Seminar 1 Algebră & Geometrie   Matrice, determinanți, sisteme

① $\det A = \Delta$

$A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$ (Matrice circulară)

$l_1 = l_1 + l_2 + l_3$

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} =$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = (a+b+c)\begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix} =$$

$$= (a+b+c)\left[(c-b)(b-c) - (a-b)(a-c)\right] =$$

$$= (a+b+c)(cb - c^2 - b^2 + cb - a^2 + ac + ba - bc)$$

$$= (a+b+c)(cb + ac + ba - a^2 - b^2 - c^2) \qquad \cancel{a\ b}$$

$$= -(a+b+c)(a^2 + b^2 + c^2 - cb - ac - ba).$$

$$= -\frac{1}{2}(a+b+c)\left[(a^2 - 2ab + b^2 + b^2 - 2cb + c^2 + c^2 - 2ac + a^2)\right]$$

$$= -\frac{1}{2}(a+b+c)\left[(a-b)^2 + (a-c)^2 + (b-c)^2\right]$$

② $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$; $\det(A) = ?$; $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} =$

$$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} = \begin{vmatrix} (b-a) & (c-a) \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} = (b-a)(c-a)(c+a) -$$

$\boxed{\det A = (c-b)(c-a)(b-a)}$ $\qquad = (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$

$$-(c-a)(b-a)(b+a) = (b-a)(c-a)\left[c+a-b-a\right] =$$

$$= (b-a)(c-a)(c-b) \quad \text{(Determinant Vandermonde)}$$

③ $A^{-1} = \frac{1}{\det A} \cdot A^*$; $\det A^{-1} = \left(\frac{1}{\det A}\right)^n \cdot \det A^*$; $A \cdot A^{-1} = I_n$

$\det A \cdot \det A^{-1} = 1$

$\det A^{-1} = \frac{1}{\det A}$

$\frac{1}{\det A} = \left(\frac{1}{\det A}\right)^n \cdot \det A^* \Rightarrow (\det A)^{n-1} = \det A^*$, $n \geq 2$

$\det A^* = (\det A)^2$

$$\begin{vmatrix} 1+a^2 & ba & ca \\ ba & 1+b^2 & bc \\ ca & bc & 1+c^2 \end{vmatrix} = \begin{array}{cccccc} C_1 & C_1' & C_2 & C_2' & C_3 & C_3' \end{array} \begin{vmatrix} 1+a^2 & 0+ba & 0+ca \\ 0+ba & 1+b^2 & 0+bc \\ 0+ca & 0+bc & 1+c^2 \end{vmatrix} =$$

$|C_1\ C_2\ C_3| +$
$|C_1\ C_2\ C_3'| +$
$|C_1\ C_2'\ C_3| +$
$\boxed{|C_1\ C_2'\ C_3'| +}$

$+ |C_1'\ C_2\ C_3|$
$\boxed{+ |C_1'\ C_2\ C_3'|}$
$\boxed{+ |C_1'\ C_2'\ C_3|}$
$\boxed{+ |C_1'\ C_2'\ C_3'|}$

Poți rupe un determinant ca să îți dea mai mulți determinanți simpli care se adună

$$|C_1\ C_2\ C_3| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|C_1\ C_2\ C_3'| = \begin{vmatrix} 1 & 0 & ca \\ 0 & 1 & bc \\ 0 & 0 & c^2 \end{vmatrix} = c^2$$

$$|C_1\ C_2'\ C_3| = \begin{vmatrix} 1 & ba & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} = b^2$$

$$|C_1'\ C_2\ C_3| = \begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ac & 0 & 1 \end{vmatrix} = a^2$$

$\Rightarrow \det A = a^2 + b^2 + c^2 + 1$

$\Rightarrow \det A^* = (\Delta A)^2 = (1 + a^2 + b^2 + c^2)^2$

4) $A=\begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

$\det A \cdot \det A^{-1} = 1 \Rightarrow$

$\det A^{-1} = \frac{1}{\det A}$

a) $m = ?$ a.î. $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

b) $m=0$, calculați A^{-1}

$\det(A^{-1}) \in \mathbb{Z}$

$A \in \mathcal{M}_3(\mathbb{Z}) \Rightarrow \det(A) \in \mathbb{Z}$

$\Rightarrow \mathbb{Z} = \frac{1}{\mathbb{Z}} \Rightarrow$

$\det(A) = \pm 1$.

$$\begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = (-1)^4 \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix} =$$

$= -3 - (3m+4)(m-1) = -3 - (3m^2 - 3m + 4m - 4) =$

$= +3 + 3m^2 + 3m + 4m + 4 = 3m^2 + m - 1$, ~~$\Delta = 1 + 1\cdot 3\cdot 4 = 1+12=13$~~

I $3m^2 - m - 1 = 1 \Rightarrow 3m^2 - m - 2 = 0$

II $3m^2 - m - 1 = -1 \Rightarrow 3m^2 - m = 0$

$m(3m-1) = 0 \Rightarrow \boxed{m=0;\ m=\frac{1}{3}}$ dar $m \in \mathbb{Z}$

I $\Delta = 1 + 24 = 25 \Rightarrow m_{1,2} = \frac{1 \pm 5}{6} \Rightarrow \boxed{\begin{array}{l} m_1 = \frac{6}{6} = 1 \\ m_2 = \frac{-4}{6} = -\frac{2}{3} \end{array}}$ $\Rightarrow m \in \{-1, 0\}$

b) $A^{-1} = \frac{1}{\det A} \cdot A^*$

(5) $\exists k \in \mathbb{N}, k \geq 2$ a.i. $A^k = O_2$, $A^2 = O_2$

(T) $A^2 - tr(A) \cdot A + det(A) \cdot I_2 = O_2$

$A^k = O_2 \Rightarrow (det A)^k = 0 \Rightarrow (det A) = 0$

$\Rightarrow A^2 = tr(A) \cdot A$

$\boxed{Obs: A^2 = \alpha \cdot A \Rightarrow A^n = \alpha^{n-1} \cdot A, \forall n \geq 2}$ Se dem cu inducție

$A^k = \underline{tr(A) \cdot A = O_2} \,|\, tr$ $tr(A) = A$??

$\boxed{OBS: tr(\alpha A) = \alpha \, tr A}$

$(tr(A))^{k-1} \cdot tr(A) = 0 \Rightarrow (tr A)^k = 0$

$tr A = 0 \Rightarrow A^2 = O_2$

(6) $f: M_2(\mathbb{C}) \to M_2(\mathbb{C})$, $f(X) = X^n$

f inj? există $X_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

f surj!

$\left.\begin{array}{l} det(X_1) = 0 \\ det(X_2) = 0 \end{array}\right\} \Rightarrow$ f nu e ~~surj~~ inj

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~~$\exists$ pt oricare $r \in \mathbb{R}$ $R_r =$~~

f surj $\Leftrightarrow$ im $f = \mathbb{R}$ $\Rightarrow \forall a \in \mathbb{R} \, \exists X = \begin{pmatrix} a & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

$det(X) = a$

$\Rightarrow$ f nu e inj dar e surjectivă

1, 2, 3, 4, 10, 11

(10) Fie $\begin{pmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$; $rg A = ?$

$$\det A = \begin{vmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ 1-a & 0 & 1-2 \\ -1-a & 0 & -1-a \end{vmatrix} = 1 \cdot (-1)^3 \begin{vmatrix} 1-a & 1-2 \\ -1-a & -1-a \end{vmatrix} =$$

$$= -(-1-a) \begin{vmatrix} 1-a & -1 \\ 1 & 1 \end{vmatrix} = (1+a)[1-a+1] = (1+a)(2-a)$$

$C_1: (1+a)(2-a) \neq 0 \Leftrightarrow a \in \mathbb{R} \setminus \{-1, 2\} \Rightarrow rg(A) = 3$

$C_1: (1+a)(2-a) = 0 \Leftrightarrow a \in \{-1, 2\}$

a) $a = -1 \Rightarrow A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \Rightarrow \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0 \Rightarrow rg(A) = 2$

b) $a = 2 \Rightarrow A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \Rightarrow \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1 \neq 0 \Rightarrow rang(A) = 2$

(11) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} = 0 \Rightarrow 0 + 6 + 0 - 0 - a - 12 = -a - 6 = 0 \Rightarrow \boxed{a = -6}$

$\begin{vmatrix} 2 & 3 & 1 \\ 0 & -6 & 1 \\ 1 & 3 & b \end{vmatrix} = 0 \Rightarrow -12b + 0 + 3 + \cancel{6} - \cancel{6} - 0 \Rightarrow -12b + 3 = 0$

$-12b = -3$

$\boxed{b = \frac{1}{4}}$

(12) $A^{2023} - 2023A = I_3$ a) $rg(A) = ?$

$A(A^{2022} - 2023 I_3) = I_3$ b) $rg(2023A + I_3) = ?$

$\det A \cdot \det(A^{2022} - 2023 I_3) = \det I_3$

$\det A \cdot \det(A^{2022} - 2023 \cdot I_3) = 1 \Rightarrow \det A \neq 0$

$\Rightarrow \operatorname{rang}(A) = 3$

le) $A^{2023} = I_3 + 2023A$

$\det(A^{2023}) = 1 + \det(A) 2023$

$(\det A)^{2023} = 1 + \det(A) \cdot 2023$

$\det A \neq 0 \Rightarrow (\det A)^{2023} \neq 0$

$\Rightarrow \det(I_3 + 2023A) \neq 0$

$\Rightarrow rg(I_3 + 2023A) = 3$

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Ex 3: $A^T = -A$, $A \in M_{2n+1}(\mathbb{R}) \Rightarrow \det A = 0$

$\det(A^T) = \det(A)$ formulă

$\Rightarrow \det(A^T) = \det(A) \cdot (-1)^{2n+1}$

$\det(A^T) = \det(A) \cdot (-1)$

$\det(A) = -\det(A)$

$\Rightarrow \det(A) = 0$