Seminar II Algebră

Forma eșalon

Th. Hamilton-Cayley

(1) a) $A=\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}$; $A^{-1}=?$ $\Rightarrow 1\cdot(-1)^{1+1}\begin{vmatrix}0&1\\1&1\end{vmatrix}=-1$

1) $A^{-1}=\frac{1}{\det A}\cdot A^*$

2) Th Hamilton-Cayley

3) Gauss-Jordan

$$(-1)\left[A^3-\sigma_1A^2+\sigma_2A-\sigma_3I_3\right]=O_3 \mid \cdot A^{-1}$$

$$(-1)\left[A^2-\sigma_1A+\sigma_2-\sigma_3A^{-1}\right]=O_3$$

$$-A^2+\sigma_1A+\sigma_2I_3-\sigma_3\cdot A^{-1}=O_3$$

$$\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&1\\0&1&2\end{pmatrix}$$

$$-A^2+\sigma_1A+\sigma_2I_3=\sigma_3\cdot A^{-1}$$

$$A^{-1}=\frac{-A^2+\sigma_1A+\sigma_2I_3}{\sigma_3}$$

$$A^{-1}=\frac{-\begin{pmatrix}1&1&1\\0&1&1\\0&1&2\end{pmatrix}+2\cdot\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}+\left(\begin{vmatrix}1&1\\0&0\end{vmatrix}+\begin{vmatrix}1&0\\0&1\end{vmatrix}+\begin{vmatrix}0&1\\1&1\end{vmatrix}\right)\cdot I_3}{-1}$$

$$A^{-1}=\frac{-\begin{pmatrix}1&1&1\\0&1&1\\0&1&2\end{pmatrix}+2\cdot\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}+0\cdot I_3}{-1}$$

$$A^{-1}=-\begin{pmatrix}2&2&0\\0&0&2\\0&2&2\end{pmatrix}+\begin{pmatrix}1&1&1\\0&1&1\\0&1&2\end{pmatrix}=\begin{pmatrix}-1&-1&1\\0&1&-1\\0&-1&0\end{pmatrix}\cdot(-1)=$$

$$A^{-1}=\begin{pmatrix}1&1&-1\\0&-1&1\\0&1&0\end{pmatrix}$$

$4-\frac{1}{20}=\cancel{?}$ $\quad \frac{80}{20}-\frac{1}{20}=\frac{79}{20}$ $\quad \frac{1}{2}+\frac{1}{8}=\frac{4}{8}+\frac{1}{8}=\frac{5}{8}$ $\quad \frac{10}{2}-\frac{9}{2}=\frac{1}{2}$

b) $A=\begin{pmatrix}2&-1&3\\0&4&1\\3&1&5\end{pmatrix}$

$$\left(\begin{array}{ccc|ccc}2&-1&3&1&0&0\\0&4&1&0&1&0\\3&1&5&0&0&1\end{array}\right)\xrightarrow{L_3-\frac{3}{2}L_1}\left(\begin{array}{ccc|ccc}2&-1&3&1&0&0\\0&4&1&0&1&0\\0&-\frac{1}{2}&\frac{1}{2}&-\frac{3}{2}&0&1\end{array}\right)\xrightarrow{L_3+L_2:8}$$

$$\left(\begin{array}{ccc|ccc}2&-1&3&1&0&0\\0&4&1&0&1&0\\0&0&\frac{5}{8}&-\frac{3}{2}&\frac{1}{8}&1\end{array}\right)\xrightarrow{\substack{L_1:2\\L_2:4\\L_3\cdot\frac{8}{5}}}$$

$$\left(\begin{array}{ccc|ccc}1&-\frac{1}{2}&\frac{3}{2}&\frac{1}{2}&0&0\\0&1&\frac{1}{4}&0&4&0\\0&0&1&-\frac{24}{10}&\frac{1}{5}&\frac{8}{5}\end{array}\right)\xrightarrow{L_2-L_1\cdot\frac{1}{4}}\left(\begin{array}{ccc|ccc}1&-\frac{1}{2}&\frac{3}{2}&\frac{1}{2}&0&0\\0&1&0&+\frac{24}{40}&\frac{79}{20}&-\frac{1}{4}\\0&0&1&-\frac{3}{2}&\frac{1}{8}&1\end{array}\right)$$

forma eșalon

(2) $A=\begin{pmatrix}3&1&2\\0&4&1\\1&1&0\end{pmatrix}\sim\begin{pmatrix}\boxed{1}&1&0\\0&4&1\\3&1&2\end{pmatrix}\xrightarrow{L_3-3L_1}\begin{pmatrix}1&1&0\\0&4&1\\0&-2&2\end{pmatrix}\sim\begin{pmatrix}1&1&0\\0&-2&2\\0&4&1\end{pmatrix}\xrightarrow{L_2\cdot-\frac{1}{2}}$

$\begin{pmatrix}1&1&0\\0&1&-1\\0&4&1\end{pmatrix}\xrightarrow{L_3-4\cdot L_1}\begin{pmatrix}1&1&0\\0&1&-1\\0&0&5\end{pmatrix}\xrightarrow{L_3\cdot\frac{1}{5}}\begin{pmatrix}1&1&0\\0&1&-1\\0&0&1\end{pmatrix}\xrightarrow{L_2+L_3}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{L_1-L_2}$

$\sim\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

$\Rightarrow$ inversabilă $\Rightarrow \det(A)\neq 0 \Rightarrow$ rang $=3$

(4) $A=\begin{pmatrix}1&-1\\2&0\end{pmatrix}$, $B=A^4-3A^3+3A^2-2A+8I_2 \quad |\cdot A^{-2}$

$B\cdot A^{-2}=A^2-3A+3I_2-2A^{-1}+8A^{-2}$

③ $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$

ⓐ Să se afle polinomul

ⓑ $A^{100} = ?$ (T.H-C)

④ făcut

a) $P_A(X) = \det(A - XI_4) =$

$= X^4 - \sigma_1 X^3 + \sigma_2 X^2 - \sigma_3 X + \sigma_4 \cdot 1$

$\sigma_1 = \operatorname{tr} A$; $\sigma_2 = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix} =$

$= 1 + 1 + 0 - 4 + 1 + 1 = 0$

$\sigma_3 = \begin{vmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix} \neq 0$

$\sigma_4 = \det(A) = 0 \Rightarrow P_A(x) = X^4$

$P_A(A) = O_4 \Rightarrow A^4 = O_4 \Rightarrow A^{100} = O_4$

⑪ $X^{2024} = A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, $X \in \mathcal{M}_2(\mathbb{R})$

a) Precizați nr de soluții

b) ——— // ———, $X \in \mathcal{M}_2(\mathbb{C})$

$\det A = 0 \Rightarrow \det(X^{2024}) = [\det(X)]^{2024} = 0 \Rightarrow \det(X) = 0$

$\Rightarrow \det(X^{2024}) = (\det(X))^{2024} = 0 \Rightarrow \det(X) = 0$

$X^2 + \operatorname{tr}(X) \cdot X + \det(X) \cdot I_2 = O_2$

$X^2 = \operatorname{tr}(X) \cdot X$; $X^{2024} = \operatorname{tr}^{2023}(X) \cdot X$

$A = tr^{2023}(X) \cdot X \ / tr$

$tr(A) = tr^{2024}(X) = 4$

$tr(X) = \pm \sqrt[2024]{4}$ => de 2 soluții

b) În $\mathbb{C}$, numărul de soluții este 2024

OBS a) $X^2 = \alpha X$

$X^n = \alpha^{n-1} \cdot X$

b) $tr(\alpha X) = \alpha\, tr(X)$

OBS $z^{2n} = 1$; 2 soluții reale

$z^{2n+1} = 1$; 1 soluție reală

$\boxed{14}$ $A, B \in M_n(\mathbb{R})$ a.î. $AB = BA$

$\Rightarrow \det(A^2 + B^2) \geq 0$

$\det(A^2 + B^2) = \det(A^2 - i^2 \cdot B^2) = \det((A - iB)(A + iB)) =$

$= \underbrace{\det(A - iB)}_{\bar{z}} \underbrace{\det(A + iB)}_{z} = z \cdot \bar{z} = |z|^2 \geq 0$

7) x_1, x_2, x_3 răd $x^3 + px + q = 0$

$s_1 = x_1x_2 + x_2x_3 + x_3x_1 = p$

$s_2 = x_1x_2x_3 = -q$

$n = x_1 + x_2 + x_3 = 0$

$A = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$

$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = \det A \cdot \det A^t = \det A^2$

$A \cdot A^t = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} =$

$\boxed{S_i = x_1^i + x_2^i + x_3^i, \ i \in \mathbb{N}}$ $= \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}$

$S_0 = 3$

$S_1 = s_1 = 0$

$S_2 = s_1^2 - 2s_2 = -2p$

$$\begin{cases} x_1^3 + p x_1 + q = 0 \\ x_2^3 + p x_2 + q = 0 \\ x_3^3 + p x_3 + q = 0 \end{cases} \quad \begin{array}{l} S_3 + p(S_1) + 3q = 0 \Rightarrow S_3 = -3q \\ S_4 + p(S_2) + q S_1 = 0 \Rightarrow S_4 = 2p^2 \end{array}$$

$S_0 = 3$ $\quad S_3 = -3q$

$S_1 = \sigma_1 = 0$ $\quad S_4 = 2p^2$

$S_2 = \sigma_1^2 - 2\sigma_2 = -2p$

$$\det(A \cdot A^t) = \begin{vmatrix} 3 & 0 & -2p \\ 0 & -2p & -3q \\ -2p & -3q & 2p^2 \end{vmatrix} = -12p^3 + 8p^3 - 27q^2 = $$
$$= -4p^3 - 27q^2$$

(9) $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$; $B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$

$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 \Rightarrow$

$\det A \cdot \det B = \det(A \cdot B)$

$$A \cdot B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix}$$

$\det(A \cdot B) = (ac - bd)^2 + (ad + bc)^2$

$$\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & l(x) \end{vmatrix} ; \quad \Delta'(x) = \begin{vmatrix} f'(x) & g'(x) \\ h(x) & l(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ h'(x) & l'(x) \end{vmatrix}$$