Analiză Curs 4

$s = \sum_{n \geq 1} x_n$, $x_n \geq 0$, $s_n = \sum_{k=1}^{n} x_k$ $s_{n+1} - s_n = x_n \geq 0$

ordine
sqrt/radical; după complexitate etc

$\Rightarrow (s_n)_{n\geq 1}$ crescător)

$\Rightarrow (s_n)_{n \geq 1}$ este convergent $\Leftrightarrow$ este mărginit

Recapitulare

$\boxed{\text{Crit}}$ $\boxed{\text{comparației}}$ $\sum_{n\geq 1} a_n \sim \sum_{n \geq 1} b_n$ dacă $\exists \lim_{n\to\infty} \frac{a_n}{b_n} \in (0, +\infty)$

$\sum_{n\geq 1} \frac{1}{n^\alpha}$ conv $\Leftrightarrow \alpha > 1$

$\sum_{n\geq 1} \frac{\sqrt{n}}{n^2+1} \sim \sum \frac{1}{n^{\frac{3}{2}}}$ conv $\alpha = \frac{3}{2} > 1$

$\boxed{\text{Crit}}$ $\boxed{\text{raportului}}$ $l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ Dacă $l > 1 \to$ serie div
și dacă $l < 1 \Rightarrow$ s. conv.

$x > 1$ div

$\underbrace{\sum_{n\geq 1} x^n \frac{\sqrt{n}}{n+1}}_{= a_n}$ $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} \cdot \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n+1}{n+2} \to x$ $x < 1$ conv

$x = 1$ $\sum_{n \geq 1} \frac{\sqrt{n}}{n+1} \sim \sum \frac{1}{n^{1+\frac{1}{2}}}$ div $\alpha = \frac{1}{2} < 1$

~~Obs~~ Obs $\lim \frac{a_{n+1}}{a_n} < 1 \Rightarrow$ s e conv

$\lim \frac{a_{n+1}}{a_n} > 1 \Rightarrow$ s. e dive

Criteriul Radicalului Fie $\sum_{n \geq 1} x_n$ cu $x_n \geq 0$ Atunci

1) Dacă $\exists \alpha < 1$ și $\exists n_0$ a.î. $\forall n \geq n_0 \Rightarrow \sqrt[n]{x_n} < \alpha \Rightarrow$ s este conv $\left(\Leftarrow \overline{\lim} \sqrt[n]{x_n} < 1 \right)$

2) Dacă $\exists x_{n_k}$ a.î. $\sqrt[n_k]{x_{n_k}} \geq 1 \Rightarrow$ seria este divergentă

$\left(\overline{\lim} \sqrt[n]{x_n} > 1 \right)$

$\sqrt[n]{x_n} < \alpha \Rightarrow x_n < \alpha^n \Rightarrow \sum x_n \leq \sum_{n \geq 1} \alpha^n = \frac{1}{1-\alpha}$

$n_0 = 1$

$\Rightarrow \sum_{n \geq 1} x_n < +\infty$

Crit $\sqrt{\ }$ $l = \lim_{n \to \infty} \sqrt[n]{x_n} \Rightarrow l > 1 \Rightarrow$ s. dive

$l < 1 \Rightarrow$ s. conv

$\lim \sqrt[n]{x_n} = \lim_{n \to \infty} \frac{x_{n+1}}{x_n}$

$\sum_{n \geq 1} x^n \left(\frac{n}{2n+1} \right)^n \frac{\sqrt{n}}{2n+1}$ Dacă $x > 2$ s. dive

dacă $x < 2$ s. conv

$x = 2$

$\sqrt[n]{a_n} = x \frac{n}{2n+1} \frac{\sqrt[2n]{n}}{\sqrt[n]{2n+1}} \to \frac{x}{2}$

$\sum_{n \geq 1} \left(\frac{2n}{2n+1} \right)^n \frac{\sqrt{n}}{2n+1} \sim \sum_{n \geq 1} \frac{\sqrt{n}}{2n+1}$

<u>Criteriul Raabe-Duhamel</u> Fie seria $\sum_{n\geq 1} a_n$ și $l=\lim_{n\to\infty} n\cdot\left(\frac{a_n}{a_{n+1}}-1\right)$ este inversul conv și div

Dacă $l>1 \Rightarrow$ este conv și dacă $l<1 \Rightarrow$ este div

<u>Algoritm</u> PAS 1 Criteriul raportului sau al radicalului

PAS 2 Criteriul comparației sau R.D.

<u>Ex</u> $\sum_{n\geq 1} x^n \frac{a(a+1)\dots(a+n)}{(n+7)!}$ $a>0,\ x>0$

$$\frac{a_{n+1}}{a_n}=\frac{x^{n+1}}{x^n}\cdot\frac{\cancel{a(a+1)\dots(a+n)}(a+n+1)}{\cancel{a(a+1)\dots(a+n)}}\cdot\frac{(n+7)!}{(n+8)!}$$

$$=x\cdot\frac{a+n+1}{n+8}\to x$$

$x>1$ div

$x<1$ conv

$$x=1\quad \sum_{n\geq 1}\frac{a(a+1)\dots(a+n)}{(n-7)!}$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=n\left(\frac{n+8}{a+n+1}-1\right)=n\,\frac{7-a}{a+n+1}\to 7-a$$

<u>$x=1$ $a=6$</u> $\sum_{n\geq 1}\frac{6\dots(6+n)}{(n+7)!}=\sum_{n\geq 1}\frac{1}{5!\dots(n+7)}\sim\sum\frac{1}{n}$ divergent

Criteriul lui Cauchy

Fie seria $\sum_{n\geq 1} x_n$ (cu $x_n \in \mathbb{R}$) Seria $\sum_{n\geq 1} x_n$ este convergentă $\Leftrightarrow \forall \varepsilon > 0\ \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon\ \forall p \in \mathbb{N}^* \Rightarrow$

$$|x_n + x_{n+1} + \dots + x_{n+p}| < \varepsilon \qquad s_n = \sum_{k=1}^{n} x_k$$

$$|s_{n+p} - s_n|$$

<u>Def</u> O serie $\sum_{n\geq 1} x_n$ s.n. absolut convergentă dacă seria $\sum_{n\geq 1} |x_n| < +\infty$ și semi convergentă dacă este con. dar nu e absolut con.

<u>Propoziție</u> O serie absolut convergentă este convergentă

<u>Dem</u> Fie seria $\sum_{n\geq 1} x_n$ care să fie absolut conv.

Crit Cauchy

$\Rightarrow \sum_{n\geq 1} |x_n|$ este conv $\Rightarrow$

$\forall \varepsilon > 0\ \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon$ $\big||x_n| + |x_{n+1}| + \dots + |x_{n+p}|\big| < \varepsilon$

$\Rightarrow \sum x_n$ este conv

<u>Ex 1</u> $\sum_{n\geq 1} \frac{\sin n^2}{n^2}$

$$\left|\frac{\sin n^2}{n^2}\right| \leq \frac{1}{n^2} \Rightarrow \sum_{n\geq 1} \left|\frac{\sin n^2}{n^2}\right| \leq \sum_{n\geq 1} \frac{1}{n^2} \text{ conv}$$

$\Rightarrow$ S. este absolut convergentă

Ex 2: $\sum_{n\geq 1} (-1)^{n+1}\cdot\frac{1}{n}$; $\sum_{n\geq 1} \left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n\geq 1}\frac{1}{n} = +\infty$

s. nu este abs. conv.

$$s_{2n} = \sum_{k=1} (-1)^{k-1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2n}-\frac{1}{2n+1} =$$

$$= \frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\dots+\frac{1}{2n(2n+1)} < +\infty$$

=> semi convergentă

C.L. $a_n \downarrow 0 \Rightarrow \sum_{n\geq 1}(-1)^n a_n$ este conv

$\frac{1}{n}\downarrow 0 \Rightarrow \sum_{n\geq 1}(-1)^n \frac{1}{n}$ este conv

$\frac{1}{\ln n}\downarrow 0 \Rightarrow \sum_{n\geq 1}(-1)^n\frac{1}{\ln n}$ este conv

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Criteriul Abel-Dirichlet   Seria $\sum_{n\geq 1} a_n x_n$ este conv. dacă

CAZ I
i) $a_n \to 0$
ii) $\sum_{n\geq 1}|a_n - a_{n+1}| < +\infty$
iii) $\exists M>0$ a.î. $\left|\sum_{k=1}^{n} x_k\right| \leq M$

CAZ II
i) șirul $(a_n)_{n\geq 1}$ este monoton și conv.
ii) seria $\sum_{n\geq 1} x_n$ este conv

$a_n \downarrow 0$ $\quad x_n = (-1)^n \quad s_{2n} = \sum_{k=1}^{2n} (-1)^k = \underbrace{-1+1}_{0} \quad s_{2n+1} = 1$

PAS 1 Studiem absolut con.

PAS 2 $\lim_{n\to\infty} x_n = 1$ $(x_n \not\to 0 \Rightarrow s$ dive.$)$

PAS 3 $\sum_{n\geq 1} |x_n| = +\infty$ Crit AD $\Rightarrow s$ semi con.

$x_n \to 0$ (C.L.)

$\sum_{n\geq 1} x^n \sin^\alpha \frac{1}{n}$, $x, \alpha \in \mathbb{R}$

PAS 1 $\sum_{n\geq 1} \overbrace{|x|^n \sin^\alpha \frac{1}{n}}^{a_n}$

$x = 0 \Rightarrow \sum 0$ este con

$x \neq 0 \quad \frac{a_{n+1}}{a_n} = \frac{|x|^{n+1} \sin^\alpha \frac{1}{n+1}}{|x|^n \sin^\alpha \frac{1}{n}} = |x| \frac{\sin^\alpha \frac{1}{n+1}}{\left(\frac{1}{n+1}\right)^\alpha} \cdot$

$\cdot \frac{\left(\frac{1}{n}\right)^\alpha}{\sin^\alpha \frac{1}{n}} \cdot \frac{n^\alpha}{(n+1)^\alpha} \to |x|$

$|x| > 1 \Rightarrow a_n \to \infty \Rightarrow s$ dive

$|x| < 1 \Rightarrow s$ este abs conv

$|x| = 1$ $(x = 1)$ $\sum_{n\geq 1} \sin^\alpha \frac{1}{n} \sim \sum_{n\geq 1} \frac{1}{n^\alpha}$ ~~conv~~ $\underbrace{\text{Conv} \Leftrightarrow \alpha > 1}_{\text{PAS 1}}$

$x=-1 \quad \sum_{n\geq 1} (-1)^n \sin^{\alpha}\frac{1}{n}$

<u>PAS 2</u> $\lim_{n\to\infty} \sin^{\alpha}\frac{1}{n} = \begin{cases} 0 & \alpha>0 \\ 1 & \alpha=0 \\ \infty & \alpha<0 \end{cases}$ | $\Rightarrow$ div.

$\alpha>1$ abs conv | $0<\alpha\leq 1$

$\alpha\leq 0$ div | $\sin^{\alpha}\frac{1}{n}\downarrow 0$ C.L. $\Rightarrow$ semi. conv.

<u>Ex 2</u> $\sum_{n\geq 1} x^n \frac{a(a+1)\dots(a+n)}{(n+5)!} \quad x\in\mathbb{R},\ a>0$

<u>PAS 1</u> $\sum_{n\geq 1} |x|^n \frac{a(a+1)\dots(a+n)}{(n+5)!}$ | $|x|>1 \Rightarrow l_{n} \to \infty \Rightarrow$ div

| $|x|<1 \Rightarrow$ abs conv

$x=0 \Rightarrow$ conv.

$x\neq 0 \Rightarrow \frac{a_{n+1}}{a_n} = |x| \frac{a+n+1}{n+6} \to |x|$

$|x|=1 \quad \sum_{n\geq 1} \frac{a(a+1)\dots(a+n)}{(n+5)!}$

R.D. $n\left(\frac{a_n}{a_{n+1}}-1\right) = n\left(\frac{n+6}{a+n+1}-1\right) = n\frac{5-a}{a+n+1} \to 5-a$

$4>a \Leftrightarrow 5-a>1$ conv

$4<a \Leftrightarrow 5-a<1$ div

$a=4 \quad \sum_{n\geq 1} \frac{4\cdot 5\cdots(4+n)}{(5+n)!} = \sum_{n\geq 1}\frac{1}{3!(5+n)} \Rightarrow$ div

$$* \sum_{n\geq 1} (-1)^n \overbrace{\frac{a(a+1)\dots(a+n)}{(n+5)!}}^{b_n} \Rightarrow \sum_{n\geq 1} (-1)^n b_n \quad n\left(\frac{b_n}{b_{n+1}}-1\right)\to l$$

$4>a$ $5-a>1$ abs conv $\qquad l>1$ s abs con.

$4=a$ semiconv $\qquad l=1$ rau

$5>a>4$ $0<5-a<1$ semiconv $\qquad 0<l<1$ semi con.

$5=a$ $\qquad l=0$

$a>5$ $5-a<0$ dive $\qquad l<0$ s dive

$a=5$

$$\sum_{n\geq 1} (-1)^n \frac{5\cdot 6\cdots 5+n}{(5+n)!} = \sum_{n\geq 1} (-1)^n \underbrace{\left(\frac{1}{4}\right)}_{\not\to 0} \text{ dive}$$

Serii de Puteri

Def O serie de forma $\sum_{n\geq 0} a_n(x-a)^n$ s.n. serie de puteri

$D = \{x \mid s(x)$ este convergenta$\}$ – domeniul de convergentă

$\rho = \frac{1}{\lim \sqrt[n]{|a_n|}} \in [0, +\infty]$ – raza de convergentă

<u>Teorema Cauchy-Hadamard</u> Fie $s(x) = \sum_{n \geq 0} a_n x^n$, Atunci

1) i) dacă $\rho = 0 \Rightarrow D = \{0\}$

ii) dacă $\rho = +\infty \Rightarrow D = \mathbb{R}$

iii) $0 < \rho < +\infty \Rightarrow (-\rho, \rho) \subset D \subset [-\rho, \rho]$

2) Serie $s_1(x) = \sum_{n \geq 1} a_n n x^{n-1}$ are $\rho_1 = \rho \Rightarrow s = s_1$ pe $(-\rho, \rho)$

3) $s^p = s_p(x) = \sum_{n \geq p} a_n n(n-1) \dots (n-p+1) x^{n-p}$

$s^p(0) = a_n \cdot n!$

<u>Ex 1</u> $\overline{\lim} \sqrt[n]{|a_n x^n|} = \overline{\lim} |x| \sqrt[n]{|a_n|} = \frac{|x|}{\rho}$

$|x| < \rho$ abs con.

$|x| > \rho$ s. div

<u>Ex</u> $\sum_{n \geq 0} a^n \quad s_n = \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \to \frac{1}{1-a} \quad |a| < 1$

$|a| \geq 1$

$\sum_{n \geq 1} n \cdot a^{n-1} \; ; \; s_{1,n} = \sum_{k=1}^{n} k \cdot a^{k-1} = \sum_{k=1}^{n} (a^k)' =$

$$= \left(\sum_{k=1}^{n} a^k\right)' = \left(\frac{1-a^{n+1}}{1-a}\right)' = \left(\frac{1}{1-a}\right)' - \left(\frac{a^{n+1}}{1-a}\right)' \to 0$$

$$\left(\frac{1}{1-a}\right)' = \frac{1}{(1-a)^2}$$

Ex 1 $\sum_{n\geq 1} x^n n^n \quad \sqrt[n]{n^n} = n \to +\infty \Rightarrow \rho = \frac{1}{\infty} = 0 \Rightarrow D = \{0\}$

Ex 2 $\sum_{n\geq 1} \frac{x^n}{n^\alpha} \quad \sqrt[n]{\frac{1}{n^\alpha}} \to 1 \; ; \; \rho = \frac{1}{1} = 1 \Rightarrow (-1,1) \subset D \subset [-1,1]$ $\quad D = [-1,1] \; \alpha > 1$ $\quad D = (-1,1) \; \alpha < 0$

Ex 3 $g(x) = \sum_{n\geq 0} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \quad D = [-1,1) \; 0 < \alpha \leq 1$

$\sqrt[n]{\frac{1}{n!}} \to +0 \quad \rho = \frac{1}{0} = +\infty \Rightarrow D = \mathbb{R}$

$$g'(x) = \sum_{n\geq 1} \left(\frac{x^n}{n}\right)' = \sum_{n\geq 1} \frac{n x^{n-1}}{(n-1)!\, n} = (n-1 = m)$$

$$= \sum_{m\geq 0} \frac{x^m}{m!} = g(x)$$

$h(x) = g(x) \cdot e^{-x}$

$h'(x) = g'(x) e^{-x} - e^{-x} g(x) = 0 \Rightarrow h = c$

$g'(x) = g(x)$

$g(0) = c e^0 \Rightarrow c = 1$, $g(0) = 1$, $e^0 = 1$