Seminar 11 A-G

Spații vectoriale. Transformări ortogonale

(4) (S11) $(\mathbb{C}, +, \cdot)/_{\mathbb{R}}$, $g : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ mat. asociată g în raport cu $\mathcal{R}_0 = \{1, i\}$

(a) $(\mathbb{C}, g)$ s.v.e.r.?

g prod scalar?

(b) $u = 2 - i$ versor în raport cu g?

(c) $u^{\perp}$

(d) Să se ortonormeze $\mathcal{R}_0$ în rap cu g

(e) Să se afle intersecția dintre cercul unitate în $(\mathbb{C}, g_0)$ și în $(\mathbb{C}, g)$

(a) $z = x_1 + x_2 i$; $z' = y_1 + y_2$

$g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$g(z, z') = ((x_1, x_2), (y_1, y_2)) = x_1 y_1 + 2 x_1 y_2 + 2 y_1 x_2 + 5 x_2 y_2$

din ipoteza formă biliniară (1)

$G = G^T \Rightarrow g$ simetrică (2)

$Q(z) = Q(x_1, x_2) = x_1^2 + 4 x_1 x_2 + 5 x_2^2 =$

$= (x_1 + 2x_2)^2 + x_2^2 = x_1'^2 + x_2'^2$

$x_1' = x_1 + 2x_2$

$x_2' = x_2^2$ $\qquad S(3, 0)$

$\Rightarrow g$ este strict pozitiv (3)

Din (1), (2) și (3) $\Rightarrow g$ este produs scalar

(a) $g_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$g_0((x_1, x_2); (y_1, y_2)) = x_1 y_1 + x_2 y_2$

Produsul scalar canonic

T.

$\|(x_1, x_2)\| = \sqrt{g((x_1, x_2)(x_1, x_2))}$ $= y_1 + i y_2$

$= \sqrt{Q(x_1, x_2)}$

$Q((x_1, x_2)) = x_1^2 + 4x_1x_2 + \cancel{5}x_2^2$

$g((x_1, x_2), (y_1, y_2)) =$

(b) $Q_0(2, -1) = 2^2 + 1 = 5$

$\|u\|_g^2 = Q(u) = Q(2, -1) = 0 + 1 = 1 \Rightarrow u$ este versor în raport cu g

dacă îți dă $1 \Rightarrow$ e versor

(c) $u^\perp = \left\{ y \in \mathbb{R}^2 \mid g(u, y) = 0 \right\} \equiv \mathbb{R}$; $u^\perp = \{(y_1, 0) \mid y_1 \in \mathbb{R}\}$

$u = (2, -1)$, $y = (y_1, y_2)$

$2y_1 + 2 \cdot 2 y_2 + 2 \cdot (-1) y_1 + 5(-1) y_2$

$= 2y_1 + 4y_2 - 2y_1 - 5y_2 = -y_2 = 0$

$\mathbb{R}^2 = \langle\{u\}\rangle \oplus \mathbb{R} = \langle\{u\}\rangle^\perp$

d) Gram-Schmidt

$f_1 = (1,0)$, $g(f_2, e_1) = g((0,1),(1,0)) = 2$

$f_2 = (0,1)$

$e_1 = f_1 = (1,0)$

$$e_2 = f_2 - \frac{g(f_2,e_1)}{g(e_1,e_1)} e_1 = (0,1) - \frac{2}{1}(1,0) = (-2,1)$$

$\|e_1\| = \sqrt{g(e_1,e_1)} = 1$ $\quad\quad$ $\{e_1, \underset{-u}{\underset{\|}{e_2}}\}$ reper ortonormat în $\overset{\text{rap. cu}}{g}$

$\|e_2\| = \sqrt{g(e_2,e_2)} = 1$

e) $\overset{\mathbb{R}^2}{\underset{go}{\overset{V_1}{S}}} = \{z \in \mathbb{C} \mid |z| = 1\} = \{(x_1,x_2) \in \mathbb{R}^2 \mid g_0((x_1,x_2),(x_1,x_2)) = 1\}$

$Q_0((x_1,x_2)) = x_1^2 + x_2^2$

$$S_g^1 = \{(x_1,x_2) \in \mathbb{R}^2 \mid \underset{\underset{x_1^2 + 4x_1x_2 + 5x_2^2 = 1}{\cap}}{Q}(x_1,x_2) = 1\}$$

$$\begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^2 + 4x_1x_2 + 5x_2^2 = 1 \end{cases}$$

$4x_1x_2 + 4x_2^4 = 0$

$x_1x_2 + x_2^4 = 0$

$x_2(x_1 + x_2) = 0$

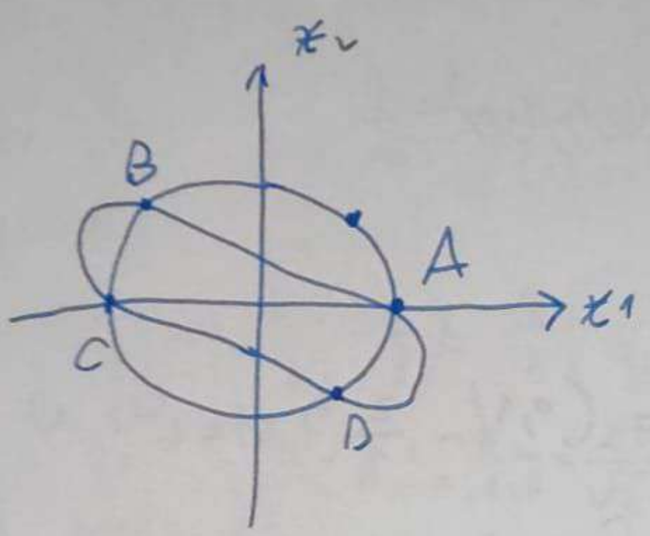
I $x_2 = 0 \Rightarrow x_1^2 = 1$

$\Rightarrow x_1 = \pm 1$

II $x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$x_2^2 + x_2^2 = 1 \Rightarrow x_2 = \pm \frac{1}{\sqrt{2}}$

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$



$S^1_{g_0} \cap S^1_g = \{A(1,0), C(-1,0), B(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), D(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\}$

Există mai mulți produși scalari

(14) S10 $(M_2(\mathbb{R}), g)$, $g(A,B) = tr(A^T \cdot B)$ $\forall A, B \in M_2(\mathbb{R})$

a) g prod scalar

b) $\mathcal{R} = \left\{ \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Să se ortonormeze

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \longrightarrow (x_1, x_2, x_3, x_4)$$

$$B = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

$$A^T \cdot B = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1 y_1 + x_3 y_3 & x_1 y_2 + x_3 y_4 \\ x_2 y_1 + x_4 y_3 & x_2 y_2 + x_4 y_4 \end{pmatrix}$$

$$tr(A^T \cdot B) = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

$$g(A,B) = g_0((x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4))$$

$f_1 = (1,0,2,1),\ f_2 = (0,-1,1,0),\ f_3 = (1,2,1,0),\ f_4 = (0,0,0,1)$

$e_1 = f_1$

$e_2 = f_1 - \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = (0,-1,1,0) - \dfrac{2}{6}(1,0,2,1) =$

$= \left(-\dfrac{1}{3}, -1, \dfrac{1}{3}, -\dfrac{1}{3}\right) = \dfrac{1}{3}(-1,-3,1,-1)$

$e_1' = \dfrac{e_1}{\|e_1\|} = \dfrac{e_1}{\sqrt{6}},\ e_2' = \dfrac{e_2}{\|e_2\|} = \dfrac{e_2}{\sqrt{2}}$

$e_3 = f_3 - \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \dfrac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 = (1,2,1,0) - \dfrac{1}{2}(1,0,2,1) -$

$- \dfrac{\frac{1}{3}(-1-6+1)}{\frac{1}{9}\cdot 12} \cdot \dfrac{1}{3}(-1,3,1,-1) = (1,2,1,0) - \dfrac{1}{2}(1,0,2,1) +$

$\dfrac{1}{2}(-1,-3,1,-1) = \dfrac{1}{2}(0,1,1,-2)$

$e_4 = f_4 - \dfrac{\langle f_4 e_1\rangle}{\langle e_1 e_1\rangle} e_1 - \dfrac{\langle f_3 e_2\rangle}{\langle e_2 e_2\rangle} e_2 - \dfrac{\langle f_2 e_3\rangle}{\langle e_3 e_3\rangle} e_3$

$e_4 = (0,0,0,1) - \left(\dfrac{1}{6}, 0, \dfrac{1}{3}, \dfrac{1}{6}\right) + \left(-\dfrac{1}{12}, -\dfrac{3}{12}, \dfrac{1}{12}, -\dfrac{1}{12}\right) + \left(0, \dfrac{1}{3}, \dfrac{1}{3}, \dfrac{-2}{3}\right) =$

$= \left(-\dfrac{1}{4}, \dfrac{1}{12}, \dfrac{1}{12}, \dfrac{1}{12}\right) = \dfrac{1}{12}(-3,1,1,1)$

$\cancel{e_3' = \dfrac{e_3}{\|e_3\|} = \dfrac{\frac{1}{2}(0,1,1,-2)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}} = \dfrac{\frac{1}{2}(0,1,1,-2)}{\sqrt{\frac{6}{4}}}}$

$e_3' = \dfrac{e_3}{\|e_3\|} = \dfrac{(0,1,1,-2)}{\sqrt{6}}$

$e_4' = \dfrac{1}{\sqrt{12}}(-3,1,1,1)$

Ex 1) $(\mathbb{R}^3, g_0)$ s.v.e., cu str. euclidiană canonică

$f \in End(\mathbb{R}^3)$, $A = [f]_{R_0, R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$

R_0 = reperul canonic

a) Să se arate că $f \in O(\mathbb{R}^3)$, de speța 2 i.e. $f = s \circ R_\varphi$

b) Să se det $\sphericalangle$ rotației și axa de simetrie

c) Să se det un reper $R = \{e_1, e_2, e_3\}$ ortonormat a.î.

$[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} = A'$

(a) dem că $A \cdot A^T = I_3$ (ortogonală)

și $\det(A) = -1$

$\frac{1}{9^2}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix} = \frac{1}{9^2}\begin{pmatrix} 64+1+16 & 8+8-16 & -32+4+28 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \Rightarrow A$ matrice ortogonală

$A \in O(3)$

$\det(A) = \begin{vmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{vmatrix} = -7 \cdot 64 - 16 - 16 - 64 - 64 + 7$

$= (-14)32 - 32 \cdot 5 + 7$

$= -19 \cdot 32 + 7 = -1$?!?!

$\Rightarrow f$ este speța 2

$$\varphi \in [-\pi, \pi)$$

(b) $tr\, A = tr\, A'$

$\left(\frac{1}{9}\right)(16 - 7) = -1 + 2\cos\varphi$; $\cos\varphi = 1 \Rightarrow \varphi = 0$

$f(x) = -x$

$AX = -X$

$(A + I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} 8x_1 + x_2 - 4x_3 = -9x_1 \\ x_1 + 8x_2 + 4x_3 = -9x_2 \\ -4x_1 + 4x_2 - 7x_3 = -9x_3 \end{cases} \Rightarrow \begin{cases} 17x_1 + x_2 - 4x_3 = 0 \\ x_1 + 17x_2 + 4x_3 = 0 \\ -4x_1 + 4x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 17 & 1 & -4 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{pmatrix} = 2 \begin{vmatrix} 17 & 1 & -2 \\ 1 & 17 & 2 \\ -4 & 4 & 1 \end{vmatrix} = 0$$

$$\begin{cases} x_1 + 17x_2 = -4x_3 & \Big| \cdot 4 \\ -4x_1 + 4x_2 = -2x_3 \\ \quad 4 \cdot 18x_2 = -18x_3 & \Big| :18 \end{cases}$$

$4x_2 = -x_3$

$x_2 = -\frac{1}{4}x_3 \Rightarrow 4x_1 - x_3 = -2x_3$

$-4x_1 = -1x_3$

$x_1 = \frac{1}{4}x_3$

versorul axei

$\Rightarrow \left(x_3 \frac{1}{4}, -\frac{1}{4}x_3, x_3\right) = \frac{x_3}{4}(1, -1, 4)$ $\quad e_1 = \frac{1}{3\sqrt{2}}(1, -1, 4)$

am aflat e_1 din bază e_2, e_3 se află din Gram-Schmidt

dar mai întâi

$$e_1^{\perp} = \{x \in \mathbb{R}^3 / x_1 - x_2 + 4x_3 = 0\}$$

$$= \{(x_1, x_1 + 4x_3, x_3) \mid x_1, x_3 \in \mathbb{R}\}$$

$\underset{f_2}{\underset{\shortparallel}{(1,1,0)}}, \underset{f_3}{\underset{\shortparallel}{(0,4,1)}}$ $\{f_2, f_3\}$ bază în $e_1^{\perp}$

$e_2 = f_2$

$e_3 = f_3 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 = (0,4,1) - \frac{4}{2}(1,1,0) = (-2,2,1)$

$e_2' = \frac{1}{\sqrt{2}}(1,1,0)$

$e_3' = \frac{1}{3}(-2,2,1) \Rightarrow \mathcal{R} = \{e_1', e_2', e_3'\}$ bază ortonormată

$$A' = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$