REYNOLDS OPERATOR, OMEGA PROCESS AND WHY IT ALL WORKS

1. Group actions

Let G be an algebraic group and V an (algebraic) representation, i.e., we have an algebraic map $m : G \times V \to V$. This gives a 'pullback' map on the coordinate rings $\mu : K[V] \to K[V] \otimes K[G]$. Here $K[G]$ denotes the coordinate ring of the algebraic group. Since representations are 'linear' actions, the pull back map preserves the degree of polynomials in V, i.e., we get $\mu : K[V]_d \to K[V]_d \otimes K[G]$, where $K[V]_d$ denotes degree d polynomials.

2. The Reynolds operator

For any representation V of a reductive group G, let $V^G := \{v \in V \mid g \cdot v = v \ \forall g \in G\}$ denote the invariants. Clearly V^G is a subrepresentation of G. By reductivity, we have a G-equivariant projection called Reynolds operator $R : V \twoheadrightarrow V^G$.

It is very easy to create a projection from V to V^G, but harder to guarantee G-equivariance. If we have an inner product that is G-invariant, then the Reynolds operator is just the orthogonal projection. For a complex reductive group like GL_n, it's not possible to get a GL_n-invariant form, but we can use a U_n-invariant form, and mumble something about Zariski dense to finish the argument.

3. The Master Reynolds operator

Let G be a reductive algebraic group, and let $K[G]$ denote the coordinate ring. G acts on G by left multiplication. This gives an action of G on the coordinate ring $K[G]$. It turns out that $K[G]^G = K$, where K denotes the constant functions on G. This follows from Peter-Weyl theorem (there's probably a simpler explanation)

In any case, $K[G]$ is a representation of G, and so we have a Reynolds operator. I will call this $R_G : K[G] \twoheadrightarrow K$ the "master Reynolds operator."

Now, let us justify why R_G is the master Reynolds operator. Given any representation V of G, we have the pull back map on coordinate rings $\mu : K[V] \to K[V] \otimes K[G]$ (see Section 1). If we compose this with the master Reynolds operator R_G, we get the Reynolds operator for $K[V]$. Precisely, we have

$$R : K[V] \xrightarrow{\mu} K[V] \otimes K[G] \xrightarrow{\mathrm{Id} \otimes R_G} K[V].$$

The condition $f \in K[V]^G$ is the same as $\mu(f) = f \otimes 1$. So, it is clear that $R(f) = f$ if $f \in K[V]^G$. Moreover, it is clear that R is G-equivariant, because both maps are. We want to show that $R(f) = R(\sigma f)$ for $\sigma \in G$. Suppose $\mu(f) = \sum_i f_i \otimes a_i$, then $\mu(\sigma f) = \sum_i f_i \otimes \sigma a_i$. So,

$$R(f) = \sum_i f_i \otimes R_G(a_i) = \sum_i f_i \otimes R_G(\sigma a_i) = \mu(\sigma f).$$

This means that image of R is $K[V]^G$. This proves that R is the Reynolds operator.

As discussed in Section 1, the map μ restricts to degree d homogeneous polynomials nicely, so we have:

$$R : K[V]_d \xrightarrow{\mu} K[V]_d \otimes K[G] \xrightarrow{\mathrm{Id} \otimes R_G} K[V]_d.$$

4. The master Reynolds operator via the defining the representation

Reductive groups often come with a defining representation, which gives an embedding $G \hookrightarrow GL_n \subseteq \mathrm{Mat}_{n,n}$. The group GL_n is slightly annoying as a variety, but $\mathrm{Mat}_{n,n}$ is a vector space, which is the best kind of variety there is. The best situation is when the embedding $G \hookrightarrow \mathrm{Mat}_{n,n}$ is a Zariski-closed subset, i.e., defined by some algebraic equations.

Let's not get bogged down in the general setup. So, let's just do the case of SL_n and then O_n.

4.1. Special linear group.

The group $\mathrm{SL}_n \subseteq \mathrm{Mat}_{n,n}$ is defined by $Det = 1$. In other words, the coordinate ring is $K[SL_n] = K[x_{i,j}]/(Det - 1)$, so we have a surjection $K[\mathrm{Mat}_{n,n}] = K[x_{ij}] \twoheadrightarrow K[\mathrm{SL}_n]$. There is a master Reynolds operator $R_G : K[\mathrm{SL}_n] \to K$. Moreover, the defining action (i.e., left multiplication of SL_n on $\mathrm{Mat}_{n,n}$) gives a Reynolds operator $R : K[x_{ij}] \to K[x_{ij}]^{\mathrm{SL}_n}$. These fit together nicely in a commutative diagram

$$\begin{array}{ccc} K[x_{i,j}] & \longrightarrow & K[x_{i,j}]^{\mathrm{SL}_n} \\ \downarrow & & \downarrow \\ K[\mathrm{SL}_n] & \longrightarrow & K \end{array}$$

- The top horizontal map is the Reynolds operator for the defining action.
- The bottom horizontal map is the master Reynolds operator.
- The two vertical maps are taking quotients by the relation (Det - 1).

4.1.1. *Reynolds operator for the defining action and Omega process.* First, let us note that $K[x_{ij}]^{\mathrm{SL}_n} = K[Det]$. This follows from FFT and SFT for SL_n. Now, let us restrict to degree dn. The invariants $(K[x_{ij}]^{\mathrm{SL}_n})_{dn}$ are just scalar multiples of Det^d, and so 1-dimensional. To write the Reynolds operator as a projection, we just need an $SU(n)$-invariant inner product. One such inner product is given by the Bombieri norm, which we will denote $< -, - >$.

So, for $f \in K[x_{ij}]_{dn}$, the orthogonal projection is then just the formula

$$f \mapsto \left(\frac{< f, Det^d >}{< Det^d, Det^d >} \right) Det^d.$$

Now, a little thought with the Bombieri norm will tell you that upto a scalar, $\Omega^d =< -, Det^d >$. The $< Det^d, Det^d >$ term is then the renormalization that we see in the Omega process.

4.1.2. *Reynolds operator for defining repn to coefficient bounds.* Suppose we have an action of SL_n on W. This gives a map $\mu : K[W] \to K[W] \otimes K[\mathrm{SL}_n]$. This map is actually not mysterious, and let me tie it back to how we have gotten used to seeing the Omega process work. Identify W with K^m and let the coordinate functions be $x_1, \dots, x_m$. The repn is given by a map $\mathrm{SL}_n \to \mathrm{Mat}_{m,m}$, which we (Avi in particular) like to interpret as an $m \times m$ matrix of functions on SL_n, i.e. a matrix whose entries are in $K[\mathrm{SL}_n]$. The map μ is quite simple. Suppose the first column consists of the functions $a_1, \dots, a_m$. Then, we have $\mu(x_1) = \sum_i x_i \otimes a_i$. Then, applying the master Reynolds operator on the second tensor factor, so we get $R(x_1) = \sum_i R_{\mathrm{SL}_n}(a_i) x_i$. So, the coordinates of the invariant in the basis $x_1, \dots, x_m$ is given by R_{SL_n} applied to the functions in the first column. Similarly coefficient bounds is just bounding $R_{\mathrm{SL}_n}(a_i)$.

Now, how does one describe functions in $K[\mathrm{SL}_n]$? We can represent it by a polynomial that maps to it under the surjection $K[x_{ij}] \twoheadrightarrow K[\mathrm{SL}_n]$.

Suppose $a \in K[\mathrm{SL}_n]$, and suppose $f \in K[x_{ij}]$ that represents a. Look at the above commutative diagram. To compute $R_{\mathrm{SL}_n}(a)$ amounts to starting with $f \in K[x_{ij}]$, going down first, and then right on the diagram. But this is the same as going right, and then down. But now suppose f is homogeneous of degree dn (otherwise split into homogeneous components first), then first going right is given the formula for orthogonal projection above. Then going down is the same as replacing Det by 1, so this tantamounts to

$$R_{\mathrm{SL}_n}(a) = \left(\frac{< f, Det^d >}{< Det^d, Det^d >} \right) = \frac{\Omega^d(f)}{\Omega^d(Det^d)},$$

Remark 4.1 (Upshot)**.** *In order to compute a Reynolds operator, we need to be able to compute the master Reynolds operator on the entries of the matrix describing the representation. This master Reynolds operator is defined on $K[\mathrm{SL}_n]$. But elements in $K[\mathrm{SL}_n]$ are naturally represented by polynomials in $K[x_{ij}]$ (Think about it, this is how they will almost always be given). So, for $a \in K[\mathrm{SL}_n]$, which is represented by $f \in K[x_{ij}]$, we can follow the commutative diagram the other way.*

4.2. **Orthogonal group.** The ideas are precisely the same as summarized in the above remark. So, it all boils down to understanding the commutative diagram.

$$\begin{array}{ccc} K[x_{i,j}] & \longrightarrow & K[x_{i,j}]^{O_n} \\ \downarrow & & \downarrow \\ K[O_n] & \longrightarrow & K \end{array}$$

We have to look closer at our commutative diagram. $K[O_n] = K[x_{i,j}]/(p_{ij} - \delta_{i,j})$, where p_{ij} is the inner product between the i^{th} and the j^{th} column, i.e., $p_{ij} = \sum_k x_{ki}x_{kj}$. So $p_{ij} - \delta_{i,j} = 0$ for all i, j is a fancy way of writing $XX^t = \mathrm{Id}$. Now, what is $K[x_{i,j}]^{O_n}$? By the FFT and SFT for O_n, we have $K[x_{ij}]^{O_n} = K[p_{ij}]$, which is a polynomial ring with no relations.

Given $a \in K[O_n]$, it will be represented by $f \in K[x_{ij}]$. To compute $R_{O_n}(a)$, we should take f, go right, and then down. So, let us analyze those two maps.

The going down map is easy. It takes $p_{ii} \mapsto 1$ and $p_{ij} \mapsto 0$ if $i \neq j$. The top horizontal map seems harder to write. The only thing that prevents us from having coefficient bounds is the ability to analyze this map. First, the top horizontal map can be split into degree. In odd degrees, the map is zero. In even degrees, the map is non-zero (because p_{ij} have degree 2).

If we restrict to degree $2d$ polynomials, we have $R : K[x_{ij}]_{2d} \to (K[x_{ij}]^{O_n})_{2d} = K[p_{ij}]_d$. So, if $R(f) = \sum_I \lambda_I p^I$, where p^I denotes a monomial in the p_{ij}'s, then $R_{O_n}(a) = \sum_J \lambda_J$, where J is the subset of monomials that do not contain a p_{ij} with $i \neq j$. This is because the right vertical map kills all p_{ij}'s for which $i \neq j$. Roughly speaking, you start with this matrix of representation (or rather the $2d^{th}$ symmetric power), then pick one entry - say it is $a \in K[O_n]$, and represented by $f \in K[x_{ij}]$, you replace it with $\sum_J \lambda_J$. This is the analogue of the procedure for SL_n.

Remark 4.2. *The problem therefore is to compute (or bound) $\sum_J \lambda_J$. If the p^Is were orthogonal, then you could compute each λ_J as $< f, p^J >$ upto some scalar. But this is unfortunately not true. The monomials in the p_{ij}'s are not orthogonal under the Bombieri norm. Funnily enough the smaller subset of just the monomials in p_{ii}'s are orthogonal under the Bombieri norm, but I don't how this might be helpful.*

One vague idea that just kind of floated in my head was to show (quantitatively) that p^Is are almost orthogonal (i.e., their inner products are small), and then use that to conclude that λ_J and $\frac{<f,p^J>}{<p^J,p^J>}$ do not differ by much. We would probably get slightly larger bounds, but I wouldn't be surprised if they stayed within the same complexity class.

Remark 4.3. *One can do this for other classical groups as well – SP(n), SO(n) etc, but it would be as complicated as their invariant theory is.*

5. An Omega-Like Process for any reductive group

Suppose G is a (connected?) reductive group. In order to understand Reynolds operators for various representations, we need to understand the master Reynolds operator $R_G : \mathbb{C}[G] \to \mathbb{C}$. From a computational complexity perspective, there is the issue that one needs to identify how to represent elements of $\mathbb{C}[G]$. To overcome this issue, we propose the following:

Suppose V is a faithful representation of G. In other words, we have an injective homomorphism $\pi : G \hookrightarrow \mathrm{GL}(V)$. Now $\mathrm{GL}(V)$ acts on $\mathrm{End}(V)$ in many ways, but here we consider the following action: For $\phi \in \mathrm{GL}(V)$ and $\psi \in \mathrm{End}(V)$, we have $\phi \cdot \psi = \phi \circ \psi$, the composition of the two functions $\phi, \psi : V \to V$. This means that we have an action of G on $\mathrm{End}(V)$ given by $g \cdot \psi = \pi(g) \cdot \psi = \pi(g) \circ \psi$.

If V is n-dimensional, we can identify $V = \mathrm{Mat}_{n,n}$, thus $\mathbb{C}[\mathrm{End}(V)] = \mathbb{C}[x_{ij}]$ where x_{ij} denotes the $(i,j)^{th}$ coordinate of $\mathrm{Mat}_{n,n}$.

Lemma 5.1. *$\pi(g)$ is a closed subgroup of GL_n*

Proof. It is a basic result that the image of a homomorphism of algebraic groups is closed. $\square$

Thus, $\mathbb{C}[G]$ is a quotient of $\mathbb{C}[x_{ij}][1/\det]$.

5.1. The case when $\pi(g)$ is a closed subgroup of SL_n.

In this case, we get that $\mathbb{C}[G]$ is a quotient of $\mathbb{C}[\mathrm{SL}_n]$ which is a quotient of $\mathbb{C}[x_{ij}]$. So, we have an inclusion $\iota : G \hookrightarrow \mathrm{Mat}_{n,n}$, which gives a surjection $\iota^* : \mathbb{C}[x_{ij}] \twoheadrightarrow \mathbb{C}[G]$

Lemma 5.2. *Let I_π be the ideal of all polynomials in $\mathbb{C}[x_{ij}]$ which vanish on $\pi(G)$. Then I_π is generated by invariant polynomials.*

Proof. Consider the action of G on $\mathrm{Mat}_{n,n}$ as described previously. Under this action, the stabilizer for any point in $p \in \mathrm{GL}_n \subseteq \mathrm{Mat}_{n,n}$ is trivial. Further, we claim that the orbit of p is closed. In other words, we are claiming that all G-orbits of points in GL_n are closed. First, it suffices to prove this for any one orbit because if $G \cdot p$ is closed, then clearly so is $G \cdot pA$ for any $A \in \mathrm{GL}_n$. But the latter is the orbit of pA, and every point in GL_n is of the form pA for some $A \in \mathrm{GL}_n$.

Now, $\pi(G)$ is the orbit of identity, which is closed by the assumption that $\pi(G)$ is a closed subgroup of SL_n.

Recall Mumford's result that invariants can separate orbit closures. Since GL_n consists of closed orbits of maximal dimension, we conclude that under the quotient $j : \mathrm{Mat}_{n,n} \twoheadrightarrow \mathrm{Mat}_{n,n}//G$, every closed orbit in GL_n is mapped to a single point, whose preimage consists of nothing else. Suppose $p \in \mathrm{GL}_n$, and $j(p) = q$ in the quotient. Let us describe the scheme-theoretic fiber. Suppose $\mathbb{C}[\mathrm{Mat}_{n,n}]^G = \mathbb{C}[f_i]$. Let $f_i(A) = c_i$. Then it is clear that the maximal ideal of functions on $\mathbb{C}[\mathrm{Mat}_{n,n}//G] = \mathbb{C}[f_i]$ that vanish at q is given by $I_q = \{f_i - c_i\}$. The scheme theoretic fiber over q is the ideal in $\mathbb{C}[\mathrm{Mat}_{n,n}] = \mathbb{C}[x_{ij}]$ generated by $\{f_i - c_i\}$, let us call it J. Note that this is bigger than I_q even though it has the same generators because we are in a different ring.

It is clear now that $J = \{f_i - c_i\} \subseteq \mathbb{C}[x_{ij}]$ set-theoretically cuts out $G \cdot A$. The only issue is whether J is radical, equivalently, whether the fiber is reduced. This now comes from two arguments. First, generically the fibers are reduced. Second for $A, B \in \mathrm{GL}_n$, the fiber over $j(A)$ is reduced if and only if the fiber over $j(B)$ is reduced since $\mathrm{GL}_n //G$ is a homogeneous space and $\mathrm{GL}_n //G \hookrightarrow \mathrm{Mat}_{n,n}//G$. $\square$

Corollary 5.3. *Suppose $G \hookrightarrow \mathrm{SL}_n$ is a closed embedding, as above. Let $\mathbb{C}[\mathrm{Mat}_{n,n}]^G = \mathbb{C}[f_i]$, and let I denote the identity matrix. Then, we have a commutative diagram, where both vertical arrows are surjections got by quotienting out the relations $\{f_i - f_i(\mathrm{I})\}$, and both horizontal arrows are Reynolds operators.*

$$\begin{array}{ccc} \mathbb{C}[x_{i,j}] & \longrightarrow & \mathbb{C}[x_{i,j}]^G \\ \downarrow & & \downarrow \\ \mathbb{C}[G] & \longrightarrow & \mathbb{C} \end{array}$$

Let W be a representation of G. Thus, we have a map $G \to \mathrm{GL}(W)$ or equivalently a map $G \times W \to W$. To describe this map in coordinates, we will think of G as a closed subgroup of SL_n. Suppose W is m dimensional with coordinate functions $y_1, \dots, y_m$. Thus, giving a map $\rho : G \times W \to W$ amounts to giving a description that tells us what $\rho(g, w)$ is where $g = (g_{ij})$ and $w = (w_1, \dots, w_m)$. Now, $\rho(g, w)$ is an m-dimensional vector, so $\rho(g, w) = (\rho_1(g, w), \dots, \rho_m(g, w))$, where $\rho_k : G \times W \to \mathbb{C}$ is a regular function on $G \times W$. But since G is a closed subset of $\mathrm{Mat}_{n,n}$, any regular function ρ_k on $G \times W$ can be described as a polynomial p_k in the variables x_{ij} for $\mathrm{Mat}_{n,n}$ and $y_1, \dots, y_m$ for W.

From the perspective of arithmetic complexity, this is how one must represent the action $\rho : G \times W \to W$. Now, the action ρ gives a map on the coordinate rings $\mu : \mathbb{C}[W] \to \mathbb{C}[W] \otimes \mathbb{C}[G]$. where $y_i \mapsto \rho_i$. But given that we represent ρ_i as polynomials p_i, we can in fact write μ as a composition of maps

$$\mathbb{C}[W] \to \mathbb{C}[W] \otimes \mathbb{C}[\mathrm{Mat}_{n,n}] \to \mathbb{C}[W] \otimes \mathbb{C}[G],$$

where the first map is given by $y_i \mapsto p_i$, and the second map is $\mathrm{id} \otimes \iota^*$.

Putting it all together, we summarize things as the following.

Proposition 5.4. *The Reynolds operator R_W is given as the following map:*

$$\mathbb{C}[W] \to \mathbb{C}[W] \otimes \mathbb{C}[\mathrm{Mat}_{n,n}] \to \mathbb{C}[W] \otimes \mathbb{C}[Mat_{n,n}]^{\mathrm{SL}_n} \to \mathbb{C}[W] \otimes \mathbb{C} = \mathbb{C}[W],$$

where the first map is given by $y_i \mapsto p_i$, the second map is $\mathrm{id} \otimes R_{\mathrm{Mat}_{n,n}}$ and the third map is given by $f_i \mapsto f_i(\mathrm{I})$.

Let us note that in the proposition, the first map will be part of the description of the representation W. Thus, to understand the Reynolds operator, one must understand the composition of the second and third map. But this really is just about understanding $\mathbb{C}[\mathrm{Mat}_{n,n}] \to \mathbb{C}[\mathrm{Mat}_{n,n}]^{\mathrm{SL}_n} \to \mathbb{C}$ and then tensoring with id.

The coefficient bounds will depend on the map $\mathbb{C}[\mathrm{Mat}_{n,n}] \to \mathbb{C}[\mathrm{Mat}_{n,n}]^{\mathrm{SL}_n} \to \mathbb{C}$ and the polynomials p_i, just like we exhibited in the case of SL_n and O_n previously.

5.2. **The case when G is a closed subgroup of GL_n, but not a closed subset of $\mathrm{Mat}_{n,n}$.** This part is silly (which I realized much later). Then, one can just do $G \hookrightarrow \mathrm{GL}_n \hookrightarrow \mathrm{SL}_{n+1}$ where both are closed embeddings.

6. Omega-like process for a parametric group

Let $G = \left\{ g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid \det(g) = 1 \right\} \subseteq \mathrm{SL}_{2n}$. Then, we will describe an Omega process for this group. First, we must write $G = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = -C, A = D, \det = 1 \right\}$. Thus $\mathbb{C}[G] = \mathbb{C}[a_{ij}, b_{ij}, c_{ij}, d_{ij}]/\mathcal{J}$ where $\mathcal{J} = \{a_{ij} - d_{ij}, b_{ij} + c_{ij}, \det\}$.

$$\begin{array}{ccc} \mathbb{C}[a_{ij}, b_{ij}, c_{ij}, d_{ij}] & \longrightarrow & \mathbb{C}[a_{ij}, b_{ij}, c_{ij}, d_{ij}]^G \\ \downarrow & & \downarrow \\ \mathbb{C}[G] & \longrightarrow & \mathbb{C} \end{array}$$

Or equivalently, quotienting $a_{ij} = d_{ij}$ and $b_{ij} = -c_{ij}$, we get the diagram

$$
\begin{array}{ccc}
\mathbb{C}[a_{ij}, b_{ij}] & \longrightarrow & \mathbb{C}[a_{ij}, b_{ij}]^G \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}[G] & \longrightarrow & \mathbb{C}
\end{array}
$$