Curs 5 Analiză

Topologie

Def → O funcție $d : X \times X \to [0, \infty)$ a.î.

1) $d(x,y) = 0 \iff x = y$

2) $d(x,y) = d(y,x)$, $\forall x, y \in X$

3) $d(x,y) + d(y,z) \geq d(x,z)$ $\forall x, y, z \in X$ s.n. distanță

(X, d) s.n. spațiu metric

$B(a, r) = \{x \mid d(a,x) < r\}$

→ $x_n \to a$ $\forall \varepsilon > 0$ $\exists n_\varepsilon$

→ O mulțime $D \subset X$ s.n. deschisă dacă $\forall x \in D \Rightarrow r_x > 0$ a.î. $B(x, r_x) \subset D$

$D = \bigcup_{x \in D} B(x, r_x)$ $\quad \tau = \tau_d = \{D \subset X \mid D \text{ deschisă}\}$ - topologia asociată lui d

→ V vecinătate a lui a dacă $\exists \varepsilon > 0$ a.î. $B(a, \varepsilon) \subset V$;

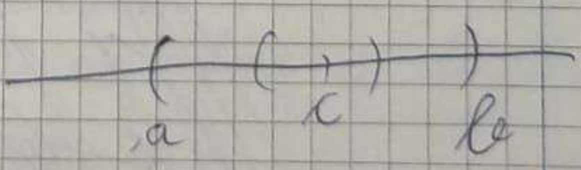
$\mathcal{V}_a = \{V \subset X \mid \exists \varepsilon > 0 \text{ a.î. } B(a, \varepsilon) \subset V\}$

→ O mulțime F s.n. închisă dacă $X \setminus F \in \tau$

$\mathcal{F} = \{F \subset X \mid F \text{ închisă}\}$

$(\mathbb{R}, d)$ $d(x,y) = |x - y|$ $\quad (a, b)$ $\quad c \in (a, b)$

$B(a, r) = (a - r, a + r)$



$r = \min(c - a, b - c)$

$B(c, r) \subset (a, b)$

$B(a,r) = (a-r, a+r)$

$$(a,b) = \bigcup_{x \in (a,b)} B(x, r_x)$$

Prop 1) $\emptyset, X \in \mathcal{T}_d$ $\quad X = \bigcup_{x \in X} B(x,1)$

2) $D_1, D_2 \in \mathcal{T}_d \Rightarrow D_1 \cap D_1 \in \mathcal{T}_d$

3) $(D_i)_{i \in J} \in \mathcal{T}_d \Rightarrow \bigcup_{i \in J} D_i \in \mathcal{T}_d$

Dem 2) $x \in D_1 \cap D_2 \Rightarrow x \in D_1, D_1 \in \mathcal{T}_d \Rightarrow \exists B(x, r_1) \subset D_1$

$x \in D_2, D_2 \in \mathcal{T}_d \Rightarrow \exists B(x, r_2) \subset D_2$

r - minimul dintre $r_1, r_2 > 0$

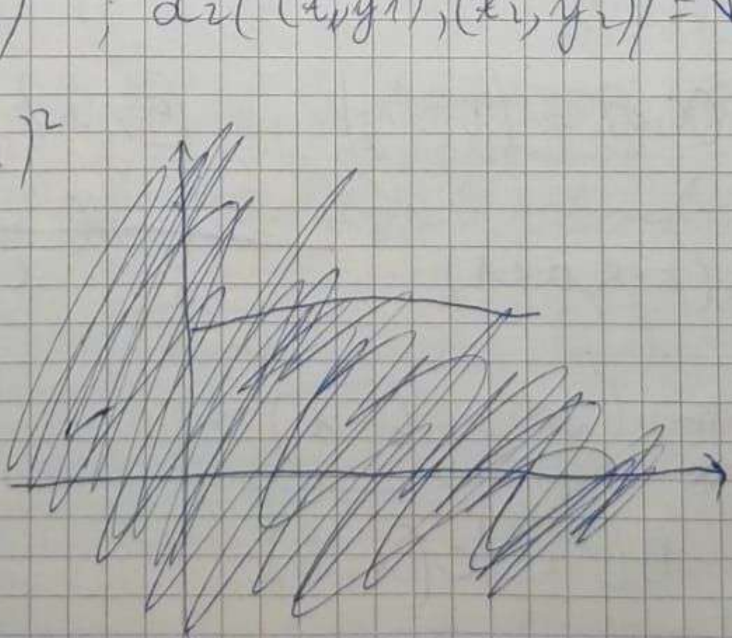
$\Rightarrow B(x,r) \subset D_1 \cap D_2$

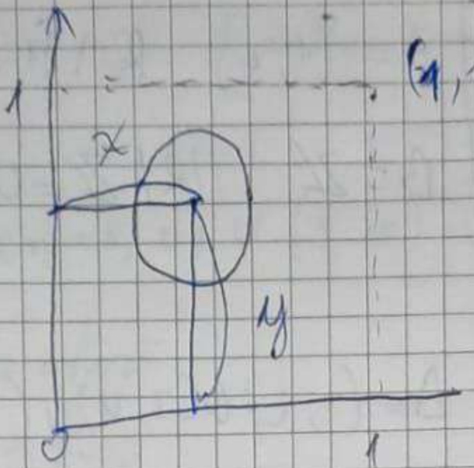
Dem 3) $x \in \bigcup_{i \in J} D_i \Rightarrow \exists i$ a.î. $x \in D_i \Rightarrow \exists r > 0$ a.î. $B(x,r) \subset D_i$

$B(x,r) \subset D_i \subset \bigcup_{i \in J} D_i$

Ex 1 $(\mathbb{R}^2, d_2)$; $d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$D = (0,1)^2$





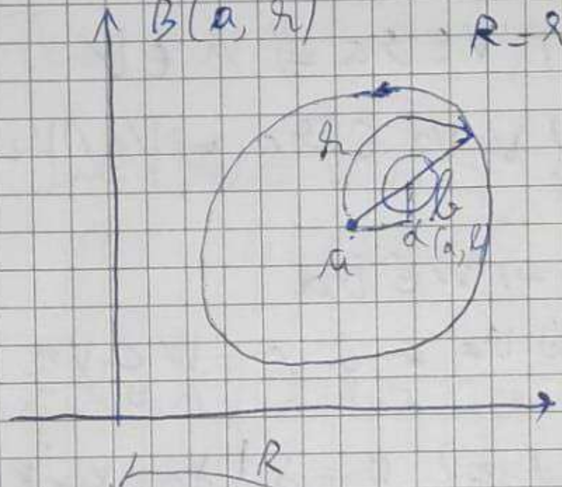
$B((a,b), r) =$

$r_{x,y} = \min(x, y, 1-x, 1-y)$

$B((x,y), r_{x,y}) \subset (0,1)^2$

$(0,1)^2 = \bigcup_{x,y \in (0,1)} B((x,y), r_{x,y})$

<u>Ex 2</u>



$R = r - d(a,b)$

$B(b,R) \subset B(a,r)$

$y \in B(b,R) \Leftrightarrow d(y,b) < R$

$y \in B(a,r) \Leftrightarrow d(a,y) < r$

$d(a,y) \leq d(a,b) + d(b,y) <$

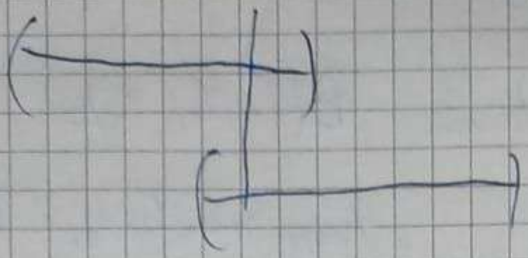
$< d(a,b) + r - d(a,b) = r$

<u>OBS</u> D este deschisă $\Leftrightarrow D = \bigcup_{i \in I} B(x_i, \alpha)$

$D \in \mathcal{T} \Rightarrow D = \bigcup_{x \in D} B(x, r_x)$

<u>Teoremă</u> O mulțime din $\mathbb{R}$ este deschisă $\Leftrightarrow$ se poate scrie ca o ~~reuniune~~ reuniune al mult numărabilă de intervale deschise și disjuncte.

$\left(D = \bigcup_{n \geq 1} J_n \right.$, unde $J_n = (a_n, b_n)$ și $\left. J_n \cap J_m = \emptyset \ \forall n \neq m \right)$



$A = [1,2]$ $\quad \mathbb{R} \setminus [1,2] = (-\infty, 1) \cup (2, \infty)$

$B = \mathbb{Z}$ $\quad \mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \geq 1} (n, n+1) \Rightarrow \mathbb{Z}$ închisă

deschisă

$D = (1,2) \cup (2,4) \cup (6,7)$

<u>Prop vecinătăți</u> 1) $V \in \mathcal{V}_a \Rightarrow a \in V$

2) $V_1, V_2 \in \mathcal{V}_a \Rightarrow V_1 \cap V_2 \in \mathcal{V}_a$

3) $V \in \mathcal{V}_a$; $V \subset W \Rightarrow W \in \mathcal{V}_a$

4) $V \in \mathcal{V}_a \Rightarrow \exists\ W \in \mathcal{V}_a$ a.î. $a \in W \subset V$ și $W \in \mathcal{V}_y\ \forall y \in W$

<u>Prop</u> 1) $\emptyset, X \in \mathcal{F}$ (închisă) $\emptyset = \underbrace{X \setminus X}_{\in \mathcal{G}}$ $\quad X = X \setminus \emptyset \in \mathcal{G}$

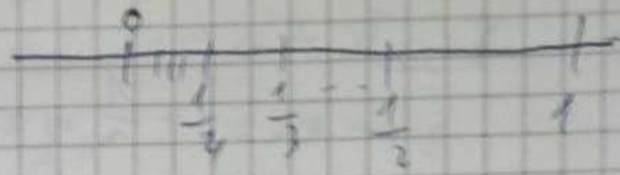
2) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cup F_2 \in \mathcal{F}$

$$X \setminus (F_1 \cup F_2) = \underbrace{(X \setminus F_1)}_{\in \mathcal{G}} \cap \underbrace{(X \setminus F_2)}_{\in \mathcal{G}}$$

3) $(F_i)_{i \in J} \subset \mathcal{F} \Rightarrow \bigcap_{i \in J} F_i \in \mathcal{F}$

$$X \setminus \left(\bigcap_{i \in J} F_i \right) = \bigcup \underbrace{(X \setminus F_i)}_{\in \mathcal{G}} \in \mathcal{G}$$

Ex $A = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$



$\mathbb{R} \setminus A = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{n \geq 1} \left(\frac{1}{n+1}, \frac{1}{n}\right)$

- m. deschisă

$\Rightarrow$ A închisă

$\cup(a,b)$ = reuniune de [illegible]

Def Fie (X, d) un spațiu metric și $A \subset X$

$$\mathring{A} = \text{Int}(A) = \bigcup_{D \subset A,\, D \in \mathcal{T}} D = \{x \in X \mid A \in \mathcal{V}_x\}$$

interiorul lui A

$$A' = \{x \in X \mid \exists (x_n)_n \subset A \text{ a.î. } x_n \to x \text{ și } x_n \neq x\} =$$

$$= \{x \mid \forall V \in \mathcal{V}_x \Rightarrow V \cap A \setminus \{x\} \neq \emptyset\}$$

închiderea $\bar{A} = \bigcap_{F \text{ închisă},\, A \subset F} F = \{x \in X \mid \exists (x_n)_n \subset A \text{ a.î. } x_n \to x\} =$

$$= A \cup A' = \{x \mid \forall V \in \mathcal{V}_x \Rightarrow V \cap A \neq \emptyset\}$$

$\text{Fr}(A) = \bar{A} \setminus \mathring{A}$ - frontiera lui A ($\cancel{\partial A}$)

$\text{iz}(A) = A \setminus A'$ izolate

Ex $A = (2, 3]$

$\mathring{A} = (2, 3)$

$A \subset A' = [2, 3]$

$\bar{A} = A \cup A' = [2, 3]$

$\text{Fr}(A) = \bar{A} \setminus \mathring{A} = \{2, 3\}$

$\text{Iz}(A) = \emptyset = A \setminus A'$

~~x~~A $[2,3] \subset A'$

$2 \leftarrow \underset{\in A}{x_n} = 2 + \frac{1}{n} \Rightarrow 2 \in A'$

$3 \leftarrow \underset{\in A}{x_n} = 3 - \frac{1}{n+1} \to 3 \Rightarrow 3 \in A'$

$x_n \neq 3$

$a \in (2,3)$

$2 < a < x_n = a + \left(\frac{1}{n+1}\right)(3-a) < \cancel{a} + 3 - a - 3$

$x_n \in A \quad x_n \to x$

$x_n \neq x \Rightarrow x \in A'$

$A' \subset [2,3]$

$a \in A' \Rightarrow \exists (x_n)_n \subset A$ a.î. $x_n \to a$ și $x_n \neq a$

$x_n \in A \Leftrightarrow 2 < x_n \leq 3 \Rightarrow 2 \leq a \leq 3$

$\Rightarrow A' \subset [2,3]$

$(2,3) \subset A \Rightarrow (2,3) \subset A^\circ$

$\hookrightarrow$ m. deschisă

$A^\circ \subset (2,3)$

$x \in (\mathbb{R}) \setminus (2,3) \Rightarrow x \notin A^\circ$

dacă $A^\circ \subset A$

$x \in A \setminus (2,3) \Rightarrow x \notin A^\circ$

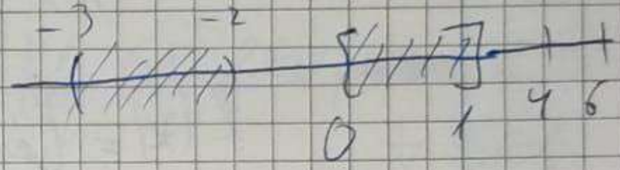
$a \in A^\circ \iff \exists (x_n)_n \subset \mathbb{R} \setminus A$ a.î. $x_n \to a$

$3 < x_n = 3 + \frac{1}{n} \to 3 \Rightarrow 3 \notin A^\circ$

$\cancel{\overline{A}}$

Ex 2 $A = (-3,-2) \cup [0,1) \cup \{4,6\}$

$A^\circ = (-3,-2) \cup (0,1)$



$A' = [-3,-2] \cup [0,1]$

$\bar{A} = A \cup A' = [-3,-2] \cup [0,1] \cup \{4,6\}$

$Fr(A) = \bar{A} \setminus A^\circ = \{-3,-2,0,1,4,6\}$

$Iz(A) = A \setminus A' = \{4,6\}$

Ex 3 $A = (1,2) \cup ((3,4) \cap \mathbb{Q})$

$A^\circ = (1,2)$

$A \subset A' = [1,2] \cup [3,4]$

$\bar{A} = A \cup A' = A'$

$Fr(A) = \bar{A} \setminus A^\circ = \{1,2\} \cup [3,4]$

$Iz(A) = \emptyset$

$\lim_{n \to \infty} \frac{x + e^{nx}}{2 + e^{nx}}$, $\lim_{n \to \infty} nx = \begin{cases} +\infty & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases}$

$\lim_{n \to \infty} e^{nx} = \begin{cases} +\infty, & x > 0 \\ 1 = e^0, & x = 0 \\ 0 = e^{-\infty} & x < 0 \end{cases}$

(C1) $x>0$ $f(x)=\lim\limits_{n\to\infty} \frac{e^{nx}\left(1+\frac{x}{e^{nx}}\right)}{e^{nx}\left(\frac{2}{e^{nx}}+1\right)}=\frac{1}{1}=1$

(C2) $x=0$ $f(0)=\lim\limits_{n\to\infty}\frac{1}{3}=\frac{1}{3}$

(C3) $x<0$ $f(x)=\lim\limits_{n\to\infty}\frac{x+e^{nx}}{2+e^{nx}}=\frac{x}{2}$

$$f(x)=\begin{cases}1, & x>0\\ \frac{1}{3}, & x=0\\ \frac{x}{2}, & x<0\end{cases}$$

$f_n:\mathbb{R}\to\mathbb{R}$

$f_n(x)=\frac{x+e^{nx}}{2+e^{nx}} \Rightarrow f_n\to f \Leftrightarrow \lim\limits_{n\to\infty} f_n(x)=f(x)\ \forall x\in\mathbb{R}$

f_n continuă f discontinuă în 0

<u>Definiții</u> Fie $f_n, f: A\to\mathbb{R}$

Spunem că șirul $(f_n)_{n\geq1}$ converge simplu (punctual) la f dacă $\lim\limits_{n\to\infty} f_n(x)=f(x)\ \forall x\in A$

$\forall x\in A$ și $\forall \varepsilon>0\ \exists n_{\varepsilon,x}$ a.î. $\forall n\geq n_{\varepsilon,x} \Rightarrow$

$\Rightarrow |f_n(x)-f(x)|<\varepsilon$

Spunem că $(f_n)_n$ converge uniform la f și notăm

$f_n \xrightarrow{u} f$ dacă

$\forall \varepsilon > 0 \ \exists \ n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon \Rightarrow |f_n(x) - f(x)| < \varepsilon$

Obs 1) $f_n \xrightarrow{u} f \Rightarrow f_n \xrightarrow{s} f$

2) $f_n \xrightarrow{u} f \Leftrightarrow a_n = \sup_{x \in A} |f_n(x) - f(x)| \leq \varepsilon$
$\forall x \in A$

$\left(a_n \Leftrightarrow \sup_{x \in A} |f_n(x) - f(x)| \leq \varepsilon \right)$

Teoremă Fie $f_n, f : (a,b) \to \mathbb{R}$ și $c \in (a,b)$

Dacă $f_n \xrightarrow{u} f$ și funcțiile $f_n |_{n \geq 1}$ sunt continue în $c \Rightarrow f$ este continuă în c

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Ex 1 $f_n : [0,1] \to \mathbb{R}$ $f_n(x) = x^n$

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} = f(x) \Rightarrow f_n \xrightarrow{s} f$$

f disc
f_n cont $\Rightarrow f_n \not\xrightarrow{u} f$

Ex 2 $f_n : [0,1] \to \mathbb{R}$ $f_n(x) = x^n(1-x)$

$$\lim_{n\to\infty} x^n(1-x) = \begin{cases} 0, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases} \Rightarrow f_n \xrightarrow{s} f = 0$$

f_n converge simplu la f

$$a_n = \sup |f_n(x) - \underset{\overset{\|}{0}}{f(x)}| = \sup_{x \in [0,1]} f_n(x)$$

$$f_n'(x) = nx^{n-1}(1-x) - x^n = x^{n-1}(n - nx - x) = 0$$

$= 0$ ⟶ $x = 0$

⟶ $x = \frac{n}{n+1}$

	0		$\frac{n}{n+1}$		1
f'	$+ + + +$		0	$— — —$	
f	0	$\nearrow$		$\searrow$	0

$$a_n = f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \quad \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \xrightarrow{u} f$$

$$\leq \frac{1}{n+1}$$

Ex 3 $f_n : [0,1] \to \mathbb{R}$ $f_n(x) = x^n(1 - x^{2n})$ $f_n \xrightarrow{s} 0$

$$a_n = \sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} f_n(x)$$

$$f_n'(x) = nx^{n-1}(1 - x^{2n}) + x^n(-2n)x^{2n-1} =$$

$$= nx^{n-1} - nx^{3n-1} - 2nx^{3n-1} =$$

$$= nx^{n-1} - 3nx^{3n-1} = nx^{n-1}(1 - 3x^{2n}) = 0$$

$x_1 = \sqrt[2n]{\frac{1}{3}}$

$x_2 = \sqrt[2n]{\frac{1}{3}}$ X

x	0		$\sqrt[2n]{\frac{1}{3}}$		1
f'		$+ + + +$	0	$- - - -$	
f	0	$\nearrow$	$f_n\left(\sqrt[2n]{\frac{1}{3}}\right)$	$\searrow$	0

$a_n = f_n\left(\sqrt[2n]{\frac{1}{3}}\right) =$

$= \sqrt{\frac{1}{3}}\left(1 - \frac{1}{3}\right) \not\to 0$

$\Rightarrow f_n$ nu converge la f

$f_n \not\rightrightarrows f$

T. Dini Fie $f_n, f : [a, b] \to \mathbb{R}$ a.î.

1) $f_n \xrightarrow{s} f$

2) $(f_n)_n$ să fie str monoton și

3) funcțiile f_n și f să fie continue

Atunci $f_n \xrightarrow{u} f$

Ex 2 $f_n(x) = x^n(1-x)$ $\quad f_n > f_{n+1}$

$f_n = [0, 1] \to \mathbb{R}$ $\quad f_n \xrightarrow{s} 0 = f \Rightarrow f_n \xrightarrow{u} f$ din T. Dini

f_n și f cont

Ex 3 $f_n, f : (0,1) \to \mathbb{R}$ $\quad f(x) = \frac{1}{x}$ $\quad f_n(x) = \frac{n}{n+1}$

$f_n \xrightarrow{s} f$ $\quad a_n = \sup_{x \in (0,1)} |f_n(x) - f(x)|$

$= \sup_{x \in (0,1)} \frac{1}{x} \cdot \frac{1}{n+1} = +\infty$

Ex 3 $f_n, g_n : \mathbb{R}^2 \to \mathbb{R}$ $f_n(x,y) = \frac{x^2y^2 \cdot n^2}{x^6+y^6+n^6}$

$$g_n(x,y) = \frac{x^2y^2 \cdot n}{x^6+y^6+n^6}$$

$\lim\limits_{n\to\infty} f_n(x,y) = 0 = \lim\limits_{n\to\infty} g_n(x,y)$ $f_n \xrightarrow{\Delta} 0 = f$

$g_n \xrightarrow{\Delta} 0 = g$

$a_n = \sup\limits_{(x,y)\in\mathbb{R}^2} |f_n(x,y) - 0| = \sup\limits_{x,y\in\mathbb{R}} \frac{x^2y^2n^2}{x^6+y^6+n^6} \geq f_n(n,n)$

$\Rightarrow f_n(n,n) = \frac{1}{3} = \frac{n^6}{n^6+n^6+n^6} \Rightarrow f_n \not\xrightarrow{u} f$

$b_n = \sup\limits_{x,y\in\mathbb{R}} |g_n(x,y)| = \sup\limits_{x,y\in\mathbb{R}} \frac{x^2y^2 \cdot n}{x^6+y^6+n^6}$

$x^6+y^6+n^6 \geq 3x^2y^2n^2$

$\frac{1}{3} \geq \frac{x^2y^2n^2}{x^6+y^6+n^6}$

$\frac{1}{3n} \geq \frac{x^2y^2n}{x^6+y^6+n^6}$

$\frac{1}{3n} \geq b_n \geq 0$

$\Rightarrow b_n \to 0 \Rightarrow \underline{\underline{g_n \xrightarrow{u} 0}}$