Seminar
Analiză 6

$f: \mathbb{R} \to \mathbb{R} \quad f(x) = \begin{cases} x^3 + 4x, & x \in A \\ 4x^2, & x \notin A \end{cases}$

1) $A = [0,2]$; 2) $A = \mathbb{Q}$; 3) $A = \left\{ \frac{2}{n} \middle| n \geq 1 \right\}$

1) $\lim\limits_{\substack{x \to 0 \\ x \in A}} f(x) = \lim\limits_{\substack{x \to 0 \\ x > 0}} x^3 + 4x = 0$

$\lim\limits_{\substack{x \to 0 \\ x \notin A}} f(x) = \lim\limits_{\substack{x \to 0 \\ x < 0}} 4x^2 = 0$

$\Rightarrow f$ cont în 0

$f(0) = \{0\}$

$\lim\limits_{\substack{x \to 2 \\ x < 2}} f(x) = \lim\limits_{\substack{x \to 2 \\ x < 2}} x^3 + 4x = 8 + 8 = 16$

$\lim\limits_{\substack{x \to 2 \\ x > 2}} f(x) = \lim\limits_{\substack{x \to 2 \\ x > 2}} x^3 + 4x = 8 + 8 = 16$

$\Rightarrow f$ cont în 2

$f(2) \in \{16\}$

$\lim\limits_{\substack{x \to 0 \\ x > 0}} \frac{f(x) - f(0)}{x - 0} = \frac{x^3 + 4x - 0}{x} = \lim\limits_{x \to 0} x^2 + 4 \to 4$

$\lim\limits_{\substack{x \to 0 \\ x < 0}} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{x^2 - 0}{x} \to 0$

$$\lim_{\substack{x \to 2 \\ x < 2}} \frac{f(x) - f(2)}{x - 2} = \lim_{\substack{x \to 2 \\ x < 2}} \frac{x^3 + 4x - 8 - 8}{x - 2} = 16$$

$$\lim_{\substack{x \to 2 \\ x > 2}} \frac{f(x) - f(2)}{x - 2} \to 16 \Rightarrow \text{e derivabilă în } 2$$

$$\mathring{A} = (0, 2)$$

$$\mathbb{R} \setminus \mathring{A} = (-\infty, 0) \cup (2, \infty)$$

$f(0,2) = x^3 + 4x \Rightarrow f$ continuă, derivabilă $(0,2)$

$A = \mathbb{Q} \Rightarrow \mathbb{Q}' = \mathbb{R} = (\mathbb{R} \setminus \mathbb{Q})' \ni a$ $\quad$ A' mulțimea punctelor de acumulare

$$\lim_{\substack{x \to a \\ x \in \mathbb{Q}}} f(x) = \lim_{x \to a} x^3 + 4x = a^3 + 4a \Rightarrow f \text{ cont. } a \Leftrightarrow a^3 + 4a = 4a^2$$

ecuație de gradul 3

$\lim_{x \to a}$

f continuă pe $a \notin \{0,2\} \Rightarrow$ f o licc în $a \Rightarrow$

$a = 0$ $\lim\limits_{\substack{x \to a \\ x \notin \mathbb{Q}}} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{x^3 + 4x - 0}{x - 0} = 4$

$\lim\limits_{\substack{x \to 0 \\ x \in \mathbb{Q}}} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{4x^2 - 0}{x} \to 0$

$\lim\limits_{\substack{x \to 2 \\ x \in \mathbb{Q}}} \frac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \frac{x^3 + 4x - 8 + 8}{x - 2} \to 16$

$\lim\limits_{\substack{x \to 2 \\ x \in \mathbb{Q}}} \frac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \frac{4x^2 - 16}{x - 2} = 16$

$f(x) = \begin{cases} x^3, & x \in \mathbb{Q} \\ 3x^2, & x \notin \mathbb{Q} \end{cases}$

$\left.\begin{array}{l} \lim\limits_{\substack{x \to a \\ x \in \mathbb{Q}}} f(x) = \lim\limits_{x \to a} x^3 = a^3 \\ \lim\limits_{\substack{x \to a \\ x \notin \mathbb{Q}}} f(x) = \lim\limits_{x \to a} 3x^2 = 3a^2 \end{array}\right\} \Rightarrow$ f cont. $\Leftrightarrow a^3 = 3a^2$

$\Rightarrow a = \{0, 3\}$

$\lim\limits_{\substack{x \to 3 \\ x \in \mathbb{Q}}}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f_1(x,y) = \begin{cases} x^2, & x, y \in \mathbb{Q} \quad A = \mathbb{Q}^2 \\ 1-y^2, & \text{în rest} \quad B = \mathbb{R}^2 \setminus \mathbb{Q}^2 \end{cases}$

$f = (f_1, f_2)$ $\quad f_2(x,y) = \begin{cases} x^2+y^2, & x, y \in \mathbb{R}\setminus\mathbb{Q} \quad C = (\mathbb{R}\setminus\mathbb{Q})^2 \\ 2x, & \text{în rest} \quad D = \mathbb{R}^2 \setminus (\mathbb{R}\setminus\mathbb{Q})^2 \end{cases}$

$\lim\limits_{\substack{(x,y)\to(a,b) \\ (x,y)\in A}} f_1(x,y) \to a^2$

$\lim\limits_{\substack{(x,y)\to(a,b) \\ (x,y)\in B}} f_1(x,y) \to 1-b^2$

$\Rightarrow f_1$ cont $\Leftrightarrow a^2+b^2=1$

$f(x) = \begin{cases} x^3 & x \in A \\ x^2 & x \notin A \end{cases}$ $\quad A = \left\{\frac{1}{n} \mid n \geq 1\right\}$

$\mathring{A} = \emptyset$

$f|_{\mathbb{R}\setminus A} \Rightarrow f$ este cont și deriv pe $\mathbb{R}\setminus A$ deschisă

$A' = \{0\}$

$\bar{A} = A' \cup A = \{0\} \cup \left\{\frac{1}{n}\right\} = \text{Fr}(A)$

$\text{Iz}(A) = A \setminus A' = \left\{\frac{1}{n}\right\}$

$\text{Ext}(A) = \mathbb{R} \setminus \bar{A} = \mathbb{R}\setminus A = (-\infty, 0) \cup (1, +\infty) \cup \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $n \geq 1$

$$f: \mathbb{R}^2 \to \mathbb{R} \qquad f(x,y) = \begin{cases} \dfrac{x^4 y}{x^4+y^4} & , x^2+y^2 \neq 0 \\ 0 & , x=y=0 \end{cases}$$

$$\lim_{\substack{x\to 0 \\ y=ax}} \frac{x^4 y}{x^4+y^4} = \lim_{\substack{x\to 0 \\ y=ax}} \frac{x^4 ax}{x^4+a^4x^2} = \lim_{x\to 0} \frac{ax}{1+a^4} = 0$$

$$|f(x,y)-f(0,0)| = \left|\frac{x^4 y}{x^4+y^4} - 0\right| = \left|\frac{x^4}{x^4+y^4}\right| \cdot |y| \leq |y| \underset{\substack{x\to 0 \\ y\to 0}}{\to} 0$$

$a = 0$

$a \in A'$, $a \in (\mathbb{R} \setminus A)'$

$\lim\limits_{\substack{x \to 0 \\ x \in A}} f(x) = \lim\limits_{x \to 0} x^3 = 0^3$

$\lim\limits_{\substack{x \to 0 \\ x \notin A}} f(x) = \lim\limits_{x \to 0} x^2 = 0$

$f(0) = 0$

$\Rightarrow f$ cont în 0

$\lim\limits_{\substack{x \to 0 \\ x \in A}} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{x^3 - 0}{x - 0} = \lim\limits_{x \to 0} 3x^2 = 0$

$\lim\limits_{\substack{x \to 0 \\ x \notin A}} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{x^2 - 0}{x} = \lim\limits_{x \to 0} x = 0$

$\Rightarrow f$ deriv în $0 \Rightarrow f'(0) = 0$

$a = \frac{1}{n}$ $\quad a \notin A'$

$a \in (\mathbb{R} \setminus A)'$

$\lim\limits_{\substack{x \to a \\ x \notin A}} f(x) = \lim\limits_{x \to a} x^2 = a^2$

$f(a) = a^3$

$\Rightarrow$ f cont. în $a \Leftrightarrow a^2 = a^3$

$\Leftrightarrow a^2(a-1) = 0$

$a \in \{\cancel{0}, 1\}$

$a = \frac{1}{n}$ $n > 1 \Rightarrow f$ e discontinuă în a

$a = 1$

$$\lim_{\substack{x \to 1 \\ x \notin A}} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

f deriv în 1

$$f(x) = \begin{cases} \frac{1}{1 + \left[\frac{1}{x}\right]^2}, & x > 0 \\ x^4, & x \le 0, x \in \mathbb{Q} \\ x^2, & x < 0, x \notin \mathbb{Q} \end{cases}$$

$x > 0$

$\frac{1}{x} > 0$

$\left[\frac{1}{x}\right] = n$

$n \le \frac{1}{x} < n + 1$

$\frac{1}{n+1} < x \le \frac{1}{n}$ $\quad f(x) = \frac{1}{1 + n^2}$

$f\left(\left(\frac{1}{n+1}, \frac{1}{n}\right)\right) = \frac{1}{1 + n^2} \Rightarrow f$ e cont. și oblive.

pe $\left(\frac{1}{n+1}, \frac{1}{n}\right]$ pt $\forall n \in \mathbb{N}$

$$\left.\begin{aligned} &\lim_{\substack{x \to \frac{1}{n} \\ x > \frac{1}{n}}} f(x) = \frac{1}{1 + (n-1)^2} \quad x \in \left(\frac{1}{n}, \frac{1}{n-1}\right) \\ &\lim_{\substack{x \to \frac{1}{n} \\ x < \frac{1}{n}}} f(x) = \frac{1}{1 + n} \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \end{aligned}\right\} = 1$$

f e discontinuă în $\frac{1}{n}$