the Hurwitz formula one also excludes the cases $(d, g, p) = (4, 3, 5)$, $(6, 10, 17)$, and $(10, 36, 59)$. Looking at the table above, we conclude then that said l.c.m. divides 216, 168, 60, 1080, 2520, 48, 102060, 240 respectively for d equal to 3, 4, 5, 6, 7, 8, 9, 10 respectively.

These bounds seem to be pretty good: by the above, the actual l.c.m. equals 108 (resp. 168) for $d = 3$ (resp. 4); it is not unreasonable to expect that the bound is sharp for $d = 5$, 8, and 10 (perhaps there even exist curves with automorphism groups of this order); the Valentiner sextic has only simple flexes and 360 automorphisms (cf. [BHH]), so the bound for $d = 6$ is sharp if and only if there exists a sextic with only simple flexes and with 27 dividing the order of its stabilizer. Finally, for $d = 9$ the bound is probably not optimal.

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CONSTRUCTION OF SURFACES IN $\mathbb{P}_4$

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Introduction

All smooth projective surfaces can be embedded in $\mathbb{P}_5$, but only few of them in $\mathbb{P}_4$. Indeed the numerial invariants of a smooth surface in $\mathbb{P}_4$ have to satisfy the double point formula (1.2). Moreover Ellingsrud and Peskine proved

Theorem [EP]. *The family of smooth surfaces $X \subset \mathbb{P}_4$ of Kodaira dimension $\kappa \leq 1$ is bounded.*

This poses the problem of classifying the smooth nongeneral type surfaces in $\mathbb{P}_4$ completely.

In this paper we present a method to construct smooth surfaces in $\mathbb{P}_4$, and thus establish the existence of certain families of such surfaces. The most remarkable families we discovered are:

(1) rational surfaces of degree $d = 11$ and sectional genus $\pi = 11$.

(2) Enriques surfaces with $d = 10$, $\pi = 8$, $d = 11$, $\pi = 10$ and $d = 13$, $\pi = 16$.

The basic idea of our method is an application of Beilinson's spectral sequence: to construct the ideal sheaf $\mathcal{J}_X$ and thus X we construct the cohomology modules $H^1_*\mathcal{J}_X$ and $H^2_*\mathcal{J}_X$ first. From these graded modules of finite length we obtain vector bundles $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}_4$ with $\operatorname{rk}\mathcal{G} = \operatorname{rk}\mathcal{F} + 1$, and a morphism $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$, which drops rank precisely along X.

The crucial part in the construction of X by this method is to determine the module structure of the $H^i_*\mathcal{J}_X$. This is often very easy; sometimes however it is quite subtle.

Once the module structure and hence also $\mathcal{F}$ and $\mathcal{G}$ are chosen appropriately it remains

(1) to establish that the dependency locus of a generic morphism $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ is indeed a smooth surface X, and

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(2) to analyse how this surface fits into the Enriques-Kodaira classification.

Concerning (1) there is Kleiman's result [Kl]: If the bundle $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is generated by its global sections, then the generic $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ gives rise to a smooth surface (see also [Ch] for another result in this direction). However in many of our examples Kleiman's condition is not satisfied.

Another possibility to verify the existence of a family of smooth surfaces is to check one example explicitly: Our description of $\mathcal{F}$ and $\mathcal{G}$ is explicit, and we can derive from a given φ an explicit system of homogeneous equations for its dependency locus X. The smoothness of X can then be checked via the implicit function theorem, i.e. by a straightforward computation.

Although this computation is too extensive to be carried through by ourselves, it is within the range of a personal computer and nowadays computer algebra systems. We use the computer algebra system Macaulay [Mac] on a Macintosh Mac II with 8 MB main memory and a Sun SPARC-station 2 with 64 MB.

For (2) we apply adjunction theory (cf. [SV]); i.e., we study the linear system $| H + K |$ on X, where H denotes the hyperplane class and K the canonical divisor. By the general theory we know (neglecting some well-known exceptions): $| H + K |$ is basepoint free, unless $X \subset \mathbb{P}(H^0(X, \mathcal{O}_X(H)))$ is ruled by lines, and then

$$\varphi_{H+K} : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(H+K)))$$

is birational onto its image X', unless $X \subset \mathbb{P}(H^0(X, \mathcal{O}_X(H)))$ is ruled by conics or is a Del Pezzo surface. X' is again a smooth surface and $\varphi_{H+K} : X \to X'$ blows down all (-1)-curves which are embedded in $\mathbb{P}(H^0(X, \mathcal{O}_X(H)))$ as lines (exceptional lines). Studying, if necessary, $| H' + K' |$ and so on will finally lead to a minimal model of X.

To keep track of the numerical data of X' and the second adjoint surface X'', it is crucial to know the number of exceptional lines on X. By Le Barz's 6-secant formula [LB], the number of 6-secants to X (if finite) plus the number of exceptional lines equals an expression $N_6(d, \pi, \chi)$ which depends only on the degree d, the sectional genus π and the Euler characteristic $\chi = \chi(\mathcal{O}_X)$. On the other hand in most cases the number of 6-secants is precisely the number of sextic generators of the homogeneous ideal of X, and this number is known to us by the construction.

We can thus give a rough description of the linear system H and X in terms of a minimal model of X.

This paper is organized as follows: In §1 we explain our method. Section 2 contains a detailed description of a series of examples:

2.1. Rational surfaces with $d = 10$, $\pi = 9$;
2.2. Rational and Enriques surfaces with $d = 10$, $\pi = 8$;
2.3. Rational and Enriques surfaces with $d = 9$, $\pi = 6$;
2.4. Rational surfaces with $d = 11$, $\pi = 11$;
2.5. Enriques surfaces with $d = 11$, $\pi = 10$;
2.6. Enriques surfaces with $d = 13$, $\pi = 16$;
2.7. K3-surfaces with $d = 12$, $\pi = 14$;
2.8. Bielleptic and abelian surfaces with $d = 10$, $\pi = 6$.

There are two appendices. Appendix A contains some remarks about our computations done in Macaulay. In Appendix B we give a list of all smooth nongeneral type surfaces $X \subset \mathbb{P}_4$ known to us. We include enough information to enable the reader to construct these surfaces by our method. Moreover we make our programs accessible via anonymous ftp as explained at the end of Appendix A.

1. The method

1.1. Notation. If not otherwise mentioned, X denotes a smooth surface in $\mathbb{P}_4$ and

H its hyperplane class,
$d = H^2$ its degree,
K its canonical divisor,
$\pi = \frac{1}{2}H.(H + K) + 1$ its sectional genus, and
$\chi = \chi(\mathcal{O}_X) = 1 - q + p_g$ the Euler characteristic of its structure sheaf.

1.2. The numerical invariants of X satisfy the *double point formula*

$$d^2 - 10d - 5H.K - 2K^2 + 12\chi = 0$$

(cf. [Ha2, Appendix A, 4.1.3]).

1.3. We will construct X as the determinantal locus of a map between vector bundles. Let $\mathcal{F}$ and $\mathcal{G}$ be vector bundles on $\mathbb{P}_4$ of rank $\mathrm{rk}\mathcal{F} = f$, $\mathrm{rk}\mathcal{G} = f + 1$ and let $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ be a morphism. Then

$$V(\varphi) = \{p \in \mathbb{P}_4 \mid \mathrm{rk}\,\varphi(p) < f\}$$

has codimension ≤ 2, and we expect that equality holds (cf. [ACGH, Chapter II]). If equality holds, then $X = V(\varphi)$ is a locally Cohen-Macaulay surface, and the *Eagon-Northcott complex*

$$0 \leftarrow \mathcal{O}_X(m) \leftarrow \mathcal{O}(m) \cong \bigwedge^{f} \mathcal{F}^* \otimes \bigwedge^{f+1} \mathcal{G} \leftarrow \mathcal{G} \xleftarrow{\varphi} \mathcal{F} \leftarrow 0$$

is exact and identifies $\operatorname{coker}\varphi$ with the twisted ideal sheaf

$$\operatorname{coker}\varphi \cong \mathcal{J}_X(m), \qquad m = c_1\mathcal{G} - c_1\mathcal{F},$$

of X [BE1].

1.4. Furthermore a mapping cone

$$\begin{tikzcd}
 & \mathcal{J}_X(m) & & & & & \\
0 & \mathcal{G} \arrow[l] \arrow[u] & \mathcal{G}_0 \arrow[l] \arrow[ul] & \mathcal{G}_1 \arrow[l] & \mathcal{G}_2 \arrow[l] & \mathcal{G}_3 \arrow[l] & 0 \arrow[l] \\
0 & \mathcal{F} \arrow[l] \arrow[u] & \mathcal{F}_0 \arrow[l] \arrow[u] \arrow[ur, "\oplus"] & \mathcal{F}_1 \arrow[l] \arrow[u] \arrow[ur, "\oplus"] & \mathcal{F}_2 \arrow[l] \arrow[u] \arrow[ur, "\oplus"] & \mathcal{F}_3 \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

between minimal free resolutions of $\mathcal{F}$ and $\mathcal{G}$ is a (not necessarily minimal) free resolution of $\mathcal{J}_X(m)$.

1.5. To construct a surface X with the desired numerical invariants we have to find appropriate $\mathcal{F}$ and $\mathcal{G}$. One easy way to construct vector bundles is to sheafify syzygy modules.

Proposition. *Let $M = \bigoplus_{m\in\mathbb{Z}} M_m$ be a graded module of finite length over the homogeneous coordinate ring $S = k[x_0, \cdots, x_n]$ of $\mathbb{P}_n$, and let*

$$0 \leftarrow M \leftarrow P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} P_2 \leftarrow \cdots \leftarrow P_n \xleftarrow{d_n} P_{n+1} \leftarrow 0$$

be its minimal free resolution. Then for $1 \le i \le n-1$ the sheafified syzygy module

$$\mathcal{F}_i = \mathcal{S}yz_i(M) = (\ker d_{i-1})^{\sim} = (\operatorname{im} d_i)^{\sim}$$

is a vector bundle on $\mathbb{P}_n$ with intermediate cohomology

$$\bigoplus_{m\in\mathbb{Z}} H^j(\mathbb{P}_n, \mathcal{F}_i(m)) \cong \begin{cases} M, & i = j, \\ 0 & i \ne j, \quad 1 \le j \le n-1. \end{cases}$$

Conversely for any vector bundle $\mathcal{F}$ on $\mathbb{P}_n$ with this intermediate cohomology there exists a direct sum of line bundles $\mathcal{L}$ such that

$$\mathcal{F} \cong \mathcal{F}_i \oplus \mathcal{L}.$$

Proof. For the first assertion cut the sheafified minimal free resolution of M into short exact sequences and take cohomology; for the converse part compare it with the dual of a minimal free resolution of $\mathcal{F}^*$. q.e.d.

1.6. Another way frequently used to construct vector bundles (or more generally coherent sheaves) is to construct the differentials of Beilinson's spectral sequence first.

Theorem [Bei]. *For any coherent sheaf $\mathcal{S}$ on $\mathbb{P}_n$ there is a spectral sequence*

$$E_1^{pq} = H^q(\mathbb{P}_n, \mathcal{S}(p)) \otimes \Omega^{-p}(-p) \Rightarrow \mathcal{S}.$$

Here $\Omega^k(k)$ denotes the kth exterior power of the twisted cotangent bundle $\Omega^1(1)$. Note that all the E_1-terms are in the 2nd quadrant and that only finitely many of them are different from zero. The differentials

$$\begin{aligned} d_1^{pq} &\in \mathrm{Hom}(H^q(\mathbb{P}_n, \mathcal{S}(p))\otimes\Omega^{-p}(-p), H^q(\mathbb{P}_n,\mathcal{S}(p+1))\otimes\Omega^{-p-1}(-p-1)) \\ &\cong \mathrm{Hom}(H^0(\mathbb{P}_n, \mathcal{O}(1))\otimes H^q(\mathbb{P}_n, \mathcal{S}(p)), H^q(\mathbb{P}_n, \mathcal{S}(p+1))) \end{aligned}$$

coincide with the natural multiplication maps, they are determined by the module structure of $\bigoplus_p H^q(\mathbf{IP}_n, \mathcal{S}(p))$. The higher differentials

$$d_r^{pq} : E_r^{pq} \to E_r^{p+r, q-r+1}, \qquad r \geq 2,$$

are induced by maps $E_1^{pq} \to E_1^{p+r, q-r+1}$. These maps however are not canonically given.

1.7. We shall apply Beilinson's spectral sequence to the twisted ideal sheaf $\mathcal{J}_X(4)$. The terms in the E_1-diagram are determined by the dimensions $h^i\mathcal{J}_X(m)$, $0 \leq i, m \leq 4$. First information about these dimensions can be obtained from Riemann-Roch.

Proposition. *Let $X \subset \mathbf{IP}_4$ be a smooth surface of degree d, sectional genus π, irregularity q and geometric genus p_g. Then*

$$\chi(\mathcal{J}_X(m)) = \chi(\mathcal{O}_{\mathbb{P}_4}(m)) - \binom{m+1}{2}d + m(\pi - 1) - 1 + q - p_g.$$

Some of the cohomology groups vanish

$$h^4\mathcal{J}_X(m) = h^4\mathcal{O}_{\mathbb{P}_4}(m) = 0 \quad \text{for } m \geq -4;$$

$$h^3\mathcal{J}_X(m) = h^2\mathcal{O}_X(m) = h^0\omega_X(-m) = 0 \quad \text{for } m \geq 1,$$

if X has Kodaira dimension $\kappa \leq 1$;

$$h^2\mathcal{J}_X(m) = h^1\mathcal{O}_X(m) = h^1\omega_X(-m) = 0 \quad \text{for } m \leq -1$$

by Kodaira vanishing;

$$h^1\mathcal{J}_X(1) = 0,$$

if X is not the Veronese surface by Severi's theorem [Se];

$$h^0\mathcal{J}_X(1) = 0$$

if X is nondegenerate, i.e., does not lie on any hyperplane.

In particular the *speciality* of a nondegenerate, nongeneral type surface different from the Veronese surface is given by

$$s = h^1\mathcal{O}_X(1) = h^2\mathcal{J}_X(1) = \chi(\mathcal{J}_X(1)) = \pi + 3 - d + q - p_g .$$

Moreover

$$h^0\mathcal{J}_X(2) = 0 , \qquad h^0\mathcal{J}_X(3) = 0$$

mean that X does not lie on a quadric and cubic hypersurface resp. Smooth surfaces on quadrics and cubics resp. are completely classified [Au], [Ko]. So in order to construct new surfaces we will suppose that these groups vanish.

Summary: For a smooth nondegenerate, nongeneral type surface $X \subset \mathbb{P}_4$ different from the Veronese surface we have the following *Beilinson cohomology table*:

i

0	0	0	0	0	0
N+1	p_g	0	0	0	0
0	q	s	$h^1\mathcal{O}_X(2)$		
0	0	0	$h^1\mathcal{J}_X(2)$	$h^1\mathcal{J}_X(3)$	$h^1\mathcal{J}_X(4)$
0	0	0			$h^0\mathcal{J}_X(4)$

m

where

$$N = \pi + q - p_g - 1 .$$

1.8. Later on in the examples we wish to construct a surface with fixed numerical invariants and with fixed Beilinson cohomology table. We will interpret one part of the Beilinson spectral sequence of the ideal sheaf $\mathcal{J}_X(4)$ (to be constructed) as the spectral sequence of a vector bundle $\mathcal{F}$, the other part as that of a vector bundle $\mathcal{G}$. The differential between the two parts will give the morphism $\varphi : \mathcal{F} \to \mathcal{G}$ whose cokernel is the desired $\mathcal{J}_X(4)$.

1.9. After having constructed by this method a family of smooth surfaces in $\mathbb{P}_4$ it remains to analyse how the surfaces fit into the Enriques-Kodaira classification.

The numerical invariants d, π, p_g and q are known by construction and K^2 can be calculated by the double point formula (1.2).

To obtain further information we apply adjunction theory [SV]:

1.10. Theorem (Adjunction mapping). *Let $X \subset \mathbb{P}_n$ be a smooth surface, H the hyperplane class and K the canonical divisor. Then $|H+K|$ is nonspecial and has dimension N. Furthermore*

(A) $|H+K| = \phi$ *iff*

(1) $X \subset \mathbb{P}_n$ *is ruled by lines, i.e., $X \subset \mathbb{P}_n$ is a scroll over a curve of genus $q = \pi$*

or

(2) $X = \mathbb{P}_2$, $H = \mathcal{O}_{\mathbb{P}_2}(1)$ *or* $H = \mathcal{O}_{\mathbb{P}_2}(2)$.

(B) *If $|H+K| \neq \phi$ then $|H+K|$ is basepoint free. In this case*

(i) $(H+K)^2 = 0$ *iff*

(3) X *is a Del Pezzo surface, i.e., $H = -K$ (in particular X is rational)*

or

(4) $X \subset \mathbb{P}_n$ *is a conic bundle.*

(ii) *If $(H+K)^2 > 0$ then the map*

$$\varphi_{H+K} : X \to X' \subset \mathbb{P}_N$$

defined by $|H+K|$ is birational onto a smooth surface X' of degree $(H+K)^2$ and blows down all (-1)-curves E with $E.K = -1$, $E.H = 1$, unless

(5) $X = \mathbb{P}_2(p_1, \cdots, p_7)$, $H = 6L - \sum_{i=1}^{7} 2E_i$.

(6) $X = \mathbb{P}_2(p_1, \cdots, p_8)$, $H = 6L - \sum_{i=1}^{7} 2E_i - E_8$.

(7) $X = \mathbb{P}_2(p_1, \cdots, p_8)$, $H = 9L - \sum_{i=1}^{8} 3E_i$.

(8) $X = \mathbb{P}(\mathcal{E})$, *where $\mathcal{E}$ is an indecomposable rank 2 bundle over an elliptic curve, and $H = 3B$, where B is an effective divisor on X with $B^2 = 1$.*

1.11. We also need

Theorem (Hodge-Index). *Let $X \subset \mathbb{P}_n$ be a smooth surface, H the hyperplane class and K the canonical divisor. Then the intersection matrix*

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix}$$

is indefinite, unless H and K are numerical dependent.

1.12. To apply (1.10) repeatedly we have to control the numerical data of the first adjoint surface $X' \subset \mathbb{P}_N$. We know

$$(H')^2 = (H+K)^2, \qquad H'.K' = (H+K).K.$$

For

$$(K')^2 = K^2 + a$$

however, we have to compute the number a of *exceptional lines on* X, i.e. of curves $E \subset X$ with $E \, . \, K = E^2 = -1$ and $E \, . \, H = 1$.

1.13. Here *Le Barz's* 6-*secant formula* is useful: Let $X \subset \mathbb{P}_4$ be a smooth surface of degree d, sectional genus π and Euler characteristic χ. Consider the double curve of a general projection of X to $\mathbb{P}_3$ and denote by

$$\delta = \binom{d-1}{2} - \pi$$

its degree, by

$$t = \binom{d-1}{3} - \pi(d-3) + 2\chi - 2$$

its number of triple points and by

$$h = \frac{1}{2}(\delta(\delta - d + 2) - 3t)$$

its number of apparent double points. Let

$$\begin{aligned} N_6(d, \pi, \chi) = & -\frac{1}{144} d(d-4)(d-5)(d^3 + 30d^2 - 577d + 786) \\ & + \delta\left(2\binom{d}{4} + d\binom{d}{3} - 45\binom{d}{2} + 148d - 317\right) \\ & - \frac{1}{2}\binom{\delta}{2}(d^2 - 27d + 120) - 2\binom{\delta}{3} \\ & + h(\delta - 8d + 56) + t(9d - 3\delta - 28) + \binom{t}{2}. \end{aligned}$$

Theorem [LB]. *Suppose that there are no lines on* X *with self-intersection* ≥ 0. *Then the number of* 6-*secants* (*if finite*) *plus the number of exceptional lines on* X *is equal to* $N_6(d, \pi, \chi)$.

1.14. On the other hand it is a plausible guess and in many cases true that the number of 6-secants to X is precisely the number of sextic generators of the homogeneous ideal I_X of X. For example, if the ideal is generated by quintics, then there are no 6-secants at all.

In any case by construction and 1.4 we know the number of sextic generators of I_X and their syzygies. Hence we can compute all 6-secants easily. As it turned out, this is just enough information to analyse in our examples how the surfaces fit into the Enriques-Kodaira classification.

2. Examples

$S = k[x_0, \cdots, x_4]$ will denote the homogeneous coordinate ring of $\mathbb{P}_4$.

2.1. Surfaces with $d = 10$, $\pi = 9$ **and** $q = p_g = 0$. A family of rational surfaces with these invariants which all lie on two quartics has been constructed by Ranestad [Ra]. We give an example of a family of surfaces which lie on a single quartic:

2.1.1. A plausible Beilinson cohomology table in view of (1.7) is

$i \uparrow$

	2			
			2	
				1

$\rightarrow m$

(we omit entries which are zero).

2.1.2. For any surface X with this cohomology Beilinson's spectral sequence yields a presentation

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{I}_X(4) \to 0$$

with

$$\mathcal{F} = 2\Omega^3(3), \qquad \mathcal{G} = 2\Omega^1(1) \oplus \mathcal{O}.$$

Conversely one can check on a computer: If $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ is general enough then $\operatorname{coker} \varphi$ is the twisted ideal sheaf $\mathcal{I}_X(4)$ of a smooth surface $X \subset \mathbb{P}_4$ with the desired invariants.

2.1.3. Starting from the Koszul resolutions of $\Omega^3(3)$ and $\Omega^1(1)$ we obtain the syzygies of $\mathcal{I}_X(4)$ via a mapping cone (1.4)

$$\begin{tikzcd}
0 & \mathcal{G} \arrow[l] & 20\mathcal{O}(-1) \oplus \mathcal{O} \arrow[l] & 20\mathcal{O}(-2) \arrow[l] & 10\mathcal{O}(-3) \arrow[l] & 2\mathcal{O}(-4) \arrow[l] & 0 \arrow[l] \\
0 & \mathcal{F} \arrow[l] \arrow[u] & 10\mathcal{O}(-1) \arrow[l] \arrow[u] \arrow[ur, phantom, "\oplus"] & 2\mathcal{O}(-2) \arrow[l] \arrow[u] \arrow[ur, phantom, "\oplus"] & 0 \arrow[l] \arrow[u] & &
\end{tikzcd}$$

Thus a minimal resolution of $\mathcal{I}_X$ is of type

$$0 \leftarrow \mathcal{I}_X \leftarrow \mathcal{O}(-4) \oplus 10\mathcal{O}(-5) \leftarrow 18\mathcal{O}(-6) \leftarrow 10\mathcal{O}(-7) \leftarrow 2\mathcal{O}(-8) \leftarrow 0;$$

the ideal I_X of X is generated by one quartic and 10 quintics.

2.1.4. By the double point formula (1.2) $K^2 = -9$ and the intersection matrix of X is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & -9 \end{pmatrix}.$$

Since I_X is generated by quartics and quintics, X has no 6-secants and

$$N_6(10, 9, 1) = 7$$

is the number of exceptional lines.

2.1.5. By (1.10) the adjunction map

$$X \to X' \subset \mathbb{P}_8$$

is birational onto a smooth surface X' with intersection matrix

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 13 & -3 \\ -3 & -2 \end{pmatrix}.$$

In particular no multiple of K' is effective and since $q = 0$, X is rational.

The next adjunction yields a smooth surface $X'' \subset \mathbb{P}_5$ of degree 5. Hence X'' is a Del Pezzo surface and

$$\begin{pmatrix} (H'')^2 & H''.K'' \\ H''.K'' & (K'')^2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}.$$

It follows that X is a blow-up

$$X = \mathbb{P}_2(p_1, \cdots, p_{18})$$

of $\mathbb{P}_2$ in 18 (possibly infinitesimal near) points and that

$$H = 9L - \sum_{i=1}^{4} 3E_i - \sum_{i=5}^{11} 2E_i - \sum_{i=12}^{18} E_i$$

with L the preimage of a line and E_i the exceptional divisors. The points however lie in special position since $h^1(\mathcal{O}_X(H)) = 2$ (compare [PR] for a detailed discussion).

Let us also consider the surfaces constructed by Ranestad.

2.1$'$.1. The Beilinson cohomology table is

i

	2			
			2	1
				2

m

2.1′.2. The minimal free resolution of the graded S-module

$$\bigoplus_{m\in\mathbb{Z}} H^1\mathcal{I}_X(4+m)$$

has the form

$$\begin{array}{ccccccccccc}
0 \leftarrow \bigoplus_{m\in\mathbb{Z}} H^1\mathcal{I}_X(4+m) \leftarrow 2S(1) & \xleftarrow{\psi} & 9S & & 15S(-1) & & 11S(-2) & & 3S(-3) & & \\
 & & & \nwarrow & \oplus & \leftarrow & \oplus & \leftarrow & \oplus & & \\
 & & & & S(-2) & & 3S(-3) & & 3S(-4) & \nwarrow & S(-5) \leftarrow 0
\end{array}$$

and we have a presentation

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{I}_X(4) \to 0$$

with

$$\mathcal{F} = 2\Omega^3(3), \qquad \mathcal{G} = 2\mathcal{O} \oplus (\ker\psi)^{\sim}.$$

2.1′.3. Via a mapping cone (1.4) we obtain the minimal resolution

$$\begin{array}{ccccccc}
 & 2\mathcal{O}(-4) & & & & & \\
 & \oplus & & & & & \\
0 \leftarrow \mathcal{I}_X \leftarrow & 5\mathcal{O}(-5) & \nwarrow & 9\mathcal{O}(-6) & & 3\mathcal{O}(-7) & \\
 & \oplus & & \oplus & \leftarrow & \oplus & \\
 & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & \nwarrow\ \mathcal{O}(-9) \leftarrow 0
\end{array}$$

2.1′.4. This time there is precisely one sextic generator of I_X, and we guess that there is precisely one 6-secant. Indeed the three linear forms

$$\mathcal{O}(-6) \leftarrow 3\mathcal{O}(-7)$$

in the minimal resolution of $\mathcal{I}_X$ annihilate the module $I_X/(I_X)_{\leq 5}$. Hence this module is supported on a line L. This line is just the line distinguished by the module $H^1_*\mathcal{I}_X$: L is defined by the forms in the kernel of the natural map

$$H^0(\mathbb{P}_4, \mathcal{O}(1)) \to \mathrm{Hom}(H^1\mathcal{I}_X(3), H^1\mathcal{I}_X(4)).$$

Notice that the kernel is 3-dimensional since in Beilinson's spectral sequence the term $E_\infty^{0,1}$, which lies outside the diagonal, has to be zero. Consequently there are 6 exceptional lines since

$$N_6(10, 9, 1) = 7 = 1 + 6.$$

2.1′.5. Going through the adjunction process it follows [Ra] that X is a blow-up

$$X = \mathbb{P}_2(p_1, \cdots, p_{18})$$

and that

$$H = 8L - \sum_{i=1}^{12} 2E_i - \sum_{i=13}^{18} E_i.$$

2.2. Surfaces with $d = 10$, $\pi = 8$ and $q = p_g = 0$. A rational surface with these invariants has been constructed by Ranestad [Ra]. We reconstruct this surface with our method, and construct also an Enriques surface with these invariants.

2.2.1. In both cases the Beilinson cohomology table is

i

	1			
		2	5	3

m

The difference between the two surfaces resides in the different module structures of their H^1-modules. We are going to analyse that in detail:

We assume that $M = \bigoplus_{m\in\mathbb{Z}} H^1\mathcal{J}_X(m)$ is generated as S-module by $H^1\mathcal{J}_X(2)$. Then by considering the Hilbert function of M we find that M has a minimal free presentation of type

$$0 \leftarrow M \leftarrow 2S(2) \xleftarrow{\psi} 5S(1) \oplus r\,S$$

with $r \geq 2$. The morphism $\psi = (\psi_1, \psi_2)$ is given by a $2 \times (5+r)$ matrix with linear entries in ψ_1 and quadratic entries in ψ_2. $r > 2$ iff ψ_1 has some nontrivial linear syzygies. For a generic choice of ψ_1 this does not occur: in that case the cokernel of ψ_1 is supported on 5 points and its minimal free resolution

$$0 \leftarrow \operatorname{coker}\psi_1 \leftarrow 2S(2) \xleftarrow{\psi_1} 5S(1) \leftarrow 10S(-1) \leftarrow 10S(-2) \leftarrow 3S(-3) \leftarrow 0$$

is given by the Eagon-Northcott complex of ψ_1 (compare again [BE1]). Assuming that the 5 points are

$$p_0 = (1:0:\cdots:0), \cdots, p_4 = (0:\cdots:0:1) \in \mathbb{P}_4$$

we find that

$$\psi_1 = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ a_0x_0 & a_1x_1 & a_2x_2 & a_3x_3 & a_4x_4 \end{pmatrix}$$

with pairwise distinct $a_i \in k$. After column operations on $\psi = (\psi_1, \psi_2)$ we obtain that

$$\psi = \left(\begin{array}{ccc|cc} x_0 & \cdots & x_4 & 0 & 0 \\ a_0x_0 & \cdots & a_4x_4 & q_1 & q_2 \end{array}\right)$$

with quadrics

$$q_i = \sum_{j=0}^{4} b_j^i x_j^2 .$$

For all j the scalars b_j^1, b_j^2 are not both zero since otherwise $\operatorname{coker}\psi$ would not have finite length.

The idea is now to take

$$\mathcal{F} = \Omega^3(3), \qquad \mathcal{G} = \ker(2\mathcal{O}(2) \xleftarrow{\psi} 5\mathcal{O}(1) \oplus 2\mathcal{O}).$$

However for the generic choice of ψ (i.e. of q_1 and q_2), $\mathcal{G}$ has a resolution of type

$$\begin{array}{ccccccccccc} 0 \leftarrow \mathcal{G} \leftarrow 15\mathcal{O}(-1) & & 10\mathcal{O}(-2) & & 3\mathcal{O}(-3) & & & & \\ & \nwarrow & \oplus & \leftarrow & \oplus & & & & \\ & & 10\mathcal{O}(-3) & & 10\mathcal{O}(-4) & \nwarrow & 3\mathcal{O}(-5) \leftarrow 0, \end{array}$$

and no $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ would lead to a smooth surface: φ is covered by a morphism $(\varphi, \varphi_0, \varphi_1)$ of complexes

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathcal{G} & \longleftarrow & 15\mathcal{O}(-1) & \longleftarrow & \begin{array}{c}10\mathcal{O}(-2)\\ \oplus \\ 10\mathcal{O}(-3)\end{array} & \longleftarrow & \cdots \\ & & \Big\uparrow{\scriptstyle\varphi} & & \Big\uparrow{\scriptstyle\varphi_0} & & \uparrow{\scriptstyle\varphi_1} & & \\ 0 & \longleftarrow & \mathcal{F} & \longleftarrow & 5\mathcal{O}(-1) & \longleftarrow & \mathcal{O}(-2) & \longleftarrow & 0 \end{array}$$

which necessarily maps to the subcomplex

$$\begin{array}{ccccccccccc} 0 & \leftarrow & \ker\psi_1 & \leftarrow & 10\mathcal{O}(-1) & \leftarrow & 10\mathcal{O}(-2) & \leftarrow & 3\mathcal{O}(-3) & \leftarrow & 0 \\ & & \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi_0} & & \uparrow{\scriptstyle\varphi_1} & & & & \\ 0 & \leftarrow & \Omega^3(3) & \leftarrow & 5\mathcal{O}(-1) & \leftarrow & \mathcal{O}(-2) & \leftarrow & 0 & & \end{array}$$

defined by the Eagon-Northcott complex of ψ_1. So φ cannot be injective since $\ker\psi_1$ has only rank 3. The trick is to choose ψ_2 special to obtain some extra syzygies and hence a larger space $\operatorname{Hom}(\Omega^3(3), \mathcal{G})$.

2.2.3. If we take

(a) $$q_1 = b_0^1 x_0^2 + b_1^1 x_1^2 + b_2^1 x_2^2, \qquad q_2 = b_2^2 x_2^2 + b_3^2 x_3^2 + b_4^2 x_4^2$$

with $b_j^i \in k^*$, then the corresponding vector bundle $\mathcal{G}_a$ has a minimal resolution of type

$$\begin{array}{ccccccccc} & 15\mathcal{O}(-1) & & 12\mathcal{O}(-2) & & 3\mathcal{O}(-3) & & & \\ 0 \leftarrow \mathcal{G}_a \leftarrow & \oplus & \leftarrow & \oplus & \leftarrow & \oplus & & & \\ & 2\mathcal{O}(-2) & & 10\mathcal{O}(-3) & & 10\mathcal{O}(-4) & \nwarrow & 3\mathcal{O}(-5) \leftarrow 0 & \end{array}$$

and this time a general $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G}_a)$ gives a smooth surface $X_a \subset \mathbb{P}_4$ with minimal resolution

$$\begin{array}{ccccccccc} & & 10\mathcal{O}(-5) & & 11\mathcal{O}(-6) & & 3\mathcal{O}(-7) & & \\ 0 \leftarrow \mathcal{J}_{X_a} \leftarrow & & \oplus & \leftarrow & \oplus & \leftarrow & \oplus & & \\ & & 2\mathcal{O}(-6) & & 10\mathcal{O}(-7) & & 10\mathcal{O}(-8) & \nwarrow & 3\mathcal{O}(-9) \leftarrow 0 \end{array}$$

On the other hand taking

(b) $$q_1 = b_0^1 x_0^2 + b_1^1 x_1^2 + b_2^1 x_2^2, \quad q_2 = b_3^2 x_3^2 + b_4^2 x_4^2$$

with $b_j^i \in k^*$ leads to a vector bundle with resolution

$$\begin{array}{ccccccccc} & & 15\mathcal{O}(-1) & & 14\mathcal{O}(-2) & & 4\mathcal{O}(-3) & & \\ 0 \leftarrow \mathcal{G}_b \leftarrow & & \oplus & \leftarrow & \oplus & \leftarrow & \oplus & & \\ & & 4\mathcal{O}(-2) & & 11\mathcal{O}(-3) & & 10\mathcal{O}(-4) & \nwarrow & 3\mathcal{O}(-5) \leftarrow 0 \end{array}$$

and the generic $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G}_b)$ defines a smooth surface $X_b \subset \mathbb{P}_4$ with syzygies

$$\begin{array}{ccccccccc} & & 10\mathcal{O}(-5) & & 13\mathcal{O}(-6) & & 4\mathcal{O}(-7) & & \\ 0 \leftarrow \mathcal{J}_{X_b} \leftarrow & & \oplus & \leftarrow & \oplus & \leftarrow & \oplus & & \\ & & 4\mathcal{O}(-6) & & 11\mathcal{O}(-7) & & 10\mathcal{O}(-8) & \nwarrow & 3\mathcal{O}(-9) \leftarrow 0 \end{array}$$

The families of surfaces of type X_a and X_b resp. lie in different components of the Hilbert scheme:

Bundles of type $\mathcal{G}_a$ specialize into bundles of type $\mathcal{G}_b$ as $b_2^2 \to 0$, but

$$\dim_k \operatorname{Hom}(\Omega^3(3), \mathcal{G}_a) = 12$$

while

$$\dim_k \operatorname{Hom}(\Omega^3(3), \mathcal{G}_b) = 14.$$

2.2.4. What kind of surfaces did we get? By construction $d = 10$, $\pi = 8$, $p_g = q = 0$, $K^2 = -4$ and the intersection matrix is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & -4 \end{pmatrix}.$$

Moreover

$$N_6(10, 8, 1) = 6.$$

Applying the principle (1.14) we guess that X_a has two 6-secants and hence four exceptional lines, and that X_b has four 6-secants and hence two exceptional lines. That this is indeed true can be checked in an example or by studying the syzygies of $\mathcal{G}_a$ and $\mathcal{G}_b$ resp. in more detail.

2.2.5. By (1.10) we obtain first adjoint surfaces

$$X'_a, X'_b \subset \mathbb{P}_7$$

with intersection matrices

(a) $$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & 0 \end{pmatrix}$$

and

(b) $$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & -2 \end{pmatrix}.$$

In case (a) K' is numerically trivial by (1.11); hence $X_{\min} = X'_a$ is a minimal Enriques surface since $p_g = q = 0$. X_a is then an Enriques surface with 4 points blown up:

$$X_a = X_{\min}(p_1, \cdots, p_4) \quad \text{and} \quad H_a = H_{\min} - \sum_{i=1}^{4} E_i .$$

In case (b) X_b has negative Kodaira dimension since $H'.K'_b = 0$ and $(K'_b)^2 = -2$. So X_b is rational since $q = 0$. It follows [Ra] that

$$X_b = \mathbb{P}_2(p_0, \cdots, p_{12})$$

and

$$H_b = 14L - 6E_0 - \sum_{i=1}^{9} 4E_i - 2E_{10} - E_{11} - E_{12}.$$

Remark. The two components of the Hilbert scheme of surfaces with $d = 10$, $\pi = 8$ and $p_g = q = 0$ have nonempty intersection, conformable with the fact that the Hilbert scheme is connected [Ha1]. One can check that a general point in the intersection corresponds to a surface with a nonisolated singularity.

2.3. Surfaces with $d = 9$, $\pi = 6$ and $q = p_g = 0$. Two families of such surfaces are known:

(a) Rational surfaces found by Alexander [Al1]:

$$X = \mathbb{P}_2(p_1, \cdots, p_{10})$$

is the blow-up of $\mathbb{P}_2$ in 10 points in general position embedded by

$$H = 13L - \sum_{i=1}^{10} 4E_i .$$

(b) Nonminimal Enriques surfaces

$$X \cong X_{\min}(p)$$

obtained by projecting a minimal Enriques surfaces $X_{\min} \subset \mathbb{P}_5$ of degree 10 from a general point p on the surface, cf. [Cos], [CV].

2.3.1. The surfaces in both families have the Beilinson cohomology table

		3	5	1

i.e., they are nonspecial and do not lie on any quartic.

2.3.2. In both cases the minimal free presentation of the H^1-module of the ideal sheaf is of type

$$0 \leftarrow H^1_* \mathcal{J}_X(4) \leftarrow 3S(2) \xleftarrow{\psi} 10S(1).$$

Thus we may take the rank 7 bundle

$$\mathcal{G} = \ker(3\mathcal{O}(2) \xleftarrow{\psi} 10\mathcal{O}(1)).$$

To get $\mathcal{F}$ we compare the syzygies of $\mathcal{G}$ with Beilinson's spectral sequence for $\mathcal{G}$: In both cases $h^0\mathcal{G} = 6$; hence the cohomology table of $\mathcal{G}$ is

		3	5	1
				6

and the E_∞-filtration of $\mathcal{G}$ is an exact sequence

$$0 \to H^0\mathcal{G} \otimes \mathcal{O} \to \mathcal{G} \to \mathcal{J}_X(4) \to 0,$$

where X is the desired surface. So we take

$$\mathcal{F} = H^0\mathcal{G} \otimes \mathcal{O}$$

and

$$\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}(H^0\mathcal{G} \otimes \mathcal{O}, \mathcal{G})$$

defined by evaluation.

Hence everything is determined by $H^1_* J_X$. The difference between the two families lies again in the module structure, which most clearly comes out from the presentation of the k-dual module

$$M = \operatorname{Hom}_k(H^1_*\mathcal{J}_X(4), k) \cong \operatorname{Ext}^5_S(H^1_*\mathcal{J}_X(4), S(-5)).$$

In case (a) the minimal free resolution of the H^1-module is of type

$$0 \leftarrow H^1_* J_X(4) \leftarrow 3S(2) \leftarrow 10S(1) \leftarrow \begin{matrix} 6S \\ \oplus \\ 15S(-1) \\ \oplus \\ S(-2) \end{matrix} \leftarrow \begin{matrix} 26S(-2) \\ \oplus \\ 3S(-3) \end{matrix} \leftarrow \begin{matrix} 15S(-3) \\ \oplus \\ 3S(-4) \end{matrix} \leftarrow \begin{matrix} 3S(-4) \\ \oplus \\ S(-5) \end{matrix} \leftarrow 0$$

In case (b), at least for a general surface of the family, the syzygies are

$$0 \leftarrow H^1_* J_X(4) \leftarrow 3S(2) \leftarrow 10S(1) \leftarrow \begin{matrix} 6S \\ \oplus \\ 15S(-1) \end{matrix} \leftarrow 25S(-2) \leftarrow 12S(-3) \leftarrow S(-5) \leftarrow 0$$

Thus the difference lies in the number of minimal generators of M. In case (a) this module requires 4 generators, in case (b)

$$M \cong S/J\ ,$$

where $J = (q_1, \cdots, q_{12})$ is an ideal generated by 12 sufficiently general quadrics.

2.3.3. The minimal resolutions of the ideal sheaves are of type

$$\text{(a)} \quad 0 \leftarrow \mathcal{J}_X \leftarrow \begin{matrix} 15\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{matrix} \leftarrow \begin{matrix} 26\mathcal{O}(-6) \\ \oplus \\ 3\mathcal{O}(-7) \end{matrix} \leftarrow \begin{matrix} 15\mathcal{O}(-7) \\ \oplus \\ 3\mathcal{O}(-8) \end{matrix} \leftarrow \begin{matrix} 3\mathcal{O}(-8) \\ \oplus \\ \mathcal{O}(-9) \end{matrix} \leftarrow 0$$

and

(b) $$0 \leftarrow \mathcal{J}_X \leftarrow 15\mathcal{O}(-5) \leftarrow 25\mathcal{O}(-6) \leftarrow 12\mathcal{O}(-7) \nwarrow \mathcal{O}(-9) \leftarrow 0$$

resp.

2.3.4. To see which module gives which surface, we compute the intersection matrix

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$N_6(9, 6, 1) = 1.$$

In case (a) we spot one 6-secant from the resolution of $\mathcal{J}_X$, in case (b) the ideal is generated by quintics and there is no 6-secant. So there in none exceptional line in case (a) and one in case (b).

2.3.5. Consequently the first adjoint surface $X' \subset \mathbb{P}_5$ has the intersection matrix

(a) $$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & -1 \end{pmatrix}$$

and

(b) $$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}$$

resp. In case (b) $X' \subset \mathbb{P}_5$ must be a minimal Enriques surface; in case (a) X' must be rational and further adjunction yields that

$$X = \mathbb{P}_2(p_1, \cdots, p_{10})$$

and

$$H = 13L - \sum_{i=1}^{10} 4E_i.$$

Remark. We conclude this example with a somewhat curious observation:

We have seen that the Enriques surfaces are completely determined by their H^1-module, which in turn is given by a sufficiently general 12-dimensional subspace of $H^0(\mathbb{P}_4, \mathcal{O}(2))$ (at least for a general surface in the family). Thus a component $\mathcal{H}$ of the Hilbert scheme containing Enriques surfaces of degree 9 is birational equivalent to the Grassmannian $G(12, H^0(\mathbb{P}_4, \mathcal{O}(2)))$, isomorphic on dense open parts

$$\mathcal{H}^\circ \cong G(12, H^0(\mathbb{P}_4, \mathcal{O}(2))^\circ.$$

Passing to the dual space $\mathbb{P}_4^*$ and identifying

$$H^0(\mathbb{P}_4, \mathcal{O}(2))^* \cong H^0(\mathbb{P}_4^*, \mathcal{O}(2))$$

and

$$G(12, H^0(\mathbb{P}_4, \mathcal{O}(2))) \cong G(3, H^0(\mathbb{P}_4^*, \mathcal{O}(2)))$$

we find that for the generic choice of subspaces the intersection of the three quadrics in the dual space is a canonical curve of genus 5. Hence

$$\mathcal{H}^\circ \cong G(12, 15)^\circ \cong G(3, 15)^\circ \cong (\mathrm{Hilb}_{8t+1-5}(\mathbb{P}_4^*))^\circ$$

at least as reduced varieties. The automorphism group PGL(4) of $\mathbb{P}_4$ acts on all these spaces. Taking quotients of possibly smaller Zariski open subsets we obtain on the right-hand side a Zariski open part $\mathfrak{M}_5^\circ$ of the moduli space $\mathfrak{M}_5$ of genus 5 curves. The left-hand side gives, in view of the geometric construction of the Enriques surfaces as projection from a point, a Zariski open subset $\mathfrak{U}^\circ$ of the universal surface over a Zariski open subset $\mathfrak{M}_E^\circ$ of the moduli space $\mathfrak{M}_E$ of Reye polarized Enriques surfaces. Thus we obtain

$$\begin{tikzcd} \mathfrak{M}_5^\circ \arrow[r] \arrow[dr] & \mathfrak{U}^\circ \arrow[d] \\ & \mathfrak{M}_E^\circ \end{tikzcd}$$

The dimension of $\mathfrak{M}_E$ is 10 and the dimension of $\mathfrak{U}^\circ$ is $12 = 10 + 2$ as desired.

Problem. Describe these maps geometrically.

2.4. Surfaces with $d = 11$, $\pi = 11$ and $q = p_g = 0$.

2.4.1. A plausible Beilinson cohomology table for a surface with these invariants is

i

	3	1		
			2	1

m

2.4.2. Thus we may take

$$\mathcal{F} = 3\Omega^3(3), \qquad \mathcal{G} = \Omega^2(2) \oplus (\ker \psi)^\sim,$$

where $2S(1) \xleftarrow{\psi} 9S$ is the minimal free presentation of the H^1-module as in $2.1'$. By checking on a computer we find that a general $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ indeed defines a smooth surface X.

2.4.3. From a mapping cone

$$\begin{tikzcd}
0 & \mathcal{G} \arrow[l] & {\begin{matrix}25\mathcal{O}(-1)\\ \oplus \\ \mathcal{O}(-2)\end{matrix}} \arrow[l] & {\begin{matrix}16\mathcal{O}(-2)\\ \oplus \\ 3\mathcal{O}(-3)\end{matrix}} \arrow[l] & {\begin{matrix}4\mathcal{O}(-3)\\ \oplus \\ 3\mathcal{O}(-4)\end{matrix}} \arrow[l] & \mathcal{O}(-5) \arrow[ul] & 0 \arrow[l] \\
0 & \mathcal{F} \arrow[l] \arrow[u] & 15\mathcal{O}(-1) \arrow[l] \arrow[u] \arrow[ur, "\oplus"] & 3\mathcal{O}(-2) \arrow[l] \arrow[u] \arrow[ur, "\oplus"] & 0 \arrow[l]
\end{tikzcd}$$

we obtain the syzygies

$$0 \to \mathcal{J}_X \leftarrow \begin{matrix}10\mathcal{O}(-5)\\ \oplus \\ \mathcal{O}(-6)\end{matrix} \leftarrow \begin{matrix}13\mathcal{O}(-6)\\ \oplus \\ 3\mathcal{O}(-7)\end{matrix} \leftarrow \begin{matrix}4\mathcal{O}(-7)\\ \oplus \\ 3\mathcal{O}(-8)\end{matrix} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

2.4.4. The intersection matrix is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 11 & 9 \\ 9 & -11 \end{pmatrix}$$

and

$$N_6(11, 11, 1) = 7 = 1 + 6 .$$

From the syzygies we spot one 6-secant which is as in $2.1'.4$ the line distinguished by $H^1_* \mathcal{J}_X$.

2.4.5. Hence the first adjoint surface $X' \subset \mathbb{P}_{10}$ has the intersection matrix

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 18 & -2 \\ -2 & -5 \end{pmatrix}$$

and X is rational, since $H'.K' < 0$ and $q = 0$. Moreover the second adjoint surface $X'' \subset \mathbb{P}_8$ has the intersection matrix

$$\begin{pmatrix} (H'')^2 & H''.K'' \\ H''.K'' & (K'')^2 \end{pmatrix} = \begin{pmatrix} 9 & -7 \\ -7 & -5+b \end{pmatrix},$$

where b is the number of exceptional conics on X. We have $0 \le b \le 10$ by (1.11), on the other hand $0 \le (H''')^2 = (H'' + K'')^2 = 9 - 14 - 5 + b$. It follows that $b = 10$ and that $X'' \subset \mathbb{P}_8$ is a conic bundle $X'' \to \mathbb{P}_1$ with 3 singular fibres. So

$$X = \mathbb{F}_e(p_1, \cdots, p_{19})$$

is a Hirzebruch surface blown up in 19 points and

$$H = 6C_0 + (3e+7)f - \sum_{i=1}^{3} 3E_i - \sum_{i=4}^{13} 2E_i - \sum_{i=14}^{19} E_i .$$

Moreover $0 \le e \le 2$ since $H.C_0 \ge 1$ and we may choose $\mathbb{P}_2$ as a minimal model:

$$X = \mathbb{P}_2(p_0, \cdots, p_{19})$$

and

$$H = 10L - 4E_0 - \sum_{i=1}^{3} 3E_i - \sum_{i=4}^{13} 2E_i - \sum_{i=14}^{19} E_i \, .$$

We now come to the family with the most sophisticated construction.

2.5. Surfaces with $d = 11$, $\pi = 10$ and $q = p_g = 0$.

2.5.1. A plausible cohomology table for a surface with these invariants is

i

	2			
		1	5	5

m

2.5.2. This suggests taking

$$\mathscr{F} = 2\Omega^3(3)\,, \quad \mathscr{G} = (\ker \psi)^{\sim}\,, \quad \text{where} \quad S(2) \xleftarrow{\psi} 10S$$

is the minimal free presentation of $H^1_*\mathscr{J}_X(4) = H^1_*\mathscr{G}$. For a general choice of ten quadrics defining ψ however the module $H^1_*\mathscr{G}$ has a minimal free resolution of type

$$\begin{array}{lcccccc}
0 \leftarrow H^1_*\mathscr{G} \leftarrow S(2) \xleftarrow{\psi} 10S & & & 15S(-1) & & & \\
 & & & \nwarrow \oplus & & & \\
 & 5S(-2) & \nwarrow & 26S(-3) & \leftarrow 20S(-4) & \leftarrow 5S(-5) & \leftarrow 0
\end{array}$$

and

$$\operatorname{Hom}(\mathscr{F}, \mathscr{G}) = 0$$

since every commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 15\mathscr{O}(-1) & & & & \\
0 & \leftarrow & \mathscr{G} & \leftarrow & \oplus & & & & \\
 & & \uparrow & & 5\mathscr{O}(-2) & \leftarrow & 20\mathscr{O}(-3) & \leftarrow & \cdots \\
 & & | & & \uparrow & & \uparrow & & \\
0 & \leftarrow & 2\Omega^3(3) & \leftarrow & 10\mathscr{O}(-1) & \leftarrow & 2\mathscr{O}(-2) & \leftarrow & 0
\end{array}$$

has to be trivial. What is needed is that the ten quadrics defining ψ have at least two linear syzygies of second order. We choose ψ carefully.

Let $E \subset \mathbb{P}_4$ be an elliptic normal curve of degree 5 and let $\tau : E \to E$ be a nontrivial 2-torsion translation. For $p \in E$ denote by $\overline{p\tau(p)}$ the line in $\mathbb{P}_4$ spanned by p and $\tau(p)$. Then (cf. [BHM])

$$Q = \bigcup_{p \in E} \overline{p\tau(p)} \subset \mathbb{P}_4$$

is a quintic elliptic scroll ruled over the elliptic curve E/τ such that $E \subset Q$ is a 2-section of $Q \to E/\tau$. Corresponding to the other two unramified double covers of E/τ there are two more elliptic normal curves E' and E'' of which Q is a 2-torsion translation scroll. The curves $E, E', E'' \subset Q$ are pairwise disjoint.

Since $E \subset \mathbb{P}_4$ is arithmetically Gorenstein of codimension 3 its syzygies are of type

$$0 \leftarrow \mathcal{O}_E \leftarrow \mathcal{O} \xleftarrow{A} 5\mathcal{O}(-2) \xleftarrow{B} 5\mathcal{O}(-3) \leftarrow \mathcal{O}(-5) \leftarrow 0\ ,$$

where B is skewsymmetric and A is given by the pfaffians of B (cf. [BE 2] or [Hu]). The syzygies of E' and E'' are of the same type, given by matrices A', B' and A'', B'' resp. In particular the homogeneous ideals I'_E and I''_E of E' and E'' resp. are both generated by five quadrics. We take the map ψ above to be defined by the ten quadrics of

$$J = I'_E + I''_E.$$

To see that

$$H^1_*\mathcal{G} := (S/(I'_E + I''_E))(2)$$

has the desired Hilbert function and the desired syzygies, recall that the syzygies of Q are of type

$$0 \leftarrow \mathcal{O}_Q \leftarrow \mathcal{O} \xleftarrow{C} 5\mathcal{O}(-3) \xleftarrow{D} 5\mathcal{O}(-4) \leftarrow 5\mathcal{O}(-5) \leftarrow 0$$

(compare Appendix B). In particular the homogeneous ideal of Q is generated by 5 cubics. Thus the complex

$$0 \leftarrow H^1_*\mathcal{G} \leftarrow S(2) \xleftarrow{(A'\ A'')} 10S \xleftarrow{\begin{pmatrix} B' & 0 & -C' \\ 0 & B'' & C'' \end{pmatrix}} 15S(-1) \xleftarrow{\begin{pmatrix} D' \\ -D'' \\ D \end{pmatrix}} 5S(-2)$$

is a part of the minimal free resolution of $H^1_*\mathcal{G}$. Here C', C'' are defined by the inclusions $E', E'' \subset Q$; i.e., the five cubics defining Q may be written as linear combinations of the generators of I'_E and I''_E resp.:

$$C = A' \cdot C'\ , \qquad C = A'' \cdot C''\ .$$

D', D'' come from the fact that $C \cdot D = 0$, i.e., that

$$A'(C' \cdot D) = 0, \qquad A''(C'' \cdot D) = 0.$$

The relations between the generators of $I_{E'}$ and $I_{E''}$ are generated by B' and B'' resp.; hence there exist 5×5-matrices D', D'' with linear entries satisfying

$$C' \cdot D = B' \cdot D', \qquad C'' \cdot D = B'' \cdot D''.$$

Checking the syzygies of $H^1_* \mathcal{G}$ in an example on the computer we find that $\mathcal{G}$ has a minimal free resolution of type

$$0 \leftarrow \mathcal{G} \leftarrow \begin{matrix} 15\mathcal{O}(-1) \\ \oplus \\ 10\mathcal{O}(-2) \end{matrix} \leftarrow \begin{matrix} 5\mathcal{O}(-2) \\ \oplus \\ 26\mathcal{O}(-3) \end{matrix} \nwarrow 20\mathcal{O}(-4) \leftarrow 5\mathcal{O}(-5) \leftarrow 0$$

so

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(2\mathcal{O}(-2), 5\mathcal{O}(-2))$$

is 10-dimensional.

2.5.3. As one can check, a general $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ defines a smooth surface X with a minimal resolution of type

$$0 \leftarrow \mathcal{J}_X \leftarrow \begin{matrix} 5\mathcal{O}(-5) \\ \oplus \\ 10\mathcal{O}(-6) \end{matrix} \leftarrow \begin{matrix} 3\mathcal{O}(-6) \\ \oplus \\ 26\mathcal{O}(-7) \end{matrix} \nwarrow 20\mathcal{O}(-8) \leftarrow 5\mathcal{O}(-9) \leftarrow 0$$

2.5.4. What kind of surface is X? The intersection matrix is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 11 & 7 \\ 7 & -6 \end{pmatrix}$$

and

$$N_6(11, 10, 1) = 10.$$

2.5.5. The first adjoint surface X' has the intersection matrix

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 19 & 1 \\ 1 & -6+a \end{pmatrix},$$

where a is the number of exceptional lines on X. From $H'.K' = 1$ it follows that $\kappa(X) \leq 0$ and since $q = p_g = 0$, X is either rational or an Enriques surface. Since the number of sextic generators of I_X coincides with N_6 one might guess that $a = 0$. However, as we checked in an example, there are infinitely many 6-secants (and Le Barz's formula does not apply). The quintics containing X intersect in

$$V((I_X)_{\leq 5}) = X \cup Q.$$

The intersection $D = X \cap Q$ is a smooth curve of numerical class $D = 6H - 10f$ on Q, where f denotes the ruling on Q and H the hyperplane class. Hence the 6-secants to X are precisely the lines in the ruling of Q.

To determine the type of X we check in an example:

$X \cup Q$ is arithmetically Cohen-Macaulay with syzygies of type

$$0 \leftarrow \mathscr{O}_{X \cup Q} \leftarrow \mathscr{O} \leftarrow 5\mathscr{O}(-5) \leftarrow 3\mathscr{O}(-6) \oplus \mathscr{O}(-7) \leftarrow 0.$$

The minors of the 5×3-submatrix

$$5\mathscr{O}(-5) \xleftarrow{\alpha} 3\mathscr{O}(-6)$$

vanish along a curve of degree 10 :

$$V(I_3(\alpha)) = E \cup \bigcup_{i=1}^{5} L_i \subset Q \cup X \ ,$$

where $E \subset Q$ is the elliptic curve we started with and the $L_i \subset X$ are five lines. Actually these lines are precisely the 5 exceptional lines on X. Moreover

$$p_2 = h^0(X, \omega_X^{\otimes 2}) = 1 ;$$

i.e., X is an Enriques surfaces. As it turns out, $X = X_{\min}(p_0, \cdots, p_5)$ is a minimal Enriques surface blown up in 6 points and

$$H = H_{\min} - 2E_0 - \sum_{i=1}^{5} L_i \ .$$

2.6. Surfaces with $d = 13$, $\pi = 16$ and $q = p_g = 0$.

2.6.1. A promising Beilinson cohomology table is

i

	6	5	1	
				1

m

2.6.2. A module $M = H^2_* \mathscr{I}_X(4)$ with a minimal free resolution of type

$$\begin{array}{ccccccccccc}
0 \leftarrow M \leftarrow 6S(3) \leftarrow 25S(2) & & 36S(1) & & 16S & & & & \\
& \nwarrow_{\alpha} & \oplus & \leftarrow & \oplus & & & & \\
& & S & & 10S(-1) & \nwarrow & 9S(-2) & & \\
& & & & & & & \nwarrow_{\beta} & S(-4) \leftarrow 0
\end{array}$$

has the desired Hilbert function. To construct a sufficiently general M with these syzygies we construct as in (2.3.2) the k-dual module M^*. So we consider the resolution from the back; i.e., we are looking for nine quadrics with a single linear syzygy of second order. To obtain these we take part of the Koszul complex

$$\begin{tikzcd} & & \mathcal{O}(-2) & \\ \mathcal{O} & 5\mathcal{O}(-1) \arrow[l] & 10\mathcal{O}(-2) \arrow[l] \arrow[u, "\delta"] & 10\mathcal{O}(-3) \arrow[l, "\gamma"'] \arrow[ul, "\varepsilon"'] \end{tikzcd}$$

a general quotient δ and the composition $\varepsilon = \delta \circ \gamma$. Then a general morphism in

$$\operatorname{Hom}(\mathcal{O}(-4), \ker \varepsilon)$$

will give by composing with γ nine quadrics defining β^*. Resolving gives α^*.

We denote the linear part of α by α' and take

$$\mathcal{F} = 16\mathcal{O}, \qquad \mathcal{G} = \ker\left(25\mathcal{O}(2) \xleftarrow{\alpha'} 36\mathcal{O}(1)\right)$$

and $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ defined by the resolution of M.

2.6.3. Then $\operatorname{coker}\varphi = \mathcal{J}_X(4)$ is the twisted ideal sheaf a smooth surface X with the desired invariants and the syzygies of type

$$\begin{tikzcd} & 5\mathcal{O}(-5) & & & & \\ 0 & \mathcal{J}_X \arrow[l] & \oplus \arrow[l] & & & \\ & \mathcal{O}(-6) & 10\mathcal{O}(-7) \arrow[ul] & 6\mathcal{O}(-8) \arrow[l] & \mathcal{O}(-9) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

2.6.4. The intersection matrix is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 13 & 17 \\ 17 & -17 \end{pmatrix}$$

and

$$N_6(13, 16, 1) = 17.$$

On the other hand, as one can check, the ideal sheaf of X is generated by the quintics alone. So there are seventeen exceptional lines on X.

2.6.5. Consequently the first adjoint surface X' has the intersection matrix

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 30 & 0 \\ 0 & 0 \end{pmatrix};$$

i.e., X' is again a minimal Enriques surface.

Remark. This family was found in collaboration with Sorin Popescu.

2.7. Surfaces with $d = 12$, $\pi = 14$ and $q = 0$, $p_g = 1$.

2.7.1. A Beilinson cohomology table with these invariants is

i

1				
	4	3		

m

2.7.2. We may take

$$\mathscr{F} = \mathscr{O}(-1) \oplus 4\Omega^3(3), \quad \mathscr{G} = 3\Omega^2(2)$$

and $\varphi \in \operatorname{Hom}(\mathscr{F}, \mathscr{G})$ generic.

2.7.3. Then $\operatorname{coker}\varphi$ is the twisted ideal sheaf of a smooth surface with a minimal resolution of type

$$0 \leftarrow \mathscr{J}_X \leftarrow 9\mathscr{O}(-5) \leftarrow 11\mathscr{O}(-6) \leftarrow 3\mathscr{O}(-7) \leftarrow 0\,.$$

2.7.4. The intersection matrix of X is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 12 & 14 \\ 14 & -11 \end{pmatrix}$$

and

$$N_6(12, 14, 2) = 10$$

is the number of exceptional lines on X since I_X is generated by quintics.

2.7.5. So the intersection matrix of the first adjoint surface X' is

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 29 & 3 \\ 3 & -1 \end{pmatrix}$$

and further adjunction yields that X is the blow-up of a minimal K3-surface in 11 points:

$$X = X_{\min}(p_0, \cdots, p_{10}) \quad \text{and} \quad H = H_{\min} - 4E_0 - \sum_{i=1}^{10} E_i\,.$$

Remark. We also constructed a family of K3-surfaces in degree 11 (see Appendix B4.8). For more details and some other K3-surfaces found by Popescu we refer to [Po].

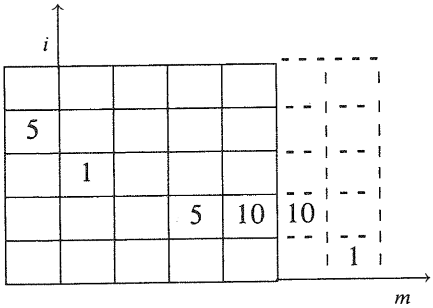
2.8. The bielliptic and the abelian surfaces of degree 10. We briefly sketch what is need to construct

(a) the bielliptic surfaces due to Serrano [Ser] and the

(b) abelian surfaces due to Comessatti [Co] and Horrocks-Mumford [HM].

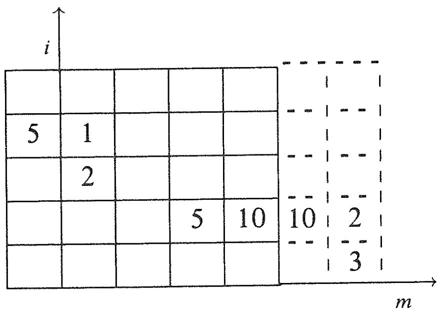
These are minimal surfaces with $d = 10$, $\pi = 6$ and Beilinson cohomology tables

(a)



and

(b)



For a description of $H^1_*\mathcal{J}_{X_b}(3)$ we refer to [De], the bielliptic surface will be treated in a forthcoming paper.

The minimal free resolution of $H^1_*\mathcal{J}_X(3)$ is of type

$$\begin{array}{ccccccc} 0 \leftarrow H^1_*\mathcal{J}_{X_a}(3) \leftarrow 5S(1) \overset{\psi_a}{\leftarrow} 15S & & 10S(-1) & & S(-2) & & \\ & \nwarrow & \oplus & & & & \\ & & S(-2) & \leftarrow & \oplus & & \\ & & \oplus & & & \nwarrow & \\ & & 25S(-3) & & 55S(-4) & & 40S(-5) \leftarrow 10S(-6) \leftarrow 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 \leftarrow H^1_*\mathcal{J}_{X_b}(3) \leftarrow 5S(1) \overset{\psi_b}{\leftarrow} 15S & & 10S(-1) & & 2S(-2) & & & & \\ & \nwarrow & \oplus & & & & & & \\ & & 4S(-2) & \leftarrow & \oplus & & & & \\ & & \oplus & & & \nwarrow & & & \\ & & 15S(-3) & & 35S(-4) & & 20S(-5) & & \\ & & & & & & & \nwarrow & \\ & & & & & & & & 2S(-7) \leftarrow 0 \end{array}$$

Comparing with the E_∞-filtration of Beilinson's spectral sequence

$$0 \to \mathcal{F}_a = 5\mathcal{O}(-1) \oplus \Omega^3(3) \to \mathcal{G}_a = \ker \psi_a \to \mathcal{J}_{X_a}(3) \to 0$$

and

$$0 \to \mathcal{F}_b = \mathcal{O}(-2) \oplus 2\Omega^3(3) \to \mathcal{G}_b = \ker \psi_b \to \mathcal{J}_{X_b}(3) \to 0,$$

we find that $\mathcal{J}_X$ has syzygies of type

$$0 \leftarrow \mathcal{J}_{X_a} \leftarrow \begin{array}{c} \mathcal{O}(-5) \\ \oplus \\ 25\mathcal{O}(-6) \end{array} \nwarrow 55\mathcal{O}(-7) \leftarrow 40\mathcal{O}(-8) \leftarrow 10\mathcal{O}(-9) \leftarrow 0$$

and

$$0 \leftarrow \mathcal{J}_{X_b} \leftarrow \begin{array}{c} 3\mathcal{O}(-5) \\ \oplus \\ 15\mathcal{O}(-6) \end{array} \nwarrow 35\mathcal{O}(-7) \leftarrow 20\mathcal{O}(-8) \nwarrow 2\mathcal{O}(-10) \leftarrow 0$$

Actually the cokernel

$$0 \to 2\Omega^3(3) \to \mathcal{G}_b \to \mathcal{E}_{HM} \to 0$$

is the (twisted) Horrocks-Mumford bundle [HM].

Le Barz's formula gives

$$N_6(10, 6, 0) = 25$$

and in both cases there are indeed 25 6-secants (for the abelian surface see [HM]).

Remark. The easiest way to obtain the presentation matrices ψ_a and ψ_b is to follow the construction of the Serrano surface [Ser] and of the Horrocks-Mumford bundle [HM]. Thus our method is not really powerful enough to cover these examples. The same holds for the nonminimal abelian surfaces of degree 15 (compare Appendix B).

Appendix A: Some remarks about the computations

To establish the existence and in particular the smoothness of our examples we used the computer algebra system Macaulay [Mac] developed by Dave Bayer and Mike Stillman. To our knowledge Macaulay is at present the only computer algebra system whose performance is strong enough to handle our examples.

Basically Macaulay allows to calculate Groebner bases and syzygies for graded S-modules, S a polynomial ring over a finite prime field. However there are a couple of further features, e.g. one can compute the codimension, the degree and the Hilbert polynomial of an ideal from its Groebner basis. Furthermore it is possible to compute exterior powers of matrices. For a detailed description we refer to the Macaulay user manual. To get an impression of the variety of possible applications, which range from the computation of the dual variety to the calculation of the cohomology of coherent sheaves on $\mathbb{P}_n$, one may look at the Macaulay scripts due to David Eisenbud and Mike Stillman.

It is no disadvantage that Macaulay works over a finite prime field. This is rather an advantage, since so the culmination of denominators during the calculations is avoided. To deduce from our computations the existence of the surfaces over the complex numbers (which is our main concern), we argue that our constructions work over a Zariski open subset of $\operatorname{Spec}\mathbb{Z}$. The actual equations dealt with may be viewed as the reduction modulo p of a surface over $\operatorname{Spec}\mathbb{Z}$, and hence the existence over the complex numbers follows from the openness of smoothness.

Occasionally we need an explicit solution of an algebraic equation (see e.g. Example 2.5, where we need a point on Bring's curve to construct a 2-torsion translation scroll of an elliptic normal curve in $\mathbb{P}_4$, compare [BHM]). In this case we pick the solution first, and then we choose the characteristic p in such a way, that the equation is valid. We may then think of the construction to be done over the ring of integers O_K of a number field K. The actual computation is done over $O_K/\mathfrak{p}$ where $\mathfrak{p} \in \operatorname{Spec} O_K$ is inert over p in $\operatorname{Spec}\mathbb{Z}$.

Sometimes during a computation we need to pick a sufficiently general element of some space, e.g. a sufficiently general $\varphi \in \operatorname{Hom}(\mathscr{F}, \mathscr{G})$. In this case we make a random choice and check that the point chosen does not lie in the subscheme of points which are bad for our purposes. The reason why explicit examples can be computed rather easily is that the components of the Hilbert scheme of (nearly all) of our examples are unirational. The unirationality follows from our construction method.

In each example our computation goes along the following steps:

1. We compute presentations of $\mathscr{F}$ and $\mathscr{G}$. Since $\mathscr{F}$ and $\mathscr{G}$ are obtained by sheafifying syzygy modules we have a presentation of both of them as a cokernel and as a kernel of a morphism between direct sums of line bundles on $\mathbb{P}_4$, e.g.

$$\bigoplus_i \mathscr{O}(a_i) \to \bigoplus_j \mathscr{O}(b_j) \to \mathscr{F} \to 0$$

and

$$0 \to \mathscr{G} \to \bigoplus_k \mathscr{O}(c_k) \to \bigoplus_l \mathscr{O}(d_l).$$

A morphism $\varphi \in \mathrm{Hom}(\mathscr{F}, \mathscr{G})$ may thus be represented by a morphism

$$\varphi: \bigoplus_j \mathscr{O}(b_j) \to \bigoplus_k \mathscr{O}(c_k),$$

i.e., by a matrix of homogenous forms. Explicit equations of X in $\mathbb{P}_4$ can be easily deduced from the syzygies of φ^t.

2. Next we check smoothness. This is the most time and space consuming step in the computation. In principle it suffices to check that the ideal generated by the 2×2–minors of the Jacobian matrix together with the original generators defines something in codimension 5. However both the computation of the minors and the computation of the standard basis are very expensive, so that we might settle to do the computation with only some of the minors. Another more sophisticated method is the following. We pick one of the equations of the surface and check that the corresponding hypersurface is singular along the surface in precisely the number of points predicted by the Chern classes of the normal bundle suitably twisted. We then check that all the 2×2–minors of the Jacobian matrix involving this equation define the same number of points on the surface. From this we deduce that the surface is singular at most at the points where the hypersurface is singular, and by semicontinuity we obtain that the surface is singular at most in the intersection of the singular loci of all the hypersurfaces containing it and having the same degree as the chosen one. At this point it roughly suffices to check that the surface has at most hypersurface singularities, which is much easier to establish.

Now we might wish to study the structure of the surface, e.g. to compute a minimal model. This can be done via adjunction theory.

3. We compute a presentation of the dualizing sheaf ω_X of X. Two methods are useful. We may compute

$$\omega_X = \mathscr{E}xt^{\operatorname{cod} X}(\mathscr{O}_X, \omega_{\mathbb{P}_n})$$

or

$$\omega_X = \mathscr{H}om(\pi_* \mathscr{O}_X, \omega_{\mathbb{P}_2}),$$

where $\pi : X \to \mathbb{P}_2$ is a generic linear projection. We apply the first isomorphsm in particular to study the first adjunction mapping. The second is very helpful at the further steps of the adjunction process, namely if the first adjoint surface X' is embedded in some higher dimensional $\mathbb{P}_n$ and if it is arithmetically Cohen-Macaulay. The latter we expect for regular higher adjoint surfaces from the general theory, cf. [EL].

4. From the presentation of ω_X we can deduce a presentation of the submodule generated by $H^0(X, \mathscr{O}_X(K+H)) \subset \bigoplus_i H^0(X, \mathscr{O}_X(K+iH))$.

Multiplying the presentation matrix with coordinates of $\mathbb{P}_m$, where $m = h^0(X, \mathcal{O}_X(K+H)) - 1$, we obtain defining equations of the graph of the morphism defined by $|H+K|$. Taking the quotient $(I_{\text{graph}} : h)$ where h defines a nonzero divisor of $X \subset \mathbb{P}_n$, we obtain equations of the image in $\mathbb{P}_m$. The number of exceptional curves of the morphism defined by $|H+K|$ is given as the degree of the intersection of the images of all (or three sufficiently general) hyperplanes under this morphism.

5. Repeating this process, we obtain a minimal model. In case of rational surfaces we might actually obtain a rational parametrization of the original surface, at least in cases where the last adjoint surface is not a Del Pezzo surface or a conic bundle.

To give the reader a chance to control our computations and to study the constructed surfaces further, we make our programs public via anonymous ftp. They can be downloaded from ftp.math.uni-sb.de (Internet 134.96.32.23) via anonymous ftp, the directory is pub/surfaces. The files include examples of all surfaces mentioned in Appendix B.

Appendix B. Overview

We give a list of all families of smooth nongeneral type surfaces $X \subset \mathbb{P}_4$ known to us.

We include enough information to enable the reader to construct these surfaces by our method. For a given surface X we write

$$M = H^1_* \mathcal{J}_X, \qquad N = H^2_* \mathcal{J}_X.$$

B1. Rational surfaces

B1.1. The plane $\mathbb{P}_2$

B1.2. The quadric $\mathbb{P}_1 \times \mathbb{P}_1$

B1.3. The cubic surface in $\mathbb{P}_3$

B1.4. The cubic scroll $\mathbb{F}_1$: $d = 3$, $\pi = 0$, $K^2 = 8$, $N_6 = 21$

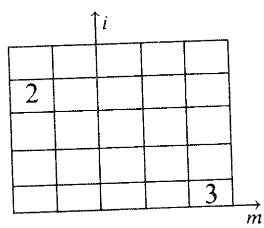
Classification

$X = \mathbb{P}_2(p_0)$
$H = 2L - E_0$

Construction

$\mathcal{F} = 2\mathcal{O}(-1)$, $\mathcal{G} = 3\mathcal{O}$

Cohomology



$h^i(\mathcal{J}_X(m))$

B1.5. The Del Pezzo surface of degree 4

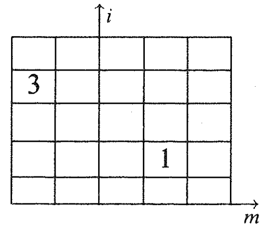
B1.6. The Veronese surface: $d = 4$, $\pi = 0$, $K^2 = 9$, $N_6 = 0$

Classification

$X = \mathbb{P}_2$

$|H| \subset |2L|$

Cohomology



$h^i(\mathcal{J}_X(m))$

Syzygies of $\mathcal{J}_X$

total:	1	7	10	5	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	7	10	5	1

Construction

$\mathcal{F} = 3\mathcal{O}(-1)$, $\mathcal{G} = \Omega^1(1)$

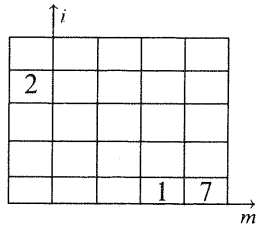
B1.7. The Castelnuovo surface: $d = 5$, $\pi = 2$, $K^2 = 1$, $N_6 = 14$

Classification

$X = \mathbb{P}_2(p_0, \cdots, p_7)$

$H = 4L - 2E_0 - \sum_1^7 E_i$

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = 2\mathcal{O}(-1)$, $\mathcal{G} = \mathcal{O}(1) \oplus 2\mathcal{O}$

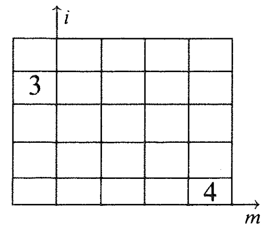
B1.8. The Bordiga surface: $d = 6$, $\pi = 3$, $K^2 = -1$, $N_6 = 10$

Classification

$X = \mathbb{P}_2(p_1, \cdots, p_{10})$

$H = 4L - \sum_1^{10} E_i$

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = 3\mathcal{O}(-1)$, $\mathcal{G} = 4\mathcal{O}$

B1.9. $d = 7$, $\pi = 4$, $K^2 = -2$, $N_6 = 5$ [Ok2], [Io1]

Classification

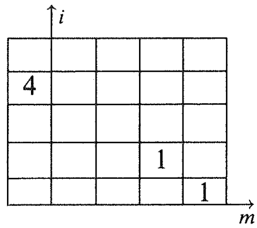
$X = \mathbb{P}_2(p_1, \cdots, p_{11})$

$H = 6L - \sum_1^6 2E_i - \sum_7^{11} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	7	10	5	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	1	-	-	-
3:	-	6	10	5	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = 4\mathcal{O}(-1)\,,\ \mathcal{G} = \Omega^1(1) \oplus \mathcal{O}$

B1.10. $d = 8\,,\ \pi = 5,\ K^2 = -2\,,\ N_6 = 1$ [Al1]

Classification

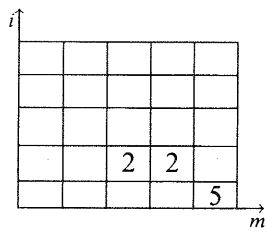
$X = \mathbb{P}_2(p_1\,, \cdots\,, p_{11})$

$H = 7L - \sum_1^{10} 2E_i - E_{11}$

Syzygies of $\mathcal{J}_X$

total:	1	9	14	8	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	5	4	-	-
4:	-	4	10	8	2

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = 2\Omega^2(2)\,,\ \mathcal{G} = 2\Omega^1(1) \oplus 5\mathcal{O}$

B1.11. $d = 8\,,\ \pi = 6\,,\ K^2 = -7\,,\ N_6 = 12$ [Ok3]

Classification

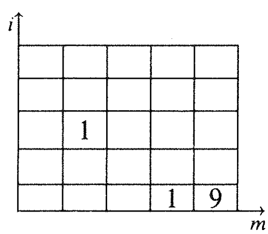
$X = \mathbb{P}_2(p_1\,, \cdots\,, p_{16})$

$H = 6L - \sum_1^4 2E_i - \sum_5^{16} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	5	5	1
0:	1	-	-	-
1:	-	-	-	-
2:	-	1	-	-
3:	-	4	5	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = \Omega^3(3)\,,\ \mathcal{G} = \mathcal{O}(1) \oplus 4\mathcal{O}$

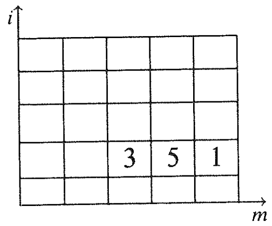
B1.12. $d = 9$, $\pi = 6$, one 6-secant, $K^2 = -1$, $N_6 = 1$ [Al1] (compare B3.1)

Classification
$X = P_2(p_1, \cdots, p_{10})$
$H = 13L - \sum_1^{10} 4E_i$

Syzygies of $\mathscr{J}_X$

total:	1	16	29	18	4
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	15	26	15	3
5:	-	1	3	3	1

Cohomology



$h^i(\mathscr{J}_X(m))$

Construction

$\mathscr{F} = 6\mathscr{O}$, $\mathscr{G} = \mathscr{S}yz_1(M)$,
where M^* has syzygies of type

total:	4	18	29	22	10	3
-5:	1	3	3	1	-	-
-4:	3	15	26	15	-	-
-3:	-	-	-	6	10	3

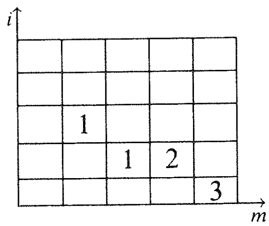
B1.13. $d = 9$, $\pi = 7$, $K^2 = -6$, $N_6 = 6$ [Al2]

Classification
$X = \mathbb{P}_2(p_1, \cdots, p_{15})$
$H = 9L - \sum_1^6 3E_i - \sum_7^9 2E_i - \sum_{10}^{15} E_i$

Syzygies of $\mathscr{J}_X$

total:	1	9	15	9	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	3	1	-	-
4:	-	6	14	9	2

Cohomology



$h^i(\mathscr{J}_X(m))$

Construction
$\mathscr{F} = \Omega^3(3) \oplus \Omega^2(2)$, $\mathscr{G} = 2\Omega^1(1) \oplus 3\mathscr{O}$

B1.14. $d = 10$, $\pi = 8$, four 6-secants, $K^2 = -4$, $N_6 = 6$ [Ra] (compare B3.2)

Classification

$X = \mathbb{P}_2(p_0, \cdots, p_{12})$

$H = 14L - 6E_0 - \sum_1^9 4E_i - 2E_{10} - E_{11} - E_{12}$

Syzygies of $\mathscr{J}_X$

total:	1	14	24	14	3
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	10	13	4	-
5:	-	4	11	10	3

Cohomology

$h^i(\mathscr{J}_X(m))$

	1			
		2	5	3

Construction

$\mathscr{F} = \Omega^3(3)$, $\mathscr{G} = \mathscr{S}yz_1(M)$, where M has syzygies of type

total:	2	7	19	25	14	3
-2:	2	5	-	-	-	-
-1:	-	2	15	14	4	-
0:	-	-	4	11	10	3

B1.15. $d = 10$, $\pi = 9$, no 6-secants, $K^2 = -9$, $N_6 = 7$ (2.1)

Classification

$X = \mathbb{P}_2(p_1, \cdots, p_{18})$

$H = 9L - \sum_1^4 3E_i - \sum_5^{11} 2E_i - \sum_{12}^{18} E_i$

Syzygies of $\mathscr{J}_X$

total:	1	11	18	10	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	1	-	-	-
4:	-	10	18	10	2

Cohomology

$h^i(\mathscr{J}_X(m))$

	2			
			2	
				1

Construction

$\mathscr{F} = 2\Omega^3(3)$, $\mathscr{G} = 2\Omega^1(1) \oplus \mathscr{O}$

B1.16. $d = 10$, $\pi = 9$, one 6-secant, $K^2 = -9$, $N_6 = 7$ [Ra]

Classification

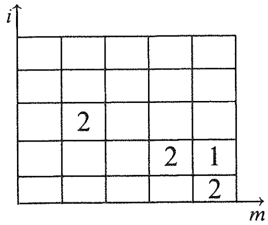
$X = \mathbb{P}_2(p_1, \cdots, p_{18})$

$H = 8L - \sum_1^{12} 2E_i - \sum_{13}^{18} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	8	12	6	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	2	-	-	-
4:	-	5	9	3	-
5:	-	1	3	3	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction
$\mathcal{F} = 2\Omega^3(3)$,
$\mathcal{G} = 2\mathcal{O} \oplus \mathcal{S}yz_1(M)$
where M has syzygies of type

total:	3	7	16	14	6	1
-1:	2	9	15	11	3	-
0:	-	-	1	3	3	1

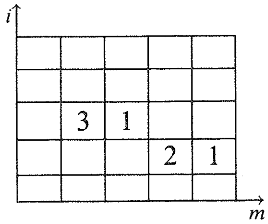
B1.17. $d = 11$, $\pi = 11$, no 6-secant, $K^2 = -11$, $N_6 = 7$ [Po]

Classification
$X = \mathbb{P}_2(p_0, \cdots, p_{19})$
$H = 11L - 5E_0 - \sum_1^6 3E_i - \sum_7^{12} 2E_i - \sum_{13}^{19} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	10	14	6	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	10	12	3	-
5:	-	-	2	3	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction
$\mathcal{F} = 3\Omega^3(3)$,
$\mathcal{G} = \ker(\mathcal{O} \xleftarrow{\psi} \Omega^2(2) \oplus 2\Omega^1(1))$
with ψ generic

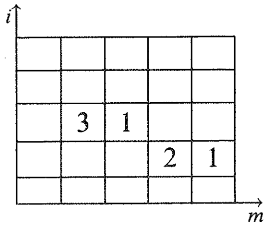
B1.18. $d = 11$, $\pi = 11$, one 6-secant, $K^2 = -11$, $N_6 = 7$ (2.4)

Classification
$X = \mathbb{P}_2(p_0, \cdots, p_{19})$
$H = 10L - 4E_0 - \sum_1^3 3E_i - \sum_4^{13} 2E_i - \sum_{14}^{19} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	11	16	7	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	10	13	4	-
5:	-	1	3	3	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction
$\mathcal{F} = 3\Omega^3(3)\,,\ \mathcal{G} = \ker(\mathcal{O} \xleftarrow{\psi} \Omega^2(2) \oplus 2\Omega^1(1))$
with ψ special

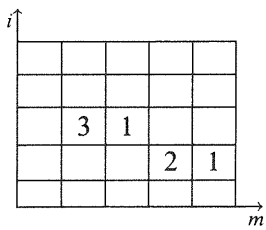
B1.19. $d = 11\,,\ \pi = 11\,,$ infinitely many 6-secants, $K^2 = -11\,,\ N_6 = 7$ [Po]

Classification
$X = \mathbb{P}_2(p_0\,, \cdots\,, p_{19})$
$H = 13L - 5E_0 - \sum_1^7 4E_i - \sum_8^{10} 2E_i - \sum_{11}^{19} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	12	19	10	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	10	14	6	1
5:	-	2	5	4	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction
$\mathcal{F} = 3\Omega^3(3)\,,\ \mathcal{G} = \ker(\mathcal{O} \xleftarrow{\psi} \Omega^2(2) \oplus 2\Omega^1(1))$
with ψ special

B.2. Ruled surfaces

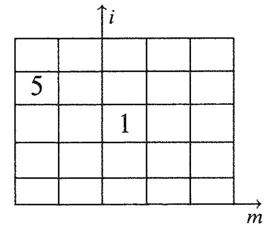
B2.1. The quintic elliptic scroll: $d = 5\,,\ \pi = 1\,,\ K^2 = 0\,,\ N_6 = 50$

Classification
$X = \mathbb{P}_C(\mathcal{E})$ with C elliptic

Syzygies of $\mathcal{J}_X$

total:	1	5	5	1
0:	1	-	-	-
1:	-	-	-	-
2:	-	5	5	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = 5\mathcal{O}(-1)\,,\ \mathcal{G} = \Omega^2(2)\,,$

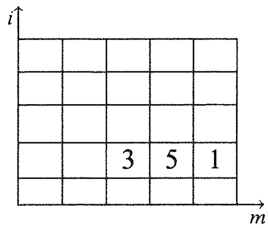
B3. Enriques surfaces

B3.1. $d = 9$, $\pi = 6$, no 6-secant, $K^2 = -1$, $N_6 = 1$ [Cos], [CV] (compare B1.12)

Classification

$X = X_{\min}(p_0)$
$H = H_{\min} - E_0$

Cohomology



$h^i(\mathcal{J}_X(m))$

Syzygies of $\mathcal{J}_X$

total:	1	15	25	12	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	15	25	12	-
5:	-	-	-	-	1

Construction

$\mathcal{F} = 6\mathcal{O}\,,\ \mathcal{G} = \mathcal{S}yz_1(M)\,,$
where M^* has syzygies of type

total:	1	12	25	21	10	3
-5:	1	-	-	-	-	-
-4:	-	12	25	15	-	-
-3:	-	-	-	6	10	3

B3.2. $d = 10$, $\pi = 8$, two 6-secants, $K^2 = -4$, $N_6 = 6$ (2.2) (compare B1.14)

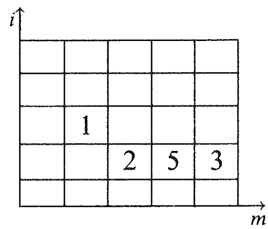
Classification

$X = X_{\min}(p_1, \cdots, p_4)$
$H = H_{\min} - \sum_1^4 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	12	21	13	3
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	10	11	3	-
5:	-	2	10	10	3

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = \Omega^3(3)$,
$\mathcal{G} = \mathcal{S}yz_1(M)$,
where M has syzygies of type

total:	2	7	17	22	13	3
-2:	2	5	-	-	-	-
-1:	-	2	15	12	3	-
0:	-	-	2	10	10	3

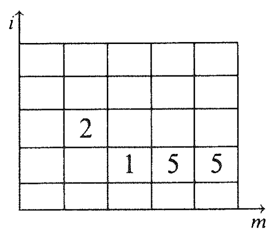
B3.3. $d = 11$, $\pi = 10$, $K^2 = -6$, $N_6 = 10$ (2.5)

Classification
$X = X_{\min}(p_0, \cdots, p_4)$
$H = H_{\min} - 2E_0 - \sum_1^4 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	15	29	20	5
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	5	3	-	-
5:	-	10	26	20	5

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction
$\mathcal{F} = 2\Omega^3(3)$, $\mathcal{G} = \mathcal{S}yz_1(M)$
where M has syzygies of type

total:	1	10	25	31	20	5
-2:	1	-	-	-	-	-
-1:	-	10	15	5	-	-
0:	-	-	10	26	20	5

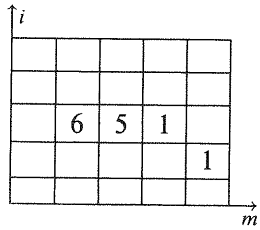
B3.4. $d = 13$, $\pi = 16$, $K^2 = -17$, $N_6 = 17$ (2.6)

Classification
$X = X_{\min}(p_1, \cdots, p_{17})$
$H = H_{\min} - \sum_1^{17} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	6	10	6	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	5	-	-	-
5:	-	1	10	6	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = 16\mathcal{O}$,

$\mathcal{G} = \ker(\mathcal{O} \leftarrow \mathcal{S}yz_2(N))$

where N^* is a generic module with syzygies of type

total:	1	9	26	37	25	6
-4:	1	-	-	-	-	-
-3:	-	9	10	1	-	-
-2:	-	-	16	36	25	6

B4. K3 surfaces

B4.1. The quartic surface in $\mathbb{P}_3$

B4.2. The complete intersection of type (2,3)

B4.3. $d = 7$, $\pi = 5$, $K^2 = -1$, $N_6 = 1$ [Ok2]

Classification

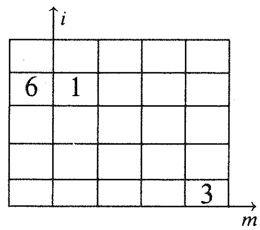
$X = X_{\min}(p_0)$

$H = H_{\min} - E_0$

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$, $\mathcal{G} = 3\mathcal{O}$

Cohomology



$h^i(\mathcal{J}_X(m))$

B4.4. $d = 8$, $\pi = 6$, $K^2 = -1$, $N_6 = 0$ [Ok3]

Classification

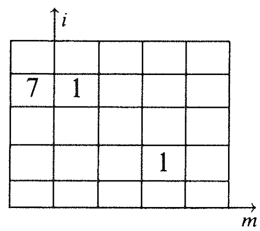
$X = X_{\min}(p_0)$

$H = H_{\min} - 2E_0$

Syzygies of $\mathcal{J}_X$

total:	1	8	11	5	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	8	11	5	1

Cohomology



$h^i(\mathcal{J}_X(m))$

Construction

$\mathcal{F} = \mathcal{O}(-2) \oplus 2\mathcal{O}(-1)\,,\ \mathcal{G} = \Omega^1(1)$

B4.5. $d = 9\,,\ \pi = 8\,,\ K^2 = -5\,,\ N_6 = 5$ (see [AR])

Classification

$X = X_{\min}(p_1\,, \cdots\,, p_5)$
$H = H_{\min} - \sum_1^5 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	6	6	1
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	6	6	1

Cohomology

$h^i(\mathcal{J}_X(m))$

1				
	1			
				6

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus \Omega^3(3)\,,\ \mathcal{G} = 6\mathcal{O}$

B4.6. $d = 10\,,\ \pi = 9\,,$ one 6-secant, $K^2 = -3\,,\ N_6 = 3$ [Po] (compare B7.4)

Classification

$X = X_{\min}(p_1\,, \cdots\,, p_3)$
$H = H_{\min} - 4E_1 - \sum_2^3 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	11	18	10	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	1	-	-	-
4:	-	9	15	7	1
5:	-	1	3	3	1

Cohomology

$h^i(\mathcal{J}_X(m))$

1				
	1			
		1	3	1
				1

total:	2	10	19	17	7	1
-5:	1	3	3	1	-	-
-4:	1	7	16	15	5	-
-3:	-	-	-	1	2	1

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus \Omega^3(3)\,,\ \mathcal{G} = \mathcal{S}yz_1(M))$
where M^* has syzygies of type

B4.7. $d = 10$, $\pi = 9$, three 6-secants, $K^2 = -3$, $N_6 = 3$ [Ra]

Classification

$X = X_{\min}(p_1, \cdots, p_3)$

$H = H_{\min} - \sum_1^3 2E_i$

Syzygies of $\mathcal{J}_X$

total:	1	9	15	9	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	2	-	-	-
4:	-	4	7	2	-
5:	-	3	8	7	2

Cohomology

$h^i(\mathcal{J}_X(m))$

1				
	1			
		1	3	2
				2

total:	1	6	14	16	9	2
-2:	1	2	1	-	-	-
-1:	-	4	10	8	2	-
0:	-	-	3	8	7	2

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus \Omega^3(3)$, $\mathcal{G} = \mathcal{S}yz_1(M) \oplus \mathcal{O}$

where M has syzygies of type

B4.8. $d = 11$, $\pi = 11$, no 6-secant, $K^2 = -5$, $N_6 = 4$

Classification

$X = X_{\min}(p_1, \cdots, p_5)$

$H = H_{\min} - 5E_1 - \sum_2^5 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	9	13	7	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	9	8	-	-
5:	-	-	5	7	2

Cohomology

$h^i(\mathcal{J}_X(m))$

1				
	2			
			3	2

total:	2	7	15	20	13	3
-5:	2	7	5	-	-	-
-4:	-	-	10	20	13	3

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus 2\Omega^3(3)$, $\mathcal{G} = \mathcal{S}yz_1(M)$

where M^* has syzygies of type

B4.9. $d = 11$, $\pi = 11$, one 6-secant, $K^2 = -5$, $N_6 = 4$ [Po]

Classification

$X = X_{\min}(p_1, \cdots, p_5)$

$H = H_{\min} - 4E_1 - 2E_2 - \sum_3^5 E_i$

Syzygies of $\mathscr{J}_X$

total:	1	10	15	8	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	9	9	1	-
5:	-	1	6	7	2

Cohomology

$h^i(\mathscr{J}_X(m))$

1				
	2			
			3	2

total:	2	8	17	21	13	3
-5:	2	7	6	1	-	-
-4:	-	1	11	20	13	3

Construction

$\mathscr{F} = \mathscr{O}(-1) \oplus 2\Omega^3(3)$,

$\mathscr{G} = \mathscr{S}yz_1(M)$

where M^* has syzygies of type

B4.10. $d = 11$, $\pi = 11$, two 6-secants, $K^2 = -5$, $N_6 = 4$ [Po]

Classification

$X = X_{\min}(p_1, \cdots, p_5)$

$H = H_{\min} - 3E_1 - \sum_2^3 2E_i - \sum_4^5 E_i$

Syzygies of $\mathscr{J}_X$

total:	1	11	17	9	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	9	10	2	-
5:	-	2	7	7	2

Cohomology

$h^i(\mathscr{J}_X(m))$

1				
	2			
			3	2

total:	2	9	19	22	13	3
-5:	2	7	7	2	-	-
-4:	-	2	12	20	13	3

Construction

$\mathscr{F} = \mathscr{O}(-1) \oplus 2\Omega^3(3)$,

$\mathscr{G} = \mathscr{S}yz_1(M)$

where M^* has syzygies of type

B4.11. $d = 11$, $\pi = 11$, three 6-secants, $K^2 = -5$, $N_6 = 4$ [Po]

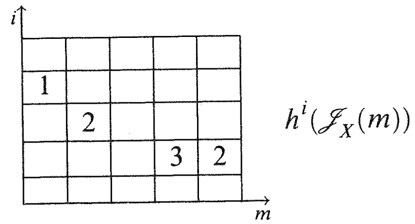
Classification

$X = X_{\min}(p_1, \cdots, p_5)$
$H = H_{\min} - \sum_1^4 2E_i - E_5$

Syzygies of $\mathcal{J}_X$

total:	1	12	19	10	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	9	11	3	-
5:	-	3	8	7	2

Cohomology



total:	2	10	21	23	13	3
-5:	2	7	8	3	-	-
-4:	-	3	13	20	13	3

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus 2\Omega^3(3)$, $\mathcal{G} = \mathcal{S}yz_1(M)$
where M^* has syzygies of type

B4.12. $d = 11$, $\pi = 12$, $K^2 = -10$, $N_6 = 9$ (Ranestad, unpublished, compare [Po])

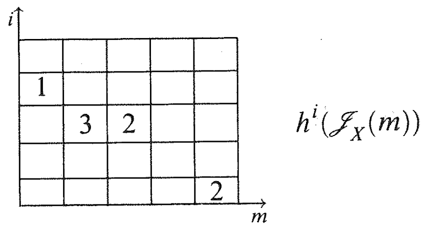
Classification

$X = X_{\min}(p_0, \cdots, p_9)$
$H = H_{\min} - 2E_0 - \sum_1^9 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	6	7	2
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	2	-	-
4:	-	4	7	2

Cohomology



Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus 3\Omega^3(3)$, $\mathcal{G} = 2\Omega^2(2) \oplus 2\mathcal{O}$

B4.13. $d = 12$, $\pi = 14$, $K^2 = -11$, $N_6 = 10$ (2.7)

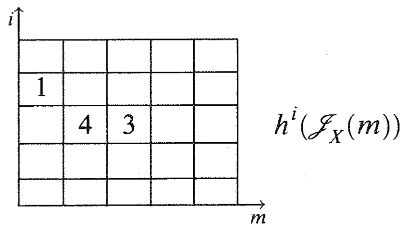
Classification

$X = X_{\min}(p_0, \cdots, p_{10})$
$H = H_{\min} - 4E_0 - \sum_1^{10} E_i$

Syzygies of $\mathcal{I}_X$

total:	1	9	11	3
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	-	-	-
4:	-	9	11	3

Cohomology



Construction

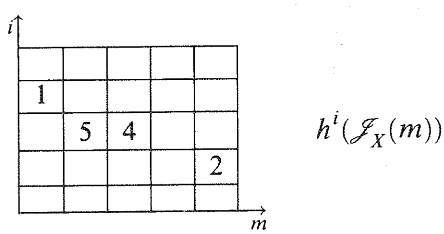
$\mathcal{F} = \mathcal{O}(-1) \oplus 4\Omega^3(3)$,
$\mathcal{G} = 3\Omega^2(2)$

B4.14. $d = 13$, $\pi = 16$, $K^2 = -11$, $N_6 = 10$ [Po]

Classification

$X = X_{\min}(p_0, \cdots, p_{10})$
$H = H_{\min} - 7E_0 - \sum_1^{10} E_i$

Cohomology



Syzygies of $\mathcal{I}_X$

total:	1	9	16	10	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	4	-	-	-
5:	-	5	16	10	2

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus 12\mathcal{O}$,
$\mathcal{G} = \ker(2\mathcal{O} \leftarrow \mathcal{S}yz_2(N))$
where N has syzygies of type

total:	5	21	32	27	15	4
-3:	5	21	30	12	-	-
-2:	-	-	2	15	15	4

B4.15. $d = 14$, $\pi = 19$, $K^2 = -15$, $N_6 = 22$ [Po]

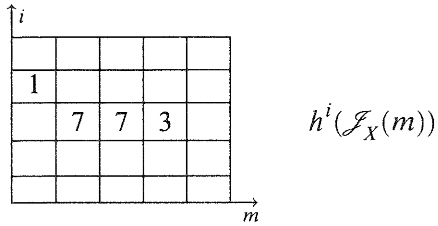
Classification

$X = X_{\min}(p_0, \cdots, p_{14})$
$H = H_{\min} - 4E_0 - \sum_1^4 2E_i - \sum_5^{14} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	8	10	3
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	-	-	-
4:	-	4	2	-
5:	-	4	8	3

Cohomology



total:	3	10	34	38	28	7
-4:	3	8	4	-	-	-
-3:	-	2	5	-	-	-
-2:	-	-	15	38	28	7

Construction
$\mathcal{F} = \mathcal{O}(-1) \oplus 15\mathcal{O}$,
$\mathcal{G} = \mathcal{S}yz_2(N))$
where N^* has syzygies of type

B5. Bielliptic surfaces

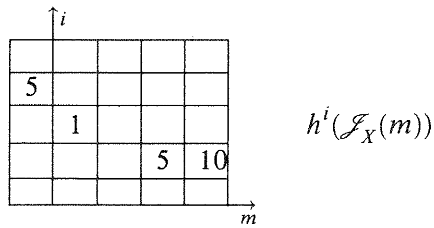
B5.1. $d = 10$, $\pi = 6$, $K^2 = 0$, $N_6 = 25$ [Se]

Classification
$X = X_{\min}$
$H = H_{\min}$

Syzygies of $\mathcal{J}_X$

total:	1	26	55	40	10
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	1	-	-	-
5:	-	25	55	40	10

Cohomology



total:	5	15	36	56	40	10
-1:	5	15	10	1	-	-
0:	-	-	1	-	-	-
1:	-	-	25	55	40	10

Construction
$\mathcal{F} = 5\mathcal{O}(-1) \oplus \Omega^3(3)$,
$\mathcal{G} = \mathcal{S}yz_1(M)$
where M has syzygies of type

B6. Abelian surfaces

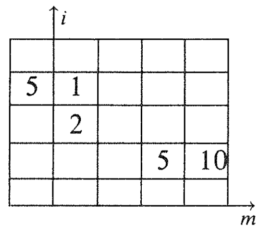
B6.1. $d = 10$, $\pi = 6$, $K^2 = 0$, $N_6 = 25$ [Co], [HM]

Classification
$X = X_{\min}$
$H = H_{\min}$

Syzygies of $\mathcal{J}_X$

total:	1	19	35	20	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	3	-	-	-
5:	-	15	35	20	-
6:	-	-	-	-	2

Cohomology



$h^i(\mathcal{J}_X(m))$

total:	5	15	29	37	20	2
-1:	5	15	10	2	-	-
0:	-	-	4	-	-	-
1:	-	-	15	35	20	-
2:	-	-	-	-	-	2

Construction
$\mathcal{F} = \mathcal{O}(-2) \oplus 2\Omega^3(3)$,
$\mathcal{G} = \mathcal{S}yz_1(M)$
where M has syzygies of type

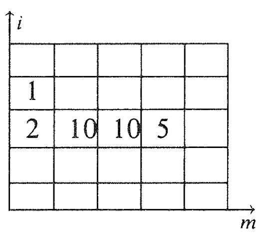
B6.2. $d = 15$, $\pi = 21$, $K^2 = -25$, $N_6 = 25$ (compare [HM])

Classification
$X = X_{\min}(p_1, \cdots, p_{25})$
$H = H_{\min} - \sum_1^{25} E_i$

Syzygies of $\mathcal{J}_X$

total:	1	8	15	10	2
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	3	-	-	-
5:	-	-	-	-	-
6:	-	5	15	10	2

Cohomology



$h^i(\mathcal{J}_X(m))$

total:	5	15	29	37	20	2
-1:	5	15	10	2	-	-
0:	-	-	4	-	-	-
1:	-	-	15	35	20	-
2:	-	-	-	-	-	2

Construction
$\mathcal{F} = \mathcal{O}(-1) \oplus 15\mathcal{O}$,
$\mathcal{G} = \ker(2\mathcal{O}(-1) \leftarrow \mathcal{S}yz_2(N))$
where N^* has syzygies of type

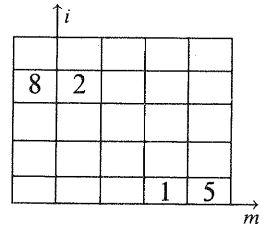
B7. Elliptic surfaces

B7.1. $d = 7, \ \pi = 6, \ K^2 = 0, \ N_6 = 0$ [Ok2]

Classification
$X = X_{\min}$
$H = H_{\min}$

Construction
$\mathscr{F} = 2\mathscr{O}(-2)$,
$\mathscr{G} = 2\mathscr{O}(-1) \oplus \mathscr{O}(1)$

Cohomology



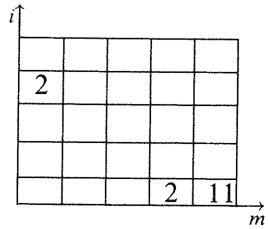
$h^i(\mathscr{I}_X(m))$

B7.2. $d = 8, \ \pi = 7, \ K^2 = 0, \ N_6 = 0$ [Ok3]

Classification
$X = X_{\min}$
$H = H_{\min}$

Construction
$\mathscr{F} = 2\mathscr{O}(-1), \ \mathscr{G} = \mathscr{O} \oplus 2\mathscr{O}(1)$

Cohomology



$h^i(\mathscr{I}_X(m))$

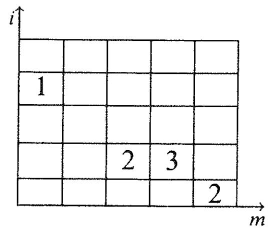
B7.3. $d = 9, \ \pi = 7, \ K^2 = 0, \ N_6 = 0$ [AR]

Classification
$X = X_{\min}$
$H = H_{\min}$

Syzygies of $\mathscr{I}_X$

total:	1	11	20	13	3
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	2	-	-	-
4:	-	9	20	13	3

Cohomology



$h^i(\mathscr{I}_X(m))$

Construction
$\mathscr{F} = \mathscr{O}(-1) \oplus 2\Omega^2(2), \mathscr{G} = 3\Omega^1(1) \oplus 2\mathscr{O}$

B7.4. $d = 10$, $\pi = 9$, no 6-secant, $K^2 = -3$, $N_6 = 3$ [Ra] (compare B4.6)

Classification

$X = X_{\min}(p_1, \cdots, p_3)$
$H = H_{\min} - \sum_1^3 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	10	15	7	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	1	-	-	-
4:	-	9	14	5	-
5:	-	-	1	2	1

Cohomology

$h^i(\mathcal{J}_X(m))$

1				
	1			
		1	3	1
				1

Construction

$\mathcal{F} = \mathcal{O}(-1) \oplus \Omega^3(3)$,
$\mathcal{G} = \mathcal{S}yz_1(M))$
where M has syzygies of type

total:	1	7	16	16	7	1
-2:	1	2	1	-	-	-
-1:	-	5	15	15	5	-
0:	-	-	-	1	2	1

B7.5. $d = 10$, $\pi = 10$, $K^2 = -2$, $N_6 = 2$ [Ra]

Classification

$X = X_{\min}(p_1, p_2)$
$H = H_{\min} - \sum_1^2 E_i$

Cohomology

$h^i(\mathcal{J}_X(m))$

2				
	1			
			1	
				3

Syzygies of $\mathcal{J}_X$

total:	1	6	9	5	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	3	-	-	-
4:	-	3	9	5	1

Construction

$\mathcal{F} = 2\mathcal{O}(-1) \oplus \Omega^3(3)$, $\mathcal{G} = \Omega^1(1) \oplus 3\mathcal{O}$

B7.6. $d = 11\,,\ \pi = 12\,,\ K^2 = -4\,,\ N_6 = 3$ [Po]

Classification

$X = X_{\min}(p_0\,, \cdots\,, p_3)$

$H = H_{\min} - 2E_0 - \sum_1^3 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	9	13	6	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	1	-	-	-
4:	-	8	13	6	1

Cohomology

$h^i(\mathcal{J}_X(m))$ (i vertical, m horizontal)

2				
	2	1		
			1	
				1

Construction

$\mathcal{F} = 2\mathcal{O}(-1) \oplus 2\Omega^3(3)\,, \mathcal{G} = \Omega^2(2) \oplus \Omega^1(1) \oplus \mathcal{O}$

B7.7. $d = 12\,,\ \pi = 13\,,\ K^2 = 0\,,\ N_6 = 10$ [Po]

Classification

$X = X_{\min}$

$H = H_{\min}$

Syzygies of $\mathcal{J}_X$

total:	1	15	30	21	5
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	3	-	-	-
5:	-	12	30	21	5

Cohomology

$h^i(\mathcal{J}_X(m))$ (i vertical, m horizontal)

2				
	2			
			4	5

Construction

$\mathcal{F} = 2\mathcal{O}(-1) \oplus 2\Omega^3(3)\,, \mathcal{G} = \ker(12\mathcal{O} \leftarrow \mathcal{S}yz_1(M))$

where M has syzygies of type

total:	4	15	27	32	21	5
-1:	4	15	15	2	-	
-0:	-	-	12	30	21	5

B7.8. $d = 12\,,\ \pi = 14\,,$ no 6-secant, $K^2 = -5\,,\ N_6 = 4$ [Po]

Classification

$X = X_{\min}(p_1\,, \cdots\,, p_5)$

$H = H_{\min} - 2E_1 - \sum_2^5 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	8	11	5	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	8	7	1	-
5:	-	-	4	4	1

Cohomology

$h^i(\mathcal{J}_X(m))$ (i vertical, m horizontal)

2				
	3	2		
			1	1

Construction
$\mathcal{F} = 2\mathcal{O}(-1) \oplus 3\Omega^3(3)$, $\mathcal{G} = \ker(\mathcal{O} \xleftarrow{\psi} 2\Omega^2(2) \oplus \Omega^1(1))$
with ψ generic

B7.9. $d = 12$, $\pi = 14$, infinitely many 6-secants, $K^2 = -5$, $N_6 = 4$ [Po]

Classification
$X = X_{\min}(p_1, \cdots, p_5)$
$H = H_{\min} - \sum_1^5 E_i$

Syzygies of $\mathcal{J}_X$

total:	1	10	14	6	1
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	-	-	-	-
4:	-	8	9	2	-
5:	-	2	5	4	1

Cohomology

$h^i(\mathcal{J}_X(m))$ (i vertical, m horizontal)

2				
	3	2		
			1	1

Construction
$\mathcal{F} = 2\mathcal{O}(-1) \oplus 3\Omega^3(3)$, $\mathcal{G} = \ker(\mathcal{O} \xleftarrow{\psi} 2\Omega^2(2) \oplus \Omega^1(1))$
with ψ special

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