A FAMILY OF NUMERICAL SEMIGROUPS THAT ARE NOT WEIERSTRASS SEMIGROUPS

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ABSTRACT. In this paper we give a new method for showing that certain semigroups cannot be the semigroup of pole orders of rational functions that are regular at all but one point of a compact Riemann surface or smooth algebraic curve, and we give many examples to which the method applies, including two of genus 13 of multiplicity 6 and 8 respectively.

INTRODUCTION

HISTORY AND DEFINITIONS

A *numerical semigroup* is a subset of the non-negative integers with finite complement that is closed under addition (and, in particular, contains 0, the empty sum.)

Suppose that X is a compact Riemann surface, or more generally a smooth projective curve over an algebraically closed field, of genus $g \geq 2$. If $p \in X$, then all non-constant meromorphic (or rational) functions that are regular everywhere but p have poles at p. Since the pole order of a product of two such functions is the sum of their pole orders, these orders form a *numerical semigroup* $S := S(X,p)$.

Weierstrass' student Schottky [Sch77] published the theorem, proved by Weierstrass in his lectures in the 1860's, that, for every $p \in X$, the semigroup $S(X,p)$ contains all but g := genus X positive integers (Modern proof: by the Riemann-Roch theorem, a number h is *not* in $S(X,p)$ if and only if there is a regular differential form on X that vanishes at p to order exactly $h-1$). The semigroup $S(X,p)$ is now called the Weierstrass Semigroup of X at p, and for any numerical semigroup S, the number of *gaps* of S, that is the positive integers not in S, is called the *genus* of S.

For all but finitely many points $p \in X$, called *Weierstrass points*, the semigroup $S(X,p)$ consists of all integers *except* 1,. . . , g. Hurwitz [Hur92] posed the problem of deciding which semigroups actually occur as Weierstrass semigroups for some X and p. The problem attracted attention: Haure [Hau96] gave an incorrect proof that not all semigroups can occur, and Hensel-Landsberg [HL65] gave an incorrect proof that, on the contrary, all semigroups do occur.

There, it seems, the matter rested until Henry Pinkham [Pin74] gave an algorithm that can in principle determine whether a given semigroup can be a Weierstrass semigroup. Then in his 1980 Thèse d'État, Ragnar Buchweitz demonstrated conclusively that the semigroup S of genus 16 that is generated by the elements $\{13, 14, 15, 16, 17, 18, 20, 22, 23\}$ is not a Weierstrass semigroup (Reason: the number of integers that can be expressed as the sum of a pair of gaps of S is 46, while the dimension of the space of quadratic differential forms on a surface of genus 16 is only 45.) Since 1980 many other semigroups have been shown not to be Weierstrass, both using Buchweitz's method and otherwise; see for example [GGMATVT21], [Kom21] [Kom13]. But until now, no non-Weierstrass semigroup of genus < 16 was known.

On the other hand Eisenbud and Harris [EH87] proved that in characteristic 0 all semigroups of low weight are Weierstrass. That result was extended to all characteristics by Osserman [Oss06] and to a wider class of semigroups by Pflueger [Pfl18]. Putting these results together with machine computation using the idea of Pinkham's theorem, we have been able to show that all semigroups of genus at most 10 are Weierstrass. We originally checked this in finite characteristic, and have shown that they lift to characteristic 0.

1. Pinkham's theorem and a structure theorem

In this section we review two known results that we will use in the proof that certain semigroups are not Weierstrass semigroups.

1.1. **Pinkham's Theorem.** If S is a numerical semigroup, and k is a field, then the *semigroup algebra* of S over k is the graded ring $k[S] := k[\{t^s \mid s \in S\}] \subset k[t]$. We generally define S by giving an ordered set of generators $\{s_0 < \cdots < s_u\}$, and we call $m := s_0$ the *multiplicity* of S. We write $k[S] = k[x_{i_1}, \dots, x_{i_m}]$ where the variables $x_{i_j} = t^{s_j}$ are labeled by the residue class i_j of s_j modulo $m := s_0$. Thus for example if the semigroup S is generated by $L = \{6, 9, 13, 16\}$ then the multiplicity is 6 and we write $k[S] = k[x_0, x_3, x_1, x_4]/I$ where I is the "semigroup ideal" of S. In the natural grading of S, the variable x_{i_j} has degree s_j.

Pinkham [Pin74, Theorem 13.9] showed that S occurs as the Weierstrass semigroup of a smooth curve over an algebraically closed field k if and only if there is a homogeneous 1-parameter smoothing A_z of the affine algebra $A_0 = k[S]$, in which case S is the Weierstrass semigroup of the point "at infinity" on the normalization of the projective closure of $\operatorname{Spec} A_z$ for general z. We will use only one implication of this result, and for the reader's convenience we give a proof:

Proposition 1.1. *Suppose that C is a smooth curve defined over a field k, and that $p \in C$ is a k-rational point with Weierstrass semigroup S. Let A be the coordinate ring of the affine curve $C \setminus \{p\}$. There is a flat family of affine rings A_z over $k[z]$ with generic fiber $A \otimes_k k(z)$ and special fiber $A_0 = k[S]$.*

Proof. The curve $C \setminus \{p\}$ is affine because the divisor p is ample, and C is the normalization of any projective closure of $\operatorname{Spec} A$. We regard p as the unique point of C "at infinity".

We filter A by ideals, letting $A_i \subset A$ be the elements having pole order $\geq i$ at p. Enumerate the elements of S as $0 = s_0 < s_1 < s_2 \dots$ and set

$$\tilde{A} := \oplus_{i \geq 0} A_i z^i.$$

The $k[z]$-algebra $\tilde{A}$ is finitely generated over k because S contains all large integers, and flat because it is torsion free. The formulas

$$k[z, z^{-1}] \otimes_{k[z]} \tilde{A} = k[z, z^{-1}] \otimes_k A$$

and

$$k[z]/(z) \otimes_{k[z]} \tilde{A} = \oplus A_{i+1}/A_i = k[S]$$

are evident, proving the Proposition. □

All the flat, graded families of $k[z]$-algebras specializing to $k[S]$ are flat subfamilies of the universal homogeneous unfolding of $k[S]$. If it happens that every fiber of this universal unfolding has a singular point, then these points will define a multi-section. In the family of semigroups that we will study, there is a natural section of the family defined everywhere over the unfolding.

Our philosophy is that *if such a family has a natural section, then it probably represents a singular point in each fiber*, and thus the semigroup is likely not to be a Weierstrass semigroup. This is what we will prove (Theorem 2.7) about the family of "degree-special" semigroups defined below. (However there are also families with section where the section actually follows the "origin", tracing out a smooth point—see Example 3.3.) See Example 3.1 for the two degree-special semigroups of genus 13, and Example 3.5 for other families of examples of degree-special semigroups that we have found.

1.2. **A structure theorem for resolutions.** Here is the special case of [BE74, Theorem 3.1] and [Bru87], that we will use. It is closely related to the "determinant of a complex" discovered by Cayley [Cay48] and also exploited in [KM76] and [Mac65].

Proposition 1.2. *Suppose that*

$$F_0 \xleftarrow{\psi_1} F_1 \xleftarrow{\psi_2} F_2 \xleftarrow{\psi_3} F_3 \longleftarrow 0$$

is the free resolution of a module of grade ≥ 2 over a local or positively graded ring R. Writing $r_i = rank(\psi_i)$, so that

$$r_1 = \operatorname{rank} F_0,\ r_1 + r_2 = \operatorname{rank} F_1,\ r_2 + r_3 = \operatorname{rank} F_2,\ r_3 = \operatorname{rank} F_3,$$

there is a commutative diagram

$$\begin{tikzcd}
\wedge^{r_1}F_1 & \cong & \wedge^{r_2}F_1^* \arrow[r, "\wedge^{r_2}\psi_2^*"] & \wedge^{r_2}F_2^* & \cong & \wedge^{r_3}F_2 \\
\arrow[from=1-1, drr, "\wedge^{r_1}\psi_1"'] & & \wedge^{r_1}F_0 \cong & R & \cong \wedge^{r_3}F_3 \arrow[ur, "\wedge^{r_3}\psi_3"] &
\end{tikzcd} .$$

In plain terms, this means that the subdeterminant of ψ_1 obtained by removing a set σ of r_2 columns, times the subdeterminant of ψ_3 obtained by removing a set τ of r_2 rows, is equal to the subdeterminant of ψ_2 involving rows σ and colums tau, up to a unit of R independent of σ and τ. $\square$

2. Special resolutions and Non-Weierstass Semigroups

The *format* of a free complex

$$F_0 \longleftarrow F_1 \longleftarrow \cdots$$

is the list of integers $\{rank\ F_1, rank\ F_2, \cdots\}$ where we suppress the ranks corresponding to the terms $F_i = 0$ in the complex.

Definition 2.1. We will say that a complex

$$*) \quad P \xleftarrow{\phi_1} P^6 \xleftarrow{\phi_2} P^8 \xleftarrow{\phi_3} P^3 \longleftarrow 0$$

over a ring P is *special of format* $\{1, 6, 8, 3\}$ if it admits an *acyclic* subcomplex

$$**) \quad P \xleftarrow{\phi_1'} P^4 \xleftarrow{\phi_2'} P^4 \xleftarrow{\phi_3'} P \longleftarrow 0.$$

that is, term by term, a summand of $*)$.

In concrete terms, this means that there is a choice of generators of the free modules of the complex $*)$ such that:

(1) The last 4 entries in the the first column of the matrix ϕ_3, are zero as in the diagram below, where we show the column as the first column:

$$\phi_3 : \quad \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

and ϕ_3' is the 4×1 submatrix corresponding to the first 4 elements in the distinguished column of ϕ_3.

(2) We may divide write ϕ_2 into two 6×4 matrices, $\phi_2 = A \mid B$, such that the 5th and 6th rows of A are zero as in the following diagram:

$$A : \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and ϕ_2' is the 4×4 matrix corresponding to the first 4 rows of A.

(3) With these choices, ϕ_1' is the matrix corresponding to the first 4 generators of I

To apply this idea to show that some semigroups with special resolution of format $\{1, 6, 8, 3\}$ are not Weierstrass, we need conditions that ensure that the $\{1441\}$ subcomplex persists in flat graded deformations.

Definition 2.2. A numerical semigroup S is *degree-special* if every flat graded deformation of the semigroup ring $k[t^S]$ has minimal free resolution that is special of format $\{1, 6, 8, 3\}$. In this case the four nonzero elements in the distinguished column of the 8×3 matrix of the resolution define a finite subscheme of each fiber, and thus a *distinuished multi-section* of the family.

To make use of this definition, we give a sufficient condition on the minimal graded resolution of the semigroup ring itself. In the matrix representing a homogeneous map of free modules $\oplus P(c_j) \longrightarrow \oplus P(r_i)$ we say that the element in position (i, j) has *formal degree* $r_i - c_j$. This is simply the degree of that element if the element is nonzero. For example, since our rings will be positively graded, if an element of a matrix has negative formal degree, then it is zero.

Proposition 2.3. *Let S be a numerical semigroup, and write the semigroup ring $k[t^S]$ of S as P/I, where P is a graded polynomial ring whose generators have degrees corresponding to the generators of S. If the minimal graded P-free resolution of the semigroup ring $k[t^S] = P/I$ has the form*

$$P \xleftarrow{\phi_1} P^6 \xleftarrow{\phi_2} P^8 \xleftarrow{\phi_3} P^3 \longleftarrow 0,$$

and (possibly after reordering the generators of the free modules) satisfies the following conditions, then S is degree-special.

(1) *The last four entries of the first column of the matrix ϕ_3 have negative formal degree.*
(2) *The first three entries of each of the last two rows of ϕ_2 have formal degree strictly less than the multiplicity (= the smallest positive element) of S.*
(3) *The induced subcomplex*

$$P \xleftarrow{\phi_1'} P^4 \xleftarrow{\phi_2'} P^4 \xleftarrow{\phi_3'} P \longleftarrow 0,$$

where ϕ_1' consists of the first 4 columns of ϕ_1, ϕ_2' consists of the first 4 rows and columns of ϕ_2 and ϕ_3' consists of the first four rows of the first column of ϕ_3, is acyclic.

Moreover, when these conditions are met, the 4 elements of the matrix ϕ_3' form a regular sequence.

Based on several thousand examples, we conjecture:

Proof. The elements of negative formal degree are of course 0.

Write $\omega := \operatorname{coker} \phi_3^*$. Since ω is, up to a shift in grading, the canonical module of P/I, the annihilator of ω is I. Let w_i be the generator of ω corresponding to the i-th column of ϕ_3. The ideal J generated by the first column of ϕ_3 is the annihilator of the image of w_1 in $\omega/(Pw_2 + Pw_3)$, so $I \subset J$. By the degree-special hypothesis, J is generated by 4 elements, whereas I requires 6 generators, so $I \subsetneq J$. As I is a prime of codimension 3 J must have codimension 4, so the 4 nonzero elements in the first column of ϕ_3 are a regular sequence, as claimed.

The degrees of these four elements are in the semigroup S, and since they form a regular sequence, any nonzero relation among them must have degrees in S too. The first 4 elements in each row of ϕ_2 form such a relation. By the degree-special hypothesis, the first 3 elements of the last two rows of ϕ_2 are zero, and since P/I is a domain, the 4th element in each of these rows must be zero as well. □

Conjecture 2.4. *If S is satisfies conditions (1) and (2), then it satisfies condition (3) as well; so S is degree-special without further hypotheses.*

Example 2.5. The semigroup S generated by $\{6, 9, 13, 16\}$ is degree-special. Indeed, the semigroup ring of S has differentials

$$\phi_1 = \begin{pmatrix} x_0^3 - x_3^2 & x_3x_1 - x_0x_4 & x_0^2x_1 - x_3x_4 & x_0x_1^2 - x_4^2 & x_0^2x_3^3 - x_1^3 & x_0x_3^4 - x_1^2x_4 \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} -x_1 & -x_4 & 0 & 0 & 0 & -x_0x_3^3 & 0 & x_0x_3^2x_4 \\ -x_3 & -x_0^2 & -x_4 & -x_0x_1 & -x_1^2 & 0 & -x_0x_3^3 & x_0^3x_3^2 \\ x_0 & x_3 & -x_1 & -x_4 & 0 & x_1^2 & 0 & 0 \\ 0 & 0 & x_0 & x_3 & 0 & 0 & 0 & -x_1^2 \\ 0 & 0 & 0 & 0 & -x_3 & x_0^2 & -x_4 & -x_0x_1 \\ 0 & 0 & 0 & 0 & x_0 & -x_3 & x_1 & x_4 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} x_4 & x_0x_3^3 & 0 \\ -x_1 & -x_0^2x_3^2 & 0 \\ -x_3 & -x_1^2 & 0 \\ x_0 & 0 & -x_1^2 \\ 0 & x_4 & x_0x_1 \\ 0 & -x_1 & -x_4 \\ 0 & -x_3 & -x_0^2 \\ 0 & -x_0 & -x_3 \end{pmatrix}$$

Proposition 2.6. *Suppose that P is a local or positively graded ring, and that*

$$P \xleftarrow{\psi_1} P^4 \xleftarrow{\psi_2} P^4 \xleftarrow{\psi_3} P \longleftarrow 0$$

is a minimal free resolution. If the 4 entries of the matrix ψ_3 form a regular sequence, then the entries of the matrix ψ_2 are in the ideal J generated by the entries of ψ_3, and the entries of ψ_1 are in J^2.

Proof. Since I_1 is an ideal of finite homological dimension, we may write $I_1 = rI_1'$ for some nonzerodivisor r and some ideal I_1' of depth ≥ 2. Thus, replacing ψ_1 by $r^{-1}\psi_1$ if necessary, we may assume that $*)$ is a resolution of a module of grade ≥ 2.

Let $I_1,\ I_2,\ I_3$ be the ideals generated by the entries of ϕ_1, the 3×3 minors of ϕ_2, and the entries of ϕ_3 respectively.

The first assertion of the Proposition holds because the columns of ψ_2^* are syzygies of the regular sequence of elements of ψ_3^*, and are thus linear combinations of the Koszul syzygies.

We may apply Proposition 1.2 to conclude that $I_2 = I_1I_3$. By the first assertion, $I_2 \subset I_3^3$. Thus

$$I_1 \subset I_2 : I_3 \subset I_3^3 : I_3.$$

Since I_3 is a complete intersection, the Rees algebra $P[I_3z]$ is isomorphic to a polynomial ring whose variables are the minimal generators of I_3. Thus

$$I_3^3 : I_3 = I_3^2,$$

completing the proof. ☐

Theorem 2.7. *If S is a degree-special semigroup then S is not a Weierstrass semigroup.*

Proof. We will show that any graded deformation of $\operatorname{Spec} k[t^S]$ is singular along the distinguished multi-section of Definition 2.2.

With notation as in the definition of degree-special, we consider the resolution of P/I. The condition that the last four entries of ϕ_3 have negative formal degree persists in any flat deformation. By Lemma **??** the first four entries form a regular sequence, and this condition too persists. Also the degree condition on the first four entries of the last rows of ϕ_2 persists, and implies that these entries are zero. If the induced subcomplex of format $\{1,4,4,1\}$ in the resolution is acyclic, then this will persist as well.

To complete the proof we will show, more generally, that if P is a positively graded polynomial ring, and P/I is a factor ring with minimal free resolution of special format $\{1,6,8,3\}$ then with ideals I_1, I_2, I_3 defined in terms of the distinguished $\{1,4,4,1\}$ subcomplex as in the proof of Proposition 2.6, $\operatorname{Spec} P/I$ is singular along $\operatorname{Spec} P/I_3$.

Indeed, since I_1 is generated by four of the 6 generators of I, and lies in I_3^2. Thus all the derivatives of the polynomials generating I_1 are in I_3, so the 3×3 minors of the Jacobian matrix of I also lie in I_3 proving the claim. Since this applies to every flat deformation of the semigroup ring, Proposition 1.1 shows that S is not Weierstrass. □

Example 2.8. Returning to the example of the semigroup generated by $\{6,9,13,16\}$ above we see that the induced subplex of format $\{1,4,4,1\}$ has differentials

$$\phi_1' = \begin{pmatrix} x_0^3 - x_3^2 & x_3x_1 - x_0x_4 & x_0^2x_1 - x_3x_4 & x_0x_1^2 - x_4^2 \end{pmatrix}$$

$$\phi_2' = \begin{pmatrix} -x_1 & -x_4 & 0 & 0 \\ -x_3 & -x_0^2 - x_4 & -x_0x_1 & -x_1^2 \\ x_0 & x_3 & -x_1 & -x_4 \\ 0 & 0 & x_0 & x_3 \end{pmatrix}, \qquad \phi_3' = \begin{pmatrix} x_4 \\ -x_1 \\ -x_3 \\ x_0 \end{pmatrix}.$$

Using the theorem of [BE73] it is easy to show that this subcomplex is acyclic. In particular S is not a Weierstrass semigroup.

The only condition in the definition of the degree-special property which is not immediately combinatorial is the acyclicity of the $\{1,4,4,1\}$ subcomplex. This is a condition on the resolution of the monomial ideal itself, since the acyclicity is an open condition. As remarked above Conjecture 2.4, this may in fact be a consequence of conditions (1) and (2). There are several ways to check condition (3) It follows from [BE73] that it suffices to show that the 3×3 minors of ϕ_2' generated an ideal of codimension ≥ 2. In all the examples we know, there are additional "coincidences" that we describe as hypotheses in the following proposition that imply this condition:

Proposition 2.9. *Suppose that S is a numerical semigroup whose minimal free resolution satisfies conditions (1) and (2) in Definition 2.2, and let*

$$P \xleftarrow{\phi_1'} P^4 \xleftarrow{\phi_2'} P^4 \xleftarrow{\phi_3'} P \longleftarrow 0$$

be the subcomplex of the minimal free resolution of $P/I = k[t^S]$ as in the proof of Proposition 2.3. Each of the following two conditions implies condition (3) of Definition 2.2.

(1) *The 2×2 minors of the 4×2 submatrix of ϕ_3 consisting of the last 4 rows of the last 2 columns generate an ideal of codimension 2.*
(2) *After possibly reordering the rows and columns of ϕ_2', the situation is as in Example 2.8 in the following 3 senses:*
   (a) *The first two elements of the last row of ϕ_2' are zero.*
   (b) *the last two elements of the last row of ϕ_2' form a regular sequence.*
   (c) *the 2×2 minors of the first two columns of phi_2' are nonzero and have formal degrees equal to the formal degrees of the first 3 generators of the semigroup ideal.*

Remark 2.10. Both conditions (1) and (2) of this Proposition hold in all our examples.

Proof. The remaining condition (3) of Definition 2.2 is the acyclicity of the induced $\{1, 4, 4, 1\}$ subcomplex, and to prove this, it suffices by [BE73], to prove that the ideal of 3×3 minors of ϕ_2' generated an ideal of codimension ≥ 2.

(1) Theorem 1.2 shows that the 5×5 minors containing the first 3 columns of ϕ_2 are the product of the last 3 generators of the semigroup ideal I with the 3×3 minors of ϕ_3 omitting the first 3 rows, and these minors are each a product of the last nonzero element of the first column of ϕ_3 with the 2×2 minors of Q. The ideal of 3×3 minors in the first 3 columns of ϕ_2 (or, equivalently, ϕ_2') contain these products. Similarly the 3×3 minors in the last 3 columns of ϕ_2' generate an ideal containing the products of the second to last nonzero element.

Since the nonzero entries of the first column of ϕ_3 form a regular sequence, this shows that the ideal of 3×3 minors of ϕ_2' contains the product of two ideals of codimension ≥ 2, completing the proof in this case.

(2) Hypothesis (2), the 2×2 minors of the nonzero 3×2 submatrix A consisting of the first two columns of ϕ_2' compose with A to 0. Since the semigroup algebra is a domain, the generators of the semigroup ideal are all irreducible polynomials, and thus the first 3 generators form an ideal of codimension ≥ 2. It follows that the 2×2 minors of A are multiples of these generators, and since they have the same degrees, they must generated the same ideal.

Because the last two elements of the first two columns of ϕ_2' are 0, the 3×3 minors of ϕ_2' involving the first 3 columns and the last row are multiples of these 2×2 minors by the last element of the third column, and similarly the 3×3 minors of ϕ_2' involving the first second and fourth columns and the last row are multiples of these 2×2 minors by the last element of the fourth column. Thus again the 3×3 minors of ϕ_2' contain the product of two ideals of codimension 2, completing the proof in this case. □

3. Examples

Example 3.1. There are no degree-special semigroups of genus < 13, but they exist in every genus from 13 to 20 *except* genus 18. Here is the complete list of the 4-tuples of generators:

```
g = 13, {6, 9, 13, 16}, {8, 9, 12, 13}
+----------------------------------------------
g = 14, {6, 9, 14, 17}
+----------------------------------------------
g = 15, {8, 10, 13, 15}
+----------------------------------------------
g = 16, {6, 9, 16, 19}, {8, 11, 12, 15}
+-----------------------------------------------
g = 17, {6, 9, 17, 20}, {9, 10, 14, 15}
+-----------------------------------------------
g = 19, {6, 9, 19, 22}, {8, 12, 13, 17}
+-----------------------------------------------
g = 20, {6, 9, 20, 23}, {6, 15, 19, 22}
```

Example 3.2. Consider the semigroup $L = \{6, 9, 9+c, 12+c\}$ for an integer $c \not\equiv 0 \mod 3$. The semigroup ideal of L has a $\mathbb{Q}[x_0, x_3, y, z]$-resolution

$$0 \longleftarrow S \xleftarrow{\phi_1} S^6 \xleftarrow{\phi_2} S^8 \xleftarrow{\phi_3} S^3 \longleftarrow 0$$

with differentials as indicated below for the case when c is even:

$$
\begin{array}{|c|c|}
\hline
\phi_1^t & \phi_2 \\
\hline
 & \phi_3^t \\
\hline
\end{array}
=
\begin{array}{|c|cccccccc|}
\hline
x_3^2 - x_0^3 & -y & -z & 0 & 0 & 0 & f & 0 & 0 \\
x_3 y - z\,x_0 & x_3 & x_0^2 & -z & y\,x_0 & -y^2 & 0 & -f & 0 \\
x_3 z - y\,x_0^2 & x_0 & x_3 & y & -z & 0 & -y^2 & 0 & f \\
z^2 - y^2 x_0 & 0 & 0 & -x_0 & x_3 & 0 & 0 & 0 & -y^2 \\
y^2 z - x_3 f & 0 & 0 & 0 & 0 & -x_0 & x_3 & -y & z \\
y^3 - x_0 f & 0 & 0 & 0 & 0 & x_3 & -x_0^2 & z & -y\,x_0 \\
\hline
 & z & -y & x_3 & x_0 & 0 & 0 & 0 & 0 \\
 & f & 0 & y^2 & 0 & -z & y & x_3 & -x_0 \\
 & 0 & -f & 0 & y^2 & yx_0 & -z & -x_0^2 & x_3 \\
\hline
\end{array}
$$

where $f \in \mathbb{C}[x_0]$. Here we use any monomial order on S which has the terms colored in red as lead terms. For example we can take the weight order of the grading on S refined by the reversed lexicographic order with $x_3 > y > z > x_0$ for the variables. The colored terms in the syzygy matrices are then the lead term with respect to the induced monomial orders, and the complex is exact by the Gröbner basis algorithm for syzygies as presented for example in [Sch25][Section 8.3]. In the universal family the entries indicated in blue remain zero for degree reasons if $c \geq 4$. For c coprime to 6 the differentials look similarly, but slightly different. For example the last two equations take the form $y^2 z - x_0 f$, $y^3 - x_3 f$ and the higher differentials change accordingly. In all cases the semigroup L has genus $c + 9$ and for $c \geq 4$ the degree conditions on ϕ_3 and ϕ_2 are satisfied. These L's give examples of non Weierstrass semigroups for any genus $g \geq 13$ with $g \not\equiv 0 \mod 3$.

Kunz and Waldi [KW17] studied a family of semigroups S such that when S is minimally generated by 4 elements the resolution of the semigroup algebra has format $\{1, 6, 8, 3\}$. Moreover the ideal of a 4-generator Kunz-Waldi semigroup can be written as a sum of 3 perfect, 3-generator ideals [SS24, Sect. 4.1]. The degree-special semigroup with generators $\{6, 9, 13, 16\}$ is Kunz-Waldi, but the degree-special semigroup $\{6, 9, 14, 17\}$, for example, is not.

Example 3.3. An example of a semigroup such that there is a flat family with smooth generic fiber, but having a section is $L = \{7, 15, 23, 39\}$. The genus of the semigroup is 30. The minimal free resolution over $S = \mathbb{Q}[x_0, x_1, x_2, x_4]$ has shape

$$0 \longleftarrow S \xleftarrow{\phi_1} S^4 \xleftarrow{\phi_2} S^6 \xleftarrow{\phi_3} S^3 \longleftarrow 0$$

with differentials as indicated below.

$$\begin{array}{|c|c|}\hline \phi_1^t & \phi_2 \\ \hline & \phi_3^t \\ \hline \end{array} = \left|\begin{array}{c|cccccc} x_1^2 - x_0x_2 & -x_2^2 + x_0x_4 & -x_2x_4 & -x_0^{10} & -x_4^2 + x_0^9x_1 & x_0^9x_2 & 0 \\ x_2^2 - x_0x_4 & x_1^2 - x_0x_2 & -x_0x_4 & -x_1x_4 & 0 & x_0^{10} & x_0^9x_1 - x_4^2 \\ x_1x_2x_4 - x_0^{11} & 0 & x_1 & x_2 & 0 & -x_4 & 0 \\ x_4^2 - x_0^9x_1 & 0 & -x_0^2 & -x_0x_1 & x_1^2 - x_0x_2 & x_1x_2 & x_2^2 - x_0x_4 \\ \hline & x_4 & -x_2 & x_1 & x_0 & 0 & 0 \\ & 0 & x_4 & 0 & -x_2 & x_1 & -x_0 \\ & -x_0^9 & 0 & -x_4 & 0 & -x_2 & x_1 \end{array}\right|$$

Again the lead terms in suitable monomial orders are indicated in red. The universal unfolding has a section because the entries indicate by blue remain zero for degree reasons. However this is a Weierstrass semigroup by [Wal80], since the semigroup ring is an almost complete intersection. Indeed a smoothing family is given by the polynomials

$$\begin{aligned}
&x_1^2 - x_2x_0 + x_2t^7 \\
&x_2^2 - x_4x_0 \\
&x_1x_2x_4 - x_0^{11} + x_0^{10}t^7 - x_0^2t^{63} + x_0t^{70} \\
&x_4^2 - x_1x_0^9 - x_1t^{63}
\end{aligned}$$

Example 3.4. There exist non Weierstrass semigroups of multiplicity m for every $m \geq 6$. Buchweitz examples yield non Weierstrass semigroups of multiplicity m and genus $m + 3$ for every $m \geq 13$. The degree-special special semigroups of multiplicity $m \in \{6, \ldots, 12\}$ of smallest genus are

```
{6, 9, 13, 16},    g = 13
---------------------------
{7, 15, 16, 24},   g = 22
---------------------------
{8,  9, 12, 13},   g = 13
---------------------------
{9, 10, 14, 15},   g = 17
---------------------------
{10, 11, 15, 16}, g = 21
---------------------------
{11, 12, 17, 18}, g = 26
---------------------------
{12, 13, 15, 29}, g = 27
---------------------------
{12, 14, 15, 25}, g = 27
```

Example 3.5. Let $L = \{6, 9, 13, 16\}$ and consider the semigroup generated by $L' = 2L \cup \{6 + 9\} = \{12, 15, 18, 26, 32\}$. The semigroup ideal $I_{L'} \subset S = \mathbb{Q}[x_0, x_3, x_6, x_2, x_8]$ is generated by the image of $I_L \subset \mathbb{Q}[x_0, x_3, x_1, x_4]$ under the map $x_i \mapsto x_{2i}$ and the equation $g = x_3^2 - x_0x_6$. Thus the resolution is a mapping cone

$$\begin{tikzcd}
S & S^6 \arrow[l] & S^8 \arrow[l] & S^3 \arrow[l] & 0 \arrow[l] \\
S \arrow[u, "g"] & S^6 \arrow[l] \arrow[u, "g"] & S^8 \arrow[l] \arrow[u, "g"] & S^3 \arrow[l] \arrow[u, "g"] & 0 \arrow[l]
\end{tikzcd}$$

By degree reason, the universal unfolding has a 2-valued section and the flat family will has an exact subcomplex of type

$$S \longleftarrow S^5 \longleftarrow S^8 \longleftarrow S^5 \longleftarrow S^1 \longleftarrow 0.$$

Perhaps this is another non Weierstrass semigroup. Similar constructions yields a lot of Weierstrass semigroups whose families have a multivalued section.

The last example is closely related to Torres covering construction of further non Weierstrass semigroups:

Theorem 3.6 (Torres,[Tor94],rough idea)**.** *Let L be a non Weierstrass semigroup. Consider the semigroup L' generated by $2L$ and some further odd generators of sufficiently large value. Then L' is not a Weierstrass semigroup.*

Proof. Suppose L' is the Weierstrass semigroup of a point p in a smooth projective curve $\widetilde{C}$. Let m be the maximal element among the minimal generators of L. We assume that the odd degree generators of L' are larger than $2m$. Consider the morphism $\varphi_{|2mp|}\colon \widetilde{C} \to \mathbb{P}^r$ where $r = |\{\ell \in L \mid \ell < m\}|$. If $\varphi_{|2mp|}$ is birational onto its image then the genus of $\widetilde{C}$ is bounded by Castelnuovo's bound for the genus of non-degenerate curves of degree $2m$ in $\mathbb{P}^r$. We can choose our odd generators so large that the genus of the semigroup L' exceeds this bound. Thus the morphism $\varphi_{|2mp|}$ factors $\widetilde{C} \to C$ over a smooth projective curve C and the degree of $\widetilde{C} \to C$ is 2 since the $\gcd(2L) = 2$. Moreover p is a branch point. The image $q \in C$ of p would have L as its Weierstrass semigroup, a contradictiom. $\square$

Example 3.7. For our semigroup $L = \{6, 9, 13, 16\}$ of genus 13 Torres theorem implies that the semigroups $L' = \{12, 18, 26, 32, 2x + 1\}$ with $x \geq 120$ is not Weierstrass: The genus of a non-degenerate curve of degree $d = 32$ in $\mathbb{P}^r$ for $r = 4$ is bounded by

$$\pi_0 = \pi_0(d, r) = (r - 1)n(n - 1)/2 + n\epsilon = 145$$

since $d - 1 = n(r - 1) + \epsilon$ with $0 \leq \epsilon < r - 2$ is satisfied $n = 10$ and $\epsilon = 1$ [Har82][Castelnuovo's bound] and the genus of L' is $g' = 2\cdot 13 + x > 145$ for $x \geq 120$. Looking to the embedding by $|52p|$ into $\mathbb{P}^{13}$ this can be improved to $g' > 82$ for $x > 56$.

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