# A Proof of Levin's Theorem

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These are informal notes by Craig Huneke, written while he was teaching a course from the Queens Lecture notes by Gulliksen and Levin. The theorem below is stated by Vasconcelos in his book "Arithmetic of Blowup Algebras", Thm 10.3.16, and attributed to Levin-personal communication. Vasconcelos gives a proof of the easy half, but it seems that no proof of the harder half was published.

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**Theorem 1.** *Let $R$ be a local ring, $I$ an ideal in $R$ of finite projective dimension. Let $J$ be the "content" of $I$ ,i.e. the ideal of all coefficients of relations on a minimal set of generators of I. If $r=\mu(I) > pd_R R/I$ then $(I : J)^r \subset I$.*

The proof depends on a lemma of Gulliksen's(Lemma 1.3.2 of Gulliksen-Levin) on the extension of derivations. If $A$ is a differential graded $R$-algebra, a derivation $j$ of $A$ is an $R$- linear mapping $A \to A$ of degree $w$ such that for $x \in A_p$ , $y \in A_q$

$$j(xy) = (-1)^{wq} j(x)y + xj(y)$$

Suppose $z$ is a cycle in $A_p$ and $B = \{A\langle S\rangle; dS = z\}$, the algebra obtained by adjoining to $A$ a variable to kill $z$. Gulliksen's lemma says that a derivation $j$ of $A$ of negative degree $-w$ may be extended to $B$ if and only if $j(d(A_{p+1})) \subset d(A_{p-w+1})$

In particular, if $A$ is an acyclic closure of the homomorphism $R \to R/I$, (begin with the Koszul complex over $R$ on a set of generators of $I$ and adjoin a sequence of variables $S_1, S_2, \ldots$ to kill all homology in degrees $> 0$), then an $R$-linear map $j : A_p \to R$ may be extended to derivation of $A$ of degree $-p$ if and only if $j$ takes the $p$-boundaries to 0- boundaries, i.e. elements of $I$, but this is precisely the condition that the composite $A_p \to R \to R/I$ be a $p$-cocycle in $Hom_R(A, R/I)$.

This leads to the following theorem about the action of $Ext_R(R/I, R/I)$ on $Tor^R(R/I, R/I)$.

**Theorem 2.** *Let $x_1, \ldots, x_s$ be elements in $Ext^1_R(R/I, R/I)$ and let $y_1, \ldots, y_s$ be elements of $Tor^R_1(R/I, R/I)$. Then*

$$(x_1 \ldots x_s)(y_1 \ldots y_s) = \det(M)$$

*where $M$ is the $s \times s$ matrix whose $i, j$-th entry is $x_i y_j$*

Here the product of the $x_i$ is the Yoneda product while the product of the $y_i$ is the usual algebra product in $Tor^R(R/I, R/I)$).

*Proof.* The usual way to compute the action of $Ext^1_R(R/I, R/I)$ on $Tor^R(R/I, R/I)$ is to take a representative cocyle in $Hom_R(A_1, R/I)$ , lift it to a map $\xi$ of degree -1 from $A \to A$ commuting with the differential and then apply $\xi \otimes 1$ to a cycle in $A \otimes_R (R/I)$ representing a class in $Tor^R(R/I, R/I)$. However, in this situation, we can take $\xi$ to be a derivation so multiplication by any $x_i$ satisfies the derivation rule. The result follows. $\square$

**Theorem 3.** *Let $I = (t_1, \ldots, t_r)$ and let $f_1, \ldots, f_r$ be any $r$ 1-cocycles in $Hom_R(A, R/I)$, and $T_1, \ldots, T_r$ a basis for $A_1$ with $dT_i = t_i$. (This means that for any relation $\sum_{i=1}^r a_i t_i = 0 \in R$, it is also true that $\sum_{i=1}^r a_i f_j(T_i) = 0 \in R/I$) Let $b_{ij} \in R$ represent the elements $f_i(T_j) \in R/I$ and let $M$ be the $r \times r$ matrix with these entries. Then all subdeterminants of $M$ of order $> pd_R R/I$ are elements of $I$.*

*Proof.* Since each $dT_i \in I$, each $T_i \otimes 1$ is a 1-cycle in $A \otimes (R/I)$. Apply the theorem above. $\square$

*Proof of main result.* If $b_1, \ldots, b_r \in (I : J)$ the maps

$$f_i(T_j) = \begin{cases} b_i & j = i \\ 0 & j \neq i \end{cases}$$

when composed with the canonical map $R \to R/I$ become 1-cocycles in $Hom_R(A, R/I)$. The matrix $M$ is then diagonal and the determinant is $b_1 \ldots b_r \in I$ since $Tor^R_r(R/I, R/I) = 0$. $\square$