Ptolemy's Mathematical Approach: Applied Mathematics in the Second Century

by

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Your file Votre référence
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For my parents,
all four of them

Abstract

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The study is an examination of the mathematical methods of Ptolemy and his predecessors. It attempts, so far as possible, to situate this work in the context of what we know about the rest of Greek mathematics and the exact sciences, with little or no reference to current scientific and mathematical knowledge.

After a brief discussion of Ptolemy's philosophy of mathematics, the first chapter gives a classification of types of mathematical text found in Ptolemy and the Greek applied mathematical tradition in general. This is followed by sections that deal with the use of ratio and tables in Ptolemy's work. In order to apply metrical methods to geometrical problems, Ptolemy uses proportions as equations and develops tables to model continuous functions. Both of these practices, although natural to us, are unusual in the context of Greek mathematics. I examine the implicit assumptions and explain how these methods serve the applied mathematician.

The second chapter is a study of the first and most crucial application of these methods: the development of the chord table and its application to trigonometric problems. It also examines the trigonometric methods of the Hellenistic mathematical astronomers and shows how these fundamentally differed from Ptolemy's practice. It develops a general picture of the mathematical practices used in the trigonometry by means of chord tables.

The third chapter is an examination of all of the evidence we have for the so-called Menelaus Theorem, the fundamental theorem of ancient spherical trigonometry. It studies the texts of Ptolemy, his predecessors and his commentators and shows that the line of transmission cannot have been as straightforward as has previously been assumed. This is followed by an investigation of Ptolemy's practices in applying the fundamental theorem. This study of Ptolemy's spherical astronomy acts as a case study which gives us insight into the deductive structure of Ptolemy's exact science. This investigation allows us to develop a sense for how the ancient mathematical astronomer used these methods to produce new results.

The final chapter is an exegesis of ancient methods of projecting the sphere onto the plane. It explores the texts of Ptolemy and his predecessors which are concerned with projecting the sphere either for the purpose of drawing maps or in order to model the sphere and solve for arc lengths. This leads to discussions of two important ancient methods of doing spherical geometry.

Each of these chapters describes a domain of Greek mathematical practice that is not witnessed in the theoretical texts and is generally left out of discussions of Greek mathematics. Moreover, in each case, I help the reader develop a sense for the methods and practices of the ancients instead of focusing simply on their results.

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Preface

It has often been claimed that Ptolemy did not produce original mathematics. Although he may have developed new applications and superior logical arrangements, and although this may have required here and there a new result or an innovative method, his overall approach and the character of his mathematical thought was thoroughly grounded in the work of his predecessors. There is no good evidence that Ptolemy did any work devoted to pure mathematics. Nevertheless, the majority of his works are highly mathematical. They are the application of mathematical methods to the production of models for explaining and describing the physical world.

This study makes no attempt to reexamine the question of Ptolemy's originality as a mathematician. Nevertheless, Ptolemy is our best evidence for almost all branches of ancient exact science and his texts preserve the mathematical methods, and many of the results, upon which these were based. In many cases, his methods go back well into the Classical and Hellenistic periods; our understanding of Greek mathematics would be incomplete without exploring them. Indeed, a study of Ptolemy and his immediate predecessors constitutes a fundamental study of applied mathematics in Greco-Roman antiquity.

In applying mathematics to the study of the physical world, Greek mathematicians not only used the results of the theoretical tradition, they also used its methods and practices, adapting them to the needs of the exact sciences. These sciences were also a domain in which new methods and practices were developed or adopted from other cultures.

Although there were clear divisions between the different fields of mathematics and the mathematical sciences, these were created by the patterns of textual tradition, not by any disciplinary division in the modern sense. Many texts in the exact sciences were written by authors who were also producing texts in pure mathematics. Ptolemy, in fact, appears to be a rare counterexample to this general trend.[1]

Because the Greek mathematicians themselves produced and practiced the exact sciences, and because Ptolemy is the best preserved author of these sciences, no account of Greek mathematical thought will be complete without including the evidence in Ptolemy's writings. This is not to say that Ptolemy's work can be used to explicate the particular concerns or ideas of his predecessors. It does, however, mean that in order to come to terms with ideas about, and approaches to, fundamental issues in Greek mathematics, we will need to take the mathematics of the exact sciences into account; and this means studying Ptolemy's approach to mathematics. It will not be possible to create a complete picture of Greek ideas about such things as *ratio*, *number*, the *relationship between geometry and arithmetic*, the *concept of given* and so on without exploring how these function in Ptolemy's writings. This is especially true if we are interested in the conceptual habits inherited by those cultures that considered the Greeks one of the sources of their own mathematical tradition.

The object of this study is not to chronicle Ptolemy's mathematical or scientific results. It is, rather, to explore how Ptolemy uses the elementary knowledge and methods of Greek mathematics to produce new knowledge in the exact sciences. To this end, the study focuses on three fundamental aspects of his mathematical approach: the linguistic expressions of mathematical discourse, the units of text that these form and the overall deductive architecture that express scientific theories. At each of these levels, Ptolemy operates in fairly well-defined and controlled ways. This fundamental methodology was adopted from the theoretical tradition of Greek mathematics.[2] Nevertheless, the math-

[1] Even Hipparchus appears to have worked in pure mathematics. See Acerbi [2003b] for a discussion of the evidence for his work in combinatorics.

[2] These aspects texts on Greek geometry are discussed at length by Netz [1999b].

ematical tools which Ptolemy has at his disposal are more diverse and powerful than those we encounter in the Greek traditions of pure mathematics. This study shows how Ptolemy's practice in the exact sciences is founded in, and grows out of, the practice of the pure traditions.

The first two chapters deal with the fundamental characteristics that distinguish Ptolemy's applied mathematics from the pure mathematical traditions. They provide a classification system for units of Ptolemy's texts and discuss the new practices that result from a desire to solve problems arithmetically, as opposed to geometrically. They treat the arithmetization of ratio, the use of tables as functions and as objects of knowledge, and the relationship between trigonometric computation and the underlying geometric objects.

The next two chapters apply these results to studies of Ptolemy's mathematical work on spherical astronomy. These furnish case studies which allow us to confront important aspects of Ptolemy's mathematical practice. They explore the various ways Ptolemy employed his predecessor's work, the overall deductive structure for which he strove and the use of various mathematical theories and methods to solve different, but related, sets of problems. This study is an account of the elementary methods underlie Ptolemy's work in the exact science. It exhibits the ways in which Ptolemy developed traditional Greek methods for the production of new knowledge in the exact sciences. A number of more advanced techniques, indeed some of Ptolemy's most creative mathematical work, has hence been left out of this study. These are, for example, methods of approximation by iteration and the mathematical modeling of motion through both geometric and numerical procedures.

Historiographic note

There have been considerable changes in the historiography of ancient mathematics in the last 30 years, and the transition has not always been smooth. Nevertheless, this period has seen much new and important work in the history of ancient mathematics and the mathematical sciences. Although few new sources have turned up during this time, a number of new methods and approaches have been developed and fruitfully applied to sources that have been available in critical editions since the beginning of the 20th century.[3]

The two methods that are most relevant to my work are textual studies and balanced reconstruction. Textual studies encompass scholarly pursuits ranging from edition and translation to investigating the interdependence between texts, or between logical units within a particular text. They analyze the linguistic structure, and rhetorical approach, exhibited by the texts. Two examples of the new approach to textual studies that I have found most useful are the *Éléments* by Vitrac and Netz's *The Shaping of Deduction in Greek Mathematics*.[4] Vitrac provides a new translation of, and commentary on, Euclid's *Elements* that sheds light on the thought processes that structure Greek mathematics by staying close to the literal meaning and grammatical structure of the text and commenting on, rather than trying to clean up, the idiosyncrasies in Euclid's expression. Netz gives a linguistic and cognitive study of core texts in the Greek mathematical corpus that allows rare insight into the conceptual methods and practices of the Greek geometers.

Studies like these show that one of the most interesting and useful ways to understand Greek mathematics is through carefully analyzing the texts themselves, in light of theories of argumentation contemporary with the texts, as opposed to translating the mathematics described in the texts into modern forms and evaluating them in the light of contemporary theories of mathematical knowledge. With good reason, elaborate reconstructions of ancient mathematical practices and theories are no longer much in favor. Nevertheless, since most of the ancient mathematical texts have been lost, in order to come to a full appreciation of the scope and depth of the field, it is still necessary to engage in some reconstruction. This reconstruction, however, must be carefully balanced against the texts we do have and it must adhere to mathematical methods and practices found

[3] Berggren [1984] and Saito [1996].
[4] Vitrac [2001] and Netz [1999b].

in related texts. Good examples of balanced reconstruction are Knorr's *The Ancient Tradition of Geometric Problems* and the essays on the lost works of geometrical analysis by Jones in his edition and translation of Pappus, *Book 7 of the Collection.*[5] Knorr helps us develop a sense for the methods and techniques that Greek mathematicians would have actually been able to use to do mathematics, to solve problems and generate new results. The essays by Jones describe the results of the lost works of geometric analysis while staying true to the spirit of the evidence we do have and avoiding the level of reconstructive detail that can only be achieved by pure speculation. These studies give us a sense of the breadth and variety of Greek mathematics by providing insight into the content and use of ancient geometric analysis.

My research combines these two approaches and applies them to a study of Ptolemy and his predecessors. The focus of most of the new work on ancient mathematics has been on the great mathematicians of the Hellenistic period: Euclid, Archimedes and Apollonius. Although the texts in Greek astronomy, optics and mechanics have often been studied from the perspective of the history of these individual sciences, they are rarely studied for the sake of explicating the mathematical methods they evince. Nevertheless, the ancient exact sciences were not produced simply by applying the results of pure mathematics to the physical world. Mathematical styles of reasoning were also imported and combined in interesting ways with empirical and observational methods. Moreover, many mathematical methods and styles of reasoning that were developed in the exact sciences had no counterpart in pure mathematical texts. By using the methods of textual study and balanced reconstruction to study the work of Ptolemy and his predecessors, I am able to develop a more complete picture of the mathematical methods and practices that ancient thinkers used to develop their models of the world.

Neugebauer has said that the best hope of the historian is to come to an understanding of facts and conditions, but never causes.[6] While balanced reconstruction may help us develop a better sense for the mathematical conditions, almost our only access to the facts

[5] Knorr [1986] and Jones [1986, 510 - 599].
[6] Neugebauer [1969, 255].

is through the ancient and medieval texts. In order to be convincing, a reconstruction must present itself as possessing the characteristic of plausibility. Our knowledge of history and our experience of life, however, assures us that those events which actually transpire are not at all inhibited by this constraint. Reconstructions will come and go. If we are to have any knowledge of ancient mathematics, it must be based securely on a close reading of the ancient texts. How we read these texts will, of course, change over time. This study is primarily an exposition of the way I read the mathematics of Ptolemy and his predecessors at the present time. Only occasionally do I engage in reconstruction to elucidate a particular passage or, more rarely, to illustrate what I believe are the underlying mathematical conditions. Through this study I hope to show that it is possible to produce a successful and explanatory reading of these authors by situating their work within the context of what we know about the rest of ancient mathematical practice with little or no reference to modern theories or knowledge.

Remarks on notation

In this study, I focus on situating the work of Ptolemy and his predecessors in the context of what we know about the rest of Greek mathematical practice, both in terms of form and content. I make little effort to translate ancient mathematics into modern forms. This is not because I do not believe that such translation is interesting or helpful, but because it has already been done so well that my own work could hardly have gone forward without the guidance of this previous scholarship.[7] Nevertheless, I introduce numerous notational changes to the original texts for the sake of the modern reader.

In some sense this is inevitable. The Greek expression for a square on line AB is "the upon the AB," τὸ ἀπὸ τῆς AB; for an angle between the lines AB and GB, "the under the ABGs," ἡ ὑπὸ τῶν $AB\Gamma$. These are obviously technical expressions.[8] I use the

[7] There are many examples of this form of scholarship. See for example Czwalina [1927], Luckey [1927], Pedersen [1974b] (with which see Toomer [1977]) and Neugebauer [1975].

[8] See Netz [1999b, 127 - 167] for a discussion of the role of such technical idioms in Greek mathematics.

equally technical, although less alien, AB^2 and $\angle ABG$. Some will object that AB^2 can indicate an arithmetical operation while "the upon the AB" always refers to an actual square. In the pure geometrical texts this is true. In the exact sciences, however, "the upon the AB" very often does imply the arithmetical operation of squaring the numerical value of the length of the line AB. For the modern reader, as for the ancient, context will determine the intended meaning of the expression.

According to Schmidt, every historian of mathematics devises a new system of notation for explaining the ancient texts.[9] Indeed, a great many systems of notation have been put forward over the years, with varying success.[10] Only recently, have accounts of Greek mathematics been written that allow the texts to speak for themselves without the use of modern notation.[11] Unfortunately, these sometimes go to the opposite extreme of presenting obscure texts with little or no commentary.[12] I attempt, as far as possible, to strike a balance between the ancient expressions and modern notation.

Burnett characterizes translation in the history of mathematics as operating at two levels: a literal translation from the text and a translation into modern notation.[13] This study is not primarily a translation of original texts. In the places where my reading depends upon the precise meaning of the text, I give a literal translation. Elsewhere, I take more liberties and translate from the Greek directly into a modern notation which is meant to serve as a shorthand for the ancient expression. When a standard English translation exists, I use this translation. Nevertheless, even here I introduce changes in

[9] Schmidt [1975, 3]. Striking examples of such systems are provided by Schmidt [1975] himself and Taisbak [1971]. Dijksterhuis [1987] introduces a notational system which, although historiographically important and conceptually helpful, introduces its own set of difficulties to an already difficult body of mathematical work.

[10] Opposite ends of this spectrum can even be furnished from the works of the same author. The notation developed in Taisbak [2003] is quite helpful for reading the *Data*. In fact, it is simply a shorthand for the expressions in the text itself. The notion that Taisbak [1971] introduces for studying *Elem.* VII - IX, on the other hand, leads me to wonder if the author and I are reading the same books.

[11] A good example is Fried and Unguru [2001].

[12] Cuomo [2001] is refreshingly free of equations and other modern formulae. On the other hand, there are too few explanatory comments, especially for a book addressed to a general audience. For example, in an already difficult passage we encounter, with no explanation, the conclusion, "and ΔE will be the double power of EZ, and $\Theta\Gamma$ of $E\Delta$, and ΘZ of $\Theta\Gamma$," Cuomo [2001, 88]. It is unlikely that anyone but a specialist would know that, where $\Theta Z = 2EZ$, this means that $EZ : \Delta E = \Delta E : \Theta\Gamma = \Theta\Gamma : \Theta Z$, so that $EZ : \Theta Z = (EZ : \Delta E)^3 = (\Delta E : \Theta\Gamma)^3 = (\Theta\Gamma : \Theta Z)^3$.

[13] Burnett [2003, 1].

notation and expression to bring these quotations into line with the rest of my discussion. I have no doubt that not all of my readers will be pleased with all of my choices.

For the ratio of A to B, I use the standard notation $A : B$. This does not generally mean that A is divided by B, although, as we will see, in the exact sciences this is sometimes the case. Again, context will help us decide. For a proportion, when the ratio of A to B is the same as C to D, I use the notation $A : B = C : D$. This notation may obscure the fact that Greek mathematicians did not think of the relation of equality as being reflexive and simply asserted the identity of the same object.[14] Nevertheless, in the exact sciences, proportions where often used to solve for one term given the other three and in this context they were implicitly used as equations.

For the Greek concept of parts, or fractions, I use $^m/_n$. For the most part, Greek mathematicians worked with unit fractions, or simple parts. Only rarely were collections of parts specified. I also use this notation for fractions which are greater than a unit and which the Greeks generally denoted verbally. For example, the sentence, "arc AG is the *epitritic* of arc AB" has undergone some ellipsis but it is clear that the epitritic part is meant.[15] In Greek mathematics a ratio is always a relation between two objects. It is never asserted that a single object is itself a ratio. Since this sentence is about the singular object "arc AG," I write it as, $\overset{\frown}{AG} = {}^4/_3\, \overset{\frown}{AB}$. For sexagesimal fractions, I use Neugebauer's notation. For the number $134 + {}^{13}/_{60} + {}^{46}/_{60^2}$, which Ptolemy would write as ρλδ ιγ μς, I write 134;13,46.

In a few cases, where the Greek author gives a verbal description of an arithmetic procedure or algorithm, I have not hesitated to summarize this with a formula that expresses the same arithmetic operations in a more concise form. These formulae should be read as abbreviations of a lengthy rhetorical procedure. They are not meant to imply any abstraction on the part of the ancient authors.

I have tried to remain faithful to the diagrams in the manuscript tradition. Only occasionally have I given a reworked figure, and this in conjunction with that found in the

[14] See page 43, n. 93.
[15] Heiberg [1916, vol. 1, p. 1, 45].

manuscripts. Although the figures in the manuscript tradition are not drawn according to current standards of accuracy, they do have their own internal logic. Moreover, the use of these figures play an important role in the deductive structure of Greek geometry.[16] Although the medieval diagrams will strike many modern readers as peculiar, they are almost always sufficient, in conjunction with the text, for forming a clear understanding of the mathematics at hand. Wherever a correction has been necessary, it has been noted.

In the course of this study, I introduce many notational conventions that I have not mentioned here. Most of them, I hope, will be sufficiently obvious. On the following page, I include a list of common symbols; many are traditional and most of the rest are adopted from Neugebauer.[17]

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[16] See Netz [1999b, 12 - 88] for discussion of the role of diagrams in Greek geometry.

[17] Neugebauer [1975, 1204 - 1206].

Notations and Symbols

Spherical Astronomy

α	Right ascension
δ	Declination
λ	Celestial longitude
β	Celestial latitude
ε	Obliquity of ecliptic ($= 23;51,20°$ for Ptolemy)
η	Ortive amplitude
n	Ascensional difference
L	Geographic longitude
φ	Geographic latitude
$s:g$	Ratio of shadow length to gnomon length
δ-circle	A circle parallel to the celestial equator
φ-circle	A circle parallel to the terrestrial equator
β-circle	A circle parallel to the ecliptic
v-circle	The greatest always visible circle
i-circle	The greatest always invisible circle
$\lambda_{\odot}$	Solar longitude
$\rho(x), \sigma(x)$	Rising and setting time of arc x
h_s	Seasonal hour
h_e or h	Equinoctial hour
M	Length of longest day
m	Length of shortest day

Motions of the Planets and Luminaries

α, $\overline{\alpha}$	Epicyclic anomaly, mean epicyclic anomaly
λ, $\overline{\lambda}$	Geocentric longitude, mean geocentric longitude; $\lambda = 0°$ is the vernal point, ♈ 0°
λ_A	Longitude of apogee
β	Geocentric latitude
κ, $\overline{\kappa}$	$\lambda - \lambda_A$, "normed" longitude, eccentric anomaly
c	Equation of center
e	Eccentricity
R	Radius of the deferent
r	Radius of the epicycle

Planetary Symbols and Zodiacal Signs

☉	Sun	☾	Moon	♈	Aries	♉	Taurus	♊	Gemini
☿	Mercury	♀	Venus	♋	Cancer	♌	Leo	♍	Virgo
♂	Mars	♃	Jupiter	♎	Libra	♏	Scorpio	♐	Sagittarius
♄	Saturn			♑	Capricorn	♒	Aquarius	♓	Pisces

Chapter 1

Ptolemy's Approach to Applied Mathematics

The purpose of this chapter is to introduce the fundamental characteristics that distinguish Ptolemy's work, and the Greek exact sciences in general, from other Greek mathematical traditions. The subjects covered belong together only in the sense that they are all introductory and prerequisite to the rest of this study.

A brief opening section will discuss Ptolemy's philosophy of mathematics. Ptolemy thought of himself as working within the larger framework of philosophy and practicing the branch of that discipline known as mathematics. He introduces, however, some interesting ideas about the relationship between mathematics and the other fields of philosophy and the status of applied mathematical knowledge. This section will discuss Ptolemy's position, making little attempt to situate his ideas in the context of other Greek theories of mathematical knowledge.

The next section is a classification of the types of mathematical text we find in Ptolemy's work. This classification will be useful for any study of applied mathematics in Greco-Roman antiquity. It will help us understand the relationship between the applied mathematical traditions and the pure traditions from which they arose. In this section, we will come to grips with the essential forms that mediated all mathematical knowledge in the Greek exact sciences.

The third section deals with special use of ratio in applied mathematics. After a general introduction to ratios and operations on ratios, it treats the uses of ratio that we find in the Ptolemaic corpus. Specifically, it traces the implicit arithmetization of ratio and the use of proportions as equalities.

The final section discusses the role of tables in Ptolemy's work. Because the use of tables as tools for calculation is dealt with in a number of other places, this section focuses on the role of tables as objects of knowledge. In particular, it addresses the relationship between tables and functions, and the use of tables to convey quantitative information.

This chapter is introductory; it lays groundwork and introduces themes that will reoccur throughout the rest of the work. Although some evidence will be presented for the claims made here, much of the evidence is best read in its own context and will be found in later sections. Wherever this is the case, references to this later material will be given.

Alm. I 10, I 13 - II 13 & VIII 5 & 6 are dense sections of mathematical argument. It is sometimes necessary to refer to units of these sections with greater precision than that provided by the numbers of the established text. For this purpose, I have divided these sections more finely. These divisions are given in Appendixes A & D. Wherever I refer to units of these sections, I use the more precise numbers.

1.1 The role of mathematics in Ptolemy's thought

Given the mathematical nature of most of his works, it should come as no surprise that Ptolemy accords mathematics a privileged place in his accounts of knowledge. Ptolemy nowhere develops a full philosophy of mathematics, but in a few places he discusses the nature of mathematical knowledge and its relation both to the objects it studies and to other types of knowledge. The most important passages, in this regard, are in the *Harmonics* and the *Almagest.*

I will discuss the passages from the *Harmonics* first for two reasons. Because of its relationship to the *Canobic Inscription*, there is reason to believe that the *Harmonics* is an early work.[1] Moreover, the *Harmonics* contains Ptolemy's most thorough discussions of the way he believes knowledge should be produced in the sciences. A reading of the *Harmonics* is essential for anyone hoping to come to grips with the methods of science Ptolemy claims to have practiced.[2]

In *Harm.* III 3 & 4, Ptolemy concludes his discussion of harmonics in music theory and turns to a philosophical discussion of the *harmonic faculty.*[3] The harmonic faculty, as Ptolemy tells us in *Harm.* III 4, is a faculty present in all things which have a source of motion in themselves; such as human souls or the celestial bodies. Moreover, the theoretical study of this faculty is a type (εἶδος) of mathematics.[4] Ptolemy approaches his discussion of this faculty, in *Harm.* III 3, by a series of trichotomies.

First, he tells us that all things require three principles (ἀρχαῖς): (A1) matter in respect to the underlying subject and the from-which, (A2) motion in respect to the cause and the by-which, and (A3) form in respect to the end and the for-the-sake-of-which. The harmonic faculty, however, is neither (A1) an underlying subject nor (A3) an end. It is, rather, (A2) a cause acting on a subject to produce some end; such as good melody and rhythm, sound law and order.

Causes are themselves taken in three ways: (B1) corresponding to nature and only to being, (B2) corresponding to reason and only to good being, and (B3) corresponding to god and to good and eternal being. The cause with respect to the harmonic function is understood as corresponding to neither (B1) nature nor (B3) god, but to (B2) reason which falls between the other two and works with them in producing the good (συναπεργάζεται τὸ εὖ).

1 Swerdlow [2004, 175 - 176]. For the case that the *Canobic Inscription* is an early work, see Hamilton, Swerdlow and Toomer [1987].

2 Barker [2000] provides an excellent study of this difficult text. More concise accounts are found in Barker [1991] and Swerdlow [2004].

3 Düring [1930, 91 - 94] and Barker [1989, 371 - 373]. See Barker [2000, 259 - 269] for a discussion of this material.

4 Düring [1930, 95] and Barker [1989, 374]. Also, see Barker [2000, 259 - 260] for a discussion of the harmonic faculty.

The cause with respect to reason has itself three aspects: (C1) like intelligence and corresponding to (B3) the more divine form, (C2) like technical skill and corresponding to (B2) reason itself, and (C3) like habit and corresponding to (B1) nature. In this trichotomy, we find the harmonic faculty in all three areas, accomplishing its own end (περαίνουσαν τὸ ἴδιον τέλος). Indeed, reason establishes correct order in audible things (D1) through the discovery of well-ordered relations (τῶν συμμετρίων), by means of theoretical inquiry, corresponding to (C1) intelligence; (D2) through the practical exhibition of them, by means of manual dexterity, corresponding to (C2) technical skill; and (D3) through experience, by means of the act of following chains of inference (διὰ τῆς παρακολουθητικῆς), corresponding to (C3) habit.

The result of all this is that reason discovers the good through theorizing, it produces what is understood through action, and it brings the underlying subject into conformity through habit. Hence, the common science of forms corresponding to reason, specifically called mathematics, not only involves the theoretical investigation of beautiful things, but also exhibition and the experiential practice (μελέτης) resulting from the very act of following chains of inference.[5]

It is clear from this passage that, for Ptolemy, the term *mathematics* refers to the whole of the mathematical sciences. No distinction is made between what we would call pure and applied mathematics. Mathematics is the science which studies all of the forms in correspondence with reason and has to do with what is good. Moreover, the proper subject of the theoretical part of mathematics are beautiful things.[6] Mathematics, however, also encompasses hands-on demonstration of these things as well as the experience that accrues as a result of engaging in deductive thought.

Harm. III 3 goes on to explain that the harmonic faculty uses, as instruments and servants (ὀργάνοις... καὶ διακόνοις), the higher senses of hearing and sight. These senses are privileged because they judge the underlying subject not only on the basis of plea-

[5] This paragraph summarizes a grammatically difficult sentence; see Düring [1930, 93]. A note by Höeg [1930, 658, n. 2], on the punctuation, supports my reading.

[6] Barker [2000, 264].

sure, but, more importantly, by the beautiful. The two most rational sciences, which correspond to these senses, are astronomy and harmonics. These sciences use arithmetic and geometry as indisputable instruments (ὀργάνοις ἀναμφισβητήτοις). It is by means of these instruments that astronomy and harmonics are produced from the faculties of sight and hearing.

In this account, the proper goal of mathematics is the production of the exact sciences, and the objects it studies are beautiful things as perceived by sight and hearing. What we would call the pure mathematical sciences of geometry and arithmetic are, for Ptolemy, mere tools. In this passage, the studies of astronomy and harmonics are the highest calling of the mathematical scientist. This is because these sciences explore the beauty which is perceived in audible and visible things.

It should not be thought, however, that beauty is apparent in these things by virtue of their being heard and seen. As Ptolemy tells us in *Harm.* III 4, beauty is perceived by the part of our soul called the harmonic faculty, which is properly studied by mathematics. The structural similarity between the harmonic faculty and the science which studies it implies that it is our capacity to mathematize which allows us to experience beauty. Moreover, this ability to mathematize functions within us on three levels: the theoretical, the practical, and the experiential. It is by virtue of this capacity that we can know, produce and experience beauty.

This discussion of mathematics in the *Harmonics* is appropriate to the subject matter of that text. In the *Almagest*, on the other hand, mathematics is explained without any reference to aesthetic judgments. In *Alm.* I 1, Ptolemy situates mathematics, and in particular astronomy, in the context of philosophy as a whole.[7] He refers his discussion vaguely to "the genuine philosophers" and more particularly to Aristotle. The Aristotelian orientation of this preface has been discussed by Boll.[8] More recently, however, Taub has pointed that, although Ptolemy clearly wanted us to read this passage with

[7] Heiberg [1916, 4 - 8] and Toomer [1984, 35 - 37].
[8] Boll [1894, 66 - 76].

Aristotle in mind, he made significant departures from Aristotle's thought.[9] This is not the place to attempt an analysis of the position of Ptolemy's philosophical views within the greater context of the ancient philosophies. For our purposes, it will be sufficient to simply examine what Ptolemy himself has to say.

Ptolemy begins *Alm.* I 1 by conferring his consent on certain traditional divisions of philosophy. The genuine philosophers, he tells us, divided philosophy into the practical and the theoretical. Aristotle divided the theoretical part of philosophy into theology, mathematics and physics. Ptolemy then states that everything which exists is composed of matter, form and motion. He never actually says as much, but his discussion of the three parts of theoretical philosophy implies that theology has more to do with form, mathematics with motion and physics with matter.

The proper subject of theology is the first motion of the universe, an invisible and motionless god completely separated from perceptible things. The subject of physics on the other hand is material and always-moving, the corruptible bodies below the lunar sphere. Mathematics, on the other hand, gives expression to a quality with respect to forms and motion from place to place. It investigates shape, quantity, size, place, time and such things. As Ptolemy says,[10]

> Its subject matter falls, as it were, in the middle between the other two, since, firstly, it can be conceived of both with and without the aid of the senses, and, secondly, it is an attribute of all existing things without exception, both mortal and immortal: for those things which are perpetually changing in their inseparable form, it changes with them, while for eternal things which have an ethereal nature, it keeps their unchanging form unchanged.

Because the subject matter of theology is unseen and undetectable, while matter is unstable and obscure, Ptolemy concludes that theology and physics result in mere conjecture. Only mathematics provides firm and unshakable (βεβαίαν καὶ ἀμετάπιστον)

[9] Taub [1993, 19 - 37].
[10] Toomer [1984, 36].

knowledge. This is because mathematics proceeds by the indisputable paths (δι' ἀναμφισβητήτων ὁδῶν) of arithmetic and geometry. Moreover, mathematics can be useful in both theology and physics. In theology, it investigates the divine objects in the heavens. In physics, it describes the motion of physical objects.

Here again, we find mathematics associated with a mean between the domains of form and matter. Mathematics, by situating itself in this middle ground, bridges the gap between these opposites. This situation gives mathematics more epistemological power than either of the other types of theoretical philosophy. Indeed, it is only insofar as they can become mathematical that theology and physics will ever produce any certain knowledge.

As in the *Harmonics*, the term *mathematics* refers to the exact sciences, in this case astronomy. Arithmetic and geometry are described as paths. They are not construed as belonging to mathematics proper, they are the means by which mathematics travels. In the case of astronomy, the objects of mathematical investigation are divine and heavenly bodies. Because these objects are eternal and unchanging, mathematics, the science that studies them, can be eternal and unchanging as well.

As we saw above, the value of mathematics is not restricted to pure theory. In the *Almagest*, we learn that mathematics is also of value to ethics. Ptolemy puts it as follows.[11]

> With regard to virtuous conduct in practical actions and character, this science, above all things, could make men see clearly; from the constancy, order, symmetry and calm which are associated with the divine, it makes its followers lovers of this divine beauty, accustoming them and reforming their natures, as it were, to a similar spiritual state.

This discussion makes it clear that, for Ptolemy, mathematics is the ultimate science, and he himself its ultimate practitioner. In fact, however, Ptolemy's discussions of mathematics are often founded on analogies and give us no understanding of how he actually

[11] Toomer [1984, 37].

produces mathematical knowledge. In order to develop this understanding, we must turn to the actual mathematics in his texts.

1.2 Categories of mathematical text in Ptolemy's works

The goal of this study is not to detail the specific mathematical results produced by Ptolemy; it is to understand how Ptolemy applies mathematics and mathematical methods to understanding the physical world. We will be interested to know the methods and practices that Ptolemy uses to produce his results.

It goes almost without saying that the exact sciences must be understood in the mathematical context in which they are carried out. Practitioners of the exact sciences use the methods, habits and results of the mathematical culture in which they work. Moreover, they often create new results or methods to handle the problems that arise in the course of their work. While new results are sometimes interesting and useful, it is the development of new methods that is most beneficial to the project as a whole.

The basic elements of Greek mathematical exposition are words, numbers and diagrams. These elements are grouped together in specific and controlled ways to produce both meaning and necessity.[12] These primitive groupings are arranged in more complex ways to solve problems and demonstrate theorems. In the theoretical tradition, the proposition, which may be either a problem or a theorem, emerges as the basic unit of mathematical prose. Although there are different types of propositions and it is not always clear what constitutes a new proposition and what is simply another case, students of Greek mathematics, since antiquity, have found it useful to treat the proposition as the basic unit of mathematical prose. This allows us to study the proposition as an object; to classify its varieties, explain its structure and expose the relationship between

[12] Netz [1999b] is a study of the production of necessity and generality that focuses on the tradition of pure geometry.

the proposition itself and the larger body of theory in which it exists.[13]

In the tradition of the exact sciences, the modes of mathematical exposition are both more diverse and less well defined than those in the theoretical tradition. Nevertheless, the overall structure of texts in applied mathematics is a deductive edifice built out of units of mathematical prose. In investigating Ptolemy's mathematical approach, it is often useful to think in terms of the function of individual units. To this end, I give a classification system for the types of mathematical prose found in Ptolemy's texts. This classification system is probably valid and useful for all surviving texts in the Greek exact sciences.

1.2.1 The classification system

I use the following six categories to describe Ptolemy's modes of mathematical discourse: (1) *theorem*, (2) *problem*, (3) *analysis*, (4) *computation*, (5) *table*, and (6) *description*. A few of these categories take their names from analogous categories found in the theoretical tradition.

The scheme omits passages which may be about mathematics but do not form part of the mathematical argument; for example, discussion of the choice of number system or exposition of the hypotheses and their empirical or philosophical justification. Such material will be called *discussion* and corresponds to the introductory material, including definitions and first principles, in the theoretical tradition.[14] In Ptolemy's works, it is not always possible to clearly differentiate this material from other aspects of the argument that are more straightforwardly non-mathematical, such as analysis of observations or discussion of experimental difficulties. I have found it simplest to leave all of this material out of the classification scheme.

I am aware of at least two other attempts to systematically classify the types of text in one of Ptolemy's works. Lejeune, in his study of the Euclidean and Ptolemaic theories

[13] Knorr [1986, 348 - 360] and Netz [1999a].
[14] Netz [1999b, 94 - 95].

of vision, denotes passages of Ptolemy's *Optics* as either a *proposition* or *expérience.*[15] It is fairly obvious how these categories are defined. Smith, in his translation of the *Optics*, labels certain sections of texts as either *theorem*, *experiment* or *example*, while much of the text remains unlabeled.[16] His numbers do not agree with those given by Lejeune. Moreover, the basis of his categorization is not always clear. The category of *theorem* is straightforward and appears to agree with the definition of *theorem* given below.[17] His division between *experiment* and *example*, on the other hand, is not always clear. This is, no doubt, partly due to the fact that Ptolemy is not always clear in reporting empirical evidence. In some cases, it seems certain he means us to understand that he actually carried out a certain empirical procedure and obtained the reported results.[18] In other cases, he seems to be more generally stating what the phenomena would be under certain presupposed conditions.[19] Because Smith does not explain how his categories are defined, it is not always clear what the difference is between an *experiment* and an *example.*[20] Some *examples* even have features of *theorems* and include theoretical justification combined with empirical exposition.[21] Moreover, the term *example* is too vague to be of much help. The only category that my system shares with Smith's is *theorem.*

Each of the six categories are defined and exemplified below. I also note which of the Ptolemaic texts contain which types of mathematical text. The scheme is meant to be descriptive; I make no claim that Ptolemy made any deliberate effort to adhere to these categories.

[15] Lejeune [1948].
[16] Smith [1996].
[17] See page 11.
[18] Clear examples of experiments are Smith's *Experiments* III.1, III.2, IV.1 - IV.2 & V.1 - V.3, Smith [1996, 134 - 136, 147 - 148 & 232 - 237].
[19] Instances of this sort are Smith's *Examples* II.1 - II.6, II.7 - II.8 & V.1, Smith [1996, 93 - 97, 103 - 105 & 231].
[20] For example, it is not obvious what experimental feature of *Experiments* II.1 - II.5 is lacking in *Example* V.1, Smith [1996, 231 - 232]. *Experiment* V.5 can hardly be said to describe a concrete procedure. At best, it vaguely suggests a whole program of experiments.
[21] For instance, *Example* V.1, discussed below on page 28. Also compare *Experiment* III.1 & III.2 with *Theorem* III.1, Smith [1996, 150 - 151].

Categories of mathematical text in Ptolemy's works

Theorem

The category of *theorem* is probably the most easily recognizable kind of mathematical text in Ptolemy's corpus. It is modeled on the *theorem* of the theoretical tradition. In a theorem, a proposition is first asserted and then demonstrated. The demonstration can rely on hypotheses asserted previously in the text, theorems demonstrated previously in the text or mathematical knowledge drawn from texts fundamental to the domain of the theorem. The mathematical knowledge most generally drawn upon is that provided by Euclid's *Elements*, the spherical geometry in the *Spherics* of Theodosius and Menelaus and a basic knowledge of how to manipulate ratios and utilize lists. The utilization of special hypotheses is one of the characteristics that distinguishes an applied mathematical theorem from a pure mathematical theorem. Although Ptolemy includes some pure mathematical theorems in his works, I have chosen two applied theorems to exemplify the category.

In *Alm.* III 3, before demonstrating the equivalence of the eccentric and epicyclic models, Ptolemy proves, individually for each model, that the difference between mean motion, $\overline{\kappa}$, and the apparent motion, κ, known as the equation of anomaly, $c = \overline{\kappa} - \kappa$, is greatest when the celestial body is at quadratures. This proof is based both on elementary geometry and deductions drawn from the kinematics of the model.

Ptolemy begins by setting out the model. He lets the circle $ABGD$ be the body's (τοῦ ἀστέρος) eccentric circle, with center E and diameter AEG. He sets out the position of the observer at Z and draws $BD \perp AG$. He points out that the body will be at quadratures at points B and D. The proof, then, must show that when the body is at points B and D the equation of anomaly, c, will be greatest. Ptolemy joins EB and ED. The first stage of the proof involves showing that $c = \angle EBZ = \angle EDZ$. Ptolemy asserts that this is the case because $\angle AEB$ subtends the arc of uniform motion, $\overline{\kappa}$, and $\angle AZB$ subtends the arc of anomalous, apparent motion, κ, and $\angle EBZ$ is the difference between them.[22]

[22] Toomer [1984, 146 - 147].

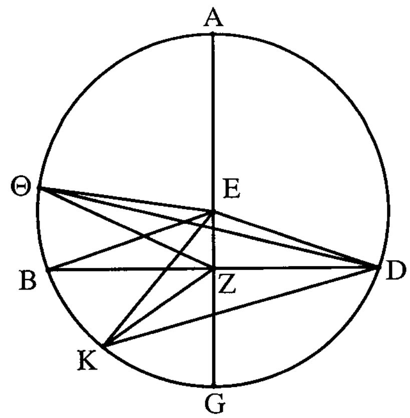

Figure 1.1: Diagram for Ptolemy's proof, in *Alm.* III 3, that the greatest equation of anomaly, for a simple eccenter model takes place at quadratures.

This assertion could be justified by drawing a line through E parallel to BD. Ptolemy, however, assumes that this is obvious enough and is content with pointing out how the features of the geometric model are associated with the astronomical phenomena. The first part of the proof simply draws out the kinematic consequences of the model. Ptolemy then turns to geometry to complete his argument:[23]

> I say, then, that no angle greater than these two [$\angle EBZ$ and $\angle EDZ$] can be constructed on line EZ at the circumference of circle $ABGD$. Construct, at points Θ and K, $\angle E\Theta Z$ and $\angle EKZ$, and join ΘD and KD. Then, since, in any triangle, the greater side subtends the greater angle [*Elem.* I 19], and $\Theta Z > ZD$ [1], therefore $\angle JDZ > D\Theta Z$ [*Elem.* I 18].[24] But, since $E\Theta = ED$ [*Elem.* III def. 15], $\angle ED\Theta = \angle E\Theta D$ [*Elem.* I 5]. Therefore, the whole $\angle EDZ$ [$= \angle\Theta DZ + \angle ED\Theta$] $=$ $\angle EBD > \angle E\Theta Z$ [$= \angle D\Theta Z +\ \angle ED\Theta$]. Again, since $DZ > KZ$ [2], $\angle ZKD >$ $\angle ZDK$ [*Elem.* I 18]. But, since $EK = ED$ [*Elem.* III def. 15], $\angle EKD = \angle EDK$ [*Elem.* I 5]. Therefore, the remainder $\angle EDZ$ [$= \angle EKD - \angle ZKD$] $= \angle EBZ >$ $\angle EKZ$ [$= \angle EKD - \angle ZKD$]. Therefore, it is impossible for any other angle to be constructed in the way defined greater than those at points B and D.

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[23] Toomer [1984, 147]. I have changed some notation and inserted some text in brackets.

[24] Toomer points out that although Ptolemy states *Elem.* I 19 he requires *Elem.* I 18, Toomer [1984, 147, n. 37].

As we see, the proposition is demonstrated by appeal both to the strictures of the model and to elementary geometry. I have provided justifications for most of the geometric points of the argument from Euclid's *Elements.* The steps which I have numbered [1] & [2] cannot be justified by any single proposition in the elementary texts of which I am aware. Nevertheless, the geometric fact is obvious and, as Heiberg notes, a proof could be easily fashioned from *Elem.* III 3 & 7.[25] In this case, the inferences drawn from the model are not explicitly asserted as drawn from the postulates of a deductive system.

The practice of modeling and then deducing on the basis of the model in the *Almagest* has a hypothetical character that should be distinguished from the strictly deductive method of producing theorems in the *Optics.* The mathematical principles of vision are explicitly asserted in the *Optics* in such a way that they can then be directly appealed to within the argument of a theorem. For example, in the beginning of *Optics* III, Ptolemy states three principles (*principia*) governing reflected vision. (P.Rl.1) A visual object seen in a mirror appears along the extension of the ray of incidence. (P.Rl.2) A visual object seen in a mirror appears along the extension of a perpendicular line dropped from the object to the mirror. (P.Rl.3) The angle of incidence equals the angle of reflection.[26] These principles are then justified by appeal both to familiar phenomena and instruments specifically designed for their verification. For our purposes, they act as starting points for mathematical reasoning, used as the justification of a step in a theorem.

In *Optics* III [68 - 72], Smith's *Theorem* III.4, Ptolemy demonstrates that a single visible object produces a single image in a plane mirror. The proof is effected by appeal to his three principles of reflected vision and the use of a toolbox of elementary geometry.[27]

> Let the straight line ABG be on the surface of a plane mirror. [See Figure 1.2.] Let D be the eye (*uisus punctus*) and E the visible object (*res uidenda*). And let the visual ray, which proceeds from D, be reflected at equal angles to E [P.Rl.3], and let it be as ray DBE. We say, therefore, that no other ray, of those proceeding from

[25] Heiberg [1916, 222].
[26] Lejeune [1989, 88] and Smith [1996, 131].
[27] Lejeune [1989, 120 - 122] and Smith [1996, 154 - 155]. I have modified a few passages and inserted some text in brackets.

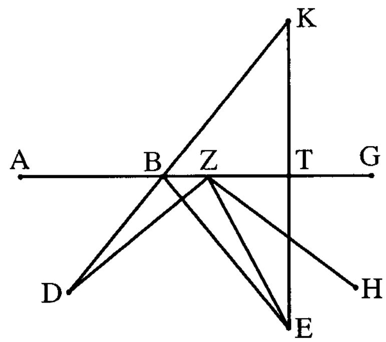

Figure 1.2: Diagram for *Optics* III [68 - 72], Smith's *Theorem* III.4, Smith [1996, 154].

> D, will be reflected by the mirror to E at equal angles. If such is possible, let the ray DZE be reflected. Hence, since $\angle ABD > \angle AZD$ [*Elem.* I 16], and $\angle ZBE > \angle GZE$ [*Elem.* I 16], and $\angle ABD = \angle ZBE$ [by hypothesis], hence $\angle GZE > \angle BZD$. Therefore, ray DZ will not be reflected on line ZE at equal angles. Therefore, from what we have said, it is clear, that if we set out $\angle GZH$ in the same way as $\angle AZD$, lines ZH and BE will not meet on either point H or E or in their direction, since $\angle ZBE > \angle GZH$ [$\angle ZBE = \angle ABD$ and $\angle GZH = AZD$]. Again, it will be demonstrated in the same way that if, from the visible object, we drop $ET \perp AG$ and produce lines ET and DB they will meet at point K. Hence, since $\angle DBA = \angle EBG$, $\angle DBA$ will be less than a right [angle] [*Elem.* I 13]. Therefore, $\angle KBT$, opposite to it, will be less than a right [angle] [*Elem.* I 15]. And for that reason, and because $\angle KTB$ is a right [angle], because it is opposite $\angle GTE$ [*Elem.* I 15], $\angle TBK + \angle KTB$ (*quod ex utrisque angulis* tbk, btk) will be less than two right [angles]. Therefore lines ET and DB meet at point K [*Elem.* I post. 5]. Therefore, the appearance of E, which the eye D sees, is at point K [P.Rl.1 & P.Rl.2]. Thus, it occurs in this mirror just as in direct vision, because of the fact that things seen with a single deflected (*non recto*) ray appear in one place.

Of Ptolemy's texts, the *Optics* is that which relies most heavily on theorems. In the *Optics*, theorems constitute Ptolemy's primary mathematical approach. Following the *Optics*, are the *Planisphaerium* and the *Almagest* which use the *theorem* intermittently to support other mathematical approaches such as analysis and computation. All of

Ptolemy's other works, including the *Analemma*, use theorems infrequently or not at all.

Problem or construction

In Ptolemy's works, there are almost no *problems* of the kind we find in the theoretical tradition. Such a *problem* enunciates a construction in the infinitive, performs the construction and then demonstrates that the construction satisfies the terms of the enunciation.[28] In Ptolemy's works, a problem is rarely solved simply by effecting a geometric construction. The issue is not merely that Ptolemy is generally seeking numerical results and hence prefers computation. Even in the *Optics* and the *Analemma*, where he is seeking respectively geometric and nomographic results, he chooses to proceed without making much use of formalized geometric constructions.

Two passages that best exemplify the way Ptolemy does use geometric construction are the first mathematical proposition of *Alm.* I 10 and the construction of a plane model of the celestial sphere inside an arbitrary circle, *Planis.* 14.[29] Both of these constructions are discussed in detail in later chapters and situated in their mathematical contexts.[30] A short construction from the *Planisphaerium* will give a sense for the way Ptolemy generally uses this type of mathematical text in his argument.

The project of the *Planisphaerium* is to develop a plane model of the celestial sphere though the combined techniques of projection and geometric construction. Points on the sphere are projected through the south celestial pole onto the plane of the celestial equator. Using these points as a basis, a model is then constructed in the plane. In *Planis.* 19, Ptolemy shows that a circle parallel to the ecliptic which passes through the south celestial pole will be represented by a straight line in the plane of the figure.

As is usual with Ptolemy, the theorem is not enunciated. It begins with the exposition. In the *Planisphaerium*, the figure often contains two planes folded into each other. In this case, the two planes contained in the figure are the plane of the solstitial colure and

[28] The most important study of problems in the theoretical tradition is Knorr [1986].
[29] Toomer [1984, 48 - 49], and Anagnostakis [1984, 89 - 91] (see also Heiberg [1907a, 249 - 251]).
[30] See pages 84 and 223.

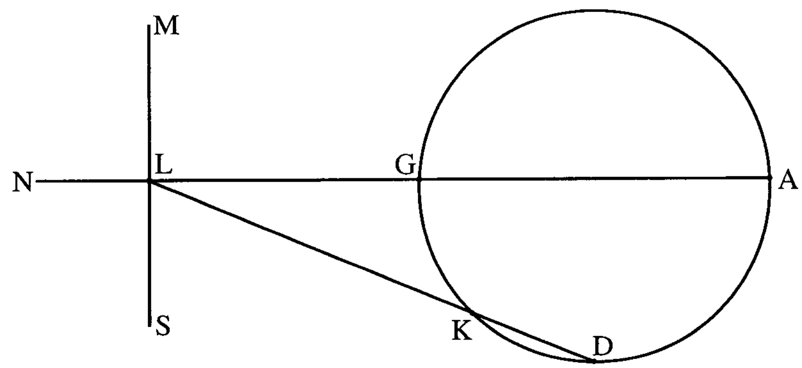

Figure 1.3: Diagram for *Planis.* 19, Anagnostakis [1984, fig. 13].

the equatorial plane, that is the plane of the model. In Figure 1.3, D is the south celestial pole, GA the diameter of the equator and KD the diameter of a circle parallel to the ecliptic. The Arabic version of Ptolemy's text proceeds as follows.[31]

> Clearly, if we consider, as in this figure [1.3], the circle parallel to the ecliptic which is drawn through point D, let it be the circle which is drawn on line DK. We produce line DK for the goal which we mentioned, and we cross, at point L, line $MLS \perp AGN$. This line on the plate corresponds to the circle whose diameter is DK. Proof: all straight lines which issue from point D passing through this circle lie in one plane, namely the plane of the circle [*Elem.* XI 1], and the intersection of this plane and the plane of the equator is the line MLS [*Elem.* XI 3]. Furthermore, the plane of the meridian circle which is on line AG is at right angles to each one of these two planes which we mentioned [1].

This example is built on the model of classical problems. The construction is followed by a proof which demonstrates that it effects what was required. In this case, a single line is projected from D through K and onto AGN; the rest of the construction is carried out in the plane of the model. The demonstration of *Planis.* 19 is interesting because it is the only one in the whole work which is based on the properties of the projection. That is, it relies on the properties of the solid figure to justify the construction and makes implicit

[31] Anagnostakis [1984, 95 - 96]. I have changed some notation and added some text in brackets. See also Heiberg [1907a, 257 - 258].

appeal to propositions in the solid books of the *Elements*. The reasoning behind the step labeled [1] is not spelled out; however, it is also based on considering the projection. The plane of the solstitial colure, the meridian through AG, meets both the plane of the equator and the plane of a circle parallel to the ecliptic at right angles. Therefore, by *Elem.* IX 19, the intersection of these planes, line MLS, will be at right angles to the plane of the solstitial colure, that is to line AGN.

The *Planisphaerium* is the only one of Ptolemy's works that makes consistent use of problems. This is partly due to the nature of the subject, the construction of a plane model of the celestial sphere. As was noted above, however, the subject matter and approach of both the *Optics* and the *Analemma* lend themselves to *problems.* Nevertheless, no *problems* are found in these works. Outside of the *Planisphaerium*, Ptolemy prefers to approach problems with the modes discussed in the next two sections.

Analysis

There are no analysis/synthesis pairs in Ptolemy's works. In fact, there are no *analyses* of the kind we find in the theoretical geometric tradition. Nevertheless, there is a category of mathematical text that uses the same linguistic idiom as the section of a theoretical *analysis* which Hankel has called a *resolution.*[32] This style of text, which I call *metrical analysis* is unique to the applied mathematical tradition. It is a form of *analysis* that provides the theoretical justification for the derivation of a numerical value given through computation as opposed to a geometric object given through construction. This form of *analysis* is also found in the works of Hero and was certainly used by earlier writers in the applied mathematical tradition such as Hipparchus, Diodorus and Menelaus. In the next chapter, I discuss the relationship between metrical analysis and theoretical geometric analysis.

There is only one complete analysis in the planetary books of the *Almagest.* After having derived the parameters of all of the planetary models using computation based

[32] Hankel [1874, 143 - 144]. Also see Berggren and Van Brummelen [2000] for a recent discussion of Hankel's division of the theoretical *analysis.*

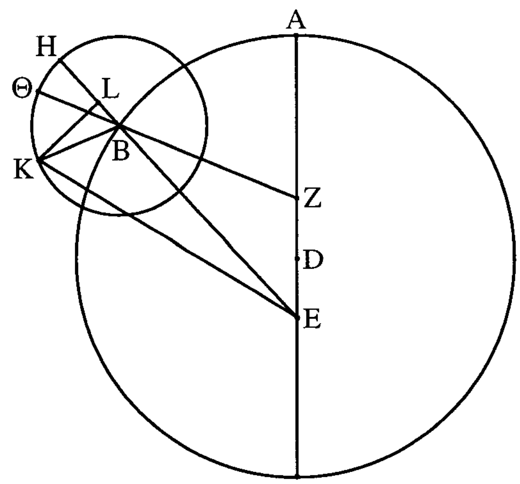

Figure 1.4: Diagram for *Alm.* XI 9, Toomer [1984, 544].

on select observations, Ptolemy uses a metrical analysis to show that for each planet, if the mean position in longitude, $\overline{\lambda}$, and anomaly, α, are given, then the apparent position of the planet, λ, will also be given.[33]

> In the simplified diagram containing [only] the eccenter and the epicycle [see Figure 1.4], we join $Z B \Theta$ and EBH. Then, if we are given the mean position in longitude [$\overline{\lambda}$], that is $\angle AZB$, from what we proved previously [3], $\angle AEB$ will be given according to both hypotheses, and so will $\angle EBZ$, (which is the same as $\angle HB\Theta$), and also the ratio of line EB to the radius of the epicycle [1]. Also, if we suppose that the planet is located on the epicycle, for example at point K, and, when EK and BK are joined, $\overset{\frown}{\Theta K}$ is given [$= \alpha$], then, if instead of dropping the perpendicular from the epicycle center B onto EK (as in the converse proof [2]), we drop $KL \perp EB$ from the planet K, then $\angle HBK$ will be given by addition [of the given angles $\angle \Theta BK$, $\angle HB\Theta$], and hence the ratios $KL : BK$ and $LB : BK$ [will be given] [*Data* 40], and also, obviously, $KL : EB$ and $LB : EB$ [will be given] [*Data* 8]. Accordingly, the ratio of the whole line EBL to LK will be given [*Data* 6 & 8]. Hence $\angle LEK$ will be given [*Data* 43] and we will have computed (ἡμῖν συνῆχθαι) the whole $\angle AEK$ which comprises the apparent distance of the planet from the

[33] Toomer [1984, 544 - 545]. I have changed some notation and added some text in brackets.

apogee.

This passage demonstrates the two primary functions that analysis performs for Ptolemy; it (1) summarizes numerical computation in a more compact idiom and it (2) establishes the theoretical possibility of carrying out further calculation. A number of steps in this analysis must be justified by referring to previous computations. For each planetary model, Ptolemy calculates the position of the planet on its epicycle, in *Alm.* IX 10 for Mercury, X 4 for Venus, X 9 for Mars, XI 3 for Jupiter and XI 7 for Saturn. The work of these sections is then used, in this analysis, to justify the steps labeled [1] and [2] and referred to in the step labeled [3]. Following Heiberg and Toomer, I have provided justifications for the other steps of the analysis from Euclid's *Data*. Nevertheless, as I argue in the next chapter, Ptolemy's use of analysis must be distinguished from the use of analysis in the theoretical tradition. Whereas in the theoretical tradition, analysis is a purely geometric tool, it is used by Ptolemy to investigate problems that have numerical solutions and often require trigonometry by chord tables.

Ptolemy uses analyses in both the *Almagest* and the *Analemma*. He seems to generally reserve analysis for relatively short derivations. The above example is one of the longest ones in the *Almagest*.[34] The two analyses in *Anal.* 9 & 10 are also fairly long but they provide the analytical determination for seven arcs. Where a more involved determination is necessary, Ptolemy prefers to use computation.

Computation

Ptolemy's works are full of implicit calculations. For a simple computational procedure, such as taking a square root or finding a fourth proportional, Ptolemy simply states the value he obtained. If the reasoning is more involved, however, Ptolemy may choose to use a *computation* to take us through the process of deriving the value in the text. Even in a *computation*, Ptolemy does not explain the methods of performing basic arithmetic procedures. In practice, a *computation* always involves geometric reasoning and usually

[34] The analysis in *Alm.* V 19 is only longer because it involves three cases, Toomer [1984, 269 - 271].

involves the chord table. Computation is closely related to analysis. Analysis provides the theoretical justification that ensures a computation can be carried out. There can be little doubt that the problems Ptolemy presents as *computations* were first solved by analysis and then carried through with actual numbers. A single example from the *Planisphaerium* will suffice to show the category.

In *Planis.* 11, Ptolemy computes the ascensional difference, n, of a quadrant at the solstices for a given latitude, $\varphi = 36°$. The ascensional difference is the difference between the rising time of a given arc of the ecliptic at *sphaera recta*[35] and the same arc at *sphaera obliqua*.[36] The determination of n is used in *Planis.* 12 & 13 to calculate the rising times of the each of the zodiacal signs at $\varphi = 36°$. In Figure 1.5, circle $ABGD$ is the equator and circle $ZBHD$ is the ecliptic projected onto the plane of the equator by means of the south pole, so that E is the north pole and T the northern pole of the ecliptic. If the stars are considered as fixed on a stationary equatorial plane, their motion with respect to the local horizon is represented by rotating the center of the horizon circle, S, on a circle with center E. The motion of the stars is considered to be clockwise from B to A; hence, the diurnal motion of S is counterclockwise. $Z\overset{\frown}{KH}L$ is the position of the horizon at $\varphi = 36°$ when the winter solstice, Z is rising. Since at *sphaera recta*, the horizon passes through the north pole, the line $ZEHG$ represents the horizon at *sphaera recta* when the winter solstice rises. Hence, $\overset{\frown}{KA}$ is the ascensional difference. Ptolemy proceeds as follows.[37]

> We lay off a figure [Figure 1.5], like this figure [Figure 4.8], with only the horizon through points Z, K and L. Let our goal be to find the size of $\overset{\frown}{KA}$. We again make point S the center of the horizon circle in this position [that is though points

[35] The upright sphere, or *sphaera recta*, is the technical term for the arrangement of the local and celestial coordinates when the observer is on the equator. In this situation, the equatorial poles lie on the horizon and the rising times of arcs of the equator are what we now call their right ascensions. The inclined sphere, *sphaera obliqua*, is the term for the situation at any horizon between the equator and the poles.

[36] Ascensional difference is introduced in *Alm.* II 7 to expedite the calculation of the table of rising times, Toomer [1984, 94 - 99].

[37] Anagnostakis [1984, 85 - 86]. I have changed some terminology and added some text in brackets. See also Heiberg [1907a, 244 - 245].

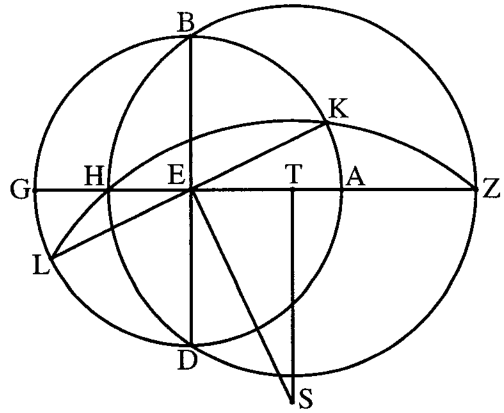

Figure 1.5: Diagram for *Planis.* 11, adopted from Anagnostakis [1984, fig. 6].

> Z, K and L] and we join the two lines ST, SE; they are perpendicular to the two lines ZH, KL [*Planis.* 10]. We have already proved that line ES (namely the line which is between the center of the equator and [the center of the] horizon circle of this klima [$\varphi = 36°$] which we laid down) is $82;35,3^{\text{p}}$, where line ET (namely the line which is between the center of this circle and the center of the ecliptic) is $26;31,58^{\text{p}}$ [*Planis.* 7]. Hence the line ET is also approximately $38;33^{\text{p}}$ (where line ES, namely the line that subtends the right angle) is 120^{p}; and the arc which is on it is $37;30^{\circ\circ}$, where $2\text{R} = 360^{\circ\circ}$ [**Chord Table**]. Hence $\angle TSE = \angle AEK = 37;30^{\circ\circ}$, where $2\text{R} = 360^{\circ\circ}$, and [$\angle AEK =$] $18;45°$, where $4\text{R} = 360°$; and since this angle is at the center of the equator, $\overset{\frown}{KA} = 18;45°$.

This is a typical short computation. Ptolemy uses the geometric properties of the diagram to solve his problem. The calculations themselves involve finding a fourth proportional and an application of the chord table, *Alm.* I 11. It is assumed that the reader will know how these basic operations are carried out.

Computations are an integral part of both the *Almagest* and the *Planisphaerium*. In most of Ptolemy's other works, the calculations, although numerous, remain implicit. There are few or no calculations in either the *Optics* or the *Tetrabiblos*.

Table or list

Ptolemy uses *tables* or lists in three different ways: (1) to catalog quantitative information, (2) to facilitate conversion between two numeric values and (3) to make it possible to convert between two numeric values. In (1) and (2), the *table* encapsulates a function, or functions, of one or more variables. In most cases the *table* defines a function and its inverse. This functional aspect of *tables* is discussed below in Section 1.4.

The first type of table, or list, simply organizes quantitative information that is important to the subject but is not necessarily intended to serve a role in the mathematical argument. Examples of this type are *Alm.* II 6, *Harm.* II 15, and the many tables in both the *Geography* and the *Harmonics.*

There are two varieties of the second type of table: (2a) helps the calculator perform a straightforward calculation and (2b) helps the calculator by substituting a straightforward calculation using a table for a more involved computation based on the geometric model. Typical examples of (2a) are the tables of mean motions for the celestial bodies in the *Almagest.* In fact, a numeric value for the daily mean motions for each body would be sufficient to solve all of the problems in the *Almagest.* In practice, this value could then be multiplied by the number of days and divided through by the number of hours and minutes in order to find the change in mean motion. Instead, Ptolemy tabulates change in the mean motions by hours, days, months, years and 18-year periods. This allows one to avoid multiplication in finding the change in mean motions; all that is required are a number of additions and one division.[38] A typical example of (2b) is the declination table, *Alm.* I 15. This table allows one to readily calculate the declination of a given degree of the ecliptic. The table is constructed on the basis of the computation given in *Alm.* I 14 and simply saves one the trouble of carrying out this computation every time one needs to convert between an ecliptic longitude and declination.

The third type of table is perhaps the most interesting; although there is only one

[38] In general, the time difference between two observations will not contain a whole number of hours. The change in mean motion in this fractional part of an hour must be found by division of the hour entry.

example. The chord table, *Alm.* I 11, makes it possible to convert between values for a chord and the angle that subtends it. In general, it is not possible to make this conversion directly using the geometric methods which the Greeks had at their disposal. The chord table gets around this difficulty by tabulating corresponding values at small intervals. Thus, it provides a numerical approximation to an intractable geometric problem. The construction of the chord table is discussed below in Section 2.1.4.

There are tables in all but one of Ptolemy's scientific works. Only the *Planisphaerium* contains no tables or lists. Although there are no preserved tables in the *Planetary Hypotheses*, there were originally tables at the end of Book II. The *Canopic Inscription* and the *Handy Tables* are almost entirely constituted by tables. Tables are equally integral to the project of the *Geography* and the *Harmonics.* The ostensible goal of the *Analemma* is the construction of a set of tables. In the *Almagest*, tables play a vital role in the mathematical argument. Tables, like theorems, must be understood as both a means and an end in Ptolemy's practice of exact science.

Description

One of the most interesting and prevalent forms of mathematical exposition in Ptolemy's work is in some sense only partially mathematical. This is the use of mathematical *description*, generally accompanied by a diagram, to guide the reader through the details of a model, experimental apparatus or argument. *Descriptions* are only partially mathematical because they do not rely on a body of assumed mathematical knowledge. The reader is directed to consider a logical or mathematical construct which more or less mirrors a physical object or process. This construct, then, becomes the focus of discourse and statements made about it are assumed to apply also to the objects it models. The reasoning followed in the course of a *description* is intended to be justified by an intuitive grasp of the properties of the constructed model.

That this mode of exposition was considered mathematical in Ptolemy's time is made clear by a section of Galen's *On the Usefulness of the Parts of Animals.* As Galen tells us in *On the Use.* X 12, he had intended to omit mathematical theories of vision from his treatment of the eyes for fear of annoying his readers with their obscurity.[39] Nevertheless, having been visited by a dream, Galen undertakes this difficult topic so as not to offend the deity. He asks mathematicians to forgive him for stating the obvious and enjoins the rest of us to follow him with great care. It is clear he considers his presentation in *On the Use.* X 12 to be mathematical. Nevertheless, the two core arguments are pure descriptions. They require no mathematical knowledge and can be followed by anyone with access to the accompanying diagrams. There can be little doubt that these descriptions would have been considered mathematical in antiquity. In fact, Galen's modern translator also considers these passages to be mathematical.[40]

Uses of mathematical description in Ptolemy's corpus are not as canonical as those of other mathematical categories. I will give four examples of *description* falling into two basic types: (1) pure modeling and (2) mixed empirical modeling.

In pure modeling, Ptolemy constructs an object which models the phenomena and becomes the focus of discourse. Usually, the model is geometric and can hence be considered analogous to some underlying object; although in a few of these cases, such as the *Analemma*, this analogy can be ambiguous. Sometimes, the model is numeric or conceptual, as often in the *Harmonics,* and it can be more difficult to identify a specific underlying object.

In mixed empirical modeling, Ptolemy represents an experimental instrument with a mathematical construction. He then uses this mathematical construction to describe the phenomena. In some cases, the difference between mixed empirical modeling and direct reporting of experimental, or observational, findings will be difficult to discern. Nevertheless, attention should be drawn to the fact that Ptolemy often presents empirical evidence in the form of a mathematical overview that exhibits the phenomena along with

[39] Helmreich [1909, v. 2, 92 - 103] and May [1968, 490 - 498].

[40] May argues that the "excellence of Galen's geometric analyses," in *On the Use.* X 12, "goes far to disprove" the claim that Galen was a poor mathematician, May [1968, 494, n. 56]. Whether or not the sight mathematics of *On the Use.* X 12 has any bearing on Galen's mathematical ability is a moot point. It is clear, however, that this text has been construed as mathematical by both ancients and moderns.

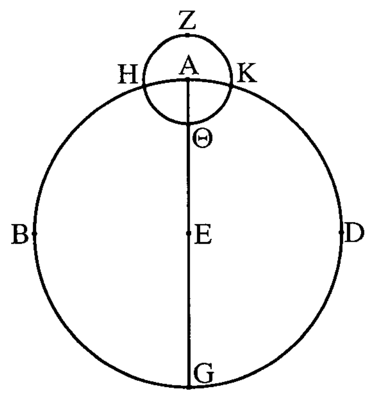

Figure 1.6: Diagram for the epicycle model of the solar motion in *Alm.* III 3, Toomer [1984, 135].

its cause, as opposed to a simple account of what he observed.

An example of pure geometric modeling can be drawn from the *Almagest.* In *Alm.* III 3, Ptolemy exhibits the simple eccentric and epicyclic models for planetary motion. It will be sufficient to look at the epicycle model. The primary assumption of this model is that a body will revolve uniformly around an epicycle while the epicycle itself revolves uniformly about a deferent circle. The purpose of the description is to exhibit the phenomena of the apparent motion of the body with respect to a central observer. Ptolemy's description of the epicyclic model is as follows.[41]

> In the epicyclic hypothesis, we imagine the circle concentric with the ecliptic as $ABGD$ on center E, with diameter AEG, and the epicycle carried on it, on which the body moves, as $ZH\Theta K$ on center A. Then, here too, it is immediately obvious that, as the epicycle traverses circle $ABGD$ with uniform motion, say from A toward B, and as the body traverses the epicycle with uniform motion, then when the body is at points Z and Θ, it will appear to coincide with A, the center of the epicycle, but when it is at other points it will not. Thus, when it is, for example, at H, its motion will appear greater than the uniform motion [of the epicycle] by $\overset{\frown}{AH}$, and similarly when it is at K its motion will appear less than the uniform motion by $\overset{\frown}{AK}$.

[41] Toomer [1984, 144]. I have changed some terminology and added some text in brackets.

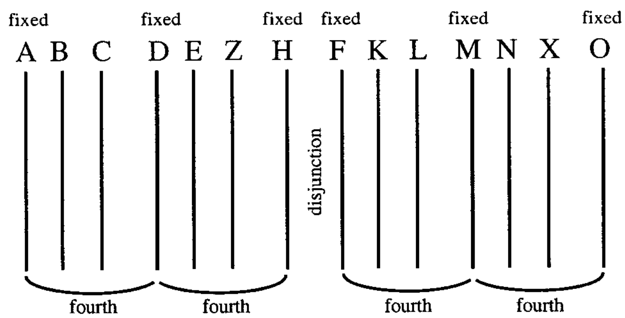

Figure 1.7: Diagram for *Harm.* II 3, Barker [1989, 323].

This is a typical example of a geometric description. The points which Ptolemy makes result directly from a consideration of the model. The description simply establishes the model and guides the reader through its consequences.

Much of the modeling in the *Harmonics* involves ratios and logical structures. While the models use diagrams, these often depict logical schema as opposed to geometrical objects. In *Harm.* II 3, Ptolemy sets out the *forms* of the first concords; that is, the fourth, fifth and octave.[42] In this case, the concords are considered as placed within a system of notes so that they are filled by other, lesser intervals composed of notes that lie between the bounding notes of the concords. The forms of a concord are given by the position of a characteristic ratio (ἰδιάζων λόγος) within the concord.[43] In the case of the fifth and the octave, these ratios are the tones; that is the disjunctions (οἱ τονιαῖοι καὶ διαζευκτικοί).[44] In the case of the fourth, this ratio is that between the two first notes. Hence, for each concord there are as many *forms* as there are places that can be occupied by a ratio: three for the fourth, four for the fifth and seven for the octave. The *forms* are named by the ordinal position of the characteristic ratio. Ptolemy uses a description to set out the concords in the complete system and show how many *forms* there for each concord between the fixed notes of the complete system. He proceeds as follows.[45]

Now it turns out that there is only one *form* of the fourth that is bounded by fixed

[42] Düring [1930, 49 - 50] and Barker [1989, 322 - 323].
[43] Düring [1930, 49].
[44] These terms are two ways of describing the same interval, Düring [1930, 49].
[45] Barker [1989, 322 - 323]. I have made slight modifications. See also Solomon [1999, 70].

> notes (the first), only two of the fifth (the first and fourth), and only three of the octave (the first, fourth and seventh). For, if we set out a fourth $ABCD$, imagining A as the highest note, and conjoins another similar fourth below, $DEZH$, then conjoin with this in the same way a tone HF, and conjoin with this, once again, a fourth $FKLM$, and with that another fourth $MNXO$, then the fixed notes will be A, D, H, F, M and O. The first *form* of the fourth will be MO, the second LX, the third KN, and it is clear that it is only MO, the first, which is bounded by fixed notes. Of the fifth, the first *form* will be HM, the second ZL, the third EK, the fourth DF, and it is clear that of these only HM, the first, and DF, the fourth, are bounded by fixed notes. Of the octave, the first *form* will be HO, the second ZX, the third EN, the fourth DM, the fifth CL, the sixth BK and the seventh AF, and, of course, once again, only HO, the first, and DM, the fourth, and AF, the seventh, are bounded by fixed notes.

In this description, the diagram serves no geometric purpose. It is a logical schema that exhibits the implicit structure of the model. The model itself is a conceptual construct. It is not clear that it is meant to directly represent any physical object or process. The mathematical objects out of which the model is built are ratios. The ratios, in turn, are used to model intervals between notes. Only in a few, particular cases is there any analogy between these ratios and physical characteristics present in the production of notes. Because the model can be shown to be a good fit in a few cases, it is assumed to be a good fit in all cases. In the description above, Ptolemy is simply explaining some of the structural elements inherent in the conventions of the complete system.

Mixed empirical descriptions combine the perspective and idiom of a mathematical approach with the context and the concerns of experimental, or observational, science. There is often little difference in presentation between a mixed empirical description and an experimental report. They are generally both construed with the same linguistic expressions and ostensibly concern a diagram. Nevertheless, a report discusses an experiment, or observation, that Ptolemy intends us to believe he actually carried out, while an empirical description discusses the phenomena in a more general and systematic way.[46] In practice, it is probably not possible to maintain an absolute distinction between the two; many passages could reasonably be argued to possess characteristics of both approaches. Ptolemy's process of mathematizing the physical world begins at the level of presenting the empirical phenomena. A few examples will suffice to support this claim.

In the beginning of his treatment of refraction, in *Optics* V, Ptolemy discusses an empirical situation which is traditionally used by ancient authors to explain the phenomena of refraction. A coin is placed in a vessel and the eye is situated such that the coin is hidden just below the lip of the vessel. When water is then poured into the vessel, with the eye remaining stationary, the coin becomes apparent under the water. A number of ancient authors present this phenomena with various degrees of precision.[47] Ptolemy first gives an overview of the situation and then proceeds to make this more precise with a mixed empirical description. In this description, he invokes two principles of refracted vision to exhibit the cause of the phenomena along with the phenomena itself. These principles are modifications of those given for reflected vision:[48] (P.Rr.1) A visual object seen through a refracting substance appears along the extension of the incidence ray, and (P.Rr.2) A visual object seen through a refracting substance appears along a perpendicular dropped from the object to the surface of the substance.[49] *Optics* V [6], Smith's *Example* V.1, proceeds as follows.[50]

> Now, let us suppose the eye (*uisum punctum*) is A, $\langle ZHE \rangle$ the common section of the plane containing the refracted ray and the surface of the vessel, ⟨and⟩ ABD the ray passing over the lip of the vessel, which is B. And let us suppose a coin in the place G, which lies toward the bottom of the vessel. Then, as long as the

[46] Many of the passages Smith [1996] labels *examples* are mixed empirical *descriptions*, as well as a few of his *experiments*

[47] See Lejeune [1989, 225, n. 9] for a full list of ancient passages concerning the coin or ring in a jar filled with water. In particular, two predecessors are Euclid and Seneca, Heiberg [1895, 286] and Corcoran [1962, vol. 1, 58]. Knorr [1985b] discusses most of these passages.

[48] See page 13 for Ptolemy's principles of reflected vision.

[49] Lejeune [1989, 224] and Smith [1996, 230].

[50] Lejeune [1989, 226 - 226] and Smith [1996, 230 - 231]. I have modified a few passages and inserted some text in brackets.

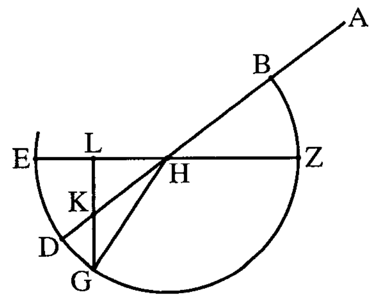

Figure 1.8: Diagram for *Optics* V [6], Smith's *Example* V.1, Smith [1996, 231].

> vessel remains empty, the coin will not be seen, because the body of the instrument at point B blocks the visual ray that could proceed directly to it. Yet, when just enough water is poured into the vessel so that its surface reaches line ZHE, line ABH is deflected to line GH, compared to which AH is higher. In that case then, the coin will appear to be located along the perpendicular dropped from G to EH [P.Rr.2], that is perpendicular LKG, which intersects AHD at point K. And the location of the image (*situs illium quod inde apparet*) will be on the ray produced from the eye and continuing rectilinearly to point K [P.Rr.1], which is higher than the actual line [HG] and nearer to the water's surface. And [the image] will appear at point K.

This example gives a clear mixture of experimental procedures and mathematical principles. It is not clear that Ptolemy has actually carried this procedure out or that he intends his readers to carry it out. The description is used in such a way as to display the phenomena along with the mathematical principles that Ptolemy claims underly them. The phenomena and the mathematical principles are presented as having a strong correspondence. Ptolemy probably thought that this correspondence was sufficient and necessary grounds for accepting the validity of the principles and their explanation of the phenomena.

The next example has a stronger basis in empirical investigation. In the course of refuting the musical theorists who follow the ideas of Aristoxenus, Ptolemy argues, in

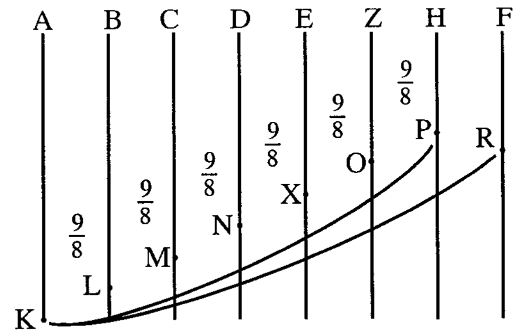

Figure 1.9: Diagram for *Harm.* II 11, Barker [1989, 299].

Harm. I 11, that the octave does not consist of six whole tones.[51] His argument is ostensibly empirical. First, he claims that if we instruct a skilled musician to construct six successive tones by ear, the first and last note will not sound an octave. Secondly, if even we construct the six successive tones by means of ratio (τῷ λόγῳ), the extreme interval of this system will be heard to differ from an octave also constructed by means of ratio. On the surface, it seems that Ptolemy intends the reader to simply carry out the experiment in order to confirm the asserted fact. In practice, however, experimental confirmation would be difficult, especially with the materials available to the ancients. Indeed, Ptolemy spends the rest of *Harm.* I 11 trying to assuage some of these concerns by arguing that dissimilar strings of equal lengths which make the same pitch will produce similar experimental effects.[52] The description in *Harm.* I 11 is as follows.[53]

> If ... we construct six tones in succession by ratio (τῷ λόγῳ), the extreme notes will make a magnitude slightly greater than the octave ... which ... comes very

[51] For a discussion of Ptolemy's arguments against the Aristoxenians, see Barker [2000, 88 - 108]. In particular, Barker [2000, 106 - 107] discusses *Harm.* I 11.

[52] Düring [1930, 26 - 28] and Barker [1989, 300 - 301]. See Barker [2000, 202 - 203] for a useful discussion of this material. By taking ἰσοτόνος as "equal tension" as opposed to "equal pitch," Solomon produces a translation that makes little or no sense, Solomon [1999, 37 - 38]. See, in particular, p.37, [27.5].

[53] Düring [1930, 25 - 26] and Barker [1989, 299 - 300]. Here as well, Solomon causes confusion by translating ἰσοτόνος as "equal tension," Solomon [1999, 37]. This is compounded with the fact that his figure is badly mislabled. My translation follows Barker, changing some passages and adding some text in brackets.

> close to being in the ratio 65 : 64 [*Harm.* I 10].[54] This sort of result will easily be understood if we fasten seven more strings on the *kanōn*, in association with the one string, according to similar preparation and position. For, if we accurately tune eight notes, such as the notes $ABCDEZHF$, at equal pitch in equal lengths of the strings, and then, with [the strings] divided into six 9 : 8 ratios in succession by an application of the measuring rod (τοῦ κανονίου), [and if] we place a similar bridge at the proper division corresponding to each note, such that the distance AK is the 9/8 [part] of BL, BL of CM, CM of DN, DN of EX, EX of ZO, and ZO of HP, while AK makes the 2 : 1 ratio to FR, then the latter notes will accurately sound the homophone of the octave, but PH will be slightly higher than FR, and always to the same degree.

This text appears to present a simple empirical proof of the fact that an octave is not composed of six whole tones. In fact, however, it is part of Ptolemy's ongoing argument against the Aristoxenian claim that musical intervals should not be considered as ratios, in this case the tone as 9 : 8. When we actually set up two monochord strings with lengths in the ratio 9 : 8, we find that slight, yet perceptible, changes in the lengths of the strings produce imperceptible changes in the musical interval. Even a discerning ear, which can detect some change in the interval will be at a loss as to which interval is the true tone. In the modern tempered system, we distribute the difference between six tones and an octave over the six tones and assert that no one can hear the difference.[55]

The Aristoxenians held that this interval, equivalent to our tempered tone, was not tempered at all but was, in fact, the true tone. Ptolemy's description does nothing to counter the Aristoxenian position. It does, however, give a representation of the phenomena which is construed in terms of his mathematical model. *Harm.* I 11 develops

[54] The reason for these numbers is as follows. In *Harm.* I 10, the difference between a semitone (= 258 : 243) and a *leimma* (= 256 : 243) is shown to be 129 : 128. Since, Ptolemy takes the octave to be two fourths and a tone, where a fourth is two tones and a *leimma*, the octave is five tones and two *leimmas*. Hence the difference between six tones and an octave is the difference between a tone and two *leimmas*, that is (129 : 128) × (129 : 128) ≈ 65 : 64. See 38 for a brief discussion of these operations on intervals.

[55] The difference between a tempered tone and a true tone, in Ptolemy's terms, is $(65 : 64)^{1/6} \approx 387 : 386$. This interval is much smaller than 129 : 128, which Ptolemy acknowledges is imperceptible in *Harm.* I 10.

the correspondence between the empirical phenomena and the mathematical model by combining the two to exhibit a particular phenomenon, the slight difference between an octave and six whole tones. The difficulty is that, in this case, the phenomenon in question is one that his opponents claim does not even exist. It is clear, however, that Ptolemy is again using a mixed empirical description to build a correspondence between the phenomena and the mathematical model which he believes explains them.

Mathematical descriptions are prevalent in every one of Ptolemy's works that applies mathematical models to the physical world. Descriptions are an essential component of Ptolemy's theoretical exposition. The only mathematical works that do not contain descriptions are the two tabular works, the *Canopic Inscription* and the *Handy Tables.*

1.2.2 Remarks on the classification of Ptolemy's mathematical text

We have reviewed examples of the six basic categories of mathematical text in Ptolemy's works. These categories represent all of the modes in which Ptolemy applies mathematics to the physical world. Whenever he encounters a problem which is susceptible to mathematical exposition, he disposes of it using one or more of these modes of discourse. The first three categories, those of *theorem*, *problem* and *analysis*, are modeled on categories of text in the pure theoretical tradition. The category of *computation* appears to have been used in both pure and applied traditions from the earliest times, although trigonometric *computation* was a later product of the applied tradition. The final two categories, *table* and *description*, appear to have originated in the exact sciences and not to have entered into the tradition of pure mathematics during the ancient period.

All six of these modes are used together in a number of ways to produce continuous chains of logical inference. In the same way that the texts of the theoretical tradition combine *theorems*, *problems* and occasionally *analyses* to produce deductive structures, so Ptolemy uses all of the different units of mathematical text in this classification scheme.

Section 3.4 provides a case study in how the various units are used to support one another. By examining the interrelation between the different modes, we expose the overall logical structure of Ptolemy's approach.

Although Ptolemy is our best evidence for the applied mathematical sciences among the Greeks, we should not suppose that any of the modes of applying mathematics to the physical world originated with him. The applied theorem goes back at least as far as Autolycus and Euclid.[56] There is an interesting applied problem and geometric analysis in the Aristotelian *Meteorology*.[57] Applied computations, albeit in a different style, are attested in the works of Aristarchus and Archimedes.[58] Although we have no extant examples, we can be certain that metrical analyses and trigonometric computations were used by Hipparchus, Diodorus and Menelaus.[59] Tables probably entered the Greek tradition with the influx of Babylonian astronomical knowledge in the middle of the Hellenistic period. Mathematical descriptions are found in the majority of Greek works on the exact sciences. Indeed, only the most technical writings, such as those of Euclid on optics or Archimedes on mechanics, eschew descriptions entirely in favor of more strictly mathematical modes of discourse.

There is probably not a single case in which we posses the earliest example of a mode of Greek mathematical text, pure or applied. The names mentioned above are not necessarily to be taken as the originators of these styles, but simply as the first incontestable practitioners. It would serve no purpose to hunt for the origin of these forms in lost texts. In the case of tables and trigonometric computation, it does seem clear that these arose in the middle of the Hellenistic period, partly under the influence of foreign methods and ideas.[60] The other modes go back well into the Classical period.

[56] Mogenet [1950], Menge [1916] and Heiberg [1895].
[57] Lee [1952, 268 - 280]. See Jones [1994], Merker [2002] and Vitrac [2002] for discussions of this material.
[58] Heath [1913, 352 - 410] and Heiberg [1973, 216 - 258]. Examples of these computations are discussed in Section 2.1.2.
[59] The arguments for this claim are made in the chapters that follow.
[60] The argument for this claim is made in the next chapter.

1.3 Operations on ratios

Ratios, proportions and ratio inequalities play a fundamental role in all fields of Greek mathematics and the exact sciences. The theoretical foundations provided for these concepts, however, do not fully justify the many uses to which they were put in practice. The most complete foundation for the theory of ratios of general magnitudes is *Elem.* V, while a number of propositions relating to ratio and proportion are proved independently for numbers in *Elem.* VII.[61] Moreover, modern scholars have detected traces of pre-Euclidean ratio theories and made attempts at reconstructing these.[62] For our purposes, however, we can ignore the issue of the foundations of ratio theory and its many difficulties in favor of a discussion of the use of a well defined set of operations on ratios, proportions and ratio inequalities.

There are six fundamental operations that are explicitly used by Greek mathematicians. In *Elem.* V, these operations are all defined for ratios but their use in the text is always in the context of proportions. In fact, however, they are also used by Greek mathematicians for ratio inequalities.[63] Four of these operations are both defined and justified in *Elem.* V; two of them, however, are only defined.

It is the regularity of the expressions that are used to indicate these operations in mathematical argumentation that makes clear that they were considered as *as operations* and establishes how fundamental they were to the working mathematician. They are consistently referred to by a specific word or phrase; either a dative of means, an adverb or a prepositional phrase.[64] This is in much the same vein as a modern mathematician might say "by integration," or "distributively," to indicate a well-known operation that the reader can be expected to know.

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[61] Mueller [1981, 118 - 151] is probably the most complete discussion of the foundational issues of the ratio theory in *Elem.* V. A recent, although less systematic, study of *Elem.* V is Acerbi [2003a].

[62] See for example Becker [1933], Knorr [1975], Knorr [1978] and Fowler [1987].

[63] There are a number of summaries of the six operations on ratios. See for example, Heath [1926, vol. 2, 134 - 136 & 164 - 184], Dijksterhuis [1987, 52 - 54], Netz [1999b, 139 - 140], Vitrac [2001, vol. 2, 50 - 56 & 61 - 65] and Taisbak [2003, 44 - 45].

[64] This reference is usually translated into Latin or, in one case, simply transliterated. This is presumably meant to mark the operations as such. I have chosen to translate the expressions into English and indicate that they are technical terms by using italics.

Another operation, that of compounding ratios, should be included with these, although certain important differences must be noted. Composition of ratios was not semantically marked in the same way as the other operations; that is, there was no expression that referred to the operation itself. Nevertheless, it was always expressed in certain well defined ways, as will be seen below. Moreover, composition of ratios appears to have been understood as an operation in a later period than the others. Composition of ratios comes into widespread usage only in texts written after *Elem.* V was composed. This operation was used to great effect by Apollonius and Archimedes and plays an important role in Ptolemy's mathematics. I give here a brief summary of each operation.

1. *Inversely* (ἀναπαλιν, *anapalin*, invertendo)[65]

   This operation is defined in *Elem.* V def. 13. It is not justified in the *Elements.* Perhaps the proof for proportions was considered too obvious. It is used by Greek mathematicians for both proportions and ratio inequalities:

   $$A : B = C : D \Longrightarrow B : A = D : C,$$

   and

   $$A : B \gtrless C : D \Longrightarrow B : A \lessgtr D : C.$$

   The effect of the inequality changes when a ratio inequality is subjected to *inversion.* I am not aware of any ancient proof of this fact, although it was was well known in practice.

2. *Alternately* (ἐναλλαξ, *enallax*, alternando *or* permutando)[66]

   This operation is defined in *Elem.* V def. 12 and demonstrated for proportions in *Elem.* V 16. It is used by Greek mathematicians for both proportions and ratio inequalities:

   $$A : B \gtreqless C : D \Longrightarrow A : C \gtreqless B : D.$$

[65] Mugler [1959, 58 - 59].
[66] Mugler [1959, 175 - 176].

3. *By composition* (συνθέντι or rarely κατὰ σύνθεσιν, *sunthenti* or *kata sunthesin*, componendo)[67]

   This operation is defined in *Elem.* V def. 14 and demonstrated for proportions in *Elem.* V 18.[68] It is used by Greek mathematicians for both proportions and ratio inequalities:

   $$A : B \gtreqless C : D \Longrightarrow (A+B) : B \gtreqless (C+D) : D.$$

4. *By separation* (διελόντι or rarely κάτα διαίρεσιν, *dielonti* or *kata diairesin*, separando)[69]

   This operation is defined in *Elem.* V def. 15 and demonstrated for proportions in *Elem.* V 17.[70] It is used by Greek mathematicians for both proportions and ratio inequalities:

   $$A > B \ \wedge \ A : B \gtreqless C : D \Longrightarrow (A-B) : B \gtreqless (C-D) : D.$$

5. *By conversion* (ἀναστρέψαντι, *anastrepsanti*, convertendo)[71]

   This operation is defined in *Elem.* V 16. It is not demonstrated in the *Elements.* In fact, *conversion* is simply successive applications of *separation*, *inversion* and *composition.* It is used by Greek mathematicians for both proportions and ratio inequalities:

   $$A > B \ \wedge \ A : B = C : D \Longrightarrow A : (A-B) = C : (C-D),$$

   and

   $$A > B \ \wedge \ A : B \gtrless C : D \Longrightarrow A : (A-B) \lessgtr C : (C-D).$$

[67] Mugler [1959, 400 - 401].
[68] *Elem.* V 18 shows that $A - B : B = C - D : D \Longrightarrow A : B = C : D$.
[69] Mugler [1959, 130 - 131].
[70] *Elem.* V 18 shows that $A + B : B = C + D : D \Longrightarrow A : B = C : D$.
[71] Mugler [1959, 60].

Because of the function of *inversion*, the effect of the inequality changes when a ratio inequality is subjected to *conversion*.

6. *Through equality* (δι' ἴσου, *di isou*, ex equali)[72]

This operation is defined in *Elem.* V def. 16 and two ways in which it functions are demonstrated for proportions in *Elem.* V 22 & 23.[73] It is used by Greek mathematicians for both proportions and ratio inequalities. *Elem.* V 22, extended to include ratio inequalities, asserts that

$$\begin{array}{rcccl} & A_1 : A_2 & \gtreqless & B_1 : B_2 & \wedge \\ & A_2 : A_3 & \gtreqless & B_2 : B_3 & \wedge \quad \ldots \\ \wedge & A_{n-2} : A_{n-1} & \gtreqless & B_{n-2} : B_{n-1} & \wedge \\ & A_{n-1} : A_n & \gtreqless & B_{n-1} : B_n & \Longrightarrow \\ & A_1 : A_n & \gtreqless & B_1 : B_n, & \end{array}$$

while, *Elem.* V 23, extended to include ratio inequalities, asserts that,

$$A_1 : A_2 \gtreqless B_1 : B_2 \ \wedge \ A_2 : A_3 \gtreqless B_3 : B_1 \Longrightarrow A_1 : A_3 \gtreqless B_3 : B_2.$$

The alternative definition of *through equality* asserts that it is "a selection of the extremes according to a removal of the means."[74] These two theorems show that (1) the means can be removed so long as they are equal pairwise and that (2) the order in which the means occur is irrelevant.

My translation of *di isou* implies that *isou* is a noun. In fact, it is an adjective and the expression has undergone ellipsis. Heath and Taisbak take the full expression to be *through an equal distance*; that is, with an equal number of intervening terms.[75] Vitrac conveys a similar reading with more precision through the expression *à*

[72] Mugler [1959, 229 - 230].

[73] *Elem.* V 22 shows that $A : B = D : E \ \wedge \ B : C = E : F \Longrightarrow A : C = E : F$; the argument makes no use of the specific number of proportions involved. *Elem.* V 23 shows that $A : B = E : F \ \wedge \ B : C = D : E \Longrightarrow A : C = D : F$.

[74] λῆψις τῶν ἄκρων καθ' ὑπεξαίρεσιν τῶν μέσων, Heiberg [1977, vol. 2, 3].

[75] Heath [1926, 136] and Taisbak [2003, 45].

égalité de rang.[76] The structure of the *through equality* arrangement, however, insures that there will always be an equal number of terms. Terms in the sequence of As are set pairwise into proportion with terms in the sequence of Bs so that the two sequences must have the same number of members. The expression *di isou* could just as easily refer to the equality of the means themselves. I have chosen to substantiate the adjective and preserve the ambiguity of the original expression.

7. Compound ratio (ὁ συγκει΄μενος λόγος or ὁ συμημμένος λόγος, *ho sunkeimenos logos* or *ho summēmmenos logos*)[77]

Compounding of ratios is not semantically marked by a phrase referring to the operation. The expression generally used is "the ratio A to B has been compounded of the ratio C to D and the ratio E to F." Two different verbs are used: συντιθέται, "to add or put together" and συνάπτειν, "to join or connect." Most authors use both terms but earlier authors tend to prefer the former term, later authors the latter. The history of compound ratios among the Greeks is not well understood and more work in this area is needed.[78]

There were three different traditions in which ratios were understood to be compounded, or joined, in distinct but related ways. These were (1) music theory, (2) geometry and (3) number theory. Although these different ways of compounding ratios were maintained as separate in the theoretical traditions, there can be little doubt that Greek mathematicians understood their functional similarity. In fact, as will become clear, it is exactly this functional similarity that was exploited by the applied tradition in order to develop an arithmetized conception of the composition of the ratios of geometric objects.

In the branch of Greek music theory which traces its roots back to the Pythagoreans of the Classical period, intervals (διαστήματα) are joined together by taking the

[76] Vitrac [2001, vol. 2, 52, n. 72].
[77] Mugler [1959, 384 - 385 & 397 - 398].
[78] Saito [1986] is the best study of the early history of compound ratio. Sylla [1984, 17 - 26] also makes a number of important points.

product of the antecedents of the ratios that express them to the product of the consequents of these ratios and expressing this new ratio in least terms. Hence the octave, which is expressed by $2:1$, may be understood as the fifth, $3:2$, joined with the fourth, $4:3$. That is,

$$(3:2) \times (4:3) = 12:6 = 2:1.$$

The arithmetic details of joining intervals are almost never discussed. The texts of Greek music theory simply assume these as understood and proceed to discuss the results of joining intervals which posses certain properties. There is a definition of compound ratio given in some manuscripts of *Elem.* VI which, despite the fact that it is generally thought to be spurious, covers the conception of compound ratio found in the music and number theoretic traditions. This definition reads, "A ratio is said to be compounded of ratios, when the sizes (πηλικότητες) of the ratios multiplied together make some [ratio]."[79] The sizes referred to here are the sizes of the individual terms of the ratios since, in *Elem.* V def. 3, a ratio is defined as "a kind of relation between magnitudes of the same kind with respect to size (κατὰ πηλικότητα)."[80] Clearly, this conception of compound ratio covers the operation of joining musical intervals.

The geometrical conception of ratio composition is first formalized in the *Elements*. *Elem.* VI 23, states that two similar parallelograms are to one another in the ratio compounded of the ratios of their sides. That is, in Figure 1.10,

$$\text{para. } AC : \text{para. } CF = (BC:CG) \times (DC:CE) = (BC:CE) \times (DC:CG).$$

The possibility of rearranging the consequents of the compound ratio is secured by the construction in the proof of *Elem.* VI 23. Although this conception of

[79] Heath [1926, 189 - 190]. An application of this definition of compound ratio to ratios between any magnitudes is discussed below. See page 146.

[80] Heath [1926, 116 - 119].

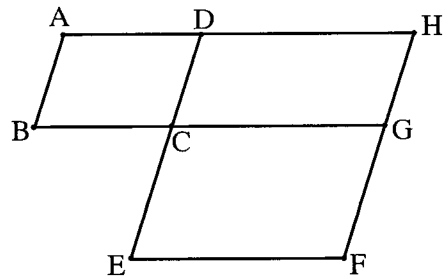

Figure 1.10: Diagram for *Elem.* VI 23, Heath [1926, 247].

compound ratio is not used in the *Elements*, it is used by Archimedes and plays a major role in Apollonius' *Conics*.[81] There is no definition of compound ratio in the *Elements* that covers the geometrical conception of *Elem.* VI 23. As Saito has shown, the operation of compounding of ratios sits uneasily in the Euclidean *Elements*.[82] Nevertheless, it is clear that during the course of the Hellenistic period the geometrical composition of ratios came to be understood as an important and versatile tool.

Elem. VIII 5 gives the number theoretic analog of *Elem.* VI 23. It states that two plane numbers are to one another in the ratio compounded of their sides. That is, where a, b, c, d are positive integers,

$$ab : cd = (a : c) \times (b : d) = (a : d) \times (b : c).$$

It is clear that the spurious *Elem.* VI def. 5 also covers this conception of compound ratio.

Another operation related to compound ratio is used both in the music theoretic

[81] Saito [1986, 41 - 54] argues that *Conics* 39, 41 & 43 rely not on *Elem.* VI 23 but on an operation related to compounding which he calls "reduction to linear ratio." His argument relies on a reconstructed analysis and the fact that he himself can write shorter proofs of these theorems which rely more extensively on *Elem.* VI 23. Nevertheless, even if we accept Saito's arguments in regard to *Conics* 39, 41 & 43, there are still numerous uses of *Elem.* VI 23 in the *Conics.* Although, Heiberg [1891] does not note these in his edition and translation of the first four books, they are almost all noted in the translation by Taliaferro and Densmore [1998] of the first three books.

[82] Saito [1986, 29 - 35].

and geometrical traditions. This is the expansion of a single ratio into a compound ratio. Thus, if A, B and C are any magnitudes of the same kind, then

$$A : B = (A : C) \times (C : B).$$

No proof of this fact has survived in the theoretical texts; however, Theon attempts to provide one in his commentary on *Alm.* I.[83] Nevertheless, it is clear that Greek mathematicians were well aware of this fact and felt free to use it in their work.

1.3.1 Ptolemy's use of the traditional operations

Considering the quantity of Ptolemy's writings devoted to mathematical argumentation, there is surprisingly little use of the operations on ratios. The six traditional operations are used only in geometrical theorems. They are found in a handful of theorems in the *Optics* and in theoretical discussions of some of the experimental instruments in the *Harmonics*.[84] They are also used in three places in the *Almagest*.[85] All of the uses in the *Almagest* can be shown to either be taken from, or motivated by, the work of Ptolemy's Hellenistic predecessors.

In the *Almagest*, Ptolemy uses operations on ratios in (1) the derivation of the chord table, (2) the derivation of the fundamental theorem of ancient spherical trigonometry and (3) the theoretical treatment of planetary retrogradation.

The operations on ratios in the derivation of the chord table are found in a theorem which is a lemma to a type of trigonometry practiced in the Hellenistic period by Aristarchus and Archimedes.[86] Moreover, the form of the theorem is very similar to a lemma attributed by Ptolemy to Apollonius in *Alm.* XII 1. The theorem clearly has its roots in the traditional geometry of the Hellenistic period.

[83] Theon's proof is discussed in Chapter 3. See page 146.

[84] These are *Harm.* II 2 & III 1, *Optics Theorem* III.2, IV.5, IV.9, IV.25 & *Experiment* V.9, Barker [1989, 320, 361] and Smith [1996, 148 - 149, 178, 181, 201, 244].

[85] These are in *Alm.* I 10.11, I 13.1 & 13.2, & XII 1, Toomer [1984, 55, 64 - 65, 558 - 561].

[86] This early Hellenistic trigonometry is discussed in Chapter 2, Section 2.1.2. This theorem is discussed below; see page 90.

Compound ratios are used by Ptolemy in a lemma necessary to his development of spherical trigonometry.[87] This lemma goes back to the work of Menelaus or Hipparchus, as I will argue in Chapter 3. The lemma itself is simple and is typical of those Euclid assumed in his more advanced works, such as the *Porisms*.[88] Again, the context is that of traditional Hellenistic geometry.

The final use of the operations on ratios in the *Almagest* is found in a section that Ptolemy tells us is derived from the work of Apollonius. Although Ptolemy explicitly takes credit for some parts of this presentation, the section as a whole is motivated by Apollonius' treatment of planetary retrogradation. Once again, the mathematical methods and the style of presentation are traditional.

On the whole, Ptolemy's uses of operations on ratio in his geometrical presentations are fairly conservative and represent no departure from tradition. Indeed, Aristarchus and Archimedes exhibit a much more interesting use of these operations as a sort of calculus for solving numerical problems.[89]

Despite the fact that Ptolemy's manipulations of ratios are traditional, his use of ratios in metrical analysis and computations represents an important new understanding of ratio which was developed in the applied tradition. We will discuss this new understanding in the next section.

1.3.2 Ptolemy's break with tradition

It should be emphasized that the conceptions of ratio in Ptolemy's work that appear to be innovations on the tradition are not due to Ptolemy himself. Indeed, the very notion of a traditional conception of ratio is problematic. In all likelihood, there were a number of competing views on ratio in the Classical and Hellenistic periods and Greek mathematicians were able to draw on these in a variety of ways. What should be emphasized is that, in the applied mathematical texts, the notion of ratio became increasingly arithmetized

[87] See page 163 for a discussion of this lemma.
[88] See page 132, note 29.
[89] This material will be discussed in the next chapter. See Chapter 2, Section 2.1.2.

until a ratio was understood as a relationship between two numerical values which could itself be used in calculations as a single numerical value. A few examples will show the range of Ptolemy's application of ratio toward producing numerical solutions to specific problems.

In the simplest sense, proportions or ratio inequalities were used to form equations or inequalities. This sort of manipulation was used extensively in the mathematical astronomy of the early Hellenistic geometers.[90] For example, in *Harm.* II 2 we encounter the assertion that $ML = 2LH$ follows from $ML : LH = BD : BZ$ and $BD = 2BZ$.[91] In *Alm.* I 10.2 we find that $AG : BA < \widehat{AG} : \widehat{AB}$ and $\widehat{AG} = 4/3\ \widehat{AB}$ allows us to assert that $AG < 4/3 AB$.[92] These deductions are obvious, and well within the strictures of Euclidean ratio theory. Nevertheless, it is clear how they may be used to derive numerical solutions to geometric problems.

Proportions were often used as equations to solve for a fourth term given the other three terms.[93] In this context, the arithmetic operations of division and multiplication and their relationship to the terms of the proportion, were well understood. This is especially clear in metrical analysis.[94] Although a ratio is expressed as a relation between two numbers, in numeric practice it would have been clear that the ratio can be represented by the quotient of the two values and that this is equal to the quotient of any other two numbers in the same ratio. The process of finding a fourth proportional, so key to metrical analysis, depends on this fact.

An especially interesting use of this is found in Ptolemy's plane trigonometry. In

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90 This material is discussed in its mathematical context below. See Section 2.1.2.

91 Barker [1989, 320].

92 Toomer [1984, 55]. The theorem from which this step is excerpted is discussed below, page 91.

93 Grattan-Guiness [1996] makes the point that Euclid never states of a proportion that the two ratios are equal. Thus, he argues that there should be a fundamental distinction between a proportion and an equality. This is true in the sense that Euclid is not asserting the equality of two quantities. A ratio is a relation between quantities; it is not, in Euclid's idiom, itself a quantity. We define the relation *equality* as being reflexive. This is a highly abstract idea and absent from Greek mathematics. A Greek mathematician might assert the equality of two different parallelograms of the same area or lines of the same length, but would simply assume the *identity* of the same object. A proportion should be read as asserting two instances of the same relation, that is of a single ratio.

94 The role of ratio in metrical analysis is discussed throughout Chapter 2. See especially Sections 2.1.4 and 2.2.

using ratios to convert from one set of units to another, Ptolemy uses an expression that differs from the traditional expressions for ratio and which more directly conveys the arithmetic conversion at issue. In *Alm.* III 4, the first plane trigonometric computation in the *Almagest*, we encounter the following expression: "Now since, EZ was shown to be $2;29\frac{1}{2}$ of those [units] of which the line ZX was 1;2, therefore also, the hypotenuse EZ is 120 of those of which the line ZX will be approximately 49;46."[95] In other words, $EZ : ZX = 2;29\frac{1}{2} : 1;2 = 120 : 49;46$. This idiom and its variants can be taken as specifying an arithmetized proportion in the context of ancient plane trigonometry. It is related to the traditional proportion in the same way as the computations of plane trigonometry are related to metrical analysis. That is, the traditional proportion establishes the theoretical possibility of using the arithmetized proportion to calculate a fourth proportional. Nevertheless, within the context of computation, a proportion has a thoroughly arithmetic function.

Probably the most interesting uses of ratio in the *Almagest* is that of compound ratio in the material on spherical astronomy. Here again, we find compound proportions being used as equations to solve for a particular term. In this context, it is clear that the inverse of compounding was also understood as an arithmetic operation. Although this operation is never discussed in the theoretical texts, it was frequently used in music theory. A single example will illustrate both points.

In *Alm.* II 7.5, Ptolemy introduces a batch calculation method based on ascensional difference to compute the rising times of 10° arcs of the ecliptic at *sphaera obliqua*.[96] In the course of this calculation, a single compound proportion is used a number of times to generate a list of ratios found by varying two terms in the proportion. In this process, we read:[97]

> From the above, if we take away from $\text{Crd}(2\,\widehat{\Theta H}) : \text{Crd}(2\,\widehat{HZ})$, that is, $48;31,55 : 109;44,53$, each of [the ratios] $\text{Crd}(2\,\widehat{LK}) : \text{Crd}(2\,\widehat{KZ})$, set out [above] for 10°

[95] The Greek and Toomer's translation of this passage are given below, see page 2.1.4, n. 94.
[96] See page 157 for a more complete discussion of this material.
[97] Heiberg [1916, vol. 1, p. 1, 129 - 130] and Toomer [1984, 97]. I have modified the beginning of Toomer's translation, at the expense of some clarity, to better reflect the Greek.

> [arcs],[98] we will be left Crd(2 $\widehat{\Theta E}$) : Crd(2 $\widehat{EL}$), the same for all latitudes. For the 10° arc it is 60 : 9; 33; for the 20° arc, 60 : 18; 57; for the 30° arc, 60 : 28; 1; for the 40° arc, 60 : 36; 33; for the 50° arc, 60 : 44; 12; for the 60° arc, 60 : 50; 44; for the 70° arc, 60 : 55; 45; for the 80° arc, 60 : 58; 55.

In other words, Ptolemy takes a series of eight ratios Crd(2 $\widehat{\Theta E}$) : Crd(2 $\widehat{EL}$) = (Crd(2 $\widehat{\Theta H}$) : Crd(2 $\widehat{HZ}$)) ÷ (Crd(2 $\widehat{LK}$) : Crd(2 $\widehat{KZ}$)), where Crd(2 $\widehat{\Theta H}$) : Crd(2 $\widehat{HZ}$) is constant and Crd(2 $\widehat{KZ}$) : Crd(2 $\widehat{LK}$) is expressed by eight different pairs of sexagesimal numbers. He then expresses this series of ratios such that the antecedent is set to 60 in each case. Although the ratios are always expressed as a relation between two terms, Ptolemy must have seen the advantage, on the computational level, of treating them as rational numbers; and he almost certainly took advantage of it. The *Almagest* is dense with computation and there is compelling evidence that Ptolemy took steps to alleviate it as much as possible.[99] In this particular instance, treating Crd(2 $\widehat{\Theta H}$) : Crd(2 $\widehat{HZ}$), or its inverse, as a rational number would allow the computer to carry out the calculation with nine divisions and nine multiplications. On the other hand, if Crd(2 $\widehat{\Theta H}$) : Crd(2 $\widehat{HZ}$) is treated strictly as a relation between two terms, the calculation requires eight divisions and 24 multiplications. While divisions were more difficult than multiplications, there can be little doubt that the advantage of substituting a single division for fifteen multiplications was perceived and exploited. There are numerous situations of this kind in Ptolemy's computations.

These examples make it clear that Ptolemy understood the relationship between proportions and ratio inequalities on the one hand and equalities and inequalities on the other; and that he knew, at least on the practical level, that a ratio could be represented as a rational number.

By the time Theon wrote his *Commentary*, the idea that a ratio could be represented

[98] These ratios are given in the previous passage.

[99] For example, the chord interpolation table, the "sixtieths" table, which was compiled with some effort, saves the computer two subtractions and one division for each use of the chord table; see Section 2.1.4. Van Brummelen has shown that there is reason for believing that Ptolemy used interpolation grids as opposed to direct calculation to fill in some of his tables. This would have been done to save computational labor. See especially Van Brummelen [1993, 40 & 377 - 378]

by a single number had become explicit. In Chapter 3, we will encounter an attempted proof by Theon in compound ratio theory.[100] In this passage, he describes a ratio as being described by a single size (πηλικότης), or value, and states that this value can be multiplied by, or found equal to, the value of another ratio.

1.3.3 Ratio in applied mathematics

We have seen a number of ways in which the use of ratio, proportion and ratio inequality in the applied tradition went beyond the foundations provided in *Elem.* V. It is possible that there were other, lost texts that gave these concepts a broader and more versatile foundation. It is more probable, however, that there were no such texts, and that working mathematicians simply held a variety of different, and not always explicit, ideas about the role and function of ratio in mathematics.

The exact sciences demonstrate the existence of applications for, and ideas about, ratio which we would not suspect if we limited our study to the theoretical texts. Although a number of examples have been mentioned above, we will find more as we discuss the texts themselves. It is important, in this regard, to realize that the foundational work of *Elem.* V was almost certainly not written in order to facilitate new practice but rather to justify an elementary subset of existing practice and establish this on acceptable first principles. As we will see in the next chapter, some of the most interesting uses of ratio manipulation which are not justified by the *Elements* are practiced by mathematicians who worked around the time of, or fairly shortly after, the composition of that book and were probably not much under its influence. We must constantly avoid reading the *Elements* as representative of all Greek mathematics.

[100] See page 146.

1.4 The role of tables as objects of knowledge

The role of tables in Ptolemy's writings is similar to that of propositions in the body of mathematical theory in which they are established. These objects are both means and ends. They are set out both because they are productive of new knowledge and because the knowledge they contain is inherently interesting. We have seen above how tables may be used in the argumentative structure of Ptolemy's works; how they help the mathematician simplify or solve particular problems. It remains to discuss the role of tables as objects of knowledge.

A clear example of this is found in the *Analemma.* One of the main goals of this text is the production of 49 tables of 42 entries, tabulating six functions in three variables. The calculation of all 2058 of these values would be so laborious that one of the other primary goals of the work is the development of nomographic techniques to lighten this work. Although these tables may be of some use to sundial makers, within the context of the *Analemma* itself, they must be seen as an end result, intrinsically valuable for the information they convey.[101] By studying these tables, one can develop a quantitative understanding of the local, diurnal motion of the sun.

There have been very few studies devoted to Ptolemy's tables. Two of these have applied statistical methods in an attempt to determine how the tables were generated.[102] A number of interesting observations have come out of this work but on the whole the results, although conflicting, have been insecure or negative.

One of the most interesting studies of Ptolemy's tables is that of O. Pedersen.[103] This work makes clear that we can understand Ptolemy's tables as representing functions and that Ptolemy's knowledge of how these functions behaved is quite sophisticated. In particular, Pedersen shows that Ptolemy separated functions of two variables into cases where the second variable was either strong or weak. If the second variable was weak, Ptolemy used this fact to simplify his tables in a systematic way with little loss of

[101] This material is discussed below, see Section 4.1.
[102] Newton [1985] and Van Brummelen [1993].
[103] Pedersen [1974b, 78 - 93] and more fully in Pedersen [1974a].

precision.

When we make these statements about functions, however, we must remember that we are using this notion for our own understanding; it not found in Ptolemy. The function concept is an abstraction, whereas Ptolemy always deals with concrete tables.[104] There is no reason to suppose that Ptolemy viewed a table as defining an abstract object. The table itself was the object. It acted as a quantitative representation of the underlying model.

An example will help to clarify these statements. After his treatment of the mathematical relationship between the solar mean motion, $\overline{\kappa}$, apparent motion, κ, and equation of anomaly, c, in *Alm.* III 5, Ptolemy discusses how he will structure the table which will numerically encapsulate these relationships. Toomer's text reads, "we prefer that form in which the argument is the mean motion and the function is the equation of anomaly."[105] Ptolemy, however, simply refers to this table as, "the [one] having the equations of difference laid out next to the uniform arcs [of mean motion]."[106] For Ptolemy, the table exhibits the numerical relationship between two quantities which he has just shown can be respectively used to find each other. The object underlying the table, for which it provides a quantitative model, is not an abstract function. It is a geometric model, which in turn models the actual motion of the sun. In fact, the design of the table is such that it allows us to form a clear idea of the way the two quantities, $\overline{\kappa}$ and c, change in relation to one another. This table is not only useful as a tool for calculation, it also provides the reader with direct knowledge about the anomalistic motion of the sun.

This table is meant to be read in conjunction with the solar mean motion table which has already been given, in *Alm.* III 2.[107] Ptolemy is quite explicit about this. Prior to

[104] Pedersen claims that we can think of a function "not as a formula, but as a more general relation associating the elements of one set of numbers ... with the elements of another set of numbers ..., Pedersen [1974a, 36]. This conception of a function, however, is even more abstract and assumes a number of concept of set theory that are not articulated in Greek mathematics.

[105] Toomer [1984, 165]. A literal translation is given in footnote 55, "which contains the equations of anomaly corresponding to the arcs of the mean motion."

[106] ἡ ταῖς ὁμαλαῖς περιφερείαις παρακειμένας ἔχουσα τὰς παρὰ τὴν ἀνωμαλίαν διαφορὰς, Heiberg [1916, vol. 1, p. 1, 251].

[107] Toomer [1984, 142 - 143].

setting it out, he explains his purpose in presenting the mean motion table.[108]

> Now, with regard to the determination of the positions of the sun and the other [heavenly bodies] for any given time, which the construction of individual tables is designed to provide in a handy and as it were readymade form: we think that the mathematician's task and goal ought to be to show all the heavenly phenomena being reproduced by uniform circular motions, and that the tabular form most appropriate and suited to this task is one which separates the individual uniform motions from the non-uniform [anomalistic] motion which only seems to take place, and is [in fact] due to the circular models; the apparent paths of the bodies are then demonstrated by the combination of these two motions into one.

This passage is one of the few places where Ptolemy addresses the purpose of tables in his work. It precedes the solar mean motion table which, as was discussed above, is not necessary for the logical structure of the work. Although the mean motion table may facilitate calculation, the uses to which it is put in the mathematical argument of the *Almagest* could also be served by a single value for the daily mean motion. Moreover, this practical advantage is not the reason Ptolemy states for setting out the table. The table is meant to display the underlying reality of the solar motion, its uniformity. Indeed, by studying the mean motion table we can develop a better intuitive sense for this motion than we can on basis of a single value.

The purpose of the two tables taken together is to provide an accurate description of the real motion of the sun. In the case of the solar motion, Ptolemy could easily construct a single table that combined the effects of the two motions of the sun and displayed the longitudinal position of the sun for any given time from a cardinal position of the ecliptic. This table would then be used in conjunction with a table which gave dates for the initial position of the solar motion.[109] Such a combination of tables, however, would obscure

[108] Toomer [1984, 140]. I have altered the last phrase of Toomer's translation which reads. "the apparent places (παρόδους) of the bodies are then displayed (ἀποδεικνύουσαν) by the combination of these two motions into one." Although this is a valid reading of the Greek, I believe it obscures Ptolemy's point.

[109] We have examples of such tables in the Oxyrhynchus papyri, see Jones [1999a, 48 - 53 & 94 - 101]. Also see Jones [1997].

the underlying uniformity. They would exhibit, instead, the most illusory level of the phenomena. Such tables, although perhaps more convenient, would tell us nothing more than a series of observations.

Another example of the use of tables to display quantitative information is provided by the more complex table of rising times, *Alm.* II 8.[110] This table tabulates the rising times of arcs of the ecliptic, ρ, as a function of two variables, geographic latitude, φ, and the longitude of the arc of the ecliptic.[111] Because both variables are strong, Ptolemy tabulates at 10° degree intervals of the ecliptic for eleven specific latitudes from the equator, $\varphi = 0°$, to $\varphi = 54;1°$.

Along with the column for accumulated time degrees, Ptolemy includes a column for the difference between the total rising time and the previous rising time; that is, the rising time for each 10° arc. As Pedersen points out, this column is irrelevant to the representation of rising time as $\rho(\lambda, \varphi)$.[112] It is also irrelevant to the use of the table of rising times as a calculation tool. The applications of the table, in *Alm.* II 9, use the values in the column of accumulated time degrees, or a value derivable from these.[113] Nevertheless, a study of the columns of differences is essential for developing an intuitive understanding of the phenomena of rising times.

Since the autumnal equinox is the midpoint of the rows, we can see at once that the rising times are symmetrical about this point. Moreover, it is clear that the greatest rising times are not at the autumnal equinox but at two points symmetrically situated before and after that point. For *sphaera recta*, we can see that the changes in rising times are periodic and symmetrical about the cardinal points of the ecliptic. As we head north on the globe, however, we notice two distinct patterns. (1) The minimum of the spring equinox decreases while the minimum of the autumnal equinox increases. (2) The maximums increase together and move from the solstices toward the spring equinox. All

[110] Toomer [1984, 100 - 103]. This table and its role in Ptolemy's spherical astronomy is discussed below, see Sections 3.3, 3.4 and 3.5.
[111] This material is discussed in Pedersen [1974b, 110 - 115] and Neugebauer [1975, 34 - 37].
[112] Pedersen [1974a, 41].
[113] Toomer [1984, 99 - 104].

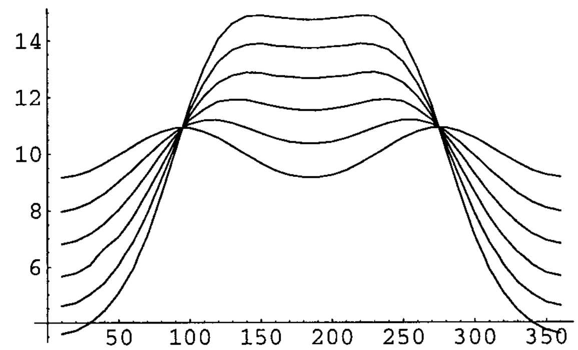

Figure 1.11: Plot of the rising times for six of the geographic latitudes in *Alm.* II 8 using the values in Ptolemy's table, Toomer [1984, 100 - 101]. The lowest minimum at the autumnal equinox ($\lambda = 180°$) is *sphaera recta* ($\varphi = 0°$). The other latitudes, rising with this local minimum are Meroe ($16;27°$), Lower Egypt ($30;22°$), Hellespont ($40;56°$), Mouths of Borysthenes ($48;32°$) and Mouths of Tanais ($54;1°$).

these changes are displayed by modern commentators using a graph depicting a series of functions.[114] These graphs generally plot the rising times of arcs of 1° but we could just as well use the values Ptolemy tabulates for 10° arcs; see Figure 1.11. Although the graph strikes us as the most agreeable way of presenting this information, there is no pattern which is discernible in the graph but which is not also obvious from an examination of the numbers in Ptolemy's table.

The tables we have discussed so far are all based on geometric models. In the astronomical works, this is generally the case. There are, however, quite a few tables that are based on a model which is numeric or governed by ratios. All of the tables in the *Harmonics* are of this kind. A series of lists that belong to this group are those of the angles of refraction in *Optics* V.

The lists in the *Optics* are particularly interesting because they are presented as the results of three elaborate experiments.[115] Ptolemy describes the use of a dioptra mounted on a degree calibrated disk and a glass semicylinder to measure the angle of refraction from air to water, air to glass and water to glass.[116] There can be no doubt

[114] See for example, Schmidt [1943, fig. 22], Pedersen [1974b, 112], and Nadal, Taha and Pinel [2004, 401].
[115] Smith [1996, 232 - 238].
[116] The sighting instrument is described in *Optics* III [8 - 12], Smith [1996, 134 - 136].

that Ptolemy intends us to understand that he actually carried out these experiments. It is equally clear that Ptolemy subjected his observations to smoothing, or extrapolation, by a straightforward numerical procedure before he published them. Although Ptolemy indicates that the numbers are approximate, each of the lists for angles of refraction have equal second differences of a semidegree.[117] The values that Ptolemy reports are as follows:[118]

Incidence Angle	**Refraction Angle**		
	Air/Water	**Air/Glass**	**Water/Glass**
10°	8°	7°	9½°
20°	15½°	13½°	18½°
30°	22½°	19½°	27°
40°	29°	25°	35°
50°	35°	30°	42½°
60°	40½°	34½°	49½°
70°	45½°	38½°	56°
80°	50°	42°	62°

As Lloyd states, we cannot know precisely what the differences are between these and the results predicted by current theory because we do not know the chemical details of the water and glass Ptolemy used.[119] Nevertheless, it is possible to form a rough estimate by calculating tables for the angles of refraction based on the sine law and current refractive indexes of glass and water. These calculations were carried out and tabulated by Govi in the introduction to his edition of the *Optics*.[120]

From the numbers Govi prints we see (1) that Ptolemy's values are fairly good overall, (2) that the fit is best for the middle values and (3) only poor for the highest values. They also, however, underscore the realization that Ptolemy's numbers are not simply the result of rounding the observed values to the nearest semidegree. Ptolemy must have

117 The first term of each list is modified by "nearly," *ad prope*, Lejeune [1989, 229, 234 & 236].

118 Smith [1996, 233, 236 & 238].

119 Lloyd [1982, 151]. Although the refractive indexes of atmospheric air and water taken from natural sources are probably about the same now as they were in the second century, we do not know whether Ptolemy used fresh or salt water. Primarily, however, we cannot know the type and purity of the glass that Ptolemy used.

120 Govi [1885, xxx].

worked systematically to produce three lists with equal second differences.[121] In other words, Ptolemy must have been looking for a numerical model which would explain the phenomena. In this regard, he was probably influenced by the methods of numerical astronomy that had been imported from the Near East and were still in use in the Imperial period.[122]

The only general principle that Ptolemy asserts about the relation between angle of incidence, i, and angle of refraction, r, is that the ratio between the two will decrease as i increases; that is $i_m : r_m > i_n : r_n$, where $0° < i_m < i_n < 90°$.[123] Although this principle would have been obvious from the experiment, Ptolemy presents it, following the experiment, along with an explanation of the cause of the ratio inequality that has to do with the density of the substances involved. On the other hand, no causal explanation is given for the numbers found in the lists.

Any number of numerical models will satisfy the stated ratio inequality but, in some sense, the one Ptolemy chooses is the simplest. The experiment would have shown Ptolemy that the angle of refraction increases as the angle of incidence increases, but that it is not increasing at the same rate. In Ptolemy's model, the angle of incidence increases at a constant rate while the angle of refraction constantly increases as well; however, it increases at a constantly decreasing rate. This information is concisely conveyed in the tables.

Both Lejeune and Smith believe that the numeric model which underlies the three lists in *Optics* V is Ptolemy's general principle for the behavior of refraction and they try to explain why Ptolemy does not state this principle in more general terms.[124] In fact, however, the general principle is the trivial ratio inequality which is stated after the experiments, presumably, because it can be accompanied by a causal explanation. In

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121 Two studies of this material that provide reconstructions of Ptolemy's approach are Lejeune [1946] and Smith [1982]. Lejeune's account has the advantages of being shorter and simpler.

122 See Jones [1991] and Jones [1996] discussions of the influence of Babylonian astronomical methods on Greek astronomy. See Jones [1999a] for the evidence that the importation of Babylonian astronomical methods was extensive and that these methods were still in use in the Imperial period.

123 Smith [1996, 244].

124 Lejeune [1946, 100] and Smith [1982, 235 - 238].

Optics V [32 - 34], Smith's *Experiment* V.4, Ptolemy gives a mathematical description of the phenomena which includes a discussion of the density differential between the two substances as the physical cause of refraction.[125] Angles from the previous lists are cited as examples of the general principle. Following this, Ptolemy states that the particular way the principle is instantiated must be determined experimentally in each case and it should be expressed as a series of numbers.[126] He seems to have considered the possibility that other substances would produce angles of refraction that would fulfill the conditions of the ratio inequality according to a different pattern. Ptolemy's own practice, however, makes it clear that he believed it was his responsibility to find some governing rule and incorporate it into the presentation of the numbers even if this rule could not be accompanied by a causal explanation.

The lists of angles of refraction in the *Optics* tell us everything Ptolemy felt we need to know about the angles in these particular cases. They are not meant to convey a general rule, applicable to all instances of refraction.

1.5 Conclusion

In the opening section, we have examined Ptolemy's views on mathematics. We have seen that Ptolemy takes the highest aim of mathematics to be the production of the exact sciences, specifically harmonics and astronomy. Geometry and arithmetic are construed as subsidiary subjects; they are described as tools or paths. Although Ptolemy promotes mathematics as both powerful and widely valuable, he does little to explain how it is actually used to produce knowledge.

In the next section, we have seen how the types of mathematical text found in the Ptolemaic corpus can be grouped into six categories. Each category was defined and exemplified. These categories will be used throughout the rest of this study to describe the fundamental forms in which Ptolemy and his predecessors operate. It will be seen that

[125] Smith [1996, 243 - 244].
[126] Smith [1996, 244].

all of the mathematical texts encountered can be well described by one of the categories presented here.

Following this, we have examined the role of ratio and operations on ratio in the applied mathematical traditions. Particular emphasis was given to the composition of ratios, an operation little supported by the foundations of ratio theory but used to good effect in both pure and applied mathematical texts. In this context, we have seen how the concept of ratio was gradually arithmetized, so that, by late antiquity, a ratio could be represented by a single number.

Finally, we have explored the role of lists as objects of knowledge. We have seen how lists can be used to describe the functional relationship between two variable quantities. In most cases where functions are tabulated, this is done in such a way as to express a function and its inverse. Ptolemy designs his tables so as to best convey quantitative information about the models which they represent. They are designed to play a role similar to the current use of graphs; they allow the reader to develop an intuitive understanding of the quantitative relationships which underlie the phenomena of the physical world.

Chapter 2

Ancient Analysis and Trigonometry: Geometrically and Numerically Given

This chapter attempts to trace the rise of trigonometric practices and situate them in the mathematical context in which they were originally used. Unfortunately, we do not have enough evidence to tell a narrative history of trigonometry among the Greeks. Nevertheless, the first section examines episodes of trigonometric practice before Ptolemy and theorems of other branches of mathematics that relate to trigonometry. Following this, Ptolemy's construction of the chord table in the *Almagest* is studied along with his trigonometric practice.

The second section of the chapter situates Ptolemy's trigonometric practice within the context of other Greek mathematical practices. This section also explains the utility of metrical analysis for solving trigonometric problems.

For the purpose of referencing units of the mathematical argument of Ptolemy's construction of the chord table in the *Almagest*, I have made a finer division of the text than that provided by the section numbers. The specifics of this division are given in Appendix A. Wherever I refer to the *Almagest* by book and section, I use these more precise numbers.

2.1 Trigonometry

The story of trigonometry among the Greeks is usually told in terms of trigonometry as we now understand it. Indeed, the trigonometry by chord tables that arose in the middle of the Hellenistic period can be readily expressed in terms of the sine function. There are a number of accounts of Ptolemy's trigonometry in terms of modern trigonometric formulae.[1] These are helpful because they allow us to situate ancient practices within the context of our own mathematical knowledge. In its fully developed state, however, Greek trigonometry had no formulae and only employed a single function and its inverse. In order to understand Ptolemy's trigonometry, it will be useful to situate it in the context of Greek mathematical practice. Although it is not possible to write a history of Greek trigonometry, nevertheless, by examining a number of different texts, we may form a picture of the trigonometric methods and techniques that were employed by Greek mathematicians at various times.

In order to sketch a picture of the use of trigonometry among the Greeks, I have chosen to look at a select group of texts that are representative of various stages in its history. I have also selected key theorems from Euclid's *Data* that give theoretical background for the mathematical objects involved in trigonometric practice. Although these theorems of the *Data*, and the whole possibility of trigonometry, rely on the elementary properties of similar triangles, such as those demonstrated in the beginning of *Elem.* VI, this investigation will show that the proper theoretical context in which to understand the development of trigonometry is the analytical tradition as codified in the *Data*. Indeed, this investigation will show that, although trigonometry was created in an applied context, its later practitioners sought to ground it in the theoretical tradition of analysis. In fact, the construction of trigonometry should be seen as a methodological, as opposed

[1] See for examples Czwalina [1927], Aaboe [1964, 101 - 126], Pedersen [1974b, 63 - 78] and Neugebauer [1975, 21 - 30].

to theoretical, development.

I have not included material from Hero, despite the fact that *On Measurement* gives witness to a merger of the theoretical and practical traditions and makes direct reference to works on chords. Rome has already argued that these works on chords do not relate to trigonometry.[2] I have also neglected the passage in Pappus which Rome claims is an example of Hipparchus' trigonometry.[3] Both Neugebauer and Toomer have argued that Pappus' reconstruction is fictitious and based on a manuscript error in a presumed Alexandrian recension of the *Almagest*.[4]

2.1.1 Triangles *given in form*

A sequence of theorems in Euclid's *Data* secures the theoretical possibility of doing trigonometry in the context of Greek mathematics. These theorems, however, like the whole of the *Data* are in the tradition of pure geometry and, hence, do not satisfy the metrical requirements that are necessary to a true trigonometric approach. *Data* 39 - 46 concern triangles that are *given in form*.[5] In the *Data* we find that there are different ways in which mathematical objects are said to be given:[6]

1. *Given in magnitude* is said of figures (χωρία) and lines and angles for which we can provide (πορίσασθαι) equals.

2. A ratio is said to be *given*, for which we can provide the same.

3. Rectilinear figures are said to be *given in form* if each angle is given and the ratios of the sides to one another are given.

4. *Given in position* is said of points and lines and angles which always hold the same place.

[2] Rome [1932] and Rome [1933b]. These papers cover the same material. The latter is the more developed.
[3] Rome [1943, 150 - 155] and Rome [1933b, 187 - 192].
[4] Neugebauer [1975, 323 - 325] and Toomer [1984, 268, n. 82].
[5] Menge [1896, 66 - 84] and Taisbak [2003, 118 - 132].
[6] Menge [1896, 2] and Taisbak [2003, 17].

On the basis of the third definition, we can see that if a side of a triangle *given in form* is *given in magnitude*, or may be taken as the unit, then the triangle is completely determined, that is *given in form and magnitude.*

The term *given*, however, has a different meaning in the tradition of pure geometry than it has in applied mathematics. In the context of the pure geometrical texts, a given object is generally one for which we can furnish an equal object through the techniques of geometric construction.[7] In applied mathematics, however, a given object is one for which we can provide a numerical value. Thus, when Euclid says that a triangle is *given in form* he means that we can construct angles that are equal to the angles of the given triangle and we can set out a series of three lines that have to one another the same ratios as the sides of the given triangle. On the other hand, if Ptolemy were to speak of a triangle *given in form* he would mean one in which we can name what part each of its angles forms of a whole circle and can state the lengths of each of its sides in terms of some assumed unit. Although Euclid's concepts and theorems lay the foundation upon which later trigonometric methods were developed, the path that led from the one to the other was not direct. As we shall see in the next section, the early Hellenistic mathematical astronomers did not make use of triangles *given in form* in their trigonometric work. Instead they relied on basic properties of the right triangle to make approximations where they could not make determinations.

In order to compare these two different uses of triangles *given in form*, it will be helpful to examine some texts from the two traditions. Euclid's *Data* is the fundamental text of geometrical analysis. It is also one of the few texts in this tradition that has survived. I will look at *Data* 40 & 43.

There are very few examples of metrical analysis before Ptolemy. Nevertheless, we have good reason to believe that other mathematical astronomers, particularly Hipparchus, Diodorus and Menelaus wrote in this vein. The text that I will use proves

[7] The concept of *given* in Greek geometric analysis is not trivial. Jones [1986, 67 - 68] discusses the role of the term in this tradition. Taisbak [2003, 19 - 36] deals specifically with the issues that the term raises in Euclid's *Data.*

that this sort of argumentation was used in the centuries before the current era. It is from Theon of Smyrna who is reporting the work of Adrastus. It is a short passage that gives a confused determination of Hipparchus' eccentric model of the solar orbit given the position of the solar apogee and a ratio related to solar distance.

Data 40 states that if the angles of a triangle are given in magnitude then the triangle is *given in form*. Let each of the angles in $\triangle ABC$ be given in magnitude; see Figure 2.1 (a). Let DE have been set out given in position and magnitude, and on DE, at points D and E, let $\angle EDZ = \angle ABC$ and $\angle ZED = \angle ACB$ have been constructed [*Elem.* I 23]. Then also $\angle DZE = \angle BAC$ [*Elem.* I 32]; and each of the angles at A, B and C is given; therefore, each of the angles at D, E and Z is given. Now since DE is given in position and, at the given point D on it, DZ has been drawn making a given angle; therefore, DZ is given in position [*Data* 29]. For the same reason, EZ is given in position. Therefore, the point Z is given in position [*Data* 25], and the points D and E are given. Therefore, the lines DZ, DE and EZ are given in position and in magnitude [*Data* 26]. Therefore, $\triangle DZE$ is *given in form*, and $\triangle DZE \sim \triangle ABC$ [*Elem.* VI 4], therefore $\triangle ABC$ is *given in form.*[8]

This proposition may be related to the later trigonometry of a right triangle by pointing out that it applies to the situation where an angle of a right triangle is given and the hypotenuse is assumed as given. It it clear, however, in reading through the proof of this proposition that it will be of no use for metrical investigations. $\triangle ABC$ is shown to be *given in form* by the fact that we can construct another triangle similar to it on a line of our choosing. The angles are only given insofar as we can construct other angles equal to them. The ratios of the sides are given by the fact that we can construct equal ratios between another set of lines through the involved process of constructing a similar triangle. Even if we had values for the angles expressed as parts of a circle, this proposition would give us no means for expressing the ratios as relations between numerical values.

[8] Menge [1896, 72] and Taisbak [2003, 124].

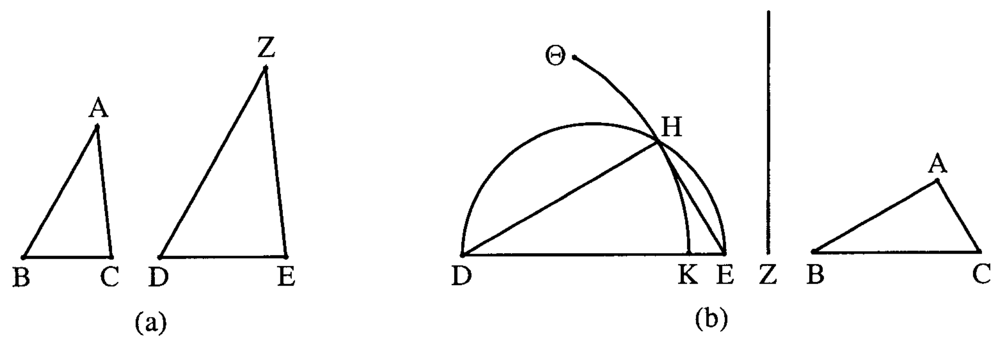

Figure 2.1: Diagrams for Euclid's *Data* 40 & 43. Taisbak [2003, 124 & 127 - 128].

It is shown in *Data* 43 that if the sides about one of the acute angles of a right triangle have a given ratio to one another then the triangle is *given in form*. In R$\triangle ABC$, let $CB : BA$ be given; see Figure 2.1 (b). Let DE have been laid out given in position and magnitude. Let the semicircle DHE have been described on DE so that it is given in position [*Data* 8]. Since $CB : BA$ is given, let Z be set out such that $CB : BA = DE : Z$ [*Elem.* VI 12]. Thus, $DE : Z$ is given. Hence, since DE is given, Z is given [*Data* 2]. But $CB > BA$ [*Elem.* I 19], therefore $ED > Z$ [*Elem.* VI 16 & 14]. Let $DH = Z$ have been fit into the semicircle [*Elem.* IV 1], and let HE have been joined. With center D and radius DH let the circle ΘHK have been drawn. Then the circle ΘHK is given in position [*Data* def. 6]. Therefore H is given in position, since semicircle DHE is also given in position [*Data* 25]. Therefore, since D and E are given in position the lines HD, DE and EH are given in position and magnitude [*Data* 26]. Hence $\triangle HDE$ is *given in form.* Now, since, in $\triangle ABC$ and $\triangle DEH$, $\angle BAC = \angle DHE$ [*Elem.* III 31], the sides about $\angle CBA$ and $\angle EDH$ are proportional while $\angle ABC$ and $\angle HDE$ are both acute [*Elem.* I 32], $\triangle ABC \sim \triangle DEH$ [*Elem.* VI 7]. But $\triangle DEH$ is *given in form*, therefore $\triangle ABC$ is *given in form.*

On the surface this theorem seems to be ideally suited to the right triangle trigonometry developed by the later mathematical astronomers. As Taisbak points out, some will be tempted to see the fact that $AC : BC = \sin\angle ABC$ as an indication that this theorem concerns the fundamental relation of ancient trigonometry.[9] Again, however, the proof shows us how far Euclid's text is from any metrical considerations. What it means for $\triangle ABC$ to be *given in form* is the fact that we can construct a similar triangle on an arbitrary, or given, side. With DE given, we set out Z in a given ratio to it. Although this step is glossed over in the presentation, it must be done through *Elem.* VI 12 which itself involves the construction of a similar triangle. The given length Z is then used to construct a triangle that is given in both form and magnitude. What it means for a ratio to be given is that if we start with an arbitrary length we can use *Elem.* VI 12 to find a given forth proportional. This theorem provides us no way of dealing with the given objects as numeric values. Even if we are given the ratio as a value or as a relation between two values the theorem provides no way to move from this value to the value of the triangle's angles and this transition is precisely what a true trigonometry demands.

In both of these examples the meaning of a triangle *given in form* is rooted in geometric construction, the principal mathematical operation of the pure geometric tradition. In the applied mathematics of the astronomers, however, a triangle which is *given in form* comes to mean one which we can use in a series of calculations to move between values for lengths and values for angles. The passage in Theon of Smyrna, mentioned above, provides one of the earliest surviving examples of this latter conception.

In going through Adrastus' discussion of the eccentric and epicyclic solar models, Theon demonstrates that, in the eccentric model, the solar orbit is given both in position and magnitude. He does this by taking the degree position of the solar apogee and a ratio related to the solar distance as given. The solar apogee is taken to be $5^1/_2{}^\circ$♊. Theon does not say how this value is derived but it is the same as that in the solar model attributed to Hipparchus by Ptolemy.[10] The ratio of the distance from the earth to the center of the sun's orbit compared to the radius of the sun's orbit is taken to be 1 : 24 and was apparently found "through the treatise concerning sizes and distances."[11] In Ptolemy's

[9] Taisbak [2003, 128].

[10] Toomer [1984, 155]. Ptolemy's presentation of Hipparchus' solar model is discussed below, see page 94.

[11] διὰ τῆς περὶ ἀποστημάτων καὶ μεγεθῶν πραγματείας, Hiller [1878, 158] and Dupuis [1892, 256].

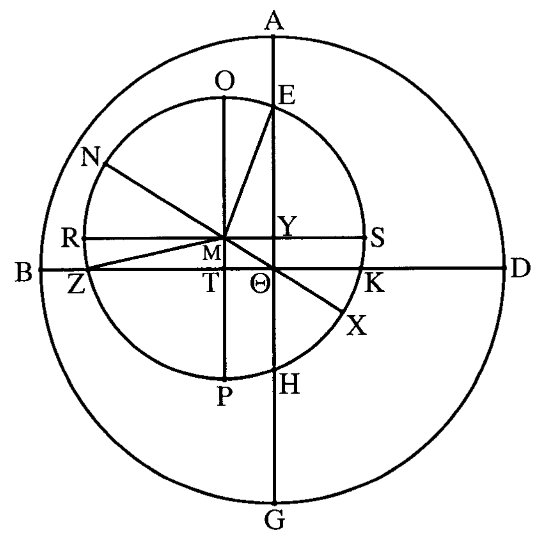

Figure 2.2: Diagram for Theon of Smyrna's figure for the eccentric model of the solar orbit. Hiller [1878, 157 - 158].

account of Hipparchus' procedure these two values are derived through trigonometric computation on the basis of observations of season lengths. Theon's presentation inverts this procedure so that we use the derived parameters of the model to determine the position and size of the solar orbit. In fact, Theon also includes numbers relating to season length, the sums of length of spring plus summer and of autumn plus winter. These numbers are not sufficient, however, to determine the parameters which Hipparchus derived.[12] Since the passage is not long and there is no careful English translation, I will quote it in full.[13] See Figure 2.2. I have numbered the sentences for later reference.[14]

> [1] The circle $EZHK$ is found, given in position and magnitude. [2] For let the perpendiculars OP and RS have been drawn through M parallel to AG and BD [*Elem.* I 31], and let ZM and ME have been joined. [3] Clearly now, since the circle $EZHK$ is divided into 365¼ days, the arc EZH will be 187 days, while [arc]

[12] According to Ptolemy, Hipparchus began with the lengths of the spring and the summer, Toomer [1984, 153].

[13] The standard text is Hiller [1878] to which Dupuis [1892] adds a few corrections and a French translation. Lawlor and Lawlor [1976] is an English translation. The Lawlors, however, often give a closer rendition of Dupuis' French than Theon's Greek.

[14] Hiller [1878, 157 - 158], Dupuis [1892, 254 - 256] and Lawlor and Lawlor [1976, 102].

> HKE is 178¼ days. [4] Then, each of the pairs EO, PH and RZ, SK are equal [by construction], while the arcs SP, PR, RO, and OS are equal to $90\frac{1}{4} + \frac{1}{16}$ [days] [$365\frac{1}{4}^{d} \div 4$]. [5] Then, the given $\angle OMN$ is equal to [$\angle$]ΘMT [*Elem.* I 15]; likewise, $\angle RMN$ is equal to [$\angle$]$YM\Theta$ [*Elem.* I 15]. [6] Then, the ratio $MT : M\Theta$, or rather, $MT : T\Theta$ is given. [7] Then, $\triangle MT\Theta$ is given in form [*Data* 40]. [8] And [the distance from] Θ, the center of the cosmos, to each of the points N and X is given, for one determines the greatest distance, the other determines the least; and ΘM is between the center of the cosmos and [the center] of the solar circle. [9] Then, the circle $EZHK$ is given in position and magnitude, since it is found through the treatise concerning sizes and distances that the ratio $\Theta M : MN \approx 1 : 24$.

This passage is something of a mess from both a mathematical and philological perspective. One of the things we notice first is the vagueness of the logical inferences when compared with other mathematical texts. It appears to have been written by someone ignorant of the common mathematical idiom. While each of the statements is valid it is not always clear from Theon's presentation why this is so.[15] It will be helpful to follow through the arguments in detail.

Theon begins in [2] by assuming the center of the sun's eccentric orbit is located at M and drawing lines through this point parallel to the lines joining the earth with the cardinal points of the ecliptic. In [3], he divides a solar year-length of $365\frac{1}{4}^{d}$ into two parts such that $\overset{\frown}{EZH} = 187^{d}$ and $\overset{\frown}{HKE} = 178\frac{1}{4}^{d}$. Theon seems to imply that these latter numbers follow as a matter of course from the year length. In fact, they come from season lengths that Hipparchus claims to have observed. According to Ptolemy, Hipparchus derived the parameters for his solar model, $\Theta M : MN$ and $\angle OMN$, under the assumption that the interval from the spring equinox to the summer solstice is $94\frac{1}{2}^{d}$

[15] In particular, the particle ἄρα, which I have translated as *then*, has a looser meaning than we find in general mathematical usage. In the mathematical corpus ἄρα is usually reserved for inferences that can be drawn directly from the previous statement and is best translated by *therefore*, Mugler [1959, 82 - 83]. For Theon, it introduces nearly every statement and rarely indicates logical dependence on the previous statement. Another example is Theon's use of τουτέστι which I have translated with *or rather* in [6]. This expression is usually translated as *that is* and indicates a strict equality. Theon, however, uses it to indicate a given ratio which is not equal but which is given for the same reason.

while that from the summer solstice to the autumnal equinox is $92^1/_2{}^d$.[16] In fact, the numbers given by Theon are insufficient for the determination of the model and it is not clear in Theon's exposition how they relate to the rest of the argument. Theon next states, in [5], that $\angle OMN$ is given. This is the case because, in the previous discussion, Theon has remarked that the solar apogee is $5^1/_2{}^\circ$♊, so that $\angle A\Theta N = \angle OMN = 65^1/_2{}^\circ$. Hence, all three angles of $\triangle MT\Theta$ are given in degrees. Theon then claims, in [6], that $MT : M\Theta$ and $MT : T\Theta$ are given. This would follow as a result of *Data* 40 and the definition of rectilinear figure *given in form*, however, Theon will next state on the bases of [6] that $\triangle MT\Theta$ is *given in form*.[17] But we saw above that *Data* 40 provides no way of treating the ratios of a triangle *given in form* other than by laying out a set of line segments. What is required here is some method, presumably by means of a chord table, of using the values of the angles in $\triangle MT\Theta$ to derive the ratios of the sides as values or relations between values. When Theon states, in [7], that $\triangle MT\Theta$ is *given in form* he means the same thing that Ptolemy would mean if he used that expression; its angles and sides have determinate numerical values for the purposes of calculation. The rest of the passage is muddled but the sense is clear. Sentence [8] asserts that $N\Theta$ and $X\Theta$ are given because, as [9] states, $M\Theta : MN = 1 : 24$. The circle $EZHK$ is then given in relation to Θ by the fact that the two ratios $MT : \Theta$ and $M\Theta : MN$ are both given. Since the fact that the ratio $M\Theta : MN$ is given means that we can express it as a relation of two values, we should understand the fact that $MT : T\Theta$ is given in the same way.

The exposition of Theon, or rather Adrastus, is not as intelligible as what we have come to expect based on our knowledge of Ptolemy and his commentators. It is either (1) a loose summary of a proposition in a work by Hipparchus, or more likely (2) an argument, based on Hipparchus' derivation of the parameters of the eccentric solar model, that the parameters imply that the model is completely determined.

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[16] Toomer [1984, 153 - 155]. Ptolemy uses a more precise year length with which he claims Hipparchus agrees in the two works *On the Length of the Year* and *On Intercalary Months and Days*, Toomer [1984, 139]. This more precise year length, however, has no bearing on the derivation of the parameters of his solar model.

[17] See page 61 for *Data* 40 and page 59 for *Data* def. 3.

It is clear that between the time of Euclid and Hipparchus the notion of *given* had changed. For Euclid *given* means determined through the techniques of geometrical construction. For Hipparchus *given* means expressible as a numeric value. A triangle which is *given in form* comes to mean one that can be used to transform between values for angles and values for lengths. This transformation was done using some trigonometric function, presumable a chord table.

Before we turn to our evidence for the construction of chord tables, it will be useful to examine the practices of mathematical astronomers before the development of tables. Well before the time of Hipparchus, applied mathematicians realized that the key to using triangles to convert between angle measures and line lengths lay in the regulating properties of the right triangle.

2.1.2 Trigonometric calculation in Hellenistic astronomy

By looking at two texts from the Hellenistic era we can develop a sense for how Greek mathematicians approached trigonometric problems before the development of chord tables. The two texts that are relevant in this regard are Aristarchus' *On the Sizes and Distances of the Sun and the Moon* and Archimedes' *Sand Reckoner.* Because Archimedes offers nothing new with respect to trigonometric methods it will be sufficient to examine two propositions from *On the Sizes.*

Both these mathematicians used a series of lemmas to produce ratio inequalities that relate angle ratios to side ratios in right triangles. Knorr has collected and studied all of the variant proofs of two of these lemmas that are extant in the Greek mathematical corpus and speculated on the existence and provenance of a source text for the lemmas.[18] For our purposes, speculation on the origin of the lemmas is less important than seeing how they were used in practice.

The first two lemmas are given a general enunciation by Archimedes. In the *Sand Reckoner*, in the course of a proof that the apparent diameter of the sun is greater than

[18] Knorr [1985a].

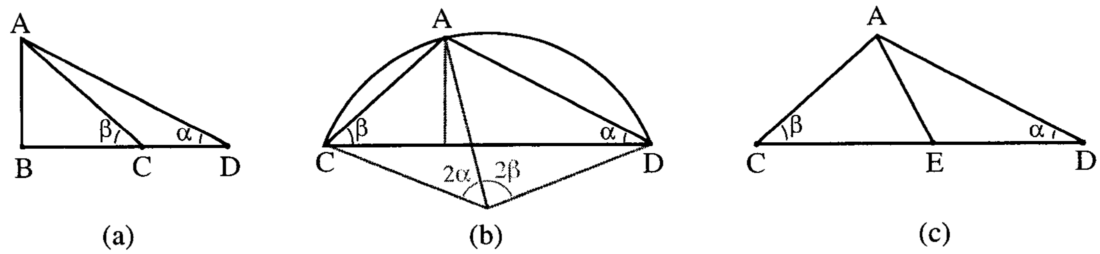

Figure 2.3: Diagrams for the trigonometric lemmas used by the Hellenistic astronomers.

the side of a regular 100-gon inscribed in a celestial great circle, Archimedes asserts a pair of ratio inequalities relating angles and sides in right triangles under the same height.[19] Consider Figure 2.3 (a). He states, in effect, that if $BD > BC$, then

$$\beta : \alpha < BD : BC, \qquad \text{(T.L. 1)}$$

and

$$\beta : \alpha > AD : AC. \qquad \text{(T.L. 2)}$$

The earliest text we have which demonstrates the first lemma is Euclid's *Optics*.[20] The second lemma is first proved, in a trivial variant, T.L. 2a, by Ptolemy in his treatment of the chord table, *Alm.* I 10.11. Consider Figure2.3 (b). Ptolemy shows that if $AD > AC$, then[21]

$$\overset{\frown}{AD} : \overset{\frown}{AC} > AD : AC. \qquad \text{(T.L. 2a)}$$

A third trigonometric lemma, T.L. 3, is used by Aristarchus in a passage we will examine below. Consider Figure 2.3 (a). The lemma Aristarchus requires amounts to

[19] Heiberg [1973, vol. 2, 232].

[20] Heiberg [1895, 164 - 166]. When Knorr studied these lemmas he still believed that the version attributed by Heiberg to Euclid was the earlier text, Knorr [1985a, 370]. Following Jones, however, he later came to view the version attributed by Heiberg to Theon of Alexandria as earlier, Knorr [1991, 195, n. 7]. Jones [1994] and Knorr [1994] present the case for this position.

[21] $\overset{\frown}{AD} : \overset{\frown}{AC} = \beta : \alpha$ so that T.L. 2a is an immediate consequence of the T.L. 2. Ptolemy, however, does not base his proof of this lemma on a previous proof of T.L. 2., Toomer [1984, 54 - 55]. See page 90 for a discussion of Ptolemy's proof.

the statement that if $BD > BC$, then

$$BD : BC > 90° - \alpha : 90° - \beta. \qquad \text{(T.L. 3)}$$

I am not aware of any ancient proof of this lemma. Heath gives a proof based on the ancient demonstrations of the other lemmas.[22]

A final lemma, attributed to Apollonius by Ptolemy in *Alm.* XII 1, should perhaps be included among these.[23] There is no evidence that it was ever used in trigonometric calculation, however, its subject matter and proof structure indicate that it belongs in the same tradition. Consider figure 2.3 (c). In the course of his work on planetary retrogradation, Apollonius showed that if $CD > CA$ and $CE \geq AC$, then $CE : DE > \alpha : \beta$.

In order to understand how the mathematical astronomers employed these lemmas to derive numerical approximations, it will be necessary to see the lemmas in use. To this end, we will look at Aristarchus' *On Sizes* 4 & 7.[24] Aristarchus' *On the Sizes and Distances of the Sun and the Moon* is a systematic mathematical text that deduces eighteen propositions from six hypothesis.[25] The hypotheses are as follows:[26]

1. That the moon receives its light from the sun.

2. That the earth has the ratio of a point and a center to the sphere of the moon.

3. That, when the moon appears to us halved, the circle which divides the dark and the bright portions of the moon inclines toward our eye.[27]

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22 Heath [1913, 377, n. 1].
23 Toomer [1984, 558 - 559].
24 Neugebauer [1975, 773 - 775] discusses the trigonometric aspect of these two theorems in modern notation.
25 There is also an implicit assumption that is later proved as *On Sizes* 8, see n. 30 and Neugebauer [1975, 635].
26 Heath [1913, 352 - 353]. I have made a few slight changes.
27 That is, the circle lies in the same plane as our eye. In the Greek text, this hypothesis is ill formed. Despite being supported by the MSS and Pappus, the adjective μέγιστον, "greatest," modifying τὸν...κύκλον should be excised. *On Sizes* 2 proves that this circle is *not* a great circle. Moreover, *On Sizes* 5, the only proposition that uses Hypothesis 3, cites it as referring to "the circle" not "the great circle." I have omitted the adjective from the English translation.

4. That, when the moon appears to us halved, its distance from the sun is then less than a quadrant by $1/13$ of a quadrant.

5. That the breadth of the shadow [of the earth] is [that] of two moons.

6. That the moon subtends $1/15$ of a sign of the zodiac.

Some of the hypotheses are physical while some seem to be grounded in crude observations. Neugebauer has shown, however, that none of the hypotheses can have had any basis in observation and he argues that they were chosen as handy numerical parameters that would lead to nice numerical results.[28] It is also possible that Aristarchus deliberately chose numbers that would lead to fairly modest lower bounds, thus underlining the magnitude of the sizes and distances of the luminaries.

In *On Sizes* 4, Aristarchus demonstrates that the circle which divides the light and the dark sides of the moon is not perceptibly different from a great circle. Because the sun is much larger than the moon, it is shown, in *On Sizes* 2, that the light side of the moon is greater than a hemisphere.[29] In *On Sizes* 3, Aristarchus proves that maximum difference between the dividing circle and a great circle occurs when the cone which contains both the sun and the moon has its vertex at "our eye." [30] Hence, in *On Sizes* 4, we only need a figure containing our eye and the moon.

Let our eye be A and the center of the moon B, see figure 2.4.[31] Let a plane have been drawn through A and B cutting the moon in the great circle $ECDF$ and the cone in lines AC, AD and DC. Then the circle about DC, which is perpendicular to AB, is the circle which divides the illuminated and the dark parts of the moon [*On Sizes* 3]. Let FE have been drawn parallel to DC, and let $\widehat{HG} = \widehat{GK} = 1/2\, \widehat{FD}$ have been set out. Let KB, BH, KA, AH and BD have been drawn. Since, by the Hypothesis 6, the moon subtends $1/15$

[28] Neugebauer [1975, 642 - 643].

[29] Heath [1913, 358 - 360].

[30] Heath [1913, 630 - 364]. The implicit assumption is that, since the moon and the sun appear to be the same size, they subtend the same angle and hence are internally tangent to the same cone. In *On Sizes* 8, Aristarchus proves that this is the case, Heath [1913, 382].

[31] The text and translation of *On Sizes* 4 is found in Heath [1913, 364 - 371].

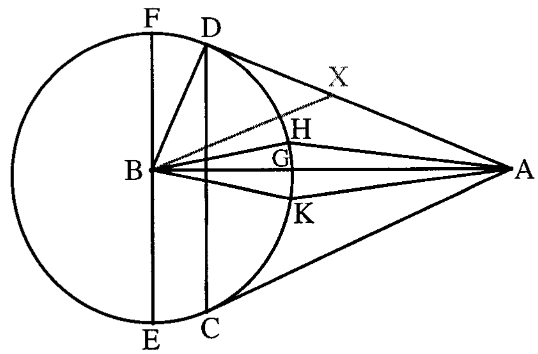


Figure 2.4: Diagram for Aristarchus' *On Sizes* 4, Heath [1913, 364]. Gray elements do not appear in Aristarchus' diagram.

of a zodiacal sign, therefore $\angle CAD = {}^1\!/_{15}Z_{sign}$ [$= 2°$].[32] But ${}^1\!/_{15}Z_{sign} = {}^1\!/_{180}C$, therefore $\angle CAD = {}^1\!/_{180}\,4R = {}^1\!/_{45}R$. And $\angle BAD = {}^1\!/_2\angle CAD$, therefore $\angle BAD = {}^1\!/_{45}\,{}^1\!/_2R$ [$= 1°$]. Now since $\angle ADB$ is right, $\angle BAD : {}^1\!/_2R > BD : DA$ [T.L. 1].

This is Aristarchus' first use of the trigonometric lemmas and it is somewhat obscure. It is a consequence of T.L. 1. Cut off $DX = DB$. Then since, $DA > DX$ [T.L. 1], $DA : DX > \angle DXB : \angle DAB$. But $DX = BD$ and $\angle DXB = {}^1\!/_2R$, therefore, *by inversion*, $BD : DA < \angle BAD : {}^1\!/_2R$ [*Elem.* V 16 applied to ratio inequalities; the inequality changes effect].

Hence, $BD < {}^1\!/_{45}\,DA$. So that $BG \ll {}^1\!/_{45}\,BA$ [$BG = BD$ and $DA < BA$], and *by separation*, $BG < {}^1\!/_{44}\,GA$.

As noted in the previous chapter, these operations are not justified by theorems in the theoretical texts. Nevertheless, they represent part of the practical tool box. The first inequality follows fairly obviously from Euclid's definition of greater ratio, *Elem.* V def. 7.[33] We have discussed the operation of *separation* above with respect to proportions

[32] Aristarchus uses three units of angular measure: the circle, the right angle and a zodiacal sign. I denote these by C, R and Z_{sign}. An estimate of 2° for the angular size of the moon is large. In this proposition, however, an overestimate for the apparent size of the moon only underscores the results of the proof. It should be noted that Aristarchus is reported to have had another figure for the apparent size of the sun which is much closer to our current value. In the *Sand Reckoner*, Archimedes reports that Aristarchus found that the apparent diameter of the sun was ${}^1\!/_{720}$ of a circle, that is ${}^1\!/_2°$, Heiberg [1973, vol.2, 248].

[33] Heiberg [1977, vol. 2, 2].

and ratio inequalities.[34] Here, however, the operation is being applied to an inequality. Or perhaps, the inequality, because it involves a fractional part of one of the terms, is being treated as a ratio inequality. That is, since $BG < 1/45\ BA$, we can assert $BA : BG > 45 : 1$, so that, *by separation*, $GA : BG > 44 : 1$, from which we can claim $BG < 1/44\ GA$. Whether Aristarchus thought of this operation as performed on the ratio inequality or directly on the inequality is immaterial. What is important is that Aristarchus worked in a tradition that saw no practical distinction between ratio inequalities and inequalities. Fowler's comments on Aristarchus' *On the Sizes* are useful but he is incorrect in his claim that the technique of manipulating ratio inequalities is completely distinct from that of manipulating parts, or fractions.[35] In fact, it appears that the early Hellenistic geometers transformed expressions involving parts, or fractions, into ratios in order to manipulate them and tried, where possible, to reduce them to a form expressible by unit fractions before transforming them back into expressions involving fractions. As Fowler points out, the manipulations that they do perform directly on fractions are not those that are familiar to us in the algebraic idiom, such as addition, multiplication and so forth. Indeed, as we see here, they are formed by analogy from the standard operations on ratios such as separation.

Since $BG < 1/44\ GA$, therefore $BH \ll 1/44\ AH$ [$BH = BG$ and $AH > AG$]. But $BH : AH > \angle BAH : \angle ABH$ [T.L. 2], therefore $\angle BAH < 1/44 \angle ABH$; while $\angle KAH = 2\angle BAH$ and $\angle KBH = 2\angle ABH$, therefore $\angle KAH < 1/44 \angle KBH$. But $\angle KBH = \angle DBF = \angle CDB = \angle BAD$ [by construction, *Elem.* I 29, *Elem.* VI 8], therefore $\angle KAH < 1/44 \angle BAD$. But $\angle BAD = 1/45\ 1/2\mathrm{R}$; hence, $\angle KAH < 1/3960\mathrm{R}$ [$= 1/44°$, $1/2 \times 1/45 \times 1/44 = 1/3960$]. But a magnitude seen under such an angle is imperceptible to our eye.[36] And $\overset{\frown}{KH} = \overset{\frown}{DF}$. Moreover, $\overset{\frown}{KH}$ is viewed dead on, whereas $\overset{\frown}{DF}$ will generally

[34] See Section 1.3.

[35] Fowler [1987, 246 - 248]. In particular, the statement, "when the language of ratios is in use, it is not mixed with the language of multiples or parts, even in the most obvious cases," is misleading, Fowler [1987, 247]. As we have seen, both the language and the operations of ratios are transferred to inequalities of multiples or parts.

[36] The angular span would be $\approx 0.0227°$.

be viewed obliquely and, hence, appear smaller still.[37]

Trigonometric calculation shows that $\angle KAH \approx 0.0178°$, whereas Aristarchus gives an upper bound of $\angle KAH < 0.0227°$, expressed as $1/3960$R. Ostensibly, the only piece of numerical information that Aristarchus uses in this derivation is the assumed angular size of the moon, $\angle DAC = 2°$. In fact, however, his use of T.L. 1 is also significant. Because of the need to derive a ratio inequality involving BD, Aristarchus is compelled to introduce a 45° angle. This angle then furnishes the primary numerical element in the comparison of $\angle KAH$ and $\angle BAD$. An angle closer to 1° would have given a better approximation but would not have produced a relation involving BD. Since Aristarchus thinks his upper bound is sufficiently small, a better approximation is unnecessary.

In order to derive quantitative information involving $\angle KAH$ from the fact that $\angle BAD = 1°$, Aristarchus has to transform a quantitative relation involving $\angle BAD$ and another given angle into a relation involving sides. He manipulates this relation involving sides from one triangle to another and then transforms it back into a relation involving angles. Both of these transformations are made with the trigonometric lemmas. Hence, the trigonometric lemmas serve a similar function in Aristarchus' trigonometry as does the chord table in later trigonometry; they allow the geometer to transform between statements relating angles and statements relating sides. The trigonometric lemmas, however, are ratio inequalities; hence, each time one is used some precision is lost.

In *On Sizes* 7, Aristarchus demonstrates that the distance of the sun from the earth, D_s, is greater than 18 times, and less than 20 times, the distance of the moon from the earth, D_m; that is $18D_m < D_s < 20D_m$. For the purpose of this proof, the sun is assumed to lie on a circle centered on the earth and the moon to be half illuminated. The three

[37] Neugebauer considers the final part of *On Sizes* to be "slightly garbled," Neugebauer [1975, 639 - 640]. He finds it strange that $\overset{\frown}{DF}$ has been not been laid off to one side of BA as it would appear at half moon, Neugebauer [1975, 640]. Aristarchus, however, probably chooses this arrangement because he wants to show that $\overset{\frown}{DF}$ is still "imperceptible" when it is seen straight on, under its greatest possible angular span. The argument Aristarchus gives about $\overset{\frown}{DF}$ appearing under a smaller angle from A than $\overset{\frown}{HK}$ is rather odd because, as Neugebauer points out, $\overset{\frown}{DF}$ cannot be seen at all from A. The intent, however, is clear and my loose summary of the conclusion captures the general sense of the argument.

bodies are treated as points. Each inequality is proven separately.

Let A be the center of the sun, B the center of the earth, and C the center of the half moon, see figure 2.5.[38] Let a plane have been drawn through AB and C, "and let it have made a section, the great circle ADE, in the sphere on which the center of the sun moves."[39] Let AC and CB have been joined and let BC have been produced to D. Obviously, since C is the center of the half moon, $\angle ACB = \mathrm{R}$. Let BE have been drawn $\perp$ BA [*Elem.* I 11]. Then, by Hypothesis 4, $\overset{\frown}{ED} = {}^1\!/_{30}\ \overset{\frown}{EDA}$, so that $\angle EBD = {}^1\!/_{30}\mathrm{R}$. Let the parallelogram AE have been completed [*Elem.* I 31], and let BF have been joined. Then $\angle FBE = {}^1\!/_2\mathrm{R}$. Let $\angle FBE$ have been bisected by BG so that $\angle GBE = {}^1\!/_4\mathrm{R}$. Therefore, $\angle GBE : \angle EBD = 15 : 2$ [$= {}^1\!/_4\mathrm{R} : {}^1\!/_{30}\mathrm{R}$], taking the unit of angular measure as ${}^1\!/_{60}\mathrm{R}$. And since, $GE : EH > \angle GBE : \angle HBE$ [T.L. 3], therefore $GE : EH > 15 : 2$. But $FB^2 = 2BE^2$ [*Elem.* I 47] and $FB^2 : BE^2 = FG^2 : GE^2$ [*Elem.* VI 3], therefore $FG^2 = 2GE^2$. But $49 < 2{\times}25$, so that $FG^2 : GE^2 > 49 : 25$.[40] Therefore, $FG : GE > 7 : 5$. Therefore, *by composition*, $FE : GE > 12 : 5 = 36 : 15$ [*Elem.* V 18 applied to ratio inequalities]. But it was shown that $GE : EH > 15 : 2$, therefore *through equality*, $FE : EH > 36 : 2 = 18 : 1$ [*Elem.* V 22 applied to ratio inequalities]. Hence, $FE > 18EH$, but $FE = BE$, therefore $BE > 18EH$ and $BH \gg 18EH$. But $BH : HE = AB : BC$, since $\triangle ABC \sim \triangle BHE$, therefore $AB > 18BC$. But AB is the distance from the earth to the sun while CB is the distance from the earth to the moon, that is $\mathrm{D}_s > 18\mathrm{D}_m$.

This is a virtuoso piece of Greek mathematics. It uses a geometric construction and a straightforward fact of arithmetic to introduce numeric values into a purely geometric problem. It then combines the third trigonometric lemma and an admirable use of ratio manipulations to produce the final inequality. In the first case, $\angle GBE = 22{}^1\!/_2°$ is constructed greater than $\angle DBE = 3°$. This approximation is not very close, but as we

[38] The text and translation of *On Sizes* 7 is found in Heath [1913, 376 - 381].

[39] ...καὶ ποιείτω τομὴν ἐν τῇ σφαίρᾳ, καθ' ἧ φέρεται τὸ κέντρον τοῦ ἡλίου, μέγιστον κύκλον τὸν ΑΔΕ, Heath [1913, 376]. Aristarchus appears to be working within the context of Eudoxus' theory of homocentric spheres. See Mendell [1998] and Mendell [2000] for recent discussions of Eudoxus' model.

[40] Aristarchus probably derives these numbers from the approximation $\sqrt{2} \approx {}^7\!/_5$, Heath [1913, 379, n. 1]. Aristarchus' statement is equivalent to $\sqrt{2} > {}^7\!/_5$.

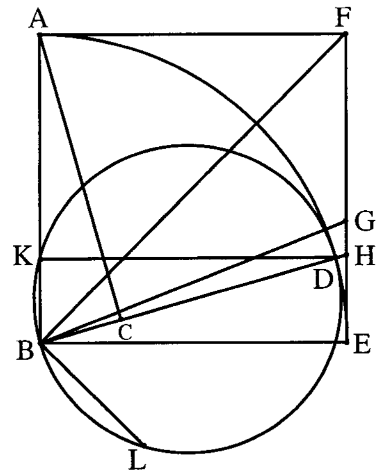


Figure 2.5: Diagram for Aristarchus' *On Sizes* 7, Heath [1913, 378]. Aristarchus' text includes the whole great circle of the solar sphere, ADE. I show only a quadrant of this circle; the rest is unnecessary for the proof.

shall see below, it is close enough. In the second case, the fact that $49 < 2 \times 25$ is used to approximate the relation $FG^2 = 2GE^2$. Here, the ratio of the numeric boundary values is much closer to that of the geometric objects. Because of the geometry of the situation, the two approximations are interdependent. In fact, if Aristarchus had continued his use of the halving procedure to introduce a closer fit on $\angle DBE$, it would have made the second approximation either less precise or much more involved.

Now to show that $D_s < 20D_m$. Let $DK \parallel EB$ have been drawn [*Elem.* I 31], and let circle DKB have been drawn around $\triangle DKB$ [*Elem.* IV 5], with DB as its diameter, since $\angle DKB$ is right [conv. *Elem.* III 31]. Let BL, the side of the hexagon, be fit into circle DKB [*Elem.* IV 1]. Then, since $\angle DBE = \angle BDK = 1/30$R, $\overset{\frown}{BK} = 1/60$C [*Elem.* III 20]. But $\overset{\frown}{BL} = 1/6$C, therefore $\overset{\frown}{BL} = 10\ \overset{\frown}{BK}$. And since $\overset{\frown}{BL} : \overset{\frown}{BK} > BL : BK$ [T.L. 2a], therefore $BL < 10BK$. And $BD = 2BL$ [*Elem.* IV 15 cor.], so that $BD < 20BK$. But $BD : BK = AB : BC$ [$\triangle BKD \sim \triangle ABC$], so that $AB < 20BC$, that is $D_s < 20D_m$.

In this proposition, the most significant value is $\angle DBE = 3°$, introduced by Hypothesis 4. Because the proposition concerns line segments while the given value is that of an angle, the trigonometric lemmas are used to transform the given angle relation into a side relation. In both cases, a geometric construction is used to introduce another given value into the ratio inequality. In the first part of the proof, $\angle GBE = 22^1/2°$ is constructed greater than 3°; in the second part, $\widehat{BL} = 60°$ is constructed greater than 6°. If Aristarchus had constructed given angles closer to the angles given by hypothesis he could have produced a tighter squeeze on D_s. On the other hand, he appears to have preferred whole numbers. When one sees that $18D_m < D_s < 20D_m$, one is inclined to suppose that $D_s \approx 19D_m$. Since, in fact, $D_s \approx 19.11D_m$, Aristarchus' expression describes the situation fairly well.

Although I have referenced many of the steps in these two proofs to propositions in Euclid's *Elements*, we should not think that Aristarchus himself thought of these steps as supported by specific theorems. The *Elements* was composed either during or shortly before Aristarchus' time.[41] In all likelihood, Aristarchus' tool box is a loosely defined body of geometric knowledge and an understanding of how proportions and ratio inequalities can be manipulated. Aristarchus also assumes the reader has the mathematical background to follow steps that have no justification in any theoretical text that we possess. In particular, the trigonometric lemmas and the basic operations on inequalities and ratio inequalities are the elementary mathematics that a reader of his text can be expected to know. Proportions and ratio inequalities can also be transformed into equalities and inequalities but this almost only occurs when the resulting expression is a simple multiple or a unit fraction.[42] These texts provide us with a tool box of theorems and manipulations that we do not find justified in the systematic treatises. Speculation on the existence and provenance of texts that proved these results is not as important as the realization that these tools formed part of the knowledge base of practicing mathematicians.

[41] Aristarchus is dated by a reported observation made in 280 BCE and the references made to his past astronomical work in Archimedes' *Sand Reckoner*, [ToomerToomer1984, 139] and Heiberg [1973, vol. 2, 218 - 220]. Concerning Euclid, the most famous mathematician in history, we know next to nothing. One of the only useful pieces of information we have is a passage in Pappus about Apollonius which states that, "he studied in Alexandria with the [people] ⟨under⟩ Euclid," σχολάσας τοῖς ⟨ὑπὸ⟩ Εὐκλείδου μαθηταῖς ἐν 'Αλεξανδρείαι, Jones [1986, 121]. All this tells us, if it is true, is that Euclid was at least a generation or two prior to Apollonius. He could well have been contemporary with Aristarchus.

[42] One exception to this rule is found in Archimedes' *Sand Reckoner*, in which he infers from $^1/_{200}R : \alpha < 100 : 99$ that $\alpha > {}^{99}/_{20,000}R$, Heiberg [1973, vol. 2, 232].

Trigonometry

Two other theorems in *On Sizes*, 11 & 12, along with a theorem in the *Sand Reckoner*, fill out the rest of our evidence for trigonometric calculation before the development of chord tables.[43] With regard to trigonometric procedures, these three theorems offer nothing we have not already seen. Again, we encounter the same basic mathematical tools: use of the trigonometric theorems to transform between given angle and side relations, manipulations of proportions and ratio inequalities, and the transformation of ratio inequalities into inequalities. We find that angles are expressed in various units whereas sides, being given no units, are simply compared.

All of this work in Hellenistic trigonometry takes place in the context of mathematical astronomy, as does the later work with metrical analysis. An important difference should, however, be noted. Whereas Hipparchus, Diodorus and Ptolemy use mathematics to serve the needs of an astronomy that is, at least ostensibly, based on careful observation and geared toward practical as well as theoretical concerns, Aristarchus and Archimedes use astronomical problems as a domain in which to demonstrate their considerable mathematical abilities.

In both Aristarchus and Archimedes, we find elaborate mathematical proofs of statements that are either obvious or irrelevant from an observational or practical perspective. *On Sizes* 4 proves that the circle which divides the light and the dark sides of the moon appears to be a great circle. An empirically minded astronomer might choose to base knowledge of this sort on observations of the moon at quadratures. In the *Sand Reckoner*, Archimedes shows at great length that the lower bound of the apparent size of the sun is not significantly affected by daily parallax.[44] This tells us little more than what we already knew from observation and the precision achieved by this mathematical treatment is at odds with the crude estimates assumed in setting up the problem.[45]

Neugebauer has pointed out the striking similarity between Aristarchus' and Archimedes'

[43] Heath [1913, 386 - 391] and Heiberg [1973, 226 - 232].

[44] Dijksterhuis [1987, 366 - 369, esp. 396, n. 1] and Neugebauer [1975, 644 - 646]. The theorem is in Heiberg [1973, 226 - 232].

[45] Neugebauer [1975, 644].

texts.[46] They are meant to display both the mathematical skills of their authors and the power of mathematics generally to analyze complex problems with great precision. Precision, rather than accuracy, is their goal. From our perspective, looking back after the development of trigonometric functions, the methods of the Hellenistic mathematicians appear to lack elegance. This should not obscure the fact that these methods were, in fact, a significant advance for applied mathematics. By using the fundamental properties of the right triangle, mathematicians were able to make approximations between angle relations and side relations. These approximations could generally be made to fit the given value more or less tightly at the mathematician's discretion.

2.1.3 Given angles and chords

Once it was realized that right triangles could be used to convert between given angles and determinate side relations, two theorems of the *Data* must have become conspicuous. *Data* 87 and its converse, 88, show that there is a direct relationship between angles in given circles and the chords that subtend them. Moreover, the geometrical object used to demonstrate this correspondence is the right triangle. These two theorems secure the theoretical possibility of a chord table and they establish the relationship between chords and right triangles.

In *Data* 87, it is shown that if, in a given circle, a chord is drawn subtending a given angle then the chord will be given in magnitude. See figure 2.6. Let the chord AC have been drawn in the given circle ABC cutting off the segment AEC admitting a given angle. Let the center of the circle, D, have been taken [*Elem.* III 1], and let AD have been drawn through to E, and let CE have been joined. Then $\angle ACE$ is given because it is right [*Elem.* III 31], and $\angle AEC$ is also given [by hypothesis], therefore $\angle CAE$ is given [*Elem.* I 32], therefore $\triangle ACE$ is *given in form* [*Data* 40], therefore $AE : AC$ is given. And EA is given in magnitude because the circle is given in magnitude [*Data* def. 5],

[46] Neugebauer [1975, 643].

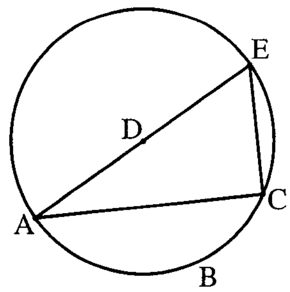


Figure 2.6: Diagrams for *Data* 87 & 88, Taisbak [2003, 226]. As is often the case, these converses share a single figure.

therefore AC is given in magnitude [*Data* 2].[47]

The converse is demonstrated in *Data* 88. If a given chord is drawn in a given circle it will subtend a given angle. See figure 2.6. Let the given chord AC have been drawn in the given circle ABC [*Elem.* IV 1], and let the figure have been completed as above. Since both EA and AC are given [by hypothesis], therefore $EA : AC$ is given [*Data* 1]. And $\angle ACE$ is right, therefore $\triangle ACE$ is *given in form* [*Data* 43], therefore $\angle AEC$ is also given.[48]

Taisbak has shown how these theorems are related to the chord table.[49] While the absolute magnitude of a chord depends on both the angle it subtends and the diameter of the circle it cuts, the ratio of the chord to the diameter depends on the angle alone. Each angle will determine a single ratio and the converse. This is made clear by the pivotal role of the right triangle in these theorems. This establishes the possibility of using the angle/chord relationship to perform the fundamental function of trigonometry, the conversion between given angles and given line segments.

These theorems, however, are still situated in the tradition of pure geometry. They depend essentially on the two theorems about triangles *given in form* discussed in Section 2.1.1. Hence, what it means for arcs and chords to be reciprocally given in this context

[47] Taisbak [2003, 225 - 226].
[48] Taisbak [2003, 226 - 227].
[49] Taisbak [2003, 226].

is for the one to be constructible from the other in a determinate fashion.[50] Within this tradition, because they concern absolute magnitudes, these theorems are strikingly trivial. The claim that they make about the possibility of using right triangles in trigonometric calculation, however, is significant. It is against this theoretical background that we should understand the development of chord tables.

The construction of chord tables should be seen originally, not as a theoretical development but as a technological one. Chord tables provide a tool for converting between values for angles and values for chords. As we shall see, however, with the introduction of a new tool came changes in methods and practices which themselves lead to reassessments of a theoretical nature.

2.1.4 Trigonometry by tables

Sometime between the composition of Archimedes' *Sand Reckoner* and the development of Hipparchus' solar model, Greek mathematical astronomers developed chord tables and the accompanying trigonometric techniques as a means of exploiting the known properties of right triangles to make determinations involving the values of angles and lengths. In the process of doing so, they tacitly expanded the meaning of the term *given*.

Although Ptolemy's is the only chord table that survives from antiquity, no one has doubted that his predecessors had already made use of tables. Theon, in his commentary on Ptolemy's treatment of the chord table, tells us that both Hipparchus and Menelaus wrote works on chords. The reported lengths of these texts, however, has always seemed incredible.[51] There are still open questions, however, with regard to the originator of the technique of trigonometry by tables and the form and construction of these early tables. Just as it is nearly impossible to imagine detailed kinematic modeling using the

[50] There are, in fact, two triangles that will produce the necessary object. The other triangle is reflected and has CD extended for its hypotenuse. Berggren has pointed out that *given in form* applies to both of these triangles, Taisbak [2003, 130, n. 97]. Since *given in form* applies to a finite set of triangles they are each determinate.

[51] Theon reports that Hipparchus composed a "treatise on chords," πραγματεία τῶν ἐν κύκλῳ εὐθειῶν, in twelve books, Menelaus in six, Rome [1943, 451].

trigonometric methods of Aristarchus and Archimedes, so it is difficult to conceive of the work attributed to Hipparchus, Diodorus and Menelaus without the use of such tables. The mathematician that remains most obscure in this transition is Apollonius. While it is known that Apollonius constructed geocentric kinematic models for the planetary orbits using either epicycles or eccentric circles, we have no examples of this work.[52] We simply do not know the level of detail that Apollonius undertook in his astronomic modeling.

There have been a couple of modern reconstructions of early Greek chord tables. Toomer gave an account of a table in steps of $7^1/2^\circ$ which he attributed to Hipparchus and derived from two parameters which Ptolemy associates with lunar models Hipparchus proposed.[53] Although this reconstruction worked well with the evidence we have of Hellenistic astronomy in Indian sources, Toomer later recanted the details of his reconstruction because he had used the wrong input values for one of his data sets.[54] Nevertheless, the basis of Toomer's reconstruction is a good fit for one of the two lunar models.

Another reconstruction was put forward by van der Waerden, based on R. Newton's discovery that Ptolemy's table for the declination of the ecliptic, *Alm.* I 15, shows slight systematic deviation from what we find if we compute directly from Ptolemy's chord table.[55] The chord table van der Waerden constructs has steps of 1° and, if we assume that its author made errors in various places in his calculations, it is able to reproduce *Alm.* I 15, for the first 90° λ with a fair degree of accuracy.[56] Van der Waerden does not explicitly calculate his table beyond 74° but, if we follow the instructions he gives, we encounter systematic deviation in this range that is equal to, or greater than, that found from Ptolemy's table.

The details of van der Waerden's reconstruction are motivated by mathematical concerns rather than ancient evidence, although he is able to derive his table using lemmas

[52] For our evidence on Apollonius' kinematic modeling see Toomer [1984, 555 - 559]. This evidence is discussed by Neugebauer [1975, 267 - 273].
[53] Toomer [1973].
[54] Toomer [1984, 215, n. 75].
[55] van der Waerden [1988] and Newton [1985, 52 - 61].
[56] van der Waerden [1988, 33].

he finds in the mathematical astronomy of Āryabhaṭa. Moreover, van der Waerden wants to attribute his reconstructed chord table to Apollonius. In fact, all that Newton's discovery suggests is that *Alm.* I 15 was not calculated with the chord table in the *Almagest* in the manner that Ptolemy's presentation suggests. It could have been calculated from a slightly different chord table or by an indirect procedure.[57] If it was calculated by a different table, this other table could have been previously calculated by any mathematical astronomer. There is no reason to assume that it goes all the way back to Apollonius.[58] Van der Waerden's claim that Hipparchus was not a good enough mathematician to create such a chord table rests on dubious evidence and recent work has shown that Hipparchus was quite capable of generating interesting mathematical results.[59]

Newton's discovery shows that there is slight systematic deviation between *Alm.* I 15 and the values we find if we calculate according to Ptolemy's instructions. The table that van der Waerden constructs, however, has no special ability to account for the pattern of deviation we find, and only comes closer to *Alm.* I 15 than Ptolemy's methods by assuming various errors on the part of the calculator. Van der Waerden's investigation shows, however, that the ancient chord table which produced *Alm.* I 15 must have been very precise and fairly accurate. See Appendix B for a more complete discussion of these tables.

It is clear that trigonometry by tables had not yet been introduced in the time of Archimedes but that by the time of Hipparchus it was in full use. While nothing rules out the possibility that these methods were introduced in the period between these two authors, nothing compels us to this conclusion. Moreover, ancient testimony agrees with

[57] Van Brummelen [1993, 97 - 101] suggests that *Alm.* II 15 may have been calculated from Ptolemy's chord table using an interpolation grid of 10° steps.

[58] It is worth pointing out that the table of $7^1/2°$ steps which Toomer attributes to Hipparchus is too crude to produce the table of declinations in the Almagest.

[59] Hipparchus' statement that it is worthy of the attention of mathematicians to show the reason that the same phenomena result from both the epicyclic and eccentric models as it is found in Theon of Smyrna does not imply that Hipparchus was unaware of, or unable to demonstrate, this equality of the hypotheses, Hiller [1878, 166]. Both Adrastus and Theon use this remark to introduce their own demonstrations of the equality of the models. In this, as in much else, they are most likely following Hipparchus. Recently, old evidence has shed new light on Hipparchus' contributions to combinatorics, Acerbi [2003b].

the general impression created by the technical literature. It seems most likely that trigonometry by tables was devised in the time of Hipparchus, if indeed it was not his own contrivance.

Ptolemy's chord table

Ptolemy introduces the material related to his chord table as a mathematical aside to the main course of his astronomical argument.[60] Strangely enough, the specific topic which he uses to motivate this digression, the determination of the obliquity of the ecliptic, ε, is one of the few topics in the *Almagest* that does not require the chord table. In *Alm.* I 12, Ptolemy sets out two observational methods for determining 2ε and claims to have devised and used one of these methods himself.[61] Both of these procedures use instruments that are designed to measure the angle directly so that no trigonometry is involved. In fact, the value Ptolemy adopts, $\varepsilon = 23;51,20$, is a traditional value which Eratosthenes and Hipparchus also used.[62] The construction of the chord table is not needed for *Alm.* I 12. Nevertheless, the chord table is one of the most important building blocks for the mathematics of the *Almagest*.

Ptolemy introduces his development of the chord table, in *Alm.* I 10.1, by setting out the conventions of his mathematical practice and explaining why he treats the material as he does. He states that he will divide the circle into 360° and express the chords in parts of which there are 120 in the diameter of the circle. He claims that he will use a small number of theorems to make a systematic and rapid (εὐμεθόδευτον καὶ ταχεῖαν) determination of the size of the chords.[63] The theorems are set out so that the reader may check the values of the table by means of the geometric method (διά τῆς ἐκ τῶν γραμμῶν μεθοδικῆς).[64] He will use the method of sixtieths (sexagesimal fractions) because of the inconvenience of the system of parts, or unit fractions, (διὰ τὸ δύσχρηστον τῶν

[60] A good introduction to this material is provided by Aaboe [1964, 101 - 126].
[61] Toomer [1984, 61 - 63].
[62] Toomer [1984, 63, n. 75] and Jones [2002].
[63] Heiberg [1916, p. 1, 31] and Toomer [1984, 48].
[64] Heiberg [1916, p. 1, 32].

μοριασμῶν); he will carry out his calculations to such precision as to produce a negligible difference from perception.[65]

This is probably as close as Ptolemy comes to addressing the topic of commensurability which would have been important to theoretical mathematicians. Indeed, Ptolemy's methods allow him to avoid the issue entirely. He has no concern with the fact that many of the square roots he takes or ratios he derives cannot be completely expressed. He simply takes these values with sufficient accuracy to solve the problem at hand. The chord table Ptolemy constructs is a computation device; it allows one to always derive a value with sufficient accuracy no matter what value one enters into the table.

The first mathematical proposition in the *Almagest* is the only straightforward problem in the whole work. Within the context of Greek mathematical practice, it is quite elegant. It relies on two remarkable theorems in *Elements* XIII. These are enunciated as follows.[66]

> *Elem.* XIII 9: If the side of the [equilateral] hexagon and that of the [equilateral] decagon inscribed in the same circle be added together, the whole straight line has been cut in extreme and mean ratio, and the greater segment is the side of the hexagon.

> *Elem.* XIII 10: If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the [equilateral] hexagon and that of the [equilateral] decagon inscribed in the same circle.

That is, $S_6 + S_{10} : S_6 = S_6 : S_{10}$, and ${S_5}^2 = {S_6}^2 + {S_{10}}^2$. *Alm.* I 10.2 sets out a construction of all three of these line segments in such a way that if S_6 is assumed as given the other two will be shown to be given as well. Moreover, in this context, *given* means *described by a determinate numerical value.* As is almost always the case, *Alm.* I 10.2 proceeds directly to the exposition, with no enunciation; see Figure 2.7 (a).

[65] See Fowler [1987, 221 - 279] for a discussion of the Greek system of parts.
[66] Heath [1926, 455 & 457].

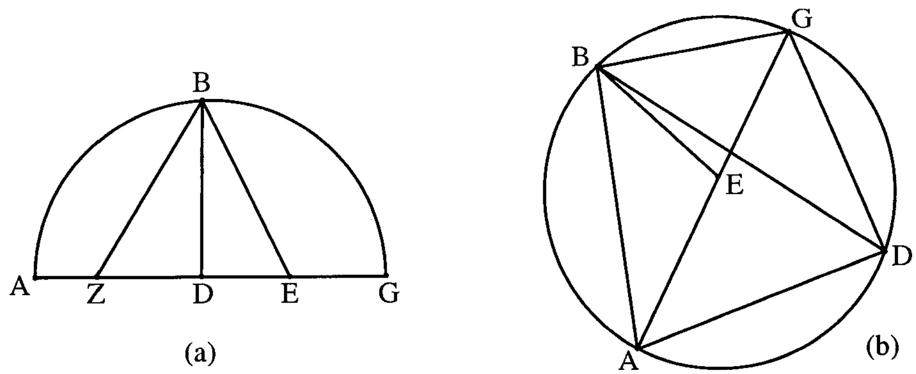


Figure 2.7: Diagrams for *Alm.* I 10.2 and *Alm.* I 10.5, Toomer [1984, 48 & 50].

Let ABG be a semicircle about center D and diameter AG, and let $BD \perp AG$. Let DG have been bisected at E, EB joined, and EZ set out $= EB$. Let ZB be joined. It remains to be shown that ZB is the side of the pentagon and ZD the side of the decagon. Since DG is bisected at E and ZD is added to it, $(GZ \times ZD) + ED^2 = EZ^2$ [*Elem.* II 6], that is $(GZ \times ZD) + ED^2 = EB^2$, since $EZ = EB$. But $EB^2 = ED^2 + DB^2$ [*Elem.* I 47], therefore $(GZ \times ZD) + ED^2 = ED^2 + DB^2$, so that, subtracting the common ED^2, $(GZ \times ZD) = DB^2 = DG^2$. Therefore, ZG has been cut in mean and extreme ratio by D and, since $DG = S_6$, $ZD = S_{10}$ [*Elem.* XIII 9]. Now since $ZD = S_{10}$ and $BD = S_6$, therefore, $BZ = S_5$ [*Elem.* XIII 10].[67]

In *Alm.* I 10.3, Ptolemy shows that, given S_6, this construction allows him to calculate values for S_5 and S_{10}. Simple geometrical considerations allow him to likewise calculate the values of the side of the square and equilateral triangle described in the same circle. *Alm.* I 10.4 shows that, in general, if the chord of an arc, $\text{Crd}(\alpha)$, is given the chord of the supplementary arc, $\text{Crd}(180° - \alpha)$, will also be given. Ptolemy calculates $\text{Crd}(180° - 36°)$ as an example. All these values are correct rounded to the seconds place.

In order to use these starting points to fill out a table in $1/2°$ intervals, Ptolemy develops methods for finding $\text{Crd}(\alpha + \beta)$, $\text{Crd}(\alpha - \beta)$ and $\text{Crd}(1/2\ \alpha)$ when $\text{Crd}(\alpha)$ and $\text{Crd}(\beta)$ are given. The solution for $\text{Crd}(\alpha + \beta)$ and $\text{Crd}(\alpha - \beta)$ are based on the same lemma,

[67] Toomer [1984, 48 - 49].

known in the secondary literature as Ptolemy's Theorem, *Alm.* I 10.5. The solution for Crd($1/2\ \alpha$), however, is based on a different lemma that is specifically introduced to handle this case, *Alm.* I 10.7. Toomer has shown that the solution for Crd($1/2\ \alpha$) can also be derived from Ptolemy's Theorem.[68]

The proof of Ptolemy's Theorem, *Alm.* I 10.5, is straightforward and presents nothing of historical interest. Ptolemy shows that, in a cyclic quadrilateral, the rectangle formed by the diagonals is equal to the sum of the rectangles formed by the opposite sides. That is, $(AG \times BD) = (AB \times GD) + (BG \times AD)$; see Figure 2.7 (b). A number of modern commentators have shown that there is a relationship between Ptolemy's Theorem and *Data* 93.[69] This relationship, however, is slight and can be attributed to the simple fact that the two authors are investigating the same mathematical object. Ptolemy and Euclid appear to have had different methods and aims in investigating the cyclic quadrilateral. Ptolemy's theorem has been designed to be useful in metrical analyses prior to the construction of a chord table. *Data* 93, however, would have been of no such use without also appealing to a chord table; and it is not at all clear what the function of the proposition was meant to be. The reason for this dependency is that *Data* 93 relies on *Data* 87 and, as we saw above, this latter theorem does not allow one to derive numeric values.[70]

The use of Ptolemy's Theorem to find the chord which is the difference of two given chords is the first example of an analysis in the *Almagest.* The use of the techniques of analysis in *Alm.* I 10.6 clearly shows the changes of meaning that the term *given* has undergone. Let $ABGD$ be a semicircle on diameter AD, and from A let AB and AG be drawn, each *given in magnitude*, where the diameter is 120^{p}; see Figure 2.8 (a). Let BG be joined. It remains to show that BG is also given. Let BD and GD be joined. [1] Then BD and GD will be given since they are the arcs of supplementary chords [*Alm.* I 10.4]. [2] Now since $ABGD$ is a cyclic quadrilateral, $(AB \times DG) + (AD \times BG) = (AG \times BD)$

68 Toomer [1973, 16 - 17].

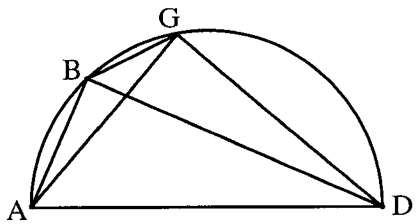
69 See, for examples, Heath [1926, vol. 2, 227] and Taisbak [2003, 2331].

70 Taisbak [2003, 233] seems to have missed this point when he claims that *Data* 93 may have been used by earlier table makers.

[*Alm.* I 10.5]. [3] But $(AG \times BD)$ and $(AB \times GD)$ are given [conv. *Data* 55], [4] therefore the remainder, $(AD \times BG)$ is given [*Data* 4]. [5] And AD is a diameter, therefore BG is given. Therefore, if the chord of two angles be given, the chord of the difference of the two angles will be given.

I have supplied justifications for the steps of Ptolemy's arguments either from the *Almagest* itself or Euclid's *Data*, the fundamental text of ancient geometric analysis. Two of the steps in Ptolemy's analysis, [3] and [5], have no justification in the *Data*. Step [3] can be justified by a converse of *Data* 55, while a proof of [5] follows almost immediately from the definition of an area *given in magnitude*, *Data* def. 1. These proofs, however, would fall within the tradition of pure geometry and would not justify the arithmetic manipulations Ptolemy intends to carry out. The justifications of [3] and [5] are both obvious, however, assuming an arithmetized conception of *given*. In this sense, [3] is justified by the well-known procedure of *multiplying* the sides of a rectangle together to find the area. Likewise, [5] is justified by the equally obvious procedure of *dividing* the area of a rectangle by one of its sides to find the other side. The logical demands of Ptolemy's analysis entails this arithmetic conception of *given*. Although in many cases throughout the *Almagest* the justification of Ptolemy's analysis can be furnished by the *Data*, in many cases it cannot. Nevertheless, even in those cases where Ptolemy seems to be calling on the *Data*, in fact, what he requires is the arithmetical equivalent of a theorem in the *Data*.

For Ptolemy, an analysis provides a theoretical justification of a calculation procedure. It fulfills a role analogous to that of our equation. I refer to the mode of arithmetized analysis we find in Ptolemy as *metrical analysis* to differentiate it from the purely geometrical analysis of the traditions of pure mathematics. We must be wary, however, of carrying the analogy between metrical analysis and algebraic equations too far. Although the fundamental conceptions and operations of Ptolemy's analysis are arithmetized, they still develop out of the tradition of geometrical analysis. The techniques for solving problems provided by a symbolic algebra are much more powerful than those available in



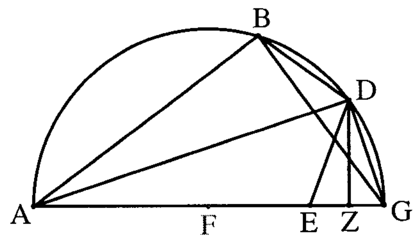


Figure 2.8: Diagrams for *Alm.* I 10.6 and *Alm.* I 10.7 & 10.8, Toomer [1984, 51 - 52]. Point F, marked in gray, does not appear in Ptolemy's diagram.

ancient analysis. While every metrical analysis can be expressed in an algebraic equation, there will be expressions in algebra that make no sense in the ancient idiom. Hence, there will be calculations which, although obvious in an algebraic translation, may be difficult or impossible within the context of ancient metrical analysis.

Before using *Alm.* I 10.5 to derive an analysis for $\mathrm{Crd}(\alpha + \beta)$, Ptolemy introduces a lemma that he will use in the analysis for $\mathrm{Crd}(^1\!/_2\ \alpha)$. As mentioned above, Toomer argued that $\mathrm{Crd}(^1\!/_2\ \alpha)$ could easily be derived from Ptolemy's Theorem, *Alm.* I 10.5, and took the fact that Ptolemy did not so derive it to be evidence of historical strata in the text.[71] Toomer's own reconstruction is too algebraic to be viable as an ancient approach; nevertheless, it is possible to rewrite the reconstruction in a manner which is much closer to ancient efforts. Even this rewrite will still be too algebraic to satisfy some readers. See Appendix C for a full discussion. This Appendix also helps illuminate the difference between metrical analysis and the methods made available by algebra.

Van Brummelen's statistical analysis of the construction of the chord table provides an alternative explanation for Ptolemy's reluctance to apply the eponymous theorem to find $\mathrm{Crd}(^1\!/_2\ \alpha)$.[72] This study shows that the procedure Ptolemy gives for finding $\mathrm{Crd}(\alpha + \beta)$ was never actually used in the calculation of the chord table. The reason for this is the fact that it invokes the procedure for $\mathrm{Crd}(180^\circ - \alpha)$, *Alm.* I 10.4, which is "highly unstable"

[71] Toomer [1973, 16 -17].
[72] Van Brummelen [1993, 60 - 101].

for small values of α.[73] The use of Ptolemy's Theorem to find Crd($^1/_2\ \alpha$) would also involve Crd($180° - \alpha$).[74] Since, as will be seen below, Ptolemy is particularly interested in using Crd($^1/_2\ \alpha$) to find the values for Crd($^3/_4°$) and Crd($1^1/_2°$) this instability would be a liability. The analysis Ptolemy gives nicely avoids this problem. Hence, it is possible to argue that Ptolemy's procedure is as roundabout as it at first appears for mathematical, as opposed to historical, reasons.

The lemma that Ptolemy uses for the analysis of Crd($^1/_2\ \alpha$), *Alm.* I 10.7, was already demonstrated in a different form by Archimedes. It is preserved in Thābit ibn Qurra's translation of Archimedes' treatise on the construction of the regular heptagon.[75] Archimedes shows that, in semicircle ABG, if $\overset{\frown}{BD} = \overset{\frown}{DG}$, then $GD^2 = (GF \times GE) = (GF \times (AG - AB))$; see Figure 2.8 (b). From this, it is clear that GD will be given if AG and AB are given. Archimedes, however, does not introduce his version of the lemma with this end in mind. Ptolemy, on the other hand, begins by showing, in *Alm.* I 10.7, that $ZG = {}^1/_2(AG - AB)$, whereas ZG does not appear in Archimedes' diagram. He then argues, in *Alm.* I 10.8, that ZG is given since both AG and AB are given. Then, since $\triangle ADG \sim \triangle DGZ$, $AG : DG = DG : ZG$ or $(AG \times GZ) = DG^2$. Therefore DG is given, since both AG and ZG are given. Toomer has plausibly argued that Ptolemy took his approach from one of his sources who in turn may have produced it on the basis of Archimedes' lemma.[76]

Ptolemy points out that a large number of chords will be determined from the chords already known by means of this theorem. In particular, he points out that we now know Crd(12°), Crd(6°), Crd(3°), Crd($1^1/_2°$) and Crd($^3/_4°$). He states that Crd($1^1/_2°$) $\approx$ $1;34,15^p$ and Crd($^3/_4°$) $\approx 0;47,8^p$.

Following this result, Ptolemy uses the theorem attributed to him, *Alm.* I 10.4, to briefly show that if Crd(α) and Crd(β) are given, then Crd($\alpha + \beta$) is also given. With these theorems, it is possible to fill in a chord table in $1^1/_2°$ steps. Unfortunately, this

[73] Van Brummelen [1993, 67].
[74] This can be seen in either of the reconstructions given in Appendix C.
[75] Schoy [1927, 81 - 82].
[76] Toomer [1973, 19].

will not suffice to achieve the goal of a table in $1/2^\circ$ steps since, as Ptolemy points out, where Crd($1 1/2^\circ$) is given, Crd($1/2^\circ$), that is Crd($1/3\alpha$), will not be given by geometric means (διὰ τῶν γραμμῶν).[77] In order to solve this problem, Ptolemy devises a method of approximating Crd(1°) by squeezing it between Crd($1 1/2^\circ$) and Crd($3/4^\circ$). As Ptolemy himself points out, the method is not generally applicable and only works in this case because the angles are relatively small.

Ptolemy begins, in *Alm.* I 10.11, with a proof of a variant of the second trigonometric lemma discussed above in Section 2.1.2. If there be a circle $ABGD$ in which two unequal chords are drawn such that $BG > BA$, the lemma shows that $BG : BA < \overset{\frown}{BG} : \overset{\frown}{BA}$; see Figure 2.9 (a). Let $\angle ABG$ be bisected by BD [*Elem.* I 9]. Join AEG, AD and GD. Then since $\angle ABG$ is bisected by BED, $AD = GD$ [conv. *Elem.* III 21] and $GE > EA$ [*Elem.* VI 3]. From D let DZ be dropped $\perp$ AZG. Then, since $AD > ED$ and $ED > DZ$, a circle drawn on center D with radius DE will cut AD and pass beyond DZ. Let it be drawn as $HE\Theta$, and let DZ be produced to Θ. Now, since $\triangle DEZ <$ sector $DE\Theta$ and $\triangle DEA >$ sector DEH, therefore $\triangle DEZ$: sec. $DE\Theta < \triangle DEA$: sec. DEH[78] and $\triangle DEZ : \triangle DEA <$ sec. $DE\Theta$: sec. DEH [*Elem.* V 16]. But $\triangle DEZ : \triangle DEA = EZ : EA$ [*Elem.* VI 1], and sec. $DE\Theta$: sec. $DEH = \angle ZDE : \angle EDA$ [1], therefore $EZ : EA < \angle ZDE : \angle EDA$. So, *by composition*, $ZA : EA < \angle ZDA : \angle EDA$ [*Elem.* V 18 applied to ratio inequalities], and doubling the first terms [of the ratios], $GA : EA < \angle GDA : \angle EDA$ [2], then *by separation*, $GE : EA < \angle GDE : \angle EDA$ [*Elem.* V 17 applied to ratio inequalities]. But $GE : EA = BG : BA$ [*Elem.* VI 3] and $\angle GDB : \angle BDA = \overset{\frown}{BG} : \overset{\frown}{BA}$ [3], therefore $BG : BA < \overset{\frown}{BG} : \overset{\frown}{BA}$.[79]

This proof almost certainly goes back at least as far as the early Hellenistic period and probably derives from the tradition that gave rise to the trigonometric methods of Aristarchus and Archimedes discussed in Section 2.1.2. It is similar in strategy to a proof

[77] Heiberg [1916, p. 1, 42 - 43]. See page 109 below for further discussion of this passage.

[78] This step appears as a scholium in the MSS Heiberg followed; however, as Theon's *Commentary* shows, it was included in the text in late antiquity, Rome [1943, 491]. Knorr has argued that it should be included in the text, Knorr [1985a, 365, n. 14].

[79] Toomer [1984, 54 - 55]. This proof is also discussed by Knorr [1985a, 364 - 367].

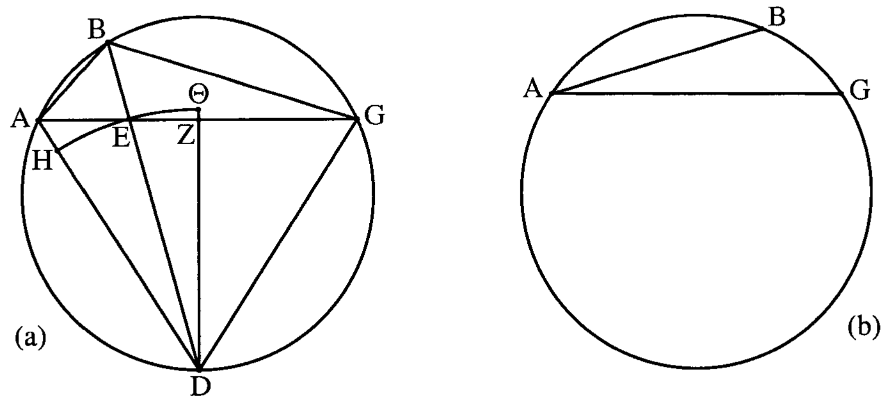


Figure 2.9: Diagrams for *Alm.* I 10.11 and *Alm.* I 10.12, Toomer [1984, 54 - 55].

attributed by Ptolemy to Apollonius later in the *Almagest* and in content to a proof given by Euclid in his *Optics*.[80] The proof exhibits the adept manipulations of ratio inequalities found in the trigonometric work of Aristarchus and Archimedes. The steps which I have numbered [1], [2] and [3] have no justification in the elementary texts but are considered by Ptolemy and his sources to be too obvious to warrant comment. These assumptions are indicative of a mathematical tradition that takes an intuitive understanding of ratios and their manipulations for granted.

With this lemma granted, Ptolemy's approach is to squeeze Crd(1°) between Crd($1^1/_2$°) and Crd($^3/_4$°). His treatment has elicited comments and criticisms from both ancients and moderns. Let circle ABG be drawn containing two chords AB and AG; see Figure 2.9 (b). If we let $AB = \text{Crd}(^3/_4{}^\circ)$ and $AG = \text{Crd}(1^\circ)$, then, since $AG : AB < \overset{\frown}{AG} : \overset{\frown}{AB}$ [*Alm.* I 10.11] and $\overset{\frown}{AG} = {}^4/_3\, \overset{\frown}{AB}$, therefore $AG < {}^4/_3 AB$, or

$$\text{Crd}(1^\circ) < {}^4/_3\ \text{Crd}(^3/_4{}^\circ). \tag{1}$$

In other words, $\text{Crd}(1^\circ) < {}^4/_3\ 0;46,8^\text{p} \approx 1;2;50^\text{p}$. On the other hand, if we let $AB =$

[80] Toomer [1984, 558 - 559] and Heiberg [1895, 14 & 164 - 165]. Knorr has studied all of the versions of these theorems and argued that they originate in a pre-Euclidean text, Knorr [1985a].

Crd(1°) and $AG = \text{Crd}(1\frac{1}{2}°)$, then, since $\widehat{AG} = \frac{3}{2}\,\widehat{AB}$, therefore $AG < \frac{3}{2}AB$, or

$$\text{Crd}(1°) > \tfrac{2}{3}\,\text{Crd}(1\tfrac{1}{2}°). \tag{2}$$

That is, $\text{Crd}(1°) > \frac{2}{3}\ 1;34,15^{\text{p}} = 1;2;50^{\text{p}}$. From the fact that $1;2;50^{\text{p}} < \text{Crd}(1°) < 1;2;50^{\text{p}}$, Ptolemy concludes $\text{Crd}(1°) = 1;2;50^{\text{p}}$.

Theon takes issue with this derivation since it is confusing (ϑορυβεῖ) that the same length can both be greater and less than the same value.[81] He begins by recalculating the values given by equations 1 and 2 and shows that, in fact, $1;2;50^{\text{p}} < \text{Crd}(1°) < 1;2;50,40^{\text{p}}$. Now, if $\text{Crd}(1°) \geq 1;2,50,30$ then, $\text{Crd}(1°) = 1;2;51^{\text{p}}$ would be a better estimate.[82] Theon proceeds by going back to *Alm.* I 10.9 and recalculating $\text{Crd}(\frac{3}{4}°)$ from $\text{Crd}(1\frac{1}{2}°) = 1;34,15^{\text{p}}$ with greater precision. Theon finds that $\text{Crd}(\frac{3}{4}°) = 0;47,7,39^{\text{p}}$.[83] He then applies this value to equation 1 and finds that $\frac{4}{3}\ 0;47,7,39^{\text{p}} = 1;2,50,12^{\text{p}}$ and states that $1;2;50^{\text{p}} < \text{Crd}(1°) < 1;2;50,12^{\text{p}}$. Hence, $\text{Crd}(1°) \approx 1;2;50^{\text{p}}$ and, he assures us, nothing absurd (οὐδὲν ἄτοπον) results.[84]

R. Newton has found fault with Ptolemy for not proceeding with enough precision.[85] Newton believes that he has detected a rounding error in Ptolemy's calculations since if we recalculate $\text{Crd}(1\frac{1}{2}°)$ and $\text{Crd}(\frac{3}{4}°)$ on the basis of *Alm.* I 10.9 with greater precision we find that $\text{Crd}(1\frac{1}{2}°) = 1;34,14,42,19^{\text{p}}$ and $\text{Crd}(\frac{3}{4}°) = 0;47,7,24,48^{\text{p}}$, whereas Ptolemy gives $\text{Crd}(\frac{3}{4}°) = 0;47,8^{\text{p}}$. Newton, however, has failed to consider that the method of *Alm.* I 10.9 is recursive and Ptolemy would have rounded as he went. Hence, Ptolemy would have calculated $\text{Crd}(\frac{3}{4}°)$ on the basis of $\text{Crd}(1\frac{1}{2}°) = 1;34,15^{\text{p}}$ not $\text{Crd}(1\frac{1}{2}°) = 1;34,14,42,19^{\text{p}}$. As pointed out in note 83, this will give $\text{Crd}(\frac{3}{4}°) = 0;47,7,33,38^{\text{p}}$ and Ptolemy's rounding is satisfactory. Newton concludes by showing that his more

81 Rome [1943, 494].

82 Rome accuses Theon of misunderstanding Ptolemy's method, but his own explanation of the method does not satisfy Theon's concerns, Rome [1943, 495, n. 1].

83 The algorithm which Ptolemy gives for $\text{Crd}(\frac{1}{2}\ \alpha)$ is equivalent to $\text{Crd}(\frac{1}{2}\ \alpha) = \sqrt{60(120 - \sqrt{120^2 - \text{Crd}(\alpha)^2})}$, Toomer [1984, 52 - 53]. Precise calculation, where $\text{Crd}(1\frac{1}{2}°) = 1;34,15^{\text{p}}$, gives $\text{Crd}(\frac{3}{4}°) = 0;47,7,33,38^{\text{p}}$.

84 Rome [1943, 494 - 495].

85 Newton [1977, 26 - 28].

precise calculations give $1;2,49,48,13^{\mathrm{p}} < \mathrm{Crd}(1^\circ) < 1;2,49,53,3^{\mathrm{p}}$, so that, in any case, Ptolemy's value is accurate rounded to the second fractional place.

Ptolemy next calculates the value of $\mathrm{Crd}(1/2^\circ) = 0;31,25^{\mathrm{p}}$ and proceeds to fill in a chord table in $1/2^\circ$ intervals. A final passage explains that he considers his method to be the fastest approach to calculating a chord table and describes the layout of his table. The chord table is set out in eight columns of 45 rows. Each row contains three entries: (1) the angle in $1/2^\circ$ steps, α_n, (2) its corresponding chord, $\mathrm{Crd}(\alpha_n)$, and (3) the "sixtieth", or $1/30$ of the difference between the chord and its successor,[86]

$$\mathrm{Sixt}(\alpha_n) = 1/30(\mathrm{Crd}(\alpha_{n+1}) - \mathrm{Crd}(\alpha_n)).$$

Ptolemy tells us that the column of sixtieths is included to make it easier to calculate the chords of angles that fall between two entries in the table. Ptolemy does not explain how to use his table, but consideration of its structure, along with comments by Theon can be used to determine Ptolemy's intent.[87]

The chord table is used throughout the *Almagest* both to find chords given angles and to find angles given chords. In other words, it allows one to define the function $\mathrm{Crd}(\alpha)$ and its inverse, $\mathrm{Arc}(x)$, where the domains α and x are respectively angles and lengths and the ranges are the converse. In order to find the chord subtending angle α, where α lies in the interval $\alpha_n < \alpha < \alpha_{n+1}$, Ptolemy uses a procedure corresponding to the following formula:[88]

$$\mathrm{Crd}(\alpha) = \mathrm{Crd}(\alpha_n) + (\alpha - \alpha_n)\mathrm{Sixt}(\alpha_n). \tag{C.T. 1}$$

The sixtieths column, however, gives no help in calculating the angle given the chord and one suspects that, in this case, Ptolemy proceeded by linear interpolation. Theon's *Commentary* confirms that in his time $\mathrm{Arc}(x)$ was found by a procedure equivalent to

86 Pedersen [1974b, 64].
87 Pedersen [1974b, 64] claims that Ptolemy explains the use of the table by means of a numerical example. I cannot find the example to which he refers.
88 Pedersen [1974b, 64].

our formula for linear interpolation:

$$\mathrm{Arc}(x) = \mathrm{Arc}(x_n) + (\mathrm{Arc}(x_{n+1}) - \mathrm{Arc}(x_n))\frac{x - x_n}{x_{n+1} - x_n}, \qquad \text{(C.T. 2)}$$

where x is a chord lying in the interval $x_n < x < x_{n+1}$. A comparison of equations C.T. 1 & C.T. 2 shows that the sixtieths column saves the computer two subtractions and a division.

In order to understand how Ptolemy uses his table to fulfill the promise of metrical analysis, it will be necessary to examine his practice in trigonometric calculations.

Ptolemy's trigonometry

The first extensive use of the chord table in the *Almagest* is the application of metrical methods to spherical astronomy. This subject is given full treatment in Chapter 3. To see the chord table applied to plane trigonometric problems, we have to look forward to the introduction of Hipparchus' solar model in Book III. One of the only attempts to understand Ptolemy's trigonometry in its own terms is the example worked through by Toomer in the introduction to his translation of the *Almagest.*[89] I will follow Toomer's approach, adding a few more examples to the one he gives. This will allow us to identify regulating practices and relate Ptolemy's trigonometry to his metrical analysis.

Ptolemy's first application of the chord table to a plane problem is in his derivation of the parameters of Hipparchus' solar model in *Alm.* III 4. Although it is not of particular mathematical interest, this material has great historical value. Based on Ptolemy's own admission and the agreement with the diagram and the values given by Theon of Smyrna, it has always been acknowledged that the essential mathematical argument of *Alm.* III 4 is due to Hipparchus and derives from the solar theory he advanced.[90] This derivation is representative of the sort of mathematical problem that chord tables, and

[89] Toomer [1984, 7 - 9].
[90] Toomer [1984, 153] and Hiller [1878, 155 - 158].

the mathematical practices that accompany them, were invented to solve.[91]

Hipparchus begins by assuming that the sun orbits the earth with constant velocity on a circle eccentric to the center of the earth and that the spring is 94½ days, the summer 92½ days. These assumptions are sufficient to model the solar motion and determine the parameters of the model. Let circle $ABGD$ be the ecliptic with the earth at its center, E, point A the vernal equinox, B the summer solstice, G the autumnal equinox and D the winter solstice; see Figure 2.10. Then, let circle ΘKLM be the eccenter of the sun arranged such that its center Z falls in the quadrant of the spring, making spring the longest season. Ptolemy derives the parameters as follows.[92]

> Now since the sun traverses circle ΘKLM with uniform motion, it will traverse $\overset{\frown}{\Theta K}$ in 94½ days, and $\overset{\frown}{KL}$ in 92½ days. In 94½ days its mean motion is $93;9°$, and in 92½ days $91;11°$ [*Alm.* III 2, **Solar Mean Motion Table**]. Therefore, $\overset{\frown}{\Theta KL} = 184;20°$ and, by subtraction of the semicircle NPO, $\overset{\frown}{N\Theta} + \overset{\frown}{LO} = 4;20°$, but $\overset{\frown}{\Theta NY} = 2\,\overset{\frown}{N\Theta}$ [*Elem.* III 3]. Thus $\mathrm{Crd}(\overset{\frown}{\Theta NY}) \approx 4;32^{\mathrm{p}}$ [$\mathrm{Crd}(\alpha)$ by the **Chord Table**], where the diameter of the eccenter $= 120^{\mathrm{p}}$,[93] and $EX = \Theta T = \frac{1}{2}\Theta Y = 2;16^{\mathrm{p}}$. Now, since $\overset{\frown}{\Theta NPK} = 93;9°$, and $\overset{\frown}{\Theta N} = 2;10°$, and quadrant $NP = 90°$, by subtraction, $\overset{\frown}{PK} = 0;59°$ and $\overset{\frown}{PKQ} = 2\ \overset{\frown}{PK} = 1;58°$, therefore $KFQ = \mathrm{Crd}(\overset{\frown}{KPQ}) = 2;4^{\mathrm{p}}$ [$\mathrm{Crd}(\alpha)$ by the **Chord Table**], where the diameter of the eccenter $= 120^{\mathrm{p}}$, and $ZX = KF = \frac{1}{2}KFQ = 1;2^{\mathrm{p}}$, in the same units.

These first applications of the chord table are not strictly trigonometric. That is, no triangle is being measured. Nevertheless, since the goal is to measure $\triangle ZXE$, it is clear that this is an apt use of the table. Notice that, when he uses the table to find the lines ΘY and XQ, Ptolemy specifies the unit of length. Ptolemy does not take a single unit for each figure and treat his trigonometric functions as ratios. Every time he uses his table, Ptolemy has to take into consideration the fact that it is built on a circle with

[91] Accounts of this material from a more astronomical perspective, using current mathematics, can be found in Pedersen [1974b, 149] and Neugebauer [1975, 57 - 58].

[92] Toomer [1984, 155]. I have changed some notation and inserted some text in brackets.

[93] Ptolemy's expression for this conversion is, "Thus chord $\Theta\Upsilon$ will be approximately 4 32 of those [units], of which the diameter of the eccenter is 120." ὥστε καὶ ἡ μὲν ὑπ' αὐτὴν εὐθεῖα ἡ ΘΥ τοιούτων ἔσται δ̄ λ̄β̄ ἔγγιστα, οἵων ἐστὶν ἡ τοῦ ἐκκέντρου διάμετρος ρ̄κ̄, Heiberg [1916, p. 1, 236].

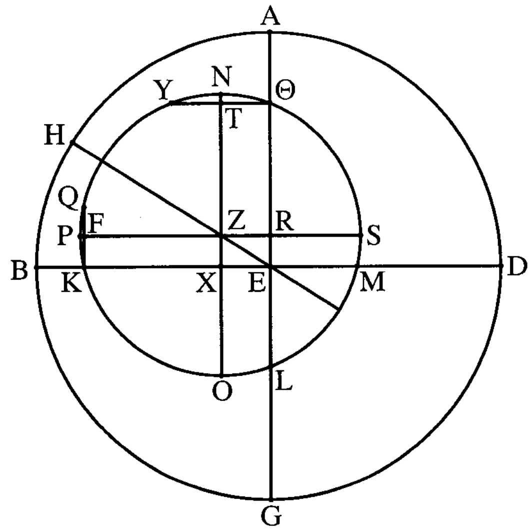


Figure 2.10: Diagram for Ptolemy's presentation of Hipparchus' solar model, *Alm.* III 4, Toomer [1984, 154]. The observer in on earth, at E, while the sun moves counterclockwise about circle $MNPO$. The vernal equinox is at A, the summer solstice at B, the autumnal equinox at G and the winter solstice at D. HZE is the line of apsides.

radius = 60ᵖ. Since the eccenter circle is the one he is interested in, no conversions are needed in these cases.

> Now since $EZ^2 = ZX^2 + EX^2$ [*Elem.* I 47], $EZ \approx 2;29\frac{1}{2}^{\text{p}}$, where the radius of the eccenter is 60ᵖ. Therefore, the radius of the eccenter is approximately 24 times the distance between the centers of the eccenter and the ecliptic [$60 : 2;30 = 24 : 1$]. Now, since $EZ : ZX = 2;29\frac{1}{2} : 1;2$, ZX will be about $49;46$ where hypotenuse $EZ = 120^{\text{p}}$.[94] Therefore, in R$\triangle EZX$, Arc(ZX) $\approx 49^{\circ}$[95] [Arc(x) by the **Chord Table**]. Therefore, $\angle ZEX = 49^{\circ\circ}$, where 2R = $360^{\circ\circ}$, and $24;30^{\circ}$ where 4R =

[94] Toomer's translation conveys the mathematical sense of this passage, but Ptolemy avoids the usual expression for a proportion, choosing to express the process of conversion more directly, Toomer [1984, 155]. "Now since *EZ* was shown to be 2 29½ of those [units] of which the line $Z\Xi$ was 1 2, therefore also, the hypotenuse *EZ* is 120 of those of which the line $Z\Xi$ will be approximately 49 46." πάλιν ἐπεί, οἵων ἡ EZ ἐδείχθη β̄ κ̄θ̄ ∠, τοιούτων ἦν καὶ ἡ ZΞ εὐθεῖα ᾱ β̄, καὶ οἵων ἄρα ἐστὶν ἡ EZ ὑποτείνουσα ρ̄κ̄, τοιούτων ἔσται καὶ ἡ μὲν ZΞ εὐθεῖα μ̄θ̄ μ̄ς̄ ἔγγιστα, Heiberg [1916, p. 1, 237]. Toomer's translation is more exact in his explanation of Ptolemy's trigonometry, Toomer [1984, 7 - 9].

[95] Because $\widehat{ZX}$ does not appear in the figure, Ptolemy does not use "the arc ZX" but rather the "arc on it," ἡ δ' ἐπ' αὐτῆς περιφέρεια, that is on ZX, Heiberg [1916, p. 1, 237]. This is his usual expression for arcs in a trigonometric computation.

> 360°.[96] So, since $\angle ZEX$ is an angle at the center of the ecliptic, $\overset{\frown}{BH}$, which is the amount by which the apogee at H is in advance of the summer solstice at B, is also $24;30^\circ$.[97]

This is the first application of the chord table to the measurement of a plane triangle and it typifies Ptolemy's procedure. In order to find an angle, Ptolemy will first find the opposite side and hypotenuse of a right triangle that contains the angle. He will then apply the chord table. Since EZ and ZX are both known in those units of which the diameter of the eccenter is 120^{p}, the ratio between them is known, and they can both be expressed in those units of which $EZ = 120^{\mathrm{p}}$. Toomer's translation conveys both the mathematical sense, and, as we shall see below, Ptolemy's theoretical conception of the situation. $\triangle ZXE$ is considered as circumscribed by a circle. The chord table is then applied to determine the arc of the circumcircle opposite the given leg, ZX. Hence, the angle which is opposite ZX, $\angle ZEX$, will be given since, by *Elem.* III 20, it is $1/2\ \overset{\frown}{ZX}$. Thus, the promise of *Data* 43 is fulfilled for metrical calculations; where the sides about an acute angle of a right triangle are given, the triangle is *given in form.*

In order to segue between $\overset{\frown}{ZX}$ on the circle circumscribing $\triangle ZXE$ and $\angle ZEX$, Ptolemy introduces semidegrees, of which two right angles are $360^{\circ\circ}$. Ptolemy always employs this device, so as, Toomer says, "to switch smoothly" between the circle and the triangle.[98] There is no mathematical advantage to this conversion. Conceptually, however, it allows Ptolemy to avoid discussing the angle at the center of the circle, which is absent in the figure.

Ptolemy's use of the chord table is always based explicitly on the circle which contains the given or sought chord. This circle is never added to the figure for the sake of employing the chord table. Drawing the circle is unnecessary because all of its essential features are

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96 The formulaic Greek expression is the same as that in note 94. It reads, "And therefore angle $ZE\Xi$ will be 49 of those [units], of which 2 right [angles] is 360, but 24 30 of those, of which 4 right [angles] is 360." See, Heiberg [1916, p. 1, 237]. See Netz [1999b, 127 - 168] for the role of formulaic expressions in Greek mathematics.

97 In this expression, "is in advance," προηγεῖται, signifies movement in the direction of the diurnal motion from B to A not in the direction of the sun's proper motion from Θ to K. Hence the apogee, H, is at $5^1/_2{}^\circ$♊.

98 Toomer [1984, 8].

already present in the figure. The diameter and chords are lines in the figure and the arcs may be referred to as standing on these lines. The chord table is a calculation tool that was designed to be adapted to specific geometric configurations. Ptolemy is careful to keep this geometric basis explicit.

The fact that the diagrams and values reported by Theon and Ptolemy are the same, argues that Hipparchus essentially proceeded along the same lines as Ptolemy. Moreover, nothing in Ptolemy's presentation vitiates this claim. Through simple applications of the chord table, equivalent to our use of the sine function, Hipparchus was able to derive the parameters of his solar model from the lengths of two seasons. It is worth noting that the chord table which Toomer proposed for Hipparchus is not accurate enough to derive the parameter values that both Ptolemy and Theon of Smyrna attest.[99] In this application of the chord table, the right triangle to which the table was applied is given in the figure. In general, this will not be the case and Ptolemy proceeds by constructing a right triangle. An example will serve to give his general approach.

In *Alm.* III 5, Ptolemy shows, for both the eccenter and epicycle models, that each of the three essential angular characteristics of solar position can be used to determine the others. The three relevant angles are (1) the mean motion, $\overline{\kappa}$, (2) the apparent motion, κ, and (3) the equation of anomaly, τὸ παρὰ τὴν ἀνωμαλίαν διάφορον, also called the positive or negative correction, προσθαφαίρεσις, c.[100] As a preliminary to setting out the table for the equation of anomaly, Ptolemy uses example calculations for each model to show that if $\overline{\kappa}$ is 30° from apogee or perigee then both κ and c are given. He then uses metrical analysis to argue that if either of the other two angles are taken as given the remaining two angles will be given as well. He begins by taking $\overline{\kappa}$ as 30° from apogee in the eccenter

[99] The table derived by Toomer [1973] gives the ratio of diameter of the eccenter to the distance between the two centers as $60 : 25;12,36$ and sets the apogee at 4;21°♊. The main reason for this is that a chord table is only as accurate as its first entry. Due to the use of linear interpolation, for applications of the table to chords, or angles, less than those in the first entry, the chord and angle are set equal to one another. In this case, the table that Toomer proposed gives chord values that are too crude for the small arcs $\overset{\frown}{\Theta Y}$ and $\overset{\frown}{QK}$.

[100] Accounts of this material from a more astronomical perspective, using current mathematics, can be found in Pedersen [1974b, 149 - 151]and Neugebauer [1975, 58 - 61]. Neugebauer also gives a mathematical summary of Ptolemy's approach in Neugebauer [1975, 25].

model.[101]

> First, let the circle concentric to the ecliptic be ABG on center D, the eccenter EZH on center Θ, and let the diameter through both centers and the apogee E be $EA\Theta DH$ [see Figure 2.11 (a)]. Cut off $\overset{\frown}{EZ}$ and join ZD, $Z\Theta$. First, let $\overset{\frown}{EZ}$ [$= \overline{\kappa}$] be given, for example as 30°. Produce $Z\Theta$ and drop the perpendicular to it from D, DK. Then, since $\overset{\frown}{EZ}$ is, by hypothesis, 30°, $\angle E\Theta Z = \angle D\Theta K = 30°$, where $4\mathrm{R} = 360°$ [*Elem.* I 15], and [$\angle D\Theta K =$] $60^{\circ\circ}$, where $2\mathrm{R} = 360^{\circ\circ}$. Therefore, in the circle about $\mathrm{R}\triangle D\Theta K$, $\overset{\frown}{DK} = 60°$, and $\overset{\frown}{K\Theta} = 120°$ (supplement) [*Elem.* III 31].[102] Therefore, the chords of these will be $\mathrm{Crd}(DK) = 60°$ and $\mathrm{Crd}(K\Theta) = 103;55°$, where hypotenuse $D\Theta = 120^{\mathrm{p}}$ [**Chord Table**].

In this example, there is no right triangle in the figure, so Ptolemy begins by constructing one. Where one of the acute angles of a right triangle is given, the length of the side opposite the given angle will be given, where the hypotenuse is set to 120^{p}. Hence, Ptolemy constructs a right triangle with a given acute angle about a given hypotenuse. He can then apply the chord table to completely determine the triangle. In applying the chord table, Ptolemy, as usual, introduces semidegrees so that he can switch from the angles in the triangle to the arcs of the circumscribing circle without reference to the angles at the center of the circle. In this case, the chord table is applied to the two acute angles to give the chords subtending the angles in terms of the diameter of the right triangle. The ratio of the diameter of the right triangle to the radii of the circles, however, is already given since this is a fundamental parameter of the model.

> Therefore, where $D\Theta = 2;30^{\mathrm{p}}$, and radius $Z\Theta = 60^{\mathrm{p}}$ [*Alm.* III 4], $DK = 1;15^{\mathrm{p}}$ and $\Theta K = 2;10^{\mathrm{p}}$. Therefore, by addition, $K\Theta Z = 62;10^{\mathrm{p}}$. Now since, $DK^2 + K\Theta Z^2 = ZD^2$ [*Elem.* I 47], the hypotenuse $ZD \approx 62;11^{\mathrm{p}}$. Therefore, where $ZD = 120^{\mathrm{p}}$, $DK = 2;25^{\mathrm{p}}$, and in the circle about $\mathrm{R}\triangle ZDK$, $\mathrm{Arc}(DK) = 2;18°$ [**Chord Table**]. Therefore $\angle DZK = 2;18^{\circ\circ}$, where $2\mathrm{R} = 360^{\circ\circ}$, and $\angle DZK = 1;9°$, where $4\mathrm{R} =$

[101] Toomer [1984, 158 - 159]. I have changed some notation and inserted some text in brackets.

[102] The expression for the arc is literally, "the arc on the [line] DK," ἡ ... ἐπὶ τῆς ΔΚ ... περιφέρια, Heiberg [1916, p. 1, 241].

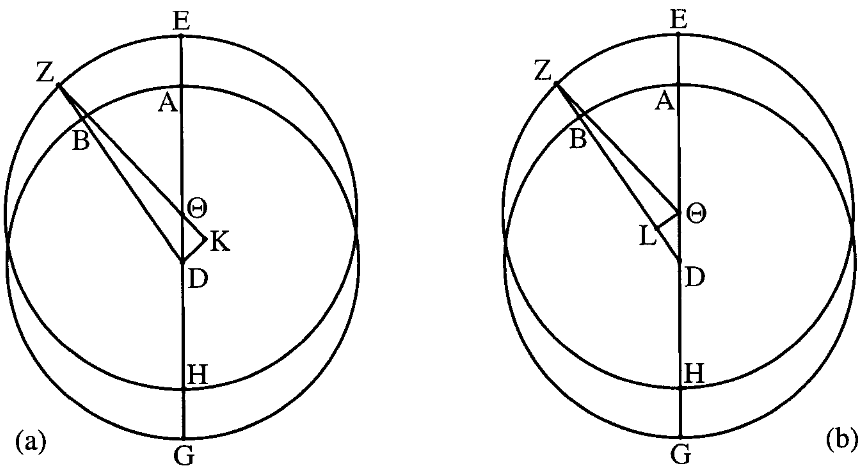

Figure 2.11: Diagrams for Ptolemy's derivation of the equation of anomaly using the epicycle model, *Alm.* III 5, Toomer [1984, 160 - 161].

> 360°. That [1; 9°] will be the amount of the equation of anomaly [c] at this position. And $\angle E\Theta Z$ was taken at 30°, therefore, by subtraction, $\angle ADB = \widehat{AB}$ (of the ecliptic) [$= \kappa$] $= 28;51°$.

Using a different right triangle in the same construction, Ptolemy applies the chord table inversely to find an angle given the ratio of the opposite side to the hypotenuse. In all of these examples we see the basic practices of Ptolemy's trigonometry. In right triangles, a given angle is used to find the ratio between the opposite side and the hypotenuse, and a given ratio between a side and the hypotenuse is used to find the opposite angle. Both of these conversions involve a shift in the unit of measure; the hypotenuse of the right triangle is set to 120^{p} and the angles of the right triangle are related to the arcs of the circumcircle by means of semidegrees.

As Toomer points out, there is also evidence that Ptolemy knows the equivalent of the modern sine formula: if the three angles of a general triangle are given, the sides of the triangle will be given in terms of the diameter of the circumcircle[103] It is rarely the case that more than one angle of a general triangle will be known, however, and the one example that Toomer cites avoids this problem because it involves an isosceles

[103] Toomer [1984, 7, n. 10].

triangle.[104]

Ptolemy's trigonometric methods are directly related to the tradition of metrical analysis. In the *Almagest* the solutions to a number of trigonometric problems are demonstrated more elegantly through metrical analysis. If we continue reading the above text we find two examples of metrical analysis which are analogous to the trigonometric calculation that we have just seen. It will be sufficient to examine the first of these.

> Furthermore, if any other of the [three relevant] angles be given, the remaining angles will be given, as is immediately obvious if, in the same figure [see Figure 2.11 (b)] we drop $\Theta L \perp ZD$ from Θ. For suppose first that $\widehat{AB}$ of the ecliptic [κ], that is $\angle \Theta DL$, is given. Then the ratio $D\Theta : \Theta L$ will be given [*Data* 40]. And since $D\Theta : \Theta Z$ is also given [*Alm.* III 4], $\Theta Z : \Theta L$ will be given [*Data* 8]. Hence, $\angle \Theta ZL$ [$= c$], the equation of anomaly, will be given [*Data* 43], and so will $\angle E\Theta Z[= \overline{\kappa} = 180^\circ - (\angle \Theta DL + \angle \Theta ZL)]$, that is $\widehat{EZ}$ of the eccenter.

Following Toomer, I have justified the steps of this analysis with references to the *Data*. If we wish to carry though a calculation on the basis of this analysis, however, only the reference to *Data* 8 will be of any help. In order to make the calculations at the steps justified by *Data* 40 & 43, we will require the chord table. The metrical analysis provides the theoretical justification of the calculation, but the calculation itself requires a new technology. The use of this technology, moreover, has changed the understanding of the theoretical foundation for the practitioners.

In Ptolemy's metrical analyses, we find no mention of different modes of *given*. Specifically, there is no mention of a triangle being *given in form*.[105] Instead, we find that if an acute angle of a right triangle is given, then the ratio of the opposite side to the hypotenuse is given; conversely, if this ratio is given, then the opposite angle is given. Clearly the only mode of *given* which is of interest in metrical analysis is *given in magnitude* and this now means expressible as a numerical value. The propositions of the

[104] Toomer [1984, 462].

[105] This is true of the *Analemma* as well as the *Almagest*, Heiberg [1907a, 203 - 210] and Edwards [1984, 108 - 117].

Data should still be regarded as the theoretical background to metrical analysis, but the conceptualization and justification of the steps of an analysis must be the computational tools of the chord table and its attendant practices.

2.1.5 Remarks on the history of Greek trigonometry

As we have seen, it is not possible to write a history of trigonometry among the Greek mathematicians. The best that we can hope with the current evidence is to collect a sufficient number of episodes to illustrate the story in broad strokes.

The first thing we notice is that trigonometry, the measurement of the angles and sides of triangles, arose within the context of mathematical astronomy and never formally entered the tradition of theoretical mathematics.[106] It appears that the applied and theoretical traditions, each with its own methods and criteria, existed in relative autonomy. Practitioners might work in both fields, but they produced texts that clearly belonged to one or the other. Applied mathematical texts would often include metrical analyses either in place, or justification, of a calculation; however, they rarely contain groups of analytical propositions that serve a purely theoretical role. A notable exception to this rule is Hero's *On Measurements*, which is a deliberate attempt to fuse theoretical and practical practices.[107] There seem to have been no texts which developed trigonometric methods from a purely theoretical perspective. It is in this context that we should read Theon's remark that Hipparchus produced a book of chords in six books, Menelaus in twelve.[108] There are reasons to believe that Theon never saw either of these works and is repeating this information from another source. Nothing in Theon's *Commentary* on *Alm.* I 10 indicates that he consulted any text beyond the *Almagest.* Moreover, his expression of wonder at Ptolemy's ability to compress so much material into so "few and

[106] There is, however, at least one indication that trigonometry was used as a heuristic tool for investigating theoretical problems. Pappus suggests that one could use Ptolemy's chord table to arithmetically verify a geometrical analysis, Hultsch [1878, 48] and Bernard [2003, 124].

[107] See especially *Meas.* III 10 - 19, a sequence of purely analytical propositions, Schöne [1903, 160 - 174] or Bruins [1964, vol. 2, 83 - 87].

[108] Rome [1943, 451].

convenient theorems" would make no sense if he had actually consulted these works to see why they were so long.[109] The most likely situation is that Hipparchus and Menelaus developed their studies on chords in the context of larger works on mathematical astronomy in general.[110] Considering that the *Almagest* is thirteen books, the lengths attributed to these earlier works are still remarkable and suspect. Nevertheless, we should consider the fact that these works may have contained other material relevant to mathematical astronomy but absent from the *Almagest*, such as projective techniques and the methods of the analemma.[111]

Within the theoretical analytic tradition, the function of angles and sides in the determination of triangles was recognized already in the Classical period. By the time Euclid compiled his *Data*, geometers had already seen that the special properties of the right triangle would make it useful in solving geometrical problems. Moreover, within this tradition, it had also been shown that the arc of a circle and the chord that subtended it were inversely related and that one could be used to geometrically determine the other. This knowledge, however, and the practices that it enabled appear to have only been applied to purely theoretical problems. There was no question of measuring angles or line segments; and within the pure theoretical tradition, there was no need for measurement.

The need for measurement arose in the application of mathematical methods to astronomical problems. By the early Hellenistic period, there were well developed methods of using the special properties of the right triangle to make approximate conversions between angles and line segments. These practices were built on a core group of trigonometric lemmas that established certain ratio inequalities relating the sides and angles of a right triangle. These lemmas may have been collected together in a canonical text or they may have simply formed a craft knowledge, along with the manipulation of ratios.

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109 Rome [1943, 451].

110 Toomer [1973, 19] put forth the "extremely tentative" suggestion that the twelve books ascribed to Hipparchus are, in fact, twelve sections of a 180-row table in 30-row parts. This is possible, but it seems to me unlikely that a Greek mathematician would publish a work that consisted simply of the derivation and presentation of a chord table. The evidence we have suggests that chord tables were not of any interest outside the context of mathematical astronomy.

111 See Chapter 4, Sections 4.1 and 4.2, for discussions of these topics.

Within these early metrical practices, the need for a unit of measure began to be felt. Line segments were simply compared to one another; they might have a known ratio to one another, or more usually, one was arbitrarily taken as the unit. Angles were compared on the basis of the whole circle and the right angle. Other arbitrary divisions of these might be taken, but there was as yet no standard unit of angular measure.

Slightly before, or during, the time of Hipparchus a number of important changes took place. Presumably as a result of the assimilation of Babylonian astronomical knowledge, Greek mathematicians began to see the possibility of a much more exact astronomy than what we find in Aristarchus or Archimedes.[112] This new astronomy, being based on carefully chosen observations and secured on the solid foundation of Greek geometric knowledge, would have to proceed by more precise mathematical methods than the approximation techniques of the trigonometric lemmas. To this end, two important mathematical tools were imported from the Near East, the 1° unit of angular measure and the system of sexagesimal fractions. The first evidence for this is Hypsicles' *Ascensions*.[113] This work uses three theorems on numerical series to justify the application of a Babylonian scheme for rising times of arcs of the ecliptic adapted to the latitude of Alexandria.[114] It uses degrees and sexagesimal fractions. At first, these foreign conventions were inconsistently applied.[115]

Around this time, an even more important tool was developed by the Greeks, perhaps by Hipparchus himself. Chord tables were created as a calculation device to convert between angles and the sides of right triangles. There was nothing theoretically innovative about chord tables, but they initiated mathematical practices that fulfilled the promise of the theoretical analytic tradition for calculation as well as geometric construction. Analysis could now be used to demonstrate the possibility of a calculation as well as of a construction. We do not know what these early chord tables looked like but they must

[112] See Jones [1991] and Jones [1996] for discussions of the influence of Babylonian methods on Greek astronomy.

[113] de Falco, Krause and Neugebauer [1966]. See Huxley [1963] for the date of Hypsicles.

[114] Hypsicles' *Ascensions* is discussed in both Evans [1998, 121 - 124] and Neugebauer [1975, 715 - 718].

[115] An example of this inconsistency is a passage from Hipparchus' *Commentary* which employs degrees along with other units of angular measure, Sidoli [2004a, 75 - 76].

have been precise enough to handle the small angles of Hipparchus' solar model.

Between Hipparchus and Ptolemy, the only known astronomer who may have introduced mathematical innovations was Menelaus. Unfortunately, the only certain evidence we have for Menelaus' work is either observational or mathematical astronomy in the pure theoretical tradition.[116] Neither of these allow us to compare Menelaus' mathematical style with Ptolemy's.

Recently, Jones has grouped together two papyri and argued that they originate in a work of theoretical astronomy in the tradition which preceded the *Almagest*.[117] Moreover, he has suggested that this treatise should be attributed to Menelaus. These papyri offer important clues about the contents of other systematic treatise of mathematical astronomy. They contain analysis of dated observations, use epicycles and eccentric circles and separate mean motion tables from tables of anomalistic motion, much as we find in the *Almagest*.[118] They use two systems of angular measure, degrees and lunar diameters. They make use of the system of sexagesimal fractions. Unfortunately, they contain no trace of trigonometric computation and, on the whole, they are are too fragmentary to make any definite statements about the mathematical methods they employ.

Ptolemy adopts the Babylonian conventions in a much more systematic way than Hipparchus. He combines these regulating tools with a highly accurate chord table to produce a standardized trigonometric practice. He applies the chord table in a number of specific ways and these applications are related to the tradition of Greek analysis.

Greek trigonometry was not established by a theoretical breakthrough but, rather, by the combined effect of a number of techniques and tools both native and imported. Nevertheless, once established, trigonometric practices influenced the underlying concepts

[116] Two observations by Menelaus are mentioned by Ptolemy, Toomer [1984, 336 & 338]. Krause [1936] published an Arabic version of Menelaus' *Spherics* with a German translation. It contains no calculations.

[117] These are *P. Oxy.* LXI 4133 and *PSI* XV 1490, Jones [2004]. Text, translation and commentary for *P. Oxy.* LXI 4133 is found in Jones [1999a, vol. 1, 69 - 81 & vol. 2, 2 - 9]. For which, see also Jones [1999b]. A tentative transcription of *PSI* XV 1490 was made by Manfredi [1966]. An English translation is given in Jones [2000b].

[118] Jones [1999b] and Jones [2000b, 81 - 85].

in the domain of theory known as analysis. Analysis changed its focus from construction to calculation and became what I have called metrical analysis. Analysis began to be viewed as a tool for solving problems that would produce results in numerical values, much as we might now manipulate an equation. The applied mathematical tradition was an important nexus for the birth and development of modes of mathematical thought that would later be called algebraic.

Because he saw the history of Greek trigonometry as leading almost inevitably to modern trigonometry, Tropfke found in Archimedes traces of later work and implied that Archimedes should be viewed as an important figure in the development of Greek trigonometry.[119] Toomer has already argued against the specifics of this interpretation.[120] What is clear, when we view Greek trigonometric practices in their own context, is that Archimedes, far from being an innovator, was one of the last practitioners of a style of trigonometry that the Greeks ultimately decided was unsuccessful and abandoned. The tools of the new trigonometry were built out of the knowledge base of the old trigonometry, but its technology was new. By applying new methods and tools to solving geometric problems, trigonometry had a lasting effect on ancient analysis, the field of theory concerned with mathematical problems.

2.2 Geometrically and numerically given

Ptolemy uses both metrical analysis and calculation but not in the form of an analysis/synthesis pair as we find in Hero's *On Measurement.* In Ptolemy, metrical analysis is used to provide the theoretical justification for a computation that, within the context of the deductive structure of the argument, need not actually be carried out. Although almost all calculations are carried out directly on the figure with no reference to an analysis, the assumption is that an analysis could have been used to justify the computation. In fact, as we will see below, some of these analyses would have been quite involved and

[119] Tropfke [1928].
[120] Toomer [1973, 21 - 23].

not all of their steps could have been justified by the methods of the *Data*.

Hero's *On Measurements* is an attempt to apply the results and methods of the theoretical tradition to practical problem solving. In *On Measurements*, an analysis is often followed with a calculation introduced by some variation of the statement, "Clearly, in accordance with the analysis, it is synthesized as follows."[121] In fact, however, the relationship between the analysis and the calculation in *On Measurements* is sometimes fairly loose.[122] Ptolemy, on the other hand, never refers to his calculations as syntheses, nor for that matter does he refer to his metrical analysis as analyses. In consideration of the other uses and descriptions of the analysis/synthesis pair in Greek mathematics, Ptolemy seems wise to avoid designating his calculations as syntheses.[123] In both ancient and modern understandings of the analysis/synthesis pair there is a reversal of the logical approach between the analysis and its corresponding synthesis.[124] There is, however, no such reversal of logical inference between Ptolemy's metrical analysis and the corresponding calculation. The metrical analysis establishes the theoretical possibility of performing the computation; while the calculation follows along precisely the same lines of inference.

There can be little doubt that Ptolemy used metrical analysis as a problem-solving tool in much the same way as we would use the manipulation of symbolic formulae. Ptolemy approaches astronomical problems through analysis of his geometric models. The trigonometric methods allow the models to produce numerical parameters and tables. Ptolemy's understanding of the relationship between geometric methods and his trigonometry is probably best expressed in his use of two phrases that we will investigate in the next section.

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[121] συντεθήσεται δὴ ἀκολούθως τῇ ἀναλύσει οὕτως, Schöne [1903, 30 ff.] or Bruins [1964, 43 ff.]. Hero's practice in this regard is discussed in Cuomo [2001, 166 - 168], who sees no reason to question Hero's terminology.

[122] See, for example, *On Meas.* III 9 where there is no problem with the analysis but the "synthesis" is both unmotivated by the analysis and incorrect, Schöne [1903, 158 - 160].

[123] See Knorr [1986, 339 - 381] and Berggren and Van Brummelen [2000] for discussions of ancient analysis that are based on the ancient mathematical texts.

[124] See, for example, Pappus' discussion of analysis in the beginning of *Coll.* VII, Jones [1986, 83], and the diagram provided by Berggren and Van Brummelen [2000, 13].

2.2.1 *Dia tōn grammōn* and *dia tōn arithmōn*

Ptolemy has two technical expressions that are related to his trigonometric approach to solving problems. These are "by means of numbers," *dia tōn arithmōn*, and "by means of lines," *dia tōn grammōn*. An investigation of how Ptolemy employs these phrases gives insight into how he thinks about his mathematical practice. In Ptolemy, these two expressions are generally used in a technical sense. Nevertheless, related expressions, such as *dia grammōn*, *di arithmōn*, and the adjectives *grammikos* and *arithmētikos* should also be considered. It will also be useful to make some comments about the use of these expressions in other technical authors.

The phrase *dia tōn arithmōn* is used twice in the *Almagest*.[125] In both cases, it refers to making trigonometric calculations in order to confirm a statement which has already been established through pure geometry, specifically the equivalence of the epicyclic and eccenter hypotheses. Here, it is best understood as "through numerical computation." It is used in the same sense by both Pappus and Theon in their commentaries, and by Pappus in *Coll.* III.[126] The phrase is also used in the title of a chapter in the *Harmonics* where it simply denotes the use of numbers to give specific instantiations of harmonic ratios in various divisions of the ecliptic.[127] It is not clear, however, that the chapter titles in Ptolemy's works were written by him. The adjective *arithmētikos* is also used by Ptolemy to refer to calculation.[128]

Pappus also uses the phrase *dia tōn arithmōn* in more literal sense in Book II of the *Collection*. A series of theorems in number theory are enunciated in general terms and then exhibited by means of a single numerical example. The example is introduced with variations on the words, "but it is obvious through the numbers," ἔστι δὲ φανερὸν διὰ τῶν ἀριθμῶν.[129] This usage clearly has a different grammatical function. It simply refers

[125] Heiberg [1916, p. 1, 241 & 339] and Toomer [1984, 157 & 211].
[126] Rome [1943, 17, 29, 57, 58, 61, 111, 123, 776, 282, 293, 890, 891 & 1084] and Hultsch [1878, 42].
[127] Düring [1930, 110] and Barker [1989, 389].
[128] For example, in *Alm.* XIII 3, a step in the calculation is justified by reference to an "arithmetical lemma," Toomer [1984, 604].
[129] Hultsch [1878, 6 ff.].

to the use of numbers to clarify the situation.

The phrase *di arithmōn* is not used by Ptolemy, but it occurs in a number of other mathematical writers. It is also used in a general sense to refer to a statement which may be exemplified, or justified, through numbers and also in the more technical meaning referring to calculation. Hero uses the phrase in both senses.[130] Pappus uses it once, along with *dia tōn arithmōn*, in a non-technical sense, however there is an interesting use of the phrase relating to the astronomical work of Hipparchus that I will argue below refers to calculation.[131] Eutocius uses the phrase to indicate Apollonius' method of finding the ratio of the circumference of a circle to its diameter with great precision.[132] He is certainly talking about some sort of calculation.

Ptolemy's use of the phrase *dia tōn grammōn* is somewhat more interesting. This expression also generally refers to a calculation, but one which is explicitly carried out by means of a geometric model.[133] It was first recognized as technical terminology by Luckey.[134] A telling instance is that mentioned above in connection with the derivation of the chord table.[135] Before he introduces the squeeze procedure for finding $\mathrm{Crd}(1/2°)$, Ptolemy says that if $\mathrm{Crd}(\alpha)$ is given then $\mathrm{Crd}(1/3\ \alpha)$ cannot be given *dia tōn grammōn*.[136] This usage makes it clear that *dia tōn grammōn* means something more specific than simply "by means of lines" or "geometrically." The ancients, in fact, had means of trisecting angles that were geometrical and effected by means of lines but none of these

[130] The two uses in *Definitions* simply mean "through numbers," Heiberg [1907b, 140]. There is a use in *On Measurements*, however, that clearly refers to calculation, Schöne [1903, 160] or Bruins [1964, 83].

[131] Hultsch [1878, 78 & 600]. See page 176 for a discussion of the passage in Pappus' *Collection* that concerns Hipparchus.

[132] Heiberg [1973, vol. 3, 258].

[133] Ptolemy uses διὰ τῶν γραμμῶν and related phrases in a technical sense sixteen times in the *Almagest* and twice in the *Analemma*, Heiberg [1916, p. 1, 32, 42, 251, 335, 349, 380, 383, 416, 449; p. 2, 193, 198, 201, 210, 321, 426, 427 & 429], Toomer [1984, 48, 54, 165, 209, 233, 235, 252, 264, 410, 413, 414, 421, 484, 544, 545 (twice)], Heiberg [1907a, 202 & 203] and Edwards [1984, 107 & 108]. In his description of the construction of the star globe, *Alm.* VIII 3, Ptolemy uses the phrase "by means of these lines," διὰ τούτων τῶν γραμμῶν, to indicate the literal use of lines to effect the desired fabrication, Heiberg [1916, p. 2, 181] and Toomer [1984, 405].

[134] Luckey [1927, 17 - 46].

[135] See page 89.

[136] Toomer translates this with "by geometric means," Toomer [1984, 54]. Taisbak [2003, 13 & 29] briefly discuses this passage in connection with the ancient concept of *given*.

methods are susceptible to numerical computation.[137]

Ptolemy uses the phrase to signify geometrical means that permit numeric computation, either in the form of metrical analysis or numeric calculation. In the two uses of the phrase in the *Analemma*, it refers to metrical analyses that have no accompanying computations.[138] On the other hand, it can also be used to refer to a trigonometric calculation which is not accompanied by an analysis. Before he begins his discussion of Mars, Ptolemy states that he determined the parameters of the lunar model *dia tōn grammōn*.[139] These computations are not supported by an analysis.[140] Pappus and Theon, in their commentaries, use the phrase numerous times in the same manner as Ptolemy.[141] It is first used as a technical phrase by Hipparchus in his *Commentary on the* Phaenomena *of Aratus and Eudoxus.* I have argued elsewhere that Hipparchus' idiom is the same as Ptolemy's and that Ptolemy is following Hipparchus.[142]

Less technical uses of *dia tōn grammōn* are preserved by both Galen and Pappus. Galen, in the section of *On the Usefulness of the Parts of the Body* referred to in the previous chapter, uses the expression to indicate reasoning which is based on a diagram but is not strictly deductive. In *On the Use.* X 12, which is based in the same mathematical tradition as Ptolemy's *Optics*, Galen introduces diagrams to explain some geometric aspects of his visual theory, including some properties of binocular vision.[143] He then refers to the statements made with the help of these diagrams as being shown *dia tōn grammōn*.[144] Here the expression simply means "geometrically" or, literally, "by means of lines."

Pappus, in his *Collection*, uses the phrase in a somewhat more peculiar manner. In *Coll.* II prop. 15, Pappus states that a number theoretic proposition has been demon-

[137] See Hultsch [1878, 272 - 288] for various angle trisections.
[138] Heiberg [1907a, 202 & 203] and Edwards [1984, 107 & 108].
[139] Toomer [1984, 484]. Here, Toomer translates διὰ τῶν γραμμῶν as "geometrically."
[140] Toomer [1984, 190 - 204].
[141] Rome [1943, 38, , 59, 61, 171, 445, 449, 451, 453, 488, 497, 498, 498, 499, 504, 520, 527, 544, 577, 584, 596, 597, 668, 857, 872, 878, 895, 896, 907, 925, 965, 968, 977, 988 & 1984].
[142] Sidoli [2004a].
[143] Helmreich [1909, v. 1, 92 - 103] and May [1968, 490 - 498].
[144] Helmreich [1909, v. 1, 99 & 101]. May translates literally as "by means of lines," May [1968, 495 - 496].

strated *dia tōn grammōn*.[145] Although no figures appear in *Coll.* II, Pappus must be referring to a proof in the tradition of Greek number theory which represented numbers as lines, in the vein of *Elem.* VII - IX.[146] In this tradition of Greek mathematics, the geometrical characterization of the objects is not essential. The reference is to a statement of number theory that is embedded in a rigorous, systematic text. In this case, Neugebauer's translation, "by rigorous methods," probably best conveys Pappus' meaning.[147]

For Ptolemy, *dia tōn grammōn* seems to be the broader expression. It is used for the whole process of making determinations based on a geometric model. It signifies the use of either metrical analysis, trigonometry or simple computation based on geometric properties, such as the construction of the chord table. On the other hand, *dia tōn arithmōn* is restricted to the application of numbers to the geometric model. Ptolemy's use of these expressions shows that he sees his trigonometry as a calculating tool, useful for obtaining concrete results from his geometric models, not as a problem-solving method in its own right.

Since methods which are *dia tōn arithmōn*, or one of its cognates, must be read as a subset of methods which are *dia tōn grammōn*, it will be useful to investigate what kinds of scientific, or mathematical, methods should be understood as opposed to these. There are two passages in Ptolemy's work that help us in this regard.

In *Harm.* I 5, Ptolemy reproduces two proofs which the Pythagoreans advanced for associating (προσάπτουσι) the octave with 2 : 1, the fifth with 3 : 2 and the fourth with 4 : 3.[148] The first of these proofs is described as *logikōteron*, "more rational," while the second is *grammikōteron*, "more geometrical," (literally "more linear"). The first proof involves making analogies between the intervals and the ratios both with regard to both magnitude and value judgments as to which is best. The second proof uses diagrams but

[145] Hultsch [1878, 4].
[146] Ver Eecke is of the same opinion and assumes that the proposition to which Pappus refers is in the lost text of Apollonius, Ver Eecke [1982, 2, n. 5].
[147] Neugebauer [1975, 302 & 771, n. 1].
[148] Düring [1930, 11 - 12] and Barker [1989, 285 - 286].

only in the same superficial way as the numerical books of the *Elements.* There is nothing geometrical about this second proof. On the other hand, it is a perfectly rigorous series of deductions given the hypotheses which the Pythagoreans put forward concerning the nature of the ratios associated with consonants; hypotheses with which Ptolemy does not entirely agree. In this passage we should read *grammikōteron* as meaning "more rigorous." It is juxtaposed to reasoning which is "more rational," that is based on analogy and an appeal to ideas about what is inherently right and good. This material is helpful but it does not suggest any mathematical alternatives to the geometrical approach.

A passage in the *Tetrabiblos* is more useful for our purposes. The context is a critical discussion of the "Egyptian" method of assigning arcs of the zodiacal signs to each of the five planets.[149] According to Ptolemy, some authors had claimed that the sum of the rising times of the set of arcs assigned to any given planet would be the same for all latitudes. Ptolemy is critical of this view on two points; only the first concerns us here. He proceeds as follows.[150]

> For first, by following the common practice (τῇ κοινῇ πραγματείᾳ) and constructing (συνισταμένῃ) [the rising times] by uniform differences of the ascensions (which is not even slightly close to the truth), and according to which, at the latitude of Lower Egypt, they want both the zodiacal signs of Virgo and Libra to rise in 38⅓ time degrees, and Leo and Scorpio in 35, while it was shown, by means of lines (*dia tōn grammōn*), that the latter [signs] rise in more than 35 time degrees, and Virgo and Libra in less.

The practice referred to in this passage is almost certainly that found in Hypsicles' *Ascensions.*[151] As mentioned above, this work uses theorems on numerical series to justify the application of a Babylonian scheme for rising times of arcs of the ecliptic adapted to the latitude of Alexandria. The numbers given above are exactly those found in *Ascensions.* As Ptolemy's remark asserts, however, the values in Hypsicles' series are

[149] Robbins [1994, 90 - 97].
[150] Robbins [1994, 94]. The translation is my own.
[151] The text is de Falco, Krause and Neugebauer [1966]. Evans [1998, 121 - 125] and Neugebauer [1975, 715 - 718] provide good discussions.

poor and the overall pattern of the numbers is not a good model of the phenomena. Both Hypsicles' linear zigzag function and Ptolemy's full geometric solution are discussed in more detail in the next chapter.[152]

It is sufficient here to say that Ptolemy's understanding of rising times is based on the combined applications of trigonometric methods and spherical geometry. His solution to the problem uses both metrical analysis and computation. Moreover, Ptolemy's approach "by means of lines" produces fairly accurate results. Hypsicles, on the other hand, bases his results on three proposition that have to do with the sums of numerical series. While these propositions are correct, they have little or no bearing on the issues at hand. In fact, they are simply a theoretical justification of a received Babylonian zigzag function. For these reasons, Ptolemy's geometrical methods must be read as juxtaposed to the methods of numerical astronomy which had been received from the Near East and were still widely practiced by Greek astronomers in the Roman period.

2.2.2 The geometrical determination of the first lunar anomaly

The relationship between Ptolemy's trigonometric computations and the underlying metrical analyses is not always straightforward. Some of his computations are so involved as to make the steps of the corresponding analysis hard to justify within the context of the analytic corpus as it has come down to us. Nevertheless, I have no doubt that these more difficult problems were also solved through analysis before calculation was carried out. It will be helpful to look at one such computation as an example. I will sketch the overall approach and provide a metrical analysis along the lines of those given in Ptolemy's writings. This will allow us to see how analysis functioned as a problem solving medium within applied Greek mathematics.

In *Alm.* IV 6, Ptolemy uses two sets of eclipse observations to determine the parameters of his preliminary lunar model and to demonstrate that these parameters are

152 See Section 3.5. Ptolemy's presentation of the rising times of arcs of the ecliptic was also discussed above, see page 50.

constant over long periods. Neugebauer gives a good summary of Ptolemy's method; I follow his notation.[153] The method that Ptolemy uses was not devised by him and goes back at least as far as Hipparchus. In fact, it is a useful, general method that allows one to determine the basic structure of either an epicyclic or eccentric model, given three observations and a value for the mean motion. In *Alm.* IV 6, Ptolemy first applies the procedure to a set of ancient observations made in Babylon and then again to a set of contemporary observations which he claims to have made himself in Alexandria. I will discuss the second, Alexandrian application of the procedure.

Ptolemy chooses three eclipse observations over a three-year period. Mid-eclipse is calculated, presumably based on measurements of the beginning and end.[154] The lunar longitude it left to the reader to infer from a calculation of the solar longitude at the moment of mid-eclipse. The three eclipses are as follows:[155]

Eclipse	**Date**	**Magnitude**	**Position**
I	Hadrian 17, Pauni 20/21 [133 May 6/7]	total	$\lambda_{\mathrm{I}} = 43;15°$
II	Hadrian 19, Choiak 2/3 [134 Oct 20/21]	5/6	$\lambda_{\mathrm{II}} = 205;10°$
III	Hadrian 20, Pharmouthi 19/20 [136 Mar 5/6]	1/2	$\lambda_{\mathrm{III}} = 344;5°$

Because the moon moves rapidly, the time interval, Δt, is calculated according to Ptolemy's discussion of the equation of time in *Alm.* III 9.[156] The change in lunar longitude, $\Delta\lambda_{☾}$, is calculated directly from the three lunar positions. The change in lunar anomaly and mean longitude, $\Delta\alpha_{☾}$ and $\Delta\overline{\lambda}_{☾}$, are found by entering Δt in the table provided in *Alm.* IV 4. For our two intervals this gives the following results:[157]

Interval	Δt	$\Delta\lambda_{☾}$	$\Delta\overline{\lambda}_{☾}$	$\Delta\alpha_{☾}$
I → II	$1^y166^d23^5/_8{}^h$	161; 55°	169; 37°	110; 21°
II → III	$1^y137^d5^1/_2{}^h$	138; 55°	137; 34°	81; 36°

153 Neugebauer [1975, 73 - 78]. See also Pedersen [1974b, 172 - 179].
154 Steele [2000, 102].
155 Toomer [1984, 198].
156 Toomer [1984, 171 - 172].
157 Toomer [1984, 198 - 199]. The values that Ptolemy gives for $\alpha_{☾}$ and $\overline{\lambda}_{☾}$ are off by 0; 1° in three cases from the values we find through *Alm.* IV 4.

On the assumption of a single lunar anomaly, the two arcs $\Delta\lambda_{☾\ \mathrm{I}\to\mathrm{II}}$ and $\Delta\lambda_{☾\ \mathrm{II}\to\mathrm{III}}$ are made up of the motions of the center of the epicycle, $\Delta\overline{\lambda}_{☾\ \mathrm{I}\to\mathrm{II}}$ and $\Delta\overline{\lambda}_{☾\ \mathrm{II}\to\mathrm{III}}$, and the apparent motion of the moon on the epicycle, $\delta_{\mathrm{I}\to\mathrm{II}}$ and $\delta_{\mathrm{II}\to\mathrm{III}}$; that is $\Delta\lambda = \Delta\overline{\lambda} + \delta$. Ptolemy considers all three observations rotated onto a single position of the epicycle by taking $\delta = \Delta\lambda - \overline{\Delta}\lambda$ as the apparent angular distance between any two eclipses. The arcs on the epicycle, $\overline{\delta}$, will then simply be equal to $\Delta\alpha$. This gives two sets of three angles which will be used to determine the parameters of the model. These angles are as follows:

Interval	δ	$\overline{\delta}$
$\mathrm{I} \to \mathrm{II}$	$161;55^\circ - 169;37^\circ = -7;42^\circ$	$110;21^\circ$
$\mathrm{II} \to \mathrm{III}$	$138;55^\circ - 137;34^\circ = 1;21^\circ$	$81;36^\circ$
$\mathrm{III} \to \mathrm{I}$	$7;42^\circ - 1;21^\circ = 6;22^\circ$	$360^\circ - (110;21^\circ + 81;36^\circ) = 168;3^\circ$

These given angles are then applied to the model. Let A, B and G be the three eclipses on epicycle ABG as seen from an observer on earth at D; see Figure 2.12. The epicycle rotates clockwise from A to B while the center of the epicycle, K, is carried counterclockwise on the deferent circle with center D.

The position of the three eclipses on the model is found on the basis of the direction, or sign, of δ. Ptolemy considers a δ which we would call negative to have a retrograde effect with respect to the motion of K on the deferent circle; while a δ that we would call positive has a progressive effect. Neugebauer gives a complete analysis of how this information can be used to situate the apogee and perigee of the model.[158] Ptolemy, however, appears to base his considerations on two fairly obvious consequences of the model: (1) perigee cannot be on an arc that is less than 180° and produces a retrograde effect and (2) apogee cannot be on an arc that is less than 180° and produces a progressive effect.[159] In Figure 2.12, if M is the perigee then *any* arc less than 180° which contains M must start somewhere on $\overset{\frown}{LBM}$, in other words in the region below DL. In order for this arc to contain M it must have an end point somewhere on $\overset{\frown}{MAL}$, in other words in

[158] Neugebauer [1975, 75].
[159] Toomer [1984, 193 & 200].

the region above DL. Hence this arc begins somewhere below DL and ends somewhere above it so that the arc always produces a progressive effect with respect to the motion of K on the deferent circle. Similar reasoning demonstrates Ptolemy's rule for apogee. It is clear that these two rules are sufficient for placing the apogee and perigee on the correct arc of the epicycle or eccenter.

In the present situation, both $\delta_{\mathrm{II}\rightarrow\mathrm{III}} = \overset{\frown}{BG} = 1;21°$ and $\delta_{\mathrm{III}\rightarrow\mathrm{I}} = \overset{\frown}{GA} = 6;21°$ produce a progressive effect and cannot contain apogee. Hence, apogee must lie on $\overset{\frown}{AB}$. We are now given the values and geometric configuration of $\overset{\frown}{AB}$, $\overset{\frown}{BG}$, $\overset{\frown}{GA}$, $\angle ADG$, $\angle BDG$ and $\angle GDA$. It remains to determine the parameters of the model. Ptolemy finds both the ratio of the radius of the deferent, R, to the radius of the epicycle, r, and the location of the apogee with respect to one of the eclipses. For our purposes, it will suffice to look at the determination of the first parameter, $R : r$.

Since all three arcs of circle ABG are given, the ratios of the chords that subtend these angles to the radius of the circle, r, will be given, by *Data* 87. On the other hand, since the three angles at D are given, right triangles can be constructed containing these angles and the sides of these triangles to one another can be found, by *Data* 40 - 43. The basic strategy is to relate these two sets of given lengths through the properties of lines that fall on a circle from an external point, *Elem.* III 36, and the addition of a line to a bisected line, *Elem.* II 6. There are no metrical analyses treating a problem as involved as this in any ancient text; nevertheless, we can reconstruct one following Ptolemy's approach and based on the metrical analyses that are preserved.

In Figure 2.12, if $\overset{\frown}{AB}$, $\overset{\frown}{BG}$, $\overset{\frown}{GA}$, $\angle ADB$, $\angle BDG$ and $\angle GDA$ are given then $DK : KL$ is also given. We begin by constructing the right triangles that will be used to determine the lengths of the lines falling on the epicycle. Join BG, EB and EG. Drop $EH \perp DG$, $EZ \perp DB$ and $G\Theta \perp EB$. Draw DL through the epicycle such that K is its center, L the apogee and M the perigee. Now, since $\overset{\frown}{AB}$ is given, $\angle AEB$ is given [*Elem.* III 20]; and hence $\angle EBD$ is given [*Elem.* I 32]. Therefore, in R$\triangle EDZ$, $EZ : DE$ is given [*Data* 40], while, in R$\triangle EZB$, $EZ : BE$ is given [*Data* 40], therefore $BE : DE$ is given [*Data*

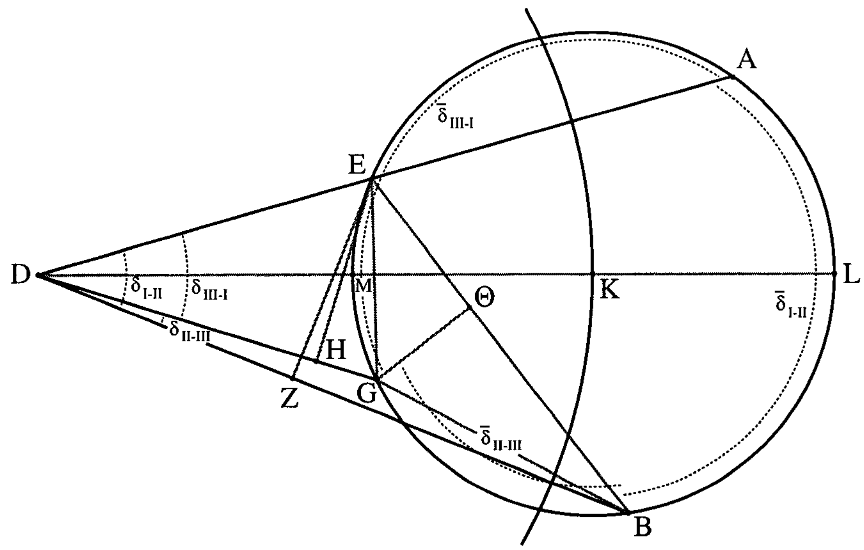

Figure 2.12: Diagram for a metrical analysis deriving the parameters of the simple lunar model from three observations. This figure combines elements of Toomer's Figs. 4.8 & 4.9, Toomer [1984, 199 & 202]. The primary lines of the model are drawn in black. Grey lines are constructions made for the sake of the mathematical derivation. The given angles are denoted by dashed lines.

8]. Now, since $\overset{\frown}{GA}$ is given, $\angle AEG$ is given [*Elem.* III 20]; and hence $\angle EGH$ is given [*Elem.* I 32]. Therefore in R$\triangle EDH$, $EH : DE$ is given [*Data* 40], while in R$\triangle EHG$, $EH : GE$ is given [*Data* 40], therefore $GE : DE$ is given [*Data* 8].

Again, since $\overset{\frown}{BG}$ is given, $\angle BEG$ is given [*Elem.* III 20]. Hence, in R$\triangle EG\Theta$, $\Theta G : GE$ and $\Theta E : GE$ are given [*Data* 40]. Therefore both $\Theta G : DE$ and $\Theta E : DE$ are given [*Data* 8]. Therefore, if DE is assumed to be given, ΘB will be given [*Data* 2 and $\Theta B = BE - \Theta E$], and hence BG will be given [*Elem.* I 47].

Now, since $\overset{\frown}{BG}$ is given, $BG : KL$ is given [**Chord Table**, Taisbak's *Data* 87*],[160] and therefore $DE : KL$ and $GE : KL$ are given [*Data* 8]; and so $\overset{\frown}{GE}$ is given [**Chord Table**, Taisbak's *Data* 88*].

Again, since $\overset{\frown}{GA}$ is given, $\overset{\frown}{EA}$ is given [$\overset{\frown}{EA} = \overset{\frown}{GA} - \overset{\frown}{GE}$], and therefore $EA : KL$ is given [**Chord Table**, Taisbak's *Data* 87*]. Now, if KL is assumed to be given, then

[160] See Taisbak's note on the relation between *Data* 87 & 88 and the mathematics of the chord table, Taisbak [2003, 226]. Taisbak's propositions *Data* 87* & 88* are precisely those needed for metrical analysis.

EA, DE and AD are given [*Data* 2 and $AD = DE + EA$]. Then, since $(AD \times AE) = (LD \times LM)$ [*Elem.* III 36], therefore $(LD \times LM)$ is given. Again, since $(LD \times LM) + KM^2 = DK^2$ [*Elem.* II 6] and $KM = KL$, therefore DK is given. Therefore $DK : KL$ is given, which was to be shown.

This analysis is modeled on the computation in *Alm.* IV 6 and I make no claim that it represents the heuristic approach of Ptolemy or any other ancient mathematician in actually solving this problem. Nevertheless, I believe it is more accurate than our trigonometric formulation in representing how a Greek mathematician would have thought about the solution to this problem. It demonstrates the general structure of the solution. It shows how angles and arcs are used to relate all line segments to two essentially arbitrary lines, DE and KL, which are finally related to one another. Propositions of the *Data* and *Elements* are combined with the mathematics of the chord table, exhibiting the way in which these practices affected analytic problem solving.

It will not advance our purposes to follow through Ptolemy's actual calculations. It will be sufficient to remark that the usual practices of ancient trigonometry are employed. Instead of dealing with given ratios, Ptolemy works with given lengths where the hypotenuse of a right triangle is assumed to be 120^{p}. For the first part of the computation all lengths are stated in terms of $DE = 120^{\text{p}}$, in the second part they are all stated in terms of $KL = 60^{\text{p}}$. As usual, semidegrees are used to move between angles and arcs without actually constructing circumcircles.[161]

If we wish to understand the trigonometry of Hipparchus, Menelaus and Ptolemy in the context in which it was originally practiced, we must situate it in the problem solving techniques of metrical analysis. Just as we would solve a problem in plane trigonometry using trigonometric equations in both known and unknown variables, the ancient mathematicians would proceed using diagrams and the techniques of metrical analysis. In investigating Ptolemy's more advanced models, such as those for the planets, it is often

[161] In terms of the final results that Ptolemy states for the ratio $R : r$ in both the Babylonian and Alexandrian eclipse trio, it makes no difference whether one calculates through with the initial values that Ptolemy states or those derived from the table of lunar mean motions, *Alm.* IV 4.

helpful to begin by producing an analytical summary of his solution. This allows the underlying structure of his reasoning to surface.

2.3 Conclusion

In the first part of this chapter, we have examined episodes in the development of Greek trigonometry. We have seen how two, originally separate, applied and theoretical traditions were eventually fused in the combined techniques of trigonometric computation and metrical analysis. In the second part of this chapter, we have situated ancient trigonometric practices within the broader context of Greek mathematics.

The theoretical problem solving techniques developed by Greek mathematicians in the Classical period and first canonized in Euclid's *Data* provide no means of coping with mensuration problems. Although there was a tradition of craft knowledge involving mensuration techniques as later represented by Hero's *On Measurement* and the Heronian *Geometry*, these techniques could not handle the fundamental problem of ancient trigonometry and appear to have been unsatisfactory to the theoretical Greek mathematicians.[162] Presumably in response to this unsatisfactory situation, geometers in the early Hellenistic period developed a rigorous tradition of trigonometric approximation techniques that were used to solve problems in mathematical astronomy. These approximation techniques, however, were both cumbersome and inexact and the need must have been felt for more precise computational tools. In the middle of the Hellenistic period, this need was answered by importation of Babylonian arithmetical methods and the construction of chord tables. It became clear that chord tables allowed trigonometric computations to be carried out for problems that could previously only be solved through geometric construction. This led to the mature stage of Greek trigonometry in the fusion of trigonometric calculation with metrical analysis. In this sense, the early development of trigonometry through chord tables was simply the production of new

[162] Høyrup [1996] and Høyrup [1997].

tools which generated arithmetic solutions to problems which could already be settled geometrically.

This understanding of the relationship between the numerical methods of trigonometry and the geometric methods of metrical analysis is preserved in Ptolemy's use of the phrases *dia tōn grammōn* and *dia tōn arithmōn.* The techniques of trigonometry through chord tables are generally seen in light of the later development of the classical trigonometric relations among the Indians and Arabs which led to our modern trigonometric functions. In fact, however, the extent of the similarity lies in the trivial observation that $\mathrm{Crd}(\alpha) = 2\sin\frac{\alpha}{2}$, where the radius of the circle is 1. This observation can be inferred directly from the medieval geometric definition of $\sin\alpha$. By examining the relationship between Ptolemy's trigonometric practice and metrical analysis for one of the canonical problems of ancient mathematical astronomy we get a glimpse of the methods that the ancients would have actually used in solving metrical problems. This has allowed us to see more clearly the relationship between ancient trigonometry and its theoretical underpinnings.

The material in this chapter, together with that in Chapter 1, lays the foundation for all the mathematics we encounter in the Ptolemaic corpus. Together with spherical geometry, which will be the subject of the next two chapters, ancient trigonometry and metrical analysis are the mathematical techniques that Ptolemy uses to solve all mathematical problems in the sciences.

Chapter 3

Spherical Geometry and Spherical Astronomy

This chapter investigates Ptolemy's approach to spherical astronomy and situates his work in the context of other Greek methods treating the same set of problems. The chapter begins with an overview of the spherical astronomy of *Alm.* I 13 - II 13 & VIII 5 - 6 followed by a lengthy discussion of the major versions of the Menelaus Theorem now in print. Ptolemy's own use of the Menelaus Theorem is analyzed as the core theorem of his metrical methods. The logical and mathematical cohesion of the spherical astronomy in the *Almagest* is investigated in order to show how the text situates itself within the context of systematic mathematical treatises. A final section deals with the important historical issue of the relationship between Ptolemy's methods and the methods of previous Greek mathematical astronomers.

For the purpose of referencing units of the mathematical and logical argument of Ptolemy's spherical astronomy, I have provided a division of the text of *Alm.* I 13 - II 13 & VIII 5 & 6. The specifics of this division are given in Appendix D. Wherever I refer to the *Almagest* by book and section, I use these more precise numbers.

3.1 Ptolemy's approach to spherical astronomy

The first astronomical topic that Ptolemy treats systematically is what we call spherical astronomy. This is a disparate collection of topics and problems which we group under the name of spherical astronomy because most of Ptolemy's Greek predecessors and Ptolemy himself handled them through various uses of spherical geometry. Topics grouped in spherical astronomy include the successive rising and setting of stars, the length of the longest and shortest periods of daylight at various geographic latitudes, and the rising and setting times of the signs of the zodiac. Ptolemy's approach to these topics differs from that of all the other texts which we still possess. Most Greek approaches to these topics produced qualitative results established by geometric reasoning. Examples of these texts are the works of Autolycus; Euclid's *Phaenomena*; the *Spherics*, *On Habitations* and *On Days and Nights* of Theodosius; and the *Spherics* of Menelaus.[1] Hypsicles is the one exception to this general practice. His *Ascensions* is a slim work which gives numerical results for the rising times of the zodiacal signs at the latitude of Alexandria derived from a few theorems on the sums of ordered series.[2] The works by Hipparchus and Menelaus on spherical astronomy have been lost. This is unfortunate; because these are almost certainly the works that Ptolemy studied and used as his point of departure.

Ptolemy's treatment of spherical astronomy in the *Almagest* makes use of a set of mathematical techniques which modern scholars refer to as spherical trigonometry, despite the fact that the fundamental figure of Ptolemy's approach is a concave, spherical quadrilateral. Ptolemy makes use of a single fundamental theorem, the so-called Menelaus Theorem, combined with the techniques of the chord table and metrical analysis to obtain numerical solutions to all of the problems which came under this branch of Greek astronomy. In this regard, Ptolemy's approach is unique among the texts that have come down to us. This makes Ptolemy our primary witness for the development

[1] The text of Autolycus is Mogenet [1950], a sometimes poor English translation is given in Bruin and Vondjidis [1971]; the text of Euclid is Menge [1916], for an English translation see Berggren and Thomas [1996]; for Theodosius see Heiberg [1927] and Fecht [1927]; for Menelaus see Krause [1936] and Björnbo [1902].

[2] de Falco, Krause and Neugebauer [1966].

of the techniques of spherical trigonometry among the Greeks. In shorter works, the *Planisphaerium* and the *Analemma*, Ptolemy deals with related topics using other mathematical techniques. These treatises are discussed in the following chapter.

It will be helpful to give an overview of Ptolemy's method of investigating the topics of spherical astronomy in the *Almagest*.[3] Because the *Almagest* is a systematic treatise, the material is not treated in a haphazard order; the treatment follows a logical chain of inference, each new topic building upon what has gone before. The few purely geometric theorems which are employed are introduced as preliminaries to the discussion of particular astronomical problems. Tables are calculated with just as much precision as will be required for that accuracy which is desired when they are used later in the work. The sections on spherical astronomy can be read as a model of the logical structure of the entire *Almagest* insofar as they constitute a complete treatment of the motion of points in the tropical and equatorial coordinate systems with respect to the local horizon.[4] The ordering of this material is as follows.

1. Geometric preliminaries. *Alm.* I 13.

   Ptolemy solves all problems of spherical astronomy by reducing them to the solution of the arcs of great circles which form a concave quadrilateral know as the Menelaus Configuration. The Menelaus Theorem then allows him to solve for unknown arcs given known arcs. In this chapter, Ptolemy proves the Menelaus Theorem, some lemmas which are used in his proof and some corollaries of these which pertain to metrical analysis.

2. Inclination of the ecliptic to the equator. *Alm.* I 14 & 15.

   In general, Ptolemy prefers the ecliptic coordinate system, however, for the purposes of spherical astronomy the equatorial system is more convenient. In *Alm.* I 14, he shows how to determine the distance of the ecliptic from the equator along a great

[3] There are useful discussions of these topics including exposition through modern spherical trigonometry in Pedersen [1974b, 94 - 121], Neugebauer [1975, 26 - 52] and [75 - 127].

[4] These sections, however, take no account of the motion of the tropics with respect to the equatorial coordinates; they do not treat the precession of the equinoxes.

circle drawn through the equatorial pole for any given longitude. He then tabulates these arc lengths at 1° intervals for a quadrant of the ecliptic from an equinox to a solstice in the table of inclination, *Alm.* I 15. This table gives the declination, δ, of the ecliptic at 1° intervals.

3. Rising times of arcs of the ecliptic at *sphaera recta*. *Alm.* I 16.

   Because *sphaera recta*[5] acts as a norm in many calculations and problems, and because the rising times of the ecliptic at *sphaera recta* can be calculated with no discussion of the inclination of the local horizon to the equator, Ptolemy calculates, and lists, these rising times at 10° intervals for a quadrant of the ecliptic from an equinox to a solstice in the list of rising times at *sphaera recta*. This list gives the right ascension, α, of 10° intervals along the ecliptic. Along with the table of inclination, this list serves as the foundation upon which the rest of the calculations of Ptolemy's spherical astronomy are built.

4. Characteristics of geographic latitude. *Alm.* II 2, 3, 5 & 6.

   For Ptolemy, geographic latitude can be defined in terms of one of the following characteristics: (1) the length of the longest day, M, (2) the elevation of the pole, φ, (3) the arc of the horizon between the rising places of the sun at the equinox and at the solstice, the *ortive amplitude*, η, at a solstice,[6] or (4) the ratio of the length of the shadow cast by a gnomon at noon on a solstice to the length of the gnomon itself, $s : g$. These characteristics are interchangeable. *Alm.* II 2 shows how to calculate η on the solstice given M. *Alm.* II 3 shows, using *Alm.* II 2, how to calculate φ given M and vice versa. *Alm.* II 5 shows how to calculate $s : g$ given φ and vice versa. *Alm.* II 6 then proceeds to enumerate 36 specific geographic latitudes from the equator to the pole giving a number of the characteristics for each.

[5] See page 20, note 35 for a definition of *sphaera recta*.

[6] *Ortive amplitude*, η, is defined as the arc of the horizon between sunrise on any given day and sunrise on the equinox; hence, η changes throughout the year. Only the value of η on the solstice can be taken as a characteristic of latitude.

5. Rising times at *sphaera obliqua*. *Alm.* II 7 & 8.

   Based on the forgoing discussion and a few preliminaries on the symmetry of the zodiacal signs, Ptolemy calculates the rising times of 10° arcs of the ecliptic for the latitude of Rhodes, $\varphi = 36;0°$. This serves as an example for the calculation of rising times in general. He then presents the table of rising times in *Alm.* II 8 which tabulates rising times as a function of two variables: 10° arcs of the ecliptic and eleven geographic latitudes.

6. Problems in spherical astronomy. *Alm.* II 9.

   Ptolemy demonstrates the versatility of his table of rising times by showing how it can be used to solve a number of problems in spherical astronomy. For instance, he shows how to find the length of a given period of daylight, or how to find the degree of the ecliptic which is rising at any given time.

7. Angles between the ecliptic and local great circles. *Alm.* II 10 - 13.

   Ptolemy calculates, at 30° intervals of the ecliptic, the angles that the ecliptic makes with the meridian, the horizon and an altitude circle. *Alm.* II 10 calculates the angles between the ecliptic and the meridian at the beginning of each sign. These calculations are valid for all geographic latitudes. *Alm.* II 11 calculates the angles between 30° intervals of the ecliptic and the horizon for the latitude of Rhodes. *Alm.* II 12 shows how to calculate (1) the zenith distance along a vertical circle cut off by the ecliptic at the beginning of a zodiacal sign for the latitude of Rhodes one hour before noon, and (2) the angle between the vertical and ecliptic circles at this same point. With these as examples of the method of calculation, Ptolemy gives a table of zenith distances and angles between the ecliptic and vertical circles, *Alm.* II 13. This is one of the most complex tables in the *Almagest*. It tabulates the stated angles and zenith distances as a function of three variables: geographic latitude, 30° intervals of the ecliptic, and hours from sunrise (or sunset) to noon. This table is used by Ptolemy in his treatment of lunar parallax.

8. The movement and visibilities of the fixed stars. *Alm.* VIII 5 & 6.

   The last section of Ptolemy's spherical astronomy comes later in the *Almagest* after the fixed star catalog. *Alm.* VIII 5 shows how to calculate the points of the ecliptic that rise, set and culminate with any given star. *Alm.* VIII 6 sketches a general theory of stellar visibility. Ptolemy shows how to calculate the *arcus visionis* from a given longitudinal distance of the sun below the horizon and he then shows how to use the *arcus visionis* to calculate first and last visibilities for any geographical latitude. Ptolemy's treatment of this material is cursory and he probably only discusses it here in deference to the traditional topics of spherical astronomy.[7] In fact, this material is the primary topic of his *Phases of the Fixed Stars*.[8]

One of the striking things about Ptolemy's exposition of spherical geometry is how little he makes use of the theorems in the *Spherics* of Theodosius and Menelaus. The major exception to this is Menelaus' *Spher.* III 1, known as the Menelaus Theorem. Ptolemy reduces every problem in spherical astronomy to the use of this fundamental theorem.

The fundamental theorem concerns a figure composed of four arcs of great circles arranged in a spherical concave quadrilateral; see Figure 3.1. The theorem relates the legs of this figure by asserting a compound ratio composed of the chords that subtend these legs. The two forms of the fundamental theorem that Ptolemy states in *Alm.* I 13 are

$$\begin{aligned} \mathrm{Crd}(2\,\overset{\frown}{GA}) : \mathrm{Crd}(2\,\overset{\frown}{EA}) \;=& \\ (\mathrm{Crd}(2\,\overset{\frown}{GD}) : \mathrm{Crd}(2\,\overset{\frown}{DZ})) \;\times\;& (\mathrm{Crd}(2\,\overset{\frown}{ZB})) : \mathrm{Crd}(2\,\overset{\frown}{BE})), \end{aligned} \tag{M.T. I}$$

[7] Ptolemy closes his sketches of these methods with the statement, "We think that the above suffices as an indication of the methods in this type of theoretical investigation, enough [at least] so that it cannot be said that we have neglected this topic." Toomer [1984, 416].

[8] Heiberg [1907a, 3 - 67]. A translation of this text is included in the Project Hindsight astrological series, Schmidt and Hand [1993].

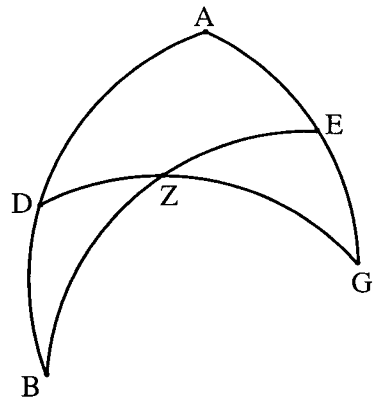

Figure 3.1: The Menelaus Configuration

and

$$\begin{aligned}\mathrm{Crd}(2\,\overset{\frown}{GE}) : \mathrm{Crd}(2\,\overset{\frown}{EA}) \;&= \\ (\mathrm{Crd}(2\,\overset{\frown}{GZ}) : \mathrm{Crd}(2\,\overset{\frown}{DZ})) \;&\times\; (\mathrm{Crd}(2\,\overset{\frown}{DB})) : \mathrm{Crd}(2\,\overset{\frown}{BA})),\end{aligned} \qquad \text{(M.T. II)}$$

where $\mathrm{Crd}(x)$ is the chord that subtends arc x.

The first of these relationships, which Neugebauer called Menelaus Theorem I,[9] relates the parts of an outer leg to the parts of both of the inner legs. The second of these theorems, Menelaus Theorem II, relates different parts of the same outer leg to the parts of one of the inner legs and one of the outer legs. Theon, in his *Commentary on the Almagest,* refers to these two versions of the theorem as the "spherical theorem according to composition," κατὰ σύνθεσιν, and "according to separation," κατὰ διαίρεσιν.[10] Rome has pointed out that this is because, in the plane case, the first term of M.T. II is the "composition," σύνθεσις, of the first term of M.T. I, while the first term of M.T. I is the "separation," διαίρεσις, of the first term of M.T. II, according to Euclid's definition of these operations.[11]

By means of the fundamental theorem and the chord table, Ptolemy is able to solve

[9] Neugebauer [1975, 28].
[10] Rome [1943, 558 f.]
[11] Rome [1933a, 40, n. 1]. Euclid's definitions of the "composition" and "separation of a ratio" are given in *Elem.* V def. 14 & 15, Heiberg [1977, vol. 2, 3]. See also Section 1.3, pages 36 - 36.

problems on the sphere for which we would today use spherical trigonometry. The theorem itself is introduced as an auxiliary to the calculation of the declination, δ, of the points of the ecliptic; however it serves as the primary mathematical tool for the whole of Ptolemy's spherical astronomy.

Although Ptolemy mentions "the geometer Menelaus" in connection with two observations made in Rome in 98 CE , he does not credit him with the theorem that now carries his name.[12] Nor does Theon, in his *Commentary*, mention Menelaus in connection with this theorem, although he was certainly familiar with the *Spherics*. Theon simply refers to the theorem as the "spherical theorem," or the "spherical proof." Menelaus is credited with this theorem by modern scholarship because some form of it exists in all versions of his *Spherics*. In order to develop a sense for how Ptolemy employed the mathematical tools available to him, we will need to look more closely at the text of Menelaus' *Spherics*.

3.2 The fundamental theorem of ancient spherical trigonometry

The theorem generally attributed to Menelaus is found as the first theorem of the third and final book of the *Spherics*. This text develops the geometry of the sphere in the pure geometrical tradition with special attention to astronomical applications.[13] Menelaus' innovations over Theodosius are based on using great circles and the spherical triangles formed by them as the fundamental objects of investigation. *Spher.* I deals with the basic properties of spherical triangles. *Spher.* II uses spherical triangles to give new proofs for a number of Theodosius' theorems with applications to spherical astronomy. *Spher.* III goes beyond the results of Theodosius and develops theorems useful for giving metrical solutions to many of the problems of spherical astronomy. These are results that we would now describe with spherical trigonometry.

[12] See *Alm.* VII 3, Toomer [1984, 336 & 338], for the reports of these observations.
[13] See Nadal, Taha and Pinel [2004] for a good overview of the astronomical applications of the *Spherics*.

The fundamental theorem of ancient spherical trigonometry

Ptolemy's use of the fundamental theorem of ancient spherical trigonometry appears to be one of the few places where we have enough of his predecessors' work to attempt a study of the ways in which Ptolemy turns the results of a systematic geometrical treatise to his own more practical ends. Only a small fragment of the first book of Menelaus' *Spherics* survives in Greek, preserved by Theon of Alexandria.[14] Hence, our principal evidence for the text comes through a number of Arabic translations and editions.[15] We also possess both a Latin and Hebrew translation of one of the lost Arabic texts. All of the versions of the text which will concern us are presented in Figure 3.2; sigla enclosed in a box indicate texts which we still possess.[16] A more complete discussion of the medieval tradition of Menelaus' *Spherics* has been given by Krause and Hogendijk.[17]

The first Arabic translation, **Ü**$_1$, was made from a Syriac version in the 8$^{\text{th}}$ century CE by an unknown translator. No extant copies of this text are known. In the 9$^{\text{th}}$ century CE, an incomplete revision of the text, **Ma**, was made by Abū ʿAbdallāh Muḥammad ibn ʿĪsā al-Māhānī. This text, now lost, ended abruptly with a misunderstood attempt at *Spher.* III 5. In the same century, the great translator, Isḥāq ibn Ḥunayn, made a new translation from the Greek, **bḤ**. This text is also lost. At an unknown date, **H**, a revised version of **Ü**$_1$ was composed by Aḥmad ibn Abī Saʿd al-Harawī. This recension is extant but unpublished.

A Latin translation, **G**, was made by Gerard of Cremona in the 12$^{\text{th}}$ century CE followed in the next century by a Hebrew translation, **J**, by Jacob ben Māḥir. Both of these texts are known to exist but neither has been published. Krause was able to show that both of these translations depended upon the same Arabic text, which he called **D**.[18] He further argued, on stylistic grounds, that **D** was an edition made from two different Arabic versions of the text. He thought that the whole of I, I 1 - 8 & III 1 came from al-Māhānī's revision, **Ma**. The rest of the text was based on Isḥāq ibn

[14] This text is reproduced in Björnbo [1902, 22 - 26].
[15] The most thorough discussion of these texts is still found in Krause [1936, 20 - 86].
[16] The schema provided is an adaptation of that given in Hogendijk [1996, 19].
[17] Krause [1936, 1 - 98] and Hogendijk [1996, 18 - 20]. See also Yussupova [1995] for a short note on two versions of the *Spherics* composed by aṭ-Ṭūsī.
[18] Krause [1936, 10 - 12].

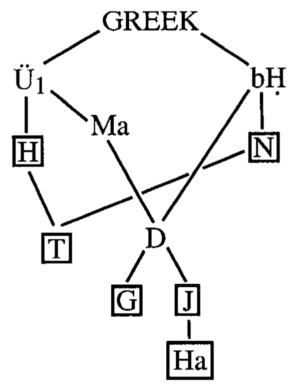

Figure 3.2: Schema for Menelaus' *Spherics*. Adapted from Krause [1936, 86]. Sigla enclosed in a box indicate texts which we still possess.

Ḥunayn's translation, **bḤ**.[19]

Only one Arabic version of the text has been published in a critical edition. Abū Naṣr Manṣūr made a revision of **bḤ** and this has been printed by Krause with a German translation. Krause designated this text as **N**.[20] Abū Naṣr's revisions included a renumbering of the propositions and the addition of commentaries relating Menelaus' geometric theorems to statements of spherical astronomy. In the 13th century, Naṣīr ad-Dīn aṭ-Ṭūsī made an edition of the *Spherics*, **T**, based on the two versions **H** and **N**. An uncritical version of this recension was printed in the Hyderabad editions of Arabic texts.[21] Neither the Latin nor the Hebrew translation have ever been published. In 1758, Edmond Halley published a free Latin translation, **Ha**, made primarily from ben Māḥir's Hebrew with the help of an unknown Arabic exemplar.[22] The two versions of the text which will be used in this study are **N** and **Ha**.

In a recent paper, Nadal, Taha and Pinel study the theorems of the *Spherics* that have direct relevance to spherical astronomy.[23] Their work is based on a French translation of aṭ-Ṭūsī's edition which is being prepared by Taha.[24] They provide the enunciations of

19
20 Krause [1936].
21 Aṭ-Ṭūsī [1940].
22 Halley [1758]. See also Björnbo [1902, 17 - 19].
23 Nadal, Taha and Pinel [2004].
24 Nadal, Taha and Pinel [2004, 393].

the relevant theorems followed by commentary based on that of Abū Naṣr and aṭ-Ṭūsī. Unfortunately, in their remarks, they often utilize trigonometric functions which were unknown to the ancients, so that more work is required if one wants to understand how Menelaus would have proceeded.

The only monograph study of Menelaus' *Spherics* is still Björnbo [1902]. Because he could not read Arabic and he felt that in general Halley's edition was too presumptuous, Björnbo based his study on Halley's version corrected by the manuscripts of Gerard's translation.[25] **N** and **D**, with its derivatives **G** and **J**, diverge with respect to the division and numbering of the propositions. Björnbo has argued that the original numbering can best be recovered by following **G** and **Ha**. The majority of modern scholars however, use the numbering of **N** and I will follow this convention.

For the most part, the differences between the surviving versions of Menelaus' *Spherics* are mathematically trivial; however, in just that theorem which interests us most they show the widest divergence. In *Spher.* III 1, not only are different cases presented in the various versions, but even those cases which are the same have subtle differences in the mathematical exposition.

Although in general Björnbo followed **G** and **Ha**, for this one theorem he followed **N** because he thought that it was the more original version. Moreover, he pointed out that with regards to *Spher.* III 1 there was no essential difference between **G** and **Ha**.[26] Thus, **N** and **Ha** are the two versions of *Spher.* III 1 which will be compared to the proof of the fundamental theorem which Ptolemy presents.

3.2.1 Abū Naṣr's version of *Spherics* III 1

Abū Naṣr introduces Book III of the *Spherics* with a short section titled, "Preface to Book III: A preface through which knowledge of the third part is eased." This introduction contains five lemmas, four of which are used in Abū Naṣr's version of *Spher.* III 1. The

[25] Björnbo [1902, 10 & 14].
[26] Björnbo [1902, 88].

first three lemmas are similar to those found in the *Almagest* in terms of content, ordering and lettering. The last two concern compound ratios and are so rudimentary that it is surprising to find them outside the work of a commentator.

The first lemma which Abū Naṣr gives proves the plane case of M.T. I and is equivalent to *Alm.* I 13.1. The lemma has no enunciation; like Ptolemy's proofs, it is written directly about the figure concerned. The exposition is as follows; see Figure 3.3 (a). If two lines, AB and AG, meet one another in point A, and two lines, BE and GD, are drawn from the two points B and G, intersecting one another in point Z, I say that $GA : AE = (GD : DZ) \times (ZB : BE)$.[27] The proof that follows is similar to Ptolemy's. It follows the same logical argument and the same basic steps. Abū Naṣr supplies a few justifications for steps that Ptolemy assumes as obvious. One example will suffice. The first step of the proof, in either case, is to draw EH parallel to GD. Ptolemy immediately deduces that $GA : AE = GD : EH$.[28] Abū Naṣr gives more detail. He points out that $\angle HEA = \angle DGE$ and that $\angle GAB$ is common to both $\triangle AGD$ and $\triangle AEH$. He then points out that the remaining angle must also be equal so that the triangles are similar. He then asserts the proportion. On the whole, the lemma is not more complicated than many that Greek geometers assumed without proof. Schmidt has pointed out the close relation between this lemma and a lemma which Pappus proves in *Coll.* VII to justify a step in Euclid's *Porisms* that was assumed without proof.[29] Abū Naṣr neither uses nor needs this lemma. He probably included it because it is the first lemma in the *Alm.* I 13.

The second of Abū Naṣr's lemmas is what Theon calls the first circular lemma, *Alm.* I 13.3. Again, the exposition and proof are almost identical to those in Ptolemy. If, in circle ABG with center D, points G, B and A are taken arbitrarily on the circumference such that $\overset{\frown}{GA}$ is less than a semicircle, then $\sin \overset{\frown}{AB} : \sin \overset{\frown}{BG} = AE : EG$; see Figure 3.3 (b).[30] Here $\sin \overset{\frown}{AB}$ is to be taken literally as AZ, $\sin \overset{\frown}{BG}$ as GH. The proof is very close to Ptolemy's with the exception that the switch from $\mathrm{Crd}(2\alpha)$ to $\sin\alpha$ makes the

[27] Krause [1936, 192].
[28] *Alm.* I 13. Toomer [1984, 64].
[29] Schmidt [1943, 68]. For Pappus' proof see Jones [1986, 270].
[30] Krause [1936, 192 - 193]. The medieval tradition uses $\sin\alpha$ in place of $\mathrm{Crd}(2\alpha)$. See page 134.

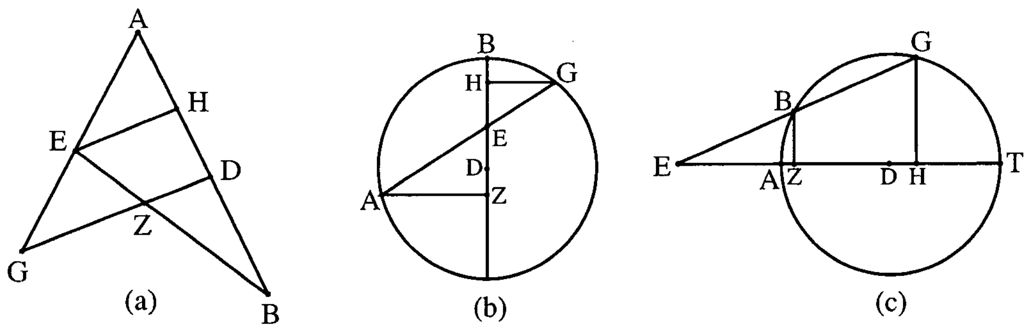

Figure 3.3: Diagrams for Lemmas 1, 2 & 3 of Abū Naṣr's Preface to *Spher.* III 1, Krause [1936, 192 - 193], Tafel V, 6, 7 & 8.

geometry more straightforward. Other than this simplification, Abū Naṣr's proof again goes into slightly more detail than Ptolemy's.

The third lemma is Theon's third circular lemma, *Alm.* I 13.4. If, in circle ABG with center D, the two lines TDE and GBE meet one another at point E, then $\sin \overset{\frown}{GA} : \sin \overset{\frown}{BA} = GE : BE$; see Figure 3.3 (c).[31] The proof is essentially Ptolemy's with the same slight differences as in the previous two lemmas.

The last two lemmas concern compound ratios.[32] They are trivial and of the sort that a Greek geometer would assume without proof. In fact, the first of these lemmas is assumed by Apollonius in the *Conics* and is demonstrated by Pappus in his lemmas to the *Conics*.[33] Lemma 4 shows that if $A : B = (C : D) \times (E : F)$, then $C : D = (A : B) \times (F : E)$. The second of these lemmas is even more obvious than the first. It shows that if $A : B = C : D$ and $E : F$ is the ratio of equality, then $A : B = (C : D) \times (E : F)$. I have not found a demonstration of this lemma in any of the commentators.

Spher. III 1 is demonstrated by using solid geometry to reduce the arcs of the spherical quadrilateral to a plane version of the theorem, as represented by Lemma 1. Lemmas 2 & 3 are used to effect this reduction of arcs of the great circles to the chords, or lines, that subtend them.

[31] Krause [1936, 193].
[32] Krause [1936, 193 - 194].
[33] Jones [1986, 300 - 301].

Following the lemmas, a title introduces Book III, "The Third Part of Menelaus' Book." The first theorem of Book III in Abū Naṣr's text of the *Spherics*, like only a few other theorems, does not begin with an enunciation.[34] It begins with the exposition; see Figure 3.4 (b).[35]

> The two arcs GE and BD meet one another in point A, and from points G and B the two arcs GD and BE are drawn, cutting one another at point Z. Each of these arcs belongs to the circumference of a great circle, and each is less than a semicircle. I say that $\sin \overset{\frown}{GE} : \sin \overset{\frown}{EA} = (\sin \overset{\frown}{GZ} : \sin \overset{\frown}{ZD}) \times (\sin \overset{\frown}{BD} : \sin \overset{\frown}{BA})$.

In the medieval Arabic tradition, *Spher.* III 1 is often expressed in terms of the sines of arcs as opposed to the ancient expression involving the chords of double arcs.[36] This change is a result the medieval definition of the sine. Menelaus would have used expressions involving the chord of the double arc.

M.T. II is then proved in three cases; see Figures 3.4 & 3.5. Case 1 proves the theorem where DA meets BH in the direction of D, Case 2 where DA meets BH in the direction of A and Case 3 where $DA \parallel BH$. M.T. I is then shown from M.T. II, with no division into cases, using a simple proof also found in Theon. Since the theorem proceeds along slightly different lines in Abū Naṣr and Halley, we will follow the proof in detail.

Let H be the center of the sphere. We draw the lines HZ, HB, HE and connect AD. Then the chord AD and the radius BH lie in the same plane. Either AD is parallel to BH, or, it meets BH in either the direction of D or the direction of A.

M.T. II, Case 1; see Figure 3.4 (b): If AD and BH meet in the direction of D, let them meet at point T. Then we draw chord AG which will meet the radius EH at point K, and chord GD which will meet the radius ZH at point L. Now, the lines HE, HZ and HT go from the center to the same circumference, so they lie in the

[34] If we follow the numbering of **D**, which Björnbo believes is closer to the original, then *Spher.* III 1 is the only theorem that does not begin with an enunciation.

[35] Krause [1936, 194].

[36] See page 120 - 120. **N** uses the expression involving sines, while **D** seems to have preserved the original expression involving chords. Björnbo attests that **G**, which like **J** and **Ha** depends on **D**, preserves the expression *corda dupli arcus* in this proposition. Björnbo [1902, 89, n. 152a].

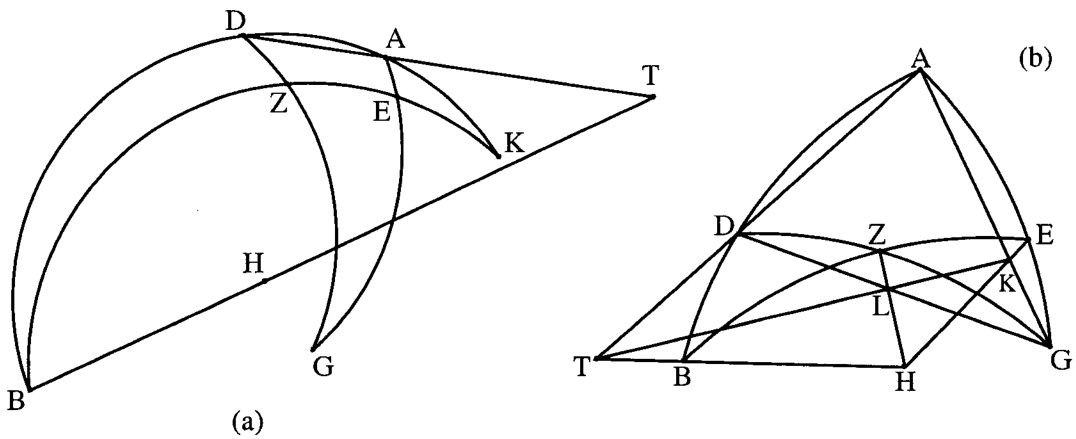

Figure 3.4: Diagrams for Abū Naṣr's *Spher.* III 1. Figure (b) is for M.T. II Case 1, Krause [1936, 194 - 195], Tafel V, 11. Figure (a) is for M.T. II Case 2; it does not appear in the MSS.

same plane. Therefore, points K, L and T lie in the same plane. $\triangle AGD$, the lines AG, AD and the point T all lie in a single plane. The points K, L, and T all lie in a second plane, the plane of the circle EZB, which is a different plane from that of $\triangle AGD$. Therefore, these planes intersect in a straight line. Hence, the line through the points K, L and T is a straight line. Then, between two intersecting lines, AG and AT, two other lines, GD and TK, have been drawn meeting one another at L; therefore $GK : KA = (GL : LD) \times (DT : TA)$ [by the plane case of M.T. II]. But $GK : KA = \sin\overset{\frown}{GE} : \sin\overset{\frown}{EA}$, $GL : LD = \sin\overset{\frown}{GZ} : \sin\overset{\frown}{ZD}$, and $DT : TA = \sin\overset{\frown}{BD} : \sin\overset{\frown}{BA}$ [Lemmas 2 & 3]. Therefore, $\sin\overset{\frown}{GE} : \sin\overset{\frown}{EA} = (\sin\overset{\frown}{GZ} : \sin\overset{\frown}{ZD}) \times (\sin\overset{\frown}{DB} : \sin\overset{\frown}{BA})$.[37]

In Abū Naṣr's proof, as in Ptolemy's, the line AD is assumed to meet line BH extended, whereas, as we will see, in Halley's proof it meets KL extended. Hence, Abū Naṣr must prove that KLT is a straight line while Halley must prove that HBT is a straight line. Once these lines are established as straight the proofs continue along similar lines. It should be noted that Abū Naṣr requires the plane case of M.T. II, whereas he has only shown the plane case of M.T. I.

M.T. II, Case 2; see Figure 3.4 (a): If AD and BH meet in the direction of A, let them

[37] Krause [1936, 194 - 195].

meet at point T. We extend the two arcs BDA and BZE so that they meet at point K. Now, by the previous case, $\sin\overset{\frown}{GZ} : \sin\overset{\frown}{ZD} = (\sin\overset{\frown}{GE} : \sin\overset{\frown}{EA}) \times (\sin\overset{\frown}{AK} : \sin\overset{\frown}{KD})$ [1]. Therefore, $\sin\overset{\frown}{GE} : \sin\overset{\frown}{EA} = (\sin\overset{\frown}{GZ} : \sin\overset{\frown}{ZD}) \times (\sin\overset{\frown}{KD} : \sin\overset{\frown}{KA})$ [Lemma 4]. Therefore, $\sin\overset{\frown}{GE} : \sin\overset{\frown}{EA} = (\sin\overset{\frown}{GZ} : \sin\overset{\frown}{ZD}) \times (\sin\overset{\frown}{DB} : \sin\overset{\frown}{BA})$ [2].[38]

There are a couple of problems with this proof. The first step, labeled [1], is justified by the previous case. The compound proportion which is asserted, however, has not been demonstrated in Case 1.[39] Moreover, even if it had been, the proof would be incomplete since it is not made clear that the results of M.T. II Case 1 can be applied to the spherical quadrilateral $GDKE$. For this proof to be successful one would have to show that the quadrilateral $GDKE$ can only be of the type handled by M.T. II Case 1, because otherwise there is the possibility that it too is the type handled by M.T. II Case 2, which we are in the process of trying to prove. The proof that Theon gives avoids these difficulties by applying a variant of the plane case of M.T. II to the chords of the original quadrilateral $GABZ$.[40] The final step in this proof, [2], relies on the lemma that if $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are two parts of a semicircle then, $\sin\overset{\frown}{AB} = \sin\overset{\frown}{BC}$.[41]

M.T. II Case 3; see Figure 3.5 (a): When $AD \parallel BH$ we extend the semicircle BAT and we draw the two chords AG and DG. From D we drop $DS \perp BT$ and from A we drop $AO \perp BT$. Then $DS = AO$, therefore $\sin\overset{\frown}{BD} = \sin\overset{\frown}{BA}$. From the center, H, we join the line EH, which cuts the line AG at point L. Now the diameter BT, $\overset{\frown}{EZB}$, the line EH and the point L all lie in the same plane, so we can draw a line in the plane of EZH which is parallel to the diameter. It will also be parallel to AD. Furthermore, in the plane ADG, we can draw a line through L parallel to AD. I say that it is the line LK. For if not, let the parallel through L in the plane EZB be the line LM and in the plane ADG let it be the line LN. Then, the lines LM and LN are parallel to each other and meet one another at the same point, which is absurd. Therefore, no line

[38] Krause [1936, 195].
[39] In Case 1, the final ratio is the "separation" of the two terms, in Case 2, it is the "composition".
[40] Rome [1943, 160 - 162].
[41] In other words $\sin(\alpha) = \sin(180° - \alpha)$. This lemma is never shown in Abū Naṣr's version of the *Spherics*; probably because it follows directly from the definition of the sine. Theon proves it for chords of double arcs, Rome [1943, 567].

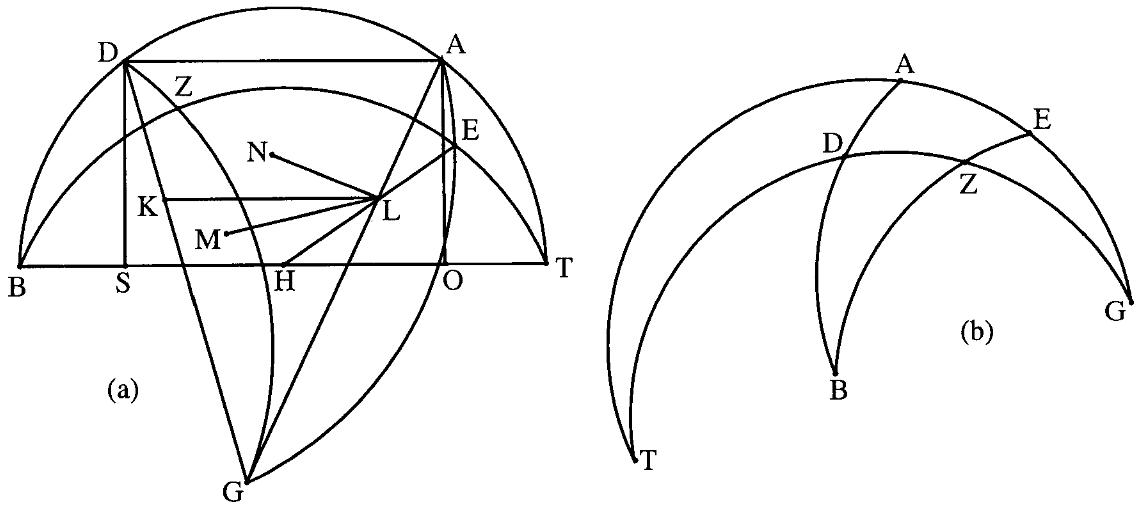

Figure 3.5: Diagrams for Abū Naṣr's *Spher.* III 1. Figure (a) is for M.T. II Case 3, Krause [1936, 195 - 196], Tafel V, 12. Figure (b) is for M.T. I; it does not appear in the MSS.

goes through L which is parallel to AD other than the line LK. Therefore, in $\triangle ADG$, a line has been drawn parallel to the base. Hence, $GL : LA = GK : KD$ [*Elem.* VI 2]. Therefore, $\sin \overset{\frown}{GE} : \sin \overset{\frown}{EA} = \sin \overset{\frown}{GZ} : \sin \overset{\frown}{ZD}$ [Lemma 2]. But, $\sin \overset{\frown}{BD} : \sin \overset{\frown}{BA}$ is the ratio of equality; therefore, $\sin \overset{\frown}{GE} : \sin \overset{\frown}{EA} = (\sin \overset{\frown}{GZ} : \sin \overset{\frown}{ZD}) \times (\sin \overset{\frown}{BD} : \sin \overset{\frown}{BA})$ [Lemma 5].[42]

This proof leaves a few things unsaid. K must be constructed as the intersection of DG and ZH for Lemma 2 to be used. Only if K is so constructed does it make sense to argue that LK is the only line that can be drawn through L parallel to DA in the two planes of EZH and ADG. The proof uses an indirect argument to establish that LK is parallel to AD and in the plane of $\triangle ADG$. Krause doubted the authenticity of this part of the proof because Menelaus criticizes his predecessors for using indirect proofs.[43] Halley's version of the proof avoids this *reductio ad absurdum* argument by assuming that AD is parallel to LK and showing that they must both then be parallel to BH. Case 3, the parallel case, appears in neither the *Almagest* nor in Theon's *Commentary.*

[42] Krause [1936, 195 - 196].
[43] Krause [1936, 196, n. 2]. For Menelaus' statement concerning indirect proofs see Krause [1936, 118].

M.T. I; see Figure 3.5 (b): We extend GA and GD so that they meet at the end point of a diameter of the sphere. Let it be T. Therefore, the two arcs TZ and BA cut one another between the two arcs TE and BE. Therefore, $\sin \overset{\frown}{TA} : \sin \overset{\frown}{AE} = (\sin \overset{\frown}{DT} : \sin \overset{\frown}{DZ}) \times (\sin \overset{\frown}{BZ} : \sin \overset{\frown}{BE})$ [by M.T. II]. But, $\sin \overset{\frown}{TA} = \sin \overset{\frown}{GA}$[44] and $\sin \overset{\frown}{TD} = \sin \overset{\frown}{GD}$. Therefore, $\sin \overset{\frown}{GA} : \sin \overset{\frown}{AE} = (\sin \overset{\frown}{GD} : \sin \overset{\frown}{DZ}) \times (\sin \overset{\frown}{BZ} : \sin \overset{\frown}{BE})$.[45]

This proof again makes use of the lemma that if $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are two parts of a semicircle then, $\sin \overset{\frown}{AB} = \sin \overset{\frown}{BC}$. Theon includes this version of M.T. I in his comments to *Alm.* I 13 almost as an afterthought. It should be noted that Abū Naṣr never makes use of the plane version of M.T. I.

3.2.2 Halley's version of *Spherics* III 1

Halley's translation of Menelaus' *Spherics* was primarily made from Jacob ben Māḥir's Hebrew translation, which Krause has shown depended on the same Arabic version as Gerhard's Latin translation. On stylistic grounds, Krause held that the Arabic edition from which **J** and **G** were made was put together from two different variants of the Arabic text. Krause identified *Spher.* III 1 as having derived from **Ma**, al-Māhānī's correction of the anonymous translation $\mathbf{\ddot{U}}_1$.

In this version of the text there are no lemmas to *Spher.* III 1. The lemmas which appear in italics in Halley's text are his own additions.[46] He says they are demonstrated in the Arabic text, which he consulted, in the same manner as in Ptolemy; however, he does not say whether he is following Ptolemy or the Arabic.[47] Halley's Lemma I proves the plane occurrence of both M.T. I & II, despite the fact that the text only requires the plane occurrence of M.T. II. The figure which he includes with Lemma I is his own. It is a combination of the figures for both *Alm.* I 13.1 & 13.2.[48] The figure that he includes

[44] Krause [1936] has BE for TA, 197, l. 4.
[45] Krause [1936, 196 - 197].
[46] Halley [1758, 83 - 84].
[47] Halley [1758, 82]. In aṭ-Ṭusī's edition, the lemmas also follow the theorem, Aṭ-Ṭūsī [1940, 62 - 66]. Halley appears to have reworked whatever version he took as his source.
[48] Toomer [1984, 64 - 65].

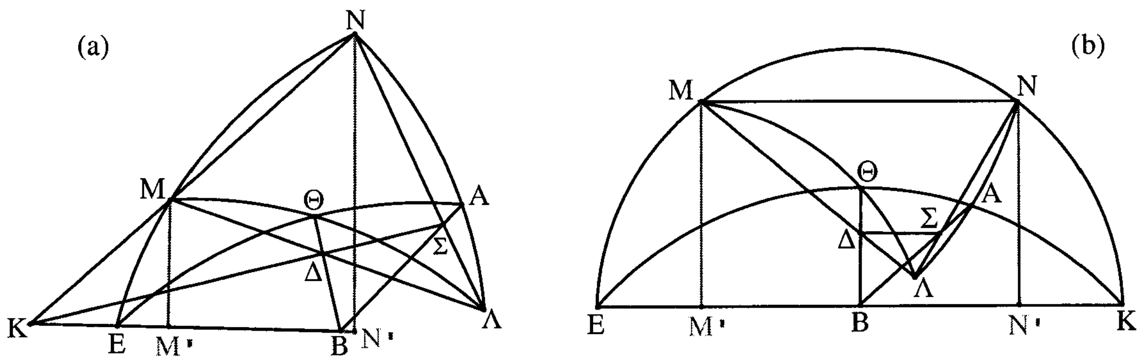

Figure 3.6: Diagrams for Halley's *Spher.* III 1, Halley [1758, 80 - 81]. Figure (a) is for *Spher.* III 1 Case 1, (b) for *Spher.* III 1 Case 2. Lines and letters in gray do not appear in Halley's text.

with Lemma II combines the two figures of *Alm.* I 13.3 & 13.4.[49] Halley has written his own version of this lemma, which combines the two proofs in the other versions and shows that what appears, in Ptolemy, to be two distinct theorems are simply different cases of a single theorem. Following the lemmas, Halley includes some more material taken from Theon's commentary: Theon's proof of M.T. II Case 2, and the simple proof of M.T. I using M.T. II.[50]

In Halley's text, *Spher.* III 1 again has no enunciation and begins with the exposition; see Figure 3.6.[51]

> If there are two arcs of great circles on the surface of a sphere, NME and $NA\Lambda$, between which two other arcs, $E\Theta A$ and $\Lambda\Theta M$, are drawn, intersecting one another at point Θ: I say that $\sin \overset{\frown}{AN} : \sin \overset{\frown}{A\Lambda} = (\sin \overset{\frown}{NE} : \sin \overset{\frown}{EM}) \times (\sin \overset{\frown}{M\Theta} : \sin \overset{\frown}{\Theta\Lambda})$.

Halley deals with the theorem in only two cases. M.T. II Case 1; see Figure 3.6 (a): Let the point B be be set out, being the center of the circle, and join ΛN, ΘM, MN, EB, ΘB meeting the chord $M\Theta$ in Δ, and AB meeting the chord $N\Lambda$ in Σ. Let $\Delta\Sigma$ be joined and produced until it meets the line MN produced at K. Thus K will be a point in the same plane as both of the circles $A\Theta E$ and NME. But the points

[49] Toomer [1984, 66 - 67].
[50] Halley [1758, 84 - 86].
[51] Halley [1758, 80].

E and B are in these same planes, therefore KEB will be a straight line. Moreover, since point Σ is the intersection of lines AB and $N\Lambda$, and point Δ is the intersection of lines ΘB and $M\Lambda$, and point K is on line $\Sigma\Delta$ produced, and the three points Σ, Δ, K are in the same plane as $\triangle N\Lambda M$; therefore, $N\Sigma : \Sigma\Lambda = (NK : KM) \times (M\Delta : \Delta\Lambda)$ [1]. But, $NK : KM = NN' : MM'$ [2].[52] Moreover, NN' is the sine of $\overset{\frown}{EN}$ and MM' is the sine of $\overset{\frown}{ME}$, and therefore $NK : KM = \sin \overset{\frown}{NE} : \sin \overset{\frown}{ME}$. Also, it is clear that $N\Sigma : \Sigma\Lambda = \sin \overset{\frown}{NA} : \sin \overset{\frown}{A\Lambda}$, and $M\Delta : \Delta\Lambda = \sin \overset{\frown}{M\Theta} : \sin \overset{\frown}{\Theta\Lambda}$ [3]; therefore, $\sin \overset{\frown}{AN} : \sin \overset{\frown}{A\Lambda} = (\sin \overset{\frown}{NE} : \sin \overset{\frown}{ME}) \times (\sin \overset{\frown}{M\Theta} : \sin \overset{\frown}{\Theta\Lambda})$.[53]

Halley begins by asserting that MN and $\Sigma\Delta$ produced will meet at K and then showing that KEB is a straight line whereas Abū Naṣr and Ptolemy assert that MN and BE produced will meet at K and then show that $K\Delta\Sigma$ is a straight line. Because MN and $\Sigma\Delta$ lie on great circles that meet at E it is clear that either of these approaches will work.

The steps marked by [1], [2] & [3] are assumed without proof. [1] is the plane occurrence of M.T. II. [2] & [3] are justified by Abū Naṣr's Lemmas 2 & 3 which are based on trivial considerations of the properties of parallel lines and similar triangles. The lemmas Halley gives supply these steps. It would not be at all surprising if a Greek geometer assumed these steps without proof.

M.T. II Case 2; see Figure 3.6 (b): Let $\Delta\Sigma$ be parallel to MN, and let the semicircles EMN and $E\Theta A$ be be completed, meeting one another at K. Then, since the two parallel lines $\Delta\Sigma$ and MN are in the two planes ENK and $E\Theta K$, they will be parallel to the intersection of these planes, the line EBK. Moreover, since NN' is equal to MM', while NN' is the sine of $\overset{\frown}{EN}$ and MM' is the sine of $\overset{\frown}{EM}$, therefore $\sin \overset{\frown}{EN} = \sin \overset{\frown}{ME}$. But, because MN is parallel to $\Delta\Sigma$, $N\Sigma : \Sigma\Lambda = \sin \overset{\frown}{NA} : \sin \overset{\frown}{A\Lambda} = M\Delta : \Delta\Lambda = \sin \overset{\frown}{M\Theta} : \sin \overset{\frown}{\Theta\Lambda}$ [1]. Therefore, since $\sin \overset{\frown}{EN} : \sin \overset{\frown}{EM}$ is the ratio of equality, $\sin \overset{\frown}{NA} : \sin \overset{\frown}{A\Lambda} = (\sin \overset{\frown}{M\Theta} : \sin \overset{\frown}{\Theta\Lambda}) \times (\sin \overset{\frown}{EN} : \sin \overset{\frown}{EM})$ [2]. The text then

[52] Neither M' nor N' appear in Halley's diagram. His texts literally reads, "But NK is to KM as the normal falling from point N to the diameter KEB is to the normal from point M to the same," Halley [1758, 81].

[53] Halley [1758, 80 - 81].

goes on to show, through the course of an entire paragraph, that $\sin \widehat{A\Lambda} : \sin \widehat{NA} = (\sin \widehat{\Theta\Lambda} : \sin \widehat{M\Theta}) \times (\sin \widehat{EM} : \sin \widehat{EN})$. It is not clear why this simple inversion warrants so much discussion.[54]

The step denoted by [1] is justified by Halley's Lemma II, appended from the astronomical tradition. The step denoted by [2] is justified by Abū Naṣr's Lemma 5. Halley considers this step too obvious to require a Lemma.

The main features to notice about the version of *Spher* III 1 preserved in **D** are the lack of lemmas, and the fact that only one version of the theorem, M.T. II, is established. As in Abū Naṣr, the theorem is shown in different cases. In **D**, the cases are distinguished on the basis of the relationship of the line NM to the line $\Delta\Sigma$ as opposed to the line BE in **N**; see Figure 3.6 (b). Because of this difference, **D** is able to demonstrate the parallel case without using an indirect argument.

3.2.3 Ptolemy's *Almagest* I 13

Ptolemy introduces his discussion of the fundamental theorem of ancient spherical trigonometry as an aside to the astronomical task of demonstrating the size of arcs of a meridian cut off between the ecliptic and the equator. He begins with a proof of the plane case of M.T. I, despite the fact that he never actually makes use of this lemma. Ptolemy only proves M.T. II, but the fact that he presents a proof of the plane case of M.T. I shows that he was familiar with, or at least saw the possibility of, a different proof of M.T. I than the one given by Abū Naṣr. Ptolemy next gives a proof of the plane case of M.T. II, the lemma he will actually use in his proof of the fundamental theorem. Ptolemy's version of the lemma is essentially the same as that in Abū Naṣr. The only difference is that Ptolemy's proof is more concise, as though Abū Naṣr were filling out the details of Ptolemy's proof.[55]

These two lemmas are then followed by two lemmas which are equivalent to Abū

[54] Halley [1758, 81 - 82].
[55] Ptolemy's version of this lemma is discussed on page 163.

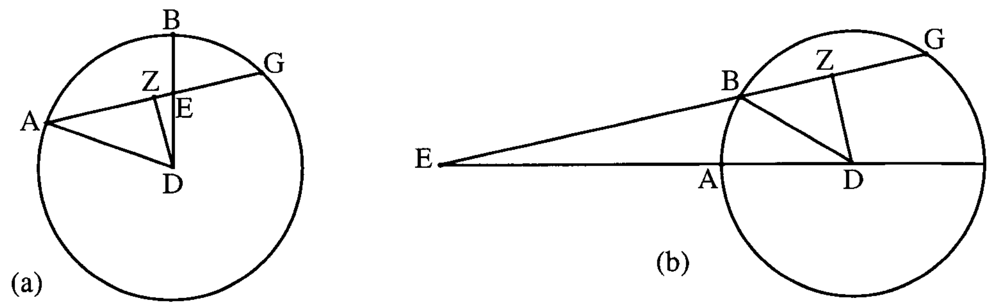

Figure 3.7: Ptolemy's diagrams for two lemmas of metrical analysis, *Alm.* I 13.3c & 13.4c, Toomer [1984, 66 - 67].

Naṣr's lemmas 2 & 3. Following each of these lemmas is a short corollary. These corollaries are metrical analyses that increase the scope of the problems that the fundamental theorem can solve. They are never used in the *Almagest.* Following the development of the chord table, these are the first theorems of metrical analysis that Ptolemy gives.

The first of these corollaries states that if $\overset{\frown}{AG}$ and Crd(2 $\overset{\frown}{AB}$) : Crd(2 $\overset{\frown}{BG}$) are given, then $\overset{\frown}{AB}$ and $\overset{\frown}{BG}$ will be given individually; see Figure 3.7 (a). Join AG and BD and drop the perpendicular DZ from D to AG. Now, if $\overset{\frown}{AG}$ is given, then $\angle ADZ$ is given, because it is $1/2$ $\overset{\frown}{AG}$ [*Data* 2]. Hence $\triangle ADZ$ will be given [*Data* 40]. Now, since the whole chord AG is given [by the **Chord Table**], and $AE : EG =$ Crd(2 $\overset{\frown}{AB}$) : Crd(2 $\overset{\frown}{BG}$) is given [*Alm.* I 13.3], then AE will be given [*Data* 7], and so will ZE by subtraction [*Data* 2 & 4]. Hence, since DZ is given in R$\triangle EDZ$ [*Elem.* I 47], $\angle EDZ$ will be given [*Data* 1 & 41],[56] and hence the whole of $\angle ADB$ will be given [*Data* 40]. Therefore, $\overset{\frown}{AB}$ will be given, and $\overset{\frown}{BG}$ will be given by subtraction [*Data* 4].[57]

The second of these corollaries states that if $\overset{\frown}{BG}$ and the ratio Crd(2 $\overset{\frown}{GA}$) : Crd(2 $\overset{\frown}{AB}$) are given, then $\overset{\frown}{AB}$ will also be given; see Figure 3.7 (b).[58] The proof follows along the same lines.

Every step of these analyses can be justified on the basis of theorems in the *Data*;

[56] This step seems to require a theorem to the following effect: if in a right triangle the two sides about the right angle are given, then the whole triangle will be given in form. There is no such theorem in the *Data*. A more general justification of this step can be fashioned out of *Data* 1 & 41.

[57] Toomer [1984, 66 - 67].

[58] Toomer [1984, 67 - 68].

nevertheless, in most cases the *Data* will not help us derive a numeric value. As the previous chapter has shown, Ptolemy means something different by *given* than Euclid does. For Ptolemy, *given* always means numerically determinate. In making use of these corollaries in practice, a Greek mathematical astronomer would have wanted to know, not merely that the final value was theoretically knowable, but *how* to actually produce the desired value. Going through a numerical calculation shows how Ptolemy's proposition can be used as a sketch of the steps that should be used. It also shows how theorems in Euclid's *Data* which treat given magnitudes and ratios can be used as steps in the course of calculating determinate values.

Consider Figure 3.7 (a). If $\overset{\frown}{AG} = 127;43°$ and $\text{Crd}(2\ \overset{\frown}{AB}) : \text{Crd}(2\ \overset{\frown}{BG}) = 1;37 : 1$, let it be required to find $\overset{\frown}{AB}$ and $\overset{\frown}{BG}$. Following Ptolemy's convention, we take $AD = 60^{\text{p}}$. AG is given by the chord table as $107;43,30^{\text{p}}$, so that $AZ = 53;51,45^{\text{p}}$. *Data* 6 & 7 show us how to to find AE. We set $x : y = \text{Crd}(2\ \overset{\frown}{AB}) : \text{Crd}(2\ \overset{\frown}{BG})$, where x is given, say $x = 1$. Thus $y = 0;37,7$ and, since $EG : AE = y : x$, therefore, *by composition*, $AG : AE = x + y : x$ [*Elem.* V 18], so that $AE = \frac{AG}{1+y} = 66;33,22^{\text{p}}$. Then, $EZ = AE - AZ = 12;41,37^{\text{p}}$, but $DZ = \sqrt{AD^2 - AZ^2} = 26;26,8^{\text{p}}$, so that $DE = \sqrt{EZ^2 + DZ^2} = 29;19,31^{\text{p}}$. Although *Data* 41 tells us that the triangle ZDE is known it gives us no help in assigning values to the triangle's angles. To find the value of $\angle ZDE$ we set DE to 120^{p} and we use the chord table to find $\overset{\frown}{ZE}$ in the circle about ZDE; this will be twice $\angle ZDE$, hence $\angle ZDE = 25;39°$. For $\angle ADZ$ we set $AD = 120^{\text{p}}$ and again use the chord table. We find that $\angle ADZ = 63;51,30°$. Hence, $\overset{\frown}{AB} = \angle ADB = 89;30,27°$ and, by subtraction, $\overset{\frown}{BG} = \angle BDG = 38;12,32°$.

This example shows both how Euclid's *Data* can be of use in Ptolemy's metrical analysis, as in the case of *Data* 6 & 7, and how it falls short, as with *Data* 41. At the stage in the proposition where I have cited *Data* 41, Ptolemy is expecting his readers to be familiar with metrical techniques that are only used later in the *Almagest.* Such techniques must have existed in other technical treatises and would have been familiar to mathematicians and mathematical astronomers.

These corollaries are not used in proving the fundamental theorem. They are meant to serve as auxiliaries to the fundamental theorem. In some cases, it may be possible to reduce a problem to a given ratio of the chords of double arcs and a given arc, where the given arc is related to, but is not one of, the arcs in the ratio. These corollaries will solve such a problem. They are never used in conjunction with the fundamental theorem in the *Almagest.* This is a rare example of Ptolemy including some mathematics out of a desire for completeness or utility which goes beyond the immediate needs of the *Almagest.* It is likely that these corollaries were present in one of the sources that Ptolemy used for this section of the *Almagest.*[59]

Following the second corollary, Ptolemy proves M.T. II. He only proves the theorem for Case 1. His proof is presented along the same line line of argument as Abū Naṣr's: he assumes that the outer leg of the plane quadrilateral, DA in Figure 3.4 (b), intersects the diameter of the sphere, BH, and then proves that the inner leg of the plane quadrilateral, TLK, is a straight line. As is the case with the lemmas, Ptolemy's proof is slightly more concise. Ptolemy does not prove M.T. I, but he provides the plane case of this theorem and following his demonstration of M.T. II he says that M.T. I can be shown "in the same way, corresponding to the straight lines in the plane figure."[60] He clearly intends us to understand that we may make a proof of M.T. I which is analogous to his proof of M.T. II but using the plane case of M.T. I, which he has provided. This is an instance of Ptolemy sketching his argument while giving us everything we need to make his argument complete. It also shows that Ptolemy was familiar with a different version of M.T. I than that found in Abū Naṣr.

[59] See Sidoli [2004a] for a discussion of the possibility that these corollaries go back to a work by Hipparchus.

[60] Toomer [1984, 69]. Κατὰ τὰ ἀυτὰ δὴ καὶ ὥσπερ ἐπὶ τῆς ἐπιπέδου καταγραφῆς τῶν ἐυθειῶν, Heiberg [1916, p. 1, 76].

3.2.4 Theon's commentary on *Almagest* I 13

Theon's comments on this material follows Ptolemy's text closely and never make any reference to Menelaus. Theon provides enunciations for all the propositions and proves an abundance of different cases for the lemmas as well as for the fundamental theorem. In particular, he provides proofs of the plane case of a number of variants of the fundamental theorem which are used, without proof, by Ptolemy in the *Almagest*.[61] Theon divides the lemmas into two rectilinear, and four circular, lemmas. He calls the fundamental theorem the "spherical theorem," or "spherical proof," and, as mentioned above, he classifies the two versions of the fundamental theorem according to the relation of the arcs in the first ratio of the proportion.[62]

Theon only deals with lemmas that appear in Ptolemy. There is no equivalent to Abū Naṣr's lemmas 4 & 5. Theon has no need of lemmas 4 & 5 because he proves M.T. II Case 2 differently than Abū Naṣr and does not prove Case 3, the parallel case, at all. Theon's proof of M.T. II Case 2 is longer, but more sound than Abū Naṣr's.[63] Theon seems to think that the parallel case, Abū Naṣr's M.T. II Case 2, is indeterminate (ἀσύστατον).[64] Up to this point there is little similarity between Theon's approach and the material included in Abū Naṣr. The last section of the commentary to *Alm.* I 13 is the only exception. Theon appends the same version of M.T. I that appears in Abū Naṣr. He proceeds this proof with the lemma mentioned above and used by both Abū Naṣr and Halley to the effect that if $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are two parts of a semicircle, then $\mathrm{Crd}(2\,\overset{\frown}{AB}) = \mathrm{Crd}(2\,\overset{\frown}{BC})$.[65]

Theon's treatment of this material makes it clear that he was not dependent on the

[61] See page 154.
[62] Throughout the section on *Alm.* I 13 but see especially Rome [1943, 557 - 558].
[63] Rome [1943, 560 - 562].
[64] See Rome [1933a, 44 - 49] and Rome [1943, 554, n. 1] for a discussion of the implications of this claim. In fact, Theon does not claim that the parallel case of the fundamental theorem is indeterminate, which is false, but rather that the parallel case of the plane version, *Alm.* I 13.4 is indeterminate, which is true. Nevertheless, the fact that Theon does not bother to prove the parallel case of the fundamental theorem along other lines, as in Abū Naṣr's M.T. II Case 3, shows that he probably thought it could not be done.
[65] Rome [1943, 567 - 570].

version of Menelaus' *Spherics* preserved in Abū Naṣr. He either did not consult Menelaus for this material, or if he did, the version of the *Spherics* which he had access to was different from Abū Naṣr's. In particular, it would be surprising if Theon knew of the parallel case of M.T. II and did not include this in his comments. Since we know that Theon was familiar with Menelaus' *Spherics*, and since both versions **N** and **D** contain the parallel case, it is strange that Theon did not include this case in his comments. Theon seems not to have consulted Menelaus' *Spherics* at this point. We should not follow Björnbo and Krause in ascribing to Menelaus the added material found in Theon.[66] It is more likely that the material that is the same in both Theon and Abū Naṣr found its way into the Arabic tradition *Spherics* from a translation of the *Commentary* itself.

There is an interesting lemma which Rome includes at the beginning of Theon's commentary on this chapter that is not found in the *Almagest*.[67] It is a strange mixture of the language of synthetic geometry and metrical analysis. It begins with an expanded version of the spurious *Elem.* VI def. 5, which actually constitutes a new reading of the definition.[68]

> One ratio is said to be compounded of two or more ratios, when the sizes (πηλικότητες) of the ratios multiplied together make some size of a ratio. For let ratio $AB : GD$ be given, and ratio $GD : EZ$; I say that $AB : EZ = (AB : GD) \times (GD : EZ)$. That is, if the size of the ratio $AB : GD$ is multiplied by the [size] of $GD : EZ$, it will make the [size] of the [ratio] $AB : EZ$.

The proof as a whole appears an attempt to show that if $A : B$ is a given ratio, then for any given C, it is the case that $A : B = (A : C) \times (C : B)$. A lemma such as this is required by *Alm.* I 13.1.[69] The lemma is proven in three cases, differentiated on the basis of the relative magnitudes of AB, GD and EZ. The proofs are structured

[66] Björnbo [1902, 88] and Krause [1936, 197, n. 2 (cor. 1)].
[67] There are some difficulties involved in reading this bit of text at the beginning of the commentary to *Alm.* I 13. Rome [1943, 532, n. 1] gives his reasons for putting it here.
[68] Rome [1943, 532 - 533]. See Heath [1926, 189 - 190]. *Elem.* VI def. 5 reads, "A ratio is said to be compounded of ratios when their sizes multiplied together make some [ratio]," λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα.
[69] See page 163 below for a discussion of *Alm.* I 13.1.

as analysis/synthesis pairs but they are really just examples worked through with actual numbers. The first of these cases will suffice to show the pattern.

First let $AB > GD$, and $GD > EZ$. And let $AB = 2GD$ and $GD = 3EZ$. Therefore $AB = 6EZ$, "since some triple doubled becomes six times the same." This is the synthesis: Since $AB = 2GD$, let AB be divided at H such that $AH = HB = GD$. And, since $GD = 3EZ$, hence $HB = 3EZ$. Therefore $AB = 6EZ$. Therefore, $AB : EZ = (AB : GD) \times (GD : EZ)$.[70]

This is a strange analysis/synthesis pair. The analysis does not assume what is to be found as given, nor does the synthesis take the result of the analysis and work back toward the assumption. The construction of point H gives the second section of the proof the appearance of playing the role of a synthesis but it really adds nothing to the logic of the demonstration. Because the first half is not a real analysis, the second half is an empty synthesis. They are both just different ways of working through the numerical example.

Nevertheless, this lemma is interesting because it shows that Theon was also unaware of any theoretical text that demonstrated the well-known fact that given any ratio $A : B$ and any magnitude C of the same kind as A and B, it is the case that $A : B = (A : C) \times (C : B)$. Moreover, this argument shows that by Theon's time, Greek mathematicians had no difficulty in applying the arithmetical conception of compound ratio in an argument that was meant to apply to ratios between any magnitudes.

Although the passage begins, like *Elem.* VI def. 5, with the plural form of the noun (πηλικότητης), meaning *size* or *value*, it quickly becomes apparent that the singular form of this noun is used to denote a characteristic of a single ratio. This is the size, or value, of the ratio itself. This value can be multiplied by the value of another ratio and the product will equal the value of a third ratio. In other words, the value of a ratio behaves like a number. This shift was doubtless the result of the applied tradition in which the arithmetic conception of ratios was consistently utilized to provide numerical solutions

[70] Rome [1943, 534].

to geometric problems.

3.2.5 Comparison of the versions of the fundamental theorem

Given the state of the evidence now in print it seems that none of the versions of *Spher.* III 1 can be taken simply as the proposition that Menelaus wrote. Both Björnbo and Krause thought that the version of the proof preserved in **N** represents a more pristine redaction when compared to that of **Ha**.[71] Rome was of the opinion that Ptolemy and Theon give witness to a version of the theorem which is perhaps incomplete but, nonetheless, closer to what Menelaus actually wrote.[72] There do appear to be fundamental differences between examples of the theorem in the theoretical tradition of the *Spherics* and in the applied tradition of the *Almagest.* Ptolemy and Theon, in fact, probably give witness to an earlier version of the theorem as found in older works on spherical astronomy. The versions of *Spher.* III 1 that go back to **Ma** have been unjustly neglected in trying to assess the form of the theorem as Menelaus wrote it.

Björnbo advanced four reasons for holding that the proof in **N** was closer to the original than that in **Ha** but I do not believe that they hold up under scrutiny.[73] These reasons may help to establish that the Latin and the Hebrew texts have suffered some corruption, but they do not imply that **N** has not undergone emendations under the influence of the astronomical tradition. In fact, as we have seen, Abū Naṣr appears to have made additions and changes to the text based on the later Greek tradition of Ptolemy and Theon.

The first reason Björnbo gives is the presence of the parallel case which is absent in both Ptolemy and Theon while present in the Latin and Hebrew versions in what Björnbo considers to be a deviant form. This is an important feature of Abū Naṣr's proof but since a version of the parallel theorem also exists in the Latin and Hebrew texts, which come from a different Arabic version, all this implies is that Menelaus' text contained a

[71] Björnbo [1902, 88] and Krause [1936, 197, n. 2 (sic.)].
[72] Rome [1933a].
[73] Björnbo [1902, 88].

proof of the parallel case. Moreover, the fact that Abū Naṣr uses an indirect proof for this case implies that the tradition of **Ma** is closer to the original text of the *Spherics*.

The second reason is the ordering of the letters. For the lemmas, the figures are lettered identically, given the conventions of transcription; while for the theorem itself the lettering is almost identical.[74] The letters in Halley diverge from the conventions of lettering in Greek diagrams.[75] It could be that this is in part influenced by the practices of ben Māḥir in transcribing the Arabic letters into Hebrew, but I have not been able to find a consistent pattern under this hypothesis. It seems that either ben Māḥir or Halley, or perhaps both, allowed some inconsistencies in their transcription. This argument does show that there is a close relationship between the lemmas and the first part of *Spher.* III 1 in **N** and *Alm.* I 13. Again, however, the more likely explanation is that Abū Naṣr took the lemmas and lettering from the tradition of the *Almagest* and its commentaries.

The third reason Björnbo advances is that the proof in the *Almagest* can be read as a concise version of that in Abū Naṣr. There is no doubt that Ptolemy's proof is more concise that that in **N**, however, Ptolemy makes explicit reference to proving M.T. I through the use of the plane version of M.T. I, as we find in Theon, whereas Abū Naṣr proves M.T. I through M.T. II. The similarity between the line of argument in *Alm.* I 13.5 and *Spher.* III 1 M.T. II Case 1 is again best explained by supposing that Abū Naṣr was working under the influence of the Ptolemaic tradition.

The final reason is that if we assume that Abū Naṣr's version is the original then we can ascribe to Menelaus the extensions to the proof included in Theon. No matter what we assume about Abū Naṣr's version of *Spher.* III 1, it is clear that the majority of Theon's comments cannot have had any basis in **N**. Theon's many variants of the plane cases of the theorem are lacking in **N**.[76] Theon neither has nor requires Abū Naṣr's lemmas 4 & 5. Theon's proof of M.T. II Case 2 is different and superior to that in **N**.

[74] The figure of lemma one in Abū Naṣr is a mirror image of that used in Ptolemy to prove the plane case of M.T. I, see Krause [1936, Tafel V 6] and Toomer [1984, 64]. In the figures for the spherical theorem, the letters L and K are switched, see Krause [1936, Tafel V 11] and Toomer [1984, 68].

[75] See Netz [1999b, 68 - 74] for a discussion of the conventions of lettering diagrams in Greek geometry.

[76] Rome [1943, 539 - 542 & 543 - 545].

Abū Naṣr is lacking the proof of M.T. I based on the plane version of M.T. I which is found in Theon. Theon is missing the parallel case, Case 3. The only similarities are M.T. II Case 1 and the concise proof of M.T. I which is based on M.T. II. It is most likely that Abū Naṣr included his version of M.T. I based on the tradition of the commentaries to the *Almagest*.

There are a number of reasons to be suspicious of the pristine character of Abū Naṣr's *Spher.* III 1. (1) Unlike almost all the other propositions in the text, *Spher.* III 1 has no enunciation. (2) Only two figures have survived, whereas the text makes implicit reference to four different figures. Indeed, the two figures that are extant are essentially the same as those in **Ha**, while the two figures that are missing can be reconstructed from Theon's *Commentary.* (3) The proof of M.T. II Case 2 is unsound as it stands, and makes use of a lemma that is not elsewhere attested. (4) **N** proves Case 3 with an indirect argument, despite the fact that Menelaus criticized his predecessors for using this type of argument. (5) All of the differences between **N** and **Ha** can be explained by supposing that Abū Naṣr made his edition under the influence of the astronomical tradition.

N and **Ha** are revisions of two different translations of Menelaus' *Spherics.* Hence, we should expect them to have some additions and emendations, but to give witness, through their common elements, to the general outline of Menelaus' presentation. The differences between these two versions of the theorem are also telling. Almost all of the differences seem to be changes and additions made by Abū Naṣr on the basis of the astronomical tradition.

There are a number of structural similarities between Halley's version of the proof and that found in Abū Naṣr. (1) Neither of them include the lemmas in the text of Book III, and (2) both of them demonstrate the theorem in more than one case. Moreover, if we take Abū Naṣr's proof of M.T. I as an insertion based on the astronomical tradition then (3) both of them prove only one version of the theorem. This would make sense in a theoretical text, since there are numerous versions of the theorem and there is nothing special about the two versions Ptolemy gives. As we will see below, they are simply the two versions he uses most often.[77] The presentation of the theorem in Menelaus' text would then be both more advanced and more complete, in keeping with its theoretical context.

The places where **N** diverges from **Ha** all seem to have been influenced by the astronomical tradition. There are three reasons for suspecting that the first three lemmas prepended to Book III have been adopted from Ptolemy. (1) Abū Naṣr only gives the proof of the plane case of M.T. I which is the first lemma that appears in the *Almagest*, whereas he only needs the proof of the plane case of M.T. II. (2) The lemmas are bracketed off from the text of book III which is introduced with the title, "The third part of the book of Menelaus." (3) These lemmas are well within the degree of simplicity that a Greek geometer could feel free to assume without proof. This last claim is fairly subjective, but a selection of the lemmas in Pappus' *Collection*, or one of the commentaries of Eutocius, will show that it is tenable. If Abū Naṣr added Lemmas 1 - 3 from the astronomical tradition to fill out the steps in the proof for *Spher.* III 1, then he may have felt compelled to include Lemmas 4 & 5 to fill out his own version of M.T. II Case 2 and to justify the final step of M.T. II Case 3.

Abū Naṣr divides the cases of the theorem on the basis of the relation of one of the outer legs of the plane quadrilateral, MN in Figure 3.6 (a), to the diameter of the sphere, BE, whereas Halley divides the cases on the basis of the relation of the outer leg of the quadrilateral, MN, to the inner leg of the quadrilateral, $\Sigma\Delta$. In this, Abū Naṣr follows the line of Ptolemy's argument, although Ptolemy only presents one case. Halley's division allows him to prove the parallel case by a direct argument whereas Abū Naṣr requires a *reductio ad absurdum* argument. Since Menelaus explicitly states that he avoided such proofs, it is likely that **Ha** preserves a version closer to what Menelaus wrote and that the presentation in **N** is the result of the influence of the *Almagest* on Abū Naṣr.

[77] See page 154 for a discussion of Ptolemy's use of the different versions of the theorem.

Abū Naṣr's proof of *Spher.* III 1 M.T. II Case 2 is incomplete and it makes use of the trivial Lemma 4, not found in any other version of the theorem. Moreover, there is no figure for this case in the text. It is probable that in Abū Naṣr's source, **bḤ**, this case was either missing or flawed. Abū Naṣr tried to correct this defect by writing the proof that he gives, and inserting Lemma 4 into his preface to support his proof.

The proof of M.T. I in **N** presents a puzzle. It seems to have been taken from Theon, especially since Abū Naṣr has given the plane occurrence for M.T. I from which a different proof could be constructed. *Spher.* III 10 requires M.T. I so we can expect some version of the theorem in the text.[78] **Ha** has no proof of M.T. I and this is the only version in **N**. Nevertheless, it seems more likely that Abū Naṣr inserted this text into **N** from the astronomical tradition than the other way around. It would be surprising if Theon, having read this theorem in Menelaus directly following the parallel case, had chosen to include the short version of M.T. I and not the parallel case. Menelaus' *Spherics* may have contained some version of this theorem but it is also possible that it did not. *Spher.* III 10 is the only use of M.T. I in the *Spherics* of which I am aware, and Menelaus could well have considered the proof of M.T. I to be sufficiently obvious, given the proof of M.T. II.

The differences in the two presentations suggest, in fact, that there were two different traditions with their own forms of the fundamental theorem: the astronomical tradition and the theoretical tradition. The difference between Ptolemy and Theon on the one hand, and Halley and Abū Naṣr on the other, are best explained by assuming that Ptolemy is working from a source other than Menelaus' *Spherics.* In Ptolemy's source both M.T. I & II were most likely demonstrated for one case only, using the plane lemmas for both versions. All of the lemmas, and in particular the unmotivated corollaries, *Alm.* I 13.3c & 13.4c, would have been contained in this source. In Menelaus' *Spherics*, on the other hand, the lemmas were probably absent. The demonstration may have dealt with only one version of the theorem but showed it in more than one case. The parallel

[78] Krause [1936, 216].

case, Case 3, was almost certainly included. Case 2 may also have been present, but was most likely in a form closer to what we find in Theon. Case 2 as it appears in Abū Naṣr was probably drafted by him, along with Lemmas 4 & 5. Abū Naṣr's text of M.T. I was probably taken by him from the astronomical tradition, although there may have been some version of this theorem in Menelaus' text.

Having surveyed the published versions of the fundamental theorem of ancient spherical trigonometry, it is possible to make some remarks about the way that Ptolemy presents the mathematics he uses. It is immediately apparent that Ptolemy is making no attempt to be comprehensive in his account. His mathematics serves the role of convincing the reader that what he says is correct and that his methods are sound but he does not strive for completeness with regard to either logic or content. Nor does he attempt to include all of the mathematical methods that may be of use to the practicing mathematical astronomer.

Ptolemy presents the mathematical text in the *Almagest* as merely preliminary to the astronomical material. The *Almagest* as a whole, nevertheless, reads like a systematic mathematical treatise; each new development depends both logically and mathematically on what has gone before. In *Alm.* I 13, Ptolemy streamlines the fundamental theorem but he gives the preliminary material in full. This serves the purpose of giving the reader everything necessary to follow the argument and to flush out passages that Ptolemy only sketches. In this way the preliminary lemmas are used to convince the reader of the fundamental theorem and they would be useful to anyone interested in completing the proof which Ptolemy has left out. Ptolemy also includes the two corollaries to the circular lemmas despite the fact that these play no role in his treatment of spherical astronomy. These were probably found in one of the texts of spherical astronomy that Ptolemy used as source for this material.

Once Ptolemy has derived the fundamental theorem of ancient spherical trigonometry, it becomes the primary tool in his approach to spherical astronomy. He combines the fundamental theorem with the use of tables and the methods of metrical analysis to solve all the problems that arise from the movement of the equatorial and ecliptic coordinate systems with respect to the local horizon.

3.3 Spherical trigonometry in the *Almagest*

Ptolemy uses the fundamental theorem along with the chord table, *Alm.* I 11, as a tool of metrical analysis to find an unknown arc of a great circle when the values of certain other arcs are known. The difficulty is that the fundamental theorem involves six terms; hence, in order to find one term, we must generally already know five. In its modern expression, the theorem itself is transparent but as a verbal expression it is involved and it can be difficult to remember how to map its terms to the legs of the spherical quadrilateral. In order to avoid confusion, Ptolemy uses the fundamental theorem in a very systematic way. In this section, I will look at the practices Ptolemy adopts in using the fundamental theorem through an investigation of an example of normal practice as well as some exceptions.

Ptolemy uses M.T. II in ten situations in Books I, II and VIII.[79] In almost every case, he is calculating the lower part of one of the outside legs of the spherical quadrilateral, corresponding to BD in Figure 3.8 (a).[80] M.T. I is used six times in Books I, II, and VIII.[81] With one exception, M.T. I is used to find a part of one of the inner legs of the spherical quadrilateral.[82] In most instances, Ptolemy uses the standard versions of the theorem which he stated in *Alm.* I 13.5 & 13.6. In other instances, the standard versions of these theorems will not allow him to calculate the desired arc. In these cases, he makes use of two alternate versions of each of M.T. I and M.T. II.[83] Although Ptolemy never demonstrates these alternate cases, Theon, in his *Commentary*, proves the plane lemmas

[79] In some instances the same arrangement is used to make two calculations. I have counted these as single instances. See, for example, *Alm.* I 16, Toomer [1984, 71 - 73].
[80] The two exceptions to this are *Alm.* II 7.5 & VIII 6, Toomer [1984, 95 & 415].
[81] Again, in some instances the same arrangement is used to make more than one calculation. Rome also counts a total of sixteen instances of either version of the theorem, Rome [1933a, 47].
[82] The exception is *Alm.* II 11.3, Toomer [1984, 113 - 114].
[83] These instances are *Alm.* I 14, II 2, 7.5, 11.3, 12.4, VIII 5.2, 6.1 & 6.2.

from which they could be derived.[84]

The expression by which we denote M.T. II may assume all of the benefit of a formalized notation:

$$\frac{\mathrm{Crd}(2\,\overset{\frown}{GE})}{\mathrm{Crd}(2\,\overset{\frown}{EA})} = \frac{\mathrm{Crd}(2\,\overset{\frown}{GZ})}{\mathrm{Crd}(2\,\overset{\frown}{DZ})} \times \frac{\mathrm{Crd}(2\,\overset{\frown}{DB})}{\mathrm{Crd}(2\,\overset{\frown}{BA})}.$$

Ptolemy's expression is purely verbal. His statement of this proposition reads,[85]

> The ratio of the [line] under the double of arc GE to the [line] under the double of arc EA is composed of the [ratio] of the [line] under the double of [arc] GZ to the [line] under the double of [arc] DZ and the [ratio] of the [line] under the double of [arc] BD to the [line] under the double of [arc] BA.

As is usual in such expressions, the Greek has undergone considerable ellipsis.[86] In this case, it is clear that the ellipsis contributes to intelligibility. We know what the statement as a whole asserts from the meaning and syntax of the expression, in reading through it we are particularly interested in the names and ordering of the mathematical objects GE, EA, GZ and so forth.[87] When we read "GZ" we already know that it is an arc and that the chord subtending its double is in a compound ratio. Ptolemy's expression imparts all of this information while keeping our attention on the arc "GZ" and its role in the overall expression. Ptolemy's expression of the fundamental theorem is what Netz calls a linguistic formula. The chords in question always appear in the same order.[88] In this, he no doubt followed his sources. This type of regulating process is not original in applied mathematics; it is borrowed from the theoretical tradition.

The most obvious example of Ptolemy's attempt to regulate his use of the fundamental theorem is in his figures. The figure which he uses to demonstrate the theorem is different

[84] The proofs are Rome [1943, 540 - 541] for *Alm.* I 14; Rome [1943, 539 - 540] for *Alm.* II 2, 11.4 & 12.4; Rome [1943, 542 - 543] for *Alm.* VIII 6.1 & 6.2; and Rome [1943, 543 - 544] for *Alm.* II 7.5 & VIII 5.2; Rome [1943, 543 - 544].

[85] Ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΕ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ λόγος συνῆπται ἔκ τε τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΖ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΖΔ καὶ τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΔΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΑ. Heiberg [1916, p. 1, 74].

[86] See Netz [1999b, 152 - 153] for a discussion of the role of ellipsis in Greek mathematical expressions.

[87] Netz [1999b, 127 - 167] provides a discussion of the ways in which the Greek geometers used linguistic structures to impart mathematical meaning.

[88] Netz [1999b, 127 - 167].

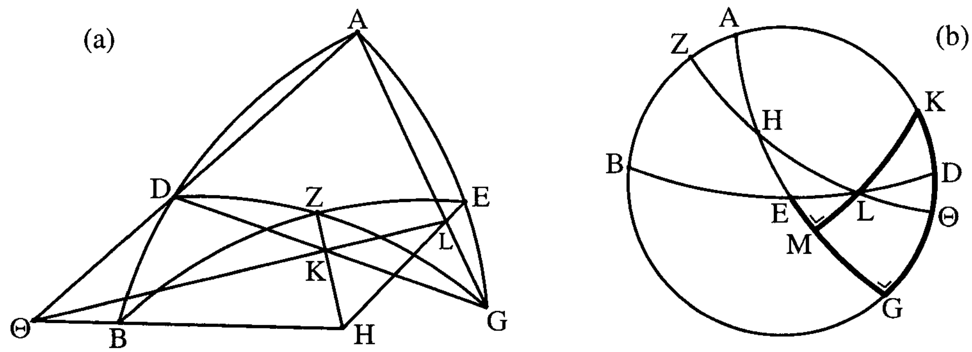

Figure 3.8: Diagrams for *Alm.* I 13.5 & II 7.3, Toomer [1984, 68 & 93].

from those he uses to apply the theorem. It features a concave spherical quadrilateral drawn on the surface of a sphere with no special reference to any circle in the sphere; see Figure 3.8 (a). Whenever the theorem is used, however, the plane of the figure contains a circle which is both a principal circle of the sphere and can be directly related to the concave quadrilateral. I call this principal circle the *reference circle*; in practice it is either the great circle through the poles of both the equator and the ecliptic, that is the solstitial colure (in I 14 - 15 & VIII 5.1), or the local meridian (in all other cases). The former is used whenever Ptolemy needs to relate the equatorial system to the ecliptic system under circumstances which are independent of the local horizon. The latter is used whenever the observer's position must be taken into account. With only two exceptions, the reference circle also contains one of the outer legs of the spherical quadrilateral.[89]

Ptolemy also attempts to regulate his figures through conventions of nomenclature. The reference circle is always called ABG, whether it is the solstitial colure or the meridian. The importance of the equator for spherical astronomy is made clear by the fact that, whenever it appears in the figure, it is called AEG. Whenever the horizon appears in the figure, it is called BED. These names may also serve for other principal great circles if the equator or the horizon are not used in the figure. The name of the ecliptic

[89] The exceptions are II 7.5 & II 12.4, Toomer [1984, 95 - 97 & 121 - 122].

varies according to whether it is being related to the equator, the horizon or both.[90]

An example will make this clear and will serve to introduce a number of Ptolemy's other regulating processes. In *Alm.* II 7.3, Ptolemy shows how to find the rising time of Aries for Rhodes; that is, where $\varphi = 36°$ and $M = 14^1/_4{}^h$; see Figure 3.8 (b). He says:[91]

> Let $ABGD$ be a meridian, BED the semicircle of the horizon, AEG the semicircle of the equator, and $ZH\Theta$ the semicircle of the ecliptic, positioned so that H represents the spring equinox. Take K as the north pole of the equator, and draw through K and L, which is the intersection of the ecliptic and the horizon, the great circle quadrant $\overset{\frown}{KLM}$ [Theo. *Spher.* I 20]. Let the problem be: given $\overset{\frown}{HL}$, to find the arc of the equator which rises with it, that is $\overset{\frown}{EH}$. First let $\overset{\frown}{HL}$ comprise the sign of Aries. Then since, in the diagram, the two great circles ED and KM are drawn to meet the two great circles EG and GK, and intersect each other at L, $\text{Crd}(2\,\overset{\frown}{KD}) : \text{Crd}(2\,\overset{\frown}{DG}) = (\text{Crd}(2\,\overset{\frown}{KL}) : \text{Crd}(2\,\overset{\frown}{LM})) \times (\text{Crd}(2\,\overset{\frown}{ME}) : \text{Crd}(2\,\overset{\frown}{EG}))$ [M.T. II]. But $2\,\overset{\frown}{KD} = 72°$ [since $\overset{\frown}{KD} = \varphi$], so $\text{Crd}(2\,\overset{\frown}{KD}) = 70;32,4^p$ [**Chord Table**]; $2\,\overset{\frown}{GD} = 108°$ [since $\overset{\frown}{GD} = 90° - \varphi$], so $\text{Crd}(2\,\overset{\frown}{GD}) = 97;4,56^p$ [**Chord Table**]. And $2\,\overset{\frown}{KL} = 156;40,1°$ [since, from the **Table of Inclination**, *Alm.* I 15, we have $\overset{\frown}{LM} = 11;39,59°$ for ♉ 0°, so that $\overset{\frown}{KL} = 90° - 11;39,59°$], so $\text{Crd}(2\,\overset{\frown}{KL}) = 117;31,15^p$ [**Chord Table**]; $2\,\overset{\frown}{LM} = 23;19,59°$, so $\text{Crd}(2\,\overset{\frown}{LM}) = 24;15,57^p$ [**Chord Table**]. Therefore, $\text{Crd}(2\,\overset{\frown}{ME}) : \text{Crd}(2\,\overset{\frown}{ED}) = (70;32,4 : 97;4,56) \div (117;31,15 : 24;15,57) = (18;0;5 : 120)$. And $\text{Crd}(2\,\overset{\frown}{EG}) = 120^p$ [since $\overset{\frown}{EG}$ is a quadrant], therefore $\text{Crd}(2\,\overset{\frown}{ME}) = 18;0;5^p$, therefore $2\,\overset{\frown}{ME} \approx 17;16°$ and $\overset{\frown}{ME} = 8;38°$. And since the whole of $\overset{\frown}{HM}$ rises with the whole of $\overset{\frown}{HL}$ at *sphaera recta*, it is $27;50°$, as was shown above [*Alm.* I 16.2, **List of Rising Times at *Sphaera Recta***]. Therefore, by subtraction, $\overset{\frown}{EH} = 19;12°$.

We see that the relevant arcs of the quadrilateral configuration are $\overset{\frown}{KG}$, $\overset{\frown}{EG}$, and $\overset{\frown}{KM}$; shown in bold in Figure 3.8 (b). $\overset{\frown}{KG}$ is a quadrant of the reference circle which in this case is the meridian. The concave quadrilateral is constructed such that all of its

[90] See Netz [1999b, 79 - 80] for some other examples of the way that names can be used to signify a particular object in Greek geometrical texts.

[91] Toomer [1984, 92 - 93]. I have changed some notation and inserted some text in brackets.

relevant legs are quadrants. This addresses the need to have five given arcs to determine one unknown arc. We saw the role of complementary angles in the calculation above. In every case where Ptolemy uses *Alm.* I 13.5 or 13.6, he constructs the concave quadrilateral to contain at least two relevant quadrants. In most cases, these are the two outer legs of the configuration.[92] In the other cases, quadrants are used as two of the relevant legs of the configuration, or as a part of one of these. Both the location of the configuration in the reference circle and the use of external quadrants serve to make the use of the fundamental theorem more consistent and therefore more transparent. The exceptions to this practice are meaningful.

For example, following the calculation that we have just cited, Ptolemy proceeds to demonstrate another method of calculating rising times, *Alm.* II 7.5.[93] He calls this method "easier and more practical" and he intends us to understand that he actually used it in calculating the table of rising times, *Alm.* II 8.[94] It introduces a method for calculating 10° intervals of the ecliptic based on the difference between rising times at *sphaera recta* and *sphaera obliqua* and a more complicated arrangement of the fundamental theorem. This batch procedure is especially useful for calculating a set of values for similar arrangements of the concave quadrilateral because it uses a single instance of the fundamental theorem, keeping one of the given ratios constant while the other is changed to produce different values in the final ratio. In this use of the theorem, neither of the outer arcs are on the reference circle, nor are they both quadrants; see Figure 3.9 (a). In fact, this use of the fundamental theorem is an exception to almost every one of Ptolemy's usual practices. Its only advantage is that it is readily adaptable to the large number of calculations that must be carried out to construct the table of rising times. Moreover, this second calculation more than reproduces the results of the first, so that the presence of *Alm.* II 7.3 in the text becomes a curiosity. The calculation in *Alm.* II 7.3 likely represents historical strata in the text. It was probably used as the only method

[92] There are four exceptions, II 7.5, II 12.4, VIII 5.1 & VIII 6.1, Toomer [1984, 95 - 99, 120 - 121, 411 - 412, 415 - 415].
[93] Toomer [1984, 94 - 99].
[94] Toomer [1984, 94].

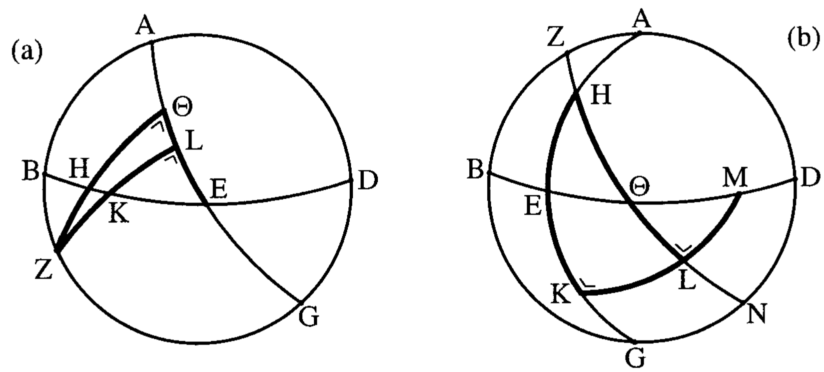

Figure 3.9: Diagrams for *Alm.* I 7.3 & II 12.5, Toomer [1984, 96 & 121].

of calculating rising times in one of Ptolemy's sources.[95] It should be noted that *Alm.* II 7.3 is not used again in the rest of the text.[96]

Another regulating practice that *Alm.* II 7.3 demonstrates is the position of the unknown arc in the statement of the fundamental theorem. In almost every case, the unknown arc is given in the penultimate position in the compound ratio. There are two exceptions, in which the unknown term appears in the antepenultimate position.[97] In both of these exceptions, the ratios could easily have been rearranged so that the unknown term fell in the penultimate position. This regulating practice, like those already mentioned, is a tendency rather than a rule.

A final regulating practice should be mentioned. We saw that, in *Alm.* II 7.3, Ptolemy used the fundamental theorem to express the ratio which contained the unknown arc in terms of 120. He does this because the final term in the final ratio is a quadrant and this allows him to immediately read off the final calculation of the compound ratio. In general, Ptolemy attempts to set a quadrant in this final position. In the actual calculations there is only one exception to this practice.[98] There are three other exceptions to this practice,

[95] Björnbo [1902, 72 - 76] was of the opinion that this method was used by Hipparchus to calculate the rising times for signs.

[96] Toomer notes that *Alm.* II 10.3 makes reference to *Alm.* II 7.3, Toomer [1984, 109, n. 91], but mathematically this reference must be to *Alm.* I 15, the table of inclinations, upon which *Alm.* II 7.3 also depends.

[97] The exceptions are *Alm.* I 14 & II 7.5, Toomer [1984, 69 - 70 & 95 - 97].

[98] This is again II 7.5, Toomer [1984, 95 - 97].

but these are all in metrical analyses where no calculation is actually carried through.[99] Ptolemy can so arrange the quadrilateral configuration in these cases because of his extensive use of quadrants.

We have seen how Ptolemy tends to use the fundamental theorem with a quadrilateral configuration that is (1) part of the reference circle and (2) consists of two or more quadrants, such that (3) the desired term is the third term in the compound ratio and (4) is in ratio with a quadrant. Against the background of these practices, any deviation can be see to be determined by the context in which it appears and can help us understand the structure of the argument.

For example, in *Alm.* II 12.5 the reference circle could have been shifted to the altitude circle so that the outer leg of the concave quadrilateral could have been in the reference circle; see Figure 3.9 (b). But this is an example calculation and it needs to be read with *Alm.* II 12.4 in which the quadrilateral is $AB\Theta H$.[100] The altitude circle $\widehat{AEG}$ rotates from Θ to B as the sun moves from sunrise to noon and the table of the ecliptic and altitude circles, *Alm.* II 13, is calculated at hourly intervals on either side of the meridian ABG. The two calculations, *Alm.* 12.4 & 12.5, show how to calculate $\widehat{AH}$ and $\angle AH\Theta$ at one hour before noon. The figure shows both the altitude circle and the meridian and hence gives the complete picture. Under these circumstances, Ptolemy diverges from his regular practice and constructs a quadrilateral configuration which is not situated in the reference circle.

The use of the fundamental theorem in *Alm.* II 7.5 breaks almost every one of the regulating practices mentioned; see Figure 3.9 (a). The concave quadrilateral does have two quadrants but the unknown arc is not part of one of these. It is only used because it facilitates the great number of calculations involved in constructing the table of rising times. The obscurity and laboriousness of *Alm.* II 7.5 is such as to make it less useful either for the calculation of individual rising times or to readily convince the reader that the rising times are determined by the principal great circles. On the other hand, *Alm.*

[99] These are *Alm.* VIII 5.2, 6.1 & 6.2, Toomer [1984, 412 & 415 - 146].
[100] Toomer [1984, 119 - 122].

II 7.3, which seems to have no logical function in the text, serves both of these purposes well, and was presumably carried over from Ptolemy's source for this reason. The purpose of *Alm.* II 7.5 is to show us how the batch calculations of the table of rising times could have actually been carried out.

We have seen how the figures and the text are used to make the application of the fundamental theorem as systematic as possible. These practices are ordering tendencies not rules. They may be changed when the astronomical or mathematical context of the argument demands it. Against the background of regularity, divergence can be read as meaningful.

3.4 The logical coherence of Ptolemy's spherical astronomy

Ptolemy's appeal to observation and empirical argumentation can obscure the fact that the *Almagest* is, as its original Greek title states, a systematic mathematical treatise. The logical structure of its thirteen books is modeled on that of the thirteen books of Euclid's *Elements*. Ptolemy's presentation of spherical astronomy can serve as an example of the architecture that informs the *Almagest* as a whole. Ptolemy is able to derive nearly all of the phenomena associated with the movement of the fixed stars and the diurnal motion of the sun by taking as his starting points only the traditional values of the obliquity of the ecliptic, $\varepsilon = 23;51,20$, and a sample given latitude; in this case, the value for Rhodes, $\varphi = 36^\circ$.[101]

Ptolemy's treatment of spherical astronomy makes use of five of the six types of mathematical texts which we discussed in Section 1.2.[102] It acts as a continuous argument built on the fundamental theorem of ancient spherical trigonometry, the table of inclination and the list of rising times at *sphaera recta*. In general, the theorems and metrical

[101] Spherical astronomy ignores the long-term phenomena of the precession of the equinoxes.
[102] There are no problems in this material.

analyses act as lemmas to calculations which in turn serve as examples of how tables are constructed. The tables are then used in further calculations. This path of implication is not, however, strictly maintained and the tables, or rather the principles that they exhibit, can be used as a justification for a step in a theorem or metrical analysis. In this way, the tables themselves take on some of the character of theorems.

The modes of mathematical discourse we encountered in Chapter 1 form the basic units of Ptolemy's mathematical argument. Although these units are of different types, they nonetheless work together in interesting and sometimes complex ways. When we study the architecture of Ptolemy's works as an abstract system, we see that the units of mathematical prose function as propositions do in the theoretical traditions. In theory, any previous mathematical unit can be assumed as known and drawn on to justify a step in a later unit. In practice, different types of mathematical text tend to draw on a subset of the other modes.

Ptolemy does not only assume his own results. He also relies on results that can be found in, or derived from, the elementary treatises in the fields of mathematics in which he works. He almost never makes any reference to these theorems in the sense that a Greek mathematician will sometimes repeat a brief version of a theorem's enunciation. He simply assumes the necessary result, sometimes giving a sketchy justification. We saw above the application of the results of Euclid's *Data* in *Alm.* I 13.3c and 13.4c.[103] We also observed that Ptolemy assumed some results that could be simply derived from, but were not found in, the *Data*. This practice is commonplace in the *Almagest*. Ptolemy's toolbox is not a strict group of theorems found in earlier work, but the basic knowledge of geometry, analysis and ratio manipulation which these texts furnish. The texts upon which Ptolemy's spherical astronomy relies are Euclid's *Elements* and *Data* and the *Spherics* of Theodosius and Menelaus. The justification for every step in Ptolemy's spherical astronomy, which is not provided by the *Almagest*, can be either found in, or furnished by, the theorems in these texts.

[103] See page 142.

A couple of examples will show how Ptolemy employs the results of his predecessors. We have already seen Abū Naṣr's proof of the plane version of M.T. I. Ptolemy's is very similar; see Figure 3.10 (a). *Alm.* I 13.1 reads as follows.[104]

> Let two lines, BE and GD which are drawn to meet two straight lines, AB and AG, cut each other at point Z. I say that $GA : AE = (GD : DZ) \times (ZB : BE)$. Let EH be drawn through E parallel to GD. Then, since $GD \parallel EH$, $GA : AE = GD : EH$ [1]. If we bring in ZD, $GD : EH = (GD : DZ) \times (DZ : EH)$ [2]. Therefore, $GA : AE = (GD : DZ) \times (DZ : HE)$. But $DZ : HE = ZB : BE$ $(EH \parallel ZD)$ [*Elem.* VI 2]. Therefore $GA : AE = (GD : DZ) \times (ZB : BE)$.

This proof makes one sketchy appeal to *Elem.* VI 2 and assumes two elementary results which are not found in Euclid's work. The first assumed result, [1], is the fact that since EH is drawn parallel to the base of $\triangle AGD$, $\triangle AEH \sim \triangle AGD$ and thus $GA : AE = GD : EH$. Ptolemy assumes this as an obvious result of the parallelism, as indeed it is; nevertheless, it is worth pointing out that Abū Naṣr fleshed this out and demonstrated the similarity of the triangles. The second assumption, [2], involves compound ratios and is equally transparent. It is the statement that, for any A, B and C, $A : B = (A : C) \times (C : B)$. There is no proof of this in the elementary texts, yet Ptolemy assumes that it is obvious. The Greek expression that Toomer translates as "if we bring in ZD," is ἔξωθεν δε ἡ ΔΖ, "but DZ is from outside."[105] It is ellipsis for ἀλλα τῆς ΖΔ ἔξωθεν λαμβανομένης, "but with DZ taken from without," given in the next lemma.[106] It is clear that these are technical expressions for a familiar operation. This step is what Theon attempted to justify with his analysis/synthesis lemma on compound ratios.[107] It should come as no surprise that Ptolemy assumes this step without justification since compound ratios were used extensively by Greek geometers despite having slight theoretical foundations.[108] Here, we see that Ptolemy's toolbox is a body

[104] Toomer [1984, 64]. I have changed some notation and inserted some text in brackets.
[105] Heiberg [1916, p. 1, 69].
[106] Heiberg [1916, p. 1, 70].
[107] See page 146 - 148.
[108] Saito has discussed the role of compound ratios in Euclid and Apollonius, Saito [1986]. See also Section 1.3, page 38.

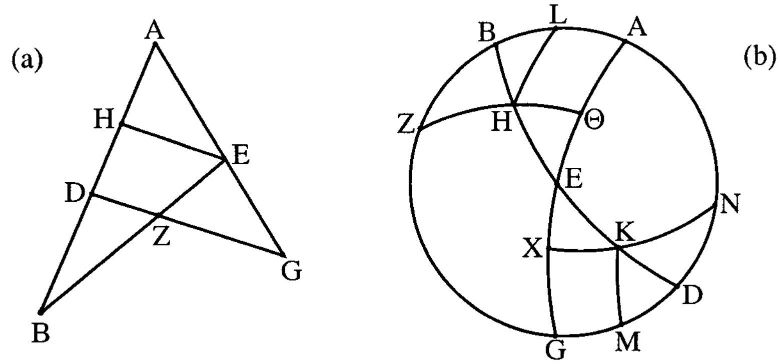

Figure 3.10: Diagrams for *Alm.* I 13.1 & II 3.3c, Toomer [1984, 64 & 79].

of mathematical knowledge that is informed by, and derived from, but not necessarily confined to the elementary texts of the theoretical tradition.

The second example shows how Ptolemy makes use of the results of Theodosius and Menelaus. At the end of the corollary on ortive amplitude, η, *Alm.* II 3.3c, there is a proof that points of the ecliptic which are equidistant from an equinox, that is, points on equal parallel circles, rise so as to cut off equal arcs of the horizon on either side of the equinox, η. In Figure 3.10 (b), this amounts to showing that $\overset{\frown}{HE} = \overset{\frown}{EK}$. The proof reads as follows.[109]

> For in the figure ... we put K as the point in which the parallel circle equal to the parallel through H cuts the semicircle BED of the horizon; we draw in arcs HL and KM of the parallel circles: these will, clearly, be equal and opposite [1]. We draw through K and the north pole the quadrant NKX [Theo. *Spher.* I 20]. Then $\overset{\frown}{\Theta A} = \overset{\frown}{XG}$ ($\overset{\frown}{\Theta A} \parallel \overset{\frown}{LH}$ and $\overset{\frown}{XG} \parallel \overset{\frown}{MK}$) [Theo. *Spher.* II 10]. Therefore $\overset{\frown}{E\Theta} = \overset{\frown}{EX}$ (complements [of $\overset{\frown}{\Theta A}$ and $\overset{\frown}{XG}$]). Then, in the two similar spherical triangles $EH\Theta$ and EKX, we have two pairs of corresponding sides equal, $E\Theta = EX$, and $H\Theta = KX$ [by Theo. *Spher.* II 18], and both of the angles at Θ and X are right, so the base EH equals the base KE [Men. *Spher.* I 4].

The crux of the argument amounts to a proof of the converse of Theo. *Spher.* III 13 through the properties of spherical triangles investigated by Menelaus. This proof draws

109 Toomer [1984, 79]. I have changed some notation and inserted some text in brackets.

on theorems from the *Spherics* of both Theodosius and Menelaus because it involves both parallel circles and spherical triangles, which are the characteristic objects around which each of these texts builds its theory of spherical geometry. The first step in the argument, [1], may be obscure, despite Ptolemy's remark to the contrary. It draws upon a theorem such as the following: arcs of equal parallel circles which are cut off between a great circle drawn through the poles of the parallels and a great circle oblique to the greatest of the parallels will be equal. No theorem to this effect is found in Theodosius' *Spherics*. Such a theorem can be derived from Theodosius' work; however, in general, Theodosius has little interest in the properties of parallel circles which are symmetrical about the greatest parallel. Ptolemy may have simply drawn his conclusion from the symmetry of the figure. This example shows how Ptolemy makes ready use of both his major predecessors in spherical geometry, employing them individually for his treatment of parallel circles and spherical triangles. This is indicative of Ptolemy's use of the two *Spherics* throughout his treatment of spherical astronomy.

We have already seen an example of how a calculation relies on previous theorems and tables in *Alm.* II 7.3.[110] Two more examples will suffice to show how Ptolemy uses his own theorems and previous results. In *Alm.* II 11.1, Ptolemy shows that "points on the ecliptic equidistant from the same equinox produce equal angles at the same horizon." His proof reads as follows; see Figure 3.11 (a).[111]

> Let $ABGD$ be a meridian circle, AEG the semicircle of the equator and BED the semicircle of the horizon. Draw two segments of the ecliptic, $ZH\Theta$ and KLM, such that points Z and K both represent the autumnal equinox, and $\overset{\frown}{ZH} = \overset{\frown}{KL}$. I say that $\angle EH\Theta = \angle DLK$. This is immediately obvious. For spherical triangle EZH is congruent to spherical triangle EKL; since, from what was proven above, the corresponding sides are equal: $\overset{\frown}{ZH} = \overset{\frown}{KL}$ [by hypothesis], $\overset{\frown}{HE} = \overset{\frown}{EL}$ ([arcs cut off by] the intersection of the horizon [with the ecliptic]) [*Alm.* II 3.3c], and $\overset{\frown}{EZ} = \overset{\frown}{EK}$ (rising time arcs [*Alm.* II 7.1]). Therefore, $\angle EHZ = \angle ELK$ [Men. *Spher.* I 18].

[110] See page 157.

[111] Toomer [1984, 110]. I have changed some notation and inserted some text in brackets.

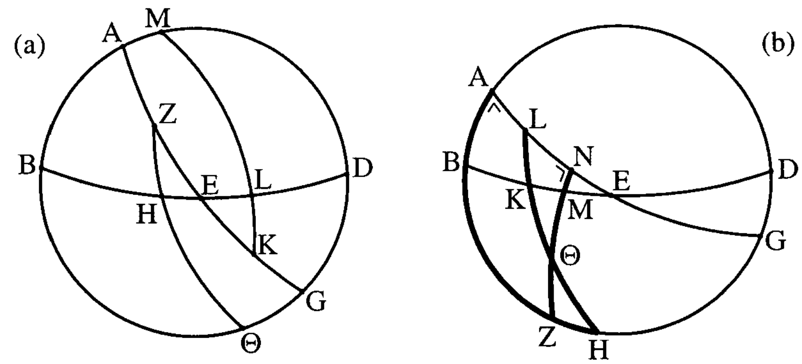

Figure 3.11: Diagrams for *Alm.* II 11.1 & VIII 5.1, Toomer [1984, 110 & 411].

Therefore, $\angle EH\Theta = \angle DLK$ (supplements).

Here we have a short theorem in which Ptolemy makes use of two of his own previous results and one of Menelaus' results. This theorem again shows the influence of Menelaus' *Spherics* in the use of the properties of spherical triangles. Ptolemy refers to his own results with very brief, almost obscure, appositive genitives that are meant to tell us what the arcs in question represent and hence what their properties are. The Greek literally reads, "For again, the three-sided figure EZH is equiangular to the [three-sided figure] EKL, since, by what was proven above, it has the three sides equal to the three sides, each to each, ZH to KL; HE, the section of the horizon, to EL; EZ, the rising time, to EK."[112] Cryptic references of this sort are typical of Ptolemy when referring to his own results. In this theorem, both of the previous results referred to are also theorems and and they are demonstrated in a similar vein. They use the previous results of spherical geometry, especially the properties of spherical triangles investigated by Menelaus.

A final example will show how all of the various types of mathematical text can be used in a metrical analysis. In *Alm.* VIII 5.1, Ptolemy gives a metrical analysis which shows that the arc of the ecliptic culminating with a given star will be given. This amounts to showing how to calculate either $\overset{\frown}{BM}$ or $\overset{\frown}{ME}$ since B is a solstice and E is

[112] Ἰσογώνιον γὰρ πάλιν γίνεται τὸ EZH τρίπλευρον τῷ ΕΚΛ, ἐπεὶ διὰ τὰ προδεδειγμένα καὶ τὰς τρεῖς πλευρὰς ταῖς τρισὶ πλευραῖς ἴσας ἔχει ἑκάστην ἑκάστῃ, τὴν μὲν ZH τῇ ΚΛ, τὴν δὲ HE τῆς τομῆς τοῦ ὁρίζοντος τῇ ΕΛ, τὴν δὲ EZ τῆς ἀναφορᾶς τῇ EK. Heiberg [1916, p. 1, 155].

an equinox; see Figure 3.11 (b). The proof reads as follows.[113]

> Let the circle through both poles, that of the equator and that of the ecliptic, be $ABGD$. Let AEG be a semicircle of the equator about pole Z, and BED a semicircle of the ecliptic about pole H. Draw through the poles of the ecliptic the great circle segment $H\Theta KL$ [Theo. *Spher.* I 20], and take on it point Θ as the required fixed star (for it is with respect to such circles [that is, in ecliptic coordinates] that we have observed and recorded the positions of the fixed stars). Also, draw through the poles of the equator and the star at Θ the great circle segment $Z\Theta MN$ [Theo. *Spher.* I 20]. Now it is obvious that the star at Θ culminates simultaneously with points M and N of the ecliptic and the equator. But these points and $\overset{\frown}{\Theta N}$, are given, as will be clear from the following considerations. From what we proved at the beginning of our treatise, since the great circle arcs $\overset{\frown}{HL}$ and $\overset{\frown}{NZ}$ have been drawn to meet the two great circle arcs $\overset{\frown}{AH}$ and $\overset{\frown}{AN}$, $\mathrm{Crd}(2\,\overset{\frown}{HA}) : \mathrm{Crd}(2\,\overset{\frown}{AZ}) = (\mathrm{Crd}(2\,\overset{\frown}{HL}) : \mathrm{Crd}(2\,\overset{\frown}{L\Theta})) \times (\mathrm{Crd}(2\,\overset{\frown}{N\Theta}) : \mathrm{Crd}(2\,\overset{\frown}{ZN}))$ [M.T. I, *Alm.* I 13.6]. But, immediately by hypothesis, each of the arcs $\overset{\frown}{AZ}$, $\overset{\frown}{ZN}$ and $\overset{\frown}{HK}$ are given as quadrants; from the [star] catalog, $\overset{\frown}{K\Theta}$ is given as the star's latitude and $\overset{\frown}{KB}$ as its longitude [*Alm.* VII 5 - VIII 1]; and $\overset{\frown}{ZH}$ and $\overset{\frown}{KL}$ are given from the demonstrated obliquity of the ecliptic [$\overset{\frown}{ZH} = \varepsilon$, $\overset{\frown}{KL}$ is given by an application of *Alm.* II 8 and I 15].[114] Hence, it is clear that, of the arcs in question, $\overset{\frown}{HA}$ [$= 90° - \varepsilon$], $\overset{\frown}{AZ}$, $\overset{\frown}{HL}$ [$= 90° + \overset{\frown}{KL}$], $\overset{\frown}{L\Theta}$ [$= 90° + \beta_\star$] and $\overset{\frown}{NZ}$ are given. Hence the remaining arc, $\overset{\frown}{N\Theta}$, will also be given. Again, since $\mathrm{Crd}(2\,\overset{\frown}{ZH}) : \mathrm{Crd}(2\,\overset{\frown}{HA}) = (\mathrm{Crd}(2\,\overset{\frown}{Z\Theta}) : \mathrm{Crd}(2\,\overset{\frown}{\Theta N})) \times (\mathrm{Crd}(2\,\overset{\frown}{NL}) : \mathrm{Crd}(2\,\overset{\frown}{LA}))$ [M.T. II, *Alm.* I 13.5], and by the above, of the arcs in question, $\overset{\frown}{ZH}$, $\overset{\frown}{HA}$, $\overset{\frown}{Z\Theta}$ [$= 90° - \overset{\frown}{\Theta N}$], and $\overset{\frown}{\Theta N}$ are given, and $\overset{\frown}{LA}$ is given from $\overset{\frown}{KB}$, by means of [the arcs of] the equator which rise together with [those of] the ecliptic at *sphaera recta* [*Alm.* I 16], the remaining arc, NL, will also be given. Similarly $\overset{\frown}{MB}$ will be given from $\overset{\frown}{NA}$ [by means of the rising-times at *sphaera recta*, *Alm.* I 16], the sum

[113] Toomer [1984, 411 - 412]. I have changed some notation and inserted some text in brackets.

[114] Toomer's note here is wrong, Toomer [1984, 411, n. 201]. $\overset{\frown}{KL}$ is not the δ of point K. Nevertheless, the methods of *Alm.* I 15 & 16 can still be used to find $\overset{\frown}{KL}$, as Neugebauer has made clear, Neugebauer [1975, 32 - 33]. We derive $\overset{\frown}{LE}$ from $\overset{\frown}{KE}$ by an inverse application of *Alm.* I 16.2. We then enter $\overset{\frown}{LE}$ into *Alm.* I 15 to find $\overset{\frown}{KL}$.

$[= \widehat{NL} + \widehat{LA}]$.

This metrical analysis is a fine example of Ptolemy's spherical astronomy at work. It demonstrates the use of both versions of the fundamental theorem and makes reference to the two fundamental tables, *Alm.* I 13.5, 13.6, 15, & 16.2. Nevertheless, Ptolemy's treatment of the movements of the fixed stars is cursory and seems to be only a gesture toward a traditional topic of spherical astronomy. He is only interested in showing how his methods are generally applicable to solving these problems. He has no interest in actually carrying these calculations out. This is well enough because the calculations involved are laborious. We have required two applications of the fundamental theorem to find the point of the ecliptic which culminates with a given star. Ancient astronomers were mostly interested in the stars which rose and set with given points on the ecliptic. To find the rising point from the culminating point requires yet another application of the fundamental theorem. These calculations would also involve referring to the two tables that are most essential to Ptolemy's spherical astronomy, the table of inclination, *Alm.* I 15, and the list of rising times at *sphaera recta*, *Alm.* I 16.2.[115] This metrical analysis shows how Ptolemy's methods can be used to convert the ecliptic coordinates of the star catalog to the equatorial coordinates of spherical astronomy, but there is no indication that Ptolemy had any interest in actually performing such transformations.

We have now seen examples of how Ptolemy uses the mathematical results of his predecessors along with those of his own to build a single deductive argument. I have made a table that shows the interdependence of *Alm.* I 13 - II 13 & VIII 5 - 6, see Tables 3.1 & 3.2. This table can be compared with tables made for other systematic treatises to reveal the similarity of deductive structure.[116] The logical structure of Ptolemy's spherical astronomy exhibits the same characteristics as that of the systematic geometric

[115] This analysis could also be taken to refer to *Alm.* II 8, but since *Alm.* I 16.2 is prior and simpler it seems the clear choice.

[116] Berggren gives tables for Theodosius' *Spherics* II, and Archimedes' *Plane Equilibria* I and *On Spirals*, Berggren [1991b, 24] and Berggren [1976, 3 - 94]. Taisbak gives a table for Euclid's *Elem.* VII and a number of tables for the *Data*, Taisbak [1971, 118] and Taisbak [2003, 55, 92, 114, 144 - 145, 176, 206 & 238 - 239].

Table 3.1: Logical structure in Ptolemy's spherical astronomy. This table shows how *Alm.* I 13 - II 16 are used to support the units that follow. A unit in the right hand column is supported by each unit marked •. *Alm.* I 11 is the chord table. Theorems from the *Spherics* of Theodosius and Menelaus are all simply denoted "Theo." or "Men." respectively.

	I 11	I 13.1	I 13.2	I 13.3	I 13.3c	I 13.4	I 13.4c	I 13.5	I 13.6	I 14	I 15	I 16.1	I 16.2	II 2	II 3.1	II 3.2	II 3.3	II 3.3c	II 4	II 5.1	II 5.2	II 6	Theo.	Men.
I 13.1																								
I 13.2																								
I 13.3																								
I 13.3c	•			•																				
I 13.4																								
I 13.4c	•					•																		
I 13.5			•	•		•																		
I 13.6		•		•		•																		
I 14	•								•															
I 15										•														
I 16.1	•							•																
I 16.2												•												
II 2	•								•															
II 3.1	•							•																
II 3.2	•							•																
II 3.3	•								•															
II 3.3c																							•	•
II 4											•													
II 5.1	•																							
II 5.2	•																							
II 6	•														•		•		•	•				
II 7.1																								•
II 7.2																		•						
II 7.3	•							•				•	•											
II 7.4																							•	
II 7.5	•							•			•		•			•								
II 8																						•		
II 9.1																								
II 9.2																								
II 9.3																								
II 9.4																								
II 9.5																								
II 9.6																								
II 10.1											•													•
II 10.2																								•
II 10.3	•							•																
II 11.1																		•						•
II 11.2																								
II 11.2c																								
II 11.3	•								•		•													
II 12.1																							•	•
II 12.2																							•	•
II 12.3											•													
II 12.4	•							•	•															
II 13																								

treatises. We see certain groups of units being used only as lemmas to other units (I 13, II 7 & II 12). We see how some units can act as key results for short stretches of the text (II 7.1 & II 8) while other units are fundamental to the entire project (I 13.5, 13.6 & I 15).

We see at once that there are very few units that are not used again in the text. In

Table 3.2: Continuation of Table 3.1. This table shows how units II 7 - II 12 are used to justify the units that follow.

	II 7.1	II 7.2	II 7.3	II 7.4	II 7.5	II 8	II 9.1	II 9.2	II 9.3	II 9.4	II 9.5	II 9.6	II 10.1	II 10.2	II 10.3	II 11.1	II 11.2	II 11.2c	II 11.3	II 12.1	II 12.2	II 12.3	II 12.4	Theo.	Men.
II 7.1																									•
II 7.2																								•	
II 7.3																									
II 7.4	•	•																						•	
II 7.5				•																					
II 8	•	•			•																				
II 9.1						•																			
II 9.2						•																			
II 9.3						•																			
II 9.4						•																			
II 9.5						•																			
II 9.6						•																			
II 10.1	•																								•
II 10.2																									•
II 10.3													•	•											
II 11.1	•																								•
II 11.2																									
II 11.2c																•	•								
II 11.3											•														
II 12.1														•										•	•
II 12.2																								•	•
II 12.3															•				•						
II 12.4										•	•														
II 13																				•	•	•	•		

particular, there is no table which is not later employed in calculations.[117] *Alm.* II 2 which introduces the concept of ortive amplitude and calculates it for winter solstice at the latitude of Rhodes, is not used again in the spherical astronomy but does play an important role in Ptolemy's theory of eclipses, *Alm.* VI. Likewise *Alm.* II 9.1-3, which handle conversions between seasonal and equinoctial hours, are not used in the spherical astronomy; nevertheless, they are necessary whenever Ptolemy uses an observation recorded in seasonal hours.

A few units really do seem to serve no logical purpose in the text. We have already taken note of the two lemmas of metrical analysis, *Alm.* I 13.3c & 13.4c and argued that they have been included because they were found in one of Ptolemy's sources where they still played a role in the deductive structure. *Alm.* II 11.2c is another short corollary which is not used elsewhere and is probably included for the sake of completeness. I have already discussed *Alm.* II 7.3 and argued that it is included because it was used by one

[117] The table found in *Alm.* II 13 is one of the most impressive tables in the *Almagest.* It is used in Ptolemy's theory of eclipses in *Alm.* VI.

of Ptolemy's sources and makes a clearer presentation of the matter at hand than *Alm.* II 7.5. The other units that are not used may also be easily explained. *Alm.* II 5 is a gesture toward a traditional approach; however, as Ptolemy points out, this approach follows an unsound method. *Alm.* II 9.6 is simply a secondary method which reproduces the result of *Alm.* II 9.5. Moreover, all of these seemingly needless units are quite short, and even if they do not add to the text as a whole, they do not detract from the flow of the argument.

The exposition of spherical astronomy in the *Almagest* is quite economical. All of the theorems are derived from four key works in elementary and spherical geometry. All of the calculations depend on only one theorem and two tables. The only empirical basis is the two traditional values $\varepsilon = 23;51,20°$ and $\varphi = 36°$.[118] Ptolemy himself tells us that it is his desire to "carry out most demonstrations involving spherical theorems in the simplest and most methodical (μεϑοδικώτερον) way possible."[119] There is little doubt that Ptolemy's practice is methodical; however, Björnbo, and more recently Nadal, Taha and Pinel, have made it clear that a number of Ptolemy's calculations could have been simplified if he had made use of more of the theorems in Menelaus' *Spher.* III which are devoted to metrical methods.[120] This might seem true at first glance until we take into consideration Ptolemy's actual practices. In order to use any non-elementary theorems from the *Spherics*, Ptolemy would have to prove them, along with any other non-elementary theorems which they require. Moreover, the uses of these theorems in the *Almagest* are all served by a single application of the fundamental theorem. It is not clear how much computation would really be saved. In order to make use of these other theorems, Ptolemy would have had to lay out more than twice again as many preliminary theorems to facilitate a small fraction of his computational work. Given the ancient methods, Ptolemy's practice is both simpler and more methodical.

[118] Despite Ptolemy's discussion of how to find ε by observation in *Alm.* I 12, the value he actually adopts comes from Eratosthenes, as Toomer [1984, 63, n. 75] points out.
[119] Toomer [1984, 64], with slight modifications.
[120] Björnbo [1902, 93 - 94 & 115] and Nadal, Taha and Pinel [2004, 403 - 418].

3.5 Spherical astronomy in the Hellenistic and Imperial Periods

There is no other text in the extant Greek corpus that approaches spherical astronomy in the same way as Ptolemy. This does not mean that there were not others before Ptolemy who developed the techniques that he uses; it simply means that none of this material is known today. The majority of the other surviving texts which deal with spherical astronomy treat rising times as the fundamental problem and give a purely qualitative description. Menelaus' *Spherics* lays the foundation for an advance metrical study of spherical astronomy but it goes no further than this foundation. Because they use no calculations, Theodosius and Menelaus are only able to state how arcs of the ecliptic rise relative to one another, and not even for all four quadrants. Schmidt, Berggren and Nadal, Taha and Pinel have drawn out the astronomical implications of these authors and shown how they address the same issues as Ptolemy.[121] The only other Greek text devoted to these issues is Hypsicles' *Ascensions*. This text is not geometric, so that, whereas it deals with the astronomical topic of rising times using mathematical methods, it cannot be called spherical astronomy. It attempts to give mathematical justification to a different, arithmetic tradition of treating the rising times of the signs of the zodiac.[122]

We should not characterize the *Spherics* of Theodosius and Menelaus as works of purely astronomical content. They are texts on spherical geometry that were motivated by astronomical study and contain theorems directly applicable to certain specialized astronomical problems; nevertheless, they are meant as studies of the properties of geometric objects in the sphere. Theodosius focuses on the relationships between, and properties of, great circles and sets of parallel circles. Menelaus is interested in the properties of spherical triangles formed by the arcs of great circles. This geometric motivation is shown in the fact that Men. *Spher.* II 10 - 13 provide alternative proofs of theorems

[121] Schmidt [1943], Berggren [1976] and Nadal, Taha and Pinel [2004].
[122] Evans [1998, 121 - 124] and Neugebauer [1975, 715 - 717].

that are already found in Theodosius.[123] These texts provide the toolbox for, and develop the intuition of, the mathematical astronomer. Ptolemy makes no use of those theorems in these writers which can be read as complex statements about the two-sphere astronomical model; nevertheless, every one of his theorems about geometrical objects on the sphere takes its point of departure from these authors.

What we really want to know is the provenance of Ptolemy's mathematical methods and techniques. We know from a remark which Pappus makes in his *Collection* that Menelaus wrote a book on rising times but, although it has never been doubted that this text employs the metrical methods of the *Almagest*, we know next to nothing of the book's contents.[124] In particular, we should like to know if Hipparchus had access to Ptolemy's trigonometric methods or if Menelaus was really the author of the eponymous theorem. It is generally assumed, following Neugebauer, that Hipparchus did not have access to the metrical methods of ancient spherical trigonometry. Neugebauer's reasons for believing this, however, are not clear. He simply says that it is evident from "everything we know about Greek 'Spherics' and Menelaus' role in it."[125] Björnbo, however, was of the opinion that the fundamental theorem went back at least as far as Hipparchus, and that Hipparchus had used the theorem in his work on rising times.[126] I have recently argued that there is a calculation in Hipparchus' *Commentary on the* Phaenomena *of Eudoxus and Adrastus* that can only be carried through by an application of the fundamental theorem of ancient spherical trigonometry.[127] Moreover, this calculation requires one of the two unused lemmas *Alm.* I 13.3c & 13.4c. This would imply that one of Ptolemy's sources went back at least as far as Hipparchus, if it was not by Hipparchus himself. In

[123] Björnbo [1902, 54 - 56] and Schmidt [1943, 50 - 60].
[124] Hultsch [1878, 600 - 601].
[125] Neugebauer [1975, 301]. He supports this statement with a footnote which refers to his own discussion of the Menelaus Theorem and a passage in Pappus' *Collection* that mentions Menelaus. Neugebauer's discussion of the Menelaus Theorem is purely mathematical and does not argue that Menelaus was the author of the theorem, Neugebauer [1975, 26 - 29]. The passage in Pappus only states that Menelaus wrote a book on rising times that mentioned, in particular, that the logic of proofs concerning setting times fell within the specifications (διορισμοί) of those for rising times, Hultsch [1878, 600 - 601]. Again, this has no bearing on whether or not Menelaus was the originator of his theorem.
[126] Björnbo [1902, 72 ff.]. Rome [1933a, 42] agrees with Björnbo that this is a real possibility. Tannery [1893, 63], in fact, was of the opinion that it went back to Apollonius.
[127] Sidoli [2004a].

fact, I think there are a number of reasons, albeit somewhat circumstantial, for thinking that Menelaus did not originate the theorem that bears his name.

We know that Hipparchus wrote a text on the rising times of the signs of the zodiac and we have good reason to believe that this was written for more latitudes than his own at Rhodes, $\varphi = 36°$. Pappus, in the midst of his lemmas on the texts of spherical astronomy, mentions this text and gives a few hints as to its contents. He says,[128]

> Hipparchus, in *On the Rising Times of the Twelve Signs of the Zodiac*, demonstrates, by means of numbers (δι' ἀριθμῶν), that [1] equal arcs of the semicircle following cancer rise and set in the same way, having a certain relationship (σύγκρισιν) of time to one another. [2] For there are some vicinities (τινας οἰκήσεις), in which, equal arcs of the semicircle following cancer which are nearer the ecliptic always rise in more time than those toward the points of the tropic. By this [reasoning], then, he says, [3] that arcs equally distant from the ecliptic make their rising in the same time.

This is a dense passage. The three statements seem to be about three different phenomena; and they do not have the logical dependency which Pappus suggests that they should have. The first sentence, [1], states that if $\alpha_1 = \alpha_2$ are arcs in the semicircle [♋0° → ♑0°], then $\sigma(\alpha_1) : \sigma(\alpha_2) = \rho(\alpha_1) : \rho(\alpha_2)$, where σ is setting time and ρ is rising time. This statement is trivial and does not commit Hipparchus to any particular scheme for rising times.

The second sentence, [2], asserts that there are certain latitudes in which if $\alpha_1 = \alpha_2$ are arcs in the semicircle [♋0° → ♑0°] and α_1 is closer to ♋0° than α_2, then $\rho(\alpha_2) > \rho(\alpha_1)$. This statement would seem to commit Hipparchus to a linear zigzag function of the sort found in Hypsicles' *Ascensions*. In that case, however, $\rho(\alpha_2) > \rho(\alpha_1)$ would be true for all latitudes and the expression *some vicinities* makes no sense.[129] This statement does

[128] Hultsch [1878, 600].

[129] Hypsicles' series of rising times for the latitude of Alexandria, $M = 14^h$, is ρ(♈) = 21;40°, ρ(♉) = 25°, ρ(♊) = 28;20°, ρ(♋) = 31;40°, ρ(♌) = 35°, ρ(♍) = 38;20°, ρ(♎) = 38;20°, ρ(♏) = 35°, ρ(♐) = 31;40°, ρ(♑) = 28;20°, ρ(♒) = 25°, ρ(♓) = 21;40°, which is a linear zigzag function with its maximum at the autumnal equinox and its minimum at the spring equinox, de Falco, Krause and Neugebauer [1966, 38].

not have any logical dependency on the first.

The final sentence, [3], claims that if α and β are arcs which are equidistant from either ♈0° or ♎0°, then they will rise in the same time. This statement is also true in any of the ancient schemes for rising times so long as the arcs are taken equidistant from the same equinoctial point.[130] This statement is again logically independent of the second.

Because of the second statement, Neugebauer took the passage to imply that Hipparchus advanced a linear numerical scheme, similar to that of Hypsicles.[131] A linear scheme, of course, is in conflict both with Ptolemy's table of rising times, *Alm.* II 8, and what we would find if we were to conduct such an exercise today. In fact, it is not the case for any latitude that the greatest rising times are at the autumnal equinox. At the equator the maximums are at the solstices and they increase together and move toward the autumnal equinox as one goes northward on the terrestrial globe.[132] Under a linear scheme, however, the maximum value is at ♎0° so that the statement would be true but trivial for all latitudes and the expression we find in Pappus would be odd.

Björnbo's attempt to explain the passage is equally unsatisfactory.[133] He argues that Hipparchus must have come to this conclusion based on calculations made for the northerly latitudes using an inexact chord table which indicted a continuous increase in the rising times from the solstices toward the autumnal equinox. We saw, however, in the previous chapter, that Hipparchus' chord table must have been fairly accurate to obtain the parameter values for the solar model which both Ptolemy and Theon of Smyrna report.[134] It is just as likely that Hipparchus looked at a table of numbers loosely similar to Ptolemy's table of rising times and, seeing that the local minimum at the autumnal equinox was rising and was nearly equal to the maximums for northerly latitudes, simply extrapolated what the pattern suggests; that the rising times would

[130] Schmidt [1943, fig. 22], Pedersen [1974b, 112], Nadal, Taha and Pinel [2004, 401] and Figure 1.11 for graphs of the rising times.
[131] Neugebauer [1975, 301].
[132] See the table of rising times, *Alm.* II 8, or Figure 1.11 to confirm these claims.
[133] Björnbo [1902, 74 - 75].
[134] See page 98.

have a single maximum farther north, in the arctic regions. Whatever the case, I don't think this passage gives us enough information to know much about the contents of Hipparchus' *On the Rising Times of the Twelve Signs of the Zodiac.*

The other reason Neugebauer gives for reading this passage as referring to a work in the tradition of Hypsicles' *Ascensions* is the fact that Pappus tells us Hipparchus made his demonstration "through numbers," *di arithmōn.* Neugebauer reads *di arithmōn* as "through arithmetical methods" and, taking these to be in the tradition of Babylonian mathematical astronomy, he sets them in contrast to geometrical methods, *dia tōn grammōn.*[135] In fact, however, as we saw above, the related phrase *dia tōn arithmōn* is a technical expression that means "through numerical calculation."[136] It is used in this sense by Ptolemy and both Pappus and Theon, following him.[137] In these authors, however, the calculations are always carried out with the aid of geometrical models. Neither of the expressions *di arithmōn* or *dia tōn arithmōn* are used by any mathematical astronomer as technical terminology to denote the construction or use of tables. Indeed, we saw that *di arithmōn* is used by mathematical writers, as Hero and Eutocius, to mean "through calculation." It is likely that here, Pappus also simply means "through calculation," not necessarily to the exclusion of geometric models.

In the second book of his *Commentary on the* Phaenomena *of Aratus and Eudoxus*, Hipparchus three times makes mention of his work on the subject of simultaneous risings.[138] One of the few things Hipparchus tells us about this work is that it contained mathematical demonstrations which allow one to determine the degrees of the equator and the ecliptic rising with a given star. Moreover, it allows one to work in more than one latitude, so that it is possible, "in almost every part of the inhabited world, to trace the differences of the simultaneous risings and settings."[139] We have no evidence in the Greek world of a text that handles this sort of material in the manner of a numerical

[135] Neugebauer [1975, 302, n. 8].
[136] See Section 2.2.1.
[137] Heiberg [1916, p. 1, 239 & 339] and Rome [1943, 17 ff. & 890 ff.].
[138] Twice he explicitly refers to a work, or works, on simultaneous rising times, Manitius [1894, 128, 148], but Manitius [1894, 184] must also be a reference to the same material.
[139] Manitius [1894, 184].

scheme. Hipparchus must have used analemma techniques, projection techniques, or metrical methods similar to those found in Ptolemy.

A horoscope from the Oxyrhynchus papyri gives us some further clues about Hipparchus' work in spherical astronomy. *P. Oxy.* 4276 tells us that a "compilation (συντάγματος) of Hipparchus" was used to calculate the rising point of the ecliptic, the ascendant, for a given time at the latitude of Egypt.[140] The calculation was carried out to the precision of minutes. Since Hipparchus himself lived at the latitude of Rhodes, and all of the work in his *Commentary* was carried out for this latitude, this other "compilation" must have provided the mathematical tools for finding the rising point of the ecliptic as a function of both date and latitude. Moreover, we have no evidence of the ancients advancing numerical schemes to solve this particular problem.

We cannot say with any certainty what methods Hipparchus used to calculate the rising times of the signs of the zodiac; nonetheless, it seems unlikely that he would have produced a series of numerical schemes when he had available whatever geometrical methods he used in "the treatise on simultaneous risings" and the "compilation." Moreover, if he set out a number of proofs that would make his methods generally applicable to any latitude, as the language of the *Commentary* suggests, then he probably demonstrated his methods by using the latitude of Rhodes as an example and Ptolemy would be following in this tradition.

There are a number of reasons for being suspicious of the standard view of Menelaus' role in the history of spherics. (1) There is nothing in the mathematics or presentation of the fundamental theorem that compels us to believe it could not have been written before Menelaus' time. (2) It makes use only of rudimentary lemmas that might well have been assumed by Euclid in his more advanced works and relies on no other theorem in the *Spherics*. (3) The theorem makes no use, or mention, of spherical triangles, the hallmark of Menelaus' approach to spherical astronomy. (4) The theorem appears as the first proposition in the third book and introduces a number of metrical theorems that all

[140] Jones [1999a, 418].

concern spherical triangles, which are the proper subject of the text. The first proposition in a book is usually a construction or a simple auxiliary theorem, one which may be well known and introduces the more advanced material of the book. It may be fundamental, but it is almost never exciting. *Spher.* III 1 serves precisely this subsidiary function in the context of book III as a whole. (5) If we assume that the theorem was well known in Menelaus' time it would explain the fact that his treatment of the theorem is both more advanced and more complete than what we find in the astronomical tradition.

We saw that Ptolemy's own approach to spherical astronomy was indebted to both Theodosius and Menelaus. In particular, Ptolemy makes use of the concept and properties of the spherical triangle. Menelaus is accepted as the originator of the ancient theory of the spherical triangle because of the extensive treatment of the properties of this figure in his work and because we are told by Pappus that he gave the figure its name, "threeside," τρίπλευρον.[141] Nevertheless, in just those places in the *Almagest* that deal with rising times and simultaneous risings, *Alm.* I 14 & 16, II 7.3, VIII 5 & 6, we find that only the fundamental theorem, which makes no use of spherical triangles, is used. There is no reason in either Ptolemy or Menelaus to deny that Hipparchus could have written a work in spherical astronomy that made use of the theorem ascribed to Menelaus.

Although we do not know the methods of Hipparchus' writing on rising times and simultaneous risings, it seems clear that Ptolemy intended his own writing to be compared to these. The fact that Ptolemy gives all of his example calculations using the arbitrary latitude of Rhodes, $\varphi = 36°$, as opposed to that of Alexandria or Rome where he and Menelaus worked, indicates that he either adopted his methods from Hipparchus or intended that his methods and figures should be directly compared with those of Hipparchus. This is the best explanation of fact that Ptolemy has two methods of calculating the rising times at *sphaera obliqua*, *Alm.* II 7.3 & 7.5, the first of which he uses to give the rising times of all twelve signs of the zodiac for $\varphi = 36°$.

141 Hultsch [1878, 476].

Ptolemy's presentation of spherical astronomy is a comprehensive account that uses and adapts the methods of his predecessors with no mention of authors or works. It is, as he says, both concise and methodical. It is, no doubt, intended to be read as the culmination of a tradition of similar works. Unfortunately, it is the only work of its kind that has survived.

3.6 Conclusion

We have examined the available evidence on the fundamental theorem of ancient spherical trigonometry and the way that it functions in Ptolemy's *Almagest.* We have seen that there is good reason for being doubtful of the pristine character of the versions of *Spher.* III 1 in the Arabic tradition.

There is no way to be certain what form the theorem took in Menelaus' work but he seems to have shown the theorem in cases, in particular the parallel case. The lemmas, on the other hand, were probably absent from Menelaus and found their way into the text through the tradition of the *Almagest.*

Ptolemy uses the fundamental theorem of ancient spherical trigonometry in a systematic way to handle all topics of spherical astronomy, even topics in which he himself has little interest. The fundamental theorem, the chord table, the techniques of metrical analysis and the toolbox of elementary and spherical geometry are the only mathematical methods employed in Ptolemy's spherical astronomy. This section of the *Almagest* shows the concision and logical arrangement typical of a systematic geometric treatise. It is one of the more polished sections of the *Almagest* and has the appearance of following, and perfecting, a long tradition. We cannot know what mathematical methods Hipparchus employed to investigate rising times and simultaneous risings but there is no reason to assert that he did not have metrical methods essentially similar to those found in the *Almagest.* Ptolemy's use of the latitude of Rhodes, $\varphi = 36°$, for the latitude at which he makes all calculations, as opposed to his own latitude of Alexandria, $\varphi = 30;58°$,[142] shows that he meant his work to be read in the context of a tradition that extended all the way back to Hipparchus.

[142] Toomer [1984, 247].

Chapter 4

Flattening the Sphere: Models and Maps

This chapter explores three of Ptolemy's texts which present methods for flattening the sphere. This flattening is effected through what we would call a projection or a mapping. Ptolemy's methods should not, however, be confused with the later procedures that bear these names. The bulk of the chapter will be an exposition of the methods that Ptolemy actually employs. The three texts examined are the *Analemma*, the *Planisphaerium* and the map constructions in the *Geography*.

In some sense, these texts are heterogeneous and are brought together only by the superficial fact that they discuss spherical surfaces on the plane. The *Analemma* and the *Planisphaerium* are mathematical texts that develop models pertaining to spherical astronomy while the map constructions in the *Geography* present a set of instructions for drawing a portion of the globe of the earth on a plane. The *Analemma* and the *Planisphaerium* present the best evidence for the only ancient techniques for studying spherical geometry besides those investigated in the last chapter. In order to get a better sense of the purpose and scope of these texts, they may be fruitfully compared with the material from the *Geography*.

The chapter begins with a study of the *Analemma* along with all of the relevant ancient material pertaining to analemma constructions. This is followed by an analysis of the *Planisphaerium*. The final section is a summary of the *Geography* material, chiefly for the sake of comparison with the other two texts.

4.1 The Analemma

Ptolemy's *Analemma* is a short work that deals with a single topic: the daily motion of the sun with respect to the local horizon. The Greek text of the *Analemma* only survives in a mathematical fragment written in Greek capitals of the 6th or 7th century which was palimpsested in a late 8th century codex of Isidore's *Etymologies*. This text is only partially readable. William of Moerbeke, however, made a Latin translation which is preserved in his own hand. The most recent edition has been made by Edwards, along with an English translation.[1] By using Moerbeke's Latin and referring at key points to what is left of the Greek, we can get a good sense of Ptolemy's text.

The work provides a mathematical model for stating the position of the sun for a given geographic latitude, date and seasonal hour. It does this through a construction known as the analemma. The analemma is a model of a sphere on a receiving plane that preserves key angles and lengths. In the known examples of ancient analemma constructions, the receiving plane is most often the meridian.[2] Twice, in Ptolemy's *Analemma*, one of the other two principal planes, the vertical and the horizon, acts as an implicit receiving plane.[3] Analemma constructions can be used to address a number of topics in spherical astronomy and geography but they originate from the study of gnomons. For this reason

[1] The relevant MSS are referenced in Edwards' dissertation, Edwards [1984, 21 - 32]. A more recent text was printed by Sinisgalli and Vastola along with an Italian translation, but they do not include any apparatus, Sinisgalli and Vastola [1992b].

[2] Edwards includes Diodorus' determination of the meridian line as an example of an analemma construction simply because it was found in Diodorus' *Analemma*, Edwards [1984, 6]. There is no reason to do so, however, which leaves us with Vitruvius, Hero and Ptolemy as the only ancient authors who preserve analemma constructions.

[3] Ptolemy never explicitly states that the horizon or the vertical act as the receiving plane but this can be drawn out of two of his constructions, see page 206. A single analemma construction in the *Planisphaerium* uses the equatorial plane, see page 226. Analemma constructions on the horizon are common in the medieval Arabic tradition, particularly in determinations of the *qibla*; see, for example, Kennedy [1974] and Berggren [1980].

they have often been associated with the manufacture of sundials. There is no doubt that the analemma was part of the theoretical framework that surrounded sundial construction but we have no ancient text which shows how the analemma was actually put to practical use.[4]

Throughout the text, Ptolemy situates his treatment within the tradition of Greek works on gnomonics. He constantly compares his methods with those of certain unnamed predecessors, whom he refers to merely as the ancients (τοῖς παλαιοῖς).[5] In some places he criticizes their work while in other places he omits certain proofs because he found those provided by these predecessors to be sufficient.[6] His criticisms, however, primarily concern a matter of convention, so that Ptolemy continues to give deference to the assumptions of the previous writers throughout his treatment. On the whole, Ptolemy seems approving of this earlier work, saying that he, "very much accepted it, but did not everywhere agree."[7] These practices differentiate the *Analemma* from most of Ptolemy's other works. Because Ptolemy does not dismiss the views of his predecessors once and for all and also seems to assume that the reader will make use of their works to fill in the details of certain arguments, the *Analemma* does not read as a single deductive argument of the sort we encountered in our study of Ptolemy's treatment of spherical astronomy.

It is not possible to date the *Analemma* with respect to Ptolemy's other works. The only exact figures given are the geographical latitudes, φ, of seven key parallels and the declinations, δ, of the starting points of the zodiacal signs. While the figures for φ given in the *Analemma* agree exactly with those in the *Geography* and differ slightly from those

[4] There have been a number of demonstrations that the analemma can be used to construct planar dials and spherical dials with a central gnomon point. Most recently see Evans [1998, 133 - 140] and Neugebauer [1975, 855 - 856]. Drecker has shown that the analemma can be used to make certain cylindrical dials, Drecker [1925, 4]. All of this, however, only accounts for about 40% of the dials listed in Gibbs [1976].

[5] Heiberg [1907a, 195] and Edwards [1984, 136].

[6] Heiberg [1907a, 194 - 195] and Edwards [1984, 95].

[7] Heiberg [1907a, 189] and Edwards [1984, 79].

in the *Almagest*, this is likely the result of rounding.[8] The numbers for δ are again the figures that would result from rounding those found in the Almagest.[9] Moreover, such figures need not have been the result of rounding from one of Ptolemy's own works. Any reasonably accurate work on spherical astronomy, such as that attributed to Menelaus, would have produced similar numbers. On the other hand, they may have been, or come from, standard values in the literature on gnomonics.

Ptolemy's deference to his predecessors as well as his failure to make the *Analemma* mathematically complete, may incline us to the belief that it is the work of a younger scholar. We should, however, also bear in mind the audience for whom the work was intended. Ptolemy, doubtless, hoped that the *Analemma* would be of interest to dialers as well as mathematical astronomers. Dialers would not likely have been concerned with mathematical completeness; especially if the omitted proofs could be found elsewhere in the gnomonics literature. Moreover, they would probably be attached to their time-proven methods of construction and would have equipment designed for the traditional conventions, so Ptolemy may have thought it wise to include these older conventions in his treatment.

In order to model the daily motion of the sun with respect to the local horizon, Ptolemy considers a sphere concentric with the point of a gnomon and tangential to the horizon; see Figure 4.1 (a). This sphere models our local position, so that the cardinal directions, S, E, N and W, can be represented on the plane parallel to the horizon through the tip of the gnomon, while the top and bottom of the sphere, A and B, represent the zenith and nadir. The proper motion of the sun is not considered to have any effect on its daily motion so that we simply imagine the sun carried about a circle coincident with, or parallel to, the celestial equator at a uniform rate. We model

[8] Although the numbers given in the *Analemma* are expressed in unit fractions, they are simply the numbers that would result if one rounded the numbers in the *Almagest* to the nearest 0;05$^{\text{th}}$ of a degree. Edwards compares the numbers from the *Analemma* with those of the *Almagest*, Edwards [1984, 211, n. 574]. The latitudes for these parallels are listed in *Geo.* I 23. In the *Geography* they are the 4$^{\text{th}}$, 6$^{\text{th}}$, 8$^{\text{th}}$, 10$^{\text{th}}$, 12$^{\text{th}}$, 14$^{\text{th}}$ & 15$^{\text{th}}$ parallels, Berggren and Jones [2000, 85].

[9] Edwards compares the numbers in the *Analemma* with those in the *Almagest*, Edwards [1984, 122, n. 577]. The *Planisphaerium* uses the same numbers as the *Almagest* in sections 4 - 6, Heiberg [1907a, 232 - 236] and Anagnostakis [1984, 74 - 77].

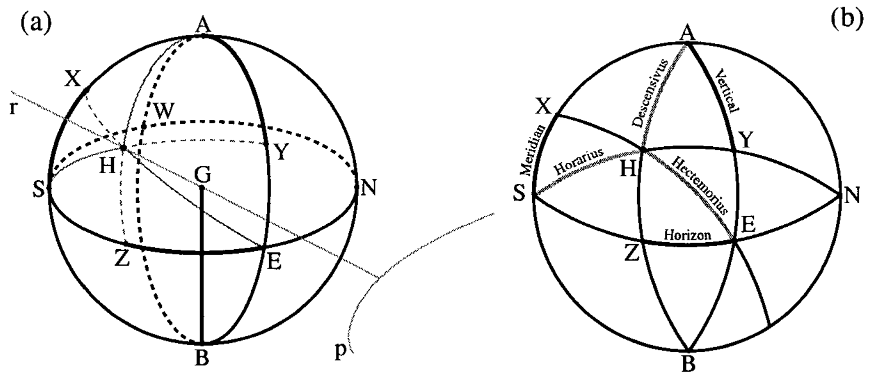

Figure 4.1: Ptolemy's model of the diurnal motion of the sun. Figure (a) is a schematic rendition of the model. Figure (b) is based on Moerbeke's text, Edwards [1984, Fig. 1]. The letters have been changed to agree with Figure 4.1 (a).

this motion by tracking a solar ray (ἀκτίς)[10] which falls upon the surface of the sphere. Ptolemy does not say so, but he must mean the solar ray which falls from the upper circumference of the solar disk upon the tip of the gnomon, denoted r in Figure 4.1 (a).[11]

In order to specify the position of this ray, Ptolemy establishes a system of coordinate planes. He takes as his fixed coordinates the circles of the meridian, $ASBN$, the horizon, $SENW$, and the vertical, $AEBW$. These are not the actual meridian, horizontal and vertical circles, but the circles in the sphere which model those circles. The position of the solar ray is then determined by considering three other circles which are rotations of the fixed circles about an axis. The meridian circle rotates around the axis NS, the vertical circle rotates around the axis AB, and the horizon circle rotates around the axis EW. Ptolemy then takes pairs of arcs, one from a fixed and one from a moving circle, to specify the position of the ray. The arc of the rotated horizon, $\overset{\frown}{EH}$, is called the *hectemorius* and it is taken together with $\overset{\frown}{SX}$ of the meridian. The arc of the rotated meridian, $\overset{\frown}{SH}$, is called the *horarius* and it is taken together with $\overset{\frown}{AY}$ of the vertical. The arc of the rotated vertical, $\overset{\frown}{AH}$, is called the *descensivus* and it is taken together with $\overset{\frown}{EZ}$ of the horizon.

[10] Heiberg [1907a, 197 & 199] and Edwards [1984, 138 & 140].

[11] It should be noted that this is a theoretical concept. The eye cannot actually perceive a distinct edge to the penumbral shadow.

The project of the *Analemma* is to determine the sizes of these arcs for any given place and time. Ptolemy uses the analemma construction to make these determinations. He presents two methods for doing this. The first is the exact, trigonometric method (*dia tōn grammōn*) for which he gives a metrical analysis but no calculation.[12] The second is a nomographic method and seems to be the centerpiece of the whole work. For the purpose of this nomographic method an analemma instrument was made and its construction is described.

We no longer possess any work before Ptolemy that uses analemma constructions for the same purpose as Ptolemy. There are only two other occurrences of an analemma in ancient texts. Vitruvius discusses the analemma in connection with the construction of sundials and Hero uses an analemma in his determination of the distance between two cities. As well as these, we have three accounts of a graphical method for determining the meridian which are attributed, in one of our sources, to the *Analemma* of Diodorus. A discussion of all this material will inform our reading of Ptolemy.

4.1.1 Diodorus' determination of the meridian

Only one construction from Diodorus' *Analemma* has been preserved. It is not an analemma construction, but it is interesting because it shows that ancient writers on gnomonics investigated the cone formed with the tip of the gnomon as its vertex and the daily path of the sun as its base. The problem is to determine the meridian line through the base of a gnomon, given three shadows cast by the same gnomon on a single day. There are three versions of this method of determining the meridian line. The first is from Hyginus Gromaticus and, lacking any proof, will not concern us here.[13] The second is from Abū Saʿīd aḍ-Ḍarīr al-Jurjānī.[14] The treatment by aḍ-Ḍarīr is complete and supplies two figures but he does not attribute it to Diodorus. He only says that it comes from a book called the *Analemma*. Finally, Abū ar-Rayḥān al-Bīrūnī gives a version of the

[12] Heiberg [1907a, 202] and Edwards [1984, 141].
[13] Blume, Lachmann and Rudorff [1848, 189 - 91], Dilke [1967, 17 - 18] and Edwards [1984, 172 - 173].
[14] Schoy [1922, 265 - 267] and Edwards [1984, 173 - 177].

same determination which he attributes to the *Analemma* of Diodorus with a complete proof but only one figure.[15] Aḍ-Ḍarīr and al-Bīrūnī give the same construction and proof in basic outline, but they differ enough in the details so that, as Kennedy states, if one of them is a faithful translation of Diodorus the other cannot be.[16]

Because we cannot claim to get closer to Diodorus' actual approach by preferring one treatment to another, I have not tried to adhere to the details of either of these authors in my summary of the construction and proof.[17] I follow al-Bīrūnī's ordering of the material but I make free use of aḍ-Ḍarīr in places. I use the medieval diagrams, taking one from each writer.[18] The presentation is a classical problem in the theoretical tradition; a construction is furnished, followed by a proof that it completes the stated problem.

The construction; see Figure 4.2 (a): Let the given shadows be the lines EYA, EMB and ESG. They are taken such that $EA > EB > EG$. We describe circle YMS about center E with radius equal to the height of the gnomon. We erect $ZE \perp AE$ [*Elem.* I 11], and take points H and T such that $\overset{\frown}{ZH} = \overset{\frown}{YM}$ and $\overset{\frown}{HT} = \overset{\frown}{MS}$ [*Elem.* I 23], hence $HE \perp EB$ and $ET \perp EG$ [$\angle ZEA = \mathrm{R} = \angle HEA + \angle ZEH = \angle HEA + \angle YEM = \angle HEB$ and $\angle HEB = \mathrm{R} = \angle TEM + \angle HET = \angle TEM + \angle MES = \angle TEG$]. We join AZ, BH and GT. With center A and radius AZ we draw $\overset{\frown}{ZO}$. With center B and radius BH we draw $\overset{\frown}{HO}$. With center O and radius GT we draw $\overset{\frown}{CF}$ intersecting the lines AO and BO. We extend the lines AB and CF to meet at L. We join GL, which will be a line parallel to the equatorial line. We drop $KE \perp GL$ [*Elem.* I 12] so that KE is the meridian.

The proof; see Figure 4.2 (b): The given shadows are the lines EA, EB and EG; $\triangle EZA$, $\triangle EZB$ and $\triangle EZG$ are the triangles of the shadows; and their hypotenuses

[15] Kennedy [1976, vol. 1, 157 - 166] and Edwards [1984, 177 - 179]. Al-Bīrūnī implies that there must have been two figures but only one survives in the manuscript, Kennedy [1976, vol. 1, 166]

[16] Kennedy [1976, vol. 2, 93].

[17] Kennedy [1959] gives a summary of al-Bīrūnī's approach.

[18] Modern diagrams can be found in Neugebauer [1975, 841 - 842] and Kennedy [1976, 63 & 65]. The best figures are given by Edwards, who consulted the manuscripts, Edwards [1984, 177 & 179, Ill. 17 & 19].

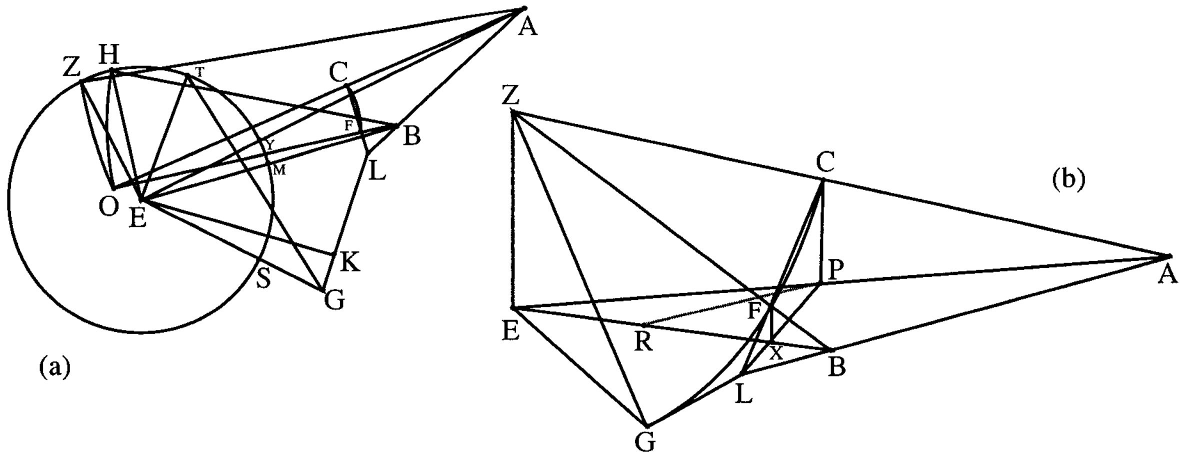

Figure 4.2: Diagrams for Diodorus' determination of the meridian line. Figure (a) is from al-Bīrūnī while Figure (b) is from aḍ-Ḍarīr, Edwards [1984, Ill. 17 & 19]. In Figure (b), the letters have been changed to agree with Figure (a).

AZ, BZ and GZ are on the surface of a cone whose vertex is the tip of the gnomon. It is clear that the intersection of the plane of a circle parallel to the equator and the plane of the horizon will form a line parallel to the equatorial line. We draw circle GFC on the surface of the shadow cone with vertex Z and distance ZG [1].[19] Since circle GFC is drawn perpendicular to the axis of the shadow cone, it will be parallel to the equator. Hence the intersection of circle GFC with the horizon will be parallel to the equatorial line and the meridian will be perpendicular to this line. We drop $CP \perp EA$ and $FX \perp EB$ [*Elem.* I 12]. We join PX and extend it. It will meet AB in the direction of G at some point L: since $AZ > BZ$ and $CZ = FZ$, $AC > BF$ and $AC : CZ > BF : FZ$, or *by composition*, $AZ : CZ > BZ : FZ$ [2]; but $AZ : AC = AE : AP$ and $BZ : FZ = BE : BX$ [*Elem.* VI 2]; hence, $AE : AP > BE : BX$; we join $PR \parallel AB$ so that $AE : AP = BE : BR$ [*Elem.* VI 2], therefore, $BR > BX$; but $\angle PAB + \angle APR = 2\text{R}$ [*Elem.* I 29] and $\angle PAB + \angle APR > \angle XPA + \angle PAB$, hence the lines PX and AB meet

[19] Schoy uses the expression "um x als Mittelpunkt mit dem Halbmesser y" and Edwards uses "with center x and radius y" but it is clear that these must derive from a mistranslation, Schoy [1922, 265 - 266] and Edwards [1984, 174]. The Greek likely used the term διάστημα which can be translated as *radius* for constructions in the plane but must be rendered as *distance* for solid constructions. See Sidoli [2004b] for a discussion of the use of this term in spherical constructions.

in the direction of G at L. Now, since the two lines PL and AL are in the planes $CPXF$ and $ABFC$, L is on the intersection of these planes. But CF is also the intersection of these two planes; therefore, CFL is a straight line [*Elem.* XI 3]. But L and G are both in the plane of the horizon and in the plane of the circle GFC. Therefore, GL is an equatorial line and the meridian is perpendicular to it through the base of the gnomon. Comparing the two diagrams in Figure 4.2, $\triangle AOB_{(a)} = \triangle AZB_{(b)}$ and the lengths AC, BF, CL and AL are all equal in the two diagrams so that the point L is similarly situated in both cases. Therefore the line EK, which is perpendicular to GL in 4.2 (a), is the meridian.

The most striking feature of this problem is the way that the construction is in the plane using planar methods while the proof involves a solid figure and uses a solid construction.[20] The motivation for the plane construction is found in the proof. The assumption is that the sun moves, on a given day, in a circle parallel to the equator so that the locus of the tip of the shadow on the plane of the horizon is a hyperbola. The hyperbola is the section that the horizon makes with the cone which has the sun as its generating circle and the tip of the gnomon as its vertex. This allows us to find an equatorial line as the intersection of the horizon with a generating circle of the shadow cone. The remaining part of the proof fills in the details of this determination. No analysis has been given because the motivation for the solution is clear.[21]

I have presented the construction first, following al-Bīrūnī, because this is in accordance with the general practice of presenting a problem in synthetic form. The construction itself is made entirely in the plane. We can think of triangle $AOB_{(a)}$ as being triangle $AZB_{(b)}$ folded down into the horizon, but the text does not speak of any folding. Moreover, the aim of the solution is to show how to use elementary means to construct the figure in the plane so that it can be shown that $AOB_{(a)}$ equals triangle $AZB_{(b)}$. This

[20] Pappus twice discusses the division of problems into planar, solid and linear in the *Collection* III & IV, Hultsch [1878, vol. 1, 54 - 56 & 270 - 272]. For a modern discussion of this issue in Pappus see Knorr [1986, 341 - 348].

[21] See Netz [2000] for a discussion of the motivations that Greek geometers may have had for publishing their analyses.

allows us to determine L in the plane using only elementary means, despite the fact that the solution was obtained using a solid construction.

At [1], Diodorus draws a circle on the conic surface with the vertex of the cone as the fixed point and the distance from the vertex as the determinate length. This construction is analogous to the way that circles are constructed on the sphere.[22] There was apparently no postulate for this construction in antiquity but this does not seem to have bothered anyone. Perhaps it was assumed that such solid constructions were covered under the Euclidean postulate, *Elem.* I post. 3. At [2], Diodorus applies one of the fundamental operations on ratios that are demonstrated in *Elem.* V for proportions but used by ancient geometers for ratio inequalities as well.[23] In both these cases, and in combining the use of solid and planar methods, Diodorus shows a basic disregard for the methodological niceties which ancient and modern commentators will sometime claim are fundamental to Greek geometry. In this regard, Diodorus sets himself in the company of the best geometers of the Hellenistic period.

Diodorus' construction is not an analemma construction but it was found in his work entitled the *Analemma.* We have no way of knowing what role this construction played in the work as a whole but it is not hard to see how determining the local meridian could fit into a work on the theory of sundials. It could be used to align sundials themselves or to align the sphere which will, in turn, be used to model the motion of the sun.

This construction shows both that Diodorus' text was greater in scope and more mathematically sophisticated than Ptolemy's and that ancient mathematical astronomers were aware of the role of conics in the theory of sundials.

[22] See for example Autolycus' *On a Moving Sphere* 6 and Theodosius' *Spherics* I 19, Mogenet [1950, 203 ff.] and Heiberg [1927, 34 ff.].

[23] See Section 1.3.

4.1.2 The analemma in Vitruvius

The earliest presentation of the analemma figure which we possess is in Vitruvius' *Architecture* IX 7.[24] Prior to his discussion of water clocks and sundials, Vitruvius gives a description of what should be be carried out, "whenever a sundial is to be laid out."[25] The construction that follows is that of the analemma figure. Vitruvius has adopted his account from a Greek source, perhaps through an intermediary, and he does not actually show how the analemma figure is to be used in the construction of dials for fear of boring his readers with too much detail. Nor does he do anything else interesting with the analemma; nevertheless, the construction serves as a good introduction to the figure and we will use it for that purpose. As there are a number of translations of this material available, there is little point in following the text closely.[26]

Before we look at the figure itself, however, a few idiosyncratic features of Vitruvius' exposition should be mentioned. Vitruvius presents his construction in a manner that might have been useful to draftsmen and engineers but would have been unsatisfactory to a geometer. This may have been a result of the fact that there was no native tradition of mathematics in Latin, but it also reflects the intended audience of the *Architecture*. In a number of places, Vitruvius gives descriptions that explain how to draw the figure but do not give a clear indication of the mathematical relationships involved. Moreover, his whole discussion is focused around the use of a compass and set-square. For instance he draws perpendiculars by describing how the set-square should be aligned and circles by indicating where the points of the compass should be placed.

Vitruvius begins by stating that "whenever a sundial is to be laid out, the lengths of the equinoctial shadow particular to that region should be determined."[27] What he means is that the ratio of the length of the equinoctial shadow to the length of the gnomon, $s : g$, must be determined before the analemma can be drawn. This ratio was a

[24] Rowland and Howe [1999, 115 - 116].
[25] Rowland and Howe [1999, 115].
[26] See also Evans [1998, 132 - 135] for this text with mathematical commentary.
[27] Rowland and Howe [1999, 115].

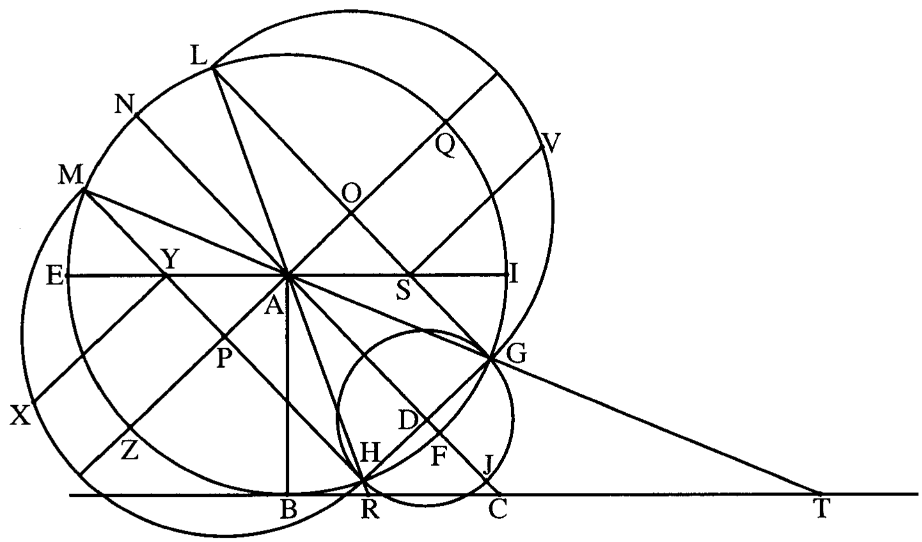

Figure 4.3: The analemma figure in Vitruvius, Rowland and Howe [1999, 115 - 116].

standard measure of geographic latitude in classical antiquity and he has already given a list of ratios for cities important to Greco-Roman astronomy.

The ratio of Rome, 8 : 9, is taken as an example and a horizontal line is laid out on a plane surface; see Figure 4.3. Then, with a set-square, the gnomon is erected and nine parts are measured off with a compass to form AB. The meridian is drawn with A as center and AB as radius. BC is laid out on the horizontal line such that it is eight parts. CA is joined through to N and represents the solar ray at the equinox. The points E and I are found by a peculiar construction and EI, the horizon, is joined.[28] An arc of 24° is taken as an approximation of ε and used to lay off $\overset{\frown}{FG}$ and $\overset{\frown}{FH}$. Through these points and the center, A, lines MT and LR are joined which represent the solar ray at the solstices. The diameters GL and HM,[29] representing the summer and winter "portions" respectively, are joined and bisected at O and P. These points are joined through A and

[28] *Tunc, a centro diducto circino ad lineam planitiae, aequilatatio signetur, ubi erit littera E sinisteriore parte, et I dexteriore, in extremis lineae circinationis*, Soubiran [1969, 27]. Evans [1998, 133] translates, "Then, extending the compass from the center to the line of the plane, mark off the equidistant points E on the left and I on the right, on the two sides of the circle."

[29] Rowland and Howe translate with "chord" but Vitruvius has *diametroe* with διαμέτρω glossed, Rowland and Howe [1999, 115] and Soubiran [1969, 28]. The lines are the diameters of circles parallel to the equator.

extended to Q and Z. This line will be perpendicular to the equinoctial ray and is called the axis of the world. Two semicircles are drawn about centers O and P. Points S and Y denote the place where the parallel lines meet the horizon. The lines SV and YX are drawn parallel to the axis from the parallel lines, GL and HM, to the semicircles. These lines are called the *loxotomus.* A circle is drawn with center D and radius DH. This circle is called the *menaeus.* In this way, we obtain the figure of the analemma.

Vitruvius' account simply tells us how the figure is drawn and what the different components are called and represent. In many places his text is cryptic and unhelpful. The analemma is drawn for a particular latitude and is based on one of the characteristics of latitude most used in the ancient world, $s : g$. The plane of the figure is the plane of the meridian. The semicircles GVL and HXM are actually normal to the meridian but they are drawn in the plane of the figure. These are the diurnal paths of the sun at the summer and winter tropics respectively. The lines of the loxotomus, SV and YX, are the intersections of these circles with the plane of the horizon, EI. In this way, $\overset{\frown}{VL}$ represents the motion of the sun from rising to noon and from noon to setting on the summer solstice. Likewise, $\overset{\frown}{VG}$ represents the motion of the sun from setting to midnight and from midnight to rising. Hence, in order to find the seasonal hours from day to night we need to divide these arcs into six equal parts. That Diodorus' *Analemma* included such a division on the figure is confirmed by Pappus. By his own report, Pappus showed how to trisect a given angle in his commentary on this lost work.[30]

The description that Vitruvius gives is purely schematic and only exhibits the cardinal positions of the sun. Nevertheless, it is easy to see how this model can be extended. The menaeus circle, HCG, which Vitruvius mentions at the end of his account, can be used to find the position of the circle of the sun's diurnal motion, given its longitude, λ. If we mark off λ from J in the direction of G and draw a line through the endpoint of this arc parallel to the ray of the ecliptic, this line will be the diameter of the circle of the sun's

[30] Hultsch [1878, vol. 1, 244 - 246].

daily motion and will be at the correct celestial latitude, β.[31] On the other hand, the word *menaeus* means "pertaining to months" and the menaeus circle may simply have been divided into twelfths to produce schematic "months" which were each a twelfth of the solar year.

Through the use of the menaeus circle and the circle of the sun's diurnal motion, the position of the sun at any time can be modeled on the analemma. It has often been shown how this model can be used to determine the length and direction of the shadow of the gnomon on the plane of the horizontal line.[32] All of these constructions are accomplished by considering the sphere and then drawing all figures necessary to the construction in the plane of the drawing so that no distortion takes place in either length or direction. In this way, the plane of the drawing acts as the locus of a number of different planes. These considerations will become more apparent when we examine Ptolemy's use of the analemma figure.

The analemma as presented by Vitruvius acts as a nomographic procedure for modeling the motion of the sun with respect to the local horizon. In particular the menaeus acts as a nomographic means of translating $\Delta\lambda$ into a given β. Thus, given $s : g$, λ and h_s, the analemma figure allows us to represent the position of the sun. There is no indication in Vitruvius that any calculations or metrical analyses were carried out on the analemma figure. The figure, as it appears in Vitruvius, is the same as that which we find in Hero and Ptolemy and it must have been similar to the one given by Diodorus.

4.1.3 The analemma in Hero

The analemma is not only useful for determining the direction and length of the gnomon's shadow. It can also be used to measure the arc distances of the sun from the cardinal points of the celestial sphere in either the local or the equatorial coordinate system. In his *Dioptra*, Hero uses the analemma to determine the arc distances between the sun and

[31] Neugebauer [1975, 845] provides a proof of this assertion that has the advantage of being motivated by considering the analemma figure as a sphere.

[32] A recent derivation is given in Evans [1998, 137 - 139].

both its rising point and the local meridian. In *Diop.* 35, Hero gives a method for finding the great circle distance between two cities based on a simultaneous observation of the same eclipse.[33] The text describes a nomographic procedure which is accomplished on the inside of a hemispherical basin. The analemma is used as an auxiliary construction which allows one to find certain arc lengths and then transfer them to the hemisphere.

The text of this chapter of the *Dioptra* is difficult and corrupt in places.[34] The first mathematical interpretation of the text was given by Rome.[35] Neugebauer independently gave his own reading of the text which agreed with Rome's in general but differed in a few important details.[36] Both of these investigations were made without the benefit of any manuscript figures since Schöne had neglected to include any in his edition, finding those in the manuscript insufficient. After being alerted to the existence of figures in the oldest manuscript, Neugebauer revisited the issue, correcting his previous mistakes and coming to the same reading as Rome.[37] My reading follows Rome and Neugebauer except where noted. I use the manuscript figures that Neugebauer printed.

Hero begins with a couple of assumptions. He takes the circumference of the earth, as measured by Eratosthenes, to be 252000 stades so that a single degree is 700 stades, and he assumes that an eclipse has been observed on the night 10 days before the vernal equinox at $h_s = 5$ in Alexandria and $h_s = 3$ in Rome.

A hollow hemisphere is constructed such that the circles of the tropics and the equator are inscribed having the same position as they have in the visible half of the celestial sphere. The local horizon is assumed to be at Alexandria. The circle $AB\Gamma\Delta$ is the rim of the hemisphere; see Figure 4.4 (a). The meridian is $\widehat{BEZH\Delta}$ and the equator is $\widehat{AH\Gamma}$. The south pole is E while the nadir, the lower pole of the hemisphere, is Z. Hero does not say how these two great circles are determined but it is clear that they can readily be marked off in an actual hemisphere. $\widehat{BEZH\Delta}$ is simply drawn in the plane of the

[33] Schöne [1903, 302 - 306].
[34] Schöne [1903, 303, n. 1] called it "schwierige und stark verderbte."
[35] Rome [1923].
[36] Neugebauer [1938].
[37] Neugebauer [1939].

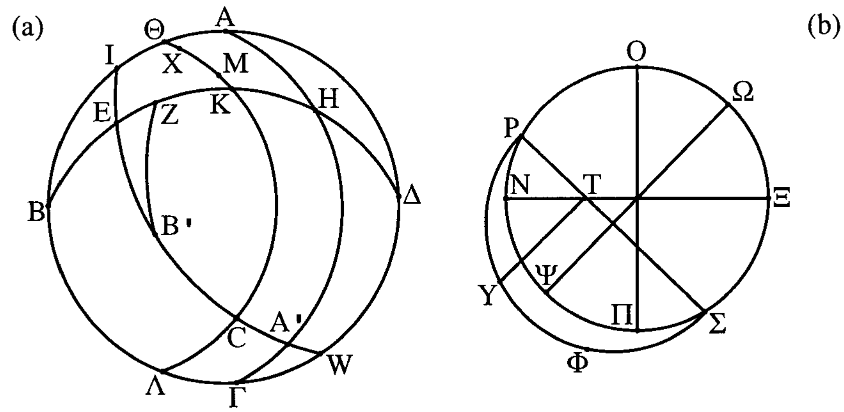

Figure 4.4: Diagrams for Hero's *Diop.* 35. The diagrams are from Neugebauer [1939, 6 - 7] with editorial cautions suppressed and a few letters changed. The letter X is missing in Neugebauer's diagram, but is required by the text.

local meridian; the north and south points are joined through Z. $\overset{\frown}{AH\Gamma}$ joins the east and west points through H. $\overset{\frown}{\Delta H}$ can be determined by means of an analemma which is arranged for Alexandria with diameter equal to the diameter of the hemisphere.[38]

The parallel circle on which the sun is carried on the night 10 days before the equinox is set out in the hemisphere; it is $\overset{\frown}{\Theta K\Lambda}$. Hero does not explain how to find this arc, but $\overset{\frown}{KH}$ can be found by means of the analemma;[39] and then the circle $\Theta K\Lambda$ can be drawn with center E and distance EK.[40] The arc of parallel circle $\Theta K\Lambda$ is divided into twelve equal parts. Again, we can use the analemma to perform this construction.[41] We take $\overset{\frown}{\Theta M}$ as five of these parts and, since the eclipse occurred at $h_s = 5$ in Alexandria, M will be the position of the sun when the eclipse occurred.

The object now is to locate the nadir of Rome in the same hemisphere. This will

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[38] If the analemma in Figure 4.3 is set up for Alexandria, $s : g = 3 : 5$, then $\overset{\frown}{FI}$ in the analemma will equal $\overset{\frown}{\Delta H}$ in the hemisphere.

[39] If the analemma in Figure 4.3 is set up for Alexandria and $\lambda_{\odot} \approx 10°$ is marked off on the menaeus circle from J toward H, then the line from the endpoint of this arc parallel to the equator represents the diameter of the parallel of the sun on the day in question. The arc between F and the endpoint of this diameter on the meridian will equal $\overset{\frown}{HK}$ in the hemisphere.

[40] This method of drawing a circle is the standard in Greek works on spherical geometry; see, for example, Heiberg [1927, 34 ff.].

[41] We divide the nighttime arc for the day in question into six equal parts by means of one of the many ancient methods for trisecting an arc; see page 110, note 137.

then give us the great arc distance between the two cities as the arc between the nadir of Rome and the nadir of Alexandria, the lower pole of the hemisphere. The determination of Rome's nadir is effected by means of an auxiliary analemma.

Hero constructs the analemma for Rome, and draws the circle corresponding to the circle $\Theta K \Lambda$ in the hemisphere; see Figure 4.4 (b). Although he does not mention it, this circle can be found by means of the menaeus circle. The diameter of the horizon is $N\Xi$, the gnomon is $O\Pi$, the axis is $\Psi\Omega$, and the diameter of the monthly parallel of the sun is $P\Sigma$.[42] The intersection of the horizon and the monthly parallel is TY. The arc of the nighttime, $\overset{\frown}{Y\Sigma}$, is divided into six equal parts and $\overset{\frown}{Y\Phi}$ is set out as three of these parts, since the eclipse occurred at $h_s = 3$ in Rome. In the hemisphere, $\overset{\frown}{MX}$ is set out similar to $\overset{\frown}{Y\Phi}$. Thus, the point X will represent the horizon at Rome. $\overset{\frown}{XKC}$ is set out similar to $\overset{\frown}{Y\Phi\Sigma}$, so that C is in the plane of the meridian at Rome. But E is the south pole, so that the great circle EC will be the meridian at Rome.[43] And $\overset{\frown}{A'B'}$ is set out similar to $\overset{\frown}{\Xi\Omega}$, but $\overset{\frown}{A'B'}$ is set out from the quadrant $\overset{\frown}{EA'}$,[44] therefore B' will be the nadir at Rome. But Z is the nadir at Alexandria. The points B' and Z are joined by an arc of a great circle, $\overset{\frown}{B'Z}$, and it is seen how many parts this arc is of the whole circle $AB\Gamma\Delta$.

Hero does not tell us how to find a value for $\overset{\frown}{B'Z}$. In fact, he simply assumes a value "at random" such that $\overset{\frown}{B'Z} = 20°$. In this way, the distance between Alexandria and Rome is computed to be 14,000 stades. $\overset{\frown}{B'Z}$ can be found either by computation or by nomographic means. Using ancient methods, however, these computations would be quite involved so that the nomographic procedure would be of considerably use from a

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42 Ptolemy calls this circle the "monthly [parallel]," μηνιαῖος, in *Anal.* 6 and Moerbeke has *parallelum mensis*, in *Anal.* 8, Edwards [1984, 137 & 48]. One might have expected "day circle" or "day parallel." Ptolemy's term probably reflect the fact that the diameter of the monthly parallel is found by means of Vitruvius' *menaeus*, see page 193. The term also reflects the degree of accuracy that the analemma obtains. Because of the size of the angles involved, in any actual analemma construction, it would be difficult to obtain anything like accuracy to the day even in the signs on either side of the equinoxes.

43 Theo. *Spher.* I 20 provides a construction for drawing a great circle through two given points.

44 This passage is corrupt. Schöne reads καὶ τῇ ΞΩ περιφερείᾳ ὁμοία κείσθω ἡ < A′B′, > ἀπὸ δὲ τοῦ CA′ τετραγώνου κείσθω ἡ A′B′Z and Rome reads καὶ τῇ ΞΩ περιφερείᾳ ὁμοία, ἀπὸ δὲ τοῦ A′, τετραγώνου κείσθω ἡ A′B′. Both of them translate τετράγωνος as "quadrilateral," a meaning not elsewhere attested in the mathematical corpus, Schöne [1903, 306] and Rome [1923, 7]. I read the text as καὶ τῇ ΞΩ περιφερείᾳ ὁμοία κείσθω ἡ < A′B′, > ἀπὸ δὲ τοῦ EA′ τετραγώνου κείσθω ἡ A′B′ which allows us to take τετράγωνος to mean "quadrant," a common meaning in the mathematical literature, Mugler [1959, 419 - 420].

practical perspective.

This construction shows how the analemma can be used to solve problems of spherical astronomy. An analemma is explicitly used to find key arcs in relation to Rome, but we saw as well how another analemma is implicitly used to effect the construction of the hemisphere for Alexandria. It is clear that this whole solution is designed around the use of the analemma as a tool of descriptive geometry. The analemma allows the ancient mathematical astronomer to work directly with gnomon shadow lengths and seasonal hours by translating them into arcs on the celestial sphere. The language of *Diop.* 35 is that of a construction but not that of a metrical analysis. There is no mention of any magnitudes being given. The purpose of this text seems to have been to construct the arc between the two nadirs but not necessarily to calculate its value. The value of this arc was most likely read off a scale set equal to the rim of the hemisphere and marked with degrees and perhaps fractions of degrees.

On the other hand, one might well ask if this method of finding great circle distances had ever actually been used. The observed eclipse that Hero used is generally thought to have occurred during his lifetime because the example is a poor choice due to the proximity of the sun to the equinox.[45] Nevertheless, it is likely that the value given for the observation at Rome was simply assumed rather than reported so that Hero's treatment is merely an illustration of the technique rather than a valid determination of the distance.[46] Hero states that a set of observations should be taken from the tables, or if there is none, that a new observation should be made.[47] Since he uses his own set, we may presume that there were no simultaneous eclipse observations for Alexandria and Rome that were available to him. Ptolemy mentions the fact that there were very few simultaneous eclipse observations, citing only one.[48] It is likely that lack of a suitable set of observations prevented this construction from becoming anything more than a theoretical device.

[45] Neugebauer [1938, 23].
[46] Neugebauer [1975, 848].
[47] Schöne [1903, 302].
[48] Berggren and Jones [2000, 63].

It is very unlikely that this material originated with Hero. Rome has attributed these methods to Hipparchus on the basis of a passage in Strabo's *Geography*.[49] According to Strabo, Hipparchus claimed it was not possible to make an accurate determination of north-south intervals without making use of latitudes, nor of east-west intervals without a comparison of eclipse times.[50] Ptolemy, in *Geog.* I 4, also associates Hipparchus with the attempt to use astronomical observations as a means of determining geographic position and in this connection discusses eclipse observations as a means of finding east-west intervals.[51] The constructions used in *Diop.* 35 can be used for these purposes. The latitude of Alexandria and Rome can be found by means of trigonometry on the analemma and, by comparison of these values, the north-south interval between the cities can be found. The east-west interval between the two cities is given by $\widehat{A'H}$ in Figure 4.4 (a) which can be found either nomographically or through metrical methods on the sphere. As Rome has already made clear, however, the fact that we can trace these methods back to Hipparchus should not lead us to the conclusion that Hipparchus made any more use of them than Hero. In fact, we have reports of only one eclipse in antiquity for which simultaneous observations were made. This is the famous eclipse seen before the battle at Arbēla which is reported by both Ptolemy and Pliny.[52] These reports, however, are inaccurate and most likely entered the technical literature through the historical tradition.[53] If this material comes from Hipparchus, it was most likely described as a theoretical technique with a single example, perhaps the eclipse seen at Arbēla.[54]

Hero preserves a method of finding the distance between two geographic locations based on simultaneous eclipse observations which relies extensively on the analemma. The method likely goes back to Hipparchus. The text, as we have it, indicates that Hero conceived of the solution as purely nomographic since no computations are performed and no metrical analysis is carried out. Here again, we see the analemma being used to

[49] Rome [1923, 246 - 251].
[50] See Strabo, *Geography* I 1.12, Jones [1969, vol. 1, 22 - 24] or Dicks [1960, 64].
[51] Berggren and Jones [2000, 62 - 64].
[52] Berggren and Jones [2000, 63] and Rackham [1949, 312].
[53] Dicks [1960, 122].
[54] Rome [1923, 256].

model the motion of the sun with respect to the local horizon, but now this is done as an auxiliary construction in a more complex problem of descriptive spherical astronomy.

4.1.4 Ptolemy's *Analemma*

After a brief introduction concerning the relationship between the theory of nature (*naturali theorie*) and mathematics, Ptolemy gives a description of his coordinate system from first principles in *Anal.* 2. He argues that there can be only three dimensions for any mass since only three straight lines can be set up at right angles to each other. Consequently, in a sphere only three diameters are set up at right angles to each other and only three great circles meet on another at right angles. These great circles in the "world sphere," are called the *horizon circle*, the *meridian circle*, and the *vertical circle*; see Figure 4.5 (a). The diameters are the *meridional diameter*, which is intersection of the horizon with the meridian; the *equatorial diameter*, which is the intersection of the horizon and the vertical; and the *gnomon*, which is the intersection of the meridian and the vertical.

The three primary circles are understood to move "with the sun" about the diameters which act as axes. Ptolemy's exposition is obscure but his intention is clear.[55] Since each great circle contains two of the principal diameters, he needs to establish the rotation of each circle such that it has a unique diameter as its axis.

The horizon circle is set to rotate about the equatorial diameter, the meridian circle about the meridional diameter and the vertical circle about the gnomon. The motion of the horizon produces the *hectemorius circle*, the motion of the meridian produces the *horarius circle* and the motion of the vertical produces the *descensivus circle.* When each of these mobile circles is aligned above the terrestrial horizon so as to contain the solar

[55] The two possible motions of each circle are described. The first motion [a.] is said to be about one diameter and toward the poles which the fixed circle divides, while the second motion [b.] is said to be about the other diameter and toward poles which the fixed circle occupies. For example, the horizon is said to rotate either [a.] about the equatorial diameter toward the nadir and zenith, or [b.] about the meridional diameter toward the east and west; see Figure 4.5 (a). The first motion in each case is chosen as "more fitting" (*conuenientiorem*) since it is "in the directions between which the fixed circle acts as a divider," Heiberg [1907a, 190] and Edwards [1984, 84, n. 434]. This distinction, however, is specious. Both movements are in the same *direction*; it is simply a question of which diameter is chosen as an axis. The choice is arbitrary, so long as each circle has a unique axis.

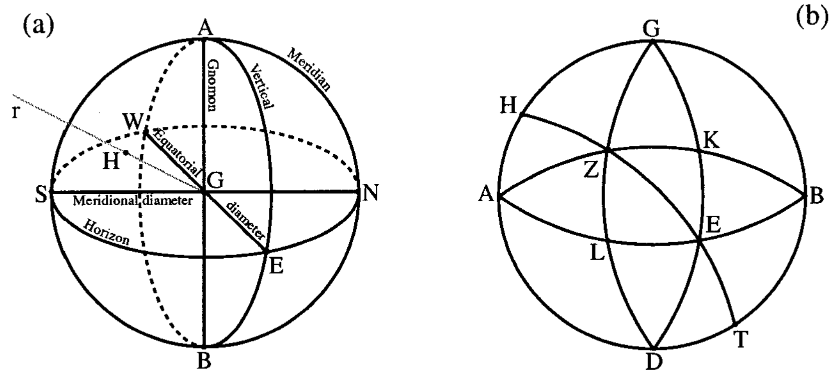

Figure 4.5: The coordinate system of Ptolemy's *Analemma.* Figure (a) is a schematic rendition showing the cardinal circles and diameters. Figure (b) is from Moerbeke's text, Edwards [1984, Fig. 1].

ray, it will make two angles which will be sufficient to determine the position of the sun. The two angles are as follows: (1) the rectilinear angle between the solar ray and the axis of the mobile circle and (2) the solid angle between the mobile circle and its fixed coordinate. For example, in the case of the hectemorius circle, the two angles are (1) the rectilinear angle which the solar ray makes with the equatorial diameter, $\angle HGE$ in Figure 4.5 (a), and (2) the solid angle which the hectemorius circle makes with the horizon, $\overset{\frown}{SX}$ in Figure 4.1 (a). The hectemorius circle produces the rectilinear *hectemorius angle* and the solid *meridian angle*, the horarius circle produces the rectilinear *horarius angle* and the solid *vertical angle*, and the meridian circle produces the rectilinear *descensivius angle* and the solid *horizon angle*; see Figure 4.1 (b). In this way, each great circle determines two angles which serve as the coordinate pairs described above.[56] In fact, as Luckey has pointed out, there are fifteen possible combination of these six angles that will determine the position of the sun.[57] Ptolemy's pairs are, however, sufficient and he has derived them in a manner which he maintains follows naturally from the phenomena.

The treatment to this point is a description of a method for determining the position

[56] See page 185.
[57] Luckey [1927, 21 - 22].

of the real sun with respect to the coordinates of the actual local horizon. The great circles here are circles of the "world sphere" and those that move, move "with the sun." The angles that these great circles form, which will determine the position of the sun, are described with respect to a local observer at the center of the sphere. In *Anal.* 3, Ptolemy moves from the global description given above to the construction of a sphere which will model the movements of the "world sphere" on a more manageable scale. He does this so that the material discussed above "may fall more within our view."[58] In Figure 4.5 (b), circle $AGBD$ is the meridian, the semicircle AEB is the horizon and semicircle GED is the vertical. The six key angles are set out as arcs of great circles on the sphere and they determine the position, not of the sun, but of the solar ray. The rest of the *Analemma* deals only with this spherical model.

In *Anal.* 4, Ptolemy provides a brief aside which states the differences between his conventions and those of "the ancients."[59] The next section gives the final conventions which follow from the fact that Ptolemy has determined that each of the key arcs should be no greater than a quadrant. Since, for most places in the *oikoumenē*, the solar ray can appear in all four sectors of the hemisphere above the horizon, three of the six arcs will have two different origins depending on the solar position and the other three will have two different directions from the same origin. All of the possibilities are explicated in *Anal.* 5.

The mathematical treatment by means of the analemma begins in *Anal.* 6. This section, in fact, contains the only synthetic theorem in the text. The general procedures for determining the sizes of the arcs on the sphere are called "mechanical determinations," *instrumentales acceptiones.*[60] This is an odd expression but it is probably a reference to

[58] Heiberg [1907a, 193] and Edwards [1984, 89].
[59] Neugebauer [1975, 849 - 850] discusses these details.
[60] Heiberg [1907a, 194] and Edwards [1984, 42]. The Greek for this passage is lost; however, Edwards conjectures that the original expression would have been ὀργανικαὶ λήψεις, based on the Greek that survives for the following passage.

the model as an instrument.[61] If we treat the spherical model as an instrument which can be analyzed by means of the analemma for given times and places, then we can see how laying out the six arcs as angles in the analemma can be thought of as a mechanical determination.

It will be helpful to look at the proof in *Anal.* 6 in detail because this will give some insight into Ptolemy's conception of the analemma as a whole. Moreover, this demonstration represents the only synthetic proof on the analemma in the Greek mathematical corpus. After dismissing the case where where the sun is at the equinoxes as trivial, Ptolemy takes up the case for any other longitude of the sun as follows; see Figure 4.6 (a):[62]

> Let there be a meridian circle $ABGD$ in which AB is the diameter of the horizon and GD is at right angles to this and corresponds to the gnomon, while the center of the solar sphere is E, and ZHT is the diameter of one of the monthly parallels north of the equator. On this diameter, in the same plane, let ZKT be imagined (νοείσθω)[63] as the eastern semicircle of the monthly parallel. Let KH be drawn perpendicular to ZT, so that ZK forms the portion of the parallel circle above the earth; and, cutting off $\widehat{KL}$, let perpendicular LM be drawn from L to ZT. With center M and distance ML let point X be taken on the meridian,[64] let EL, EMN, EX, and MX be joined and let EO be drawn perpendicular to EN. I say that $\angle XEO$ is the angle in question.[65] For let semicircle ZLT be imagined (νοείσθω) as turned (ἐπεστραμμένον) to its proper position, that is, at right angles to the plane of the meridian, and let EP be produced from E perpendicular to the same plane to represent the equatorial diameter. Then, since LM is also perpendicular

[61] Edwards has taken this expression to refer to justifications that the six arcs are equivalent to certain angles in the analemma, Edwards [1984, 94, n. 454]. Ptolemy, however, makes it clear that he will only supply a proof (ἀποδειξις) for one determination (λῆψις), that of the new hectemorius arc which he has introduced, Heiberg [1907a, 194 - 195] and Edwards [1984, 94 - 95 & 136]. This is presumably because the proofs in his predecessors' work is sufficient for the determinations of the other angles. The instrumental determinations must be understood as independent of the proof of their validity.

[62] Edwards [1984, 96 - 98]. I have made a few changes for consistency.

[63] The MS reads νοείσθαι, Heiberg corrects to νοείσθω, Heiberg [1907a, 196].

[64] As Edwards has shown, this point can also be found by taking the perpendicular to NE at M, Edwards [1984, 97, n. 466].

[65] That is the hectemorius.

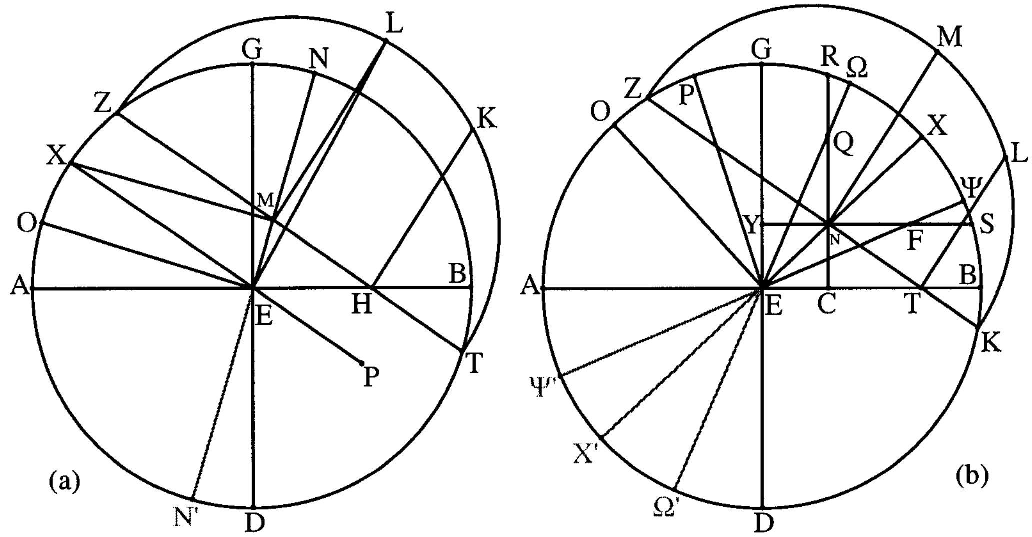

Figure 4.6: Diagrams for *Anal.* 6 & 8 from Moerbeke's text, Edwards [1984, Fig. 2 & 4]. Lines and letters in gray do not appear in Moerbeke.

> to the meridian, it is clear that the lines EN, ML, and EP lie in one plane perpendicular to $ABGD$. Likewise, it is also clear that EN is the common section of the hectemorius circle and the meridian, and EL is in line with the solar ray, and the angle in question, which is contained by the ray and the equatorial diameter, is LEP. For since, $EL = EX$ [Theo. *Spher.* I def. 1], $ML = MX$ [by construction], and EM is common, therefore, $\angle MEL = \angle MEX$. But $\angle MEP$ is right, as well as $\angle MEO$, and, therefore, the remainder [$\angle$]LEP is equal to the remainder [$\angle$]XEO. Q.E.D.

The key to understanding Ptolemy's construction is to consider the analemma figure as a representation of the solid sphere. There are a number of clues that this was Ptolemy's intention. The expression that he uses when he speaks of the monthly parallel, "let it be imagined," νοείσθω, is a standard expression in solid constructions that cannot be fully or accurately represented by the plane figure.[66] The semicircle of the monthly parallel, or β-circle, is drawn in the plane of the meridian but, in fact, it needs to be

[66] See for examples *Elem.* XII 13 & 16, *Conics* I 52, 54 & 56, Theo. *Spher.* I 19. The expression is also found hundreds of times in Archimedes' corpus, Heiberg [1973].

regarded as perpendicular to the plane of the figure. Ptolemy refers to this semicircle as having been rotated (ἐπεστραμμένον) down onto the plane of the figure. This rotation is done so that we can work with the arcs along this semicircle as well as the arcs on the meridian.

As soon as we consider semicircle ZKT in its "proper position" the construction becomes obvious. The position of the sun is marked off as point L on $\overset{\frown}{ZK}$. In its proper position, L is above the plane of the meridian on line LM perpendicular to point M. The angle of the hectemorius is given by a great circle arc through L and P, where EP is the equatorial diameter perpendicular to the plane of the meridian. The diameter of the great circle through L and P is NN'. If we consider this great circle as rotated into the plane of the meridian, EP will map to $EO \perp EN$, ML will map to $MX = ML$ and $\perp EN$, the solar ray LE will map to EX, and $\overset{\frown}{LP}$ will map to $\overset{\frown}{OX}$. This is the situation Ptolemy describes. Ptolemy does not talk of the great circle on diameter NEN' as rotated, he simply constructs the equivalent figure in the plane. This procedure agrees with what we saw in Diodorus' construction of the meridian line on the plane of the horizon.[67]

The Figure 4.6 (a) represents three circles in three different planes superimposed upon one another in the plane of the figure. The circle of the meridian, $ADBG$, lies in the plane of the figure. The circle of the hectemorius, NON', is perpendicular to the plane of the figure and intersects it in line NEN'. The monthly parallel of the sun, $TKLZ$, is perpendicular to the plane of the figure and intersects it in line $THMZ$.

The next two sections of the *Analemma* set out the determinations for the remaining five angles with no proofs. Since these five angles were covered in the works of Ptolemy's predecessors, we may presume that they provided proofs for these angles and that Ptolemy found these proofs satisfactory. *Anal.* 7 sets out the remaining angles when the sun is at an equinox and *Anal.* 8 takes up all other cases. Following the lines of Luckey's account, it will be useful to see how the determinations of the other angles may

[67] See section 4.1.1.

also be made apparent by considering the analemma as a sphere; see Figure 4.6 (b).[68]

The hectemorius circle forms the hectemorius and meridian angles. We have already seen how the hectemorius angle, $\angle OEP$, is found. The meridian angle is $\angle BEX$ between the diameter of the hectemorius circle, XX', and the meridional diameter, AB.

The horarius circle forms the horarius and vertical angles. The horarius angle is the arc of a great circle that goes through B, M and A, where NM is considered perpendicular to the plane of the meridian. If this great circle is considered as rotated into the plane of the horizon the point M will map to point C, and if it is rotated into the plane of the meridian, M will map to R. Hence, $\angle BER$ is the horarius. The vertical angle is the angle that the diameter of the horarius circle makes with the gnomon in the plane of the vertical. If we consider the plane of the figure to be the vertical circle oriented such that G is the zenith and B is the east point, then the position of the solar ray on the sphere is modeled by a point which is below the plane of the figure on a segment $= YN$, above the diameter of the horizon on a segment $= NC$ and east of the meridian on a segment $= MN$, since these distances determine the position of the solar ray with regards to the fixed planes. Thus, if we cut off YNF on YS such that $YF = NM$ and $\perp GD$, then the solar ray will be on a perpendicular dropped from point F and $\Psi FE\Psi'$ will be the diameter of the horarius circle that lies in the plane of the vertical. Hence, $\angle GE\Psi$ will be the vertical angle.

The descensivus circle forms the descensivus and horizon angles. The descensivus angle is the arc of a great circle that goes through G, M and D, where NM is considered perpendicular to the plane of the meridian. If this great circle is considered as rotated into the plane of the vertical the point M will map to Y, and if it is rotated into the plane of the meridian, M will map to S. Thus, $\angle GES$ is the descensivus angle. The horizon angle is the angle that the diameter of the descensivus circle, which lies in the plane of the horizon, makes with the equatorial diameter. If we consider the circle $AGBD$ to be the horizon circle oriented such that B is the north point and G is the east point,

[68] Luckey [1927, 25 - 26].

then the position of the solar ray is modeled by a point which is above the figure on a segment $= NC$, north of the vertical on a segment $= YN$ and east of the meridian on a segment $= MN$. Thus, if we cut off $CNQ = MN$ and $\perp AB$, the solar ray will be on a perpendicular raised from point Q and $\Omega QE\Psi'$ will be the diameter of the descensivus circle in the plane of the horizon. Hence, $\angle GE\Omega$ will be the horizon angle.

The Figure 4.6 (b) represents four circles in three different planes superimposed upon one another in the plane of the figure. The circle of the meridian, $ADBG$, lies in the plane of the figure. The circles of the vertical, $\Psi G\Psi'$, and the horizon, $\Omega G\Psi'$, which are perpendicular to the meridian are both rotated into the plane of the figure. The monthly parallel of the sun, $KLMZ$, is perpendicular to the plane of the figure and intersects it in line $KTNZ$.

In this way, the six principal arcs are found in the plane through considering the analemma as a solid construction. Constructions which may appear obscure when viewed only in the plane become clear when we conceive of the figure as a plane depiction of a solid model. We do not have the ancient proofs of any of the angles other than the hectemorius but, if Ptolemy's proof for the hectemorius angle may be taken as our starting point, we may form a good idea of how these proofs proceeded. The angle would have first been constructed in the plane and then it would have been shown how this angle is equal to the solid angle on the sphere. The latter part of the proofs would have followed the outlines of the descriptions given above. The similarity between this approach and that found in the meridian determination attributed to Diodorus is striking.

Now that the determinations are set out in the plane, it is simply a matter of finding these angles as arc lengths of the great circle $ADBG$. Ptolemy gives two methods for finding the arc lengths. The first is the method of metrical analysis which Ptolemy refers to with the phrase "by means of lines," *dia tōn grammōn*.[69] As was discussed in Section 2.2.1, this is a technical expression which means something like "through exact geometric methods." Here, it means through a metrical analysis which demonstrates the possibility

[69] Heiberg [1907a, 202] and Edwards [1984, 141].

of a trigonometric computation. The analysis is given on the plane with no reference to the sphere. *Anal.* 9 gives the analysis for the case when the sun is at an equinox and *Anal.* 10 for all other cases. No computations are actually given and it is clear that Ptolemy considers this method tedious and has included it only for the sake of logical and thematic completeness.[70] The second method is the nomographic method and its treatment fills out the work.

The final four sections constitute, along with the introduction of the hectemorius angle, Ptolemy's primary contribution to gnomonics. Here, Ptolemy describes the physical construction of an analemma plate which can be inscribed with certain permanent lines and used to make ready determinations of the principal arcs. These constructions are carried out with a compass and a set-square and devised in such a way that no lines need be drawn on the plate in the process of producing the angles. This material is interesting for the history of nomographic techniques.[71] These constructions are used for the production of a series of 49 tables containing a total of 2058 computed values. While these values could be calculated, each would involve a lengthy trigonometric procedure and filling in the entire table would be a monumental labor. Ptolemy's contribution is the discovery of a means of computing these values with much less work while maintaining as much accuracy as is necessary to the task at hand.

Following the text, Ptolemy presents a series of tables which tabulate the six angles as a function of three variables: (1) the position of the sun at the start of each zodiacal sign and (2) at the start of each seasonal hour for (3) seven geographic latitudes. If all of the angles had been tabulated for each of these variables it would have resulted in 49 tables, each of 42 entries. Only one table survives.[72] This table is not followed by the date as is typical at the conclusion of Moerbeke's translations. Moreover, the extant text ends at the bottom of the verso of the last page in a quire. Hence, there are good reasons

[70] Heiberg [1907a, 202 & 210] and Edwards [1984, 107 - 108 & 118]. Edwards [1984, 115 - 117] gives an example of a calculation of all six angles for $\varphi = 40;55°$, $\delta_{\odot} = 20;30°$ ($\lambda_{\odot} =$ ♊ $0°$, ♌ $0°$, ♐ $0°$ or ♒ $0°$) and $h_s = 2$.

[71] Luckey [1927, 32 - 39] has given a complete account of the nomographic method.

[72] Heiberg [1907a, 223] and Edwards [1984, 76].

for believing that the missing tables dropped out of the manuscript at a later date.

It has been suggested that the work originally contained further material following the tables pertaining to the construction of sundials.[73] Ptolemy, however, introduces his text as an investigation of what is "reasonable or not reasonable in the angles taken up in gnomonics."[74] This is precisely the scope and aim of the work as we have it. There is no internal reason to think that anything is missing beyond the tables.

4.1.5 Remarks on the *Analemma* and its sources

Ptolemy's text, although our best evidence for the methods of the analemma among the Greeks, is not greatly innovative from a theoretical perspective. His improvements on the previous work are the introduction of the hectemorius angle with the superior organization that follows and the development of nomographic techniques to circumvent the tedium of repeated calculations.

The *Analemma* is meant to be read as a complement to the previous tradition in gnomonics, particularly the work of Diodorus. Ptolemy does not name his predecessors but there are a number of reasons for thinking that he has Diodorus in mind.[75] We know that Diodorus' work included material that is not found in Ptolemy, but we may assume that all of the material in Ptolemy's text, with the exception of the two innovations mentioned above, were also found in Diodorus. We may further assume that Diodorus included proofs of all six of his angles and that he arrived at the figures for his angles by means of trigonometric calculation.

The methods of the analemma were developed in the context of gnomonics but provided a general way of addressing certain problems in spherical astronomy. The analemma is particularly suited to the ancient understandings of geographical latitude, $s : g$, and seasonal time, h_s.[76] It is useful for (1) relating the equatorial system to the system of

[73] Delambre [1817, vol. 2, 471].
[74] Heiberg [1907a, 189] and Edwards [1984, 79].
[75] See Edwards' essay on the ancient evidence for Diodorus, Edwards [1984, 152 - 182].
[76] Wilson has shown how the analemma can be used to derive φ from M, Wilson [1997]. Berggren shows how M can be derived from φ, Berggren [2002, 45 - 46].

the local horizon, (2) relating the ecliptic system to either of these two systems when an equinox is rising, or (3) making determinations on parallel circles. Examples of problems solved using (1) are Hero's determination of the distance between two cities, in Section 4.1.3, and medieval Arabic determinations of the *qibla*.[77] By considering individual points of the ecliptic, Ptolemy's *Analemma* uses (2) to determine the local coordinates of a given point of the ecliptic at a given time. By combining (1) and (2) the analemma can be used to transform between equatorial and ecliptic coordinates. Although we have no examples of such a transformation in antiquity, the mathematically equivalent problem of finding the azimuth of a distant city was solved using an analemma a number of times by Arabic mathematicians.[78] Although there are no ancient examples of (3), this would seem to be an obvious use for the analemma since it allows for the solution of problems that cannot be handled by ancient spherical trigonometry.[79]

In all analemma constructions, however, the circles involved must be perpendicular to the plane of the figure and either the equinox must be rising or only one point of the ecliptic can be handled. Hence, the analemma cannot in general be used to calculate the position of the ecliptic with respect to the local coordinates.[80] At some point, this limitation was considered awkward and the metrical methods of the spherical proof were developed. The metrical methods eventually found their proper place in a new spherical geometry that studied the intersections of great circles on the sphere and gave a central role to the spherical triangle.

The analemma provides a systematic way of translating the arcs of a sphere into the plane so that they can then be calculated or measured. The analemma figure represents a number of different circles in the sphere all rotated down into the plane of the figure or superimposed upon one another such that lengths and arcs are not distorted. One does this by constructing the arcs in the plane using plane methods and then proving

[77] For determinations of the *qibla* see, for example, Kennedy [1974] and Berggren [1980].
[78] Berggren [1980] summarizes four of these. Also see Carandell [1984] and Berggren [1992].
[79] The reconstructions given by Neugebauer [1975, 301 - 304 & 850 - 852] and Wilson [1997] solve problems through this use of the analemma.
[80] See Appendix E.

that these arcs are equal to arcs on the sphere by considering the analemma construction as a solid sphere. Once the arcs have been provided by geometric construction, they can then be determined by metrical analysis or simply measured through nomographic techniques. Although Ptolemy is our best evidence for the analemma methods, they were developed as early as the middle of the 2nd century BCE if not earlier.[81] The analemma methods were developed in connection with the older spherical geometry but because of their intrinsic applicability to the coordinates of the local horizon and their ability to provide metric solutions to problems involving parallel circles, they secured their usefulness alongside the newer metrical methods.

4.2 The *Planisphaerium*

Ptolemy's *Planisphaerium*, as it is know in Latin, or the *Simplification of the Sphere*,[82] is a short work that gives a plane model of the sphere. The model is produced by a mixture of projective techniques and constructions in the plane which produce the same result as pointwise stereographic projection. Berggren, however, has pointed out that Ptolemy's procedure is different from any modern approach based on pointwise projection.[83] Ptolemy, in fact, only uses projective techniques to find key points in the plane. The rest of the construction is then carried out in the plane and the proofs aim to show that the principal circles have the same properties and relationships in the plane that they have on the "solid sphere." Almost all of the proofs take place completely within the plane. Some of the few cases where Ptolemy gives an analemma construction or reveals a more projective approach will be discussed below.

[81] Proclus states that Diodorus was one of the first writers on analemmata, Manitius [1909, 112].

[82] The Suidas gives the title Ἅπλωσις ἐπιφανείας σφαίρας, *Simplification of the sphere*, in a list of Ptolemy's works, Adler [1938, 254]. Kauffmann thinks that this is a mistake for the title 'Εξάπλωσις ἐπιφανείας σφαίρας, *Unfolding of the sphere*, based on the usage of later writers describing the plane astrolabe, Pauly and Wissowa [1974, vol. 2, 1801]. Neugebauer has accepted this correction, Neugebauer [1975, 971, n. 2]. 'Εξάπλωσις, however, can also be taken as a stronger expression of ἅπλωσις. Even if we take the two terms to have different meanings, the term ἐξάπλωσις does not add much new information.

[83] Berggren [1991a, 138 - 143].

The subject matter of the text naturally falls into two parts. It is clear that the sectioning which is transmitted in the text is due to a later editor. There are a number of places where the sectioning breaks up the natural flow of the argument. *Planis.* 2 & 3 belong together, the break between *Planis.* 8 & 9 is awkward, and separating *Planis.* 5 - 7 from *Planis.* 4 is needless. The first thirteen sections introduce the plane model, discuss the representation of horizons and the ecliptic and demonstrate the validity of the model by a treatment of rising times. The final seven sections introduce and solve problems that would arise in the course of actually drawing the model on a plate. Here, we encounter the construction of a suitable southernmost circle and of the ecliptic parallels. Despite the fact that the numbering of the text does not signal a transition, Ptolemy begins *Planis.* 14 with a brief summary of the results so far, following his usual practice at the start of a new book.[84] Although the second part is concerned with practical issues, the treatment is still mathematical and we find few detailed instructions of the sort encountered in many of Ptolemy's other works.

Most of the text in the *Planisphaerium* is mathematical and consists of theorems, metrical analyses and calculations. The distinction between these types of mathematical prose is not as sharp as in the *Almagest* and the *Analemma.* For example, the demonstration that the rising times in the plane model conform to the situation on the sphere is made up of all three types of arguments. The theorems generally relate to the plane figure and make few references to the situation that they model on the sphere. In a number of cases, the theorems prove results in the plane that are obvious or axiomatic on the sphere.

The *Planisphaerium* is the only ancient text we have that deals with this plane model of the sphere and, like Ptolemy's *Optics*, it makes no reference to any previous work. As in the case of the *Optics*, however, Ptolemy's silence should not lead us to believe that

[84] Neugebauer's division into four sections is accurate with respect to the content but neglect the remarks Ptolemy makes at the beginning of *Planis.* 14, Neugebauer [1949, 247].

there was no previous mathematical tradition.[85] The usual story is that this tradition of modeling goes back to Hipparchus and that it was used to solve problems in spherical astronomy before the development of metrical methods on the solid sphere.[86] Ptolemy's text, however, does not provide any evidence of this problem solving tradition and the main aim of the first half of Ptolemy's text is to show that the plane model is consistent with the results in the *Almagest* found on the sphere. Ptolemy's text confirms that there was a tradition of mapping the celestial sphere to the plane using these methods and that part of this tradition involved demonstrating that the map was mathematically sound.

The *Planisphaerium* was certainly composed after the *Almagest*. Throughout the first part, Ptolemy makes repeated references to the *Almagest*. The *Planisphaerium* uses a number of exact values which are also found in the *Almagest* and the stated goal of the sections on rising times is to show the numerical agreement between the plane model and the results on the "solid sphere." Nothing in the text, however, allows us to situate the time of its composition with respect to any of Ptolemy's other works.

The Greek text of the *Planisphaerium* is no longer extant. We have, however, versions in both Arabic and Latin. The Arabic version was presumably made in Baghdad around or before 900 CE.[87] A Latin translation made in the late 10th century is preserved only in fragments.[88] The earliest complete Latin version was made by Hermann of Carinthia in the 12th century on the basis of a later Arabic edition with comments by Maslama al-Majrīṭī.[89] One manuscript of the Arabic text has been published in facsimile with English translation by Anagnostakis, who shows that this text cannot have been the

[85] The principal texts of ancient mathematical optics before Ptolemy are the two versions of the Euclidean *Optics*, the Euclidean *Catoptrics*, and a *Catoptrics* wrongly attributed to Ptolemy, Heiberg [1895] and Jones [2001]. There are also a number of significant passages of mathematical optics in the Aristotelean *Meteorology* III, Lee [1952].

[86] Neugebauer [1975, 858].

[87] Kunitzsch [1995, 150 - 155] and Kunitzsch [1993, 97, n. 2].

[88] Kunitzsch [1993].

[89] Hermann's text has been edited by Heiberg and included in Ptolemy's *Opera minora*, Heiberg [1907a, 227 - 270]. A German translation of Hermann's text was made by Drecker [1927].

version used by Hermann.[90] Despite minor differences in the two published versions of the text, they essentially agree, and by comparing them against each other we may form a fair view of Ptolemy's intentions and work through most of the obscurities. There are no figures in the Arabic manuscript which Anagnostakis edited. Anagnostakis drew his own figures for his edition by consulting the figures in the Latin print tradition. For the most part, these figures are in agreement with Hermann's.[91] I have used Anagnostakis' figures with minor changes.

4.2.1 Construction and test of the model

After a terse introduction in which Ptolemy states the goals of the text, the first section deals with the basic construction of the plane model including the north pole, the equator, the equatorial parallels, or δ-circles, and great circles tangent to two parallels which are equidistant from the equator, for example the ecliptic or a horizon. Berggren has argued that Ptolemy does not proceed by projection.[92] In fact, he uses projective considerations to construct certain points in the plane of the figure and then proceeds to write proofs that generally take place within the plane.

The first section shows how to arrange δ-circles such that the great circle which is tangent to these two parallels bisects the equator. Ptolemy constructs these equidistant parallels by drawing circles through points found by a projective construction. He then goes on to prove that another circle which is drawn tangent to the δ-circles will bisect the equator. This way of proceeding is mathematically equivalent to the assumption that circles on the sphere will map to circles on the plane. It is not clear on the basis of the text we have, however, whether or not Ptolemy actually made this implicit assumption. Thus, Ptolemy uses projective techniques and constructions in the plane to produce his

[90] Anagnostakis [1984, 55 - 57]. A second Arabic manuscript is studied by Kunitzsch [1994]. Sinisgalli and Vastola have printed a text with an Italian translation, Sinisgalli and Vastola [1992a]. Their book, in fact, includes three different versions of the text from 16th century editions. As with their *L'Analemma* there is no apparatus.

[91] Notable exceptions are the last two figures, Heiberg [1907a, 256 - 257] and Anagnostakis [1984, figs. 12, 12a, 13 & 13a].

[92] Berggren [1991a, 138 - 143].

plane model. He then demonstrates that the model he has so drawn is mathematically analogous to the situation on the sphere.

We consider circle $ABGD$ as the equator about center E and we let the two diameters AG and BD intersect one another at right angles; see Figure 4.7. We consider these lines as meridians and point E as the north pole. Clearly then, δ-circles which are in the northern hemisphere will be inside the circle $ABGD$ while those in the southern hemisphere will be outside it. We cut off two equal arcs on the circle $ABGD$, $\overset{\frown}{GZ} = \overset{\frown}{GH}$, and join lines DTZ and DHK falling on AG produced. With center E and radii ET and EK we draw the two circles TL and KM. Circle TL and KM will then represent parallel δ-circles, equidistant from the equator. Taking R as the midpoint of TM, with R as center and radius RT we draw circle TM. Since the circle TM is tangent to the circles which represent parallel δ-circles in the plane, a proof that TM bisects the circle which represents the equator will show that TM represents the inclined great circle tangent to the δ-circles. The proof is as follows. We join line DNM, where D is now the midpoint of $\overset{\frown}{AG}$ in the plane of the figure and, since $\overset{\frown}{AN} = \overset{\frown}{GH} = \overset{\frown}{GZ}$, $N\overset{\frown}{D}Z$ is a semicircle [$G\overset{\frown}{D}A - \overset{\frown}{AN} + \overset{\frown}{GZ} = N\overset{\frown}{D}Z$]. Hence, $\angle MDT$ is right and the circle which is drawn through points T, M and D has TM for a diameter and passes through point D. Thus, circle TMD is tangent to the circles TL and KM and bisects the circle $ABGD$. Hence, it represents a great circle inclined to the equator and tangent to two parallel δ-circles. From this discussion, Ptolemy concludes that equidistant pairs of δ-circles are drawn by taking $\overset{\frown}{GZ} = \overset{\frown}{GH}$.[93]

Ptolemy does not make all of his assumptions explicit. He does not make it clear that the plane of projection and the plane of the drawing have been folded together. Moreover, he does not maintain a clear distinction between the circles on the sphere and those that represent them in the plane. This dissociation allows the sphere to fall into the background so that although it is occasionally implicit in his constructions it is rarely invoked in the proofs.

[93] Heiberg [1907a, 227 - 229] and Anagnostakis [1984, 68 - 71].

Although it is not so stated in the text, point D is considered as the south pole and circle $ABGD$ represents the solstitial colure as well as the equator. In Figure 4.7 (a), the plane of the figure contains two different planes: (1) the plane of the equator upon which the planar model is drawn and (2) the plane of the solstitial colure which is used for determining the lengths of the various diameters in the model.

Next Ptolemy considers the special case where $\widehat{GZ} = \widehat{GH} \approx 23;51°$ and points out that, in this case, circle TL is the tropic of Cancer, circle KM is the tropic of Capricorn, while circle $TBMD$ is the ecliptic and is arranged such that B is the vernal equinox, T the summer solstice, D the autumnal equinox and M the winter solstice.[94]

Ptolemy concludes the opening section by explaining that the signs, just as the seasons, do not cut off equal arcs on the circle of the ecliptic. In order to find the beginnings of the signs, we need to first find the δ-circles at the proper declinations and draw the circles through these; because, only in this case, will the meridian lines through E meet the ecliptic at points which correspond to points which are diametrically opposite on the sphere.[95] The proof of this final remark is taken up in the next section.

The next two sections introduce horizon circles. They show that horizon circles which bisect the equator also bisect the ecliptic. The treatment is in two parts. *Planis.* 2 is a lemma which shows that if a meridian line is drawn through the visible pole, it will meet the ecliptic at two points which correspond to points which are diametrically opposite on the sphere.[96] *Planis.* 3 then uses this lemma to show that a horizon circle which is drawn in the plane such that it bisects the equator will also bisect the ecliptic. The proof in *Planis.* 2 is interesting because it contains a construction similar to what we saw in the *Analemma* and shows Ptolemy considering the solid sphere. Moreover, it is one of the few places in the *Planisphaerium* where we get a taste of projective procedures. The proof in *Planis.* 3 is quite simple and takes place entirely within the plane. It will be

[94] Heiberg [1907a, 229] and Anagnostakis [1984, 71].
[95] Heiberg [1907a, 229 - 230] and Anagnostakis [1984, 71 - 72].
[96] Lorch takes *Planis.* 2 to be a proof that a horizon circle bisects the ecliptic at *sphaera recta*, Lorch [1995, 273]. The figure could be taken to present the situation at *sphaera recta*; however, the text only speaks of the meridian and precisely this lemma concerning the meridian is needed for the proof in *Planis.* 3.

useful to compare the two.

The enunciation of *Planis.* 2 introduces the material of both sections.[97] The theorem which follows, however, shows that a meridian circle, ZH, which is drawn as a straight line through the visible pole, E, will intersect the ecliptic circle, $ZBHD$, at points which correspond to points which are diametrically opposite on the sphere; see Figure 4.7 (b).

Ptolemy sets out this theorem by constructing the equator $ABGD$ and the ecliptic $ZBHD$ as above; see Figure 4.7 (b). Through E, a line representing the meridian, ZH, is drawn. We drop $TE \perp AG$ and join TG, TA, THL and TKZ. Clearly, $\angle ATG$ is right because $\overset{\frown}{ATG}$ is a semicircle. Since $(ZE \times EH) = ED^2$ [*Elem.* III 35],[98] $ZE : ET = ET : EH$ $[ED = ET]$. Hence, $\triangle ZTH$ is right and $\angle ZTH$ is the right angle [*Elem.* VI 5]. Thus, $\angle ZTH - \angle ATH = \angle ATG - \angle ATH = \angle ZTA = \angle GTL$. If we consider circle $ABGD$ as also representing the meridian rotated into the plane of the figure about diameter AG such that point T is the south pole, then the lines joining the south pole with the end points of the diameter ZH cut off equal arcs of the meridian circle on either side of the equatorial diameter AG. Hence, from *Planis.* 1, we conclude that H and Z lie upon circles concentric with E which represent two δ-circles equidistant from the equator. Thus, points H and Z are diametrically opposite on the solid sphere, and ZEH passes through points which lie on the diameter of the ecliptic.[99]

Ptolemy himself makes no reference to the fact that point T represents the south pole or that circle $ABGD$ represents the meridian as well as the equator. Moreover, this figure is the only figure in the work which represents the south pole with any point other than the one indicated by D, and through this feature it gives one of the few hints that Ptolemy conceived of these constructions as projections of a solid sphere. The plane of the meridian which contains the meridian circle, $AKTG$, the south pole, T, and the lines TE, TA, TG, TKZ and THL must be considered as perpendicular to the plane of the model. This allows us to show that ZEH has the properties of a diameter of the ecliptic

[97] Heiberg [1907a, 230] and Anagnostakis [1984, 72].

[98] In accordance with the Arabic convention, this is expressed as an equality between products not between a rectangle and a square, Heiberg [1907a, 230 - 231] and Anagnostakis [1984, 72 - 73].

[99] Heiberg [1907a, 230 - 231] and Anagnostakis [1984, 72 - 73].

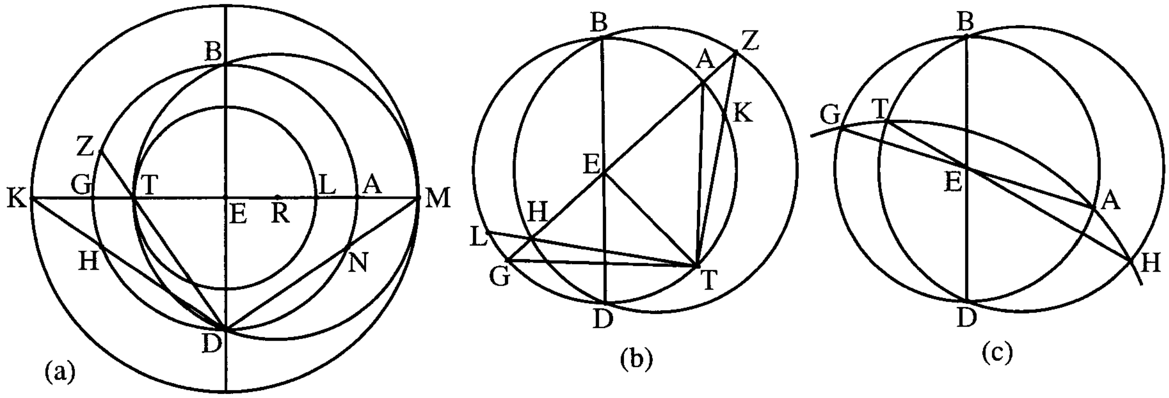

Figure 4.7: Diagrams for *Planis.* 1 - 3, Anagnostakis [1984, figs. 1 - 3]. I have added point R in (a) and line BD in (b) since these are found in the other versions of the diagrams.

on the sphere.

Planis. 3 introduces a horizon circle drawn on the plane of the figure and shows that if this circle bisects the equator, it also bisects the ecliptic. This theorem is given entirely in the plane. As in the previous sections, $ABDG$ is the equator and $HBTD$ represents the ecliptic; see Figure 4.7 (c). Circle $HATG$ represents the horizon circle and it bisects the equator along line AG. It remains to show that the intersections of circle $HATG$ and circle $HBTD$ lie on points which are joined by a straight line through E, since according to *Planis.* 2 these points will be diametrically opposite on the sphere. We join line HE and produce it until it cuts the horizon at point T. It must be shown that point T also lies on circle $HBTD$. Since HT and AG intersect in circle $HATG$, $(HE \times ET) = (AE \times EG)$ [*Elem.* III 35], but $(AE \times EG) = (BE \times ED)$ [*Elem.* III 35], so that $(HE \times ET) = (BE \times ED)$ and points B, T, and D lie on a circle [conv. *Elem.* III 35].[100] Hence, the point T lies on the ecliptic, $BTDH$. Since, in *Planis.* 2, we showed that a meridian line cuts the ecliptic at diametrically opposite points, HET represents a diameter of the ecliptic and bisects the ecliptic, as on the sphere. The case of *sphaera recta* is not mentioned, but, since in this case the horizon will coincide with a

[100] I am not aware of an ancient proof of this converse, but it can be readily shown to follow from *Elem.* III 35 by an indirect proof.

meridian line, a demonstration could be provided along the lines of the proof and diagram in *Planis.* 2.

These three sections introduce the planar representation of the principal great circles and they provide a good sense of Ptolemy's procedure. *Planis.* 2 & 3 give plane proofs of theorems that are obvious on the sphere. Since the horizon is a great circle, we know from Theodosius' *Spher.* I 11 that it will bisect the ecliptic. As Berggren has argued, if Ptolemy had proceeded strictly by means of pointwise projection he would only have had to show that the principal circles map to the relevant circles and lines of the plane figure and these theorems would then follow from our knowledge of the sphere.[101] Instead, he works within the plane to show that the figures constructed in the plane, which represent the spherical figures, are related to each other in the same ways as the corresponding figures on the sphere.

The next four sections use the techniques shown above to calculate values for the diameters of the declination circles at the beginning of each sign and the horizon circle at the latitude of Rhodes, $\varphi = 36°$. *Planis.* 4 gives the metrical analysis and *Planis.* 5 - 7 provide the computations. These values will be used in Ptolemy's calculation of the rising times.

Planis. 8 - 13 provide the fundamental test of the plane model of the sphere. By demonstrating that trigonometric computation produces the same results in the plane as it does on the sphere, Ptolemy shows that the plane model is mathematically consistent with the sphere. *Planis.* 8 - 9 handle the case of *sphaera recta*; *Planis.* 10 - 13 treat *sphaera obliqua* for an example case of $\varphi = 36°$. Ptolemy's intent is not to calculate the rising times exhaustively as in the *Almagest* but merely to show agreement between the models. He calculates the rising time for each of the zodiacal signs at $\varphi = 0°$ and $\varphi = 36°$. In order to calculate the oblique rising times, Ptolemy uses the same general approach as that found in *Alm.* II 7.4 - 7.5, the second calculation of oblique rising times in the *Almagest*. In this method, the rising times at *sphaera recta* are used along with

[101] Berggren [1991a, 139 - 141].

the ascensional difference, n, to calculate the oblique rising times.[102]

Planis. 8 gives the metrical analysis and part of the calculation for the rising times at *sphaera recta*. *Planis.* 9 completes the calculation.[103] In these sections, Ptolemy shows himself to be something of a geometrical algebraist and states that the ratio of a square to a length determines a length.[104] He then uses this relationship as an equation to calculate the said length. In this single relation, Ptolemy treats the difference of two squares as a length, a ratio as a number and violates the principle of homogeneity. He does these things without so much as a comment. It is possible that Ptolemy is being innovative and breaking with tradition in these steps but it is more likely that he is working in a tradition of applied mathematics that used the tools of geometrical analysis to obtain numerical results using quasi-algebraic manipulations that treat proportions as equations.

Planis. 10 describes the model for $\varphi = 36°$. By examining the two symmetrical cases when the solstices are rising and setting, Ptolemy argues for the symmetry of rising times of arcs of the ecliptic about the equinoxes and introduces the arcs which represent the ascensional differences in the plane model.[105]

Ptolemy begins by setting out the equator $ABGD$ around center E and the ecliptic $ZBHD$ around center T; see Figure 4.8. Since E represents the north pole, we imagine the motion of the sphere as clockwise about E. We draw two positions of the horizon circle, $Z\widehat{K}HL$ and $Z\widehat{M}HN$, both passing through the solstices. Hence, when the horizon is in the position of $Z\widehat{K}HL$, points Z and K are rising; and when the horizon is in the position of $Z\widehat{M}HN$, points H and N are rising. By the converse of *Planis.* 2 & 3, we

[102] Toomer [1984, 94 -99].

[103] Neugebauer provides a discussion of these calculations, Neugebauer [1975, 861 - 863].

[104] Ptolemy gives no indication of how he derives this identity but any reconstruction would involve some algebraic manipulation since nothing in Euclid will allow for a result of this kind, Heiberg [1907a, 238], Anagnostakis [1984, 80 & 121] and Sinisgalli and Vastola [1992a, 128, n. 79].

[105] The ascensional difference is the arc of the equator cut off by two meridian lines drawn through a rising point of the ecliptic, A, and the rising point of the equator, B. The rising time of this arc is the difference between the time that it takes the ecliptic arc bounded by A to rise at *sphaera recta* and at *sphaera obliqua*. This arc is introduced for the purpose of simplifying the calculations in *Alm.* II 7. For a discussion of the concept, see Neugebauer [1975, 36 - 37].

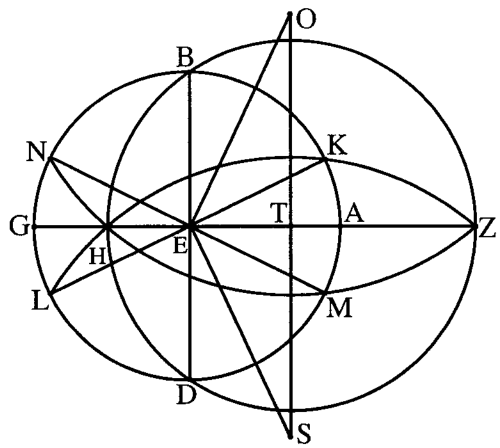

Figure 4.8: Diagram for *Planis.* 10, Anagnostakis [1984, fig. 6]. I have added lines BD and GH since these appear in the other texts.

know that the lines joining points K, L and M, N will pass through the pole.[106] Hence, $\overset{\frown}{KA} = \overset{\frown}{LG}$ and $\overset{\frown}{AM} = \overset{\frown}{GN}$. But $\overset{\frown}{AM} = \overset{\frown}{AK}$: since the centers of the horizon circles, O and S, lie on $OTS \perp ZH$ [*Elem.* III 1 cor.],[107] and, joining SE, OE, and STO, $SE \perp KL$ and $OE \perp MN$ [*Elem.* III 3] so that $\triangle OET$ is equal to $\triangle ETS$ [*Elem.* I 4]; but $\angle MEO$ and $\angle KES$ are right, hence $\angle MEO - \angle OET = \angle KES - \angle TES = \angle MEA = \angle KEA$. Hence the four arcs $\overset{\frown}{AK}$, $\overset{\frown}{AM}$, $\overset{\frown}{GN}$ and $\overset{\frown}{GL}$ are equal, as are their four supplements. In this way, since $\overset{\frown}{BH}$ of the ecliptic rises with $\overset{\frown}{BN}$ of the equator while $\overset{\frown}{ZB}$ of the ecliptic rises with $\overset{\frown}{BK} = \overset{\frown}{BN}$ of the equator, and again, since $\overset{\frown}{ZD}$ of the ecliptic rises with $\overset{\frown}{DK}$ of the equator while $\overset{\frown}{DH}$ of the ecliptic rises with $\overset{\frown}{DN} = \overset{\frown}{DK}$ of the equator, then arcs of the ecliptic which are equidistant from one of the equinoxes rise in equal times.[108]

In fact, Ptolemy does not demonstrate this symmetry in general but only when the rising arc is a quadrant. Nevertheless, he expects that this argument will hold for all cases, since he will use it in the calculations which follow. This is another indication that

[106] Ptolemy does not demonstrate this converse but a proof can be provided along the lines of that given in *Planis.* 3.

[107] The locus of points O and S is a circle concentric with E and determined by the metrical analysis in *Planis.* 4. A value for the radius of this circle when $\varphi = 36°$ can be derived from the results in *Planis.* 7. When the solstices are on the horizon, the center of the horizon circle is on the line $OTS \perp ZG$, whereas when the equinoxes are rising, it is on ZG produced.

[108] Heiberg [1907a, 241 - 244] and Anagnostakis [1984, 83 - 85].

Ptolemy did not intend these methods to be used as a rigorous means of calculating the rising times, but rather calculated the rising times with these methods to show agreement with the metrical results on the sphere.

The remainder of *Planis.* 10 deals with the ascensional difference and daylight times. Ptolemy points out that $\widehat{BZ}$ rises in less time at $\varphi = 36°$ than it does at *sphaera recta* by $\widehat{KA}$. Likewise, $\widehat{DH}$ rises in less time at $\varphi = 36°$ than it does at *sphaera recta* by $\widehat{GN}$. Since these two differences are equal, Ptolemy argues that in general the differences between rising times at *sphaera obliqua* and *sphaera recta* are symmetrical about the equinoxes. Again, he argues for the general case based on the situation at the quadratures. The arcs $\widehat{KA}$ and $\widehat{GN}$ are shown to be the ascensional differences and they are used to calculate oblique rising times given the rising times at *sphaera recta.* Ptolemy closes *Planis.* 10 by noting that the longest daylight exceeds, and the shortest daylight falls short of, the mean daylight by $\widehat{KA} + \widehat{GN}$.

Planis. 11 uses the results of *Planis.* 10 to calculate the rising times of the quadrants of the ecliptic and the difference between the longest or shortest daylight and the mean for $\varphi = 36°$. *Planis.* 12 & 13 use the ascensional difference to calculate the rising times of the zodiacal signs. In every case the results are found to agree with those given in the *Almagest.*[109]

The first part of the *Planisphaerium* gives the mathematical treatment of the plane model for spherical astronomy. In basic outline, Ptolemy's approach is the same as what we find in the treatment of the spherical models of either the *Almagest* or the *Analemma.* The assumptions and first principles are laid out and from these a model is deduced which is then shown to agree with the phenomena. In the case of the *Planisphaerium,* the phenomena are the numerical results of the *Almagest.*

[109] Neugebauer [1975, 864 - 865] provides a discussion of these calculations.

4.2.2 Mathematics for practical purposes

With the validity of the model established, the rest of the treatise takes up matters related to instrumental construction. It has been argued that the instrument Ptolemy has in mind is either the plane astrolabe or a star map, perhaps as the plate of an anaphoric clock.[110] There is nothing in Ptolemy's account that commits his description to the plane astrolabe and he is certainly concerned to show how the stars can be drawn on a plate.[111] Ptolemy himself, however, is not describing the production of any particular instrument but writes his treatise with the awareness that some of his readers may wish to use his methods in practice. In this regard, the *Planisphaerium* is similar to the *Analemma*; both texts provide mathematical theory underlying the production of instruments.

The second part of the treatise deals with constructing the plane model within a southernmost circle of reasonable size and drawing circles parallel to the ecliptic, or β-circles. Although Ptolemy tells us that we want to be able to draw β-circles in order to map the stars by their ecliptic coordinates, no construction is given for the great circles orthogonal to the ecliptic which would be needed for mapping stellar longitudes.[112]

Since δ-circles south of the equator are represented by increasingly large circles, practical considerations require that we take some southernmost δ-circle as the boundary of our plate and draw the rest of the figure within it. *Planis.* 14 shows how to draw a δ-circle within the framework of a given southernmost δ-circle. We set out circle $ABGD$ as the southernmost parallel about center E with AG and BD as perpendicular meridians; see Figure 4.9 (a). We cut off $\overset{\frown}{GZ}$ equal to the declination of the circle we have chosen as the southernmost parallel. In order to construct the equator, we produce $GH \perp GE$, join DZH, drop $HT \perp ED$, and join DKG. Then, if we draw circle SLM with center E and radius KT it will be the equator. Ptolemy proves this by geometric analysis. He assumes circle SLM is the equator and he draws the southernmost circle by means of its declina-

[110] The case for the plane astrolabe is made by Neugebauer [1949] and Drachmann [1954]; also see Neugebauer [1975, 868 - 879]. Berggren [1991a] argues for a star map.

[111] Berggren [1991a, 142 - 143].

[112] Heiberg [1907a, 251 & 258 -259] and Anagnostakis [1984, 91 & 97]. Neugebauer gives a construction of these orthogonal circles, Neugebauer [1975, 866 - 867].

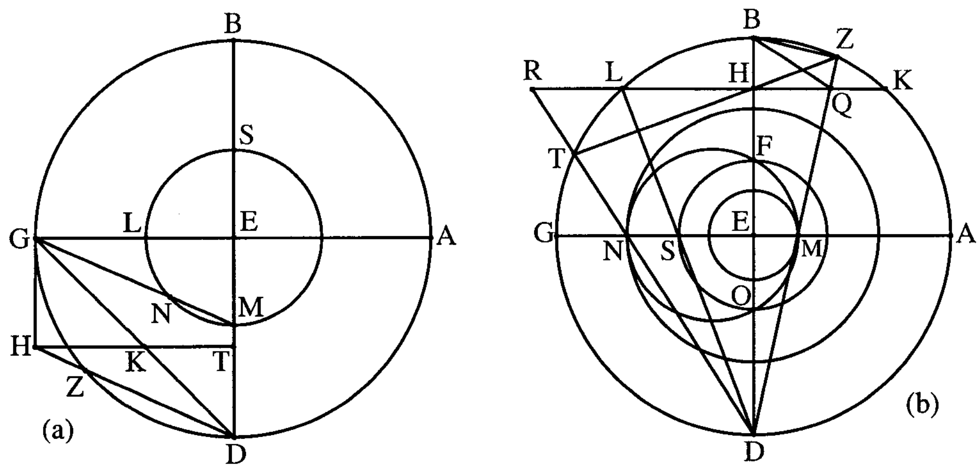

Figure 4.9: Figures for *Planis.* 14 & 16, Anagnostakis [1984, figs. 8 & 10].

tion $\widehat{LN}$. It remains to show that $\widehat{MN}$ is similar to $\widehat{DZ}$. Since, $DE : EG = DT : TK$ [from *Elem.* VI 2], and $DE = EG$ [*Elem.* III def. 1], therefore, $DT = TK$. But $TK = EM$ [by construction], hence $DT = EM$. But $TH = EG$ and they are parallel [by construction], hence $DZ \parallel MN$ [*Elem.* I 4]. Hence, $\angle EMN = \angle EDZ$ [*Elem.* I 29], so that $\widehat{SLN}$ is similar to $\widehat{BGZ}$ and the complementary arcs $\widehat{MN}$ and $\widehat{DZ}$ are also similar. Any other δ-circle is drawn following the example of the equator. We set out the declination of the parallel from G in the appropriate direction. We join a line from D through the endpoint of its declination meeting GH produced. We drop a perpendicular from this meeting point to BD. We join GD, or GB in the case of parallels north of the equator. The intersection of this line with the perpendicular gives the radius of the parallel on the plate.[113]

Planis. 15 shows how to find the northern pole of the ecliptic. This is a straightforward projection. The pole of the ecliptic is projected to the south pole and is represented on the plane of the model as the intersection of the projection line and the plane.

Planis. 16 shows how to produce a β-circle. This construction again shows Ptolemy considering the situation on the sphere and projecting the diameter of the β-circle in the

[113] Heiberg [1907a, 249 - 251] and Anagnostakis [1984, 89 - 91].

sphere to find its representation on the plane. Circle $ABGD$ is set out as the solstitial colure[114] around center E with D as the south pole and perpendicular diameters AG and BD; see Figure 4.9 (b). Line ZT is the diameter of the β-circle on the sphere, the intersection of the parallel circle with the solstitial colure. Ptolemy then shows that if ZD and TD are joined then the intersection of these lines with AG will yield the diameter of the β-circle on the plane. He does this by showing that the circle on diameter MN is tangent to two equal δ-circles at points Z and T. He does not, however, consider this sufficient proof. Ptolemy goes on to show that circle on MN bisects the circle which represents the δ-circles about diameter LK, just as on the sphere. Thus, it is necessary to show that the circle on MN passes through points F and O. We join lines BZ, BQ and produce lines KL and DT to R. Since $\angle BZQ = \angle BHQ$ are right [*Elem.* III 31], points B, H, Q and Z lie on a circle [conv. *Elem.* III 31]. Hence, $\angle BQR = \angle BZT$ [*Elem.* III 21], while $\angle BZT = \angle BDR$ [*Elem.* III 21]; therefore, B, R, D and Q lie on a circle [conv. *Elem.* III 31]. So then, $(QH \times HR) = (BH \times HD) = HL^2$ [*Elem.* III 35]. Now since, $MN \parallel QR$, $(ME \times EN) = ES^2 = EF^2 = (EF \times EO)$ [RQ and MN are proportionally cut by DQ, DH, DL and DR].[115] Hence, points M, F, N and O lie on a circle [conv. *Elem.* III 35].[116] This concludes Ptolemy's proof. He does not consider the case where the β-circle is in the direction of the south pole.[117] While projective techniques are used to find the plane representation of the parallel circle, the proof again betrays how far Ptolemy is from a deductively projective approach. The object of Ptolemy's approach is to show, through elementary means, that the circles in the plane have the same geometric relationships as the corresponding circles on the sphere and it relies on the assumption

[114] Ptolemy's expression is "the meridian through both the poles." Both Lorch and Neugebauer refer to this circle simply as "the meridian," Neugebauer [1975, 866 ff.] and Lorch [1995, 271 ff.].

[115] I follow Hermann for this step since the Arabic version that Anagnostakis translates does not convey the logic of the proof, Heiberg [1907a, 253 - 254] and Anagnostakis [1984, 93].

[116] Heiberg [1907a, 252 - 254] and Anagnostakis [1984, 92 - 93].

[117] Maslama proves the case where the β-circle is entirely south of the equator, Kunitzsch and Lorch [1994, 26 - 28] and Lorch [1995, 280].

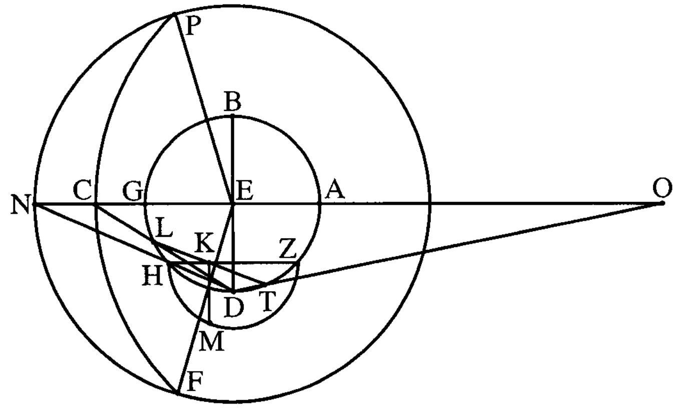

Figure 4.10: Figure for *Planis.* 18.

that δ-circles are represented by concentric circles.[118]

Planis. 17 proves that the point which represents the pole of the ecliptic is not the center of the circles which represents the ecliptic parallels.

Planis. 18 shows how to construct β-circles which intersect the southernmost equatorial parallel.[119] This section contains an interesting analemma construction. The two published versions of this proof are different and neither of them is complete. The treatment below loosely follows Hermann since the figure and exposition in the Arabic are needlessly involved.

We set out the solstitial colure as $ABGD$ around center E with point D as the south pole and perpendicular diameters AG and BD, see figure 4.10. We set out ZH as the diameter of the southernmost δ-circle and line TKL as the diameter of the β-circle which intersects the southernmost parallel. We draw the semicircle ZMH on line ZH. We erect $KM \perp HZ$ at K, the intersection of the diameters of the β-circle and the southernmost δ-circle. As above, we produce DHN and DLC to meet AG produced. Hence, the circle

[118] Lorch takes this proof, or a proof very like it, to be a demonstration of the circle preserving property of stereographic projection, Lorch [1995, 280 - 281]. The circle preserving property, however, is implicit in Ptolemy proof because his model is equivalent to stereographic projection. Ptolemy, however, does not set out to show that $MFNO$ is a circle. He does, in fact, show that $MFNO$ is a circle but only by assuming that FSO is a circle. He is only able to assume this because he assumes without proof, in *Planis.* 1, that equatorial parallels are represented by concentric circles.

[119] The text refers to this circle as the "always invisible" circle. This is may be a slip of the translator. Nowhere else in the treatise is this circle so named and, unless the plate is going to be used as an analog computer, there is no theoretical reason why the southernmost parallel must be the i-circle.

PNF, drawn with radius EN, will be the southernmost parallel. Therefore, the circle which represents the ecliptic parallel on diameter TL will go through the point C and cut circle PNF at points P and F such that $\overset{\frown}{NF} = \overset{\frown}{PN}$ are similar to $\overset{\frown}{HM}$. This will be $\overset{\frown}{PCF}$ on the plate.[120]

The reason for these final assertions is not explicitly stated in the text but is supplied from the analemma construction. If we consider the circle of the southernmost parallel as perpendicular to the plane of the figure, then it will intersect theβ-circle on TL at point M. It is then assumed that an arc on circle HMZ will be represented by a similar arc on the circle which represents it in the plane. This is true so long as the diameter of the circle in question is parallel to the plane of the figure. Here, we again see the analemma construction used as an intuitive means of modeling the sphere and reducing arcs on the sphere to arcs on the plane. No proof is offered for this construction.[121] In both versions of this section, the justification that is offered is mere hand-waving. In the Arabic version, we find an elaborate construction of $\angle NEF$ that adds nothing to the exposition.[122] Hermann's text makes the claim that C, P and O lie on the same circle, but no proof is given.[123]

Planis. 19 demonstrates that the ecliptic parallel which goes through the south pole is represented by a straight line. The final section summarizes the text and makes a few remarks pertinent to mapping stars on a plate by either the ecliptic coordinates or the older system of equatorial coordinates used by Hipparchus.[124]

The second part of the *Planisphaerium* contains the mathematics that Ptolemy thinks we will need for putting the methods of the plane model to instrumental use. A few details of practical construction are raised, such as the density of the grid lines in the two coordinate systems; however, we do not find complete instructions like those given for

120 Heiberg [1907a, 255 - 257] and Anagnostakis [1984, 94 - 95].
121 Maslama gives a proof that points C, P, F and O lie on a circle, Kunitzsch and Lorch [1994, 24 - 26] and Lorch [1995, 278 - 279]. His proof is performed in the plane and modeled on the proof given by Ptolemy in *Planis.* 16.
122 Anagnostakis [1984, 95].
123 Heiberg [1907a, 257].
124 For the case that Hipparchus used equatorial coordinates for the stars see Duke [2002].

the nomographic plate in *Anal.* 11.[125] The *Planisphaerium* is similar to the *Planetary Hypotheses* in that it gives formal treatment to a model which is used by instrument makers without actually describing the process of making any apparatus.[126] It is written in light of the practical tradition and for its benefit but remains a theoretical work concerned with the mathematics of the plane model.

4.2.3 Remarks on the *Planisphaerium*

The plane model of the *Planisphaerium* is generally held to date back at least as far as Hipparchus. We know that it was applied to instrumental use in the anaphoric clock but it is also held to have been used as a device for solving problems in spherical astronomy such as calculating the rising times.[127] Certainly, it could have been so used. Neugebauer has shown how the analemma can be used to find the declinations which are needed for these calculations.[128] For any exact calculation in spherical astronomy the model would have to be used in conjunction with the analemma methods.

The model is not presented by Ptolemy, however, as a calculation device nor is it associated with the analemma methods. This because in general it is not very useful for solving problems in spherical astronomy. There are a number of reasons for this. Any point which is give on the sphere by its ecliptic coordinates cannot be specified on the plane model without transforming it into equatorial coordinates. This transformation itself, however, would require the analemma. This means that as a calculation device the plane model would have to be used in conjunction with the analemma. Any relations between the equatorial system and the horizon system that can be calculated on the plane model can much more readily be calculated on the analemma. The only use for the plane model would be calculating the position of the ecliptic relative to the local horizon.

[125] Heiberg [1907a, 210 - 216] and Edwards [1984, 118 - 123]. The construction of the plate is discussed by Luckey [1927, 33 - 35] and Neugebauer [1975, 852 - 853].
[126] Heiberg [1907a, 110 - 145] and Goldstein [1967]. The best discussion of the formal nature of the models in the *Planetary Hypotheses* is found in Murschel [1995].
[127] Neugebauer [1975, 868 - 879].
[128] Neugebauer [1975, 303 - 304].

While the plane model may not have found much application in exact calculations, it could readily have been used as an analog computer using nomographic techniques like those found in the *Analemma*. If a suitably precise plate is drawn containing the grid lines of the equatorial and ecliptic systems and bounded by the i-circle, then only one point of the horizon is needed to draw it on the plate, since the v-circle and i-circle are known.[129] It is often assumed that Hipparchus derived the many values in the long list of the rising and setting constellations in his *Commentary* from a solid globe marked with the positions of the stars.[130] This assumption is put forward on the basis of the claim that Hipparchus either could not, or would not, calculate these values. While there is no reason that the metrical methods involved in these calculations would be outside the scope of Hipparchus' ability, it may be that the labor involved prompted him instead to use nomographic techniques. If, in fact, Hipparchus did not calculate these values, he might have found them by means of the plane model on a nomographic plate along with a table for converting between declination and longitude.

Whether or not the plane model was ever used for exact calculations, before Ptolemy's time it had already been superseded by methods that were both more precise and more general. Nevertheless, its value as a consistent model of the celestial sphere insured its preservation and continued use as an analog computer and star map.

4.3 The *Geography*

Ptolemy's *Geography*, or *Guide for Drawing the Earth*, Γεωγραφικὴ ὑφήγησις, is not predominantly a mathematical text. It contains no theorems, no metrical analyses and few calculations. Nevertheless, its project as a whole is the development of geography as an exact science. Moreover, in the course of this development it presents three different ways of drawing a portion of the earth's sphere in the plane. The presentation of these

[129] The problem of drawing a circle through a given point and tangent to two given circles was solved in Apollonius' lost *Tangencies*. See Jones [1986, 534 - 539] for a discussion of the content of this work.

[130] Nadal and Brunet [1984] have gone so far as to derive the parameters of Hipparchus' "star globe" through a statistical study of the data in the *Commentary*.

maps show Ptolemy taking an intricate path between quantitative and qualitative considerations. All three of Ptolemy's maps have been impugned with various mathematical inconsistencies, but mathematical consistency, in the sense of projective consistency, was not of primary interest to Ptolemy.[131] His intent was, rather, to preserve the qualitative effect of certain ways of viewing the sphere while still visually imparting quantitative information about key distances on the earth's surface.

The portion of the earth's sphere that Ptolemy wants to depict in the plane is what was then considered to be the known world, the *oikoumenē*. Its longitude measured 180° from the Isles of the Blest off the west coast of Africa to the far, and little known, reaches of East Asia. Its latitude measured about 80° from the northernmost parallel of Thulē, $\varphi = 63°$, to a parallel as far south of the equator as Meroē was north of it, $\varphi = 16^{5}/_{12}°$S. We know Ptolemy composed the *Geography* after the *Almagest* because the boundaries of his known world had expanded since writing *Alm.* II 6.[132] This presumably happened as a result of studying Marinos, his primary predecessor in geography.

For the purpose of accurately mapping his known world, Ptolemy sets out two different drawings in *Geo.* I 24. Ptolemy's first map is presented as an improvement over the map of Marinos and his second map is presented as an improvement over his first. Marinos' map was what we call a cylindrical projection; parallels being perpendicular to meridians. It was adjusted to the latitude of Rhodes, $\varphi = 36°$, so that east-west distances are to north-south distances in the ratio that they have at that latitude. The view in Marinos' map is considered as the composition of each position of the eye varying from east to west relative to the sphere such that each meridian appears in turn as a straight line.[133] In the first map, Ptolemy maintains the importance of the latitude of Rhodes and the varying position of the eye which produces straight meridians but introduces devices meant to impart visual information about three other parallels.[134] In the second map, he abandons the concept of a composition of views and fixes the position of the eye relative to the

[131] Berggren [1991a, 134 - 138].
[132] Berggren and Jones [2000, 17 - 20].
[133] Berggren and Jones [2000, 82].
[134] These are the latitudes of Thulē, the equator and anti-Meroē.

sphere. Moreover, he abandons the latitude of Rhodes as his central latitude, replacing it with the latitude of Soenē under the Tropic of Cancer, $\varphi = 25^5/_6{}^\circ$. Despite having fixed the viewpoint, Ptolemy makes no attempt to represent the image of the sphere through the techniques of linear perspective; probably, in part, because this would fail to convey the requisite visual information. The fixing of the eye is captured instead by creating the impression of curvature in the east-west direction; the meridians are arranged with increasing curvature around a central, rectilinear meridian.

Both of these maps are meant to be accurate depictions of the *oikoumenē* itself, displaying as much information as possible without cluttering the presentation. They are described as drawn on a plate with given dimensions and the exposition contains a number of detailed instructions or considerations of relevance to the draftsman.

In *Geo.* VII 6, another representation of Ptolemy's known world is given. Here the emphasis has shifted so that the drawing now contains the *oikoumenē* situated on a globe which is ringed by the five principal circles of the celestial sphere. The position of the eye has been moved and the presentation is more mathematical in the sense that Ptolemy pays more attention to deriving the construction. The final drawing is a combination of projective techniques and devices chosen to insure that the map visually portrays suitable relationships between distances on the sphere. The globe and its rings are projected onto the plane of the drawing so that they will appear in linear perspective while the *oikoumenē* itself is produced by concerns similar to those exhibited in the first two maps. The general description of the final drawing in *Geo.* VII 7 indicates that the presentation of the *oikoumenē* itself is given with much less detail than in the two maps presented in *Geo.* I 24.

Both the mathematics and the general approach behind all three maps have been adequately discussed in the literature.[135] I have no intention of replicating this material; however, by giving a brief sketch of each map we may compare the methods of the *Geography* to those of the *Planisphaerium* and the *Analemma.* By observing the difference

[135] Neugebauer [1975, 879 - 890], Andersen [1987], Berggren [1991a, 134 - 138] and Berggren and Jones [2000, 31 - 40].

of approach and presentation we may get a better sense of Ptolemy's intent in each of these works.

There is no complete critical edition of Ptolemy's *Geography*. The only complete text, by Nobbe, lacks an apparatus. The English translation by Berggren and Jones is the best translation of those sections of the text that concern us.[136] This translation is based on a revision of Nobbe's text collated with several manuscripts.[137]

4.3.1 Two maps of the known world

The key features that Ptolemy wants to preserve in his first map of the *oikoumenē* are a direct result of his criticism of Marinos' map.[138] He sets these features out in *Geo.* I 24. The meridians will be straight lines but the parallels will be concentric circles. The extreme parallels of Thulē and the equator will have the same ratio to one another as they have on the sphere. The overall longitudinal and latitudinal dimensions of the map will have the same ratio that the semicircle of the latitude through Rhodes has to the quadrant of the meridian, $5:4$,[139] so that the "more familiar longitudinal dimension of the *oikoumenē* is in proper proportion to the latitudinal dimension."[140]

Following a discussion of the meridians and parallels that will be included in the map, Ptolemy sets out the instructions for drawing the map, starting with the plate on which the map will be drawn. The rectangular plate $ABDG$ is set out such that $AB = 2AG$; see Figure 4.11. AB is bisected and EZ is drawn perpendicular to AB. EZ is extended above AB to H such that $EH = 34^p$ of those units of which $HZ = 131^5/_{12}{}^p$, or $EZ = 97^5/_{12}{}^p$. Then, with center H and radius $HK = 72^p$, circle ΘKL is drawn as the latitude of Rhodes. Ptolemy provides no derivation of these numbers; nevertheless, we can get some sense out of them once more of the instructions are given.[141] All we

136 Berggren and Jones [2000].
137 A list of variant readings is given in Berggren and Jones [2000, 163 - 167].
138 Berggren and Jones [2000, 82 - 83].
139 Ptolemy does not tell us how he arrived at this value, but it is readily found using the chord table.
140 Berggren and Jones [2000, 83].
141 Neugebauer [1975, 881 - 882] provides a mathematical reconstruction of this derivation.

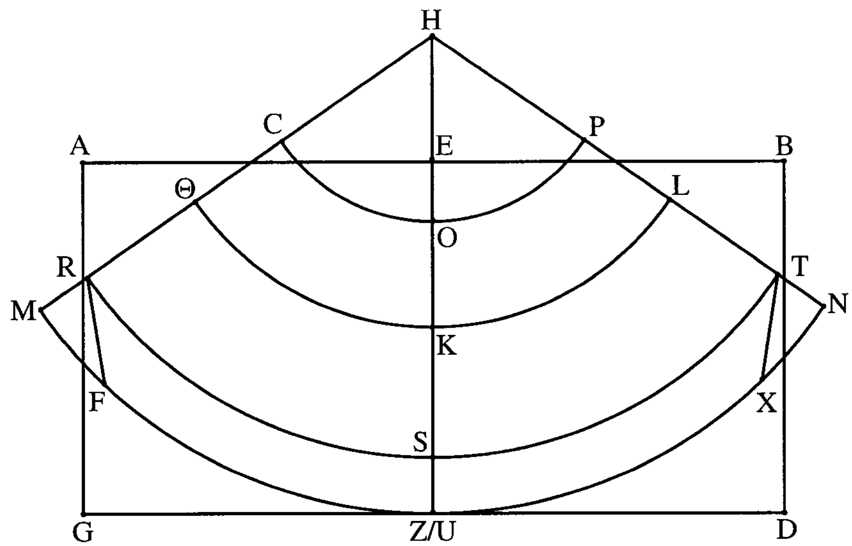

Figure 4.11: Diagram for Ptolemy's first map of the *oikoumenē*, Berggren and Jones [2000, 86].

need note at this point is that only the points S, K, O and the arc $\widehat{\Theta KL}$ are determined by the initial assumptions. Point H must have been determined from these in such a way that $\widehat{COP} : \widehat{RST}$ will be close to the ratio that the parallel through Thulē has to the equator. The fact that Ptolemy gives no indication how he found point H probably means that he found it through trial and error.[142]

In order to determine the length of $\widehat{\Theta KL}$, an interval of 4^p is taken and marked off 18 times along the circle ΘKL on each side of K. Then, since $\widehat{\Psi KL}$ is a 12^h period in which there are 36 intervals, these will mark off meridians at $1/3^h$ intervals. Moreover, since there are 180° in $\widehat{\Theta KL}$, the 36 meridians so taken will mark off 5° intervals and the ratio of the longitudinal distance of the *oikoumenē* through Rhodes to the quadrant of the great circle, 5 : 4, will be maintained.

The remaining principal parallels are determined by taking $HO = 52^p$, $HS = 115^p$

[142] In the presentation of the second map, the center of the parallels is found through geometric analysis, see page 235.

and $HU = 131^5/_{12}{}^{\mathrm{p}}$.[143] These are chosen such that $SO = 63^{\mathrm{p}}$, $SK = 36^{\mathrm{p}}$ and $SZ = 16^5/_{12}{}^{\mathrm{p}}$ since the φ of Thulē, Rhodes and anti-Meroē are respectively 63°, 36° and $16^5/_{12}$°S. Hence, the point H was not chosen to fall on the north pole but in such a way that the arcs $\overset{\frown}{COP}$ and $\overset{\frown}{RST}$ will accurately represent the size of these latitudes on the sphere. Ptolemy then does a check of the overall dimensions of the map by pointing out that $\overset{\frown}{\Psi KL} = 144^{\mathrm{p}}$, since it was set out with 36 units of 4^{p}, so that $OU : \overset{\frown}{\Psi KL} = 79^5/_{12} : 144$. This ratio is found to be in approximate agreement with the figures previously derived for the latitude of the *oikoumenē* and the longitude of the parallel through Rhodes, 40,000 and 72,000 stades.[144]

In order to portray the equator as the greatest parallel of latitude, the meridians south of equator are inflected back toward the central meridian. This is done by taking $\overset{\frown}{FUX}$ as proportional to the longitudinal length of the parallel through anti-Meroē. Ptolemy does not actually give us a value for this this length, but it can be determined using the chord table. This arc is then divided into 36 parts and these divisions are joined with the meridians at the equator. This discontinuity in the map emphasizes the role of the equator as the greatest parallel of latitude and maintains the principal that the size of the key parallels of latitude should be proportional to what they are on the sphere.

Following this material, Ptolemy includes some instructions for drawing the details of the map that are clearly meant for the draftsman. These include suggestions for making a mechanical aid for marking off points by longitude and latitude. The instructions are similar in kind to those Ptolemy gives in the final section of the *Planisphaerium*.

For Ptolemy's second map, he applies two different criterion for construction. The first is the condition that the ratio of longitudinal distances to latitudinal distances should be maintained not only at the latitude of Rhodes but roughly for all other latitudes as well.

143 Despite the fact that HZ was initially set out as $131^5/_{12}{}^{\mathrm{p}}$, from this point on Ptolemy will refer to the lowest point of the *oikoumenē* as U. Berggren and Jones have suggested that this indicates a separation between the *oikoumenē* as bounded by U and the rectangular plate as bounded by Z. Ptolemy, however, originally defines H by setting $HZ = 131^5/_{12}{}^{\mathrm{p}}$, Berggren and Jones [2000, 87, n. 69].

144 Ptolemy's derivation of these figures, based primarily on a critical reading of Marinos, is given in *Geo.* I 8 - 14, Berggren and Jones [2000, 67 - 77]. More specifically they are stated in Berggren and Jones [2000, 70 & 77].

In practice, this means he will determine the correct ratios for the latitudes of Thulē, Soēnē and anti-Meroē and join these points with a curved meridian so that all other latitudes only approach their correct values. The second criterion is that the map give the visual impression of looking at a globe from a fixed viewpoint. Hence, the meridians will appear more curved the farther they are from the central meridian.

Ptolemy begins the discussion of his second map by introducing a static eye and the central line of sight, an optical concept which is developed in his own *Optics* but not otherwise attested in the ancient treatises on optics.[145] The eye is located on the normal above the midpoint of the parallel under the tropic of Cancer, known as the parallel through Soēnē, $\varphi = 23^5/_6$. The central line of sight joins the eye with a point on the sphere at the intersection of this parallel and the central meridian, point E in Figure 4.12 (a). As in the previous map the parallels will be concentric circles. Ptolemy's first step is to find the position of the center of the parallels.

Circle $ABGD$ is the sphere of the world; see Figure 4.12 (a). Let lines $AEZG$ and BED be imagined (νοείσθωσαν) to be arcs of great circles; $AEZG$ is the central meridian and BED is the great circle between E and the east and west points.[146] Point Z is chosen such that $BE : EZ = 90^p : 23^5/_6{}^p$. A short geometric analysis is used to find the point H. This analysis is mixed with calculation to find the length HZ. Point H is assumed to be found on GA extended such that H is the center of the circle through B, Z and D. Line ZB is joined and bisected at Θ, hence $BZ = 93^1/_{10}{}^p$ [*Elem.* I 47]. After some irrelevant angle calculations using the chord table, Ptolemy points out that $HZ : Z\Theta = 181^5/_6{}^p : 46^{11}/_{20}{}^p$ [since $\triangle BEZ \sim \triangle H\Theta Z$, $HZ : Z\Theta = BE : EZ$]. In fact, Ptolemy will round HZ to 180^p.

After the basic dimensions of the map have been established, Ptolemy gives drawing instructions similar to those he gives for the first map. Rectangle $ABDG$ is set out with $AB = 2AG$; see Figure 4.12 (b), and this time we are explicitly given the unit; a line equal to EZ is divided into 90^p corresponding to the degrees of the quadrant. Again

[145] Berggren and Jones [2000, 88, n. 74].
[146] Nobbe [1966, 53] and Berggren and Jones [2000, 89]

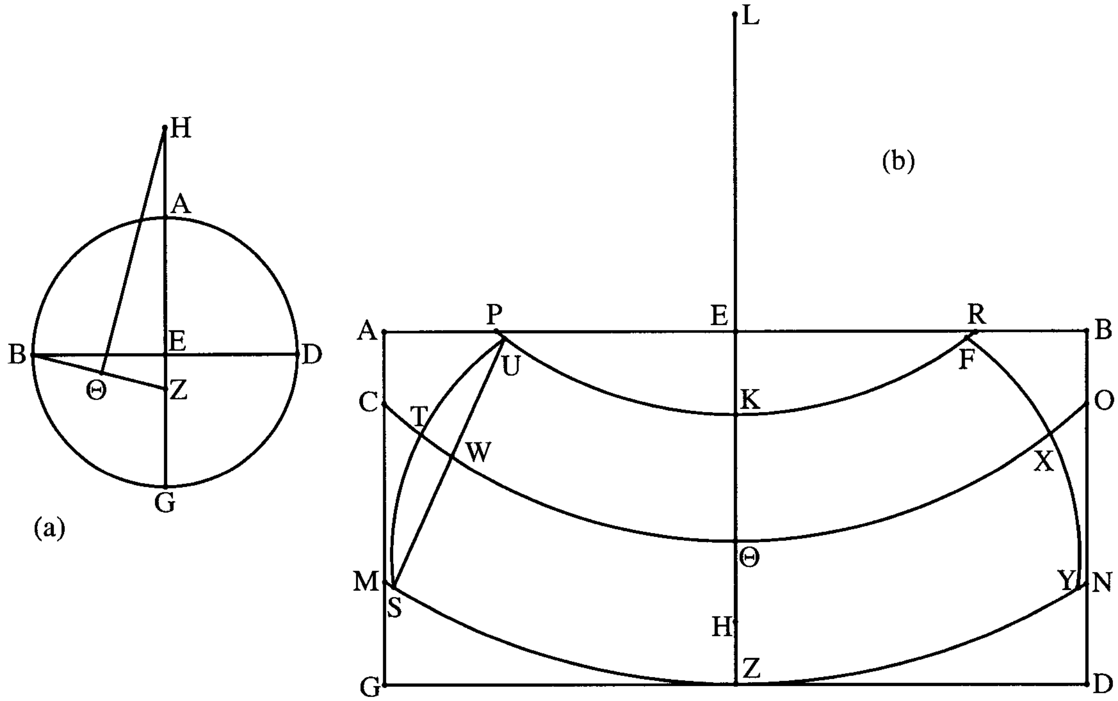

Figure 4.12: Diagram for Ptolemy's second map of the *oikoumenē*, Berggren and Jones [2000, 91].

points H, Θ and K are determined by the geographic circumstances: $HZ = 16^5/_{12}{}^p$, $H\Psi = 23^5/_6{}^p$ and $HK = 63^p$; since these correspond respectively to the φ of anti-Meroē, Soēnē and Thulē. Taking $HL = 180^p$, according to the derivation above, L will be the center of the parallels which may then be drawn through H, Θ and K and so forth.

For the determination of the meridian lines, the 36 points marking 5° or $^1/_3{}^h$ intervals must be laid out along the arcs $\overset{\frown}{PR}$, $\overset{\frown}{CO}$ and $\overset{\frown}{MN}$. Since, on the globe, the ratios of the equator to the parallels of Thulē, Soēnē and Meroē are respectively $5 : 2^1/_4$, $5 : 4^7/_{12}$ and $5 : 4^5/_6$,[147] 18 intervals of $2^1/_4{}^p$, $4^7/_{12}{}^p$ and $4^5/_6{}^p$ are respectively laid out on the arcs $\overset{\frown}{PR}$, $\overset{\frown}{CO}$ and $\overset{\frown}{MN}$ on either side of the central meridian KZ. The end points of these 18 intervals are joined by a circular arc and form the boundaries of the *oikoumenē*, $\overset{\frown}{FXY}$ and $\overset{\frown}{PTM}$. The rest of the meridians are drawn in a similar fashion.

Ptolemy then points out the advantages of his second map over his first. He notes

[147] These values are again presumably found using analemma techniques.

that it maintains the visual impression which a globe makes while also preserving the proportionality of the distances of the parallels from the equator and the ratios of the total dimensions of the *oikoumenē*. He concludes by remarking that, for all its obvious theoretical advantages, the second map will be harder to draw since one will not be able to use mechanical aids. Nevertheless, it should be preferred.

The presentation of Ptolemy's maps may be better characterized as a set of instructions than a mathematical construction in the traditional sense. With the exception of the determination of the center of the parallels in the second map, the mathematics involved in these figures takes place behind the scenes. Nevertheless, using the chord table and the techniques of metrical analysis one could give plausible reconstructions of the omitted steps.[148]

The essential features which Ptolemy tries to preserve in these maps are both visual and quantitative. Visually, he seeks to preserve the basic impression formed by different ways of viewing the globe. Quantitatively, he hopes to convey to the viewer an impression of the actual distances involved on the surface of the globe. It is this attention to two distinct and sometimes contradictory aims that has caused modern readers to find fault with Ptolemy's methods.

4.3.2 The drawing of the ringed globe

In *Geo.* VII 6, Ptolemy gives instructions for drawing a picture of the *oikoumenē* on a ringed globe that is in "as close agreement as possible with the rules of optics (ταῖς ὀπτικαῖς διατυπώσεσι)."[149] Ptolemy does not explicitly say as much, but we may assume that the reason that he cannot strictly adhere to the "rules of optics" is that he intends to preserve the longitudinal and latitudinal distances approximately as they are on the

[148] Neugebauer's analysis gives the essential mathematical features of each map, Neugebauer [1975, 880 - 886].
[149] Nobbe [1966, 182] and Berggren and Jones [2000, 113].

sphere and a drawing in linear perspective would distort these unacceptably.[150]

A globe of the earth is viewed though two of the rings of a ringed sphere. The ringed sphere itself is composed of seven interlocking rings consisting of the equinoctial colure, the equator, the ecliptic, the summer and winter tropics, and the arctic and antarctic circles. The drawing depicts the globe in such a way that the whole of the *oikoumenē* can be seen through the rings of the equator and the summer tropic. The ecliptic is arranged such that the equinoxes are on the sides of the sphere, the winter solstice faces the viewer and the summer solstice is hidden from the viewer behind the sphere.[151] Apparently, such drawings were traditional, and although many people had tried to provide a demonstration (δεῖξις) of the method best pursued for these figures, they had, according to Ptolemy, all proceeded irrationally.[152]

Ptolemy begins by situating the eye on the line which forms the intersection of planes through the parallel of Soēnē and the central meridian of the *oikoumenē* which is also the solstitial colure. He points out that when the eye is so positioned, the central meridian and the parallel through Soēnē will appear as perpendicular straight lines and that the other meridians and parallels will appear curved in toward these, and the farther they are from these central lines the more curved they will appear.

The determination of the ratio of the diameter of the globe to the diameter of the ringed sphere involves a few implicit assumptions. Let circle $ABGD$ be the ring of the equinoctial colure and circle $PFRX$ the globe of the earth; see Figure 4.13. (1) The latitude of Soēnē is taken, initially, at the round value of 24°. (2) Degrees along the central meridian will be represented by lengths along PR through orthogonal projection, as on an analemma; so that, where E lies on the equator and S lies on the parallel through Soēnē, $ES = 24^p$. (3) As a first approximation, the eye is taken to be so situated that S lies on the midpoint of EO where O is the orthogonal projection of the ring of the summer

150 It becomes clear when Ptolemy introduces the parallels and meridians that he intends to preserve in his drawing the approximate distances that these circles have on the sphere, Berggren and Jones [2000, 115 - 116]. Nevertheless, he never explicitly introduces this proportionality as a criterion of accuracy.

151 A modern rendition of the drawing Ptolemy describes was made by Jones, Berggren [1991a, 137].

152 Nobbe [1966, 181] and Berggren and Jones [2000, 112].

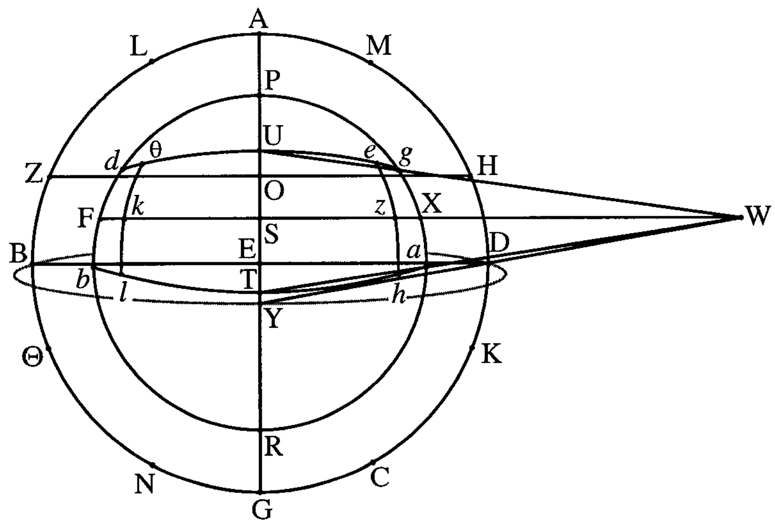

Figure 4.13: Diagram for Ptolemy's drawing of the ringed globe, Berggren and Jones [2000, 115]. The ellipse BYD, marked in gray, does not appear in the manuscripts.

tropic. Assumption (2) implies that whereas H is a point on the summer tropic of the ringed sphere, S is a point on the summer tropic of the globe. To determine the relative sizes of the globe and the sphere we simply have to compare the ratio $ES : EP$ on the globe to ratio $ES : EA$ on the sphere. Assumptions (1) & (2) imply that on the globe $ES : EP = 24 : 90 = 4 : 15$. On the ringed sphere, since $\overset{\frown}{HD} = \angle EHO = 24°$, according to the chord table, $EO = 48;48,30^{\text{p}}$, where $HE = 120^{\text{p}}$. Thus, where $HE = 60^{\text{p}}$, $EO = 24;24,15^{\text{p}}$ so that $EO : EA = 24;24,15 : 60 \approx 4 : 10$. Hence, by assumption (3), $ES : EA = 4 : 20$. Thus, since $ES : EP = 4 : 15$ and $ES : EA = 4 : 20$, $EP : EA = 3 : 4$ [*Elem.* V 22]. Ptolemy does not spell out these steps, but all of the numbers that appear in his text are reproduced here. Based on reasoning that must have followed, at least broadly, along these lines Ptolemy asserts that the radius of the ringed globe should be 4 : 3 of the earth's radius.[153]

The midpoints of the principal parallels are laid out in the same way as in the two maps of the *oikoumenē* so that we have $ET = 16^{5}/_{12}{}^{\text{p}}$, $ES = 23^{5}/_{6}{}^{\text{p}}$ and $EU = 63^{\text{p}}$. In

[153] Andersen first put forward a derivation of these numbers in response to Neugebauer's claim that this passage exhibited circular reasoning, Andersen [1987, 108 - 110] and Neugebauer [1959, 23, n. 7].

this way U, S, E and T are respectively the midpoints of the parallels through Thulē, Soēnē, the equator and anti-Meroē.

Ptolemy next fixes the position of the eye on the line SX and shows how to find the midpoints of the rings on the plane of the figure using projection. The eye is chosen at point W along SX such that the line through W and D, the nearest point on the ring of the winter tropic, falls at point Y just below point T. This means that the line WU will also fall inside the ring of the tropic circle.[154] Here, the plane of the figure depicts two perpendicular planes folded into each other. The plane of the drawing containing the circle $PFKR$ is actually perpendicular to the plane containing line SW. In the plane containing SW, the plane of the drawing is represented by line PR. Ptolemy points out that the midpoints of the rings are found by joining W with L, M, H, Z, D, B, C and N and taking the intersections of these lines with PR. Thus Ptolemy projects the points of the three dimensional sphere onto a plane perpendicular to the line of sight. This is in accordance with the principles of linear perspective. Ptolemy only gives one example in his figure, WDY.

The end points of the parallels are determined by taking the intersection the circle PXR and the line joining W with their midpoints. Thus the end points of the parallel through U are g and d. These three points are then presumably joined by the arc of a circle. This construction ensures that the further a parallel is from FSX the more inclined it will be. Only the end points of the parallels through Thulē, Soēnē and anti-Meroē are shown in the figure. The lengths of the meridians are laid out on either side of UT "in the ratios that pertain to the three parallels."[155] Through each set of three points circular arcs are drawn so that the boundaries of the *oikoumenē* are ezh and ϑkl. This probably means that the lengths of the parallels cut off by the meridians are chosen in the same way as in the second map of the *oikoumenē*.

Ptolemy finally makes it clear that, although the *oikoumenē* was drawn in such a way that distances on the drawing are approximately proportional to those on the sphere,

[154] Andersen [1987, 110] gives a proof of this statement.
[155] Berggren and Jones [2000, 116].

the rings of the outer sphere are meant to be drawn according to the principles of linear perspective. The four cardinal points of each ring are projected onto the plane of the figure and then joined with an ellipse. He ends his treatment of the drawing of the ringed globe with some remarks on how the rings and the globe should be drawn and shaded and what labels and numbers should be included in the finished product. In Figure 4.13, the ellipse BYD represents the ring of the equator drawn according to Ptolemy's specifications.

The presentation of the ringed globe contains a mixture of the techniques of linear perspective and the methods of map construction set out in *Geo.* I 24. Ptolemy's adept treatment of the linear perspective of the rings of the outer sphere only highlights the fact that he deliberately chose not to use this method for representing the surface of the earth. Despite the fact that the drawing produced here was not meant to be a detailed map of the *oikoumenē*, as in *Geo.* I 24, Ptolemy still thought that the primary role of a visual image of the earth was to portray to the viewer a visual sense of the distances as they are on the globe along with the fact that these distances are indeed on the surface of a sphere.

There is still debate about what the presentation of Ptolemy's third map tells us about the possibility of a theory of linear perspective in antiquity.[156] Historians of art are not in agreement about whether or not there was a theoretical approach to linear perspective in the ancient world.[157] Historians of science are uncertain if linear perspective is even compatible with ancient theories of optics.[158] seems that Euclid's theory of size perception is incompatible with some aspects of visual perception as implied by linear perspective.[159] Ptolemy's theory of size perception is more complicated than Euclid's and seems to have been developed to address some of the inherent difficulties of Euclid's

[156] Edgerton [1975, 106 - 113] and Valerio [1998].
[157] Andersen [1987], Richter [1970] and White [1967, 249 - 262].
[158] Panofsky [1927] is the origin of this debate. More recently, Brownson [1981] and Tobin [1990] have argued for the compatibility of Euclid's *Optics* and linear perspective. Knorr [1991], however, has shown that the text and figures these arguments have rested on are likely due to a late edition by Theon of Alexandria. See page 68 note 20.
[159] Andersen [1987], Knorr [1991] and Knorr [1992].

theory.[160] Ptolemy's treatment of vision makes it clear that he believes that the eyes are capable of making direct perceptions that will not be reproducible on a plane surface. Here, however, Ptolemy is not dealing with optical theory but with image creation.

Whatever Ptolemy's opinion about how we actually see the ringed globe, it is quite clear that he thinks we should create an image of the rings on a plane surface by means of pointwise projection. This projection will produce the two basic components of linear perspective: (1) an image-plane perpendicular to the line of sight, and (2) the resulting vanishing point in the image-plane as the orthogonal projection of the eye. I am only aware of two other places in the ancient technical literature where we find evidence of either of these concepts.[161] The first is an orthogonal plane, or line, of projection encountered in Euclid's *Optics* 10 & 11; however, Knorr has argued that these constructions and the proofs built around them are due to a later edition of the text by Theon of Alexandria.[162] The second is a discussion of a circle viewed obliquely in Pappus' commentary on Euclid's *Optics* 35.[163] Jones has pointed out that Pappus' treatment involves a vanishing point; however, Pappus chooses a vanishing point that is incompatible with linear perspective.[164] The lack of consistency in the visual evidence together with the confusion in the technical literature argue that although the projective techniques that would have led to a theory of linear perspective were available in antiquity they were never fully and consistently developed along these lines. Ptolemy's projective procedure in the third map is more likely rooted in the mathematical tradition of the *Planisphaerium* than a tradition of the visual arts.

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160 Lejeune [1989, 25 - 26 & 35 - 46] and Smith [1996, 81 - 82 & 90 - 98].

161 Vitruvius also makes some tantalizing, but vague, remarks about the practice of sceneography in his *Architecture*, Granger [1998, 26], Liou and Zuinghedau [1995, 5 - 6] and Rowland and Howe [1999, 25 & 86]. These are discussed in White [1967, 250 - 257] and Andersen [1987, 84 - 86].

162 Heiberg [1895, 16 - 20 & 166 - 168] and Knorr [1991].

163 Hultsch [1878, 588 - 592].

164 Jones [2000a, 55 - 56]. For a circle on a horizontal plane which is viewed obliquely from above, Pappus chooses a vanishing point which is below and behind the eye. In linear perspective, however, the vanishing point of a horizontal circle viewed from above would generally appear above the circle and directly in front of the eye. Unfortunately, Pappus doesn't define the image plane so we cannot determine the vanishing point by orthogonal projection.

4.3.3 Remarks on Ptolemy's maps

Although Ptolemy does not present his map constructions through the traditional modes of Greek mathematical discourse, it is clear that they are mathematical in the sense that they build upon one another, they adhere to definite criteria and they are developed from prior assumptions. This is to say that Ptolemy's maps are only mathematical in a loose sense. In fact, Ptolemy probably avoids presenting his maps through theorems, analyses and computations because this approach would make all too obvious how many of his procedures are arbitrary from a strictly mathematical perspective.

The reason Ptolemy's map constructions appear to contain mathematical inconsistencies, in the sense of projective inconsistencies, is that the criteria Ptolemy applies to his maps are visual as opposed to mathematical. The discussion of the rings in *Geo.* VII 6 and the whole of the *Planisphaerium* make it clear that mathematical consistency was well within Ptolemy's grasp. When we compare the *Planisphaerium* with the map constructions, it becomes clear how far the *Planisphaerium* as a whole is from being simply a description of a method for drawing a star map. Implicit in Ptolemy's approach to the geographic maps is the idea that the fundamental function of a map is to impart visual information about both the shape and the size of the area portrayed. This dual goal of the map maker means that Ptolemy must abandon projective consistency in an attempt to portray the proper distances of the earth while still preserving the visual impression of curvature.

Once these visual criteria are expressed, it becomes clear how Ptolemy's map constructions are both consistent and sequential. The first map considers the case where the eye is assumed to occupy all the points in a circular arc at a fixed distance from the surface of the earth along a given parallel circle. The second and third maps fix the eye in two different positions and consider the changes that these positions will produce on the visual impression of the sphere. Through all of these shifts of perspective, however, the same proportions between the distances on the sphere are approximately preserved so that the quantitative information that the maps contain act as an invariant under all three views. Unlike the *Planisphaerium*, which seeks to obtain and demonstrate mathematical consistency, the maps of the *Geography* seek to preserve a certain set of quantitative data under changes of visual perspective.

4.4 Conclusion

We have studied three different approaches to treating the sphere in Greco-Roman antiquity. The *Analemma* demonstrates a method for modeling the celestial sphere which is of great use in mathematical astronomy. The plane model of the *Planisphaerium* is a mathematically consistent representation of the celestial sphere that may have been used for maps and analog computation. The maps of the *Geography* exhibit ways of depicting the sphere on the plane that strike a balance between quantitative and visual criteria.

We looked at all three of the ancient treatments of the analemma and saw that it is used for both nomographic and trigonometric determinations on the sphere. Within the context of Greek geometry, the analemma was a powerful tool for solving problems on the sphere. As well as solving problems that involve two great-circle coordinate systems perpendicular to each other, the analemma was the only method in antiquity for solving problems involving parallel circles. Moreover, as Hero's *Diop.* 35 shows, it was ideally suited to problems that involve the local horizon. Although the analemma techniques have limitations in precisely those problems that the metrical methods of the last chapter solve well, their strengths are in areas that the metrical methods on the sphere cannot handle. Hence, these two methods were the two fundamental tools of ancient spherical astronomy.

Our reading of the *Planisphaerium* found that the two primary goals of the text was to demonstrate that a mathematically consistent model of the celestial sphere could be developed on the plane and that this model could be used for practical purposes. Ptolemy's aim was not to produce a calculation device, rather he performed calculations on the model to demonstrate its mathematical consistency. Nevertheless, the plane model could have been used as a calculation device for certain problems in spherical astronomy provided it was used in conjunction with the analemma. On its own, the plane model is not a serviceable tool for exact calculation. On the other hand, a precisely drawn plane model would have been useful as an analog calculator and may well have been so used. What is certain is that there was a practical tradition that used the plane model for making star maps for one purpose or another and Ptolemy is addressing his text to this tradition.

Our investigation of the analemma and the plane model of the celestial sphere have shown that these models were used to effect nomographic solutions to problems that would have been laborious if carried through by means of calculation. This shows that there was a Greek tradition of applied mathematics that used descriptive geometry to develop nomographic procedures for analog calculation.

A comparison of Ptolemy's maps in the *Geography* with these two shorter works makes it clear that the *Analemma* and the *Planisphaerium* are mathematical works that develop and test mathematically consistent models while the *Geography* operates according to a different set of criteria. The maps of the *Geography* aim to visually portray certain quantitative information and spatial relations. Once we understand the criteria of these maps it becomes clear how they exhibit a consistency of their own.

Appendix A:

Division of *Alm.* I 10: The Development of the Chord Table

For the purposes of studying Ptolemy's development of the chord table, I have divided *Alm.* I 10 into mathematical units. A unit may be either (1) a theorem, (2) a problem, (3) a metrical analysis, (4) a computation, (5) a table, (6) a sketch of a theorem, problem or computation or (7) a discussion. I use page and line numbers from Heiberg's text. The division of the text may seem arbitrary in places.

I 10.1 (Discussion) - [H31, 9 - H32, 9] • Introduction to the mathematical treatment of the chord table.

I 10.2 (Problem) - [H32, 10 - H43, 4] • Construction of the sides of the pentagon and the decagon in a given circle.

I 10.3 (Computation) - [H34, 5 - 35, 16] • Calculation of Crd($36°$), Crd($72°$), Crd($60°$) and Crd($90°$).

I 10.4 (Metrical Analysis with Computation) - [H35, 17 - H36, 8] • Method for finding Crd($180° - \alpha$) with an example calculation of Crd($180° - 36°$).

I 10.5 (Theorem) - [H36, 9 - H37, 18] • *Ptolemy's Theorem* - In a cyclic quadrilateral, the sum of the rectangles composed of opposite sides is equal to the rectangle composed of the diagonals.

I 10.6 (Metrical Analysis) - [H36, 19 - H39, 3] • Proof that if Crd(α) and Crd(β) are given, then Crd($\alpha - \beta$) is also given. Note that Crd(12°) is given.

I 10.7 (Theorem) - [H39, 4 - H40, 8] • Proof that if $BG = 2DG = 2BD$, then $ZG = {}^1\!/_2(AG - AB)$, see figure 2.9 (b).

I 10.8 (Metrical Analysis with Sketch) - [H40, 4 - H41, 4] • Proof that if Crd(α) is given then Crd(${}^1\!/_2\alpha$) will be given. Note that Crd(6°), Crd(3°), Crd(1${}^1\!/_2$°) and Crd(${}^3\!/_4$°) are given.

I 10.9 (Analysis) - [H41, 4 - H42, 6] • Proof that if Crd(α) and Crd(β) are given, then Crd($\alpha + \beta$) is also given.

I 10.10 (Discussion) - [H42, 7 - H43, 5] • Discussion of the squeeze theorem. We want to find Crd(${}^1\!/_2$°). Crd(${}^1\!/_3\alpha$) is not given *dia tōn grammōn*. We will derive Crd(1°) from Crd(1${}^1\!/_2$°) and Crd(${}^3\!/_4$°).

I 10.11 (Theorem) - [H43, 6 - H45, 8] • Proof of T.L. 2a, see figure 2.3. If $AD > AB$, then $\widehat{AD} : \widehat{AC} > AD : AC$.

I 10.12 (Theorem) - [H45, 9 - H46, 20] • Proof that $1;02,50^{\text{p}} >$ Crd(1°) $< 1;02,50^{\text{p}}$. Examples of how this is used to fill out the table.

I 10.13 (Discussion) - [H46, 21 - H47, 21] • Discussion of the table.

Appendix B:

The Origin of *Alm.* I 15

R. Newton discovered that there was slight systematic deviation between the declination values for the degrees of the ecliptic as tabulated in *Alm.* I 15 and those found by direct computation using the methods Ptolemy sets out in *Alm.* I 14.[1] Newton calculated values based on the methods given in *Alm.* I 14 and compared these with *Alm.* I 15 in the single manuscript **A**, Paris 2389. In *Alm.* I 14, we learn the equivalent of the formula

$$\frac{\mathrm{Crd}(180^\circ)}{\mathrm{Crd}(2\varepsilon)} = \frac{\mathrm{Crd}(2\lambda)}{\mathrm{Crd}(180^\circ)} \times \frac{\mathrm{Crd}(180^\circ)}{\mathrm{Crd}(2\delta)},$$

so that

$$\delta = \frac{\mathrm{Arc}(\frac{\mathrm{Crd}(2\lambda)\mathrm{Crd}(2\varepsilon)}{\mathrm{Crd}(180^\circ)})}{2}. \tag{B.1}$$

Using equation B.1 and equations C.T. 1 and C.T. 2, in Section 2.1.4, for $\mathrm{Crd}(\alpha)$ and $\mathrm{Arc}(x)$, I calculate a table of declinations, rounding to the nearest sixtieth. These calculated values are tabulated as column **B** in Tables B.1 & B.2. These values are compared with the table Toomer prints as *Alm.* I 15, which is tabulated as column **A**.[2] Column **B-A** gives the difference between the two columns.

It is clear that a few of the discrepancies are caused by errors in calculation.[3] The

[1] Newton [1985, 47 - 59].

[2] Toomer [1984, 72]. There are only two quantitative discrepancies between the table Toomer prints and the one Newton prints, Newton [1985, 57 - 58]. Both of Toomer's corrections are based on manuscript evidence.

[3] That is, $\lambda_{27}, \lambda_{51}, \lambda_{81}$ and, probably, $\lambda_{55}, \lambda_{62}$.

Table B.1: Table for the comparison of *Alm.* I 15 with calculated values. Column **A** is *Alm.* I 15. Column **B** is calculated according to equation B.1 using linear interpolation on *Alm.* I 11. Column **B-A** is the difference between the values in **A** and **A**. Column **W-A** is the difference between the values calculated according to van der Waerden's methods and **B**.

λ	**A** - *Alm.* I 15			**B** - Calculated values			**B-A**		**W-A**	
1	0;	24,	16	0;	24,	16	0; 0,	0	0; 0,	0
2	0;	48,	31	0;	48,	31	0; 0,	0	0; 0,	0
3	0;	12,	46	1;	12,	46	0; 0,	0	0; 0,	0
4	1;	37,	0	1;	37,	0	0; 0,	0	0; 0,	0
5	2;	1,	12	2;	1,	12	0; 0,	0	0; 0,	0
6	2;	25,	22	2;	25,	22	0; 0,	0	0; 0,	1
7	2;	49,	30	2;	49,	30	0; 0,	0	0; 0,	1
8	3;	13,	35	3;	13,	36	0; 0,	1	0; 0,	1
9	3;	37,	37	3;	37,	38	0; 0,	1	0; 0,	2
10	4;	1,	38	4;	1,	38	0; 0,	0	0; 0,	-1
11	4;	25,	32	4;	25,	33	0; 0,	1	0; 0,	1
12	4;	49,	24	4;	49,	25	0; 0,	1	0; 0,	0
13	5;	13,	11	5;	13,	11	0; 0,	0	0; 0,	0
14	5;	36,	53	5;	36,	54	0; 0,	1	0; 0,	0
15	6;	0,	31	6;	0,	31	0; 0,	0	0; 0,	-1
16	6;	24,	1	6;	24,	2	0; 0,	1	0; 0,	0
17	6;	47,	26	6;	47,	27	0; 0,	1	0; 0,	0
18	7;	10,	45	7;	10,	46	0; 0,	1	0; 0,	1
19	7;	33,	57	7;	33,	58	0; 0,	1	0; 0,	1
20	7;	57,	3	7;	57,	3	0; 0,	0	0; 0,	0
21	8;	20,	0	8;	20,	1	0; 0,	1	0; 0,	1
22	8;	42,	50	8;	42,	51	0; 0,	1	0; 0,	1
23	9;	5,	32	9;	5,	32	0; 0,	0	0; 0,	0
24	9;	28,	5	9;	28,	5	0; 0,	0	0; 0,	0
25	9;	50,	29	9;	50,	29	0; 0,	0	0; 0,	0
26	10;	12,	46	10;	12,	44	0; 0,	-2	0; 0,	-3
27	10;	34,	57	10;	34,	48	0; 0,	-9	0; 0,	-9
28	10;	56,	44	10;	56,	43	0; 0,	-1	0; 0,	-2
29	11;	18,	25	11;	18,	27	0; 0,	2	0; 0,	2
30	11;	39,	59	11;	39,	60	0; 0,	1	0; 0,	1
31	12;	1,	20	12;	1,	22	0; 0,	2	0; 0,	1
32	12;	22,	30	12;	22,	32	0; 0,	2	0; 0,	2
33	12;	43,	28	12;	43,	30	0; 0,	2	0; 0,	2
34	13;	4,	14	13;	4,	16	0; 0,	2	0; 0,	1
35	13;	24,	47	13;	24,	48	0; 0,	1	0; 0,	1
36	13;	45,	6	13;	45,	7	0; 0,	1	0; 0,	2
37	14;	5,	11	14;	5,	13	0; 0,	2	0; 0,	2
38	14;	25,	2	14;	25,	5	0; 0,	3	0; 0,	3
39	14;	44,	39	14;	44,	42	0; 0,	3	0; 0,	4
40	15;	4,	4	15;	4,	5	0; 0,	1	0; 0,	1
41	15;	23,	10	15;	23,	12	0; 0,	2	0; 0,	3
42	15;	42,	2	15;	42,	4	0; 0,	2	0; 0,	3
43	16;	0,	38	16;	0,	40	0; 0,	2	0; 0,	2
44	16;	18,	58	16;	18,	59	0; 0,	1	0; 0,	2
45	16;	37,	1	16;	37,	2	0; 0,	1	0; 0,	1

Table B.2: Continuation of Table B.1.

λ	**A** - *Alm.* I 15			**B** - Calculated values			**B-A**			**W-A**		
46	16;	54,	47	16;	54,	48	0;	0,	1	0;	0,	1
47	17;	12,	16	17;	12,	16	0;	0,	0	0;	0,	0
48	17;	29,	27	17;	29,	27	0;	0,	0	0;	0,	0
49	17;	46,	20	17;	46,	19	0;	0,	-1	0;	0,	-1
50	18;	2,	53	18;	2,	53	0;	0,	0	0;	0,	0
51	18;	19,	15	18;	19,	7	0;	0,	-8	0;	0,	-7
52	18;	35,	5	18;	35,	3	0;	0,	-2	0;	0,	-2
53	18;	50,	41	18;	50,	39	0;	0,	-2	0;	0,	-2
54	19;	5,	57	19;	5,	55	0;	0,	-2	0;	0,	-2
55	19;	20,	56	19;	20,	50	0;	0,	-6	0;	0,	-6
56	19;	35,	28	19;	35,	25	0;	0,	-3	0;	0,	-4
57	19;	49,	42	19;	49,	38	0;	0,	-4	0;	0,	-4
58	20;	3,	31	20;	3,	30	0;	0,	-1	0;	0,	-1
59	20;	17,	4	20;	17,	1	0;	0,	-3	0;	0,	-3
60	20;	30,	9	20;	30,	9	0;	0,	0	0;	0,	0
61	20;	42,	58	20;	42,	55	0;	0,	-3	0;	0,	-3
62	20;	55,	24	20;	55,	18	0;	0,	-6	0;	0,	-6
63	21;	7,	21	21;	7,	19	0;	0,	-2	0;	0,	-3
64	21;	18,	58	21;	18,	56	0;	0,	-2	0;	0,	-3
65	21;	30,	11	21;	30,	8	0;	0,	-3	0;	0,	-3
66	21;	41,	0	21;	40,	58	0;	0,	-2	0;	0,	-2
67	21;	51,	25	21;	51,	23	0;	0,	-2	0;	0,	-2
68	22;	1,	25	22;	1,	24	0;	0,	-1	0;	0,	-2
69	22;	11,	1	22;	11,	0	0;	0,	-1	0;	0,	-1
70	22;	20,	11	22;	20,	11	0;	0,	0	0;	0,	0
71	22;	28,	57	22;	28,	56	0;	0,	-1	0;	0,	-1
72	22;	37,	17	22;	37,	17	0;	0,	0	0;	0,	0
73	22;	45,	11	22;	45,	11	0;	0,	0	0;	0,	1
74	22;	52,	39	22;	52,	40	0;	0,	1	0;	0,	1
75	22;	59,	41	22;	59,	42	0;	0,	1	0;	0,	1
76	23;	6,	17	23;	6,	19	0;	0,	2	0;	0,	2
77	23;	12,	27	23;	12,	28	0;	0,	1	0;	0,	1
78	23;	18,	11	23;	18,	11	0;	0,	0	0;	0,	0
79	23;	23,	28	23;	23,	27	0;	0,	-1	0;	0,	-1
80	23;	28,	16	23;	28,	16	0;	0,	0	0;	0,	0
81	23;	32,	30	23;	32,	38	0;	0,	8	0;	0,	8
82	23;	36,	35	23;	36,	33	0;	0,	-2	0;	0,	-2
83	23;	40,	2	23;	40,	1	0;	0,	-1	0;	0,	-1
84	23;	43,	2	23;	43,	1	0;	0,	-1	0;	0,	-1
85	23;	45,	34	23;	45,	33	0;	0,	-1	0;	0,	-1
86	23;	47,	39	23;	47,	38	0;	0,	-1	0;	0,	-1
87	23;	49,	16	23;	49,	15	0;	0,	-1	0;	0,	-1
88	23;	50,	25	23;	50,	25	0;	0,	0	0;	0,	0
89	23;	51,	6	23;	51,	6	0;	0,	0	0;	0,	0
90	23;	51,	20	23;	51,	20	0;	0,	0	0;	0,	0

remaining discrepancies, however, appear to be systematic; they form inversely symmetrical waves about 45°. Figure B.14 plots column **B-A** against λ. Since column **B** gives values that are close to what we would find using modern methods, the values found in *Alm.* I 15 must have been arrived at using a chord table or a method of calculation that introduced these systematic errors.

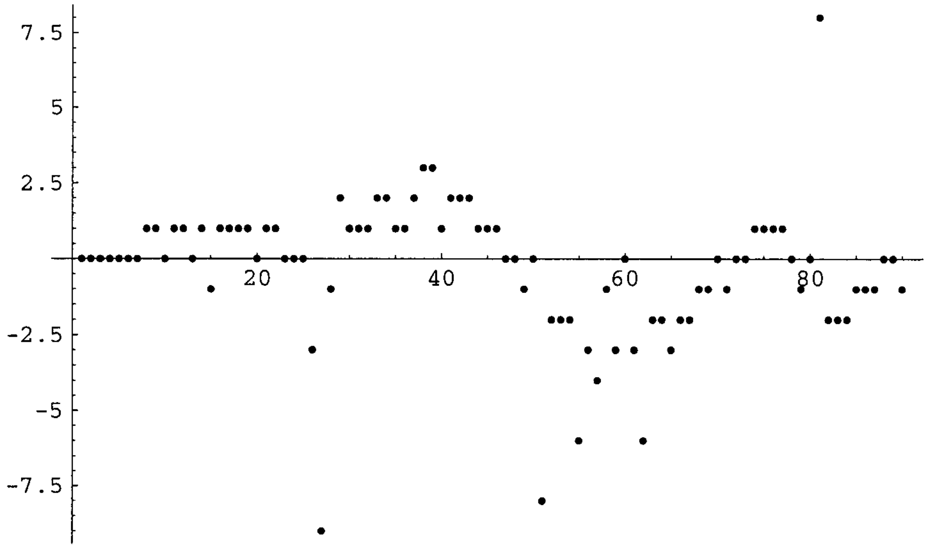

Figure B.14: Plot of the differences between the values of the declination of the ecliptic in *Alm.* I 15 and those calculated by the methods in *Alm.* I 14.

Van der Waerden constructed a table which is meant to account for these systematic errors.[4] Unfortunately, the methods that van der Waerden describes do not by themselves produce a table of declination that exhibits the pattern of errors shown in Figure B.14. It is only by introducing calculating mistakes, or "corrections," that van der Waerden is able to bring his table into agreement with *Alm.* I 15.[5] Van der Waerden only shows how this "correction" is made for the entries $\lambda_{23}-\lambda_{29}$, but he claims that further "corrections" can be made to explain the other periods of discrepancy. The introduction of supposed errors on the part of the ancient calculator confuses the matter. In order to be credible, a method of constructing the table should show some inherent tendency to explain the errors we find in *Alm.* I 15. If we are going to invoke supposed calculating errors, any number of methods of construction might be imagined.

The table is constructed by using a recursive formula for the value of a chord checked

[4] van der Waerden [1988].
[5] van der Waerden [1988, 31].

against geometric calculation for a number of values. Van der Waerden takes

$$\mathrm{Crd}(60^\circ) = r, \mathrm{Crd}(90^\circ) = \sqrt{2}r, \mathrm{Crd}(180^\circ) = 2r,$$

where, for purely mathematical reasons,

$$r = \frac{180 \times 0;0,3600}{3.14159} = 0;0,2026265.$$

By successively halving the arcs of 60° and 90° he finds Crd(30°), Crd(15°), Crd(7½°), Crd(45°), and Crd(22½°). He does not say how the ancient calculator performed these operations. Perhaps he envisions something along the lines of the method given in *Alm.* I 10.7 & 10.8. Rounding the results of the modern formula,

$$\mathrm{Crd}(x^\circ) = 2r \sin x^\circ \alpha$$

where $\alpha = 0;30$, however, yields the exact figures van der Waerden actually uses.[6]

In fact, van der Waerden does not calculate Crd(7½°) or Crd(22½°); instead he calculates Crd(8°) and Crd(23°), which are the values his reconstruction needs.[7] He does not explain how an ancient calculator would have made these calculations. They would not have been trivial using ancient means.

Van der Waerden begins his table with $\mathrm{Crd}(1^\circ) = 0;0,3600$. He then applies the following recursive formula which he takes from the work of Āryabhaṭa:

$$\mathrm{Crd}(x^\circ + 1) = \mathrm{Crd}(1^\circ) - C(\mathrm{Crd}(1^\circ) + \ldots + \mathrm{Crd}(x^\circ)) + \mathrm{Crd}(x^\circ),$$

where, $C = 4\sin^2\frac{\alpha}{2} = \frac{1}{13131}$.[8] In this way he finds the entries for Crd(2°) through Crd(7°). He then applies his correction for Crd(8°). Starting with the value calculated for Crd(8°),

[6] This equation is introduced "to simplify the notation," van der Waerden [1988, 30].
[7] van der Waerden [1988, 32 - 33].
[8] van der Waerden [1988, 31].

he again applies the recursion formula to find the values Crd(9°) through Crd(14°). In this way, by alternating between the check points Crd(8°), Crd(15°), Crd(23°), Crd(30°), Crd(45°), Crd(60°), Crd(90°), and the recursion formula, van der Waerden constructs his table. By supposing that the ancient calculator made errors in his calculations for any of the check points, he can claim that a whole run of entries in the table following the check point will be thrown off in the same direction. By modifying the calculated value for Crd(23°), van der Waerden shows how the entries for λ_{31} to λ_{37} can be made to agree with those in *Alm.* I 15. Figure B.14 shows that these entries form part of the largest positive wave of errors. Van der Waerden does not describe how his calculator would have proceeded to explain the rest of the errors in *Alm.* I 15. He simply states that he has satisfied himself that these other errors can be explained using similar devices.[9]

In order to determine if van der Waerden's table has any inherent ability to explain the errors we find in *Alm.* I 15, I calculate the entire table according to the methods he describes, without supposing any errors on the part of the ancient calculator. I then use this table to generate a table of declination using linear interpolation and the equation

$$\delta = \frac{\mathrm{Arc}(\mathrm{Crd}(2\lambda)\sin\varepsilon)}{2},$$

where, $\sin\varepsilon = .40443$.[10] To compare this table with *Alm.* I 15, I again take the difference between the calculated table and *Alm.* I 15. This difference is tabulated in column **W-A** of Tables B.1 & B.2.

Columns **B-A** and **W-A** show that the deviation of the table calculated according to van der Waerden's methods and that of the table calculated according to Ptolemy's methods is about the same. In fact, the difference between values in these two calculated tables is never more than $\pm 0;0,1$. Moreover, these differences exhibit no pattern of the sort seen in Figure B.14.

[9] van der Waerden [1988, 35].

[10] The difference between van der Waerden's equation and the ancient manipulations equivalent to equation B.1 introduces no significant deviation in the final table of declination, van der Waerden [1988, 3].

Van der Waerden's methods generate a table of declination which is as accurate as that generated by the methods set out by Ptolemy in *Alm.* I 14. In order to account for the errors in *Alm.* I 15, it is necessary to imagine mistakes in the application of the proposed method. If we are forced to imagine errors of calculation to account for the pattern of deviation in *Alm.* I 15, it is best to admit that we do not know how this table was generated.

Van Brummelen has also analyzed this material and is suspicious of both Newton's criticism and van der Waerden's reconstruction. His own statistical analysis shows that *Alm.* I 15 may well have been calculated from the chord table but using an interpolation grid of 10° intervals as opposed to direct computation using spherical trigonometry.[11] This would have been done to save the calculator some labor. Van Brummelen is undecided about how the table was filled in between the points of the grid. Nevertheless, this work shows that while *Alm.* I 15 was probably not calculated directly or consistently, it may still have been calculated on the basis of the chord table in the *Almagest*.

[11] Van Brummelen [1993, 90 - 101].

Appendix C: Toomer's Derivation of Crd(1/2 α) from Ptolemy's Theorem

Toomer derives Crd(1/2 α) from Ptolemy's Theorem using a diagram analogous to that given by Ptolemy for Crd(α+β) in *Alm.* I 10.9.[12] Because he translated his reconstruction into symbolic algebra and proceeded by obvious algebraic manipulations, his procedure is different from any extant example of ancient analysis. Consider Figure C.1. Let ACB be a semicircle on the given diameter $AB = d$. Given $\mathrm{Crd}(\alpha) = BC = s$, it remains to be shown that $\mathrm{Crd}(1/2\ \alpha) = BD = DC = x$ is also given.

Toomer first notes that the supplementary chord, $AC = s'$ is given, by a simple application of *Elem.* I 47, as

$$s' = \sqrt{d^2 - s^2}.$$

He then draws the diameter DOE and joins AE. Hence, $AE = x$. Then, by Ptolemy's Theorem,

$$(AE \times DC) + (AC \times DE) = (AD \times CE).$$

Toomer then translates this algebraically as

$$x^2 + s'd = \sqrt{d^2 - x^2}\sqrt{d^2 - x^2}.$$

[12] Toomer [1973, 16 - 17].

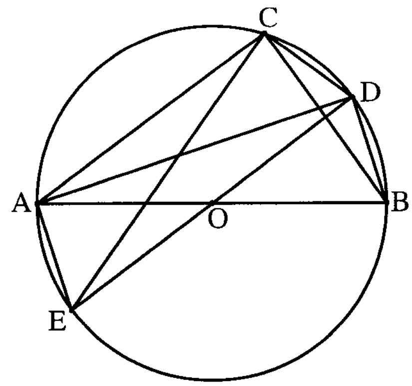

Figure C.1: Diagram for Toomer's derivation of Crd($^1/_2$ α).

Although *Elements* X gives us ways of talking about lines that are the sides of squares equal to the difference of two squares, the use of such lines in ancient analysis is unknown. Nevertheless, we can save this step by noting that, by *Elem.* I 47,

$$(AD \times CE) = AD^2 = d^2 - x^2$$

so that

$$x^2 + s'd = d^2 - x^2. \tag{C.1}$$

Even this simple translation and the manipulations it implies, however, are problematic in the context of ancient analysis. Manipulating expressions of this sort, of course, makes no sense in the tradition of geometric analysis, but one may still wonder if there is an analogous, but more acceptable, procedure in metrical analysis.

In this example, we see the fundamental difference between the algebraic approach and that of ancient analysis. The great virtue of algebra, whether symbolic or rhetorical, is that it encourages the mathematician to make statements about unknown quantities, and then carry out operations on these statements in the hope of showing that these unknown quantities are, in fact, known. In metrical analysis, the attention of the mathematician is restricted to quantities that are *given*, that is arithmetically known. Quantities can only said to be given when they are equal to other quantities that are given or can be shown to be given using the theorems of Euclid's *Data*. Hence, if we were to translate the steps of a metrical analysis into symbolic algebra we would find that at each stage one side of the the equation is entirely composed of given magnitudes. A statement analogous to equation C.1 would simply never arise in the context of metrical analysis.

We can get around this difficulty by claiming that Ptolemy could argue along the following lines. Since, $(AE \times DC) + (AC \times DE) = (AD \times CE)$ [*Alm.* I 11.9] and $(AD \times CE) = AD^2 = AB^2 - DB^2$ [*Elem.* I 47], while $AE = DC = DB$ [by construction], therefore $DB^2 + (AC \times DE) = AB^2 - DB^2$. So that $2DB^2 = AB^2 - (AC \times DE)$. Then since $AB = DE$ and AC are given, therefore $2DB^2$ is given. Then DB^2 is given [*Data* 2] and therefore DB is given [conv. *Data* 55].

This procedure, however, was found by, and modeled on, the algebraic approach. In hindsight it appears obvious but it may not have been obvious at all in a mathematical context such as metrical analysis. Although it would be hard to argue that such manipulations could not occur in the tradition of metrical analysis, they are not what the extant examples lead us to expect.

Appendix D:

Division of Ptolemy's Spherical Astronomy

For the purposes of studying the logical structure of Ptolemy's spherical astronomy, I provide a division of *Alm.* I 13 - II 13 & VIII 5 - 6 into mathematical units. A unit is either (1) a theorem, (2) a metrical analysis, (3) a computation, (4) a table, or (5) a sketch of a theorem or computation. Text which has no mathematical content, and does not contribute to the deductive structure of the argument, such as *Alm.* II 1, has not been included. I have made my numbering conform to Toomer's in all applicable places. For the sake of clarity, I use page and line numbers from Heiberg's text. The division of the text may in places be arbitrary. I have tried to divide by the type of mathematical text as well as by the content. I have always divided where Ptolemy concludes a proof with the words ὅπερ ἔδει δεῖξαι even when it is not clear what has been proven, as at the end of I 7.4, H126, 9.

I 13.1 (Theorem) - [H68, 23 - H69, 20] • Plane version of Menelaus' Theorem I.

I 13.2 (Theorem) - [H69, 21 - H70, 16] • Plane version of Menelaus' Theorem II.

I 13.3 (Theorem) - [H70, 17 - H71, 13] • Proof that $\mathrm{Crd}(2\,\overset{\frown}{AB}) : \mathrm{Crd}(2\,\overset{\frown}{BG}) = AE : EG$, where AG is the chord subtending $\overset{\frown}{AG}$ divided at E by the line joining B with the center of the circle; see Figure 3.3 (b).

I 13.3c (Metrical Analysis) - [H71, 14 - H72, 10] • Proof that if $\widehat{AG}$ and the ratio $\text{Crd}(2\,\widehat{AB}) : \text{Crd}(2\,\widehat{BG})$ are given, then $\widehat{AB}$ and $\widehat{BG}$ will be given individually; see Figure 3.3 (b).

I 13.4 (Theorem) - [H72, 11 - H73, 10] • Proof that $\text{Crd}(2\,\widehat{GA}) : \text{Crd}(2\,\widehat{AB}) = GE : BE$, where GE is a line joining G and B extended to meet a line through A and the center of the circle; see Figure 3.3 (c).

I 13.4c (Metrical Analysis) - [H73, 11 - H74, 8] • Proof that if $\widehat{GB}$ and the ratio $\text{Crd}(2\,\widehat{GA}) : \text{Crd}(2\,\widehat{AB})$ are given, then $\widehat{AB}$ will also be given; see Figure 3.3 (c).

I 13.5 (Theorem) - [H74, 9 - H76, 2] • Menelaus' Theorem II.

I 13.6 (Sketch) - [H76, 3 - H76, 9] • Menelaus' Theorem I.

I 14 (Computation) - [H76, 12 - H78, 24] • Example calculation of δ for ♉0°.

I 15 (Table) - [H80 - H81] • Table of inclination, tabulating the δ for each degree of a quadrant of the ecliptic.

I 16.1 (Computation) - [H82, 2 - H84, 17] • Example calculations of α for ♉0°, ♊0° and ♋0°.

I 16.2 (Table) - [H84, 18 - H85, 20] • List of the rising times at *sphaera recta*, listing α, for 10° intervals of the ecliptic.

II 2 (Computation) - [H89, 19 - H92, 15] • Example calculation of η at a solstice given $M = 14^1/4^{\text{h}}$ or $\varphi = 36°$.

II 3.1 (Computation) - [H92, 18 - H93, 19] • Example calculation of φ given $M = 14^1/4^{\text{h}}$.

II 3.2 (Computation) - [H93, 20 - H95, 5] • Example calculation of M given $\varphi = 36°$.

II 3.3 (Metrical Analysis) [H95, 6 - H95, 22] • To find η at a solstice given $\varphi = 36°$.

II 3.3c (Theorem) - [H95, 22 - H97, 4] • Corollary on η at various times of the year.

II 4 (Sketch) - [H97, 7 - H98, 4] • To find when and how often the sun reaches the zenith.

II 5.1 (Computation) [H98, 8 - H100, 16] • Example calculation of $s : g$ where $\varphi = 36°$ and $\varepsilon = 23;51,20°$.

II 5.2 (Sketch) - [H100, 16 - H100, 22] • To find φ and ε given $s : g$.

II 6 (Table) - [H101, 8 - H117, 9] • Exposition of the klimata.

II 7.1 (Theorem) - [H118, 5 - H119, 12] • Arcs of the ecliptic which are equal and equidistant from the same equinox have equal rising times.

II 7.2 (Theorem) - [H119, 13 - H120, 22] • The total rising time of arcs of the ecliptic which are equal and equidistant from the same solstice is the same as the corresponding arcs of the ecliptic at *sphaera recta*.

II 7.3 (Computation) - [H120, 23 - H124, 22] • Calculation of the rising times of the signs of the zodiac where $\varphi = 36°$.

II 7.4 (Theorem) - [H124, 23 - H126, 9] • Preliminary on ascensional difference, n.

II 7.5 (Computation) [H126 - 10 - K133, 3] • Example calculation of the batch calculation method, using n, of rising times made for 10° arcs of the ecliptic where $\varphi = 36°$.

II 8 (Table) - [H134 - H141] • Table of rising times, $\rho(\lambda, \varphi)$, tabulated at 10° intervals for eleven geographical latitudes.

II 9.1 (Sketch) - [H142, 9 - H142, 19] • To find the length of daylight or night, given φ and $\lambda_{\odot}$.

II 9.2 (Sketch) - [H142, 20 - H143, 7] • To find the length of the seasonal hour, given φ and $\lambda_{\odot}$.

II 9.3 (Sketch) - [H143, 8 - H143, 16] • To convert seasonal hours to equinoctial hours.

II 9.4 (Sketch) - [H143, 17 - H144, 5] • To find the rising point of the ecliptic (the horoscope), given the date and time in seasonal hours.

II 9.5 (Sketch) - [H144, 6 - H144, 13] • To find the culminating point of the ecliptic (the midheaven), given the date and time in seasonal hours.

II 9.6 (Sketch) - [H144, 14 - H145, 4] • To convert from the rising point to the culminating point and vise versa.

II 10.1 (Theorem) - [H147, 9 - H148, 10] • The angles between the meridian and the ecliptic which are equidistant from an equinox are equal.

II 10.2 (Theorem) - [H148, 10 - H149, 9] • The sum of angles between the meridian and the ecliptic, which are equidistant from a solstice, is 2R.

II 10.3 (Computation) - [H149, 10 - 153, 20] • Calculation of the angles between the meridian and the ecliptic for the beginning of each of the signs of the zodiac.

II 11.1 (Theorem) - [H154, 9 - H155, 10] • The angles between the horizon and the ecliptic which are equidistant from an equinox are equal.

II 11.2 (Theorem) - [H155, 11 - H156, 2] • The sum of angles between the horizon and the ecliptic, which have 180° longitudinal difference, is 2R.

II 11.2c (Sketch) - [H156, 3 - H156, 9] • For angles between the horizon and the ecliptic, which are equidistant from the solstice, the sum of the rising angle of one and the setting angle of the other is 2R.

II 11.3 (Computation) - [H156, 9 - H159, 22] • Example calculation of the angles between the horizon and the ecliptic at ♉0°, where $\varphi = 36°$.

II 12.1 (Theorem) - [H160, 11 - H162, 9] • The sum of angles between altitude circles and the ecliptic, which are equidistant from a solstice and symmetrical about the meridian, is 2R.

II 12.2 (Theorem) - [H162, 10 - H166, 18] • If two points of the ecliptic are symmetrical about the meridian, the arcs of altitude circles from the zenith to these points are equal, and the sum of the angles between the altitude circles and the ecliptic, which are at these points, is twice the angle of the same point of the ecliptic at the meridian.

II 12.3 (Metrical Analysis) - [H166, 19 - H167, 25] • To find the zenith arc of the altitude circle and the angle between the altitude circle with the ecliptic at the meridian and at the horizon, given φ and δ.

II 12.4 (Computation) - [H168, 9 - H171, 24] • Example calculation of the zenith arc of the altitude circle and the angle between the altitude circle and the ecliptic for ♋0° an hour before noon, where $\varphi = 36°$.

II 13 (Table) - [H174 - H187] • Table of the angles of the ecliptic, θ, and the arcs of the altitude circles, z, $\theta(\lambda, \varphi, h)$ and $z(\lambda, \varphi, h)$, tabulated at 30° intervals of the ecliptic, for hours between noon and sunrise (or sunset), at six geographic latitudes.

VIII 5.1 (Metrical Analysis) - [H194, 4 - H196, 9] • To find the degrees of the equator and ecliptic which culminate simultaneously with a given star.

VIII 5.2 (Metrical Analysis) - [H196, 10 - H197, 20] • To find the degrees of the equator and ecliptic which rise and set simultaneously with a given star.

VIII 6.1 (Metrical Analysis) - [H201, 13 - H202, 9] • To find the *arcus visionis* for a particular star from a single observation of the longitudinal distance of the sun below the Earth at the time of first visibility.

VIII 6.2 (Metrical Analysis) - [H202, 10 - H202, 23] • To calculate the longitudinal distance of the sun below the Earth at the time of first visibility for any latitude given the *arcus visionis.*

VIII 6.3 (Sketch) - [H203, 1 - H203, 6] • To find the last visibilities.

Appendix E: The Limitations of the Analemma Techniques

It is difficult to give a complete description of the ancient analemma techniques because we have so few examples of the analemma in mathematical practice. The best that we can do is form a picture of the practice based on the ancient and medieval evidence.

The central device of the analemma is the translation of arcs of the sphere into arcs on the plane of the figure. This translation takes place either through rotation or superposition. Both of these are mathematically equivalent to orthogonal projection. Hence, if the equatorial or ecliptic frames of reference can be reduced to orthogonal projection, a point on the sphere can be specified and basic problems of spherical astronomy can be solved. In practice, this means that in order to find a relation between two frames of reference the coordinate circles in these two systems must be perpendicular. Thus, since the equator is always perpendicular to the meridian, the equatorial and local coordinate systems can readily be related to one another. In the case of the ecliptic system, however, the cases that the analemma can fruitfully model are more limited: 1) the equinox must be rising or 2) only one point on the ecliptic can be treated.

Determinations on an ecliptic oblique to the meridian

In all cases where an equinox is not rising the ecliptic is oblique to the meridian. Although it is possible to characterize the motion of the ecliptic within the framework of the analemma, this section argues that precise mathematical determination of this motion goes beyond the scope of the practices we find attested for the analemma.[13] The central difficulty lies in locating one of the cardinal points (♈0°, ♋0°, ♎0° or ♑0°) with respect to the meridian. If we are given the date and time, one position of the ecliptic can always be specified in the figure by a geometrical transformation of its coordinates. This is the project of the *Analemma*. The analemma figure, however, provides no means of locating a cardinal point on the ecliptic. Moreover, a point given in general equatorial coordinates, $P(\lambda,\ \beta)$, cannot be specified on the analemma figure unless we also know the degree of the ecliptic culminating and the obliquity of the diameter of the ecliptic to the diameter of the equator in the plane of the meridian.

Consider the analemma in Figure E.1. Circle $ACBD$ is the meridian, BA the diameter of the equator, CD the diameter of the ecliptic when an equinox is rising and GF the diameter the ecliptic when neither an equinox nor a solstice is rising. The obliquity of the meridian and ecliptic planes varies from 0° to ε and back to 0° in 24 equinoctial hours. Hence, $\overset{\frown}{BG}$ varies from $\varepsilon = \overset{\frown}{BD}$ to 0° and back again in the same time. In order to use the analemma figure to make determinations on the ecliptic, we will have to specify its position in the figure; we will have to know $\overset{\frown}{BG}$ and $\overset{\frown}{GO}$, where O is one of the cardinal points. Although it is possible to determine $\overset{\frown}{BG}$, by means of a declination table similar to *Alm.* I 15, determining $\overset{\frown}{GO}$ generally requires more information than we are given.

While, in theory, it is possible that we would have enough information to specify the ecliptic in the figure, in practice we would almost never actually have this information. Specific problems in spherical astronomy might be, for example, to determine the degree of the ecliptic rising when a given star is culminating, or the degree of the ecliptic

[13] The characterization of the instantaneous positions of the ecliptic for any position and time is taken up by Ptolemy in the last four sections of *Almagest* II, Toomer [1984, 105 - 130]. He uses the spherical geometry of Menelaus' *Spherics* and the metrical methods of spherical trigonometry.

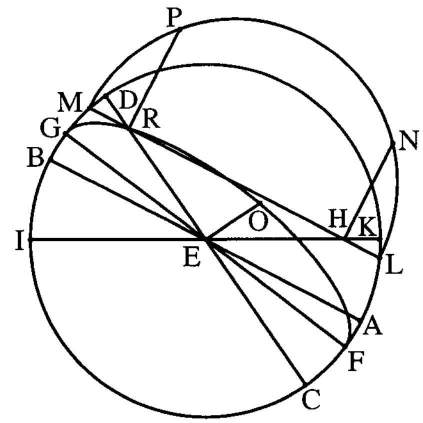

Figure E.1: Analemma figure for an oblique ecliptic. Circle $ACBD$ is the meridian, AB is the diameter of the equator, CD the diameter of the ecliptic, IK the diameter of the horizon, and ML is the diameter of a δ-circle. The circumference of the oblique ecliptic is GOF on diameter GF.

culminating at a given time.

Suppose, for example, we want to determine the rising or culminating point of the ecliptic, given the solar longitude, $\lambda_{\odot}$, time, h, and geographic latitude, φ. We assume that during the course of the day, the sun moves parallel to the equator on a single δ-circle. Let this δ-circle be MNL. Let the sun be at point P. If we consider δ-circle MNL as perpendicular to the plane of the figure then we know that the ecliptic goes through point R at the time when the sun is at P just as it will go through M when the sun crosses the meridian. The analemma, however, provides no way of determining $\overset{\frown}{GR}$. Moreover, although we know $\lambda = \overset{\frown}{RO}$, without knowing $\overset{\frown}{GO}$ we have no way of specifying these two points on the analemma. Moreover, without knowing the obliquity of the ecliptic to the meridian, even if we determine the rising point we cannot determine the culminating or setting points. This problem is equivalent to trying to determine the rising or culminating point of ecliptic when a star, given in equatorial coordinates, is on the horizon. Because of the difficulty with locating a cardinal point on the ecliptic, stars given in ecliptic coordinates cannot even be specified on the analemma.

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