Seminar Analiză 9

① $\frac{\partial f}{\partial v}(c) = \lim_{t \to 0} \frac{f(c+tv)-f(c)}{t} = \lim_{t \to 0} \frac{f(a+tm, b+tn)-f(a,b)}{t}$

$\lim_{t \to 0} \frac{(a+tm)^4 e^{3(b+tn)} - a^4 e^{3b}}{t}$ l'H $= \lim_{t \to 0} 4m(a+tm)^3 e^{3(b+tn)} +$

$+ (a+tn)^4 3n e^{3(b+tn)} =$

$= 4m a^3 e^{3b} + a^4 3n e^{3b} = m \frac{\partial f}{\partial x}(a,b) + n \frac{\partial f}{\partial y}(a,b)$

② $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x,y) = (\underbrace{x^3 e^{3y}}_{g_1}, \underbrace{x^3 + e^{3y}}_{g_2})$

$\frac{\partial g_1}{\partial x} = 3x^2 e^{3y}$ $\qquad \frac{\partial g_2}{\partial x} = 3x^2$

$\frac{\partial g_1}{\partial y} = 3x^3 e^{3y}$ $\qquad \frac{\partial g_2}{\partial y} = 3e^{3y}$

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 e^{3y} & 3x^3 e^{3y} \\ 3x^2 & 3e^{3y} \end{pmatrix}$$

③ $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{2x} \sin y$, $c = (a,b)$

$\frac{\partial f}{\partial v}(c) = \lim_{t \to 0} \frac{f(c+tv)-f(c)}{t} = \lim_{t \to 0} \frac{f(a+tm, b+tn) - f(a,b)}{t} =$

$$\frac{\partial f}{\partial v}(c) = \lim_{t\to 0}\frac{f(c+tv)-f(c)}{t} = \lim_{t\to 0}\frac{f(a+tm, b+tn)-f(a,b)}{t}$$

$$= \lim_{t\to 0}\frac{e^{2(a+tm)}\sin(b+tn)-e^{2a}\sin b}{t} \overset{\frac{0}{0}\,L'H}{=}$$

$$= \lim_{t\to 0} 2m e^{(a+tm)}\sin(b+tn) + te^{2(a+tm)}\sin(a+tn)$$

$$= m2e^{a}\sin b + ne^{2a}\cdot\cos b$$

$$\frac{\partial f}{\partial x} = 2e^{2x}\sin y$$

ca la examen

$$f:\mathbb{R}^2\to\mathbb{R}\quad f(x,y) = \begin{cases}\frac{x^4y^4}{x^6+y^6}, & x^2+y^2\neq 0\\ 0, & x=y=0\end{cases}$$

a) f cont

b) $\exists$ + cont $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

c) dediv funcției

a) f cont pe $\mathbb{R}^2/\{0\}$ (funcții elementare)

$$\lim_{\substack{x\to 0\\ y=ax}} f(x,y) = \lim_{x\to 0} f(x,ax) = \lim_{x\to 0}\frac{x^4(ax)^4}{x^6+a^6x^6} = \lim_{x\to 0}\frac{x^2a^4}{1+a^6} =$$

Arătăm că funcția este continuă prin majorare

$$|f(x,y)-f(0,0)| = \left|\frac{x^4y^4}{x^6+y^6}\right| \cdot |x\cdot y| \leq \frac{1}{2}|x\cdot y| \xrightarrow[\substack{x\to 0\\ y\to 0}]{} (0,0)$$

$x^6+y^6 \geq 2|x^3y^3| \Rightarrow \frac{1}{2} \geq \left|\frac{x^3y^3}{x^6+y^6}\right| \Rightarrow f$ este cont în punct $(0,0)$

b) $\frac{\partial f}{\partial x} = \frac{4x^3y^4(x^6+y^6)-x^4y^4(6x^5)}{(x^6+y^6)^2} = \frac{-2x^9y^4+4x^3y^{10}}{(x^6+y^6)^2}$

$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = 0$

$\lim_{\substack{x\to 0\\ y=ax}} \frac{\partial f}{\partial x} = \lim_{x\to 0}\frac{-2x^9a^4x^6+4x^3a^{10}x^{10}}{(x^6+a^6x^6)} =$

$= \lim_{x\to 0}\frac{x^{13}(-2a^4+4a^{10})}{x^{12}(1+a^6)^2} = 0$

$\left|\frac{-2x^9y^4}{(x^6+y^6)^2}\right| = \underbrace{\frac{x^6}{x^6+y^6}}_{\leq 1}\cdot\underbrace{\left|\frac{x^3y^3}{x^6+y^6}\right|}_{\leq \frac{1}{2}}|2y| \leq |y| \xrightarrow[\substack{y\to 0\\ x\to 0}]{} 0$

$\frac{4x^3\cdot y^{10}}{(x^6+y^6)^2} = \underbrace{\left(\frac{x^6}{x^6+y^6}\right)^{\frac{3}{6}}}_{\leq 1}\cdot\left(\frac{y^6}{x^6+y^6}\right)^{\frac{10}{6}}\cdot(x^6+y^6) =$

$= (x^6+y^6) \xrightarrow[\substack{x\to 0\\ y\to 0}]{} 0$

c) f este derivabilă în $(0,0) \Leftrightarrow (\exists) T: \mathbb{R}^2 \to \mathbb{R}$ liniară a[i]

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{f(x,y) - f(0,0) - T(x,y)}{\sqrt{x^2+y^2}} = 0$$

$$\Rightarrow$$

$$T(x,y) = \alpha x + \beta y$$

$$\alpha = \frac{\partial f}{\partial x}(0,0) = 0$$

$$\beta = \frac{\partial f}{\partial y}(0,0) = 0$$

$$\Rightarrow T(x,y) = 0$$

$$\Rightarrow \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\frac{x^4 y^4}{x^6+y^6}}{\sqrt{x^2+y^2}} = 0$$

$$\frac{x^4 \cdot y^4}{(x^6+y^6)\sqrt{x^2+y^2}} = \underbrace{\left|\frac{x^3 y^3}{x^6+y^6}\right|}_{\leq \frac{1}{2}} \cdot \left|\frac{x}{\sqrt{x^2+y^2}}\right| \cdot |y| \leq \frac{|y|}{2} \xrightarrow[\substack{x \to 0 \\ y \to 0}]{} 0$$