Seminar Analiză 3

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-\frac{2}{3}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$$

$\frac{4}{3} > 1 \Rightarrow$ s. convergentă

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{(n+1)-n}{n(\sqrt{n+1}+\sqrt{n})} \approx \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \text{ convergent}$$

$\sqrt{n+1}+\sqrt{n}$

$$\frac{a_n}{b_n} \to \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1+\sqrt{2}+\dots+\sqrt{n}}{n^2} = A$$

$$\lim_{n\to\infty} \frac{1+\sqrt{2}+\dots+\sqrt{n}}{n\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{(n+1)^{\frac{3}{2}}+n^{\frac{3}{2}}}{\sqrt{n+1}}}{(n+1)^{\frac{3}{2}}-n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{\left((n+1)^{\frac{3}{2}}+n^{\frac{3}{2}}\right)\sqrt{n+1}}{(n+1)^3-n^3}$$

limită finită

$$A \sim \sum_{n=1}^{\infty} \frac{n\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \Rightarrow \text{divergentă}$$

$$\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{\sqrt{n}}\right) \sim \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^3 = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{3}{2}}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim_{n\to\infty} \frac{\sin^3\left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)^3} \cdot \left(\frac{1}{\sqrt{n}}\right)^3 = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n!}} \sim \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \text{divergent}$$

$$\lim_{n\to\infty} \sqrt[n]{n!} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{n} = n+1$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e^{-1}$$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} \sim \sum_{n=2}^{\infty} \frac{1}{n} = +\infty$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$= \lim_{n\to\infty} \frac{\ln (n+1) - \ln n}{n+1-n} = \lim_{n\to\infty} \ln \frac{(n+1)}{n} = 0$$

$$\frac{\ln n}{n} > \frac{1}{n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \sum_{n=2}^{\infty} \frac{2^n \cdot 1}{2^n \ln 2^n} = \frac{1}{\ln 2} \sum_{n=2} \text{dir} \frac{1}{n} \longrightarrow +\infty$$

$\Rightarrow$ divergent

$$\lim_{n\to\infty} \frac{1}{n \ln n} \to 0$$

$$\sum_{n\geq 2} \frac{1}{n(\ln n)^\alpha} \sim \sum_{n\geq 2} 2^n \frac{1}{2^n(\ln 2^n)^\alpha} = \sum_{n\geq 2} \frac{1}{(n\ln 2)^\alpha}$$

$$\frac{1}{n(\ln n)^\alpha} \searrow 0 \qquad \frac{1}{(\ln 2)^\alpha}\sum_{n\geq 2} \frac{1}{n^\alpha} \text{ conv} \Leftrightarrow \alpha > 1$$

$$\sum \frac{1}{n^\alpha} \sim \sum \frac{2^n}{(2^n)^\alpha} = \sum_{n\geq 1} \frac{2^n}{2^{n\alpha}} = \sum_{n\geq 1} 2^{n(1-\alpha)}$$

$$= \sum_{n\geq 1} \Big(\underbrace{2^{1-\alpha}}_{= a}\Big)^n \; ; a < 1$$

$$\sum_{n\geq 2} \frac{1}{n \ln n (\ln\ln n)^\alpha} \sim \sum_{n\geq 2} 2^n \cdot \frac{1}{2^n \ln 2^n (\ln\ln 2^n)^\alpha}$$

$$= \sum_{n\geq 2} \frac{1}{n \ln 2 (\ln(n\cdot \ln 2))^\alpha} \sim \sum_{n\geq 2} \frac{1}{2^n(\ln n + \ln\ln 2)^\alpha}$$

$$\sim \sum_{n\geq 2} \frac{1}{2^n (\ln n)^\alpha} \qquad \alpha > 1 \Rightarrow \text{convergent}$$

$$\sum_{n\geq 1} \underbrace{x^n \frac{\sqrt{n}}{n+1}}_{a_n} = A$$

$$\frac{a_{n+1}}{a_n} \to l \quad \begin{matrix} l < 1 \text{ convergent} \\ l > 1 \text{ divergent} \end{matrix}$$

$$\frac{a_{n+1}}{a_n}=\frac{\cancel{x}^{n+1}\,\frac{\sqrt{n+1}}{n+2}}{\cancel{x}^{n}\,\frac{\sqrt{n}}{n+1}}=x\underbrace{\frac{\sqrt{n+1}}{\sqrt{n}}}_{\to 1}\cdot\underbrace{\frac{n+1}{n+2}}_{\to 1}\to x$$

$$\sum_{n\geq 1}\frac{\sqrt{n}}{n+1}\sim\sum_{n\geq 1}\frac{1}{n^{1-\frac{1}{2}=\frac{1}{2}}}\quad \alpha=\frac{1}{2}<1 \text{ convergent}$$

$x>0$

$$\sum_{n\geq 1}x^n\frac{2n}{n^4+1}$$

$$\frac{a_{n+1}}{a_n}=\frac{x^{n+1}}{x^n}\cdot\frac{2n+2}{2n}\cdot\underbrace{\frac{n^4+1}{(n+1)^4+1}}_{\to 1}\to x$$

$$\cancel{x}=1\Rightarrow\sum_{n\geq 1}\frac{2n}{n^2+1}\sim\sum_{n\geq 1}\frac{1}{n^3}\quad 3>1\Rightarrow \text{divergent}$$

$$\sum_{n\geq 1}\frac{2\cdot 5\cdot 8\cdot\ldots(3n+2)}{2\cdot 6\cdot 10\cdot\ldots(4n+2)}$$

$$\frac{a_{n+1}}{a_n}=\frac{\frac{2\cdot 5\cdot\ldots(3n+2)(3n+5)}{2\cdot 6\cdot\ldots(4n+6)}}{\frac{2\cdot 5\cdot\ldots(3n+2)}{2\cdot 6\cdot\ldots(4n+2)}}=\frac{3n+5}{4n+6}\to\frac{3}{4}<1$$

convergent

$$\left(\lim_{x\to 0}\frac{\sin x}{x}=1\right.$$

$$\sum_{n\geq 1} x^n \sin\frac{1}{\sqrt{n}}$$

$$\frac{x_{n+1}}{x_n}=\frac{x^{n+1}\cdot\sin\frac{1}{\sqrt{n+1}}}{x^n\sin\frac{1}{\sqrt{n}}}=x\,\frac{\sin\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}}}\cdot\frac{\frac{1}{\sqrt{n}}}{\sin\frac{1}{\sqrt{n}}}\cdot\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}$$

$\Rightarrow x\cdot 1\cdot 1\cdot 1\to x$ $\quad x>1\Rightarrow$ divergentă

$x<1\Rightarrow$ convergentă

$x=1\Rightarrow\sum_{n\geq 1}\sin\frac{1}{\sqrt{n}}\sim\sum_{n\geq 1}\frac{1}{\sqrt{n}}$ - dive

$\alpha=\frac{1}{2}<1$

$$\sum_{n\geq 1} x^n\cdot tg^3\frac{1}{\sqrt{n}}$$

$$\frac{a_{n+1}}{a_n}=\frac{x^{n+1}}{x^n}\cdot\frac{tg^3\frac{1}{\sqrt{n}}}{tg^3\frac{1}{\sqrt{n+1}}}=x\cdot\frac{tg^3\frac{1}{\sqrt{n+1}}}{tg^3\frac{1}{\sqrt{n}}}\cdot=$$

$$=x\cdot\underbrace{\frac{tg^3\left(\frac{1}{\sqrt{n+1}}\right)}{\left(\frac{1}{\sqrt{n+1}}\right)^3}}_{\to 1}\cdot\underbrace{\frac{\left(\frac{1}{\sqrt{n}}\right)^3}{tg^3\frac{1}{\sqrt{n}}}}_{\to 1}\cdot\underbrace{\frac{\left(\frac{1}{\sqrt{n+1}}\right)^3}{\left(\frac{1}{\sqrt{n}}\right)^3}}_{\to 1}\longrightarrow x$$

$x>1\Rightarrow$ divergentă

$x<1\Rightarrow$ convergentă

$$\sum_{n=1} tg^3\left(\frac{1}{\sqrt{n}}\right) \sim \sum_{n\geq 1} \left(\frac{1}{\sqrt{3}}\right)^3$$

$$\sum_{n\geq 1} \frac{1}{n^2} x^n \left(\frac{2n+1}{3n+1}\right)^n$$

$$\sqrt[n]{\frac{1}{n^2} x^n \left(\frac{2n+1}{3n+1}\right)^n} = \frac{1}{\sqrt{n^2}} \cdot x \cdot \frac{2n+1}{3n+1} \to \frac{2}{3} x$$

$x > \frac{3}{2} \to$ div

$\Rightarrow$ $x < \frac{3}{2}$ – conv.

$$x = \frac{3}{2} \Rightarrow \sum_{n\geq 1} \left(\frac{3}{2}\right)^n \frac{1}{n^2} \left(\frac{2n+1}{3n+1}\right)^n$$

$$\sum_{n\geq 1} \left(\frac{6n+3}{6n+2}\right)^n \cdot \frac{1}{n^2}$$

$$\lim_{n\to\infty} \left(\frac{6n+3}{6n+2}\right)^n = \left(\left(1 + \frac{6n+3-6n-2}{6n+2}\right)^{6n+2}\right)^{\frac{n}{6n+2}} = e^{\frac{1}{6}}$$

$\Rightarrow$ convergent

$$\sum_{n\geq 1} x^n \frac{n!}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)}$$ criteriul Raabe-Duhamel

R.D. $\boxed{n\left(\frac{a_n}{a_{n+1}}-1\right)\to l \quad \begin{matrix} l>1 \text{ conv} \\ l<1 \text{ div} \end{matrix}}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{x^{n+1}}}{\cancel{x^n}} \cdot \frac{\cancel{(n+1)!}}{\cancel{n!}} \cdot \frac{\cancel{1\cdot 3\cdot\ldots\cdot(2n+1)}}{\cancel{1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)}(2n+3)} =$$

$$= x\cdot n\cdot\frac{1}{2n+3} \to \frac{x}{2}$$

$x>2 \Rightarrow$ convergentă

$x<1 \Rightarrow$ divergentă

$$x=2 \quad \sum_{n\geq 1} 2^n \frac{n!}{1\cdot 3\cdot 5\cdot\ldots(2n+1)}$$

$$\lim_{n\to\infty} n\left(\frac{2n+3}{2n+2}-1\right) = \lim_{n\to\infty} n\cdot\frac{1}{2n+2} = \frac{1}{2} < 1 \Rightarrow$$ divergentă

$$\sum_{n\geq 1} x^n \frac{a(a+1)\ldots(a+n)}{(n+4)!} \quad ; x>0; a>0$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} \frac{\cancel{a(a+1)\ldots(a+n)}(a+n+1)}{\cancel{a(a+1)\ldots(a+n)}} = \frac{x\cdot(a+n+1)}{n+5} \to x$$

$x>1$ - divergentă

$x<1$ - convergentă

$x=1$

$$\sum_{n\geq 1} \frac{a(a+1)\cdots(a+n)}{(n+1)!}$$

$$n\left[\frac{\cancel{a(a+1)\cdots(a+n)}}{\cancel{a(a+1)\cdots(a+n)}(a+n+1)}\cdot\frac{(n+5)!}{(n+4)!}-1\right]=n\left(\frac{n+5}{a+n+1}-1\right)$$

$$=n\cdot\frac{\cancel{n}+5-a-\cancel{n}-1}{a+n+1}=n\cdot\frac{5-a-1}{a+n+1}=4-a$$

$a<3$ $4-a>1$ conv.

$a>3$ $4-a<1$ div

$a=3$

$$a=3\Rightarrow\sum_{n\geq 1}\frac{3\cdot 4\cdots(3+n)}{(n+4)}=\sum_{n\geq 1}\frac{1}{2(n+4)}\to \text{div}$$

$$\lim_{n\to\infty} n\left(\frac{2n+3}{2n+2}-1\right)=\lim_{n\to\infty} n\cdot\frac{1}{2n+2}-\frac{1}{2}<1\Rightarrow \text{s. div.}$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right)\leq 1 \text{ s.div}$$

$$\frac{(4-a)n}{a+n+1}\overset{a=3}{=}\frac{n}{n+4}\leq 1\Rightarrow \text{div}$$

$$\sum_{n\geq 1} x^n\frac{(n+10)!}{a(a+1)\cdots(a+n)}$$

$$\frac{x_{n+1}}{x_n}=\frac{x^{n+1}(n+11)!}{\cancel{a(a+1)\cdots(a+n)}(a+n+1)}\cdot\frac{\cancel{a(a+1)\cdots(a+n)}}{x^n\cancel{(n+10)!}}=$$

$$= \frac{k \cdot n + 11}{a + n + 1} \to k$$

R.D

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = l$$

$k > 1 \Rightarrow$ dive

$k < 1 \Rightarrow$ conve

$k = 1$

$$\Rightarrow \sum_{n \geq 1} \frac{(n+10)!}{a(a+1)\cdots(a+n)}$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{a+n+1}{n+11} - 1\right) = n\left(\frac{a + \cancel{n} + 1 - \cancel{n} - 11}{n+11}\right) =$$

$$= n\left(\frac{a-10}{n+11}\right) \to a - 10$$

$a > 11 \quad a - 10 > 1$ - conve

$a < 11 \quad a - 10 < 1$ - dive

$a = 11$

$$\sum_{n \geq 1} \frac{(n+10)!}{11 \cdot 12 \cdots (n+11)} = \sum_{n \geq 1} \frac{10!}{n+11} \text{ divergent}$$

$\sum_{n\geq 1} x_n$ $\to$ C_1 $\sum_{n\geq 1} |x_n|$ conv   s. absolut conv
$\Downarrow$
conv

$x_n \in \mathbb{R}$

$\to$ C_2 $\sum_{n\geq 1} |x_n| = +\infty$ semi conv.

$\sum_{n\geq 1} x_n$ conv

$\to$ C_3 s. div. $(x_n \not\to 0)$

$a_n \searrow 0 \Rightarrow \sum_{n\geq 1} (-1)^n \cdot a_n$ conv

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Ex

$\sum_{n\geq 1} \frac{x^n}{n^2}$ $x \in \mathbb{R}$

$\sum_{n\geq 1} \frac{|x|^n}{n^2}$

$x = 0 \Rightarrow$ s. conv.

$x \neq 0 \Rightarrow \frac{a_{n+1}}{a_n} = |x| \cdot \underbrace{\frac{n^2}{(n+1)^2}}_{\to 1} \to |x|$

$|x| > 1 \Rightarrow a_n \to \infty \Rightarrow$ div

$|x| < 1 \Rightarrow$ absolut conv

$|x| = 1$

$\sum \frac{1}{n^2}$ conv $\Rightarrow$ s. absolut conv.

$$\sum_{n\geq 1} x^n \sqrt{n}$$

P1 abs. convergentă $\sum_{n\geq 1} |x^n \sqrt{n}|$

$x = 0 \Rightarrow$ s. abs. conv.

$x \neq 0$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{|x^{n+1}\sqrt{n+1}|}{|x^n \sqrt{n}|} = |x| \frac{\sqrt{n+1}}{\sqrt{n}} \to |x|$$

$|x| > 1$ $\quad$ $|x| = 1$

$|x| < 1$ $\quad$ $\sum_{n\geq 1} \sqrt{n}$ dive

$|a_n| = \sqrt{n} \to \infty$ - s. dive

$$\sum_{n\geq 1} \frac{x^n}{n} \qquad x \in \mathbb{R}$$

$|x| = 1$

$\sum \frac{1}{n^2}$ conv $\Rightarrow$ s. abs. conv.

$|x| > 1 \Rightarrow a_n \to \infty \Rightarrow$ dive

$|x| < 1 \Rightarrow$ abs conv

abs. conv.

$$\sum_{n\geq 1} \left|\frac{x^n}{n}\right|$$

$x = 0 \Rightarrow$ conv

$$x \neq 0 \Rightarrow \frac{a_{n+1}}{a_n} = \left|\frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1}\right| \to |x|$$

$|x| = 1 \Rightarrow \sum_{n\geq 1}^{x=1} \frac{1}{n} \Rightarrow$ dive

$x = -1 \sum_{n\geq 1} (-1)^n \cdot \frac{1}{n} \rightarrow$ semiconvergentă

$|x|>1$ din $x=1$

$\frac{1}{n} \rightarrow 0$

$|x|<1$

$x=-1$

$$\sum_{n\geq 1} x^n \cdot tg\frac{1}{\sqrt{n}}$$

abs conv $\sum_{n\geq 1} \left| x^n \cdot tg\frac{1}{\sqrt{n}} \right|$

$x=0 \Rightarrow$ abs. conv.

$x \neq 0 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{x^{n+1}\, tg\frac{1}{\sqrt{n+1}}}{x^n\, tg\frac{1}{\sqrt{n}}} = x \cdot \left| \frac{tg\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}}} \cdot \frac{\frac{1}{\sqrt{n}}}{tg\frac{1}{\sqrt{n}}} \cdot \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} \right| =$

$\rightarrow |x| \Rightarrow |x|>1 \Rightarrow$ dive

$|x|<1 \Rightarrow$ abs conv

$x=-1$
$x=1$ $\Leftrightarrow |x|=1$

$\sum_{n\geq 1} tg\frac{1}{\sqrt{n}} \sim \sum_{n\geq 1} \frac{1}{\sqrt{n}}$ dive $(x=1)$

$x=-1 \Rightarrow \sum_{n\geq 1} (-1)^n tg\frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{n}} \searrow 0$; $tg\frac{1}{\sqrt{n}} \downarrow 0$

$a_n \downarrow 0 \Rightarrow \sum (-1)^n \cdot a_n$ conv

$x=-1$ semi conv.

$\sum_{n\geq 1} x^n \left(\frac{n}{2n+1}\right)^n \cdot \frac{1}{n^2}$

obs conv. $\sum_{n\geq 1} \left| x^n \left(\frac{n}{2n+1}\right)^n \cdot \frac{1}{\sqrt{n}} \right|$

$\Rightarrow x=0 \Rightarrow$ convergent

$x \neq 0 \Rightarrow \cancel{\left|\frac{a_{n+1}}{a_n}\right| = x^{n+1} \cdot \left(\frac{n+1}{2n+3}\right)^{n+1} \cdot \frac{1}{(n+1)^2}}$

crit rad

$\left|\sqrt[n]{a_n}\right| = \sqrt[n]{|x|^n \cdot \left(\frac{n}{2n+1}\right)^n \cdot \frac{1}{n^2}} = |x| \cdot \frac{n}{2n+1} \cdot \sqrt[n]{\frac{1}{n^2}} \to \frac{|x|}{2}$

$|x|>2 \Rightarrow$ div

$|x|<2 \Rightarrow$ n. obs conv

$|x|=2 \Rightarrow \sum 2^n \left(\frac{n}{2n+1}\right)^n \frac{1}{n^2}$ ~~crit rad~~

$\Rightarrow \lim_{n\to\infty} 2^n \left(\frac{n}{2n+1}\right)^n \cdot \frac{1}{n^2} =$

$= \lim_{n\to\infty} \left(\frac{2n+1-1}{2n+1}\right)^n = \underbrace{\left[\left(1-\frac{1}{2n+1}\right)^{-(2n+1)}\right]}_{e}^{-\frac{1}{2n+1}\cdot n}$