Gaussian mixture models

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Method of moments

- **Problem :** Given data $\{y_1, \ldots, y_N\} \sim X$, where

$$X \sim \frac{1}{k}\mathcal{N}(\mu_1, 1) + \ldots + \frac{1}{k}\mathcal{N}(\mu_k, 1)$$

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- When $k = 4$ the moment equations are
$$\begin{aligned}
\overline{m}_1 &= \frac{1}{4}\mu_1 + \frac{1}{4}\mu_2 + \frac{1}{4}\mu_3 + \frac{1}{4}\mu_4 \\
\overline{m}_2 &= \frac{1}{4}\mu_1^2 + \frac{1}{4}\mu_2^2 + \frac{1}{4}\mu_3^2 + \frac{1}{4}\mu_4^2 + 1 \\
\overline{m}_3 &= \frac{1}{4}(\mu_1^3 + 3\mu_1) + \frac{1}{4}(\mu_2^3 + 3\mu_2) + \frac{1}{4}(\mu_3^3 + 3\mu_3) + \frac{1}{4}(\mu_4^3 + 3\mu_4) \\
\overline{m}_4 &= \frac{1}{4}(\mu_1^4 + 6\mu_1^2) + \frac{1}{4}(\mu_2^4 + 6\mu_2^2) + \frac{1}{4}(\mu_3^4 + 6\mu_3^2) + \frac{1}{4}(\mu_4^4 + 6\mu_4^2) + 3
\end{aligned}$$

Project specifics

We are interested in multiple aspects of the polynomial moment equations:

1. Find closed form solution for unique solution (up to symmetry) to moment equations
2. Compute/understand discriminant conditions for various k. i.e. find conditions where the moment equations don't have $k!$ regular, isolated points
3. Compute/understand *statistically meaningful* regions in the moment space (moments that give rise to real solution $\mu_1, \ldots, \mu_k$)
4. Investigate various projection schemes for projecting sample moments onto statistically meaningful ones
5. Investigate the problem of how to pick k from data