, n(k)\}$ therefore yields $J_i \subseteq \mathfrak{m}^k$ and $J_i \not\subseteq \mathfrak{m}^k$, a contradiction.

As for the general case: we have $\widehat{\mathbf{I}^{\text{stab}}(M, 1)} = \bigcap_{j=\inf(F)}^{\infty} \widehat{J_j} = \mathbf{I}^{\text{stab}}(\widehat{M}, 1)$, where the first equality follows since completion commutes with intersections, and the second is due to the flatness of completion. Since $\mathbf{I}^{\text{stab}}(\widehat{M}, 1) \neq 0$, we must have $\mathbf{I}^{\text{stab}}(M, 1) \neq 0$. ■

Let R be a local ring. For all $n \geq 0$, we let

$$\text{ca}^n(R) = \bigcap_{M, N \in \text{mod}(R)} \text{ann}_R(\text{Ext}_R^{\geq n}(M, N))$$

Following Iyengar-Takahashi [IT15, Definition 2.1], the *cohomology annihilator* of R is defined to be the ideal $\bigcup_{n \geq 0} \text{ca}^n(R)$. By Noetherianity, $\text{ca}(R) = \text{ca}^s(R)$ for some $s \gg 0$.

Before stating the next result, we recall that $\text{C}^{+,f}(\text{mod } R)$ denotes the category of bounded below complexes of finitely generated R -modules with bounded homology.

Proposition 2.9. *Let (R, \mathfrak{m}, k) be a local ring.*

- (1) Given $M \in C^{+,f}(\text{mod } R)$, we have $\text{ca}(R) \subseteq \mathbf{I}^{\text{stab}}(M, 1)$. Thus, $\text{ca}(R) \subseteq \mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1) = \mathbf{I}^{\text{stab}}(\text{mod}(R), 1)$. In particular, if $\text{ca}(R) \neq 0$, then $\mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1) \neq 0$.
- (2) If R is not regular, then $\mathbf{I}^{\text{stab}}(k, 1) = \mathfrak{m}$, and $\mathbf{I}^{\text{stab}}(L, 1)$ is \mathfrak{m} -primary for any R -module L of finite length.

Proof. Let F be the minimal free resolution of M . Given $i \geq 0$, it suffices to show $\text{ca}(R) \subseteq I := \mathbf{I}^{\text{tot}}(F_{\geq i}, 1)$; recall that $F_{\geq i}$ denotes the smart truncation of F at i . Choose $s \geq 0$ such that $\text{ca}(R) = \bigcap_{A, B \in \text{mod}(R)} \text{ann}_R(\text{Ext}_R^{\geq s}(A, B))$. Choose also $t \geq 0$ such that $H_{\geq t}(M) = 0$, so that $F_{\geq j}$ is the minimal free resolution of a module N_j for all $j \geq t$. Let $r = \max\{i, s + t\}$. Observe that $\text{Tor}_r^R(M, R/I) = \text{Tor}_r^R(N_r, R/I)$ is a free R/I -module, and so $\text{ann}(\text{Tor}_r^R(N_r, R/I)) = I$. We therefore have

$$\text{ca}(R) \subseteq \text{ann}_R(\text{Ext}_R^r(N_r, R/I)) = \text{ann}_R(\text{Tor}_r^R(N_r, R/I)) = I,$$

where the first equality follows by [Wan94, Corollary 1.6(3)] (and taking syzygies). This proves (1). As for (2): setting $\text{ca}(L) := \bigcup_{j \geq 1} (\bigcap_{N \in \text{mod}(R)} \text{ann}_R(\text{Ext}_R^j(L, N)))$, we clearly have that $\text{ann}(L) \subseteq \text{ca}(L)$. Arguing as in the proof of (1), we conclude $\text{ca}(L) \subseteq \mathbf{I}^{\text{stab}}(L, 1)$. \blacksquare

Given a local ring R , an object $M \in C^{+,f}(\text{mod } R)$ with minimal free resolution F , and $r \geq 1$; we say $\mathbf{I}^{\text{stab}}(M, r)$ is *finitely determined* if it is equal to $\mathbf{I}^{\text{tot}}(F_{\geq k}, r)$ for some $k \gg 0$. Similarly, we say $\mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), r)$ is *finitely determined* if it is equal to the intersection of finitely many ideals of the form $\mathbf{I}^{\text{stab}}(M_i, r)$, where each $\mathbf{I}^{\text{stab}}(M_i, r)$ is finitely determined. The following is immediate from Proposition 2.9(1):

Corollary 2.10. *Let (R, \mathfrak{m}) be a local ring such that $\text{ca}(R)$ is \mathfrak{m} -primary. For any $M \in C^{+,f}(\text{mod } R)$, $\mathbf{I}^{\text{stab}}(M, 1)$ is finitely determined. Moreover, $\mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1)$ is finitely determined.*

We also have:

Corollary 2.11. *Suppose R is a localization of a finitely generated algebra over a field or an equicharacteristic excellent local ring. If R is an isolated singularity, then for any $M \in C^{+,f}(\text{mod } R)$, $\mathbf{I}^{\text{stab}}(M, 1)$ is \mathfrak{m} -primary and finitely determined. Moreover, $\mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1)$ is \mathfrak{m} -primary and finitely determined.*

Proof. Apply [IT15, Theorems 5.3 and 5.4], Proposition 2.9(1), and Corollary 2.10. \blacksquare

It follows from Theorems 3.23 and 4.9 below that, when R is a complete intersection or Golod, $\mathbf{I}^{\text{stab}}(M, r)$ is finitely determined for any $r \geq 1$ and $M \in C^{+,f}(\text{mod } R)$. We ask:

Question 2.12. *Let R be a Noetherian local ring and M a finitely generated R -module. Is $\mathbf{I}^{\text{stab}}(M, r)$ finitely determined for any $r \geq 1$?*

Theorem 2.13. *Let R be a local ring, $\text{Sing}(R)$ the singular locus of R , and $\text{Jac}(R)$ the Jacobian ideal of R ; see [IT21, §3] for the definition of $\text{Jac}(R)$. We have:*

- (1) $\text{Jac}(R)^t \subseteq \mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1)$ for some $t \geq 1$.
- (2) If R is a localization of a finitely generated algebra over a field, or if R is an equicharacteristic excellent local ring, then $V(\mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1)) \subseteq \text{Sing}(R)$. Thus, if $\text{Sing}(R) \neq \text{Spec}(R)$ (e.g. if R is reduced), then $\mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1) \neq 0$.

- (3) Suppose R is complete, Cohen-Macaulay and contains a field. We have $\mathfrak{R}^R \subseteq \mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1)$, where \mathfrak{R}^R is the Noether different of R ; see [Wan94, Definition 5.5].

Proof. By [IT21, Theorem 3.4], some power of $\text{Jac}(R)$ is contained in $\text{ca}(R)$. Part (1) therefore follows from Proposition 2.9(1). For (2), [IT15, Theorems 5.3 and 5.4] imply that $V(\text{ca}(R)) = \text{Sing}(R)$ under the given hypotheses. The claims therefore follow from Proposition 2.9(1). For (3), by [Wan94, Corollary 5.13], \mathfrak{R}^R annihilates the functor $\text{Ext}_R^1(M, -)$ for any maximal Cohen-Macaulay module M . It follows that $\mathfrak{R}^R \subseteq \text{ca}(R)$ and we conclude by Proposition 2.9(1). \blacksquare

Remark 2.14. Under extra hypotheses, [Wan94, Theorem 5.2] implies Theorem 2.13(1).

For $r > 1$, the ideals $\mathbf{I}^{\text{stab}}(M, r)$ can be 0. For example, if the Betti numbers of the minimal free resolution of $M \in C^{+,f}(\text{mod } R)$ are bounded, then $\mathbf{I}^{\text{tot}}(M, r) = \mathbf{I}^{\text{stab}}(M, r) = 0$ for $r \gg 0$. Such is the case for any finitely generated module M over a hypersurface ring, although the choice of r is dependent on M . Moreover, even if the Betti numbers of M are unbounded, $\mathbf{I}^{\text{stab}}(M, r)$ could vanish for other reasons. For instance, if R is an Artinian ring, then $\mathbf{I}^{\text{tot}}(M, r) = \mathbf{I}^{\text{stab}}(M, r) = 0$ for $r > \ell\ell(R)$, the Loewy length of R . However, we have the following result on ideals of higher rank minors of the residue field:

Theorem 2.15. *Let (R, \mathfrak{m}, k) be a local ring and K the Koszul complex on a minimal generating set of \mathfrak{m} .*

- (1) *If $\dim_k H_1(K) \geq 2$, then $\mathbf{I}^{\text{stab}}(k, r) = \mathfrak{m}^r$ for all $r \geq 1$.*
- (2) *If $R = S/I$, where (S, \mathfrak{n}) is a regular local ring and $I \subseteq \mathfrak{n}^2$, then $\mathbf{I}^{\text{stab}}(k, r) = \mathfrak{m}^r$ for all $r \geq 1$ unless R is a hypersurface.*

Proof. Let (F, d) be the free resolution of k constructed in [Tat57] (which is minimal by a result of Gulliksen [Gul68]). The resolution F is a dg R -algebra: as an R -algebra, it is a tensor product of a divided power algebra in some collection of even (and nonnegative) degree variables and an exterior algebra in some collection of odd (and positive) degree variables. Equip F with an R -basis given by monomials in these variables. Denote the exterior variables comprising our basis of F_1 by X_1, \dots, X_n , and set $x_\ell := d(X_\ell)$; the elements x_1, \dots, x_n form a minimal generating set for \mathfrak{m} . Since $\dim_k H_1(K) \geq 2$, we may also choose two distinct divided power variables T_1, T_2 in our basis of F_2 .

Fix $r \geq 1$, and let x_{j_1}, \dots, x_{j_r} be a set of r (not necessarily distinct) elements in $\{x_1, \dots, x_n\}$. To prove (1), it suffices to show that, given an integer $N \geq 1$, there is some $M \geq N$ such that the matrix $d_M : F_M \rightarrow F_{M-1}$ (with respect to our chosen basis of F) has an $r \times r$ diagonal submatrix with the elements x_{j_1}, \dots, x_{j_r} on the diagonal. Notice that, for any $k_1, k_2 \geq 0$ and $1 \leq \ell \leq n$, we have

$$(2.16) \quad d(X_\ell T_1^{k_1} T_2^{k_2}) = x_\ell T_1^{k_1} T_2^{k_2} + (\text{terms with } X_\ell \text{ as a factor}).$$

Without loss of generality, assume N is odd. Set $M := \max\{N, 2(r-1)+1\}$ and $s := \frac{M-1}{2}$. The relation (2.16) implies that the following is a submatrix of d_M :

$$\begin{matrix} & X_{j_1}T_1^s & X_{j_2}T_1^{s-1}T_2 & \cdots & X_{j_{r-1}}T_1^{s-(r-2)}T_2^{r-2} & X_{j_r}T_1^{s-(r-1)}T_2^{r-1} \\ \begin{matrix} T_1^s \\ T_1^{s-1}T_2 \\ \vdots \\ T_1^{s-(r-2)}T_2^{r-2} \\ T_1^{s-(r-1)}T_2^{r-1} \end{matrix} & \begin{pmatrix} x_{j_1} & 0 & \cdots & 0 & 0 \\ 0 & x_{j_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{j_{r-1}} & 0 \\ 0 & 0 & \cdots & 0 & x_{j_r} \end{pmatrix} \end{matrix}.$$

This proves (1). To prove (2), we observe that $\dim_k H_1(K)$ is equal to $\dim_k(I/\mathfrak{n}I)$, the minimal number of generators of I (see e.g. [GL69, Lemma 1.4.15]). Thus, R is a hypersurface if and only if $\dim_k H_1(K) = 1$. Now apply (1). \blacksquare

Remark 2.17. When $\mathbb{Q} \subseteq R$ and $\text{depth}(R) > 0$, one may use [EG94, Theorem 1] to prove Theorem 2.15.

2.5. Stable ideals of minors and the conductor.

Definition 2.18. Let R be a reduced ring. The *conductor* of R is defined to be $\mathfrak{c}_R = \{r \in R : r \cdot \overline{R} \subseteq R\}$, where \overline{R} denotes the integral closure of R .

The main goal of this subsection is to prove Theorem 2.25, which illustrates a relationship between stable ideals of 1×1 minors and the conductor in the case of a 1-dimensional analytically unramified local ring (cf. [Wan94, Corollary 3.3]). We first recall some background on trace ideals:

Definition 2.19. Given a module M over a commutative ring R , the *trace ideal* of M over R , denoted $\text{Tr}_R(M)$, is the ideal in R generated by the images of all morphisms $f \in M^\vee := \text{Hom}_R(M, R)$. Alternatively, $\text{Tr}_R(M)$ is the image of the evaluation map $M \otimes_R M^\vee \rightarrow R$.

Remark 2.20. If M is reflexive (i.e. the natural map $M \rightarrow (M^\vee)^\vee$ is an isomorphism), then $\text{Tr}_R(M) = \text{Tr}_R(M^\vee)$.

The connection between trace ideals and ideals of 1×1 minors is provided by the following result:

Proposition 2.21. *Let R be a Gorenstein local ring and M a finitely generated R -module whose minimal free resolution F is infinite.*

- (1) *If $n > \dim(R) + 1$, then $\mathbf{I}_n^1(F) = \text{Tr}_R(\text{Syz}_n(M))$.*
- (2) *If M is maximal Cohen-Macaulay (MCM), then $\mathbf{I}_1^1(M) = \text{Tr}_R(\text{Syz}_1(M))$.*

Proof. Let $i \in \{n-2, n-1\}$. Observe that $\text{Ext}_R^1(\text{Syz}_i(M), R) = \text{Ext}_R^{i+1}(M, R) = 0$, since R is Gorenstein and $i+1 \geq n-1 > \dim(R)$. Dualizing the exact sequence $0 \rightarrow \text{Syz}_{i+1}(M) \rightarrow F_i \rightarrow \text{Syz}_i(M) \rightarrow 0$ therefore gives the exact sequence $0 \rightarrow \text{Syz}_i(M)^\vee \rightarrow F_i^\vee \rightarrow \text{Syz}_{i+1}(M)^\vee \rightarrow 0$. Combining these exact sequences for $i = n-2$ and $i = n-1$ yields a free presentation $F_{n-2}^\vee \rightarrow F_{n-1}^\vee \rightarrow \text{Syz}_n(M)^\vee \rightarrow 0$. It thus follows from [Lin16, §2.1.3] (see also [Vas91, Remark 3.3]) that $\text{Tr}_R(\text{Syz}_n(M)^\vee) = \mathbf{I}_n^1(F)$. Since R is Gorenstein and $n > \dim(R)$, $\text{Syz}_n(M)$ is MCM and therefore reflexive. By Remark 2.20, we conclude that $\text{Tr}_R(\text{Syz}_n(M)) = \text{Tr}_R(\text{Syz}_n(M)^\vee) = \mathbf{I}_n^1(F)$; this proves (1). For (2): we

may assume without loss of generality that M has no free summand. (2) therefore follows from (1), along with the observation that maximal Cohen-Macaulay modules with no free summands over Gorenstein rings are infinite syzygies. ■

Corollary 2.22. *Let (R, \mathfrak{m}) be a Gorenstein local ring and M a maximal Cohen-Macaulay R -module.*

- (1) *The ideal $\mathbf{I}^{\text{tot}}(M, 1)$ is a trace ideal.*
- (2) *If $\mathbf{I}^{\text{stab}}(M, 1)$ is \mathfrak{m} -primary, then it is also a trace ideal.*

Proof. Observe that $\sum_{i \in I} \text{Tr}_R(M_i) = \text{Tr}_R(\bigoplus_{i \in I} M_i)$ for any collection $\{M_i\}_{i \in I}$ of R -modules. Part (1) therefore follows from Proposition 2.21(2). Let us now prove (2). Let F be the minimal free resolution of M . Since $\mathbf{I}^{\text{stab}}(M, 1)$ is \mathfrak{m} -primary, it is equal to $\mathbf{I}^{\text{tot}}(F_{\geq n}, 1)$ for $n \gg 0$ (recall that $F_{\geq n}$ denotes the smart truncation). Now apply (1). ■

The following fact is well-known to experts, but due to lack of an exact reference, we include a proof.

Lemma 2.23. *Let (R, \mathfrak{m}) be a 1-dimensional analytically unramified local ring, and denote by \overline{R} (resp. \widehat{R}) the integral closure (resp. \mathfrak{m} -adic completion) of R .*

- (1) *\overline{R} is a finitely generated R -module.*
- (2) *The integral closure of \widehat{R} is isomorphic to $\overline{R} \otimes_R \widehat{R}$.*

Proof. For (1): since R is reduced, we have $\overline{R} \cong \prod \overline{R/P}$, where the product ranges over the minimal primes of R . Moreover, for every minimal prime P of R , R/P is analytically unramified; see e.g. [HS06, Proposition 9.1.3]. The conclusion of (1) then follows from [Nag58, Appendix 1, Proposition 1]. For (2), note that \overline{R} is a normal semi-local ring by (1). Since $\overline{R} \otimes_R \widehat{R}$ is birational and finitely generated over \widehat{R} , it suffices to show $\overline{R} \otimes_R \widehat{R}$ is normal. Since $\widehat{\overline{R}}$ is a finitely generated module over \widehat{R} by (1), there exists a non-zero-divisor $c \in \widehat{\overline{R}}$ such that $c\widehat{\overline{R}} \subseteq \widehat{R}$; in particular, $\mathfrak{c}_{\widehat{R}} \neq 0$. Since $\mathfrak{c}_{\widehat{R}}$ is primary to the maximal ideal, we have $\mathfrak{m}^n \subseteq \mathfrak{c}_{\widehat{R}}$ for some n . Thus, we can choose a non-zero-divisor $t \in \mathfrak{c}_{\widehat{R}} \cap R$. In particular, t is a nonzero, nonunit element of \overline{R} . Now apply [Nag62, 37.2] to the ring \overline{R} and the element $t \in \overline{R}$ to conclude that $\overline{R} \otimes_R \widehat{R}$ is normal (the assumption on tR in this result of Nagata is satisfied since t is a non-zero-divisor and $\dim(R) = 1$). ■

We recall that the *order* of an ideal I in a local ring (R, \mathfrak{m}) is $\text{ord}(I) := \sup\{k : I \subseteq \mathfrak{m}^k\}$.

Lemma 2.24. *Let R be an Artinian local ring and $M \in \mathbf{C}^+(\text{mod } R)$. We have $\text{soc}(R) \subseteq \mathbf{I}^{\text{stab}}(M, 1)$, and $\text{ord}(\mathbf{I}^{\text{stab}}(M, 1)) \leq \lfloor l/2 \rfloor$, where l is the Loewy length of R . In particular, $\mathbf{I}^{\text{stab}}(\mathbf{C}^+(\text{mod } R), 1) \neq 0$.*

Proof. Fix an integer $n \geq 2 + \inf(M)$. Let (F, d) be the minimal free resolution of M , and write $\beta_{n-1} = \text{rank } F_{n-1}$. The submodule $\text{soc}(R)^{\oplus \beta_{n-1}} \subseteq F_{n-1}$ is contained in $\ker(d_{n-1}) = \text{im}(d_n)$, since F is minimal. We therefore have $\text{soc}(R) \subseteq \mathbf{I}^1(v)$ for every row v of d_n ; in particular, $\text{soc}(R) \subseteq \mathbf{I}_n^1(M)$. Thus, $\text{soc}(R) \subseteq \mathbf{I}^{\text{stab}}(M, 1)$. It follows that $\text{soc}(R) \subseteq \mathbf{I}^{\text{stab}}(\mathbf{C}^+(\text{mod } R), 1)$; in particular, $\mathbf{I}^{\text{stab}}(\mathbf{C}^+(\text{mod } R), 1) \neq 0$.

Since R is Artinian, $\mathbf{I}^{\text{stab}}(M, 1) = \mathbf{I}^{\text{tot}}(F_{\geq k}, 1)$ for some $k \geq \inf(M)$. Fix $n > k$. If $\text{ord}(\mathbf{I}_n^1(F)) \leq \lfloor l/2 \rfloor$, we are done. If not, $\mathbf{I}_n^1(F) \subseteq \mathfrak{m}^{\lfloor l/2 \rfloor + 1}$, and hence $\mathfrak{m}^{\lfloor l/2 \rfloor}$ annihilates $\mathbf{I}_n^1(F)$. Thus, the submodule $(\mathfrak{m}^{\lfloor l/2 \rfloor})^{\oplus \beta_n} \subseteq F_n$ is contained in $\ker(d_n) = \text{im}(d_{n+1})$. This implies that $\mathfrak{m}^{\lfloor l/2 \rfloor} \subseteq \mathbf{I}_{n+1}^1(F)$, and so $\text{ord}(\mathbf{I}^{\text{stab}}(M, 1)) \leq \text{ord}(\mathbf{I}_{n+1}^1(F)) \leq \lfloor l/2 \rfloor$. ■

Theorem 2.25. *Let (R, \mathfrak{m}) be a 1-dimensional analytically unramified local ring and \mathfrak{c}_R the conductor ideal of R .*

- (1) *For any $M \in C^{+,f}(\text{mod } R)$, we have $\mathfrak{c}_R \subseteq \mathbf{I}^{\text{stab}}(M, 1)$. Consequently, $\mathfrak{c}_R \subseteq \mathbf{I}^{\text{stab}}(C^{+,f}(\text{mod } R), 1) = \mathbf{I}^{\text{stab}}(\text{mod}(R), 1)$.*
- (2) *Assume R is Gorenstein with infinite residue field. The following are equivalent:*
 - (a) *There is a finitely generated R -module M such that $\mathbf{I}^{\text{stab}}(M, 1) = \mathfrak{c}_R$.*
 - (b) *R is regular or an abstract hypersurface of multiplicity 2.*

Proof. Let us prove (1). Let F be the minimal free resolution of M , and let \overline{R} and \widehat{R} be as in Lemma 2.23. It follows directly from Lemma 2.23 that $\mathfrak{c}_R \widehat{R} \subseteq \mathfrak{c}_{\widehat{R}}$, which implies $\mathfrak{c}_R \subseteq \mathfrak{c}_{\widehat{R}} \cap R$. Now, suppose $\mathfrak{c}_{\widehat{R}} \subseteq \mathbf{I}^{\text{stab}}(\widehat{M}, 1)$. We have

$$\begin{aligned} \mathfrak{c}_R \subseteq \mathfrak{c}_{\widehat{R}} \cap R &\subseteq \mathbf{I}^{\text{stab}}(\widehat{M}, 1) \cap R = \left(\bigcap \mathbf{I}^{\text{tot}}(\widehat{F}_{\geq \inf(F)+i}, 1) \right) \cap R \\ &= \bigcap (\mathbf{I}^{\text{tot}}(\widehat{F}_{\geq \inf(F)+i}, 1) \cap R) \\ &= \bigcap (\mathbf{I}^{\text{tot}}(F_{\geq \inf(F)+i}, 1) \widehat{R} \cap R) \\ &= \bigcap \mathbf{I}^{\text{tot}}(F_{\geq \inf(F)+i}, 1) \\ &= \mathbf{I}^{\text{stab}}(M, 1), \end{aligned}$$

where the third and fourth equalities hold due to flatness and faithful flatness of the completion map, respectively. We may therefore assume R is complete and reduced. Since R is a 1-dimensional reduced ring, it is Cohen-Macaulay. Thus, any first syzygy module over R is maximal Cohen-Macaulay (MCM). By [Wan94, Proposition 3.1], we have $\mathfrak{c}_R \cdot \text{Ext}_R^1(N, N') = 0$ for any finitely generated MCM R -module N and finitely generated R -module N' . It follows immediately that $\mathfrak{c}_R \subseteq \text{ca}(R)$. Applying Proposition 2.9(1), we conclude that $\mathfrak{c}_R \subseteq \mathbf{I}^{\text{stab}}(M, 1)$; this proves (1).

We now prove (2). We need only consider the case where R is not regular. Since R is analytically unramified, the integral closure \overline{R} of R is a finitely generated R -module by Lemma 2.23(1). The assumptions imply that \mathfrak{c}_R is an \mathfrak{m} -primary ideal.

Assume (a), and choose a finitely generated module M such that $\mathbf{I}^{\text{stab}}(M, 1) = \mathfrak{c}_R$. Since the residue field of R is infinite, we can choose $x \in \mathfrak{m}$ such that it is a minimal reduction of \mathfrak{m} ; see [NR54]. We have $x\mathfrak{c}_R = \mathfrak{m}\mathfrak{c}_R$, and so $\mathfrak{c}_R \subseteq (x : \mathfrak{m})$. Computing over $A := R/xR$, we have $\mathbf{I}^{\text{stab}}(M/xM, 1) = (\mathfrak{c}_R + (x))/(x) \subseteq \text{soc}(A)$. It follows from Lemma 2.24 that the order of $\text{soc}(A)$ is at most $\lfloor l/2 \rfloor$, where l is the Loewy length of A . Since A is Gorenstein, we have $\text{soc}(A) \subseteq \mathfrak{m}_A^{l-1}$. We conclude that $l \leq 2$, and so $\text{soc}(A) = \mathfrak{m}_A$. Thus, the embedding dimension of R , and hence \widehat{R} , is two. Since \widehat{R} is also reduced and one-dimensional, it is a hypersurface. Moreover, since $\ell(R/xR) = 2$, the multiplicity of R is two [HS06, Proposition 11.2.2]. Thus, R is an abstract hypersurface of multiplicity two.

Assume (b). Suppose R is an abstract hypersurface of multiplicity 2. Let $M = \overline{R}$. As R is Gorenstein, we have $\ell(M/\mathfrak{c}_R M) = 2\ell(R/\mathfrak{c}_R)$ [HS06, Theorem 12.2.2]. As the minimal number of generators of M as an R -module is 2, we get that $M/\mathfrak{c}_R M \cong (R/\mathfrak{c}_R)^2$, implying $\mathbf{I}_1^1(M) \subseteq \mathfrak{c}_R$. By Proposition 2.21, it follows that $\text{Tr}_R(\text{Syz}_1(M)) \subseteq \mathfrak{c}_R$. Since \overline{R} is a reflexive R -module, we have that $\text{Syz}_1(M)$ is an \overline{R} -module (see [DMS23, 2.9]). The module $\text{Syz}_1(M)$ has rank and is rank 1. It is therefore a regular ideal in R and hence

isomorphic to \overline{R} . The minimal free resolution of \overline{R} is therefore one-periodic; by part (1), we conclude that $\mathbf{I}^{\text{stab}}(M, 1) = \mathfrak{c}_R$, as desired. \blacksquare

2.6. Examples. We now compute some examples of stable ideals of 1×1 minors. Throughout this subsection, k denotes a field.

Example 2.26. Let $R = k[[x, y]]/(x^2, y^2)$ and M a finitely generated non-free R -module. We claim that

$$(2.27) \quad \mathbf{I}^{\text{stab}}(M, 1) \in \begin{cases} \{(x), (y), \mathfrak{m}\}, & \text{char}(k) \neq 2; \\ \{(x), (y), (x+y), \mathfrak{m}\}, & \text{char}(k) = 2. \end{cases}$$

It is easy to show that each of the ideals on the right side of (2.27) arises as $\mathbf{I}^{\text{stab}}(M, 1)$ for some M , and so it follows from (2.27) that $\mathbf{I}^{\text{stab}}(\text{mod}(R), 1) = \text{soc}(R)$. To prove (2.27), it suffices, since R is Artinian, to determine all possible values of $\mathbf{I}^{\text{tot}}(M, 1)$. Let $I = \mathbf{I}_1^1(M)$. If $I = \mathfrak{m}$, then $\mathbf{I}^{\text{tot}}(M, 1) = \mathfrak{m}$. If $I = \mathfrak{m}^2$, then $\mathbf{I}_2^1(M) = \mathfrak{m}$ and so $\mathbf{I}^{\text{tot}}(M, 1) = \mathfrak{m}$. We may therefore assume that I contains exactly one minimal generator of \mathfrak{m} , say t . We have $I = (t)$, and so $\text{ann}(t) \subseteq \mathbf{I}_2^1(M)$, which means $(t) + \text{ann}(t) \subseteq \mathbf{I}^{\text{tot}}(M, 1)$. One easily checks:

$$(t) + \text{ann}(t) \in \begin{cases} \{(x), (y), \mathfrak{m}\}, & \text{char}(k) \neq 2; \\ \{(x), (y), (x+y), \mathfrak{m}\}, & \text{char}(k) = 2. \end{cases}$$

Example 2.28. Let $R = k[[t^3, t^4]] \cong k[[x, y]]/(x^4 - y^3)$ and M a finitely generated module of infinite projective dimension. We now show that $\mathbf{I}^{\text{stab}}(M, 1)$ is either (x^2, y) or (x, y) , and both possibilities occur. We may assume M is MCM and indecomposable. Since R is a hypersurface, $\mathbf{I}^{\text{stab}}(M, 1)$ is equal to $\mathbf{I}^1(A) + \mathbf{I}^1(B)$, where (A, B) is the matrix factorization associated to M . Moreover, since R is an ADE singularity, there are only finitely many such matrix factorizations up to isomorphism. One concludes by observing the list of these matrix factorizations in [Yos90, 9.13]. It follows that $\mathbf{I}^{\text{stab}}(\text{mod}(R), 1) = (x^2, y)$. Observe that $\mathfrak{c}_R = (t^6, t^7, t^8) = (x, y)^2$, which, as predicted by Theorem 2.25, is properly contained in $\mathbf{I}^{\text{stab}}(\text{mod}(R), 1) = (x^2, y)$.

Example 2.29. Let $R = k[[t^4, t^5, t^6]] \cong k[[x, y, z]]/(x^3 - z^2, xz - y^2)$. We now show that $\mathbf{I}^{\text{stab}}(\text{mod}(R), 1) = (x^2, xy, z)$. Let M be a non-free MCM R -module. We have $\mathfrak{c}_R = (t^8, t^9, t^{10}, t^{11}) = (x, y, z)^2$; it therefore follows from Theorem 2.25(1) that $\mathbf{I}^{\text{stab}}(M, 1)$ is \mathfrak{m} -primary. Thus, by Corollary 2.22(2), $\mathbf{I}^{\text{stab}}(M, 1)$ is a trace ideal. Given $a \in k$, let $I_a := (x - ay, z)$; we conclude from [GIK20, Example 3.4] and Theorem 2.25(2) that $\mathbf{I}^{\text{stab}}(M, 1)$ must belong to the list $\{(x, y, z), (x^2, y, z), (x^2, xy, z)\} \cup \{I_a, a \in k\}$. Each of these ideals contains (x^2, xy, z) , and so $(x^2, xy, z) \subseteq \mathbf{I}^{\text{stab}}(\text{mod}(R), 1)$. We also have $I_0 \cap I_1 = (x^2, xy, z)$; thus, to show the containment $\mathbf{I}^{\text{stab}}(\text{mod}(R), 1) \subseteq (x^2, xy, z)$, it suffices to show that $\mathbf{I}^{\text{stab}}(I_a, 1) = I_a$ for $a = 0, 1$. One may see this via an explicit calculation: the minimal R -free resolution of I_0 is $\left[R^2 \xleftarrow{A} R^2 \xleftarrow{B} R^2 \xleftarrow{A} R^2 \xleftarrow{B} \cdots \right]$, where $A = \begin{pmatrix} -x & -z \\ z & x^2 \end{pmatrix}$ and $B = \begin{pmatrix} -x^2 & -z \\ z & x \end{pmatrix}$; and the minimal R -free resolution of I_1 is $\left[R^2 \xleftarrow{A'} R^2 \xleftarrow{B'} R^2 \xleftarrow{A'} R^2 \xleftarrow{B'} \cdots \right]$, where we have $A' = \begin{pmatrix} y - x & x^2 - z \\ z & x^2 + xy \end{pmatrix}$ and $B' = \begin{pmatrix} -x^2 - xy & x^2 - z \\ z & x - y \end{pmatrix}$. Finally, we observe once again that \mathfrak{c}_R is properly contained in $\mathbf{I}^{\text{stab}}(\text{mod}(R), 1)$, in accordance with Theorem 2.25.

3. PERIODICITY OF IDEALS OF MINORS OVER COMPLETE INTERSECTIONS

We now study the eventual behavior of ideals of minors in free resolutions over complete intersections. In §3.1, we recall the notion of a Shamash resolution, and we explain how Shamash resolutions can be interpreted in terms of matrix factorizations. We prove in §3.2 (resp. §3.3) that the ideals of minors in the Shamash (resp. minimal free) resolution of a finitely generated module M over complete intersections are eventually 2-periodic. In the case of a Shamash resolution, we obtain a bound on the homological degree at which this periodicity starts that is independent of M , mirroring the behavior of minimal free resolutions over hypersurfaces (Corollary 3.20). For minimal free resolutions, we show that this is impossible (Example 3.26).

Let us establish some notation for this section. Let S be a commutative ring, $f_1, \dots, f_c \in S$, and $R = S/(f_1, \dots, f_c)$. Let \tilde{S} denote the graded ring $S[t_1, \dots, t_c]$, where $\deg(t_i) = -2$, and \tilde{D} the graded S -module $\text{Hom}_S(\tilde{S}, S)$. Observe that \tilde{D} is the divided power algebra over \tilde{S} on the degree 2 dual variables y_1, \dots, y_c corresponding to t_1, \dots, t_c ; moreover, \tilde{D} is an \tilde{S} -module via contraction.

3.1. The Shamash construction and matrix factorizations. We begin by recalling the Shamash construction on a complex of S -modules associated to $f_1, \dots, f_c \in S$, introduced by Shamash in the case where $c = 1$ [Sha71] and generalized by Eisenbud to the $c \geq 1$ case [Eis77]. Our reference is Eisenbud-Peeva's book [EP16].

Given $\mathbf{a} \in \mathbb{Z}_{\geq 0}^c$, write $t^{\mathbf{a}} := t_1^{a_1} \cdots t_c^{a_c}$, and let $|\mathbf{a}| = \sum_{i=1}^c a_i$.

Definition 3.1 ([EP16] Definition 3.4.1). Let G be a complex of free S -modules. A *system of higher homotopies* for f_1, \dots, f_c on G is a collection σ of morphisms¹

$$\sigma_{\mathbf{a}} : G \rightarrow G[2|\mathbf{a}| - 1]$$

of graded modules (not complexes) for each $\mathbf{a} = (a_1, \dots, a_c) \in \mathbb{Z}_{\geq 0}^c$ with the following properties:

- (a) σ_0 is the differential on G .
- (b) Letting e_i denote the i^{th} standard basis vector for each i , the map $\sigma_0 \sigma_{e_i} + \sigma_{e_i} \sigma_0$ is given by multiplication by f_i on G for each $1 \leq i \leq c$.
- (c) If $|a| \geq 2$, then $\sum_{\mathbf{u} + \mathbf{v} = \mathbf{a}} \sigma_{\mathbf{u}} \sigma_{\mathbf{v}} = 0$.

Construction 3.2. Let G be a complex of free S -modules equipped with a system of higher homotopies σ . The *Shamash construction* $\text{Sh}(G, \sigma)$ associated to the pair (G, σ) has underlying module $\tilde{D} \otimes_S G \otimes_S R$ and differential $\sum t^{\mathbf{a}} \otimes \sigma_{\mathbf{a}} \otimes \text{id}$.

Proposition 3.3 ([EP16] Propositions 3.4.2 and 4.1.4). *Let G be an S -free resolution of a finitely generated R -module M .*

- (1) *There exists a system of higher homotopies on G for f_1, \dots, f_c .*
- (2) *Assume that $f_1, \dots, f_c \in S$ form a regular sequence. If σ is a system of higher homotopies on G for f_1, \dots, f_c , then $\text{Sh}(G, \sigma)$ is an R -free resolution of M .*

When the complex $\text{Sh}(G, \sigma)$ in Construction 3.2 is an R -free resolution of a module M , we call it a *Shamash resolution* of M .

¹Our definition looks slightly different from [EP16, Definition 3.4.1] at first glance, as the latter involves morphisms $G \rightarrow G[-2|a| + 1]$. This is because we are using a different convention for homological shifts: in [EP16], $C[1]_i := C_{i-1}$, while for us, $C[1]_i := C_{i+1}$.

Remark 3.4. If, in the context of Construction 3.2, the maps $\sigma_{\mathbf{a}}$ in the system of higher homotopies are all minimal, then $\text{Sh}(G, \sigma)$ is also minimal. In the setting of Proposition 3.3(b), it is not always possible to choose a minimal system of higher homotopies; that is, minimal free resolutions of complete intersections are not always Shamash resolutions. However, the minimal free resolution of the residue field of a complete intersection is always a Shamash resolution [Gul68, Tat57].

Example 3.5. Let k be a field, $S = k[x, y, z, w]$, $f_1 = xz$, and $f_2 = yw$. The minimal S -free resolution of $S/(x, y)$ is the Koszul complex

$$G = (S \xleftarrow{\begin{pmatrix} x & y \end{pmatrix}} S^2 \xleftarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} S).$$

A system of higher homotopies on G is given as follows. Take σ_{e_1} (resp. σ_{e_2}) to be the nullhomotopy of multiplication by xz (resp. yw) given by the maps $\begin{pmatrix} z \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & z \end{pmatrix}$ (resp. $\begin{pmatrix} 0 \\ w \end{pmatrix}$ and $\begin{pmatrix} -w & 0 \end{pmatrix}$). For degree reasons, all the other $\sigma_{\mathbf{a}}$ must be 0. One easily checks that these maps form a system of higher homotopies. The complex $\text{Sh}(G, \sigma)$ is given by R^{i+1} in homological degree i :

$$(3.6) \quad R \xleftarrow{\begin{pmatrix} x & y \end{pmatrix}} R^2 \xleftarrow{\begin{pmatrix} -y & z & 0 \\ x & 0 & w \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} 0 & z & -w & 0 \\ x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^4 \leftarrow \dots$$

Since the maps $\sigma_{\mathbf{a}}$ in our system of higher homotopies are minimal, this complex is the minimal R -free resolution of $S/(x, y)$.

We now give an alternative take on the Shamash construction in the language of matrix factorizations, as this perspective will be useful for our study of ideals of minors in Shamash resolutions in §3.2. Our approach relies heavily on work of Martin [Mar21].

Definition 3.7. A *graded matrix factorization* of $\tilde{f} = f_1 t_1 + \dots + f_c t_c \in \tilde{S}$ is a pair (F, d) , where F is a finitely generated, graded, free \tilde{S} -module, and d is a degree -1 endomorphism of F such that $d^2 = \tilde{f} \cdot \text{id}_F$. A *morphism* of graded matrix factorizations is a degree 0 endomorphism of F that commutes with d , and a *homotopy* of such morphisms f and g is a degree 1 endomorphism h of F such that $f - g = dh + hd$. Let $\text{mf}(\tilde{S}, \tilde{f})$ denote the category of graded matrix factorizations of \tilde{f} and $\text{hmf}(\tilde{S}, \tilde{f})$ the quotient of $\text{mf}(\tilde{S}, \tilde{f})$ given by modding out null-homotopic morphisms.

Example 3.8. Let G be a bounded complex of finitely generated free S -modules equipped with a system of higher homotopies σ for f_1, \dots, f_c . Letting $F = \bigoplus_{i \in \mathbb{Z}} \tilde{S}(-i) \otimes_S G_i$ and $d = \sum t^{\mathbf{a}} \otimes \sigma_{\mathbf{a}}$ gives an object (F, d) in $\text{mf}(\tilde{S}, \tilde{f})$.

For instance, applying this to the complex in Example 3.5, we get $F = \tilde{S}(-1)^2 \oplus \tilde{S} \oplus \tilde{S}(-2)$ (the terms are out of numerical order so that the components generated in odd and even degrees are separated) and $d = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$, where $A = \begin{pmatrix} zt_1 & -y \\ wt_2 & x \end{pmatrix}$ and

$$B = \begin{pmatrix} x & y \\ -wt_2 & zt_1 \end{pmatrix}.$$

By a result of Burke-Stevenson [BS15, Theorem 7.5], when S is regular of finite Krull dimension and f_1, \dots, f_c is a regular sequence, there is an equivalence of categories

$$\mathrm{hmf}(\tilde{S}, \tilde{f}) \xrightarrow{\sim} \mathrm{D}^b(R)$$

(see also [BW15, Theorem 1] for a related result). Burke-Stevenson's proof makes crucial use of a theorem of Orlov concerning singularity categories of graded Gorenstein algebras [Orl09, Theorem 2.5]. Martin gives an alternative formulation of this equivalence in his thesis [Mar21] via a version of the BGG correspondence; in fact, Martin's result is a generalization of Burke-Stevenson's, since it does not require S to be regular of finite Krull dimension. As this equivalence is not expressed in [Mar21] in quite the way we need it, we reformulate Martin's result in the following way.

We consider the functor $\mathrm{mf}(\tilde{S}, \tilde{f}) \rightarrow \mathrm{D}^b(R)$ that sends a matrix factorization (F, d) to the complex $(F \otimes_{\tilde{S}} R, d)$ of R -modules; the component in homological degree i of $F \otimes_{\tilde{S}} R$ is $F_i \otimes_S R$. This functor respects homotopy and therefore induces a functor

$$(3.9) \quad \mathrm{hmf}(\tilde{S}, \tilde{f}) \rightarrow \mathrm{D}^b(R).$$

Given $(F, d) \in \mathrm{mf}(\tilde{S}, \tilde{f})$, let F^\vee denote the S -linear dual of F . We define another functor

$$\Phi : \mathrm{hmf}(\tilde{S}, \tilde{f}) \rightarrow \mathrm{D}^b(R)^{\mathrm{op}}$$

that sends an object (F, d) to the complex $F^\vee \otimes_S R$ with differential $d^T \otimes \mathrm{id}$. That is, Φ is given by dualizing over S and tensoring with R ; equivalently, Φ is given by applying (3.9) and dualizing over R .

The following theorem is a nearly immediate consequence of work of Martin [Mar21]:

Theorem 3.10 ([Mar21]). *Assume f_1, \dots, f_c form a regular sequence in S .*

- (1) *The functor (3.9) is an equivalence.*
- (2) *If S is Gorenstein, then Φ is an equivalence.*

Proof. (2) is immediate from (1); let us prove (1). Let K be the Koszul complex on f_1, \dots, f_c , with differential denoted d_K . We view K as an exterior algebra $\bigwedge_S(e_1, \dots, e_c)$. Let $X \in \mathrm{mf}(\tilde{S}, \tilde{f})$ be the object with underlying module $\bigoplus_{i=0}^c \tilde{S}(-i) \otimes_S K_i$ and differential $d_X = (1 \otimes d_K) + \lambda$, where λ is given by left multiplication by the element $\sum_{i=1}^c t_i \otimes e_i$. By [Mar21, Theorem 5.1], there is an equivalence $\mathrm{hmf}(\tilde{S}, \tilde{f}) \xrightarrow{\sim} \mathrm{D}^b(K)$ that sends a matrix factorization (F, d_F) to the complex $\mathrm{Hom}_{\tilde{S}}(X, F)$ with differential $\alpha d_X - (-1)^{\deg(\alpha)} d_F \alpha$. The differential squares to 0 since d_X and d_F both square to multiplication by \tilde{f} . Since f_1, \dots, f_c is a regular sequence, extension of scalars along the quasi-isomorphism $K \xrightarrow{\sim} R$ induces an equivalence $\mathrm{D}^b(K) \xrightarrow{\sim} \mathrm{D}^b(R)$; composing, we obtain an equivalence $\mathrm{hmf}(\tilde{S}, \tilde{f}) \xrightarrow{\sim} \mathrm{D}^b(R)$. Tracing through the formulas, one sees that this equivalence sends (F, d_F) to the complex $(F \otimes_{\tilde{S}} R, d)$. \blacksquare

The connection between matrix factorizations and Shamash resolutions is provided by the following result. Before stating it, we note that, given a complex G of free S -modules and a system of higher homotopies $\sigma_{\mathbf{a}} : G \rightarrow G[2|\mathbf{a}| - 1]$ for f_1, \dots, f_c , the maps $\sigma_{\mathbf{a}}^\vee : G^\vee \rightarrow G^\vee[2|\mathbf{a}| - 1]$ determine a system of higher homotopies for f_1, \dots, f_c on G^\vee .

Proposition 3.11. *Let G , σ , and (F, d) be as in Example 3.8. Applying Φ to (F, d) gives the Shamash construction $\mathrm{Sh}(G^\vee, \sigma^\vee)$.*

Proof. This is a straightforward calculation. \blacksquare

Motivated by Proposition 3.11, we make the following

Definition 3.12. We call a complex of R -modules a *Shamash complex* if it is isomorphic (as a complex, not just in $D^b(R)$) to $\Phi(F, d)$ for some object $(F, d) \in \text{mf}(\tilde{S}, \tilde{f})$.

3.2. Periodicity of ideals of minors in Shamash complexes. We now prove that ideals of minors in Shamash complexes are eventually 2-periodic (Theorem 3.16). Moreover, we give an explicit lower bound on the homological index where the periodicity begins. For ideals of 1×1 minors of Shamash resolutions of modules over complete intersections, the periodicity begins after $\dim(S)+1$ steps, mirroring the starting point for the periodicity of minimal free resolutions of modules over hypersurfaces (Corollary 3.21).

We begin with a key technical lemma. We will need the following

Notation 3.13. Let F be a graded free \tilde{S} -module. Choose a basis of F , so that we may write $F = \bigoplus \tilde{S}(-d_i)$. Given $i \in \mathbb{Z}$, the graded component F_i has a S -basis B_i indexed by monomials in \tilde{S} ; notice that the B_i are determined canonically by our choice of basis for F . We set

$$\theta_F(i) := \sup\{j : t_1^j \text{ divides an element of } B_i\}.$$

If F is bounded above, i.e. $F_\ell = 0$ for all $\ell \gg 0$, then $\theta_F(i) < \infty$ for all i .

Lemma 3.14. Assume $c > 1$. Let G and G' be graded free \tilde{S} -modules that are bounded above and equipped with bases; moreover, assume G is generated entirely in even (resp. odd) degree, and G' is generated entirely in odd (resp. even) degree. Let $A : G \rightarrow G'$ be an \tilde{S} -linear map that is homogeneous of degree -1 . Denote by $A_i : G_i \rightarrow G'_{i-1}$ the S -linear map induced by A . Suppose there is some even (resp. odd) integer $n \ll 0$ such that $\mathbf{I}^1(A_i) = \mathbf{I}^1(A_{i-2})$ for all $i \leq n$. Fix $r \geq 1$; for any even (resp. odd) integer ℓ such that $\ell \leq -2(r-1)(\max\{\theta_G(n), \theta_{G'}(n-1)\} + 1) + n$ (see Notation 3.13), we have $\mathbf{I}^r(A_\ell) = \mathbf{I}^1(A_n)^r$.

Proof. It is immediate that $\mathbf{I}^r(A_\ell) \subseteq \mathbf{I}^1(A_\ell)^r = \mathbf{I}^1(A_n)^r$; we now prove the opposite containment. We view each $A_i : G_i \rightarrow G'_{i-1}$ as a matrix with entries in S via the monomial bases of G_i and G'_{i-1} induced by our chosen bases of G and G' . Let x_1, \dots, x_r be entries of A_n ; it suffices to exhibit an $r \times r$ submatrix of A_ℓ with determinant $x_1 \cdots x_r$. Each of our entries x_k of A_n corresponds to monomial basis elements m_k and n_k of G_n and G'_{n-1} , respectively. Let $s := (n - \ell)/2$, $p := \max\{\theta_G(n), \theta_{G'}(n-1)\} + 1$, and $g_i := t_1^{s-(i-1)p} t_2^{(i-1)p} \in \tilde{S}$ for $1 \leq i \leq r$. Notice that each $s - (i-1)p$ is indeed nonnegative, by our assumption on ℓ . Observe also that the degree of each g_i is $-(n - \ell)$.

By our choice of p , we have $m_i g_i = m_j g_j$ if and only if $i = j$. Indeed, if $m_i g_i = m_j g_j$ for $i > j$, then $m_i t_1^{s-(i-1)p} = m_j t_2^{(i-j)p}$; this means t_1^p divides m_j , which is impossible. Similarly, $n_i g_i = n_j g_j$ implies $i = j$. We therefore have the following $r \times r$ submatrix P of A_ℓ :

$$\begin{array}{ccccc} & m_1 g_1 & m_2 g_2 & \cdots & m_{r-1} g_{r-1} & m_r g_r \\ \begin{array}{c} n_1 g_1 \\ n_2 g_2 \\ \vdots \\ n_{r-1} g_{r-1} \\ n_r g_r \end{array} & \left(\begin{array}{ccccc} p_{1,1} & p_{1,2} & \cdots & p_{1,r-1} & p_{1,r} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,r-1} & p_{2,r} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_{r-1,1} & p_{r-1,2} & \cdots & p_{r-1,r-1} & p_{r-1,r} \\ p_{r,1} & p_{r,2} & \cdots & p_{r,r-1} & p_{r,r} \end{array} \right) \end{array}.$$

Since A is \tilde{S} -linear, we have $p_{i,i} = x_i$ for all i . It thus suffices to show that P is lower triangular (in fact, we will see that it is diagonal). We have $A(m_j g_j) = g_j A(m_j)$, which is a linear combination of the form $\sum_k a_k g_j b_k$, where each a_k is in S , and each b_k is a basis element of G'_{n-1} . It is impossible that $g_j b_k = g_i n_i$ for some $i < j$, since $t_1^{s-(i-1)p}$ divides the right side but not the left side. It follows that $p_{i,j} = 0$ for all $i < j$ (in fact, a similar argument shows $p_{i,j} = 0$ for all $i > j$ as well, i.e. P is diagonal). \blacksquare

Proposition 3.15. *Assume $c > 1$. Let $(F, d) \in \text{mf}(\tilde{S}, \tilde{f})$, and fix $r \geq 1$. Write $d_j : F_j \rightarrow F_{j-1}$ for the degree j component of d . Let N_0 (resp. N_1) be the largest even (resp. odd) degree in which F is generated and n_0 (resp. n_1) the smallest even (resp. odd) such degree. Set*

$$M_i := -(r-1)(N_i - n_i + 4) + n_i, \quad i = 0, 1.$$

- (1) *If $\ell \in \mathbb{Z}$ is even, and $\ell \leq M_0$, then $\mathbf{I}^r(d_\ell) = \mathbf{I}^1(d_{n_0})^r$.*
- (2) *If $\ell \in \mathbb{Z}$ is odd, and $\ell \leq M_1$, then $\mathbf{I}^r(d_\ell) = \mathbf{I}^1(d_{n_1})^r$.*

Proof. Let us prove (1); the proof of (2) is the same. Choose a basis of F , so that each graded component F_i may be equipped with a monomial basis over S . Let F_{even} (resp. F_{odd}) be the submodule of F given by even (resp. odd) degree elements. We will apply Lemma 3.14 to the \tilde{S} -linear map $d : F_{\text{even}} \rightarrow F_{\text{odd}}$.

By \tilde{S} -linearity, it is clear that $\mathbf{I}^1(d_i) = \mathbf{I}^1(d_{i-2})$ for all $i \leq n_0$. Notice that $\theta_{F_{\text{even}}}(n_0) = \frac{N_0 - n_0}{2}$. Let us now show $\theta_{F_{\text{odd}}}(n_0 - 1) \leq \theta_{F_{\text{even}}}(n_0) + 1$. If $\theta_{F_{\text{odd}}}(n_0 - 1) = 0$, we are done, so suppose we have a monomial basis element $m \in F_{n_0-1}$ of the form $m = t_1^q m'$ for some $q \geq 1$ and monomial m' . Write $d(m') = a_1 m_1 + \cdots + a_t m_t$, where $a_i \in S$ and each m_i is an element of the monomial basis of F_{n_0+2q-2} . We have $t_1^{q-1} m_i \in F_{n_0}$ for all i , and so $q-1 \leq \theta_{F_{\text{even}}}(n_0)$, which implies the inequality we seek.

Putting everything together, we have:

$$\begin{aligned} \ell &\leq -(r-1)(N_0 - n_0 + 4) + n_0 \\ &= -2(r-1)(\theta_{F_{\text{even}}}(n_0) + 2) + n_0 \\ &\leq -2(r-1)(\max\{\theta_{F_{\text{even}}}(n_0), \theta_{F_{\text{odd}}}(n_0 - 1)\} + 1) + n_0. \end{aligned}$$

Applying Lemma 3.14 finishes the proof. \blacksquare

Theorem 3.16. *Assume $c > 1$, and let $(F, d) \in \text{mf}(\tilde{S}, \tilde{f})$. Let n_i and M_i for $i = 0, 1$ be as in Proposition 3.15. For all $r \geq 1$ and $\ell \in \mathbb{Z}$, we have:*

- (1) *If ℓ is even, and $\ell \geq -M_1 + 1$, then $\mathbf{I}_\ell^r(\Phi(F, d)) = \mathbf{I}_{-n_1+1}^1(\Phi(F, d))^r$.*
- (2) *If ℓ is odd, and $\ell \geq -M_0 + 1$, then $\mathbf{I}_\ell^r(\Phi(F, d)) = \mathbf{I}_{-n_0+1}^1(\Phi(F, d))^r$.*

In particular, the ideals of minors in Shamash complexes are eventually 2-periodic.

Proof. Let us prove (1); the proof of (2) is the same. We have $-\ell + 1 \leq M_1$, so Proposition 3.15(2) implies that $\mathbf{I}^r(d_{-\ell+1}) = \mathbf{I}^1(d_{n_1})^r$. Dualizing, we get $\mathbf{I}^r(d_\ell^T) = \mathbf{I}^1(d_{-n_1+1}^T)^r$. This equality still holds upon tensoring the arguments with R ; the result follows. \blacksquare

Example 3.17. Let G and σ be as in Example 3.5, and let (F, d) be the matrix factorization obtained from (G^\vee, σ^\vee) as in Example 3.8. By Proposition 3.11, the Shamash resolution C constructed in Example 3.5 is given by $\Phi(F, d)$. Let us now analyze the minors of C via Theorem 3.16.

Fix $r \geq 1$. We have $n_0 = -2$, $N_0 = 0$, and $n_1 = -1 = N_1$; thus, $M_0 = -6(r-1) - 2$, and $M_1 = -4(r-1) - 1$. Theorem 3.16 therefore implies that the ideals of minors in odd

(resp. even) homological degrees in C become 2-periodic starting—at most—at position $6(r-1) + 3$ (resp. $4(r-1) + 2$). More specifically: the ideals of $r \times r$ minors in odd (resp. even) homological degrees stabilize to $(x, y, z, w)^r$ at position $6(r-1) + 3$ (resp. $4(r-1) + 2$). When $r = 1$, these bounds are sharp. The bounds are not sharp for higher ranks; a reason for this is that the bounds for periodicity we obtain in Proposition 3.15 are for ideals of minors in the ring S ; modding out by f_1, \dots, f_c often causes additional vanishing of minors.

Assume now that S is a regular local ring, and f_1, \dots, f_c is a regular sequence in S . Let M be a finitely generated R -module and G its minimal S -free resolution. By Proposition 3.3, we may equip G with a system of higher homotopies σ for f_1, \dots, f_c . As discussed above Proposition 3.11, dualizing yields a system of higher homotopies σ^\vee on G^\vee . By Proposition 3.11, applying the construction in Example 3.8 to (G^\vee, σ^\vee) yields a matrix factorization (F, d) such that $\Phi(F, d)$ is the Shamash resolution $\text{Sh}(G, \sigma)$ of M . The numbers N_0 and n_0 (resp. N_1 and n_1) from Proposition 3.15 in this case are precisely the smallest and largest even (resp. odd) nonvanishing homological degrees of G . In particular, we have $0 \leq -N_i \leq -n_i \leq \text{pd}_S(M)$ for $i = 0, 1$, and so Theorem 3.16 implies:

Theorem 3.18. *Let S be a regular local ring, f_1, \dots, f_c a regular sequence in S with $c > 1$, M a finitely generated $R = S/(f_1, \dots, f_c)$ -module, k the residue field of R , and $\text{Sh}(G, \sigma)$ a Shamash resolution of M . Fix $r \geq 1$. We have:*

- (1) $\mathbf{I}_\ell^r(\text{Sh}(G, \sigma)) = \mathbf{I}_{\ell+2}^r(\text{Sh}(G, \sigma))$ for all $\ell \geq r \cdot \text{pd}_S(M) + 4(r-1) + 1$.
- (2) $\mathbf{I}_\ell^r(k) = \mathbf{I}_{\ell+2}^r(k)$ for all $\ell \geq r \cdot c + 4(r-1) + 1$.

Proof. Part (1) is a consequence of Theorem 3.16 and the inequalities $0 \leq -N_i \leq -n_i \leq \text{pd}_S(M)$. Part (2) follows since the minimal R -free resolution of k is given by the Shamash construction [Gul68, Tat57]. ■

Remark 3.19. The bounds obtained in Theorem 3.18 depend (linearly) on the rank of the minors; we note that such dependence is necessary. For instance, in the setting of the Theorem, let F be the minimal free resolution of the residue field k . Notice that $\mathbf{I}_\ell^r(k) = 0$ when $\text{rank } F_\ell < r$. However, by Theorem 2.15(2), for any $r > 0$, there exists $\ell \gg 0$ such that $\mathbf{I}_\ell^r(k) \neq 0$. Thus, the point at which the ideals $\mathbf{I}_\ell^r(k)$ stabilize depends on r .

Even if the higher rank minors do not vanish, they may not stabilize when the lower rank minors do. For example, in Example 3.5, we have $\mathbf{I}_i^1(\text{Sh}(G, \sigma)) = \mathbf{I}_2^1(\text{Sh}(G, \sigma))$ for all even integers $i \geq 2$, but $0 \neq \mathbf{I}_2^2(\text{Sh}(G, \sigma)) \neq \mathbf{I}_4^2(\text{Sh}(G, \sigma))$.

As a consequence, we obtain a uniform lower bound (i.e. independent of the choice of module) for where the periodicity of the ideals of minors of a Shamash resolution begins:

Corollary 3.20. *In the setting of Theorem 3.18, we have $\mathbf{I}_\ell^r(\text{Sh}(G, \sigma)) = \mathbf{I}_{\ell+2}^r(\text{Sh}(G, \sigma))$ for $\ell \geq r \cdot \dim(S) + 4(r-1) + 1$.*

A famous result of Eisenbud [Eis80, Theorem 6.1(i)] states that minimal free resolutions of finitely generated modules over local hypersurface rings become 2-periodic after at most $\dim(Q) + 1$ steps, where Q is the ambient regular ring. Since minimal free resolutions over hypersurfaces are given by the Shamash construction, the following Corollary extends this result, on the level of rank 1 minors, to complete intersections:

Corollary 3.21. *Let S be a regular local ring, f_1, \dots, f_c a regular sequence in S , M a finitely generated $R = S/(f_1, \dots, f_c)$ -module, and $\text{Sh}(G, \sigma)$ a Shamash resolution of M . We have $\mathbf{I}_\ell^1(\text{Sh}(G, \sigma)) = \mathbf{I}_{\ell+2}^1(\text{Sh}(G, \sigma))$ for all $\ell \geq \dim(S) + 1$.*

Proof. Immediate from Corollary 3.20 and [Eis80, Theorem 6.1(i)]. \blacksquare

We conclude this subsection with another periodicity result for Shamash resolutions that will be useful in §3.3.

Proposition 3.22. *Let S be a commutative ring, $f \in S$, $R = S/(f)$, G a complex of finitely generated free R -modules, σ a system of higher homotopies on G for f , and $F = \text{Sh}(G, \sigma)$. Assume R is Noetherian. For $r > 0$, we have $\mathbf{I}_k^r(F) = \mathbf{I}_{k+2}^r(F)$ for $k \gg 0$.*

Proof. Let $\tilde{S} = S[t]$, with t a degree -2 variable. We have $F = \tilde{D} \otimes_S G \otimes_S R$, where \tilde{D} is the divided power algebra $\text{Hom}_S(\tilde{S}, S)$, and the differential d on F is given by $\sum_{i \geq 0} t^i \otimes \sigma_i \otimes \text{id}$. Let $y \in \tilde{D}$ denote the dual of t . For all $k \in \mathbb{Z}$, F_{k+2} decomposes as $yF_k \oplus G_{k+2}$. The differential d_{k+2} is a square matrix with respect to these decompositions, with top-left entry $d_k: yF_k \rightarrow yF_{k-1}$. Thus, $\mathbf{I}_k^r(F) \subseteq \mathbf{I}_{k+2}^r(F)$ for all $k \in \mathbb{Z}$. The Noetherianity of R implies the result. \blacksquare

3.3. Periodicity of ideals of minors of minimal free resolutions over complete intersections. Corollary 3.20 gives a tidy description of the periodic behavior of the ideals of $r \times r$ minors of a Shamash resolution for a module M over a complete intersection: there is a lower bound—independent of M —for when the periodicity begins, and this bound depends linearly on r . Moreover, when $r = 1$, this phenomenon is an extension of the familiar periodic behavior of minimal free resolutions over hypersurfaces (Corollary 3.21).

One might expect that the situation is the same for *minimal* free resolutions over complete intersections, but the story turns out to be more complicated. On one hand, we prove in Theorem 3.23 that the ideals of minors of minimal free resolutions over complete intersections are indeed eventually 2-periodic. On the other hand, there is no bound on where the periodicity starts that is independent of the module (Example 3.26); in fact, we do not obtain any lower bound at all on where periodicity begins.

Our main tool for establishing the eventual periodicity of ideals of minors of minimal free resolutions over complete intersections (Theorem 3.23) is Eisenbud-Peeva's theory of higher matrix factorizations, as introduced in their book [EP16]. The definition of a higher matrix factorization, and the sense in which higher matrix factorizations govern the asymptotic behavior of minimal free resolutions over complete intersections, is quite intricate; but the details will not be necessary for this paper. The key point is that, as discussed in [EP16, Construction 5.1.1], the minimal free resolution of a higher matrix factorization module over a complete intersection of codimension c is given by a Shamash construction associated to a single element: this allows us to apply Proposition 3.22.

With this in mind, let us now prove our periodicity result for ideals of minors of minimal free resolutions over complete intersections. In fact, we work in slightly greater generality:

Theorem 3.23. *Let S be a local ring, $f_1, \dots, f_c \in S$ a regular sequence, R the quotient $S/(f_1, \dots, f_c)$, and M a finitely generated R -module such that $\text{pd}_S(M) < \infty$. Given $r > 0$, we have $\mathbf{I}_k^r(M) = \mathbf{I}_{k+2}^r(M)$ for $k \gg 0$.*

Proof. By [EP16, Corollary 6.4.3], a sufficiently high syzygy of M is a minimal higher matrix factorization (HMF) module with respect to some choice of generators f'_1, \dots, f'_c of the ideal (f_1, \dots, f_c) . Thus, truncating and replacing each f_i with f'_i , we may assume M is a minimal HMF module for f_1, \dots, f_c . By [EP16, Construction 5.1.1], the minimal

free resolution of M is given by applying a Shamash construction on f_c to the minimal free resolution of M over $S/(f_1, \dots, f_{c-1})$. Applying Proposition 3.22 finishes the proof. ■

Remark 3.24. By the faithful flatness of completion, Theorem 3.23 holds for modules over a ring R whose completion at its maximal ideal is isomorphic to a regular local ring modulo a regular sequence. This hypothesis is indeed weaker than assuming R itself is a quotient of a regular local ring by a regular sequence: see [HJ12].

Additionally, the statement of Theorem 3.23 holds when S is a regular local ring and M is a bounded below complex of finitely generated free R -modules with homology concentrated only in finitely many degrees. Indeed, a smart truncation of the minimal free resolution of such a complex is a minimal free resolution of a module, and so one immediately reduces to the setting of Theorem 3.23.

Remark 3.25. The proofs of Proposition 3.22 and Theorem 3.23 show a bit more. Let M be a finitely generated module over a local complete intersection. As discussed in the proof of Theorem 3.23, it follows from results of [EP16] that we may choose $i \gg 0$ such that $\text{Syz}_i(M)$ is a higher matrix factorization module associated to some regular sequence and is therefore resolved by a Shamash construction on a single element. It therefore follows from the proof of Proposition 3.22 that $\mathbf{I}_j^r(M) \subseteq \mathbf{I}_{j+2}^r(M)$ for all $r \geq 1$ and $j \geq i$.

The following example shows that, in the setup of Theorem 3.23, there is no bound on where the periodicity begins that is independent of M .

Example 3.26. Let (S, \mathfrak{m}, k) be a regular local ring, $R = S/I$ an Artinian complete intersection of codimension at least two, and F the minimal R -free resolution of k . Let r be an integer greater than $\beta_3(k)$, the third Betti number of k . We first show that there is some $\ell > 0$ such that $\mathbf{I}_\ell^r(k) \neq \mathbf{I}_{\ell+2}^r(k)$. Notice that $\mathbf{I}_2^r(k) = \mathbf{I}_1^r(k) = 0$. On the other hand, by Theorem 2.15, $\mathbf{I}^{\text{stab}}(k, r) = \mathfrak{m}^r$, and so there is some i such that $\mathbf{I}_i^r(k) \neq 0$. The existence of $\ell > 0$ with the desired property follows.

Let F^\vee denote the complex $\text{Hom}_R(F, R)$. Since R is self-injective, F^\vee is a projective co-resolution of $k^\vee \cong k$. The mapping cone C of the composition $F \rightarrow k \cong k^\vee \hookrightarrow F^\vee$ is therefore an exact, minimal complex of finitely generated free R -modules. (The complex C is called a *complete resolution* of k [Buc21].) Choosing ℓ as in the preceding paragraph, we see that taking a smart truncation of C in arbitrarily negative degree gives a minimal free resolution of an R -module whose ideals of minors are not equal in arbitrarily large even/odd degrees.

Let M be a finitely generated module over a local ring. The j^{th} *Fitting ideal* of M in homological degree i is the ideal $\mathbf{F}_i^j(M) := \mathbf{I}_i^{\beta_{i-1}-j+1}(M)$, where β_ℓ denotes the ℓ^{th} Betti number of M . When $i = 1$, this recovers the definition of the j^{th} Fitting ideal of M in [BE77]. Fitting ideals in large homological degrees over complete intersections satisfy a containment, up to radical, in the opposite direction of the containment in Remark 3.25:

Proposition 3.27. *Let S be a regular local ring with infinite residue field, $f_1, \dots, f_c \in S$ a regular sequence, and $R = S/(f_1, \dots, f_c)$. We have $\text{rad}(\mathbf{F}_{i+2}^j(M)) \subseteq \text{rad}(\mathbf{F}_i^j(M))$ for all $j \geq 1$ and $i \gg 0$, where $\text{rad}(-)$ denotes the radical.*

Proof. We have a surjection $\text{Syz}_{i+2}(M) \twoheadrightarrow \text{Syz}_i(M)$ for $i \gg 0$ by [Eis80, Proposition 1.1 and Theorem 3.1]. These maps extend to surjections $\bigwedge^j \text{Syz}_{i+2}(M) \twoheadrightarrow \bigwedge^j \text{Syz}_i(M)$ for all $j \geq 1$ and $i \gg 0$. By [BE77, Corollary 1.3], the ideals $\text{ann}(\bigwedge^j \text{Syz}_i(M))$ and $\mathbf{F}_{i+1}^j(M)$

agree up to radical for all $i \geq 0$ and $j \geq 1$. Thus, for $i \gg 0$ and $j \geq 1$, we have:

$$\text{rad}(\mathbf{F}_{i+2}^j(M)) = \text{rad}\left(\text{ann}\left(\bigwedge^j \text{Syz}_{i+1}(M)\right)\right) \subseteq \text{rad}\left(\text{ann}\left(\bigwedge^j \text{Syz}_{i-1}(M)\right)\right) = \text{rad}(\mathbf{F}_i^j(M)).$$

■

4. PERIODICITY OF IDEALS OF MINORS OVER GOLOD RINGS

Our goal in this section is to prove that minimal free resolutions of finitely generated modules over Golod rings have eventually 1- or 2-periodic ideals of minors. We prove this via an explicit calculation, relying on results of Burke [Bur15] and Lescot [Les90].

We fix some notation for this section: let S be a commutative ring, R a cyclic S -algebra, and M a finitely generated R -module. We choose S -free resolutions $(A, d^A) \xrightarrow{\sim} R$ and $(G, d^G) \xrightarrow{\sim} M$, and we assume $A_0 = S$. Let A_+ denote the brutal truncation $\bigoplus_{i \geq 1} A_i$.

4.1. The bar resolution. We now (roughly) recall the notions of A_∞ -algebra and A_∞ -module structures. A recollection of the explicit definitions won't be necessary for this paper; we refer the reader to [Bur15] for the details.

Definition 4.1. An A_∞ S -algebra structure on A is given by degree -1 S -linear maps

$$m_n : A_+[-1]^{\otimes n} \rightarrow A_+[-1]$$

for $n \geq 1$ such that $m_1 = d^{A[-1]}|_{A_+[-1]}$ and satisfying the sequence of relations in [Bur15, Definition 3.1].² Assuming that A is equipped with an A_∞ -algebra structure, an A_∞ A -module structure on G is given by degree -1 S -linear maps

$$m_n^G : A_+[-1]^{\otimes n-1} \otimes_S G \rightarrow G$$

for $n \geq 1$ such that $m_1^G = d^G$ and satisfying the relations in [Bur15, Definition 3.3].

Proposition 4.2 ([Bur15] Proposition 3.6). *There exists an A_∞ S -algebra structure on A and an A_∞ A -module structure on G .*

Let us fix an A_∞ S -module structure on A and an A_∞ A -module structure on G .

Construction 4.3 ([Bur15] Definition 3.12). Given an element $x_1 \otimes \cdots \otimes x_n \in A_+[-1]^{\otimes n}$, we denote it as $[x_1 | \cdots | x_n]$. The *bar complex associated to A and G* is the complex of free R -modules with underlying graded module $\bigoplus_{n \geq 0} A_+[-1]^{\otimes n} \otimes_S G \otimes_S R$ and differential given, on the summand $A_+[-1]^{\otimes n} \otimes_S G \otimes_S R$, by the formula:

$$d = \left(\sum_{i=1}^n \sum_{j=0}^{i-1} b_{i,j} \right) + \left(\sum_{i=1}^{n+1} b'_i \right), \text{ where, setting } \underline{x} = (-1)^{|x|+1}x \text{ for } x \in A_+[-1], \text{ we have}^3$$

$$b_{i,j}([x_1 | \cdots | x_n] \otimes g \otimes r) = [\underline{x}_1 | \cdots | \underline{x}_j | m_i([x_{j+1} | \cdots | x_{j+i}]) | x_{j+i+1} | \cdots | x_n] \otimes g \otimes r, \text{ and}$$

$$b'_i([x_1 | \cdots | x_n] \otimes g \otimes r) = [\underline{x}_1 | \cdots | \underline{x}_{n-i+1}] \otimes m_i^G([x_{n-i+2} | \cdots | x_n] \otimes g) \otimes r.$$

Theorem 4.4 ([Bur15] Theorem 3.13). *The bar construction associated to A and G is an R -free resolution of M .*

²We point out that our shifting convention is different from that of [Bur15]: we assume $C[1]_j = C_{j+1}$, while Burke's convention is $C[1]_j = C_{j-1}$.

³In [Bur15, Definition 3.12] the sum over the maps b'_i ranges from 0 to n . This is an error; the correct limits on the summation are 1 and $n+1$.

4.2. Golod rings and modules. We recall here some background on Golod rings and modules; we refer the reader to [Avr86] for a comprehensive introduction to this topic. Let us assume now that S is a regular local ring with maximal ideal \mathfrak{m} and residue field k . Let $\varphi : S \rightarrow R$ denote the canonical map. We assume that our S -free resolutions $A \xrightarrow{\sim} R$ and $G \xrightarrow{\sim} M$ are minimal. Given a local ring T and a finitely generated T -module N , let β_N^i denote the i^{th} Betti number of N and $P_N^T(t) = \sum_i \beta_N^i t^i$ its Poincaré series.

Definition 4.5. We say the R -module M is φ -Golod if we have an equality

$$(4.6) \quad P_M^R(t) = \frac{P_M^S(t)}{1 - t(P_R^S(t) - 1)}.$$

The ring R is φ -Golod if k is a φ -Golod R -module.

It is known that the notion of φ -Golodness is independent of the choice of presentation φ [Bur15, §6], and so we refer to φ -Golod rings and modules as simply Golod. The bar complex associated to choices of A_∞ -structures on A and G (Construction 4.3) has Betti numbers given by the coefficients of the power series on the right of (4.6), as observed in [Bur15, Proof of Lemma 6.3]. Thus, we always have an inequality $P_M^R(t) \leq \frac{P_M^S(t)}{1 - t(P_R^S(t) - 1)}$ coefficient-wise, and one deduces:

Lemma 4.7 ([Bur15] Lemma 6.3). *Let S be a regular local ring, R a cyclic S -algebra, M a finitely generated R -module, A the minimal S -free resolution of R , and G the minimal S -free resolution of M . A finitely generated R -module M is Golod if and only if the bar construction associated to A and G is minimal with respect to some (equivalently every) A_∞ -structure on A and G .*

We also recall the following result of Lescot:

Theorem 4.8 ([Les90]). *Let S be a regular local ring, R a cyclic S -algebra that is Golod, and M a finitely generated R -module. If $n > \dim(S)$, then the n^{th} syzygy of M is Golod.*

4.3. Periodicity of ideals of minors of minimal free resolutions over Golod rings. Our main result concerning ideals of minors over Golod rings is the following:

Theorem 4.9. *Let S be a regular local ring, R a cyclic S -algebra that is Golod, M a finitely generated R -module, and $r > 0$.*

- (1) *We have $\mathbf{I}_n^r(M) = \mathbf{I}_{n+2}^r(M)$ for $n \gg 0$.*
- (2) *If $\text{pd}_S(R) \geq 2$, we have $\mathbf{I}_n^r(M) = \mathbf{I}_{n+1}^r(M)$ for $n \gg 0$.*

Proof. By Theorem 4.8, we may assume M is Golod. Let A be the minimal S -free resolution of R and G the minimal S -free resolution of M . Applying Proposition 4.2, choose an A_∞ S -algebra structure on A and an A_∞ A -module structure on M . By Theorem 4.4 and Lemma 4.7, the associated bar construction (B, d) (Construction 4.3) is the minimal free resolution of M .

If $A_t = 0$ for all $t > 0$, then $R = S$, and so R is regular local. In this case, the statement of (1) is trivial, and (2) holds vacuously. We therefore may assume that $A_1 \neq 0$.

We now proceed much like the proof of Proposition 3.22. For each $n \geq 0$, we have:

$$(4.10) \quad B_n = \bigoplus_{k \geq 0} \bigoplus_{\substack{\ell_1, \dots, \ell_k \geq 1, m \geq 0, \\ \ell_1 + \dots + \ell_k + m + k = n}} A_{\ell_1} \otimes_S \dots \otimes_S A_{\ell_k} \otimes_S G_m \otimes_S R.$$

Suppose $t > 0$ and $A_t \neq 0$. Fix $n \geq 0$. Notice that $A_t \otimes_S B_n$ is a summand of B_{n+t+1} . Let H_{n+t+1} denote the complement of $A_t \otimes_S B_n$ in the decomposition (4.10) for B_{n+t+1} . The bar differential $d_{n+t+1} : B_{n+t+1} \rightarrow B_{n+t}$ may be realized as a 2×2 matrix

$$(4.11) \quad d_{n+t+1} : (A_t \otimes_S B_n) \oplus H_{n+t+1} \rightarrow (A_t \otimes_S B_{n-1}) \oplus H_{n+t}.$$

Claim. The upper-left entry of the 2×2 matrix (4.11) is $\text{id}_{A_t} \otimes d_n$.

Before we prove the claim, let us explain how it quickly implies the desired results. Assuming the claim, we have $\mathbf{I}_n^r(M) \subseteq \mathbf{I}_{n+t+1}^r(M)$ for all $n \geq 0$. By the Noetherianity of R , we therefore have $\mathbf{I}_n^r(M) = \mathbf{I}_{n+t+1}^r(M)$ for $n \gg 0$; that is, the ideals of minors are eventually $(t+1)$ -periodic. Notice that this holds for all $t > 0$ such that $A_t \neq 0$. Applying this with $t = 1$ proves (1). In the setting of (2), we have both $A_1 \neq 0$ and $A_2 \neq 0$, so the ideals of minors of M are eventually both 2-periodic and 3-periodic, and hence 1-periodic.

Let us now prove the claim. We consider the action of the bar differential on a summand Y of B_{n+t+1} of the form $A_t \otimes_S A_{\ell_2} \otimes_S \cdots \otimes_S A_{\ell_k} \otimes_S G_m \otimes_S R$. Referring to the notation in Construction 4.3: the component $b_{i,j}$ (resp. b'_i) of the bar differential d_{n+t+1} acts on the summand Y via the map $\text{id}_{A_t} \otimes d_n$ whenever $j \neq 0$ (resp. when $i \neq k+1$). For $1 \leq i \leq n$, the component $b_{i,0}$ of d_{n+t+1} maps Y to $A_{t+\ell_2+\cdots+\ell_i+i-2} \otimes_S A_{\ell_i} \otimes_S \cdots \otimes_S A_{\ell_k} \otimes_S G_m \otimes_S R$; since $t + \ell_2 + \cdots + \ell_i + i - 2$ cannot be equal to t , the component $b_{i,0}$ does not contribute to the upper-left entry of the matrix (4.11). Similarly, the component b'_{k+1} maps Y to $G_{n+t} \otimes_S R$, and so b'_{k+1} also does not contribute to the upper-left entry of the matrix (4.11). This proves the claim. \blacksquare

Example 4.12. The assumption in Theorem 4.9(2) that $\text{pd}_S(R) \geq 2$ is necessary. For instance, take $S = k[[x]]$ and $R = S/(x^3)$. The ring R is Golod, but one easily checks that $\mathbf{I}_n^1(k) \neq \mathbf{I}_{n+1}^1(k)$ for all $n \geq 1$.

Remark 4.13. Just as in Theorem 3.23, Theorem 4.9 holds for M a bounded below complex of finitely generated free R -modules with homology concentrated in finitely many degrees, cf. Remark 3.24.

5. COUNTEREXAMPLE TO PERIODICITY OF IDEALS OF MINORS IN MINIMAL FREE RESOLUTIONS BY GASHAROV-PEEVA

We have shown that ideals of minors of minimal free resolutions over local complete intersections and Golod local rings are eventually 2-periodic. Given $n \geq 1$, we now exhibit an Artinian Gorenstein local ring R whose modules do not necessarily have n -periodic ideals of minors. We consider a ring studied by Gasharov-Peeva in [GP90, Proposition 3.1]. Let k be a field, $\alpha \in k$, $S = k[x_1, \dots, x_5]_{(x_1, \dots, x_5)}$, and $I \subseteq S$ the quadric ideal

$$(x_1^2, x_2^2, x_3x_4, x_3x_5, x_4x_5, x_5^2, \alpha x_1x_3 + x_2x_3, x_1x_4 + x_2x_4, x_3^2 - x_2x_5 + \alpha x_1x_5, x_4^2 - x_2x_5 + x_1x_5).$$

The ring $R = S/I$ is clearly Artinian, and it is Gorenstein by [GP90, Proposition 3.1](i).

Proposition 5.1. *If the group of units of k has an element of infinite order, then there is a finitely generated R -module M such that $\mathbf{I}_i^1(M) \neq \mathbf{I}_{i+n}^1(M)$ for any $i \geq 0$ and $n \geq 1$.*

Proof. Let α be an element of the group of units of k with infinite order. For $i \geq 0$, let δ_i denote the matrix $\begin{pmatrix} x_1 & \alpha^i x_3 + x_4 \\ 0 & x_2 \end{pmatrix}$ with entries in R , and let $M = \text{im}(\delta_0)$. By [GP90,

Proposition 3.1](ii), the complex $\left[R^2 \xleftarrow{\delta_1} R^2 \xleftarrow{\delta_2} R^2 \xleftarrow{\delta_3} \cdots \right]$ is the minimal free resolution of M . Observe that $\mathbf{I}_i^1(M) \neq \mathbf{I}_{i+n}^1(M)$ for $i \geq 0$ and $n \geq 1$. \blacksquare

Remark 5.2. Note that the example above is not a counterexample to the weaker version of periodicity suggested in Question 1.1.

6. APPLICATIONS OF THE PERIODICITY THEOREMS

In this final section, we apply our periodicity results (Theorems 3.23 and 4.9) to prove several additional facts about minimal free resolutions over complete intersections and Golod rings. We begin with a proof of Corollary 1.7; let us restate the result here:

Corollary 6.1. *Let S be a regular local ring, I an ideal of S , and M a finitely generated $R = S/I$ -module. Let $\text{Syz}_i(M)$ (resp. β_i) be the i^{th} syzygy (resp. Betti number) of M over R . We have the following:*

- (1) *If R is a complete intersection, then $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong \bigwedge^{\beta_{i+2}} \text{Syz}_{i+2}(M)$ for $i \gg 0$. If, in addition, the minimal free resolution of M is a Shamash resolution, then this isomorphism holds for $i \geq \dim(S) + 1$.*
- (2) *If R is Golod, then $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong \bigwedge^{\beta_{i+2}} \text{Syz}_{i+2}(M)$ for $i \gg 0$. If, in addition, we have $\text{pd}_S(R) \geq 2$, then $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong \bigwedge^{\beta_{i+1}} \text{Syz}_{i+1}(M)$ for $i \gg 0$.*
- (3) *Suppose $\text{depth}(R) = 0$ and that the Burch index of R (see [DE23, Definition 1.2]) is at least two. For all $i \geq 5$, we have $\bigwedge^{\beta_i} \text{Syz}_i(M) \cong k$.*

Proof. Let N be a finitely generated R -module that is minimally generated by μ elements. Its top exterior power $\bigwedge^\mu N$ is cyclic and hence isomorphic to $R/\text{ann}(\bigwedge^\mu N)$. By [BE77, Corollary 1.3], we have $\bigwedge^\mu N \cong R/\mathbf{I}^1(\varphi)$, where φ is a minimal presentation matrix of N . Now apply Corollary 3.21, Theorem 3.23, Theorem 4.9, and [DE23, Theorem 0.1]. \blacksquare

Next, we use Theorems 3.23 and 4.9 to relate boundedness of Betti numbers over complete intersections and Golod rings to the vanishing of stable ideals of minors:

Theorem 6.2. *Let R be a local ring and M a finitely generated R -module with rank. Denote by $\text{Syz}_i(M)$ (resp. β_i) the i^{th} syzygy (resp. Betti number) of M over R .*

- (1) *If $\mathbf{I}^{\text{tot}}(\text{Syz}_n(M), r) = 0$ for some $r > 0$ and $n \geq 0$, then there exists $N \gg 0$ such that $\beta_i < N$ for all $i \geq 0$.*
- (2) *Suppose R is a complete intersection or Golod ring. We have $\mathbf{I}^{\text{stab}}(M, r) = 0$ for some $r > 0$ if and only if there exists $N \gg 0$ such that $\beta_i < N$ for all $i \geq 0$.*

Proof. Since M has rank, the modules $\text{Syz}_i(M)$ have rank for all $i \geq 0$. Let T be the total ring of fractions of R , and denote by G_i the free T -module $\text{Syz}_i(M) \otimes_R T$ for $i \geq 0$. Let (F, d) be the minimal free resolution of M . For each i , we have a split short exact sequence of T -modules $0 \rightarrow G_{i+1} \rightarrow F_i \otimes_R T \rightarrow G_i \rightarrow 0$. Notice that the rank of each d_i (i.e. the size of the largest nonvanishing minor of d_i) is equal to the rank of G_i . We conclude that $\beta_i = \text{rank}(d_i) + \text{rank}(d_{i+1})$ for all $i > 0$. Now, suppose the Betti numbers of M are unbounded. Choose $i > n$ such that $\beta_i \geq 2r - 1$. We have $\mathbf{I}_\ell^r(M) = 0$ for $\ell \geq n$ by assumption, and so $\text{rank}(d_\ell) < r$ for all $\ell \geq n$. Thus, $\beta_i = \text{rank}(d_i) + \text{rank}(d_{i+1}) \leq 2r - 2$, a contradiction. This proves (1). Part (2) follows from (1), Theorem 3.23, and Theorem 4.9. \blacksquare

We now use Theorems 3.23 and 4.9 to establish some constraints on stable ideals of minors over complete intersections and Golod rings:

Theorem 6.3. *Let R be a local complete intersection or Golod ring and M a finitely generated R -module with rank. Assume there exists $r \geq 1$ such that $\mathbf{I}^{\text{stab}}(M, r) \neq 0$.*

- (1) *We have $V(\mathbf{I}^{\text{stab}}(M, r)) \subseteq \{\mathfrak{p} \in \text{Spec}(R) : \text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \infty\} \subseteq \text{Sing}(R)$.*
- (2) *If R is a localization of a finitely generated algebra over a field or is equicharacteristic and excellent, then there is some $n_r \geq 1$ such that $\text{ca}(R)^{n_r} \subseteq \mathbf{I}^{\text{stab}}(M, r)$.*
- (3) *Assume R is a complete intersection. If R is a localization of a finitely generated algebra over a field and has a perfect residue field, then there is some $m_r \geq 1$ such that $\text{Jac}(R)^{m_r} \subseteq \mathbf{I}^{\text{stab}}(M, r)$.*

Remark 6.4. In the setting of Theorem 6.3(3), it follows from [Wan94, Theorem 5.1] that, when $\mathbb{Q} \subseteq R$, we have $\text{Jac}(\widehat{R})^r \subseteq \mathbf{I}^{\text{stab}}(\widehat{M}, r)$, where \widehat{R} denotes the completion of R .

To prove Theorem 6.3, we will need the following well-known fact:

Lemma 6.5. *Let R be a local ring, M a finitely generated R -module with rank, and (F, d) the minimal free resolution of M . Set $\mathbf{I}_n(F) := \mathbf{I}_n^{\text{rank}(d_n)}(F)$.*

- (1) *For $\mathfrak{p} \in \text{Spec}(R)$ and $n \geq 1$, $\text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > n$ if and only if $\mathfrak{p} \in V(\mathbf{I}_n(F))$.*
- (2) *For all $i \geq \text{depth}(R)$, we have $V(\mathbf{I}_i(F)) = \{\mathfrak{p} \in \text{Spec}(R) : \text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \infty\}$.*

Proof. By [BH93, Proposition 1.4.9] and [BH93, Lemma 1.4.11], the $R_{\mathfrak{p}}$ -module $\text{coker}(d_n)_{\mathfrak{p}}$ is free if and only if $\mathbf{I}_n(F) \not\subseteq \mathfrak{p}$; this proves (1). (2) follows from (1) and the Auslander-Buchsbaum formula. ■

Proof of Theorem 6.3. Let F be the minimal free resolution of M . Applying Theorems 3.23 and 4.9, choose an integer $i \geq \text{depth}(R)$ such that $\mathbf{I}^{\text{stab}}(M, r) = \mathbf{I}_i^r(F) + \mathbf{I}_{i+1}^r(F)$. By assumption, at least one of $\mathbf{I}_i^r(F)$ and $\mathbf{I}_{i+1}^r(F)$ is nonzero; without loss of generality, assume $\mathbf{I}_i^r(F) \neq 0$. By Lemma 6.5(2), we have

$$V(\mathbf{I}^{\text{stab}}(M, r)) \subseteq V(\mathbf{I}_i^r(F)) \subseteq V(\mathbf{I}_i(F)) = \{\mathfrak{p} \in \text{Spec}(R) : \text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \infty\} \subseteq \text{Sing}(R);$$

where, as in Lemma 6.5, $\mathbf{I}_i(F) := \mathbf{I}_i^{\text{rank}(d_i)}(F)$. This proves (1). (2) follows from (1) and [IT15, Theorems 5.3 and 5.4], and (3) follows from (1) and the Jacobian criterion. ■

Corollary 6.6. *Let (R, \mathfrak{m}) be a local complete intersection or Golod ring and M a finitely generated R -module with rank.*

- (1) *If the Betti numbers of M are unbounded, then the conclusions of Theorem 6.3 hold for all $r \geq 1$.*
- (2) *If $\text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$ for all non-maximal primes \mathfrak{p} in R , then $\mathbf{I}^{\text{stab}}(M, r)$ is either 0 or \mathfrak{m} -primary. In particular, if R has an isolated singularity, then $\mathbf{I}^{\text{stab}}(M, r)$ is either 0 or \mathfrak{m} -primary.*

Proof. (1) follows from Theorem 6.2(2), and (2) is immediate from Theorem 6.3(1). ■

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