10.10.2024

C2 - 16 AG

Teorema Hamilton-Cayley. Aplicații

$A \in \mathcal{M}_n(\mathbb{C})$

$P_A(X) = \det(A - XI_n) = (-1)^n [X^n - \sigma_1 X^{n-1} + \sigma_2 X^{n-2} - \dots + (-1)^n \sigma_n]$

polinomul caracteristic asociat matricei A.

(2) T. Hamilton-Cayley : $P_A(A) = O_n$

$A^n - \sigma_1 A^{n-1} + \dots + (-1)^n \sigma_n \cdot I_n = O_n$.

σ_k = suma minorilor diagonali de ordinul k, $k = \overline{1,n}$

Aplicații

1) Calculul lui A^{-1}

2) Calcul pentru A^n

Fie $A \in \mathcal{M}_2(\mathbb{C})$. T.H.C: $A^2 - Tr(A)A + \det(A) I_2 = O_2$

$A^2 = Tr(A)A - \det(A) I_2$.

$A^n = x_n A + y_n I_2, \ \forall n \geq 1$.

Exemplu Fie $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

a) $A^n = x_n A + y_n I_2$, $x_n, y_n = ?$

b) $B = A^4 + A^3 + A^2 + A + I_2$.
Să se det $a, b \in \mathbb{R}$ a.î. $B = aA + bI_2$.

Sol

a) $Tr A = 1 + 2 = 3$, $\det A = 2$

T.H.C: $A^2 - 3A + 2I_2 = O_2$

$A^1 = 1 \cdot A + 0 I_2$, $x_1 = 1, y_1 = 0$

$A^2 = 3A - 2I_2$, $x_2 = 3, y_2 = -2$.

$A^n = x_n A + y_n I_2$

$A^{n+1} = A^n \cdot A$

$\underline{x_{n+1} A + y_{n+1} I_2} = (x_n A + y_n I_2) A$

$= x_n A^2 + y_n A$

$= x_n (3A - 2I_2) + y_n A$

$= \underline{(3x_n + y_n) A - 2x_n I_2}$

$\begin{cases} x_{n+1} = 3x_n + y_n \\ y_{n+1} = -2x_n \end{cases} \Rightarrow y_n = -2x_{n-1}$

$x_{n+1} - 3x_n + 2x_{n-1} = 0, \ \forall n \geq 2$ Recurență de ord al 2-lea

Ec. caracteristică $t^2 - 3t + 2 = 0 \Rightarrow (t-1)(t-2) = 0 \Rightarrow t_1 = 1, \ t_2 = 2.$

$x_n = c_1 \cdot t_1^n + c_2 \cdot t_2^n \quad , c_1, c_2 \in \mathbb{R}$

$n = 1 \Rightarrow x_1 = 1 = c_1 \cdot 1 + c_2 \cdot 2$

$n = 2 \Rightarrow x_2 = 3 = c_1 \cdot 1^2 + c_2 \cdot 2^2$ $\ominus$

$2 = / = 2c_2 \qquad c_2 = 1$

$c_1 = -1.$

$x_n = -1 + 2^n$

$y_n = -2x_{n-1} = -2(-1 + 2^{n-1}) = 2 - 2^n$

Deci $\underline{A^n = (2^n - 1) A + (2 - 2^n) I_2}, \ \forall n \geq 1.$

b) $B = A^4 + A^3 + A^2 + A + I_2 = aA + bI_2$

M1 $A^4 = (2^4 - 1) A + (2 - 2^4) I_2 = 15A - 14 I_2$

$A^3 = (2^3 - 1) A + (2 - 2^3) I_2 = 7A - 6 I_2$

$A^2 = 3A - 2I_2$

$B = \underline{15A} - 14 \underline{\underline{I_2}} + \underline{7A} - 6 \underline{\underline{I_2}} + \underline{3A} - 2 \underline{\underline{I_2}} + \underline{A} + \underline{\underline{I_2}} = 26A - 21 I_2$

$a = 26 \quad , b = -21$

(M2) Fie $\underset{\substack{\parallel \\ \det(A - XI_2)}}{P_A(X)} = X^2 - 3X + 2 = X^2 - \sigma_1 X + \sigma_2$ polinomul caracteristic asociat lui A

$Q = X^4 + X^3 + X^2 + X + 1$.

$Q = P_A \cdot C + R \Rightarrow \underset{\substack{\parallel \\ B}}{Q(A)} = \underbrace{P_A(A)}_{O_2} \cdot C(A) + R(A) = \underset{\substack{\parallel \\ aA + bI_2}}{R(A)}$

$R = aX + b$

$X^4 + X^3 + X^2 + X + 1 = (X-1)(X-2)C + aX + b$

1) $X = 1 \Rightarrow 5 = a + b$

2) $X = 2 \Rightarrow \underbrace{2^4 + 2^3 + 2^2 + 2 + 1}_{\frac{2^5 - 1}{2 - 1}} = 2a + b \Rightarrow 31 = 2a + b$

$\Rightarrow \underset{\substack{\parallel \\ 26}}{a}, \underset{\substack{\parallel \\ -21}}{b}$.

3) Rezolvarea de ec. matriceale binome din $\mathcal{M}_2(\mathbb{C})$.

Exemplu $X^4 = A = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$, $X = ?$, $X \in \mathcal{M}_2(\mathbb{C})$

SOL

$\det A = 0 \Rightarrow \det(X^4) = (\det X)^4 = 0 \Rightarrow \det X = 0$

T.H-C: $X^2 - Tr(X)X + \det(X)I_2 = O_2 \Rightarrow X^2 = Tr(X)X$.

OBS $X^2 = \alpha X \Rightarrow X^n = \alpha^{n-1} X$

$X^4 = Tr(X)^3 \cdot X \Rightarrow (*)\ A = \underline{Tr(X)^3} \cdot X \quad | Tr \Rightarrow \underset{\substack{\parallel \\ 1}}{Tr A} = (Tr X)^4$

OBS $Tr(\alpha X) = \alpha Tr(X)$

$Tr(X)^4 = 1 \Rightarrow Tr X \in \{\pm 1; \pm i\}$.

$(*) \Rightarrow X = \frac{1}{Tr(X)^3} \cdot A$

$X = \pm A$ sau

$X = \frac{1}{\pm i} A = \mp i A$

$\frac{1}{i} = \frac{-i}{1}$; $\frac{1}{-i} = i$

$X \in \{A, -A, iA, -iA\}$

Teorema Laplace

Fie $A \in \mathcal{M}_n(\mathbb{C})$

a) minor de ordin p.

$$\det(A_{IJ}) = \begin{vmatrix} a_{i_1j_1} & \dots & a_{i_1j_p} \\ \vdots & & \\ a_{i_pj_1} & \dots & a_{i_pj_p} \end{vmatrix}$$

$I = \{i_1, .., i_p\}$

$J = \{j_1, .., j_p\}$

$1 \le i_1 < .. < i_p \le n$

$1 \le j_1 < .. < j_p \le n$.

b) minor complementar lui $\det(A_{IJ})$

$\det(A_{\bar{I}\bar{J}})$ se obține din A suprimând liniile $l_{i_1}, .., l_{i_p}$ și coloanele $c_{j_1}, .., c_{j_p}$.

$\bar{I} = \{1, .., n\} \setminus I$, $\bar{J} = \{1, .., n\} \setminus J$.

c) complementul algebric al minorului $\det(A_{I,J})$

$$c = (-1)^{i_1 + .. + i_p + j_1 + \dots + j_p} \det(A_{\bar{I},\bar{J}})$$

Teorema Laplace

$\det A$ = suma produselor minorilor de ordinul p cu complemenții algebrici corespunzători, pentru p linii fixate: $i_1, .., i_p$, respectiv p coloane fixate: $j_1, .., j_p$.

$$\det(A) = \sum_{1 \le j_1 < .. < j_p \le n} (-1)^{i_1 + .. + i_p + j_1 + \dots + j_p} \det(A_{I,J}) \det(A_{\bar{I},\bar{J}})$$

$$= \sum_{1 \le i_1 < .. < i_p \le n} (-1)^{i_1 + .. + i_p + j_1 + \dots + j_p} \det(A_{I,J}) \det(A_{\bar{I},\bar{J}})$$

OBS Pt $p=1$ dezv. după o linie / o coloană

Exemple

① $n=3$ $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

Calculul lui $\det A$ pt a) $p=2$, l_1, l_2 fixate.
sau b) $p=2$, c_1, c_2 fixate.

SOL

a) $I=\{1,2\}$, $J=\{1,2\}$

$\det(A_{I,J}) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$ minor de ord 2

$\det(A_{\bar{I},\bar{J}}) = |1|$ minorul său complementar

$(-1)^{1+2+1+2} |1|$ complementul său alg.

$$\det A = (-1)^{1+2+1+2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \cdot |1| +$$

$$(-1)^{1+2+1+3} \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \cdot |0| +$$

$$(-1)^{1+2+2+3} \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} \cdot |1| = 2+0-2=0.$$

b) c_1, c_2 fixate.

$$\det A = (-1)^{1+2+1+2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \cdot 1 +$$

$$(-1)^{1+3+1+2} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \cdot 2 +$$

$$(-1)^{2+3+1+2} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \cdot 0 = 2-2=0$$

2) $n=4$ $\qquad A=\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix}$

$\det A = ?$ a) $p=2$ $\quad l_1, l_2$ fixate

b) $p=2$ $\quad , c_1, c_2$ fixate

a) $\det A = (-1)^{1+2+1+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\cdot\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2+1+3}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} +$

$+(-1)^{1+2+1+4}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}\cdot\begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + (-1)^{1+2+2+3}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}\cdot\begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$

$+(-1)^{1+2+2+4}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)^{1+2+3+4}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}\cdot\begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix}$

$= 0 - 1\cdot 18 + 1\cdot 12 + 1\cdot 7 - 1\cdot 5 + (-1)1$

$= -18+12+7-5-1 = -5$

b) Analog (temă)

Sisteme de ecuații liniare de ordinul întâi cu mai multe necunoscute

(*) $AX=B$ $\qquad A\in\mathcal{M}_{m,n}(\mathbb{K}),\ X\in\mathcal{M}_{n,1}(\mathbb{K})$

$B\in\mathcal{M}_{m,1}(\mathbb{K})$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{m1} & \dots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

m- ecuații cu n necunoscute: $x_1, \dots, x_n$

$$\begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + \dots + a_{mm} x_n = b_m \end{cases}$$

Not $S(A) = \{ x = (x_1, .., x_n) \in \mathbb{K}^n \mid AX = B \}$ mult. sol. sistemului (*)

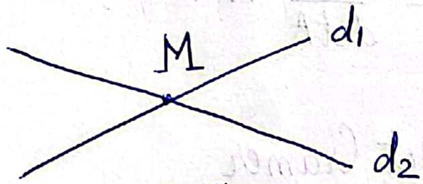
OBS

1) $S(A) \neq \phi$ — 1 sol. unică (sistem compatibil determinat) SCD
— o inf. de soluții / mai multe soluții (sistem compatibil nedeterminat) SCN

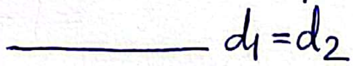
2) $S(A) = \phi$ $\nexists$ sol. (sistem incompatibil) SI

OBS

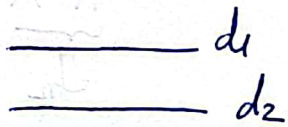
a) $n = 2$ $\begin{cases} a_{11} x_1 + a_{12} x_2 = b_1 \\ a_{21} x_1 + a_{22} x_2 = b_2 \end{cases}$ Intersecție a 2 drepte d_1, d_2 în plan.



SCD.

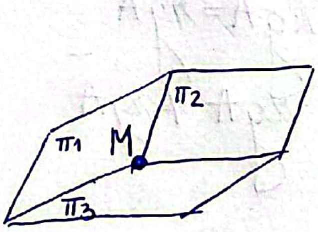


SCN

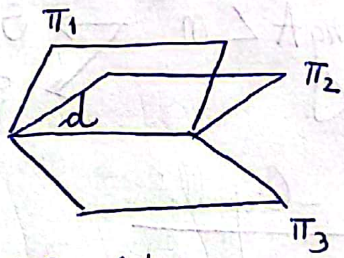


SI

b) $n = 3$ $\begin{cases} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \end{cases}$ Intersecția a 3 plane π_1, π_2 și π_3 în spațiu,

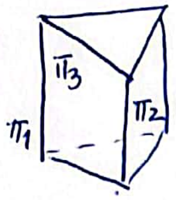


SCD.



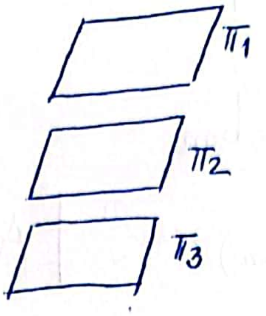
SC$\underline{s}$N

SCdN



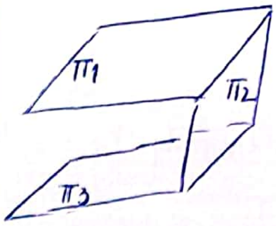
(fetele)

SI



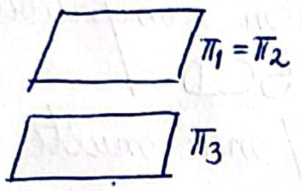
$\pi_1 \parallel \pi_2 \parallel \pi_3$

SI



$\pi_1 \parallel \pi_3$

SI



$\pi_1 \parallel \pi_3$

SI

(*) $AX = B$. $\bar{A} = (A|B)$ matrice extinsă, $A \in \mathcal{M}_{m,n}(\mathbb{K})$

(I) $m = n$ A pătratică.

a) $\Delta = \det A \neq 0 \Rightarrow \exists A^{-1} \in \mathcal{M}_n(\mathbb{K})$

$$\underbrace{A^{-1}A}_{I_n} \cdot X = A^{-1}B \Rightarrow X = A^{-1}B = \frac{1}{\det A} \cdot A^{*}B$$

$x_1 = \frac{\Delta x_1}{\Delta}, \dots, x_n = \frac{\Delta x_n}{\Delta}$ Met. Cramer

Δ_{x_k} se obține din A, înlocuind coloana C_k cu coloana termenilor liberi.

$(x_1, \dots, x_n)$ sol. unică (sistem de tip Cramer) SCD.

b) $\Delta = 0$ $rg A < n$ → SCN $rg A = rg \bar{A}$
→ SI $rg A \neq rg \bar{A}$

Teorema Kronecker - Capelli

(*) este sistem compatibil $\Leftrightarrow rg A = rg \bar{A}$

Teorema Rouché

(*) este sistem compatibil $\Leftrightarrow$ toți minorii caracteristici (dc ∃) sînt nuli

II. $m \neq n$ A nu e pătratică

$rgA = r$, $\exists \det(A_{I,J}) \neq 0$ $I = \{i_1, .., i_r\}$
$J = \{j_1, .., j_r\}$.

$= \Delta_p$ (minor principal).

Δ_c = minorul caracteristic se obține din minorul principal, prin bordare cu coloana termenilor liberi și adăugarea unei linii l_i, $i \in \{1,..,n\} \setminus \{i_1,..,i_r\}$.

1) Dc. $\exists$ un minor caract. nenul, at SI

2) Dc. toți minorii caract $= 0$ (dc $\exists$)

$rgA = rg\bar{A} = r$

(**) format din cele r ec. principale.
Eventual renumerotăm și considerăm primele r ec.

a) $m > n$ (nr. ec > nr. necunoscute)

- $rgA = rg\bar{A} = n$ SCD. ($\nexists$ var. secundare)
- $rgA = rg\bar{A} = r < n$ SCN
  r var. principale
  $n-r$ var secundare.

Cele r var principale se exprimă în funcție de cele secundare

b) $m < n$ (nr. de ec < nr. necunoscute)

$rgA = rg\bar{A} = r \leq m$ SCN.

SLO (sistem liniar și omogen)

(*) $AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ PROP Un SLO este totdeauna compatibil.

Def
2 sisteme s.n. echivalente $\Leftrightarrow$ au aceeași mulțime de to

(*) și (**) sunt sisteme echivalente.

Exemple

① $\begin{cases} ax+y+z=1 \\ x+ay+z=1 \\ x+y+az=a \end{cases}$ Să se rez. Discuție. Int. geom.

$$A=\underbrace{\left(\begin{matrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{matrix}\right|\left.\begin{matrix} 1 \\ 1 \\ a \end{matrix}\right.}_{\bar{A}}$$

$$\det A=\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}=(a+2)(a-1)^2$$

I. $\Delta\neq 0 \Leftrightarrow a\in\mathbb{R}\setminus\{-2,1\}$ $\quad rg A = rg\bar{A}=3$ SCD

$x=\frac{\Delta x}{\Delta}$, $\Delta_x=\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & 1 & a \end{vmatrix}=0 \Rightarrow x=0$

$y=\frac{\Delta_y}{\Delta}$ $\quad \Delta_y=\begin{vmatrix} a & 1 & 1 \\ 1 & 1 & 1 \\ 1 & a & a \end{vmatrix}=0 \Rightarrow y=0$

$z=\frac{\Delta_z}{\Delta}$ $\quad \Delta_z=\Delta \Rightarrow z=1$

$\exists!\,(x,y,z)=(0,0,1)$

$\pi_1\cap\pi_2\cap\pi_3=\{P(0,0,1)\}$

II. $\Delta=0$

a) $a=-2$ $\quad A=\left(\begin{matrix} \boxed{\begin{matrix}-2 & 1 \\ 1 & -2\end{matrix}} & \begin{matrix}1 \\ 1\end{matrix} \\ \begin{matrix}1 & 1\end{matrix} & -2 \end{matrix}\right|\left.\begin{matrix} 1 \\ 1 \\ -2 \end{matrix}\right.$

$\Delta_p=\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\neq 0$ $\quad rg A=2$

$x, y=$ nec. principale

$\Delta_c=\begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix}=0 \Rightarrow rg\bar{A}=2$ $\quad z=\alpha=$ nec. secundară.

$\begin{cases} -2x+y=1-\alpha \quad |\cdot 2 \\ x-2y=1-\alpha \end{cases}$ (+)

$-3x \quad\quad =3-3\alpha$

$x=\alpha-1$ $\quad \pi_1\cap\pi_2\cap\pi_3=d$ dreaptă

$y=1-\alpha+2\alpha-2=\alpha-1$

$(x,y,z)\in\{(\alpha-1,\alpha-1,\alpha)\mid\alpha\in\mathbb{R}\}$

SCN

b) $a=1$

$$A=\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \qquad rg A = rg \bar{A} = 1$$

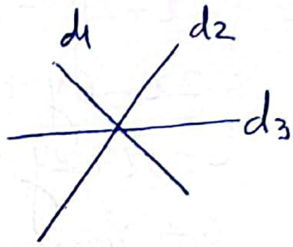
x nec. principală, $y=\alpha$, $z=\beta$ nec. secundare

$x = 1-\alpha-\beta$

$(x,y,z) \in \{(1-\alpha-\beta, \alpha, \beta) \mid \alpha,\beta \in \mathbb{R}\}$ SCdN

$\pi_1 = \pi_2 = \pi_3$

(2) $\begin{cases} x-y=1 \\ 2x-y=3 \\ ax+2y=-1 \end{cases}$ $a=?$ a.î. SCD.



$$A=\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & -1 & 3 \\ a & 2 & -1 \end{array}\right)$$

SC $\Leftrightarrow rg A = rg \bar{A}$

$\Delta_p = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \neq 0$

$\Delta_c = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ a & 2 & -1 \end{vmatrix} = 0$!

$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 3 \\ a+1 & 1 & -1 \end{vmatrix} = 0$ $\quad C_2' = C_2 + C_3$, $C_3' = C_1 - C_3$

$-1-2a-2=0 \Rightarrow \boxed{a=-\frac{3}{2}}$

(3) $\begin{cases} x+2y-z=0 \\ x+y+z=0 \end{cases}$ (2 plane care trec prin origine)

$$A=\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$rg A = rg \bar{A} = 2$

x,y nec. principale, $z=\alpha$ nec. secundară

$\begin{cases} x+2y=\alpha \\ x+y=-\alpha \end{cases}$ $\ominus$

$y=2\alpha$

$y=2\alpha$

$x = \alpha-4\alpha=-3\alpha$

$(x,y,z) \in \{\cancel{(-\frac{\alpha}{3}, \frac{2\alpha}{3}, \alpha)}\ (-3\alpha, 2\alpha, \alpha), \alpha \in \mathbb{R}\}$ $\quad \pi_1 \cap \pi_2 = d$

SCsN.

Teoremă Aplicarea transf. elementare pe linii asupra matricei extinse $\bar{A}=(A|B)$ conduce la matrice extinse ale unor sistem echivalente cu sist. inițial (*)

Metoda eliminării Gauss-Jordan

Ex
$$\begin{cases} -x+2y-3z=-2 \\ 2x-6y+9z=3 \\ -3x+2y+2z=-3 \end{cases} \qquad \bar{A}=(A|B)=\left(\begin{array}{ccc|c} -1 & 2 & -3 & -2 \\ 2 & -6 & 9 & 3 \\ -3 & 2 & 2 & -3 \end{array}\right)$$

Det. forma eșalon redusă.

$$\underset{-L_1}{\sim}\left(\begin{array}{ccc|c} \boxed{1} & -2 & 3 & 2 \\ 2 & -6 & 9 & 3 \\ -3 & 2 & 2 & -3 \end{array}\right) \underset{L_3+3L_1}{\overset{L_2-2L_1}{\sim}} \left(\begin{array}{ccc|c} \boxed{1} & -2 & 3 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & -4 & 11 & 3 \end{array}\right)$$

$$\underset{-\frac{1}{2}L_2}{\sim}\left(\begin{array}{ccc|c} \boxed{1} & -2 & 3 & 2 \\ 0 & \boxed{1} & -\frac{3}{2} & \frac{1}{2} \\ 0 & -4 & 11 & 3 \end{array}\right) \underset{L_3+L_2\cdot 4}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 5 & 5 \end{array}\right)$$

$$\underset{\frac{1}{5}L_3}{\sim}\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{\substack{L_2+\frac{3}{2}L_3 \\ L_1-3L_3}}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$\frac{1}{2}+\frac{3}{2}$

$$\underset{L_1+2L_2}{\sim}\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right) \qquad \begin{cases} x=3 \\ y=2 \\ z=1 \end{cases}$$

f. eșalon redusă