Probleme

Seminal 1 la materia Analiză

$$x_n = \frac{2n}{3n+1} \qquad a = \frac{2}{3}$$

$x_n \to a \iff \forall \varepsilon > 0 \ \exists n_\varepsilon$ a.î. $\forall n > n_\varepsilon \Rightarrow |x_n - a| < \varepsilon$

$$|x_n - a| = \left|\frac{2n}{3n+1} - \frac{2}{3}\right| = \left|\frac{2n - (2n+2)}{3n+1}\right| = \left|\frac{2n-2n-2}{3n+1}\right| \Rightarrow$$

$$\Rightarrow \left|\frac{-2}{9n+3}\right| \Rightarrow \frac{2}{9n+3} < \varepsilon$$

$$\frac{2}{\varepsilon} < 9n+3 \qquad \frac{2}{\varepsilon} - 3 < 9n \ \Big| \ :n$$

$$\frac{2}{9\varepsilon} - \frac{1}{3} < n$$

$$\Rightarrow n_\varepsilon = \left[\frac{2}{9\varepsilon} - \frac{1}{3}\right] + 1$$

$$y_n = \frac{\sqrt{n}}{2\sqrt{n}+1} \qquad a = \frac{1}{2}$$

$$|y_n - a| = \left|\frac{{}^{2)}\sqrt{n}}{2\sqrt{n}+1} - \frac{1}{2}^{(2\sqrt{n}+1)}\right| = \left|\frac{2\sqrt{n}}{4\sqrt{n}+2} - \frac{2\sqrt{n}+1}{4\sqrt{n}+2}\right| =$$

$$= \left|\frac{-1}{4\sqrt{n}+2}\right| = \frac{1}{4\sqrt{n}+2} < \varepsilon$$

$$\frac{1}{\varepsilon} < 4\sqrt{n}+2, \quad \frac{1}{\varepsilon} - 2 < 4\sqrt{n}$$

$$\Rightarrow \frac{1}{4\varepsilon} - \frac{1}{2} < \sqrt{n} \ \Big|^2 \Rightarrow \boxed{n_\varepsilon = \left[\frac{1}{4\varepsilon} - \frac{1}{2}\right] + 1}$$

$$\left(\frac{1}{4\varepsilon} - \frac{1}{2}\right)^2 < n$$

$x_n = \frac{3n}{5n+1} \qquad a = \frac{3}{5}$

$x_n \to a \iff \forall \varepsilon > 0\ \exists n_\varepsilon$ a.î. $\forall n > n_\varepsilon \Rightarrow \boxed{|x_n - a| < \varepsilon}$

$$\left|\frac{3}{5n+1} - \frac{3}{5}\right| = \left|\frac{15}{25n+5} - \frac{15n+3}{25n+5}\right| = \left|\frac{-3}{25n+5}\right| =$$

$$= \frac{3}{25n+5} < \varepsilon \Rightarrow \frac{3}{\varepsilon} < 25n+5 \Rightarrow \frac{3}{\varepsilon} - 5 < 25n$$

$$\Rightarrow \frac{3}{\varepsilon} - 5 < 25n \Rightarrow \frac{3}{25\varepsilon} - \frac{1}{5} < n \Rightarrow \boxed{n_\varepsilon = \left[\frac{3}{25\varepsilon} - \frac{1}{5}\right] + 1}$$

$x_n = \frac{n^3}{n^3+n^2+n+1} \qquad a = 1$

$$\left|\frac{n^3}{n^3+n^2+n+1} - \frac{n^3+n^2+n+1}{n^3+n^2+n+1}\right| = \left|\frac{-n^2-n-1}{n^3+n^2+n+1}\right| = \frac{n^2+n+1}{n^3+n^2+n+1} < \varepsilon$$

majorăm sau minorăm

$$\frac{n^2+n+1}{n^3+n^2+n+1} < \frac{3n^2}{n^3} = \frac{3}{n} < \varepsilon \Rightarrow \frac{3}{n} < \varepsilon \Rightarrow \frac{3}{\varepsilon} < n$$

$$\Rightarrow \boxed{n_\varepsilon = \left[\frac{3}{\varepsilon}\right] + 1}$$

$x_n = \frac{4n^2}{n^2+n} \to a = 4$

$$\left|\frac{4n^2}{n^2+n} - 4\right| = \left|\frac{16n^2}{4(n^2+n)} - \frac{16(n^2+n)}{4(n^2+n)}\right| = \left|\frac{16n^2-16n^2-16n}{4(n^2+n)}\right|$$

$$= \left|\frac{-16n}{4(n^2+n)}\right| = \frac{16n}{4(n^2+n)} = \frac{4n}{n^2+n} = \frac{4n}{n(n+1)} = \frac{4}{n+1} < \varepsilon$$

$$\frac{4}{\varepsilon} < n+1 \Rightarrow \frac{4}{\varepsilon} - 1 < n \Rightarrow \boxed{n_\varepsilon = \left[\frac{4}{\varepsilon} - 1\right] + 1}$$

$$x_n = \sqrt{n+1} - \sqrt{n-1} \to a = 0$$

$$\left|\sqrt{n+1} - \sqrt{n-1} - 0\right| < \varepsilon$$

$$\left|\frac{n+1-n+1}{\sqrt{n+1}+\sqrt{n-1}}\right| = \left|\frac{2}{\sqrt{n+1}+\sqrt{n-1}}\right| < \frac{2}{\sqrt{n+1}} < \varepsilon$$

$$\Rightarrow \frac{2}{\varepsilon} < \sqrt{n+1} \quad |^2$$

$$\frac{4}{\varepsilon^2} < n+1 \Rightarrow \frac{4}{\varepsilon^2} - 1 < n$$

$$\Rightarrow \boxed{n_\varepsilon = \left[\frac{4}{\varepsilon^2} - 1\right] + 1} \Rightarrow n_\varepsilon = \left[\frac{4}{\varepsilon^2}\right]$$

$$x_n = \frac{2^n}{2^n+1} \to a = 1$$

$$\underline{\underline{|x_n - a| < \varepsilon}} \quad \left|\frac{2^n}{2^n+1} - 1\right| = \left|\frac{2^n}{2^n+1} - \frac{2^n+1}{2^n+1}\right| = \left|\frac{-1}{2^n+1}\right| =$$

$$= \frac{1}{2^n+1} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < 2^n + 1 \Rightarrow \frac{1}{\varepsilon} - 1 < 2^n \quad |\log_2$$

$$\log_2\left(\frac{1}{\varepsilon} - 1\right) < n$$

$$\Rightarrow \boxed{n_\varepsilon = \left[\log_2\left(\frac{1}{\varepsilon} - 1\right)\right] + 1}$$

$$x_n = \frac{n^3}{2n^3-7n+1} \to a = \frac{1}{2}$$

$$|x_n - a| < \varepsilon \Rightarrow \left|\frac{n^3}{2n^3-7n+1} - \frac{1}{2}\right| = \left|\frac{2n^3}{4n^3-14n+2} - \frac{2n^3-7n+1}{4n^3-14n+2}\right| =$$

$$= \left|\frac{2n^3-2n^3+7n-1}{4n^3-14n+2}\right| = \left|\frac{7n-1}{4n^3-14n+2}\right| < \frac{7n}{|4n^3-14n|} \cancel{n_\varepsilon} <$$

$$\cancel{\frac{7n}{n(4n^2-14)} < \varepsilon \Rightarrow \frac{7}{4n^2-14} < \varepsilon \Rightarrow \frac{7}{\varepsilon} < 4n^2-14 \Rightarrow}$$

$$\cancel{\Rightarrow \frac{7}{\varepsilon} + 14 < 4n^2 \; |:4 \Rightarrow \frac{7}{4\varepsilon} + \frac{14}{4} < n^2}$$

$$< \frac{7n}{n^3} < \varepsilon \Rightarrow \frac{7}{n^2} < \varepsilon \Rightarrow \frac{7}{\varepsilon} < n^2 \Rightarrow \frac{\sqrt{7}}{\sqrt{\varepsilon}} < n \Rightarrow$$

$$\boxed{n_\varepsilon = \left[\sqrt{\frac{7}{\varepsilon}}\right] + 1}$$

__Exercițiu teoretic__

$$x_n \to a \Rightarrow y_n = \frac{x_1 + x_2 + \dots + x_n}{n} \to a$$

$\forall \varepsilon > 0 \Rightarrow \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon \Rightarrow |x_n - a| < \varepsilon$

$$|y_n - a| = \left|\frac{x_1 + x_2 + \dots + x_n}{n} - a\right| = \left|\frac{x_1 + x_2 + \dots + x_n - n \cdot a}{n}\right| =$$

$$= \left|\frac{x_1 - a + \dots + x_n - a}{n}\right| \leq \frac{|x_1 - a| + \underbrace{\dots + |x_n - a|}_{\leq \varepsilon(n - n_\varepsilon)}}{n} =$$

$$= \frac{\sum_{k=1}^{n_\varepsilon} |x_k - a| + \sum_{k=n_\varepsilon+1}^{n} |x_k - a|}{n}$$

$\varepsilon > \frac{n_\varepsilon 2M}{n}$

$< \varepsilon + \frac{\sum_{k=1}^{n_\varepsilon} |x_k - a|}{n} \le \varepsilon + \frac{n_\varepsilon 2M}{n} < \varepsilon + \varepsilon = 2\varepsilon$

$x_n \to a \Rightarrow |x_n| \le M \Rightarrow |x_n - a| \le 2M$