

Laborator Electricitate 4

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1 Problemă Seminar

O cantitate de sarcină Q este distribuită sub formă de strat subțire sferic a =raza sferei. De-a lungul unui diametr al acestei sfere, se plimbă un corp punctiform cu sarcină q . Calculați forța cu care acționează sarcina Q asupra corpului punctiform cu sarcina q .

1.1 Coordonate polare și aria de pe o sferă

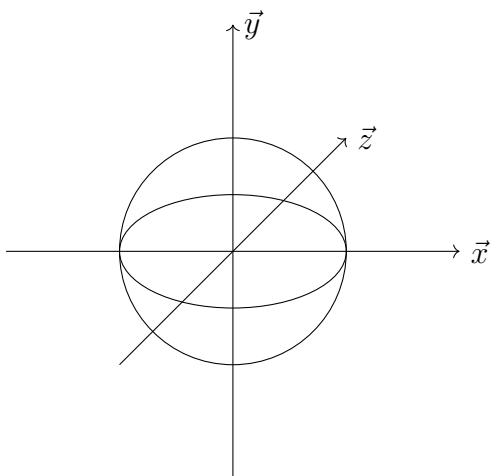


Figure 1: Cerc, elipsă și vectori

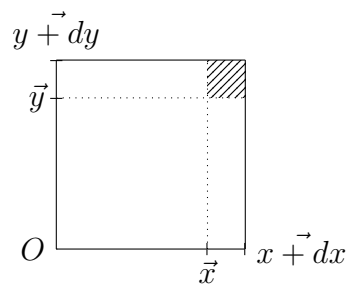


Figure 2: Coordonate Carteziane

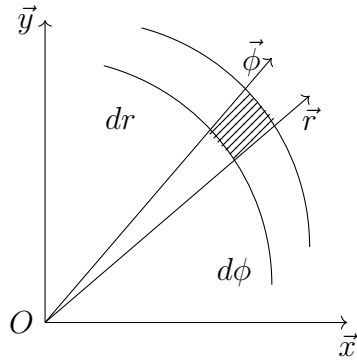


Figure 3: Coordonate Polare

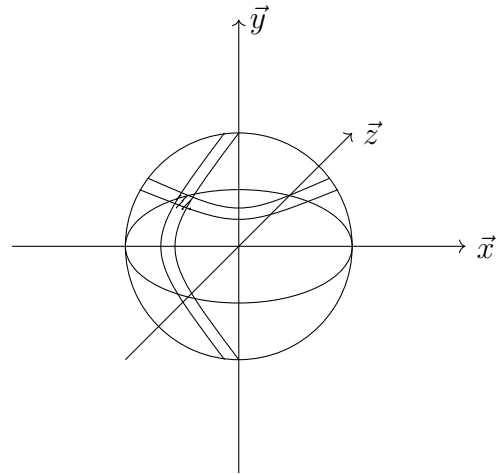


Figure 4: Arie de pe sferă

1.2 Formule Matematice

$$dS = r dr d\phi \quad \phi \in [0, 2\pi] \quad (1)$$

$$(x, y, z) \longrightarrow (r, \theta, \phi) \quad \theta \in [0, \pi] \quad (2)$$

$$\begin{cases} x = r \sin(\theta) \cos(\phi) \\ y = r \sin(\theta) \sin(\phi) \\ z = r \cos(\theta) \end{cases} \quad x, y, z \in [-r, r] \quad (3)$$

$$\begin{cases} L_1 = r \sin(\theta) d\phi \\ L_2 = r d\theta \\ dS = r^2 \sin(\theta) d\theta d\phi \end{cases} \quad (4)$$

1.3 Simplificare Integrala dublă

$$\sum_{i=1}^2 \sum_{j=1}^3 a_i b_j = \sum_{i=1}^2 (a_i b_1 + a_i b_2 + a_i b_3) = (a_1 + a_2)(b_1 + b_2 + b_3) = \left(\sum_{i=1}^2 a_i \right) \left(\sum_{j=1}^3 b_j \right) = \sum_{i=1}^2 \sum_{j=1}^3 a_i b_j \quad (5)$$

$$\implies \int_a^b \int_a^b f(x) f(y) = \int_a^b f(x) \int_a^b f(y) \quad (6)$$

1.4 Rezolvare Problemă

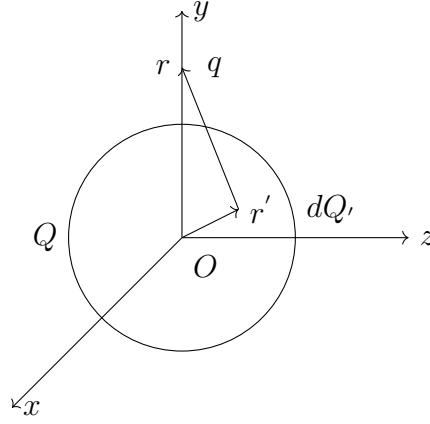


Figure 5: Schemă problemă

$$d\vec{F} = k \frac{qdQ'}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') \quad (7)$$

$$\begin{cases} Q \dots\dots\dots 4\pi a^2 \\ dQ' \dots\dots\dots dS' \\ \implies dQ' = \frac{QdS'}{4\pi a^2} \end{cases} \quad (8)$$

$$\begin{cases} \vec{r} - \vec{r}' = (-x', -y', z - z') \\ |\vec{r} - \vec{r}'| = \sqrt{x'^2 + y'^2 + (z - z')^2} \end{cases} \quad (9)$$

$$d\vec{F} = k \frac{Q \frac{dS'}{4\pi a^2} q}{\sqrt{x'^2 + y'^2 + (z - z')^2}^3} (-x' \vec{i} - y' \vec{j} + (z - z') \vec{k}) \quad (10)$$

$$= \frac{kQq}{4\pi a^2} \frac{dS'}{x'^2 + y'^2 + (z - z')^2} (-x' \vec{i} - y' \vec{j} + (z - z') \vec{k}) \quad (11)$$

$$\begin{cases} dF_x = \frac{kQq}{4\pi a^2} \frac{dS'(-x')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ dF_y = \frac{kQq}{4\pi a^2} \frac{dS'(-y')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ dF_z = \frac{kQq}{4\pi a^2} \frac{dS'(-z')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \end{cases} \quad (12)$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2} \int \frac{dS'(-x')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ F_y = \frac{kQq}{4\pi a^2} \int \frac{dS'(-y')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ F_z = \frac{kQq}{4\pi a^2} \int \frac{dS'(-z')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \end{cases} \quad (13)$$

$$\begin{cases} x' = a \sin \theta \cos \phi \\ y' = a \sin \theta \sin \phi \\ z' = a \cos \theta \end{cases} \quad (14)$$

$$\begin{aligned} &\implies (x'^2 + y'^2 + (z - z')^2)^{\frac{3}{2}} = (a^2 \sin^2 \theta \cos^2 \phi + a^2 \sin^2 \theta \sin^2 \phi + (z - a \cos \theta)^2)^{\frac{3}{2}} \\ &= (a^2 \sin^2 \theta + (z - a \cos \theta)^2)^{\frac{3}{2}} = (a^2 \sin^2 \theta + z^2 + a^2 \cos^2 \theta - 2a \cos \theta)^{\frac{3}{2}} \\ &= (a^2 + z^2 - 2a \cos \theta)^{\frac{3}{2}} = a^3 (1 + (\frac{z}{a})^2 - 2\frac{z}{a} \cos \theta)^{\frac{3}{2}} \\ &= a^3 \left(\left(\frac{z}{a} \right)^2 - \cos \theta \frac{z}{a} + 1 \right)^{\frac{3}{2}} \quad \text{notăm } \frac{z}{a} = m \\ &\implies a^3 (m^2 - \cos \theta m + 1)^{\frac{3}{2}} \end{aligned}$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2} \int_0^\pi \int_0^{2\pi} \frac{a^2 \sin \theta d\theta d\phi (-a \sin \theta \cos \phi)}{a^3 (1 + m^2 - 2m \cos \theta)^{\frac{3}{2}}} \\ F_y = \frac{kQq}{4\pi a^2} \int_0^\pi \int_0^{2\pi} \frac{a^2 \sin \theta d\theta d\phi (-a \sin \theta \sin \phi)}{a^3 (1 + m^2 - 2m \cos \theta)^{\frac{3}{2}}} \\ F_z = \frac{kQq}{4\pi a^2} \int_0^\pi \int_0^{2\pi} \frac{a^2 \sin \theta d\theta d\phi (z - a \cos \theta)}{a^3 (1 + m^2 - 2m \cos \theta)^{\frac{3}{2}}} \end{cases} \quad (15)$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2} \int_0^\pi \int_0^{2\pi} \frac{\sin^2 \theta d\theta (-\cos \phi d\phi)}{(1 + m^2 - 2m \cos \theta)^{\frac{3}{2}}} \\ F_y = \frac{kQq}{4\pi a^2} \int_0^\pi \int_0^{2\pi} \frac{\sin^2 \theta d\theta (-\sin \phi d\phi)}{(1 + m^2 - 2m \cos \theta)^{\frac{3}{2}}} \\ F_z = \frac{kQq}{4\pi a^2} \int_0^\pi \int_0^{2\pi} \frac{\sin^2 \theta d\theta d\phi (m - \cos \theta)}{(1 + m^2 - 2m \cos \theta)^{\frac{3}{2}}} \end{cases} \quad (16)$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2} \int_0^\pi \frac{\sin^2 \theta d\theta}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} \int_0^{2\pi} (-\cos\phi d\phi) = 0 \\ F_y = \frac{kQq}{4\pi a^2} \int_0^\pi \frac{\sin^2 \theta d\theta}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} \int_0^{2\pi} (-\sin\phi d\phi) = 0 \\ F_z = \frac{kQq}{4\pi a^2} \int_0^\pi \frac{\sin^2 \theta d\theta (m-\cos\theta)}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} \int_0^{2\pi} d\phi \end{cases} \quad (17)$$

$$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_z = \frac{kQq}{2a^2} \int_0^\pi \frac{(m-\cos\theta)\sin\theta d\theta}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} d\theta \end{cases} \quad (18)$$

$$\implies F_z = \frac{kQq}{2a^2} \int_{-1}^1 \frac{(m-x)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \quad (19)$$

$$F_z = \frac{kQq}{4ma^2} \int_{-1}^1 \frac{(-m^2-2mx)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \quad (20)$$

$$F_z = \frac{kQq}{4ma^2} \int_{-1}^1 \frac{(1+m^2-2mx+m^2-1)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \quad (21)$$

$$F_z = \frac{kQq}{4ma^2} \int_{-1}^1 \frac{(m^2-2mx)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \int_{-1}^1 \frac{(m^2-1)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \quad (22)$$

$$F_z = \frac{kQq}{4m^2a^2} \int_{-1}^1 (1+m^2-2mx)^{-\frac{1}{2}} \int_{-1}^1 (1+m^2-2mx)^{-\frac{3}{2}} \quad (23)$$

$$F_z = -\frac{kQq}{4m^2a^2} \left(|m-1| - |m+1| - (m^2-1) \left(\frac{1}{|m-1|} - \frac{1}{|m+1|} \right) \right) \quad (24)$$

$$m = \frac{z}{a}; m > 1 \implies z > a \text{ deasupra mingii}$$

$$\implies F_z = -\frac{kQq}{4m^2a^2} \left(m-1 - (m+1) - (m^2-1) \left(\frac{1}{m-1} - \frac{1}{m+1} \right) \right) \quad (25)$$

$$F_z = -\frac{kQq}{4m^2a^2} (-4) \quad (26)$$

$$F_z = \frac{4kQq}{4m^2a^2} = \frac{kQq}{m^2a^2} = \frac{kQq}{z^2} \quad (27)$$

$$m = \frac{z}{a}; m < 1 \implies z < a \text{ în minge}$$

$$\implies F_z = -\frac{kQq}{4m^2a^2} \left(1 - m - (m + 1) - (m^2 - 1) \left(\frac{1}{1 - m} - \frac{1}{m + 1} \right) \right) \quad (28)$$

$$F_z = -\frac{kQq}{4m^2a^2} (2m) \quad (29)$$

$$F_z = -\frac{2kQqm}{4m^2a^2} = -\frac{kQqm}{2m^2a^2} = \frac{kQq}{2za} \quad (30)$$