16.01.2025

$\boxed{C_{14} - 16\ CTI}$

Aducerea la o formă canonică a conicelor.

$(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$ sp. afin euclidian.

$\Gamma: f(x_1, x_2) = \underline{a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2} + \underline{2b_1x_1 + 2b_2x_2} + c = 0$

$A = A^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \neq O_2 \quad \tilde{A} = \begin{pmatrix} A & B^T \\ B & c \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix}$

$B = (b_1 \ b_2)$

$f(x) = X^TAX + 2BX + c = 0$

$\delta = \det A \quad , \quad \Delta = \det \tilde{A}$

$\Delta \neq 0 \quad \Gamma$ nedegenerată

$\Delta = 0 \quad \Gamma$ degenerată

I) $\delta \neq 0 \quad \Gamma$ are centru unic $P_0: \begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \quad P_0(x_1^0, x_2^0)$

$\mathcal{R}_0 = \{O; e_1, e_2\} \xrightarrow[\text{translatie}]{\theta} \mathcal{R}' = \{P_0; e_1, e_2\} \xrightarrow[\text{rotatie}]{\tau} \mathcal{R}'' = \{P_0; e_1', e_2'\}$

$\theta: X = X' + X_0 \ , \ X_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \qquad \tau: X' = RX''$

$R \in SO(2)$

$R = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$

$e_1' = (l_1, m_1)$

$e_2' = (l_2, m_2)$

II $\delta = 0 \quad \Gamma$ nu are centru unic

$\mathcal{R}_0 = \{O; e_1, e_2\} \xrightarrow[\text{rotatie}]{\tau} \mathcal{R}' = \{O; e_1', e_2'\} \xrightarrow[\text{translatie}]{\theta} \mathcal{R}'' = \{P_0; e_1', e_2'\}$

$Q: \mathbb{R}^2 \to \mathbb{R}, \quad Q(x) = X^TAX = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

Aducem Q la o f. canonică prin metoda valorilor proprii

$$P_A(\lambda) = \det(A - \lambda I_2) = 0 \Rightarrow \lambda^2 - Tr(A)\lambda + \underset{0}{\det A} = 0$$

$$\lambda(\lambda - Tr(A)) = 0 \rightarrow \lambda_1 = Tr(A) \neq 0 \ ; \ \lambda_2 = 0.$$

$V_{\lambda_1} = \langle \{e_1'\} \rangle \ ; \ V_{\lambda_2} = \langle \{e_2'\} \rangle$

$\{e_1', e_2'\}$ reper ortonormat (versori proprii, coresp. la valori proprii dist)

$e_k' = (l_k, m_k), \quad k = \overline{1,2}$

$$\tau : X = RX' \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

(putem alege $R \in SO(2)$ i.e $R \in O(2)$ și $\det R = 1$)

$$\begin{cases} x_1 = l_1 x_1' + l_2 x_2' \\ x_2 = m_1 x_1' + m_2 x_2' \end{cases}$$

$$\tau(\Gamma) : \lambda_1 x_1'^2 + 2b_1(l_1 x_1' + l_2 x_2') + 2b_2(m_1 x_1' + m_2 x_2') + c = 0$$

$$\lambda_1 x_1'^2 + 2\underbrace{(b_1 l_1 + b_2 m_1)}_{b_1'} x_1' + 2\underbrace{(b_1 l_2 + b_2 m_2)}_{b_2'} x_2' + c = 0$$

$$\tau(\Gamma) : \lambda_1 x_1'^2 + 2b_1' x_1' + 2b_2' x_2' + c = 0.$$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & b_1' \\ 0 & 0 & b_2' \\ b_1' & b_2' & c \end{vmatrix} = -b_2' \begin{vmatrix} \lambda_1 & 0 \\ b_1' & b_2' \end{vmatrix} = -b_2'^2 \cdot \lambda_1$$

1) $\Delta \neq 0 \Leftrightarrow b_2' \neq 0$ (Γ nedeg.).

$$\lambda_1 x_1'^2 + 2b_1' x_1' + 2b_2' x_2' + c = 0$$

$$\lambda_1 \left(x_1'^2 + 2 \cdot \frac{b_1'}{\lambda_1} x_1' + \left(\frac{b_1'}{\lambda_1}\right)^2 \right) + 2b_2' x_2' + \overbrace{c - \frac{b_1'^2}{\lambda_1}}^{c'} = 0$$

$$\lambda_1 \left(x_1' + \frac{b_1'}{\lambda_1} \right)^2 + 2b_2' \left(x_2' + \frac{c'}{2b_2'} \right) = 0$$

$$\begin{aligned} x_1'' &= x_1' + \frac{b_1'}{\lambda_1} \\ x_2'' &= x_2' + \frac{c'}{2b_2'} \end{aligned} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} -\frac{b_1'}{\lambda_1} \\ -\frac{c'}{2b_2'} \end{pmatrix}$$

$$\theta : X' = X'' + X_0$$

1) $\theta(\tau(\Gamma)) : \boxed{\lambda_1 x_1''^2 + 2b_2' x_2'' = 0}$ Parabolă

: $\left.\begin{array}{l} \tau : X = RX' \\ \theta : X' = X'' + X_0 \end{array}\right| \Rightarrow \theta(\tau) : X = RX'' + RX_0$.

$RX_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\qquad P_0(\alpha, \beta)$.

$\boxed{\text{OBS}}$ $\mathcal{P} : x_2^2 = 2p x_1$.

$p > 0$

2) $\Delta = 0 \Leftrightarrow b_2' = 0$ (Γ degenerată)

$\tau(\Gamma) : \underline{\lambda_1 x_1'^2 + 2b_1' x_1'} + c = 0$

$$\lambda_1 \left(x_1' + \frac{b_1'}{\lambda_1}\right)^2 + \underbrace{c - \frac{b_1'^2}{\lambda_1}}_{c'} = 0$$

$$\begin{cases} x_1'' = x_1' + \dfrac{b_1'}{\lambda_1} \\ x_2'' = x_2' \end{cases} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} -\dfrac{b_1'}{\lambda_1} \\ 0 \end{pmatrix}$$

$\theta : X' = X'' + X_0$

$\theta(\tau(\Gamma)) : \lambda_1 x_1''^2 + c' = 0. \Rightarrow x_1''^2 = -\dfrac{c'}{\lambda_1}$ $\quad \phi$ / drepte //.

$\theta \circ \tau : X = RX'' + RX_0$.

$RX_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\quad P_0(\alpha, \beta)$ (în raport cu $R_0 = \{O; e_1, e_2\}$)

Δ (natura)	δ (genul)	Tipul conicei
$\Delta \neq 0$	$\delta > 0$	ϕ Elipsă
	$\delta < 0$	Hiperbolă
	$\delta = 0$	Parabolă.
$\Delta = 0$	$\delta > 0$	punct dublu.
	$\delta < 0$	Drepte concurente.
	$\delta = 0$	ϕ, Drepte paralele.

Exemplu ($\delta = 0$)

Fie conica $\Gamma : f(x_1,x_2) = x_1^2 - 4x_1x_2 + 4x_2^2 - 6x_1 + 2x_2 + 1 = 0$.

Să se aducă la o formă canonică, efectuând izometrii.

Sol

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix};\ \delta = \det A = 0$$

$$\tilde{A} = \begin{pmatrix} \boxed{\begin{matrix} 1 & -2 \\ -2 & 4 \end{matrix}} & \begin{matrix} -3 \\ 1 \end{matrix} \\ \begin{matrix} -3 & 1 \end{matrix} & 1 \end{pmatrix},\ \Delta = \det \tilde{A} = \begin{vmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ -3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -8 & 1 & 0 \\ 1 & 3 & 0 \\ -3 & 1 & 1 \end{vmatrix}$$

$= -24 - 1 = -25 \neq 0$ (Γ nedeg) Γ = parabolă

$Q : \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = x_1^2 - 4x_1x_2 + 4x_2^2$

Aducem la o f. canonică cu met. valorilor proprii

$P_A(\lambda) = \det(A - \lambda I_2) = 0 \Rightarrow \lambda^2 - 5\lambda + 0 = 0$

$\lambda(\lambda - 5) = 0 \Rightarrow \lambda_1 = 5,\ \lambda_2 = 0$

$V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid AX = 5X\} = \{(x_1, -2x_1) \mid x_1 \in \mathbb{R}\} = \langle\{e_1'\}\rangle$ $\quad e_1' = \frac{1}{\sqrt{5}}(1,-2)$

$(A - 5I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2x_1 - x_2 = 0 \Rightarrow x_2 = -2x_1$

$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid AX = 0 \cdot X\} = \{(2x_2, x_2) \mid x_2 \in \mathbb{R}\} = \langle\{e_2'\}\rangle;\ e_2' = \frac{1}{\sqrt{5}}(2,1)$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \in SO(2)$$

$\mathcal{G} : X = RX' \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \Rightarrow \begin{cases} x_1 = \frac{1}{\sqrt{5}}(x_1' + 2x_2') \\ x_2 = \frac{1}{\sqrt{5}}(-2x_1' + x_2') \end{cases}$

$\mathcal{G}(\Gamma) : 5x_1'^2 - \frac{6}{\sqrt{5}}(x_1' + 2x_2') + \frac{2}{\sqrt{5}}(-2x_1' + x_2') + 1 = 0$

$5x_1'^2 - \frac{10}{\sqrt{5}}x_1' - \frac{10}{\sqrt{5}}x_2' + 1 = 0 \quad \Big|:5$

$$x_1'^2 - \frac{2}{\sqrt{5}}x_1' - \frac{2}{\sqrt{5}}x_2' + \frac{1}{5} = 0 \Rightarrow \left(x_1' - \frac{1}{\sqrt{5}}\right)^2 - \frac{2}{\sqrt{5}}x_2' = 0$$

$$\begin{cases} x_1'' = x_1' - \frac{1}{\sqrt{5}} \\ x_2'' = x_2' \end{cases} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$$

$\theta: \quad X' = X'' + X_0$ *translație.*

$$\theta \circ \tau: \quad \left.\begin{array}{l} X = RX' \\ X' = X'' + X_0 \end{array}\right| \Rightarrow X = RX'' + \underline{\underline{RX_0}}$$

$$RX_0 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$P_0\left(\frac{1}{5}, -\frac{2}{5}\right)$$

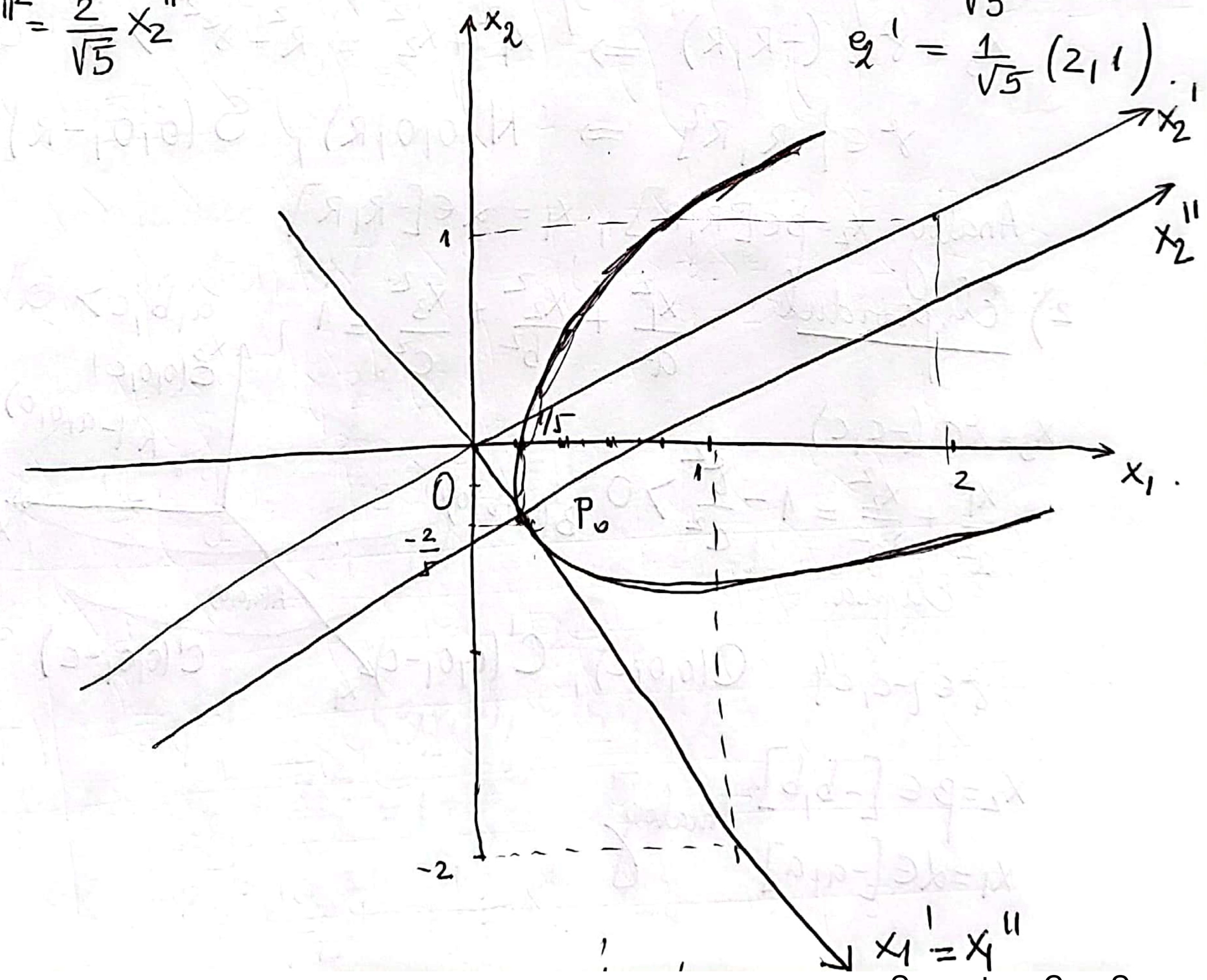
$\theta \circ \tau(\Gamma): \quad x_1''^2 - \frac{2}{\sqrt{5}}x_2'' = 0$ *Parabolă*

$$\mathcal{R}_0 = \{O; e_1, e_2\} \xrightarrow{\tau} \{O; e_1', e_2'\} \xrightarrow{\theta} \{P_0; e_1', e_2'\}$$

$e_1' = \frac{1}{\sqrt{5}}(1, -2)$

$e_2' = \frac{1}{\sqrt{5}}(2, 1)$

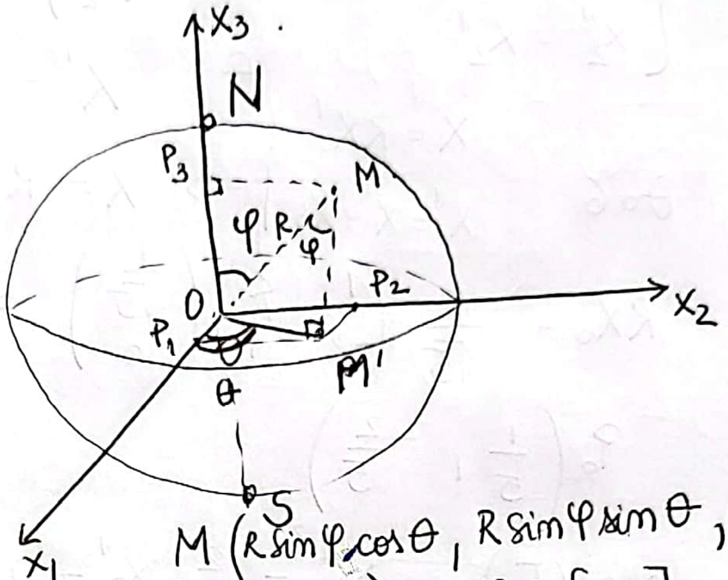
$x_1''^2 = \frac{2}{\sqrt{5}}x_2''$



Cuadrice studiate pe ec. reduse

1) $\mathcal{S}(A(a,b,c),R) : (x_1-a)^2+(x_2-b)^2+(x_3-c)^2-R^2=0$.

$\mathcal{S}(O(0,0,0),R) : x_1^2+x_2^2+x_3^2-R^2=0$.



$M(R\sin\varphi\cos\theta,\ R\sin\varphi\sin\theta,\ R\cos\varphi)$

$\theta\in[0,2\pi)$, $\varphi\in[0,\pi]$.

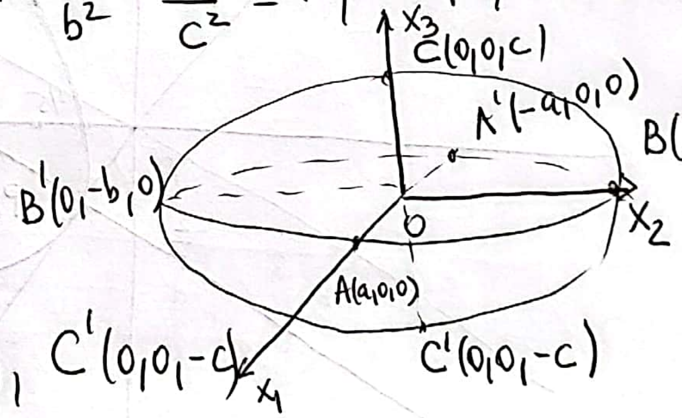
$\cap$ cu plane // cu planele de coordonate $\Rightarrow$ cercuri.

$x_3=\gamma\in(-R,R) \Rightarrow x_1^2+x_2^2=R^2-\gamma^2>0$ Cerc.

$\gamma\in\{-R,R\} \Rightarrow N(0,0,R),\ S(0,0,-R)$

Analog $x_2=\beta\in[-R,R]$, $x_1=\alpha\in[-R,R]$

2) Elipsoidul $\dfrac{x_1^2}{a^2}+\dfrac{x_2^2}{b^2}+\dfrac{x_3^2}{c^2}=1$, $a,b,c>0$



$x_3=\gamma\in(-c,c)$

$\dfrac{x_1^2}{a^2}+\dfrac{x_2^2}{b^2}=1-\dfrac{\gamma^2}{c^2}>0$

Elipsă

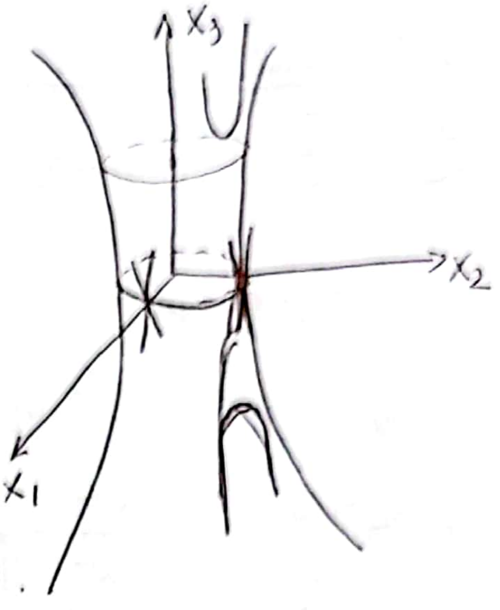
$\gamma\in\{-c,c\}$ $C(0,0,c)$, $C'(0,0,-c)$

$x_2=\beta\in[-b,b]$ Analog

$x_1=\alpha\in[-a,a]$

3) Hiperboloid cu o pânză

$\mathcal{H}_1: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1$



- $x_3 = \gamma \in \mathbb{R} \Rightarrow \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 + \frac{\gamma^2}{c^2} > 0$

Dacă $\gamma = 0 \Rightarrow \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

(Elipsa colier a lui $\mathcal{H}_1$).

- $x_2 = \beta$: $\frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 1 - \frac{\beta^2}{b^2}$ Hiperbola

$\beta \neq \pm b$

$\beta = \pm b$ $\frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 0 \Rightarrow x_3 = \pm \frac{c}{a} x_1$ 2 drepte concurente

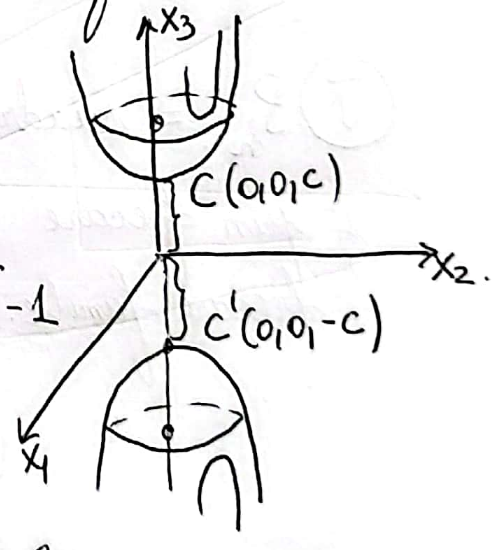
- $x_1 = \alpha$ $\frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1 - \frac{\alpha^2}{a^2}$ Hiperbola

$\alpha \neq \pm a$

$\alpha = \pm a$ $\frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 0 \Rightarrow x_3 = \pm \frac{c}{b} x_2$ 2 drepte concurente

(T) $\mathcal{H}_1$ cuadrică dublu riglată : $\exists$ 2 familii de generatoare și prin fiecare pct $\in \mathcal{H}_1$ trece câte o dreaptă din fiecare familie.

4) Hiperboloid cu 2 pânze



$\mathcal{H}_2: -\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$

$x_3 = \gamma \in (-\infty, -c) \cup (c, \infty) \Rightarrow \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = \frac{\gamma^2}{c^2} - 1$

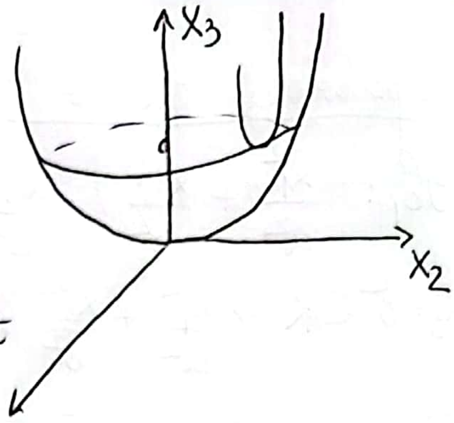
Elipsa

$\gamma \in \{-c, c\}$ $C(0,0,c)$, $C'(0,0,-c)$

$x_2 = \beta \in \mathbb{R}$ $-\frac{x_1^2}{a^2} + \frac{x_3^2}{c^2} = 1 + \frac{\beta^2}{b^2}$ Hiperbola

$x_1 = \alpha \in \mathbb{R}$ $-\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 + \frac{\alpha^2}{a^2}$ —//—

5) Paraboloid eliptic



$\mathcal{P}_e: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2x_3, \ x_3 > 0$

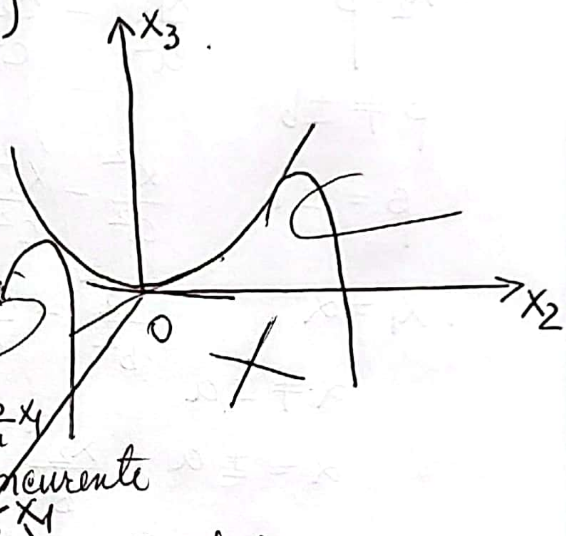
a) $x_3 = \gamma > 0 \Rightarrow \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2\gamma$ Elipsa

$\gamma = 0 \Rightarrow O(0,0,0)$

b) $x_2 = \beta$ $\quad \frac{x_1^2}{a^2} = 2\left(x_3 - \frac{\beta^2}{2b^2}\right)$ Parabola

c) $x_1 = \alpha$ $\quad \frac{x_2^2}{b^2} = 2\left(x_3 - \frac{\alpha^2}{2a^2}\right)$ Parabola

6) Paraboloidul hiperbolic



$\mathcal{P}_h: \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2x_3$

a) $x_3 = \gamma \in \mathbb{R}^* \Rightarrow \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2\gamma$ Hiperbola

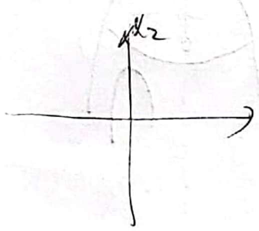
$\gamma = 0 \Rightarrow \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 0 \Rightarrow x_2 = \pm \frac{b}{a} x_1$

Drepte concurente

b) $x_2 = \beta \Rightarrow \frac{x_1^2}{a^2} = 2\left(x_3 + \frac{\beta^2}{2b^2}\right)$ Parabola

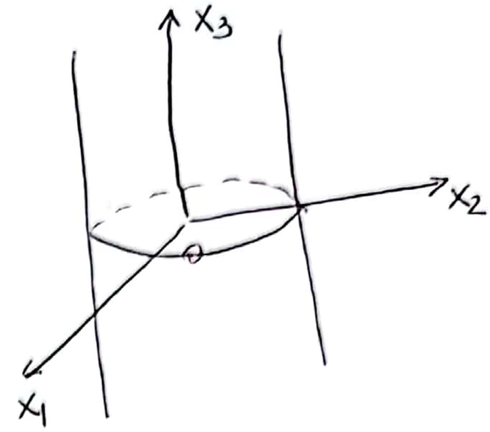
c) $x_1 = \alpha$ $\quad \frac{x_2^2}{b^2} = -2\left(x_3 - \frac{\alpha^2}{2a^2}\right)$ —||—

(T) $\mathcal{P}_h$ = cuadrică dublu riglată; $\exists$ 2 fam. de generatoare și prin fiecare pct $\in \mathcal{P}_h$ trece câte 2 dr din fiecare familie.

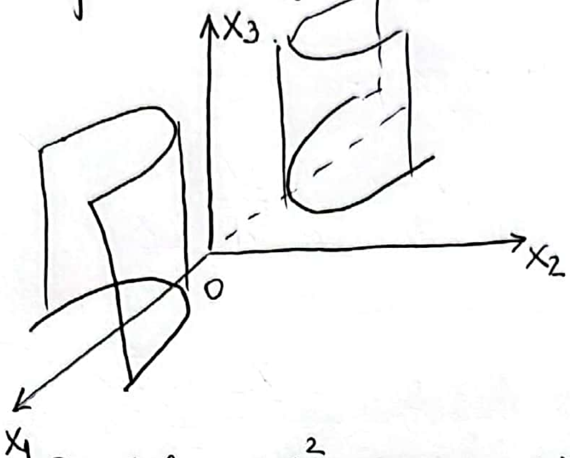


7) Cilindrul

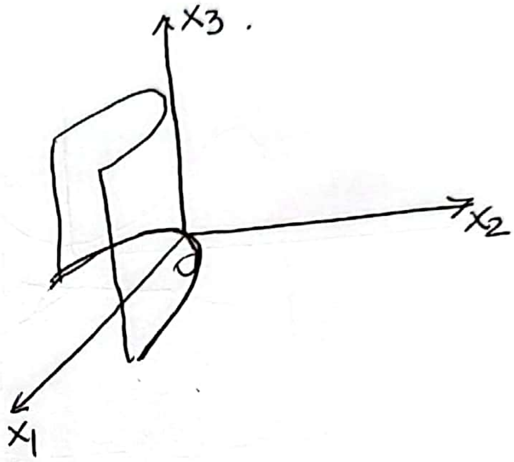
a) Eliptic: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$



b) hiperbolic $\frac{x_1^2}{a^2} - \frac{x_2^2}{b} = 1$



c) Parabolic $x_2^2 = 2px_1$, $p>0$



8) Conul pătratic

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 0$$