19.12.2024

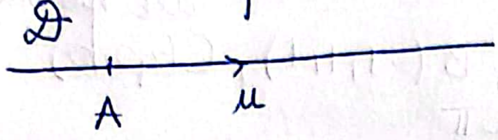
$C_{12}$ AG $\boxed{16}$

## Geometrie analitică euclidiană

$(\mathbb{R}^n, (\mathbb{R}^n, g_0), \varphi)$ sp afin euclidian $\quad g_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ g_0(x,y) = \sum_{i=1}^{n} x_i y_i$

$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \ \varphi(u,v) = v - u.$

• Ec. unei drepte afine $\mathcal{D}$ în $\mathbb{R}^n$.

a) 


$V_{\mathcal{D}} = \langle \{u\} \rangle$

$A(a_1, .., a_n)$ ; $u = (u_1, .., u_n)$

$\forall \ M(x_1, .., x_n) \in \mathcal{D} \iff \exists t \in \mathbb{R}$ a.î. $\overrightarrow{AM} = tu$

$\mathcal{D} : \ x_i - a_i = t u_i \ , \ i = \overline{1,n}$

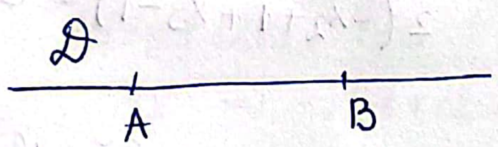
• $\mathcal{D} : \ \dfrac{x_1 - a_1}{u_1} = \ldots = \dfrac{x_n - a_n}{u_n}$

$\mathcal{R} = \{O; e_1, .., e_n\}$ reper cartezian ; $O \in \mathbb{R}^n$

$\{e_1, .., e_n\}$ - reper ortonormat în $\mathbb{R}^n$

$\overrightarrow{OA} = \sum_{i=1}^{n} a_i e_i$

$u = \sum_{i=1}^{n} u_i e_i$

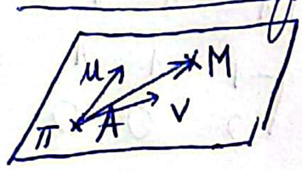
b) 


$\overrightarrow{AB} = u \qquad A(a_1, .., a_n)$

$B(b_1, .., b_n)$

$\mathcal{D} : \ \dfrac{x_1 - a_1}{b_1 - a_1} = \ldots = \dfrac{x_n - a_n}{b_n - a_n} \qquad \overrightarrow{AB} = (b_1 - a_1, .., b_n - a_n)$

• Ec unui plan afin în $\mathbb{R}^n$

a) 


$V_{\pi} = \langle \{u, v\} \rangle \qquad \{u, v\}$ SLI

$\forall M(x_1, .., x_n) \in \pi \iff \exists t, s \in \mathbb{R}$ a.î. $\overrightarrow{AM} = tu + sv$

$x_i - a_i = t u_i + s v_i , \ \forall \ i = \overline{1,n}$

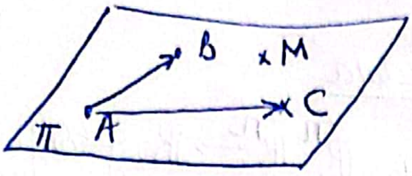
$A(a_1, .., a_n)$

$u = (u_1, .., u_n)$

$v = (v_1, .., v_n)$

Scanat cu CamScanner

b)




$A, B, C \in \pi$ (puncte necoliniare) $\{\overrightarrow{AB}, \overrightarrow{AC}\}$ SLI

$u = \overrightarrow{AB} = (b_1 - a_1, \dots, b_n - a_n)$

$v = \overrightarrow{AC} = (c_1 - a_1, \dots, c_n - a_n)$

$A(a_1, \dots, a_n)$, $B(b_1, \dots, b_n)$, $C(c_1, \dots, c_n)$

$\forall M \in \pi \iff \exists t, s \in \mathbb{R}$ a.î. $\overrightarrow{AM} = t\overrightarrow{AB} + s\overrightarrow{AC}$

$x_i - a_i = t(b_i - a_i) + s(c_i - a_i)$, $\forall i = \overline{1,n}$ ec. parametrice ale planului $\pi$.

<u>Exemplu</u> $n = 3$ $A(1,1,1)$, $B(-1,1,1)$, $C(2,0,0)$

Ec planului $\pi$ a.î. $A, B, C \in \pi$

<u>Sol</u>

$\overrightarrow{AB} = (-2, 0, 0)$, $\overrightarrow{AC} = (1, -1, -1)$ necoliniare.

$rg\begin{pmatrix} -2 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} = 2 \,(\max) \Rightarrow \{\overrightarrow{AB}, \overrightarrow{AC}\}$ SLI $\Rightarrow A, B, C$ necoliniare.

$\exists t, s \in \mathbb{R}: \overrightarrow{AM} = t\overrightarrow{AB} + s\overrightarrow{AC}$

$$\pi: \begin{cases} x_1 - 1 = t(-2) + s\,1 \\ x_2 - 1 = t \cdot 0 + s(-1) \\ x_3 - 1 = t \cdot 0 + s(-1), \ t, s \in \mathbb{R} \end{cases}$$

$$\pi: \begin{array}{c} \\ \\ \end{array}\begin{vmatrix} x_1 - 1 & -2 & 1 \\ x_2 - 1 & 0 & -1 \\ x_3 - 1 & 0 & -1 \end{vmatrix} = 0 \Rightarrow 2\begin{vmatrix} x_2 - 1 & -1 \\ x_3 - 1 & -1 \end{vmatrix} = 0 \Rightarrow$$

$A \quad \overrightarrow{AB} \quad \overrightarrow{AC}$

$$2(-x_2 + 1 + x_3 - 1) = 0$$

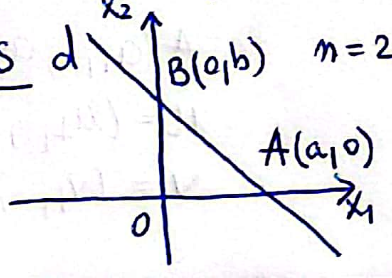
$\pi: -x_2 + x_3 = 0$

<u>OBS</u>: $\pi: ax_1 + bx_2 + cx_3 + d = 0$ ec. generală a planului.

<u>OBS</u> $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3) \in \pi$

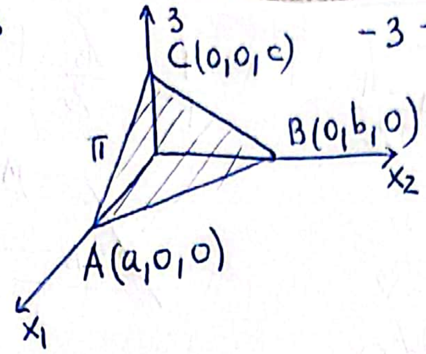
$$\pi: \begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{vmatrix} = 0 \ ; \quad \pi: \begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = 0$$

<u>OBS</u>




$n = 2$ $\quad d: \dfrac{x_1}{a} + \dfrac{x_2}{b} = 1$ (Ec. prin tăieturi)

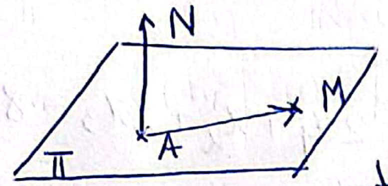
$n = 3$




$\Pi: \frac{x_1}{a} + \frac{x_2}{b} + \frac{x_3}{c} = 1$

Ec prin tăieturi a planului.

c)




$A \in \Pi$

$N \perp \Pi$ $\quad N$ = normala.

$\forall M \in \Pi \iff \langle \vec{AM}, N \rangle = 0$

$A(a_1, .., a_n)$, $M(x_1, .., x_n)$ $\quad N = (N_1, .., N_n)$

$N_1(x_1 - a_1) + N_2(x_2 - a_2) + \ldots + N_n(x_n - a_n) = 0$.

<u>Ex</u> $A(1,2,3) \in \Pi$, $u = (0,1,3)$, $v = (4,5,0)$ vectori directori.

a) $N = ?$ ; b) ec. lui $\Pi$.

<u>Sol</u>

a) $N = u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & 3 \\ 4 & 5 & 0 \end{vmatrix} = e_1 \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} - e_2 \begin{vmatrix} 0 & 3 \\ 4 & 0 \end{vmatrix} + e_3 \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix}$

$= (-15, 12, -4)$

$\boxed{\text{OBS}}$ $\Pi: ax_1 + bx_2 + cx_3 + d = 0$, $N = (a, b, c)$

$\Pi: -15x_1 + 12x_2 - 4x_3 + \boxed{d} = 0$

$A(1,2,3) \in \Pi \Rightarrow -15 \cdot 1 + 12 \cdot 2 - 4 \cdot 3 + d = 0$

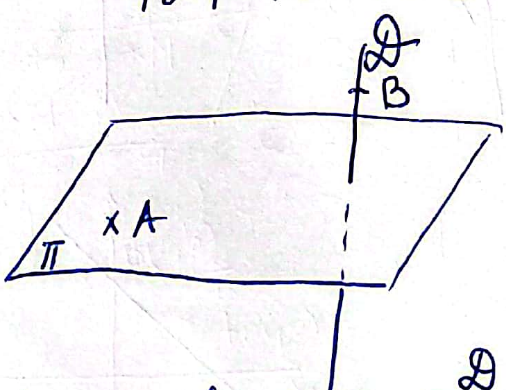
$-15 + 24 - 12 + d = 0 \Rightarrow d = 27 - 24 = 3$

$\Pi: -15x_1 + 12x_2 - 4x_3 + 3 = 0$

SAU $\Pi: (x_1 - 1)\underbrace{(-15)} + (x_2 - 2)\underbrace{12} + (x_3 - 3)\underbrace{(-4)} = 0$

$-15x_1 + 15 + 12x_2 - 24 - 4x_3 + 12 = 0$

d)




$A \in \Pi$, $\mathcal{D} \perp \Pi$ $\quad B(x_1^0, .., x_n^0)$, $A(a_1, .., a_n)$

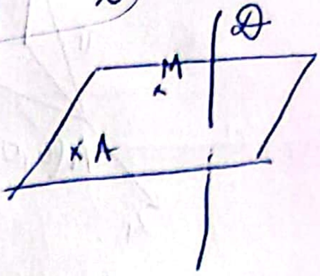
$\mathcal{D}: \frac{x_1 - x_1^0}{u_1} = \ldots = \frac{x_n - x_n^0}{u_n}$

$\mathcal{D} \perp \Pi \Rightarrow u_{\mathcal{D}} = N_\Pi = (u_1, .., u_n)$

$\Pi: (x_1 - a_1)u_1 + \ldots + (x_n - a_n)u_n = 0$

<u>Ex</u> $A(1,0,3) \in \Pi$, $\mathcal{D}: \frac{x_1-1}{2} = \frac{x_2-1}{1} = \frac{x_3}{2} = t$, $\mathcal{D} \perp \Pi$

Ec. planului $\Pi$




<u>SOL</u>

$N_\Pi = u_{\mathcal{D}} = (2,1,2)$

$\forall M \in \Pi \quad \langle \overrightarrow{AM}, N \rangle = 0$

$\overrightarrow{AM} = (x_1-1, x_2, x_3-3)$

$2(x_1-1) + 1\cdot x_2 + 2(x_3-3) = 0 \Rightarrow 2x_1 + x_2 + 2x_3 - 8 = 0$

<u>OBS</u> $\Pi: 2x_1 + x_2 + 2x_3 + d = 0$

$A(1,0,3) \in \Pi \Rightarrow 2\cdot 1 + 0 + 2\cdot 3 + d = 0 \Rightarrow d = -8$

• Intersectia unei drepte cu un plan.

$n=3 \qquad \Pi: x_1 + x_2 + x_3 - 2 = 0$

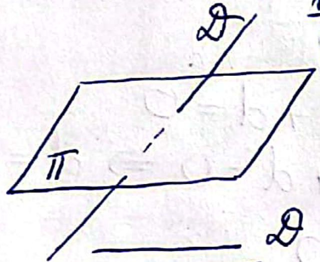
$\mathcal{D}: \frac{x_1-3}{1} = \frac{x_2}{3} = \frac{x_3}{1} = t \Rightarrow \begin{cases} x_1 = t+3 \\ x_2 = 3t \\ x_3 = t, \ t \in \mathbb{R} \end{cases}$

$\mathcal{D} \cap \Pi: \ t + 3 + 3t + t - 2 = 0 \Rightarrow 5t = -1 \Rightarrow t = -\frac{1}{5}$

$\mathcal{D} \cap \Pi = \{P\} \qquad P\left(\underbrace{-\frac{1}{5}+3}_{\frac{14}{5}}, -\frac{3}{5}, -\frac{1}{5}\right)$

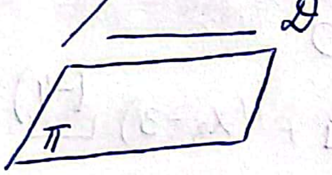
<u>OBS</u>

a)



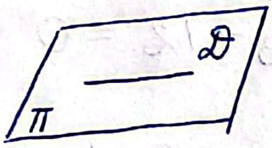

$\mathcal{D}$ secantă

b)




$\mathcal{D} \parallel \Pi$

c)




$\mathcal{D} \subset \Pi$.

## Perpendiculara comună a 2 drepte necoplanare

$n=3$

$\mathcal{D}_1 : \frac{x_1-a_1}{u_1} = \frac{x_2-a_2}{u_2} = \frac{x_3-a_3}{u_3} = t \Leftrightarrow \begin{cases} x_1 = a_1 + tu_1 \\ x_2 = a_2 + tu_2 \\ x_3 = a_3 + tu_3 \end{cases}$

$\mathcal{D}_2 : \frac{x_1-b_1}{v_1} = \frac{x_2-b_2}{v_2} = \frac{x_3-b_3}{v_3} = s \Leftrightarrow \begin{cases} x_1 = b_1 + v_1 s \\ x_2 = b_2 + v_2 s \\ x_3 = b_3 + v_3 s \end{cases}$

$u = (u_1, u_2, u_3)$ vectori directori pt $\mathcal{D}_1$, resp $\mathcal{D}_2$.

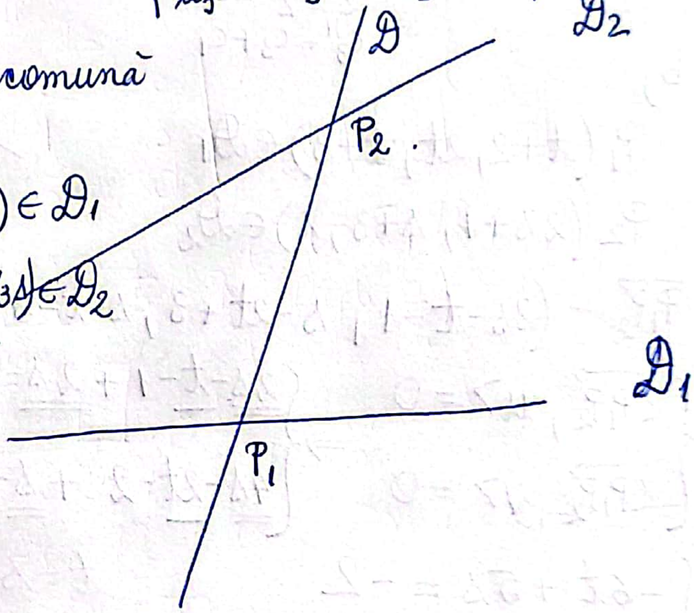
$v = (v_1, v_2, v_3)$

$A(a_1, a_2, a_3) \in \mathcal{D}_1$, $B(b_1, b_2, b_3) \in \mathcal{D}_2$

$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$

$\mathcal{D}_1, \mathcal{D}_2$ necoplanare $\Leftrightarrow \begin{vmatrix} u_1 & v_1 & b_1 - a_1 \\ u_2 & v_2 & b_2 - a_2 \\ u_3 & v_3 & b_3 - a_3 \end{vmatrix} \neq 0$

$\mathcal{D}$ = perpendiculara comună




$P_1(a_1 + tu_1, a_2 + tu_2, a_3 + tu_3) \in \mathcal{D}_1$

$P_2(b_1 + v_1 s, b_2 + v_2 s, b_3 + v_3 s) \in \mathcal{D}_2$

$\begin{cases} \langle \overrightarrow{P_1P_2}, u \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, v \rangle = 0 \end{cases} \Rightarrow t, s$

$\Rightarrow P_1, P_2$, $\mathcal{D} = P_1P_2$

### Interpretare geometrică

$N = u \times v$ directia lui $\mathcal{D}$.

$\Pi_1$ det. de $\mathcal{D}_1$ și $\mathcal{D}$ ; $A(a_1, a_2, a_3) \in \mathcal{D}_1$ ; $N_{\Pi_1} = N \times u$

$\Pi_2$ —"— $\mathcal{D}_2$ și $\mathcal{D}$ ; $B(b_1, b_2, b_3) \in \mathcal{D}_2$ ; $N_{\Pi_2} = N \times v$.

$\mathcal{D} : \Pi_1 \cap \Pi_2$.

**Exemplu** Fie dreptele $\mathcal{D}_1: \dfrac{x_1-2}{1}=\dfrac{x_2}{2}=\dfrac{x_3-3}{1}=t \Leftrightarrow \begin{cases} x_1=t+2\\ x_2=2t\\ x_3=t+3,\ t\in \end{cases}$

$\mathcal{D}_2: \dfrac{x_1-1}{2}=\dfrac{x_2-3}{1}=\dfrac{x_3}{1}=s \Leftrightarrow \begin{cases} x_1=2s+1\\ x_2=s+3\\ x_3=s,\ s\in\mathbb{R}. \end{cases}$

a) $\mathcal{D}_1, \mathcal{D}_2$ drepte necoplanare

b) Să se scrie ec. $\perp$ comune.
Aflați $dist(\mathcal{D}_1,\mathcal{D}_2)$.

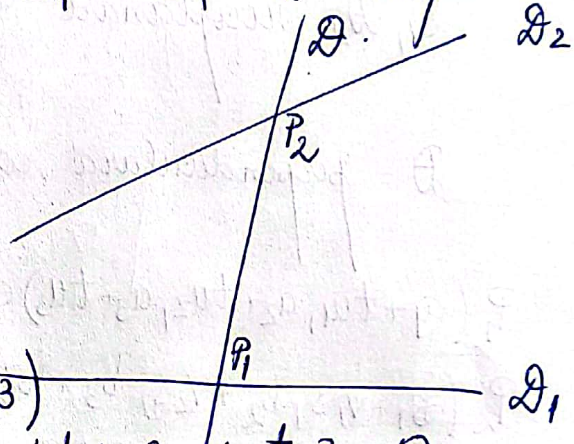
SOL

a) $u=u_{\mathcal{D}_1}=(1,2,1)$ ; $A(2,0,3)\in\mathcal{D}_1$ $\Rightarrow \overrightarrow{AB}=(-1,3,-3)$

$v=u_{\mathcal{D}_2}=(2,1,1)$ ; $B(1,3,0)\in\mathcal{D}_2$

$$\begin{vmatrix} 1 & 2 & -1\\ 2 & 1 & 3\\ 1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0\\ 2 & -3 & 5\\ 1 & -1 & -2 \end{vmatrix} \neq 0 \Rightarrow \mathcal{D}_1,\mathcal{D}_2 \text{ necoplanare.}$$

$c_2'=c_2-2c_1$
$c_3'=c_3+c_1$




b)

$P_1(t+2,2t,t+3)\in\mathcal{D}_1$

$P_2(2s+1,s+3,s)\in\mathcal{D}_2$

$\overrightarrow{P_1P_2}=(2s-t-1,\ s-2t+3,\ s-t-3)$

$$\begin{cases} \langle\overrightarrow{P_1P_2},u\rangle=0\\ \langle\overrightarrow{P_1P_2},v\rangle=0 \end{cases} \Rightarrow \begin{cases} 2s-t-1+2s-4t+6+s-t-3=0\\ 4s-2t-2+s-2t\cancel{+3}+s-t\cancel{-3}=0 \end{cases}$$

$$\begin{cases} -6t+5s=-2\\ -5t+6s=2. \end{cases}$$
$11(-t+s)=0$

$t=s$
$-t=-2 \Rightarrow \boxed{t=s=2}$

$P_1(4,4,5)$ ; $P_2(5,5,2)$ , $\overrightarrow{P_1P_2}=(1,1,-3)$

$\mathcal{D}=P_1P_2: \dfrac{x_1-4}{1}=\dfrac{x_2-4}{1}=\dfrac{x_3-5}{-3}=t$

$dist(\mathcal{D}_1,\mathcal{D}_2)=dist(P_1,P_2)=\|\overrightarrow{P_1P_2}\|=\sqrt{1+1+9}=\sqrt{11}.$

Scanat cu CamScanner

M2 $N = u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (1, +1, -3)$ direcția lui $\mathcal{D}$.

$\pi_1 = \pi(\mathcal{D}_1, \mathcal{D})$ $\quad \mathcal{D}_1 \subset \pi$ , $A(2,0,3) \in \pi_1$

$N_{\pi_1} = N \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & -3 \\ 1 & 2 & 1 \end{vmatrix} = (7, -4, 1)$

$\pi_1: \; 7x_1 - 4x_2 + x_3 + d = 0$

$A(2,0,3) \in \pi_1 \Rightarrow 7 \cdot 2 - 0 + 3 + d = 0 \Rightarrow d = -14 - 3 = -17$

$\pi_1: \; 7x_1 - 4x_2 + x_3 - 17 = 0$

$\pi_2 = \pi(\mathcal{D}_2, \mathcal{D})$ $\quad \mathcal{D}_2 \subset \pi$ , $B(1,3,0) \in \pi_2$

$N_{\pi_2} = N \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = (4, -7, -1)$

$\pi_2: \; 4x_1 - 7x_2 - x_3 + d' = 0$

$B(1,3,0) \in \pi_2 \Rightarrow 4 - 21 + d' = 0 \Rightarrow d' = 17$

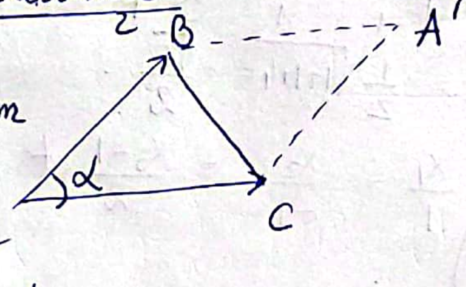
$\pi_2: \; 4x_1 - 7x_2 - x_3 + 17 = 0$

$\pi_1 \cap \pi_2 \quad \begin{cases} 7x_1 - 4x_2 + x_3 - 17 = 0 \\ 4x_1 - 7x_2 - x_3 + 17 = 0 \end{cases}$ $\quad x_3 = t \Rightarrow x_1, x_2$

## Arii, volume, distanțe

$n = 3$

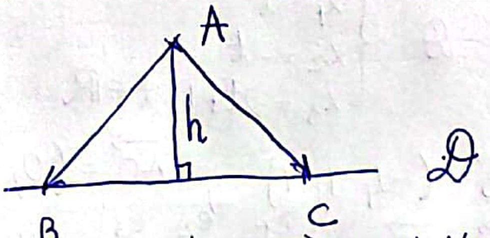
- $A_\Delta = \frac{1}{2} A_{paralelogram}$




$= \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \|$

$= \frac{1}{2} \| \overrightarrow{AB} \| \cdot \| \overrightarrow{AC} \| \cdot \sin \alpha$.

- Dist (A, dreaptă).




Fie $B, C \in \mathcal{D}$ distincte.

$A_{\Delta ABC} = \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \frac{1}{2} \cdot dist(A, \mathcal{D}) \cdot \| \overrightarrow{BC} \| \Rightarrow dist(A, \mathcal{D}) = \frac{\| \overrightarrow{AB} \times \overrightarrow{AC} \|}{\| \overrightarrow{BC} \|}$

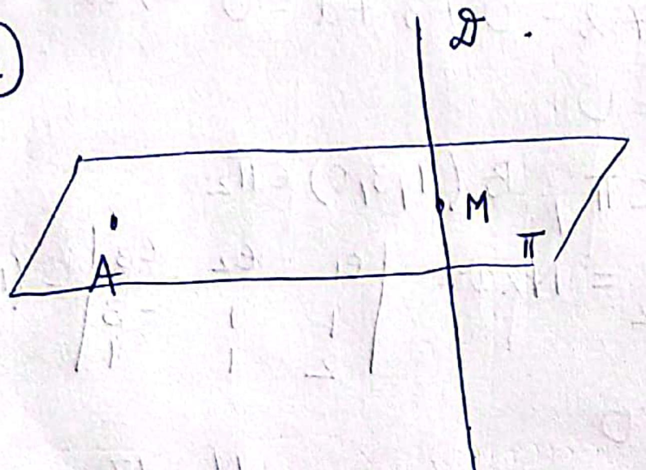
$\overrightarrow{AB} = (b_1-a_1, b_2-a_2, b_3-a_3)$ $\qquad \overrightarrow{BC} = (c_1-b_1, c_2-b_2, c_3-b_3)$

$\overrightarrow{AC} = (c_1-a_1, c_2-a_2, c_3-a_3)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \end{vmatrix} = (\Delta_1, \Delta_2, \Delta_3)$$

$$\text{dist}(A, \mathcal{D}) = \frac{\sqrt{\Delta_1^2+\Delta_2^2+\Delta_3^2}}{\sqrt{(c_1-b_1)^2+(c_2-b_2)^2+(c_3-b_3)^2}}$$

M2




$\pi \perp \mathcal{D}$

$A \in \pi$

$\mathcal{D} \cap \pi = \{M\}$

$\text{dist}(A, \mathcal{D}) = \text{dist}(A, M) = \|\overrightarrow{AM}\|$.

EX1. $A(1,0,1)$, $B(0,-1,0)$, $C(0,1,1)$

$A_{\Delta ABC}$

$\overrightarrow{AB} = (-1,-1,-1)$

$\overrightarrow{AC} = (-1,1,0)$

SOL

$A_{\Delta ABC} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & -1 & -1 \\ -1 & 1 & 0 \end{vmatrix} = (1, +1, -2)$$

$A_{\Delta ABC} = \frac{1}{2}\sqrt{1+1+4} = \frac{\sqrt{6}}{2}$.

EX2 $\quad \mathcal{D}: \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3-1}{2} = t$ ; $A(1,1,0)$

Dist $(A, \mathcal{D})$

SOL

M1 $\quad \mathcal{D}: \begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 2t+1, \ t \in \mathbb{R} \end{cases}$

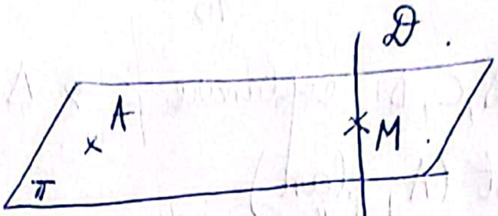
$B(0,0,1)$ $(t=0)$ $\quad B, C \in \mathcal{D}$.

$C(1,-1,3)$ $(t=1)$

$\overrightarrow{AB} = (-1,-1,1)$ $\quad \overrightarrow{AC} = (0,-2,3)$ ; $\overrightarrow{BC} = (1,-1,2)$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & -1 & 1 \\ 0 & -2 & 3 \end{vmatrix} = (-1, 3, 2) \text{ ; } \text{dist}(A, \mathcal{D}) = \frac{\sqrt{1+9+4}}{\sqrt{1+1+4}} = \sqrt{\frac{14}{6}}$$

(M2) $dist(A, \mathcal{D})$




$A \in \pi$

$\pi \perp \mathcal{D}$.

$\mathcal{D}: \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3 - 1}{2} = t$, $A(1,1,0)$.

$N_\pi = u_{\mathcal{D}} = (1, -1, 2)$

$\pi: 1 \cdot x_1 - 1 \cdot x_2 + 2x_3 + d = 0$

$A(1,1,0) \in \pi \Rightarrow 1 - 1 + d = 0 \Rightarrow d = 0$

$\pi: x_1 - x_2 + 2x_3 = 0$

$\mathcal{D} \cap \pi$:

$\mathcal{D}: \begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 1 + 2t \end{cases}$

$\Rightarrow t + t + 2 + 4t = 0$.

$6t = -2 \Rightarrow t = -\frac{1}{3}$

$\{M\} = \mathcal{D} \cap \pi$, $M\left(-\frac{1}{3}, \frac{1}{3}, \underbrace{1 - \frac{2}{3}}_{\frac{1}{3}}\right)$, $\overrightarrow{AM} = \left(-\frac{1}{3} - 1, \frac{1}{3} - 1, \frac{1}{3}\right)$

$\overrightarrow{AM} = \left(\frac{-4}{3}, \frac{-2}{3}, \frac{1}{3}\right)$.

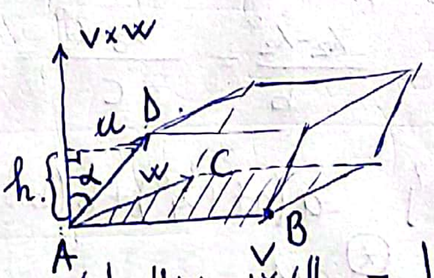
$dist(A, \mathcal{D}) = dist(A, M) = \|\overrightarrow{AM}\|$

## Volume

Fie $\{u, v, w\}$ SLI

$V_{paralelipiped} = h \cdot A_b$

$h = |\cos\alpha| \, \|u\|$




$V_{paralelipiped} = \|u\| \cdot |\cos\alpha| \, \|v \times w\| = |\langle u, v \times w \rangle|$

$= |u \wedge v \wedge w| = \left| \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right|$

$u = \overrightarrow{AD} = (d_1 - a_1, d_2 - a_2, d_3 - a_3)$

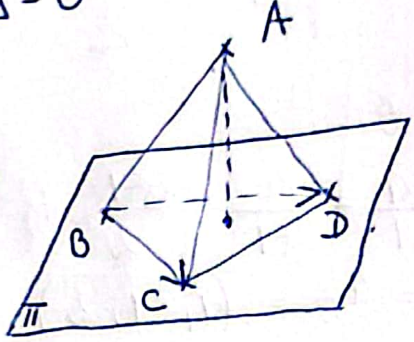
$v = \overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$

$w = \overrightarrow{AC} = (c_1 - a_1, c_2 - a_2, c_3 - a_3)$

$V_{ABCD} = \frac{1}{6} \left| \begin{vmatrix} d_1 - a_1 & d_2 - a_2 & d_3 - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} \right| = \frac{1}{6} \left| \overbrace{\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}}^{\Delta} \right|$

**Prop**

$A, B, C, D$ coplanare $\Longleftrightarrow \Delta = 0$.




- Dist $(A, \text{plan})$

$\quad\quad\quad\quad\quad\quad \parallel$

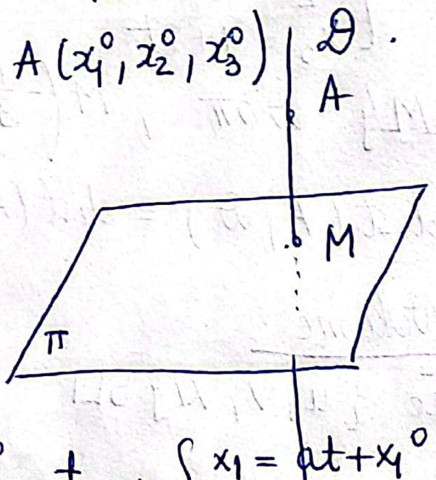
Fie $B, C, D \in \pi$ (necoliniare)

$$V_{ABCD} = \frac{1}{3} \text{dist}(A, \pi) \cdot A_{\Delta BCD}.$$

$$\frac{1}{\cancel{6}} |\Delta| = \frac{1}{\cancel{6}} \text{dist}(A, \pi) \cdot \| \vec{BC} \times \vec{BD} \|$$

$$\text{dist}(A, \pi) = \frac{|\Delta|}{\|\vec{BC} \times \vec{BD}\|} = \frac{|a x_1^0 + b x_2^0 + c x_3^0 + d|}{\sqrt{a^2+b^2+c^2}}$$

$\pi : a x_1 + b x_2 + c x_3 + d = 0$, $A(x_1^0, x_2^0, x_3^0)$




**OBS**

**M2** Fie $D \perp \pi$, $A \in D$.

$\pi : a x_1 + b x_2 + c x_3 + d = 0$

$N_\pi = (a, b, c) = u_D$.

$$D : \frac{x_1 - x_1^0}{a} = \frac{x_2 - x_2^0}{b} = \frac{x_3 - x_3^0}{c} = t \Rightarrow \begin{cases} x_1 = at + x_1^0 \\ x_2 = bt + x_2^0 \\ x_3 = ct + x_3^0 \end{cases} \circledast$$

$D \cap \pi = \{M\}$ $\quad a(at + x_1^0) + b(bt + x_2^0) + c(ct + x_3^0) + d = 0$

$\Rightarrow t \overset{\circledast}{\Longrightarrow} M$.

Dist $(A, \pi) = \text{dist}(A, M) = \|\vec{AM}\|$.

11-4

**Ex.** $\pi : x_1 - 2x_2 + 3x_3 + 1 = 0$, $A(1, 2, 3)$ $\quad 1-4+9+1$

$\text{dist}(A, \pi) = ?$ $\quad -\frac{3}{2}+3$

**SOL** Fie $D \perp \pi$, $A \in D$. $\quad u_D = N_\pi = (1, -2, 3)$

$$D : \frac{x_1 - 1}{1} = \frac{x_2 - 2}{-2} = \frac{x_3 - 3}{3} = t \Rightarrow \begin{cases} x_1 = t + 1 \\ x_2 = -2t + 2 \\ x_3 = 3t + 3, \; t \in \mathbb{R} \end{cases}$$

$D \cap \pi$: $t + 1 - 2(-2t + 2) + 3(3t + 3) + 1 = 0$

$14t + \cancel{7} = 0 \Rightarrow t = -\frac{1}{2}$ $\quad M\left(\frac{1}{2}, 1, \frac{3}{2}\right)$

$\overrightarrow{AM} = \left(\frac{1}{2}-1, 1-2, \frac{3}{2}-3\right) \Rightarrow \overrightarrow{AM}\left(-\frac{1}{2}, -1, -\frac{3}{2}\right)$

$\|\overrightarrow{AM}\| = \sqrt{\frac{1}{4}+1+\frac{9}{4}} = \sqrt{\frac{10}{4}+1} = \sqrt{\frac{5}{2}+1} = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$.

---

## Hipercuadrice în spațiul afin $\mathbb{R}^n$

**Def** $(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$ sp afin (sau $(\mathbb{R}^n, (\mathbb{R}^n/\mathbb{R}, g_0), \varphi)$ sp afin euclidian).

S.n. hipercuadrică L.G al punctelor $P(x_1, \dots, x_n)$ care verifică în rap cu $R = \{O; e_1, \dots, e_n\}$ reperul cartezian

$\Gamma: f(x_1, \dots, x_n) = a_{11}x_1^2 + \dots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + \dots + 2a_{n-1\,n}x_{n-1}x_n$
$+ 2b_1x_1 + \dots + 2b_nx_n + c = 0$

$f(x) = X^TAX + 2BX + c = 0$

$A = (a_{ij})_{i,j=\overline{1,n}} = A^T$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B = (b_1 \dots b_n)$

$\tilde{A} = \begin{pmatrix} A & B^T \\ B & c \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{n1} & \dots & a_{nn} & b_n \\ b_1 & \dots & b_n & c \end{pmatrix}$

$\delta = \det A$, $\Delta = \det\tilde{A}$, $r = \operatorname{rg} A$, $r' = \operatorname{rg}\tilde{A}$

$r \le r' \le r+2$.

**Def** $\Gamma$ s.n. degenerată $\Leftrightarrow \Delta = 0$

$\Gamma$ s.n. nedegenerată $\Leftrightarrow \Delta \ne 0$.

---

**Def** a) $(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$ sp afin.

$\Gamma_1 \sim \Gamma_2$ afin echivalente $\Leftrightarrow \exists\, \sigma: \mathbb{R}^n \to \mathbb{R}^n$ transf afină

$X' = CX + D$

$C \in GL(n, \mathbb{R})$

$r, r', \frac{\Delta}{\delta}$.

Invarianți afini: $\qquad$ a.î. $\Gamma_2 = \sigma(\Gamma_1)$

Invarianți metrici: $r, r', \delta, \Delta, \frac{\Delta}{\delta}$.

b) $(\mathbb{R}^n, (\mathbb{R}^n/\mathbb{R}, g_0), \varphi)$ sp. punctual euclidian

$\Gamma_1 \equiv \Gamma_2$ congruente metric $\Leftrightarrow \exists\, \sigma: \mathbb{R}^n \to \mathbb{R}^n$ izometrie

$X' = CX + D$, $C \in O(n)$, $\Gamma_2 = \sigma(\Gamma_1)$

$n = 2$ Conice.

$\Gamma: f(x_1, x_2) = \underline{a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2} + 2b_1x_1 + 2b_2x_2 + c = 0$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T$, $B = (b_1 \;\; b_2)$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$f(x) = X^TAX + 2BX + c = 0$

$\delta = \det A$, $\Delta = \det(\tilde{A})$

$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix}$