$\textcircled{C_7}$ - AG 16 14.11.2024

Proiecții și simetrii

Def $p: V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$ apl liniară

p s.n. proiecție pe V_1 $\Longleftrightarrow$ $p(\underset{V_1}{\overset{\cap}{v_1}}+\underset{V_2}{\overset{\cap}{v_2}})=v_1$

Prop $p \in \mathrm{End}(V)$

p proiecție $\Longleftrightarrow$ $p\circ p = p$

Dem "$\Longrightarrow$" $p: V \to V$ liniară $V = V_1 \oplus V_2$.

p proiecție pe V_1 $\Longrightarrow$ $p(v) = p(v_1+v_2) = v_1$.

$$p\circ p(v) = p(p(v)) = p(v_1) =$$
$$= p(v_1 + 0_V) = v_1 = p(v), \forall v \in V$$

$\Longrightarrow p \circ p = p$

"$\Longleftarrow$" $p \in \mathrm{End}(V)$ și $\boxed{p\circ p = p}$.

Considerăm $V = \mathrm{Im}\,p \oplus \mathrm{Ker}\,p$.

"$\supseteq$" din construcție.

"$\oplus$" Fie $v \in \mathrm{Im}\,p \cap \mathrm{Ker}\,p \Longrightarrow \exists w \in V$ a.î. $v = p(w)$ și $p(v) = 0$

$$v = p(w) \;\Big|\Longrightarrow \underset{0}{\underset{\shortparallel}{p(v)}} = p\circ \underset{v}{\underset{\shortparallel}{\underset{p(w)}{\underset{\shortparallel}{p(w)}}}} \Longrightarrow v = 0_V$$

"$\subseteq$" $$v = \underset{\mathrm{Im}\,p}{\underset{\cap}{\overset{v_1}{\overset{\shortparallel}{p(v)}}}} + \underset{\mathrm{Ker}\,p}{\underset{\cap}{\underbrace{\overset{v_2}{\overset{\shortparallel}{v - p(v)}}}}}.$$

$$p(v_2) = p(v - p(v)) \overset{\text{lin}}{=} p(v) - p(\underset{p(v)}{\underset{\shortparallel}{p(v)}}) = 0_V \Longrightarrow v_2 \in \mathrm{Ker}\,p$$

$p: V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$, $p \circ p = p$, $V_1 = \mathrm{Im}\,p$, $V_2 = \mathrm{Ker}\,p$.

p este proiecția pe V_1

$$p(\underset{\underset{\operatorname{Im} p}{\cap}}{v_1}+\underset{\underset{\ker p}{\cap}}{v_2}) = p(\overset{\overset{p(w)}{\shortparallel}}{v_1})+p(\underset{\underset{0}{\shortparallel}}{v_2}) = p(p(w)) = p(w) = v_1$$

$\Rightarrow$ p este proiecția pe $V_1 = \operatorname{Im} p$.

<u>Def</u> $s \in \operatorname{End}(V)$
s.s.n. <u>simetrie</u> $\Longleftrightarrow$ $s\circ s = id_V$
(involuție)

<u>Prop</u>. $(V,+,\cdot)/_{\mathbb{K}}$ sp vect, $\operatorname{ch}\mathbb{K} \neq 2$ $(1+1\neq 0_{\mathbb{K}})$
p este proiecție $\Longleftrightarrow$ $s = 2p - id_V$ simetrie.

<u>Dem</u>
"$\Rightarrow$" $p: V\to V$ proiecție $\Rightarrow$ $p\circ p = p$.

$$s\circ s = (2p-id_V)\circ(2p-id_V) = 4\underbrace{p\circ p}_{p} - 2p - 2p + id_V = id_V$$

"$\Leftarrow$" $s\in \operatorname{End}(V)$ $s\circ s = id_V$ $\Big|\overset{?}{\Longrightarrow} p\circ p = p$.
$s = 2p - id_V$

$$\underline{id_V} = (2p-id)\circ(2p-id_V) = 4p\circ p - 4p + \underline{id_V} \Rightarrow p\circ p = p.$$

$\boxed{\text{OBS}}$ $p: V_1\oplus V_2 \longrightarrow V_1\oplus V_2$, $p(v) = p(v_1+v_2) = v_1$
<u>proiecția pe V_1</u>

$s(v) = 2p(v) - v = 2v_1 - (v_1+v_2) = v_1 - v_2$
<u>simetria față de V_1</u>

$$p(e_1) = p(\overset{\overset{V_1}{\cup}}{e_1}+\overset{\overset{V_2}{\cup}}{0}) = e_1$$

$\boxed{\text{OBS}}$ $V = V_1\oplus V_2$, $V_1 = \operatorname{Im} p$, $V_2 = \operatorname{Ker} p$.

$\mathcal{R}_1 = \{e_1,..,e_k\}$ reper în V_1
$\mathcal{R}_2 = \{e_{k+1},..,e_n\}$ reper în V_2.

$$\begin{cases} p(e_i) = e_i & i=\overline{1,k} \\ p(e_j) = 0 & j = \overline{k+1,n}\end{cases}$$

$\mathcal{R} = \mathcal{R}_1\cup\mathcal{R}_2$ reper în $V = V_1\oplus V_2$

$$[p]_{\mathcal{R},\mathcal{R}} = A_p = \left(\begin{array}{c|c} \begin{matrix} 1 & \ddots & 0 \\ 0 & & 1\end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} I_k & O_{k,n-k} \\ \hline O_{n-k,k} & O_{n-k,n-k}\end{array}\right) \notin O(n)$$

$A \in O(n) \iff A \cdot A^T = I_n$

(matrice ortogonală)

$[s]_{R,R} = A_s = \left(\begin{array}{c|c} I_k & O_{k,n-k} \\ \hline O_{n-k,k} & -I_{n-k} \end{array}\right) \in O(n)$

$s(v) = s(v_1 + v_2) = v_1 - v_2$

$s(e_i) = e_i, \ \forall i = \overline{1,k}$

$s(e_j) = -e_j, \ \forall j = \overline{k+1,n}$

$s(e_1) = e_1 = 1 \cdot e_1 + 0 \cdot e_2 + .. + 0 \cdot e_n$

$\vdots$

$s(e_k) = e_k = 0 \cdot e_1 + \ldots + 0 \cdot e_{k-1} + 1 \cdot e_k + 0 e_{k+1} + \ldots + 0 \cdot e_n.$

$s(e_{k+1}) = -e_{k+1}$

$\vdots$

$s(e_n) = -e_n.$

Ex $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - 2x_3 = 0\}$

$\mathbb{R}^3 = V' \oplus V''$, V'' subsp. complementar lui V'.

$s, p : V' \oplus V'' \longrightarrow V' \oplus V''$ / p = proiecția pe V', s = simetria față de V'.

Să se calc $p(1,2,5)$, $s(1,2,5)$

SOL $x_1 = -x_2 + 2x_3$

$V' = \{(\underset{\overset{\parallel}{x_2(-1,1,0)+x_3(2,0,1)}}{-x_2 + 2x_3, x_2, x_3}) \mid x_2, x_3 \in \mathbb{R}\} = \langle \underbrace{\{(-1,1,0), (2,0,1)\}}_{R' \text{ e SG pt } V'} \rangle$

$\dim V' = 3 - 1 = 2 = |R'| \Rightarrow R'$ e reper în V'.

Extindem R' la un reper în $\mathbb{R}^3$.

$rg\begin{pmatrix} -1 & 2 & \boxed{0} \\ 1 & 0 & \boxed{0} \\ 0 & 1 & \boxed{1} \end{pmatrix} = 3 \max$ $\quad R = R' \cup \{e_3\}$ reper în $\mathbb{R}^3$

Considerăm $V'' = \langle \{e_3\} \rangle$

$\overset{\parallel}{e_3}$

p proiecție pe $V' \Rightarrow p(v' + v'') = v'$

s simetrie față de $V' \Rightarrow s(v' + v'') = v' + v''$

$$\begin{cases} a = 2 \\ -2 + 2b = 1 \Rightarrow b = \frac{3}{2} \\ b + c = 5 \Rightarrow c = 5 - \frac{3}{2} \end{cases}$$

$(1,2,5) = \underbrace{a(-1,1,0) + b(2,0,1)}_{v' \in V'} + \underbrace{c(0,0,1)}_{v'' \in V''} = (-a+2b, a, b+c)$

$c = \frac{7}{2}$

$2, \frac{3}{2}, \frac{7}{2})$

$v' = a(-1,1,0) + b(2,0,1) = 2(-1,1,0) + \frac{3}{2}(2,0,1) = (1,2,\frac{3}{2})$

$v'' = c(0,0,1) = \frac{7}{2}(0,0,1) = (0,0,\frac{7}{2})$

↓ $p(v) = v' = (1,2,\frac{3}{2})$

$s(v) = v' - v'' = (1,2,\frac{3}{2}) - (0,0,\frac{7}{2}) = (1,2,-2)$

Vectori proprii. Valori proprii. Diagonalizare

Problemă $(V,+,\cdot)_{/\mathbb{K}}$ sp vect finit generat, $f \in End(V)$

$\exists\, \mathcal{R} = \{e_1, .., e_n\}$ în V aî $[f]_{\mathcal{R},\mathcal{R}} = A =$ diagonală $=$

$= \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & & & \lambda_n \end{pmatrix}$?

$f(e_1) = \lambda_1 e_1$

$f(e_2) = \lambda_2 e_2$

$\vdots$

$f(e_n) = \lambda_n e_n$.

Def (vector propriu).

$f \in End(V)$

$x \neq 0_V$ s.n. vector propriu $\Leftrightarrow \exists\, \lambda \in \mathbb{K}$ aî $f(x) = \lambda \cdot x$

λ s.n. valoare proprie.

OBS $f(0_V) = f(0_{\mathbb{K}} \cdot x) = 0_{\mathbb{K}} \cdot f(x) = 0 = \lambda \cdot 0_V$

Not $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$ subspațiul propriu coresp. valorii proprii λ

Prop a) $V_\lambda \subseteq V$ subspațiu vect.

b) V_λ = subspațiu invariant al lui f i.e. $f(V_\lambda) \subseteq V_\lambda$.

Dem

a) $\forall\, x,y \in V_\lambda \Rightarrow f(x) = \lambda x$ și $f(y) = \lambda y$

$\forall\, a,b \in \mathbb{K}$.

$f(ax+by) \overset{f\ lin}{=} a \underset{\lambda x}{\underset{\parallel}{f(x)}} + b \underset{\lambda y}{\underset{\parallel}{f(y)}} = \lambda(ax+by) \Rightarrow ax+by \in V_\lambda$

$\Rightarrow V_\lambda \subseteq V$ subsp. vectorial.

b) Fie $x\in V_\lambda \Rightarrow f(x)=\underset{\underset{K}{\cap}}{\lambda}\,\underset{\underset{V_\lambda}{\cap}}{x} \in V_\lambda \Rightarrow f(V_\lambda)\subseteq V_\lambda$.

Polinomul caracteristic

$f\in End(V)$, $\mathcal{R}=\{e_1,..,e_n\}$ reper în V și $[f]_{\mathcal{R},\mathcal{R}}=A$

Fie $\underset{\underset{0_V}{\neq}}{x}$ vectorii proprii coresp. valorii proprii λ ie $f(x)=\lambda x$.

$$f(x)=f\left(\sum_{i=1}^n x_i e_i\right)=\sum_{i=1}^n x_i f(e_i)=\sum_{i=1}^n x_i\left(\sum_{j=1}^n a_{ji}e_j\right)$$

$$\| =\sum_{j=1}^n\left(\sum_{i=1}^n a_{ji}x_i\right)\boxed{e_j}$$

$$\lambda x=\lambda\sum_{j=1}^n x_j\boxed{e_j}$$

$$\Rightarrow \lambda x_j=\sum_{i=1}^n a_{ji}x_i \quad \forall j=\overline{1,n}$$

$$\sum_{i=1}^n a_{ji}x_i-\underbrace{\sum_{i=1}^n \lambda\delta_{ji}x_i}_{\lambda x_j}=0 \qquad \delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

$$\textcircled{*}\ \sum_{i=1}^n(a_{ji}-\lambda\delta_{ji})x_i=0,\ \forall j=\overline{1,n}$$

SLO $\textcircled{*}$ are și sol nenule $(x\neq 0_V)\Rightarrow \overset{\overset{P(\lambda)}{\|}}{det}(A-\lambda I_n)=0$

polinomul caracteristic

$$\Rightarrow(-1)^n\left[\lambda^n-\sigma_1\lambda^{n-1}+\sigma_2\lambda^{n-2}+\ldots+(-1)^n\sigma_n\right]=0$$

$\sigma_1=Tr(A)$ $\qquad \sigma_k$ = suma minorilor diagonali de ord k, $k=\overline{1,n}$

$\sigma_n=det(A)$

Prop Polinomul caracteristic este un invariant la schimbarea de reper $\qquad \left[det(C^{-1})=\frac{1}{det C}\right]$

Dem $\mathcal{R}=\{e_1,..,e_n\}\xrightarrow{C}\mathcal{R}'=\{e_1',..,e_n'\}$ repere în V

$A'=[f]_{\mathcal{R}',\mathcal{R}'}$, $\qquad A'=C^{-1}AC$

$$det(A'-\lambda I_n)=det(C^{-1}AC-\lambda C^{-1}I_nC)=det\left[C^{-1}\overset{\overset{det(A-\lambda I_n)}{\|}}{(A-\lambda I_n)}C\right]$$

$\boxed{\text{OBS}}$ ▨ valorile proprii = <u>rădăcinile din $\mathbb{K}$</u> ale pol. carac

<u>Ex</u>. $(\mathbb{R}^2,+,\cdot)/\mathbb{R}$, $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x_1,x_2)=(-x_2,x_1)$

$\mathcal{R}_0=\{e_1,e_2\}$ reperul canonic în $\mathbb{R}^2$

$$[f]_{\mathcal{R}_0,\mathcal{R}_0}=A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-x_2\\x_1\end{pmatrix}$$

$$P(\lambda)=\det(A-\lambda I_2)=\begin{vmatrix}-\lambda&-1\\1&-\lambda\end{vmatrix}=\lambda^2+1.$$

$\lambda^2+1=0\Rightarrow\lambda_{1,2}=\pm i\in\mathbb{C}\setminus\mathbb{R}$ $\nexists$ valori proprii

$\boxed{\text{OBS}}$ $P(\lambda)=0\Rightarrow(\lambda-\lambda_1)^{m_1}\cdot\ldots\cdot(\lambda-\lambda_r)^{m_r}=0$

$\lambda_1,\ldots,\lambda_r$ răd distincte și $m_1,\ldots,m_r$ sunt multiplicitățile coresp. și $n=m_1+\ldots+m_r$.

<u>Not</u> $\sigma(f)=\{\lambda_1,\ldots,\lambda_r\}$ spectrul lui f

$$\text{Spec}(f)=\{\underbrace{\lambda_1=\ldots=\lambda_1}_{m_1}<\underbrace{\lambda_2=\ldots=\lambda_2}_{m_2}<\ldots<\underbrace{\lambda_r=\ldots=\lambda_r}_{m_r}\}$$

<u>Prop</u> Vectorii proprii coresp. la valori proprii dist. formează un SLI.

<u>Dem</u>. Dem prin ind. după nr de vectori proprii

$\underset{\neq 0}{x}$ vect proprii $\Rightarrow\{x\}$ este SLI

Ip P_k adev : SLI

k vect proprii coresp la valori proprii dist SL

Dem P_{k+1} SLI

k+1 vect. proprii coresp la valori proprii dist

SLI

$$\circledast \quad \sum_{i=1}^{k+1} a_i v_i = 0_V \quad \Big|\, f \Rightarrow \sum_{i=1}^{k+1} a_i \underset{\lambda_i v_i}{\underset{\parallel}{f(v_i)}} = 0$$

$$\lambda_1 a_1 v_1 + \dots + \lambda_k a_k v_k + \underline{\lambda_{k+1} a_{k+1} v_{k+1}} = 0_V \qquad (1)$$

$\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ distincte. $\Rightarrow$ Putem alege (ev. renumerotăm) $\lambda_{k+1} \neq 0_{\mathbb{K}}$.

$\circledast \;\big|\cdot \lambda_{k+1}$.

$$\lambda_{k+1} a_1 v_1 + \dots + \lambda_{k+1} a_k v_k + \underline{\lambda_{k+1} a_{k+1} v_{k+1}} = 0_V \qquad (2)$$

$$(1)-(2) \Rightarrow (\lambda_1 - \lambda_{k+1}) a_1 v_1 + \dots + (\lambda_k - \lambda_{k+1}) a_k v_k = 0_V$$

$v_1, \dots, v_k \rightarrow k$ vectori proprii coresp la valorile proprii distincte $\lambda_1, \dots, \lambda_k \xRightarrow{P_k}$ SLI

$\Rightarrow a_1 = \dots = a_k = 0_{\mathbb{K}}.\ \overset{\circledast}{\Longrightarrow}\ a_{k+1} v_{k+1} = 0_V \quad \Rightarrow a_{k+1} = 0$

$v_{k+1} \neq 0_V \Rightarrow \{v_{k+1}\}$ SLI

$\Rightarrow \{v_1, \dots, v_k, v_{k+1}\}$ este SLI

EX. $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1, x_2 + x_3, 2x_3)$.

a) $A = [f]_{R_0, R_0}$; b) Aflați valorile proprii

c) subspațiile proprii

d) A se poate diagonaliza?

a) $$\underset{\underset{A}{\parallel}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + x_3 \\ 2x_3 \end{pmatrix}$$;

$f(e_1) = f(1,0,0) = (1,0,0) = e_1$

$f(e_2) = f(0,1,0) = (0,1,0) = e_2$

$f(e_3) = f(0,0,1) = (0,1,2) = e_2 + 2e$

b) Polinomul caract

$$P(\lambda) = \det(A - \lambda I_3) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & \boxed{2-\lambda} \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 (2-\lambda) = 0 \Rightarrow \begin{matrix} \lambda_1 = 1, & m_1 = 2 \\ \lambda_2 = 2, & m_2 = 1 \end{matrix}$$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x\}$ $\quad \lambda_1 = 1$

$$AX = \lambda_1 I_3 X \Rightarrow (A - \lambda_1 I_3) X = 0$$

$$\underbrace{\left(\begin{pmatrix}1&0&0\\0&1&1\\0&0&2\end{pmatrix} - \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}0&0&0\\0&0&1\\0&0&1\end{pmatrix}}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \dim V_{\lambda_1} = 3 - rg\begin{pmatrix}0&0&0\\0&0&1\\0&0&1\end{pmatrix} = 3-1 = 2.$$

$\dim V_{\lambda_1} = 2 = m_{\lambda_1}$

$x_3 = 0 \quad V_{\lambda_1} = \{\underset{x_1\underset{e_1}{(1,0,0)} + x_2\underset{e_2}{(0,1,0)}}{(x_1,x_2,0)} \mid x_1,x_2 \in \mathbb{R}\} = \langle \underbrace{\{e_1, e_2\}}_{R_1} \rangle$

R_1 reper în V_{λ_1}

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x\}$ $\quad \lambda_2 = 2$

$$AX - 2X \cdot I_3 = 0 \Rightarrow (A - 2I_3)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$A - 2I_3 = \begin{pmatrix}1&0&0\\0&1&1\\0&0&2\end{pmatrix} - \begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&-1&1\\0&0&0\end{pmatrix}$$

$$\dim V_{\lambda_2} = 3 - rg(A - 2I_3) = 3 - 2 = 1 = m_{\lambda_2}$$

$$\begin{pmatrix}-1&0&0\\0&-1&1\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{cases}-x_1 = 0\\ -x_2 + x_3 = 0 \Rightarrow x_2 = x_3\end{cases}$$

$V_{\lambda_2} = \{\underset{x_3(0,1,1)}{(0,x_3,x_3)} \mid x_3 \in \mathbb{R}\} = \langle \{\underbrace{(0,1,1)}_{R_2}\} \rangle$.

$P(\lambda) = 0$ ① $\lambda_1 = 1,\ m_1 = 2$
$\lambda_2 = 2,\ m_2 = 1$
$\lambda_1, \lambda_2 \in \mathbb{R}$

② $\dim V_{\lambda_1} = 2 = m_1$; R_1 reper în V_{λ_1}
$\dim V_{\lambda_2} = 1 = m_2$, R_2 reper în V_{λ_2}

$\Rightarrow$ A se poate diagonaliza.

$$[f]_{R,R} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix} \qquad R = R_1 \cup R_2 = \{\overset{(1,0,0)}{e_1}, \overset{(0,1,0)}{e_2}, (0,1,1)\}$$

Prop $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect finit generat, $f \in End(V)$.

λ = valoare proprie $\Rightarrow \dim V_\lambda \le m_\lambda$

unde $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$

m_λ = multiplicitatea lui λ în $P(\lambda) = \det(A - \lambda I_n) = 0$

Dem

Not $\dim V_\lambda = n_\lambda$. Dem că $n_\lambda \le m_\lambda$.

Fie $R = \{e_1, .., e_{n_\lambda}\}$ reper în V_λ ; $V_\lambda \subseteq V$ subsp. vect

Extindem la un reper în V.

$$R' = \{e_1, .., e_{n_\lambda}\} \cup \{e_{n_\lambda+1}, .., e_n\}$$

$$[f]_{R',R'} = \left(\begin{array}{ccc|c} \lambda & \cdots & 0 & \\ 0 & & \vdots & \\ \vdots & & \lambda & \\ \hline & & 0 & \\ 0 & & 0 & \end{array}\right) = A$$

$f(e_1) = e_1 \cdot \lambda$

$f(e_{n_\lambda}) = e_{n_\lambda} \cdot \lambda$

$f(e_i) = \sum_{j=1}^{n} a_{ji} e_j$

$\forall i = \overline{n_\lambda+1, n}$

$$P(X) = \det(A - X I_n)$$

$$= \left|\begin{array}{ccc|c} \lambda - X & \cdots & 0 & \\ 0 & & \vdots & \\ & & \lambda - X & \\ \vdots & & 0 & \\ 0 & & 0 & \end{array}\right| = (\lambda - X)^{n_\lambda} \cdot Q(X)$$

$\Rightarrow m_\lambda \geq n_\lambda$.

Teoremă (de diagonalizare) $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect n-dim.

$f \in End(V)$

$\exists$ un reper R în V aî $[f]_{R,R}$ = diagonală $\Leftrightarrow$

$\begin{cases} 1). \text{ toate răd. polinomului caracteristic } \in \mathbb{K}. \\ 2) \text{ dimensiunile subsp. proprii = multiplicitățile valorilor proprii coresp.} \end{cases}$

i.e. $\begin{cases} 1) \ \lambda_1, .., \lambda_r \in \mathbb{K} \\ 2) \ \dim V_{\lambda_i} = m_i, \ i = \overline{1,r} \end{cases}$, unde $\sigma(f) = \{\lambda_1, .., \lambda_r\}$ $m_1 + ... + m_r = n$

Dem

"$\Longrightarrow$" Ip: "$\exists\ \mathcal{R}=\{e_1,\dots,e_n\}$ reper în V aî

$$A=[f]_{\mathcal{R},\mathcal{R}}=\begin{pmatrix}\mu_1 & & O\\ & \ddots & \\ O & & \mu_n\end{pmatrix}\in \mathcal{M}_n(\mathbb{K})$$

Eventual renumerotăm aî $A=\begin{pmatrix}\overbrace{\lambda_1\dots\lambda_1}^{m_1} & & O\\ & \ddots & \\ O & & \overbrace{\lambda_r\dots\lambda_r}^{m_r}\end{pmatrix}$

$m_1+\dots+m_r=n$.

$$\begin{cases} f(e_1)=\boxed{\lambda_1}e_1\\ \vdots\\ f(e_{m_1})=\boxed{\lambda_1}e_{m_1}\end{cases}\qquad ⊛\begin{cases}\{e_1,\dots,e_{m_1}\}\subseteq V_{\lambda_1}.\\ \vdots\\ \{e_{m_1+\dots+m_{r-1}+1},\dots,e_n\}\subseteq V_{\lambda_r}.\end{cases}$$

$P(\lambda)=\det(A-\lambda I_n)=0 \Rightarrow$

$(\lambda_1-\lambda)^{m_1}\cdot\ldots\cdot(\lambda_r-\lambda)^{m_r}=0$, $\lambda_1,\dots,\lambda_r$ răd dist $\in\mathbb{K}$ (1) cu multiplicitățile $m_1,\dots,m_r$,

⊛ $\dim V_{\lambda_i}\geq m_i$, $i=\overline{1,r}$

dar $\dim V_{\lambda_i}\leq m_i$ (cf prop. preced) $\Bigg|\Rightarrow \dim V_{\lambda_i}=m_i$ $i=\overline{1,r}$ (2)

"$\Longleftarrow$" $f\in \mathrm{End}(V)$ $\quad P(\lambda)=(\lambda-\lambda_1)^{m_1}\cdot\ldots\cdot(\lambda-\lambda_r)^{m_r}$

1. $\lambda_1,\dots,\lambda_r$ (răd dist) $\in\mathbb{K}$
2. $\dim V_{\lambda_i}=m_i$, $i=\overline{1,r}$

$(m_1+\dots+m_r=n)$.

Considerăm $\mathcal{R}_i$ reper în V_{λ_i} , $i=\overline{1,r}$

$\mathcal{R}=\mathcal{R}_1\cup\ldots\cup\mathcal{R}_r$, $|\mathcal{R}|=m_1+\dots+m_r=n=\dim_K V$

Dem că $\mathcal{R}$ este SLI

$$\underbrace{\sum_{i=1}^{m_1}a_ie_i}_{f_1\in V_{\lambda_1}}+\ldots+\underbrace{\sum_{j=m_1+\dots+m_{r-1}+1}^{n}a_je_j}_{f_r\in V_{\lambda_r}}=0_V$$

$f_1,\dots,f_r$ sunt vect proprii coresp la val pr dist.

$\exists\, i_1, \dots, i_k \in \{1, \dots, r\}$ aî
$$\left.\begin{array}{l} f_{i_1} + \dots + f_{i_k} = 0_V \\ f_{i_1}, \dots, f_{i_k} \neq 0_V \\ \{f_{i_1}, \dots, f_{i_k}\} \text{ SLI} \end{array}\right\} \text{↯}.$$

$$\Rightarrow f_1 = 0 \Rightarrow \sum_{i=1}^{m_1} a_i e_i \xRightarrow[\mathcal{R}_1]{\text{SLI}} a_1 = \dots = a_{m_1} = 0$$
$$\vdots$$
$$f_r = 0 \Rightarrow \sum_{j=m_1+\dots+m_{r-1}+1}^{n} a_j e_j = 0 \Longrightarrow a_{m_1+\dots+m_{r-1}+1} = \dots = a_n = 0$$

$$\left.\begin{array}{l} \mathcal{R} \text{ e SLI} \\ |\mathcal{R}| = n = \dim V \end{array}\right| \Rightarrow \mathcal{R} \text{ reper în } V$$

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} \overbrace{\lambda_1 \dots \lambda_1}^{m_1} & & & 0 \\ & \overbrace{\lambda_2 \dots \lambda_2}^{m_2} & & \\ & & \ddots & \\ 0 & & & \overbrace{\lambda_r \dots \lambda_r}^{m_r} \end{pmatrix}$$

<u>OBS</u> $V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_r}$.