Seminar Analiză 14

① $\sum_{n\geq 1} \frac{x^n}{\sqrt[n]{n!}}$; $x \in \mathbb{R}$

abs conv $\sum_{n\geq 1} \left| x^n \frac{1}{\sqrt[n]{n!}} \right|$

$x = 0 \Rightarrow$ serie conv.

$x \neq 0$ Criteriul raportului

$$\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}}{|x^n|} \cdot \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} = |x| \left(\frac{n!^{\,n+1}}{(n+1)!^{\,n}} \right)^{\frac{1}{n(n+1)}} =$$

$$= \left(\frac{\cancel{n!^n} \cdot n!}{\cancel{n!^n} \cdot (n+1)^n} \right)^{\frac{1}{n(n+1)}} \cdot |x| \longrightarrow |x|$$

$$1 \leq \left(\frac{(n+1)^n}{n!} \right)^{\frac{1}{n(n+1)}} \leq (n+1)^{\frac{1}{n+1}}$$

$$\downarrow$$

$$1$$

$|x| > 1 \Rightarrow a_n \not\to 0 \Rightarrow$ serie divergentă

$|x| < 1 \Rightarrow$ serie absolut convergentă

$|x| = 1$ $\sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}} \sim \sum_{n\geq 1} \frac{1}{n} = +\infty$ divergent $x = 1$

$\sum_{n\geq 1} \frac{1}{n^{\alpha}}$ conv $\Leftrightarrow \alpha > 1$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} =$$

$$\lim_{n\to\infty} \sqrt[n]{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = n+1 \to \infty$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} =$$

$$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^{n \nearrow \infty} = \frac{1}{e}$$

$x = -1$

II $\lim\limits_{x\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$ ①

$x=-1$ ①, ③ $\Rightarrow$ seria este semiconvergentă

III $a_n \searrow 0 \Rightarrow \sum\limits_{n\geq 1} (-1)^n a_n$ conv

$$\frac{a_{n+1}}{a_n} = \left(\frac{n!}{(n+1)^n}\right)^{\frac{1}{n(n+1)}} < 1 \Rightarrow a_n \downarrow ③$$

Să se determine extremele funcției

$f(x,y,z) = xyz$ pe $A = \{x^2+y^2+4z^2=1\}$

fie $g(x,y,z) = x^2+y^2+4z^2-1$

T.M.L:

① $f, g \in C^1$

② rang $g' = 1$ max

$\hookrightarrow g' = (2x, 2y, 8z)$

pp rang $g' = 0 \Rightarrow x=y=z=0 \notin A$

③ (x_0, y_0, z_0) este un extrem pt f pe A

$\Rightarrow (\exists)\, \alpha \in \mathbb{R}$ a.î. $h_\alpha'(x_0, y_0, z_0) = 0$ unde ~~$h_\alpha = f + \alpha g$~~

$h_\alpha = f + \alpha g$

$h_\alpha = xyz + \alpha(x^2 + y^2 + 4z^2 - 1)$

$$\begin{cases} \dfrac{\partial h_\alpha}{\partial x} = yz + 2\alpha x = 0 \\ \dfrac{\partial h_\alpha}{\partial y} = xz + 2\alpha y = 0 \\ \dfrac{\partial h_\alpha}{\partial z} = xy + 8\alpha z = 0 \end{cases}$$

$\ominus \Rightarrow z(y-x) - 2\alpha(y-x) = 0$

$(z - 2\alpha)(y - x) = 0$

I $y = x$

II $z = 2\alpha$

$xyz + 2\alpha x^2 = 0 \Rightarrow xyz = -2\alpha x^2 = -2\alpha y^2 = -8\alpha z^2 \,/\cdot 2\alpha \quad \alpha \neq 0$

$\Rightarrow x^2 = y^2 = 4z^2 \Rightarrow \begin{cases} x = \pm y \\ x = \pm 2z \end{cases}$

$3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}; y = \pm\frac{1}{\sqrt{3}}; z = \pm\frac{1}{2\sqrt{3}}$

astea sunt toate punctele de extrem local posibile am determinat cu teoria multiplicitate Lagrange

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A → mărginită

A → închisă

$|x| \leq 1, |y| \leq 1, |z| \leq 1$

f continuă

$\Rightarrow$ f are un punct de maxim ~~global~~ sau minim global

Metoda 1

$\Rightarrow f\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{2\sqrt{3}}\right) = \pm\frac{1}{6\sqrt{3}}$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)$

$, \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$ punct de maxim global

Metoda 2

$\frac{1}{z^3} = -\frac{\alpha}{3} \Rightarrow \alpha = \frac{\sqrt{3}}{4}$

$$h_\alpha'' = \begin{pmatrix} 2\alpha & z & y \\ z & 2\alpha & x \\ y & x & 8\alpha \end{pmatrix}$$

$$h''_{\frac{\sqrt{3}}{4}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}\right) = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 2\sqrt{3} \end{pmatrix}$$

$\Delta_1 > 0$
$\Delta_2 = \frac{3}{4} - \frac{1}{12} > 0$ $\Big| \Rightarrow +++$ (punct maxim)
$\Delta_3 > 0$

$s(x) = \sum_{n \geq 1} \frac{1}{n^4} e^{-nx}$ $x \in (0,1)$
C^2

$f_n : (0,1) \to \mathbb{R}$

$f_n(x) = \frac{1}{n^4 e^{nx}}$

$|f_n(x)| \leq \frac{1}{n^4}$ $\sum \frac{1}{n^4}$ convergentă $\Rightarrow$ s normal convergentă

$s_1(x) = \sum_{n \geq 1} f_n'(x) = \sum_{n \geq 1} \frac{1}{n^4} \cdot -n e^{-nx} = \sum_{n \geq 1} \frac{-1}{n^3} e^{-nx}$

$|f_n'(x)| \leq \frac{1}{n^3}$, $\sum \frac{1}{n^3}$ convergent $\Rightarrow s_1$ normal convergentă

$$s_2(x) = \sum_{n\geq 1} f_n''(x) = \sum_{n\geq 1} \frac{+1}{n^{\cancel{3}2}} \cancel{x}\, e^{-nx} = \sum_{n\geq 1} \frac{1}{n^2} e^{-nx};$$

$$|f_n''(x)| \leq \frac{1}{n^2} \quad \underbrace{\sum \frac{1}{n^2}}_{\text{conv.}} \Rightarrow s_2 \text{ normal conv.}$$

$$\left.\begin{array}{l} s \text{ normal conv} \\ s_1 \text{ —"—} \\ s_2 \text{ —"—} \end{array}\right| \begin{array}{l} \Rightarrow s' = s_1 \\ \Rightarrow s_1' = s_2 \end{array} \Bigg| \Rightarrow s'' = s_2 \text{ - continuă}$$

$$f_n'' \text{ cont.} \Big| \Rightarrow s_2 \text{ continuă}$$

$$\sum_{n\geq 1} x^n \cdot \sin^2\left(\frac{1}{n^\alpha}\right), \ x \in \mathbb{R}, \ \alpha > 0$$

$$\sum_{n\geq 1} \left| x^n \cdot \sin^2\left(\frac{1}{n^\alpha}\right)\right| =$$

<u>criteriul raportului</u>

$$\frac{a_{n+1}}{a_n} = \frac{|x^{n+1}|}{|x^n|} \cdot \frac{\sin^2 \frac{1}{(n+1)^\alpha}}{\sin^2 \frac{1}{n^\alpha}} = |x| \cdot \frac{\sin^2 \frac{1}{(n+1)^\alpha}}{\frac{1}{(n+1)^{2\alpha}}} \cdot \frac{\sin^2 \frac{1}{n^\alpha}}{\frac{1}{n^\alpha}}.$$

$$\cdot \frac{n^\alpha}{(n+1)^{2\alpha}} \to |x| > 1 \overset{n\to\infty}{\Rightarrow} \text{serie divergentă}$$

$$|x| < 1 \Rightarrow \text{convergentă}$$

$|x| = 1$

$$\Rightarrow \sum_{n\geq 1} \sin^2\left(\frac{1}{n^\alpha}\right) \sim \sum \frac{1}{n^{2\alpha}} \text{ convergentă} \Leftrightarrow 2\alpha > 1 \Rightarrow \alpha > \frac{1}{2}$$

$$\boxed{\sum a_n \sim \sum b_n \text{ dacă } \lim_{n\to\infty} \frac{a_n}{b_n} \in (0, \infty)}$$

~~la~~ $x=1$ conv $\Leftrightarrow \alpha > \frac{1}{2}$

$x=-1 \quad \sum_{n\geq 1} (-1)^n \cdot \sin^2\left(\frac{1}{n^\alpha}\right) \qquad \frac{1}{n^\alpha} \searrow 0 \Rightarrow \sin^2 \frac{1}{n^\alpha} \searrow 0$

$\Rightarrow x=-1$ — $\alpha > \frac{1}{2}$ ab. conv.

— $\alpha \leq \frac{1}{2}$ semi conv.

$f_n : [0,+\infty) \to \mathbb{R}, \ f_n(x) = x^2 e^{-2nx}$

$\lim_{n\to\infty} f_n x = \lim_{n\to\infty} x^2 e^{-nx} = 0 \Rightarrow f : [0,\infty) \to \mathbb{R}$

c.s. $\Rightarrow f_n$ converge simplu la x $\Leftrightarrow$ trebuie să existe limita

c.u. $a_n = \sup_{x\geq 0}\left(|f_n(x) - \underset{0}{\underset{\parallel}{f(x)}}|\right)$

$a_n \to 0 \Leftrightarrow f_n \xrightarrow{u} f$

$a_n = \sup_{x\geq 0}(f_n(x))$

$f_n'(x) = 2xe^{-2nx} - 2nx^2e^{-2nx}$

$= 2xe^{-2nx}(1-2n) = 0$

$\Rightarrow x = 0$

$x = \frac{1}{n}$

x	0		$\frac{1}{n}$		$\frac{1}{n^2}$ ∞
$\lvert x\rvert = f_n(x)$	0	$+ + + + +$	0	$- - - - - - -$	
$f(x)$	0	$\nearrow$	a_n	$\searrow$	0

fie $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{x^7 \cdot y^5}{x^{10}+y^{10}}, & x^2+y^2 \neq 0 \\ 0, & x=y=0 \end{cases}$

, continuitatea în 0

a) $\lim\limits_{\substack{x\to 0\\ y=ax}} f(x,y) = \lim\limits_{\substack{x\to 0\\ y=ax}} \frac{x^7y^5}{x^{10}+y^{10}} = \lim\limits_{x\to 0} \frac{x^7a^5x^5}{x^{10}+a^{10}x^{10}} =$

$= \lim\limits_{x\to 0} x^2 \frac{a^5}{1+a^{10}} = 0$

$\left|\frac{x^7y^5}{x^{10}+y^{10}}\right| \leq \left|\frac{x^5y^5}{x^{10}+y^{10}}\right| \cdot x^2 \leq \frac{x^2}{2} \xrightarrow[\substack{x\to 0\\ y=ax}]{} 0$

$\frac{\partial f}{\partial x} = \frac{7x^6y^5(x^{10}+y^{10}) - x^7y^5 10x^9}{(x^{10}+y^{10})^2} =$

$= \frac{7x^{16}y^5 + 7x^6y^{15} - 10x^{16}y^5}{(x^{10}+y^{10})^2}$

$= \frac{-3x^6y^5 + 7x^6y^{15}}{(x^{10}+y^{20})^2}$

$f_x(0,0) = \lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$