Seminar Analiză 8

$f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\cos x \quad a=0$

$$f^{(n)}(x)=\cos\left(x+n\frac{\pi}{2}\right)$$

$$T_{f,n,0}(x)=\sum_{i=1}^{n}\frac{f^{(i)}(0)(x-0)^i}{i!}=\sum_{k=0}^{n}\frac{\cos\left(k\frac{\pi}{2}\right)}{k!}x^k$$

$$f(x)=T_{f,n,0}(x)+R_{f,n,0}(x)$$

$$\left|R_{f,n,0}(x)\right|=\left|\frac{f^{(n+1)}(\alpha)}{(n+1)!}x^{n+1}\right|=\left|\frac{\cos\left(\alpha+(n+1)\frac{\pi}{2}\right)}{(n+1)!}x^{n+1}\right|\le\left|\frac{x^{n+1}}{(n+1)!}\right|\searrow 0$$

$$\sum_{n\ge 0}\frac{\cos\left(n\frac{\pi}{2}\right)}{n!}x^n=\sum_{n\ge 0}\frac{\cos(n\pi)}{(2n)!}x^{2n}$$

$$n=2n$$

$$=\sum_{n\ge 0}\frac{(-1)^n\cdot x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots$$

$$f:(-1,1)\to \mathbb{R},\ f(x)=\ln(x+1),\quad a=0$$

$$f(x)=\ln(x+1$$
$$f'(x)=\frac{1}{x+1}=(x+1)^{-1}$$
$$f''(x)=-1\cdot(x+1)^{-2}$$
$$f'''(x)=1\cdot(-2)(x+1)^{-3}$$
$$f^{(4)}(x)=(-1)(-2)(-4+1)(x+1)^{4+1}$$
$$f^{(n)}(x)=-1(-2)(-n+1)(x+1)^{n+1}$$

$\Rightarrow$

$$T_{f,n,0}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}(x-0)^k=\sum_{k=0}^{n}\frac{(-1)^{k-1}\cancel{(k-1)!}}{k!}\cdot x^k=$$

$$=\sum_{k=0}^{n}\frac{(-1)^{k-1}}{k}\cdot x^k$$

$$f(x)=T_{f,n,0}(x)+R_{f,n,0}(x)$$

$$|R_{f,n,0}(x)|=\left|\frac{f^{(n+1)}(x)}{(n+1)!}x^{n+1}\right|=\left|\frac{(-1)^n\cdot n!}{(1+\alpha)^{n+1}}\cdots\frac{1}{(n+1)!}\cdot x^{n+1}\right|=$$

$$\le\frac{1}{n+1}\left|\frac{x}{1+\alpha_n}\right|^{n+1}\le\frac{1}{n+1}\longrightarrow 0$$

$$|R_{f,n,0}(x)| = \left| \frac{f^{(n+1)}(\alpha)}{(n+1)!} x^{n+1} \right| = \left| \frac{(-1)^n \cdot (3n-1)!!!}{3^{n+1}(n+1)!} \cdot x^{n+1} \cdot (1+\alpha)^{-\left(\frac{3n-1}{3}\right)} \right| = \frac{(3n-1)!!!}{(3n+3)!!!} \left| \frac{x}{1+\alpha} \right|^{n+1} (1+\alpha)^{n+1-\frac{3n-1}{3}=\frac{2}{3}}$$

$$\leq 2 \left| \frac{x}{1+x} \right|^{n+1} \leq 2 \left(\frac{|x|}{1-|x|} \right)^{n+1} \leq \left(\frac{1}{2} \right)^n \to 0$$

$$|\alpha| \leq |x| \quad \left| \frac{x}{1+\alpha} \right| \leq \frac{|x|}{1-|\alpha|} \leq \frac{|x|}{1-|x|} \leq \frac{1}{2}$$

$$|x| \leq \frac{1}{3}, \quad \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{3}$$

Algo — 1. aflii $f^{(n)}(x)$ i 2. majorezi $T_{f,n,0}(x)$; 3. modulo de $R_{f,n,0}(x)$

$s'(x) = s(x) \Rightarrow$ uniform con.

Def $f:(a,b)\to\mathbb{R}$ s.n. analitică dacă $\forall c\in(a,b)$
$\exists \varepsilon>0$ și $\exists s_c(x)=\sum_{n\geq 0} a_{n,c}\cdot(x-c)^n$ a.î.
$f(x)=s_c(x)\quad \forall x\in(c-\varepsilon, c+\varepsilon)\ (\subset (a,b))$

$$a_{n,c}=\frac{f^{(n)}(c)}{n!}$$

$$R_{f,n,a}(x)=\frac{f^{(n+1)}(\alpha)}{(n+1)!}\cdot(x-a)^{n+1}$$

$a_n=\sup\limits_{x\in(a,b)} f^{(n)}(x)$

$\exists M>0$
a.î. $a_n\leq n!\,M^n$

$$|R_{f,n,a}|\leq\frac{\cancel{(n+1)!}\,M^{n+1}}{\cancel{(n+1)!}}|x-c|^{n+1}=[M|x-c|]^{n+1}<\left(\frac{1}{2}\right)^{n+1}$$

$$|x-c|<\frac{1}{2M}$$

$f:\mathbb{R}^3\to\mathbb{R}$

$f(x,y,z)=x^3y^2+y^3e^z$

$$\frac{\partial f}{\partial x}=3x^2y^2$$

$$\frac{\partial f}{\partial y}=2yx^3+3y^2e^z$$

$$\frac{\partial f}{\partial z}=e^zy^3$$

$$f'=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=$$

$$=\left(3x^2y^2,\ 2yx^3+3y^2e^z,\ e^zy^3\right)$$

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$g(x,y) = (x^5 \cdot y^4 \, , \, x^5 + y^4)$

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 5x^4 y^4 & 4y^3 \cdot x^5 \\ 5x^4 & 4y^3 \end{pmatrix}$$