Seminar Analiză 12

Să se af extremele funcției $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = ax + by + cz$, $a^2 + b^2 + c^2 \neq 0$ pe

PAS 1) $A = \{x^2 + y^2 + z^2 = d^2\}$; $d > 0$

$g(x, y, z) = x^2 + y^2 + z^2 + d^2$

TmL

I $f, g \in C^1$ (polinomiale)

II rang g' = max

$g' = (2x, 2y, 2z) \Rightarrow$ rg $g = 1$

pt rang $g' = 0 \Rightarrow x = y = z = 0 \notin A$

III (x_0, y_0, z_0) punctele extrem pt fiecare pe A

$\Longrightarrow \lambda \in \mathbb{R}$ ai $h'_\lambda(x_0, y_0, z_0) = 0$

unde $h_\lambda = f + \lambda g = ~~ax~~ + by$

$$\frac{\partial h_\lambda}{\partial x} = a + 2\lambda x = 0 \Rightarrow x = -\frac{a}{2\lambda}$$

$$\frac{\partial h_\lambda}{\partial y} = b + 2\lambda y = 0 \Rightarrow y = -\frac{b}{2\lambda}$$

$$\frac{\partial h_\lambda}{\partial z} = c + 2\lambda z = 0 \Rightarrow z = -\frac{c}{2\lambda}$$

$$\frac{a^2}{4\lambda^2} + \frac{b^2}{4\lambda^2} + \frac{c^2}{4\lambda^2} = d^2$$

$$\frac{1}{4\lambda^2} = \frac{d^2}{a^2 + b^2 + c^2}$$

$$\frac{1}{2\lambda} = \pm \frac{d}{\sqrt{a^2 + b^2 + c^2}} \to \alpha$$

$\left.\begin{array}{l} A \text{ - mărginită închisă} \\ f \text{ continuă} \end{array}\right\} \Rightarrow$

1 pct $<$ max / min global

$$\boxed{x = \mp \frac{a\alpha}{\alpha}} \quad f\left(\mp \frac{a\alpha}{\alpha}, \mp \frac{b\alpha}{\alpha}, \mp \frac{c\alpha}{\alpha}\right) = \mp\left(\frac{a^2\alpha}{\alpha} + \frac{b^2\alpha}{\alpha} + \frac{c^2\alpha}{\alpha}\right)$$

$$= \mp \frac{\alpha}{\alpha}\left(a^2+b^2+c^2\right) = \mp \frac{\alpha}{\alpha}\cdot \alpha^2 = \mp \alpha\alpha$$

$$\boxed{y = \mp \frac{b\alpha}{\alpha}}$$

$$\boxed{z = \mp \frac{c\alpha}{\alpha}}$$

$\left(-\frac{a\alpha}{\alpha}, -\frac{b\alpha}{\alpha}, -\frac{c\alpha}{\alpha}\right)$ pct min. global

$\left(\frac{a\alpha}{\alpha}, \frac{b\alpha}{\alpha}, \frac{c\alpha}{\alpha}\right)$ pct max

<u>OBS</u> $|f(\cancel{x}, y, z)| = |ax + by + cz| \leq \alpha\alpha = \sqrt{x^2+y^2+z^2} \cdot \sqrt{a^2+b^2+c^2}$

Să se determine extremele locale ale funcției

$f : \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = xyz$ pe $A = \{x+y+z = a \mid a > 0\}$

$g(x,y,z) = x+y+z-a$

1) $f, g \in C^1$
$\in C^\infty$

2) rang g' max

$g' = (1,1,1)$

3) (x_0, y_0, z_0) extrem local pt f pe A

$\Rightarrow \exists\, \lambda \in \mathbb{R}$ a.î.

$h'_\lambda(x_0, y_0, z_0) = 0$ unde

$h_\lambda = f + \lambda g$

$= xyz + \lambda(x+y+z-a)$

$\frac{\partial h_\lambda}{\partial x} = yz + \lambda = 0 \quad \Rightarrow z(y-x) = 0 \Rightarrow z = 0$ sau $x = y$

$\frac{\partial h_\lambda}{\partial y} = xz + \lambda = 0 \quad \Rightarrow x = 0$ sau $z = y$

$\frac{\partial h_\lambda}{\partial z} = xy + \lambda = 0 \quad \Rightarrow y = 0$ sau $x = z$

$c_1 \; x = y = z \Rightarrow 3x = a \Rightarrow x = y = z = \frac{a}{2} \quad f\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right) = \frac{a^3}{27}$

$c_2 \; x = y \neq z \Rightarrow x = y = 0 \neq z \quad f(0,0,0) = 0$

$z \neq 0 \qquad f(x,y,0) = 0$

$x = y \neq z = 0 \quad x = y = \frac{a}{2} \quad z = 0$

$d^2 h_\lambda'' = 0 \cdot d^2x + 2\frac{a}{3} dx\,dy + \frac{2a}{3} dx\,dz + 0\,dy^2 +$
$+ 2\frac{a}{3} dy\,dz + 0 \cdot d^2 z = \frac{2a}{3}(dx\,dy + dx\,dz + dy\,dz) =$

Să se determine extremele funcției $f(x,y,z) = ax + by + cz$ a.î. $a^2 > b^2 + c^2$ pe mulțimea $A = \{x^2 - y^2 - z^2 = 1\}$

$g(x,y,z) = x^2 - y^2 - z^2 - 1$

1) $f, g \in C^1$

2) rang $g' = 1$

$g' = (2x, -2y, 2z)$

↑↑ că rang $g' = 0 \Rightarrow x = y = z = 0 \notin A$

3) (x_0, y_0, z_0) este un pct de extrem pe f pe A

$\Rightarrow \exists \lambda$ a.î. $h'_\lambda(x_0, y_0, z_0) = 0$ unde $h_\lambda = f + \lambda g = ax + by + cz + \lambda(x^2 - y^2 - z^2 - 1)$

$$x = \mp \frac{a}{\alpha}\ ;\ y = \pm \frac{b}{\alpha}\ ;\ z = \pm \frac{c}{\alpha}$$

$$\frac{\partial h_\lambda}{\partial x} = a + 2\lambda x = 0 \Rightarrow x = -\frac{a}{2\lambda}$$

$$h''_\lambda = \begin{array}{c|ccc} & pt\,x & pt\,y & pt\,z \\ x'' & 2\lambda & 0 & 0 \\ y'' & 0 & -2\lambda & 0 \\ z'' & 0 & 0 & -2\lambda \end{array}$$

$$\frac{\partial h_\lambda}{\partial y} = b - 2\lambda y = 0 \Rightarrow y = \frac{b}{2\lambda}$$

$$\frac{\partial h_z}{\partial z} = c - 2\lambda z = 0 \Rightarrow z = \frac{c}{2\lambda}$$

$$x^2 - y^2 - z^2 = 1$$

$$\frac{a^2}{4\lambda^2} - \frac{b^2}{4\lambda^2} - \frac{c^2}{4\lambda^2} = 1 \quad \frac{\alpha^2}{4\lambda^2} = 1 \Rightarrow \frac{1}{2\lambda} = \pm \frac{1}{\alpha}$$

$$\sqrt{a^2 - b^2 - c^2} = \alpha$$

$$d^2 h_\lambda = 2\lambda\, dx^2 - 2\lambda\, dy^2 - 2\lambda\, dz^2 = 2\lambda(dx^2 - dy^2 - dz^2)$$

$$x^2 - y^2 - z^2 - 1 = 0$$

$$dx - y\,dy - z\,dz = 0$$

$$a\,dx + b\,dy + c\,dz = 0$$

$$dx = -\frac{b\,dy + c\,dz}{a}$$

$$= 2\lambda\left(\left(\frac{b\,dy + c\,dz}{a}\right)^2 - dy^2 - dz^2\right)$$

$$= 2\lambda\left(\frac{b^2 dy^2 + 2b\,dy\,c\,dz + c^2 dz^2}{a^2} - dy^2 - dz^2\right)$$

$$= 2\lambda\left(\left(\frac{b^2}{a^2} - 1\right)dy^2 + \frac{2bc}{a^2}\,dy\,dz + \left(\frac{c^2}{a^2} - 1\right)dz^2\right)$$

$$= 2\lambda\left(\left(\frac{b^2}{a^2} - 1\right)t^2 + \frac{2bc}{a^2}\,t + \left(\frac{c^2}{a^2} - 1\right)1\right)$$

$$\Delta = \frac{4b^2c^2}{a^2} - 4\left(\frac{b^2}{a^2} - 1\right)\left(\frac{c^2}{a^2} - 1\right)$$

$$= 4\left(1 - \frac{b^2}{a^2} - \frac{c^2}{a^2}\right) > 0 \text{ punct extrem}$$