5.12.2024

C_{10} AG - 16.

Transformări ortogonale. Endomorfisme simetrice

(E, g) sp. vect. euclidian real

$g: E\times E \to \mathbb{R}$

$\left.\begin{array}{l} g \text{ biliniară} \\ g \text{ simetrică} \\ g \text{ poz def} \end{array}\right\} \Leftrightarrow g$ produs scalar.

Def (E_i, g_i) $i=\overline{1,2}$ s.v.e.r.

Aplicatia liniară $f: E_1 \to E_2$ s.n. <u>aplicatie ortogonală</u> $\Leftrightarrow$

$$\underset{\substack{\| \\ g_2(f(x), f(y))}}{\langle f(x), f(y)\rangle_2} = \underset{\substack{\| \\ g_1(x,y)}}{\langle x, y\rangle_1}, \ \forall x, y \in E_1$$

Def (E, g) s.v.e.r, $f \in End(E)$

$f: E \to E$ s.n. <u>transformare ortogonală</u> $\Leftrightarrow$

$\langle f(x), f(y)\rangle = \langle x, y\rangle, \ \forall x, y \in E$

<u>Not</u> $O(E)$ mult. transf. ortogonale

Prop $f: E_1 \to E_2$ aplicatie ortogonală

a) $\|f(x)\|_2 = \|x\|_1, \ \forall x \in E_1$ $\qquad \|x\| = \sqrt{g(x,x)}$

b) f injectivă

Dem

a) $\langle f(x), f(y)\rangle_2 = \langle x, y\rangle_1, \ \forall x, y \in E_1$.

Fie $y = x$ $\quad \underset{\substack{\| \\ \|f(x)\|_2^2}}{\langle f(x), f(x)\rangle_2} = \underset{\substack{\| \\ \|x\|_1^2}}{\langle x, x\rangle_1} \Rightarrow \|f(x)\|_2 = \|x\|_1$

b) f liniară

f injectivă $\Leftrightarrow Ker f = \{0_{E_1}\}$.

$x \in Ker f \Rightarrow f(x) = 0_{E_2}$

$\underset{\substack{\| \\ \|x\|_1}}{\|f(x)\|_2} = 0 \Rightarrow x = 0_{E_1}$ (g_1 poz def)

<u>Teoremă</u> (E,g) s.v.e.r și $f \in End(E)$

$f \in O(E) \iff \|f(x)\| = \|x\|, \forall x \in E.$

"$\Rightarrow$" cf. prop. preced.

"$\Leftarrow$" $f: E \to E$ liniară și $\|f(x)\| = \|x\|, \forall x \in E$

Dem $\langle f(x), f(y) \rangle = \langle x, y \rangle, \forall x, y \in E.$

$$\underbrace{\|f(x+y)\|^2}_{\langle f(x+y), f(x+y) \rangle} = \underbrace{\|x+y\|^2}_{\langle x+y, x+y \rangle}.$$

$$\langle f(x), f(x) \rangle + \langle f(y), f(y) \rangle + 2\langle f(x), f(y) \rangle = \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle$$

$$\underline{\|f(x)\|^2} + \underline{\underline{\|f(y)\|^2}} + 2\langle f(x), f(y) \rangle = \underline{\|x\|^2} + \underline{\underline{\|y\|^2}} + 2\langle x, y \rangle$$

$$\langle f(x), f(y) \rangle = \langle x, y \rangle, \forall x, y \in E.$$

<u>OBS</u> Matricea asociată unei transf. ortogonale.

$R = \{e_1, .., e_n\}$ reper ortonormat în $E. \iff g(e_i, e_j) = \delta_{ij}$
$\forall i,j = \overline{1,n}.$

$[f]_{R,R} = A = (a_{ij})_{i,j=\overline{1,n}}$

$f(e_i) = \sum_{j=1}^{n} a_{ji} e_j, \forall i = \overline{1,n}$

$$f \in O(E) \Rightarrow \langle \underbrace{f(e_i)}_{\sum_{r=1}^{n} a_{ri} e_r}, \underbrace{f(e_j)}_{\sum_{s=1}^{n} a_{sj} e_s} \rangle = \langle e_i, e_j \rangle = \delta_{ij}$$

$$\sum_{r,s=1}^{n} a_{ri} a_{sj} \underbrace{\langle e_r, e_s \rangle}_{\delta_{rs}} = \delta_{ij} \Rightarrow \sum_{r=1}^{n} a_{ri} a_{rj} = \delta_{ij}$$

$$A^T \cdot A = I_n \Rightarrow A \in O(n)$$

Prop. $f \in O(E)$, $U \subseteq E$ subspațiu invariant (i.e. $f(U) \subseteq U$)

a) $f(U) = U$

b) $U^{\perp} \subseteq E$ subsp. invariant

c) $f|_{U^{\perp}} : U^{\perp} \longrightarrow U^{\perp}$ transf. ortogonală

Dem a) $\left.\begin{array}{l} f|_U : U \longrightarrow f(U) \\ f \text{ inj și liniară} \end{array}\right\} \Rightarrow f|_U$ izom. de sp. vect.

$\left.\begin{array}{l} \overset{Th.}{\Longrightarrow} \dim U = \dim f(U) \\ \text{dar } f(U) \subseteq U \end{array}\right\} \Longrightarrow f(U) = U$

b) $U^{\perp} = \{x \in E \mid g(x,y) = 0, \forall y \in U\}$, $E = U \oplus U^{\perp}$

Dem că $f(U^{\perp}) \subseteq U^{\perp}$ i.e. $\forall x \in U^{\perp} \Rightarrow f(x) \in U^{\perp}$

Fie $y \in U = f(U) \Rightarrow \exists z \in U$ aî $y = f(z)$

$\langle f(x), \underset{\underset{U}{\cap}}{y} \rangle = \langle f(x), f(z) \rangle \underset{f \in O(E)}{=} \langle \underset{\underset{U^{\perp}}{\cap}}{x}, \underset{\underset{U}{\cap}}{z} \rangle = 0 \Rightarrow f(x) \in U^{\perp}$

$\Rightarrow U^{\perp}$ subsp. invariant $\overset{a)}{\Longrightarrow} f(U^{\perp}) = U^{\perp}$

c) Consecință a subpunctelor a), b)

$f|_{U^{\perp}} : U^{\perp} \longrightarrow U^{\perp}$ transf. ortogonală.

Clasificarea transformărilor ortogonale

① $\dim E = 1$. $\qquad O(E) = \{id_E, -id_E\}$

② $\dim E = 2$ $\qquad R = \{e_1, e_2\}$ reper ortonormat, $A = [f]_{R,R} \in O(2)$

OBS $\mathcal{R} \xrightarrow{C} \mathcal{R}'$, $\mathcal{R}, \mathcal{R}'$ repere ortonormate.
$C \in O(n)$; $C \cdot C^T = C^T C = I_n$
$A,\ \in O(n)$

$A = [f]_{\mathcal{R},\mathcal{R}}$, $A' = [f]_{\mathcal{R}',\mathcal{R}'}$

$$A' = C^{-1} A C = C^T A C$$

$$A'^T \cdot A' = (C^T A C)^T \cdot (C^T A C) = C^T \cdot A^T \underbrace{(C^T)^T}_{C} \cdot C^T A C = C^T \underbrace{A^T A}_{I_n} C$$

$$= C^T \cdot C = I_n. \Rightarrow A' \in O(n).$$

Prop. $f \in O(E) \Leftrightarrow \underset{[f]_{\mathcal{R},\mathcal{R}}}{A} \in O(n)$, $\forall \mathcal{R}$ reper ortonormat în E

Prop. $f \in O(E) \Leftrightarrow$ o schimbare de repere ortonormate.

"$\Rightarrow$" $f \in O(E)$, $\mathcal{R}$ = reper ortonormat

$A = [f]_{\mathcal{R},\mathcal{R}} \in O(n)$ $\quad \mathcal{R} \xrightarrow{A} \mathcal{R}'$

$\mathcal{R}'$ reper ortonormat

"$\Leftarrow$" $\underset{\{e_1,..,e_n\}}{\mathcal{R}} \xrightarrow{A} \underset{\{e_1',..,e_n'\}}{\mathcal{R}'}$ $\quad \mathcal{R}, \mathcal{R}'$ repere ortonormat

$f \in End(E)$ $\quad f(e_i) = e_i' = \sum_{j=1}^{n} a_{ji} e_j$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \underset{e_i'}{f(e_i)} = \sum_{i=1}^{n} x_i e_i' = x'$$

$f \in O(E)$.

Prop $f \in O(E) \Rightarrow$ valorile proprii $\in \{\pm 1\}$.

Dem Fie $\lambda \in \mathbb{R}$ valoare proprie $\quad f(x) = \lambda x.$, $x \neq 0_E$

$$f \in O(E) \Rightarrow \|f(x)\|^2 = \|x\|^2 \Rightarrow \langle \underset{\lambda x}{f(x)}, \underset{\lambda x}{f(x)} \rangle = \langle x, x \rangle$$

$$\lambda^2 \|x\|^2 = \|x\|^2 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

③ $\dim E = 3$. $A = [f]_{R,R}$ R reper ortonormat.

$P(\lambda) = \det(A - \lambda I_3) = 0$

(pol. de gradul al 3-lea cu coef. reali $\Rightarrow$
$\exists$ cel puțin o răd. reală: $\lambda \in \{-1, 1\}$)

Fie e_1 versorul propriu coresp. valorii proprii reale λ.

$f(e_1) = \lambda e_1$, $\lambda \in \{\pm 1\}$. $\Rightarrow <\{e_1\}>$ = subsp. invariant

$\overset{Prop}{\Longrightarrow} <\{e_1\}>^{\perp} = e_1^{\perp}$ subsp. invariant

$f/_{e_1^{\perp}} : e_1^{\perp} \longrightarrow e_1^{\perp}$ transf. ortogonală; $\mathbb{R}^3 = <e_1> \oplus <e_1^{\perp}>$

I. $\det A = 1$ (f de speță 1)

a) $\lambda = 1 \Rightarrow f(e_1) = e_1$. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{\tilde{A}}} \\ 0 & & \end{pmatrix}$

$\tilde{A} = [f/_{e_1^{\perp}}]_{R', R'}$

$\det \tilde{A} = 1 \Rightarrow f/_{e_1^{\perp}}$ = rotație $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

b) $\lambda = -1 \Rightarrow f(e_1) = -e_1$ $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\det \tilde{A} = -1 \Rightarrow f/_{e_1^{\perp}}$ = simetrie.

$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [f]_{R,R}$ $R = \{e_1, e_2, e_3\}$

$\tilde{R} = \{e_3, e_1, e_2\}$ $[f]_{\tilde{R},\tilde{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix}$

<u>Teorema</u> $\det A = 1$ (f speța 1) $\overset{\exists R \text{ reper ortonormat a.î.}}{\Longrightarrow}$ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} = [f]_{R,R}$.

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1, x_2\cos\varphi - x_3\sin\varphi, x_2\sin\varphi + x_3\cos\varphi)$

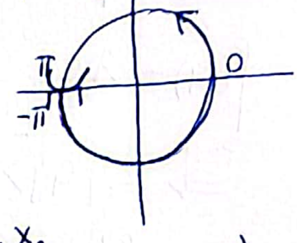
$\boxed{\mathrm{Tr} A = 1 + 2\cos\varphi}$ invariant la sch. de reper.

a) $\det A = 1$ (f este transf. ortog. de speță I)

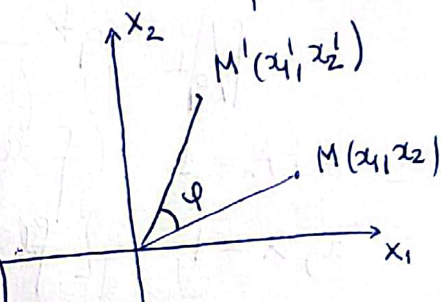
$$A_\varphi = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} \qquad A_\varphi \cdot A_\varphi^T = I_2.$$

$$f:\mathbb{R}^2\to\mathbb{R}^2,\ f(x_1,x_2) = (\overbrace{x_1\cos\varphi - x_2\sin\varphi}^{x_1'},\ \overbrace{x_1\sin\varphi + x_2\cos\varphi}^{x_2'})$$

$f = R_\varphi$ rotație de unghi φ, $\varphi \in [-\pi, \pi)$



$Tr\, A_\varphi = 2\cos\varphi$ invariant la sch. de reper.



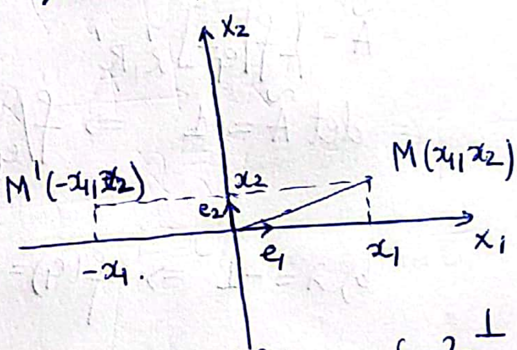
b) $\det A = -1$ $\quad A = \begin{pmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{pmatrix}$

$\exists$ un reper ortonormat a.î. $A = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$.

$f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x_1,x_2) = (-x_1, x_2)$

f este simetria ortogonală față de $e_1^\perp$

$f = s$



$E = \mathbb{R}^2 = \langle\{e_1\}\rangle \oplus \langle\{e_1\}\rangle^\perp$

Teoremă $\dim E = 2$, $f \in O(E)$, $f \neq id_E$

$\Rightarrow f$ se poate scrie ca o „$\circ$" de cel mult 2 simetrii ortogonale față de drepte

Dem $f \in O(E)$, $R = \{e_1, e_2\}$ reper ortonormat, $A = [f]_{R,R} \in O(2)$

1) $\det A_f = 1 \Rightarrow f =$ rotație

Fie s o simetrie ortog. $\Rightarrow \det(A_s) = -1$

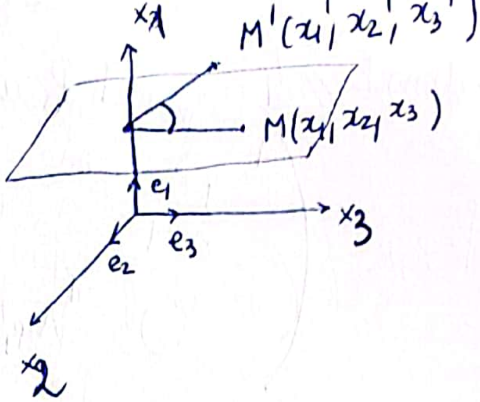
$$A_{s\circ f} = A_s \cdot A_f \Rightarrow \det(A_{s\circ f}) = -1 \Rightarrow s\circ f = s'$$

$$s\circ f = s\circ s' \Rightarrow f = s\circ s'$$

$$\underbrace{s\circ s}_{id}\circ f = s\circ s' \Rightarrow f = s\circ s'$$

2) $\det A_f = -1 \Rightarrow f = s$

f = rotație de $\measuredangle \varphi$ în planul $e_1^\perp$ și de axă $\langle e_1 \rangle$



$$\boxed{\text{Axa: } f(x) = x.}$$

II. $\det A = -1$ (f de speță 2)

a) $\lambda = 1 \Rightarrow f(e_1) = e_1$. $\quad A = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{A}$}} \\ 0 & & \end{array}\right)$

$\det \tilde{A} = -1 \Rightarrow f/_{e_1^\perp} = s$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [f]_{R,R}$$

$R = \{e_1, e_2, e_3\}$

$R' = \{e_2, e_1, e_3\}$ $\quad [f]_{R',R'} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 0 & -\sin 0 \\ 0 & \sin 0 & \cos 0 \end{pmatrix}$

b) $\lambda = -1 \Rightarrow f(e_1) = -e_1$ $\quad A = \left(\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{A}$}} \\ 0 & & \end{array}\right)$

$\det \tilde{A} = 1 \Rightarrow f/_{e_1^\perp} = R_\varphi$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

Teoremă $\det A = -1$ (f de speță 2)

$\exists R$ un reper ortonormat a.î. $A = [f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

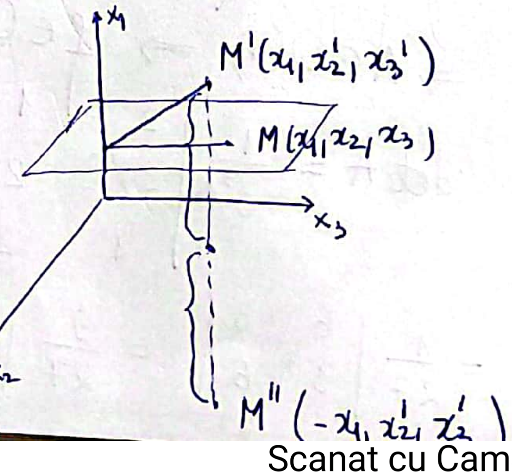
$$\boxed{\mathrm{Tr} A = -1 + 2\cos\varphi}$$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (-x_1, \overbrace{x_2\cos\varphi - x_3\sin\varphi}^{x_2'}, \overbrace{x_2\sin\varphi + x_3\cos\varphi}^{x_3'})$

$f = s \circ R_\varphi$

R_φ rotație de $\measuredangle \varphi$ în $e_1^\perp$, cu axa $\langle e_1 \rangle$; $\boxed{\text{Axa: } f(x) = -x}$

s = simetria ortogonală față de $e_1^\perp$



④ $\dim E \geq 4$ $\Rightarrow \exists\ \mathcal{R}$ un reper ortonormat aî

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & \boxed{A_{\varphi_1}} & \\ & & & & & & & \ddots \ \boxed{A_{\varphi_k}} \end{pmatrix}$$

(1 de r ori, -1 de s ori; restul O)

$n = r + s + 2k$

<u>Teorema Cartan</u> (E, g) s.v.e.r. $n \geq 2$, $f \in O(E) \setminus \{id_E\}$

$\Rightarrow f$ se poate scrie ca o "$\circ$" de cel mult n simetrii ortogonale față de hiperplane.

<u>Exemplu</u> $(\mathbb{R}^3, g_0)$ $g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

$f : \mathbb{R}^3 \to \mathbb{R}^3$ $f(x_1, x_2, x_3) = \frac{1}{3}(2x_1 + x_2 - 2x_3, -2x_1 + 2x_2 - x_3, x_1 + 2x_2 + 2x_3)$

a) $f \in O(\mathbb{R}^3)$

b) Să se det. $\mathcal{R} = \{e_1, e_2, e_3\}$ reper ortonormat în $\mathbb{R}^3$ aî

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

<u>SOL</u>

$\mathcal{R}_0 = \{\overset{(1,0,0)}{e_1}, \overset{(0,1,0)}{e_2}, \overset{(0,0,1)}{e_3}\}$ reperul canonic este un reper ortonormat

$$A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix} \qquad f(x) = y \Leftrightarrow Y = AX$$

$$A \cdot A^T = \frac{1}{9}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I_3$$

$\Rightarrow A \in O(3) \Rightarrow f \in O(\mathbb{R}^3)$

$C_1' = C_1 - 2C_2$, $C_3' = C_3 + 2C_2$

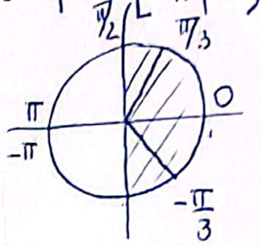
$$\det A = \frac{1}{3^3}\begin{vmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = \frac{1}{3^3}\begin{vmatrix} 0 & 1 & 0 \\ -6 & 2 & 3 \\ -3 & 2 & 6 \end{vmatrix}$$

$$= \frac{1}{27}\begin{vmatrix} 6 & 3 \\ 3 & 6 \end{vmatrix} = \frac{9}{27}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \frac{1}{3} \cdot 3 = 1 \Rightarrow f \text{ de speța } 1$$

b) $\operatorname{Tr} A = \frac{6}{3} = 2 = 2\cos\varphi + 1 \Rightarrow 2\cos\varphi = 1 \Rightarrow \cos\varphi = \frac{1}{2} \Rightarrow$

dar $\varphi \in [-\pi, \pi)$

$\varphi \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$



Axa de rotatie: $f(x) = x \Leftrightarrow AX = X \Leftrightarrow (A - I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} 2x_1 + x_2 - 2x_3 = 3x_1 \\ -2x_1 + 2x_2 - x_3 = 3x_2 \\ x_1 + 2x_2 + 2x_3 = 3x_3 \end{cases} \Rightarrow \begin{cases} -x_1 + x_2 - 2x_3 = 0 \\ -2x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases} \qquad B = \begin{pmatrix} \boxed{\begin{matrix} -1 & 1 \\ -2 & -1 \end{matrix}} & \begin{matrix} -2 \\ -1 \end{matrix} \\ \begin{matrix} 1 & 2 \end{matrix} & -1 \end{pmatrix}$$

$$\begin{vmatrix} -1 & 0 & 0 \\ -2 & -3 & 3 \\ 1 & 3 & -3 \end{vmatrix} = 0 \qquad \Delta_p = \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} \neq 0$$

$$\begin{cases} -x_1 + x_2 = 2x_3 \\ -2x_1 - x_2 = x_3 \end{cases}$$

$-3x_1 \quad / \quad = 3x_3$

$x_1 = -x_3$

$x_2 = 2x_3 - x_3 = x_3$

$\{(-x_3, x_3, x_3) \mid x_3 \in \mathbb{R}\}$.

$(-x_3, x_3, x_3) = x_3(-1, 1, 1)$

$e_1 = \frac{1}{\sqrt{3}}(-1, 1, 1)$ versorul axei

$e_1^{\perp} = \{x \in \mathbb{R}^3 \mid g_0(x, e_1) = 0\} = \{x \in \mathbb{R}^3 \mid -x_1 + x_2 + x_3 = 0\}$

$x_1 = x_2 + x_3$

$e_1^{\perp} = \{(x_2 + x_3, x_2, x_3) = x_2(1,1,0) + x_3(1,0,1) \mid x_2, x_3 \in \mathbb{R}\}$

$\{f_2 = (1,1,0), f_3 = (1,0,1)\}$ reper în $e_1^{\perp}$

Aplicăm procedeul de ortogonalizare Gram-Schmidt.

$e_2' = f_2 = (1,1,0)$, $\quad e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\sqrt{2}}(1,1,0)$

$e_3' = f_3 - \frac{\langle f_3, e_2' \rangle}{\langle e_2', e_2' \rangle} \cdot e_2' = (1,0,1) - \frac{1}{2}(1,1,0) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{2}{2}\right)$

$e_3' = \frac{1}{2}(1,-1,2)$, $\quad e_3 = \frac{e_3'}{\|e_3'\|} = \frac{1}{\sqrt{6}}(1,-1,2)$

OBS $v = \alpha v'$, $\alpha > 0$

$\frac{v}{\|v\|} = \frac{v'}{\|v'\|}$

$\left\{e_2 = \frac{1}{\sqrt{2}}(1,1,0), e_3 = \frac{1}{\sqrt{6}}(1,-1,2)\right\}$ reper ortonormat în $e_1^{\perp}$

$\mathcal{R}=\{e_1, e_2, e_3\}$ reperul ortonormat în rap cu care

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \Bigg/ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & 1/2 \end{pmatrix}$$