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Seminar 13 AG[redacted]

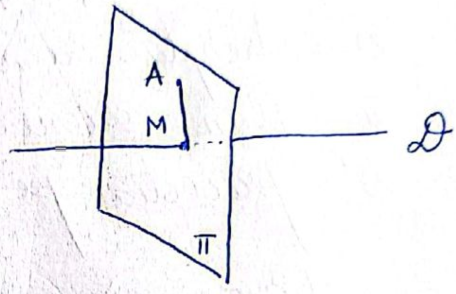
Geometrie analitică euclidiană

§1 Arii, volume, distanțe, unghiuri

- $S_{\Delta ABC} = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$

- $dist(A, \mathcal{D}) = \dfrac{\| \vec{AB} \times \vec{AC} |}{\| \vec{BC} \|}$, $B, C \in \mathcal{D}$ sau

  $dist(A, \mathcal{D}) = dist(A, M)$,
  unde $\pi \perp \mathcal{D}$, $A \in \pi$
  $\mathcal{D} \cap \pi = \{M\}$

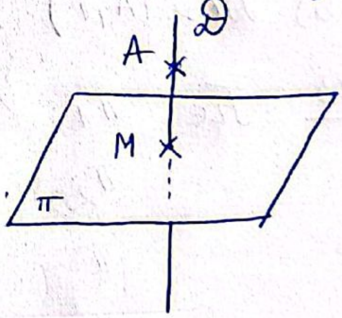


- $V_{ABCD} = \frac{1}{6} |\Delta|$, $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}$

- A, B, C, D coplanare $\Longleftrightarrow \Delta = 0$.

- $dist(A, \pi) = \dfrac{|a a_1 + b a_2 + c a_3 + d|}{\sqrt{a^2+b^2+c^2}}$, $\pi : a x_1 + b x_2 + c x_3 + d = 0$
  $A(a_1, a_2, a_3)$

  sau $dist(A, \pi) = dist(A, M)$,
  unde $A \in \mathcal{D}$, $\mathcal{D} \perp \pi$, $\mathcal{D} \cap \pi = \{M\}$.



- $dist(\mathcal{D}_1, \mathcal{D}_2) = \dfrac{|\langle \vec{AB}, N \rangle|}{\|N\|}$,
  unde $\mathcal{D}_1, \mathcal{D}_2$ drepte necoplanare, $A \in \mathcal{D}_1$, $u = u_{\mathcal{D}_1}$
  $B \in \mathcal{D}_2$, $v = u_{\mathcal{D}_2}$
  $N = u \times v$

- $\sphericalangle(\mathcal{D}_1, \mathcal{D}_2) = \sphericalangle(u_1, u_2) = \sphericalangle \varphi \in [0, \pi]$

  $\cos\varphi = \dfrac{\langle u_1, u_2 \rangle}{\|u_1\| \cdot \|u_2\|}$, unde $\mathcal{D}_k$ = dreaptă orientată de u_k, $k = \overline{1,2}$

- $\sphericalangle(\pi_1, \pi_2) = \sphericalangle(N_1, N_2) = \sphericalangle \varphi$

  $\cos\varphi = \dfrac{\langle N_1, N_2 \rangle}{\|N_1\| \cdot \|N_2\|}$, unde π_k = plan orientat de N_k, $k = \overline{1,2}$

- $\sphericalangle(\mathcal{D}, \pi) = \sphericalangle(\mathcal{D}, \mathcal{D}') = \sphericalangle \varphi$.

  $\mathcal{D}$ = dreaptă orientată de u

  π = plan orientat de N.

  $\mathcal{D}'$ = proiecția pe π a lui $\mathcal{D}$.

$(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$

Ex1 Fie $A(1,2,1)$, $B(2,1,3)$, $C(-2,1,3)$, $D(0,2,0)$

a) V_{ABCD} ; b) $S_{\Delta BCD}$; c) $dist(A, (BCD))$

Ex2 Fie $A(1,1,1)$, $\mathcal{D}: \begin{cases} x_1 + x_2 - x_3 + 1 = 0 \\ 2x_1 + x_2 - 3x_3 + 2 = 0 \end{cases}$; $\pi: x_1 + x_2 + x_3 = 0$

a) $dist(A, \mathcal{D}) = ?$

b) $Dist(A, \pi)$

Ex3 Fie $\pi_1: x_1 - 3x_2 - 1 = 0$ $\qquad \mathcal{D}_1: \begin{cases} x_1 - x_2 - 2 = 0 \\ x_1 + x_3 - 3 = 0 \end{cases}$

$\pi_2: 2x_2 + x_3 - 2 = 0$ $\qquad \mathcal{D}_2: \dfrac{x_1 - 1}{3} = \dfrac{x_2 + 1}{0} = \dfrac{x_3 - 1}{-1}$

$\pi: x_2 - x_3 - 1 = 0$ $\qquad \mathcal{D}: \dfrac{x_1 - 1}{-1} = \dfrac{x_2}{2} = \dfrac{x_3 + 1}{-5}$

a) $\sphericalangle(\mathcal{D}_1, \mathcal{D}_2)$; b) $\sphericalangle(\mathcal{D}, \pi)$; c) $\sphericalangle(\pi_1, \pi_2)$

Ex4 Fie $\mathcal{D}_1: \begin{cases} x_1 - x_2 = 2 \\ x_1 + x_3 = 3 \end{cases}$, $\mathcal{D}_2: \dfrac{x_1 - 1}{3} = \dfrac{x_2 + 1}{0} = 1 - x_3$

$Dist(\mathcal{D}_1, \mathcal{D}_2) = ?$

§2 Conice. Formă canonică

S.n. conică în $\mathbb{R}^2$ LG al punctelor $P(x_1, x_2)$ a.î.

$$\Gamma: f(x) = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 + 2b_1x_1 + 2b_2x_2 + c = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T, \quad \tilde{A} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{21} & b_2 \\ b_1 & b_2 & c \end{pmatrix} = \begin{pmatrix} A & B^T \\ B & c \end{pmatrix},$$

$B = (b_1\ b_2)$

$\delta = \det A$, $\Delta = \det \tilde{A}$

$r = rg A$, $r' = rg \tilde{A}$, $r \le r' \le r+2$

$\Delta = 0$ Γ = conică degenerată

$\Delta \neq 0$ Γ = -"- nedegenerată

1) $(\mathbb{R}^2, \mathbb{R}^2/\mathbb{R}, \varphi)$ sp afin

$\Gamma_1 \sim \Gamma_2$ conice afin echivalente $\Leftrightarrow$

$\exists\, \tau: \mathbb{R}^2 \to \mathbb{R}^2$ transformare afină a.î. $\Gamma_2 = \tau(\Gamma_1)$,

unde $\tau: X' = CX + D$, $C \in GL(n, \mathbb{R})$

Invarianți afini: $\frac{\Delta}{\delta}, r, r'$

2) $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$ sp. punctual euclidian.

$\Gamma_1 \equiv \Gamma_2$ conice congruente metric $\Leftrightarrow$

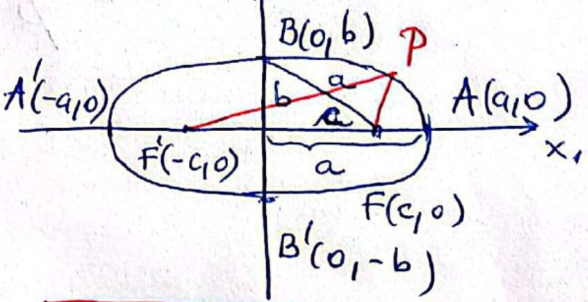
$\exists\, \tau: \mathbb{R}^2 \to \mathbb{R}^2$ izometrie a.î. $\Gamma_2 = \tau(\Gamma_1)$, unde

$\tau: X' = CX + D$, $C \in O(n)$

Invarianți metrici: $\frac{\Delta}{\delta}, r, r', \Delta, \delta$

$a^2 = b^2 + c^2$

1 Elipsa: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, $a, b > 0$



A, A', B, B' = vârfuri

F, F' = focare

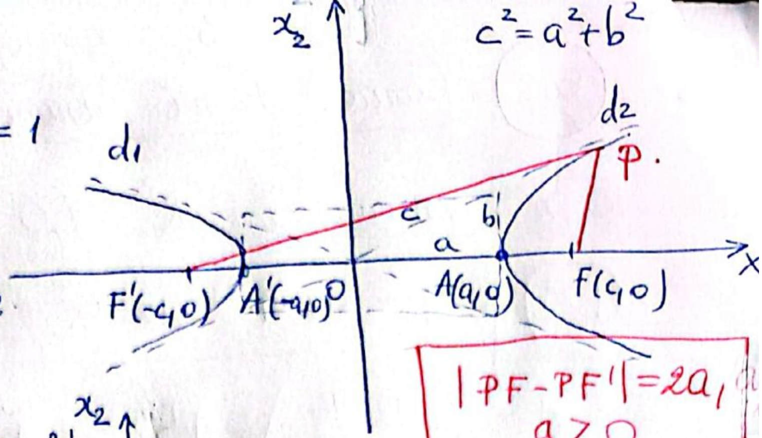
$\boxed{PF + PF' = 2a}$, $a > 0$

② **Hiperbola** : $\dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 1$ $\quad c^2 = a^2 + b^2$

$a, b > 0$

$d_1 \cup d_2 : x_2 = \pm \dfrac{b}{a} x_1$ asimptote

A, A' = vârfuri, F, F' = focare

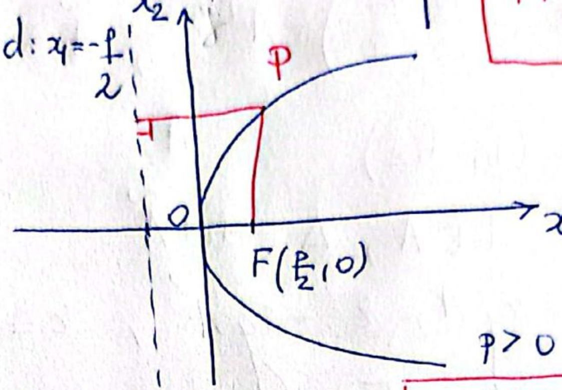


$\boxed{|PF - PF'| = 2a, \ a > 0}$

③ **Parabola** : $x_2^2 = 2p x_1$ $\quad d: x_1 = -\dfrac{p}{2}$

F = focar

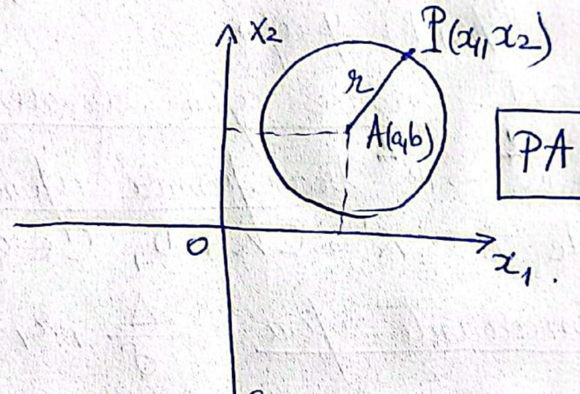
d = dreapta directoare



$\boxed{\mathrm{dist}(P, d) = PF}$

Cercul

$\mathcal{C}(A(a,b), r) : (x_1 - a)^2 + (x_2 - b)^2 = r^2$



$\boxed{PA = r}$

Conice nedegenerate: elipsa, hiperbola, parabola

Conice degenerate $\begin{cases} \text{punct dublu} \\ \text{drepte concurente} \\ \text{drepte paralele} \\ \text{dreapta dublă} \\ \emptyset \end{cases}$

• P_0 s.n. centru pt $\Gamma \Leftrightarrow [\forall P \in \Gamma \Rightarrow S_{P_0}(P) \in \Gamma]$

$\delta \neq 0 \Rightarrow \Gamma$ are centru unic. $P_0(x_1^0, x_2^0)$

$\delta \neq 0 \Rightarrow f(x_1^0, x_2^0) = \dfrac{\Delta}{\delta}$.

(I) Aducerea la f. canonică pt $\boxed{\delta \neq 0}$.

(1) $(\mathbb{R}^2, \mathbb{R}^2/_{\mathbb{R}}, \varphi)$ sp. afin.

$\mathcal{R} = \{O; e_1, e_2\} \xrightarrow[\text{translație}]{\theta} \mathcal{R}' = \{P_0; e_1, e_2\} \xrightarrow[\text{transf. afină}]{\tau} \mathcal{R}'' = \{P_0; e_1', e_2'\}$.

a) $\theta = X = X' + X_0$. $X_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$, P_0 = centru: $\begin{cases} \dfrac{\partial f}{\partial x_1} = 0 \\ \dfrac{\partial f}{\partial x_2} = 0 \end{cases}$

$\theta(\Gamma): X'^T A X' + \dfrac{\Delta}{\delta} = 0$.

b) $Q: \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = X'^T A X'$ formă pătratică

Aducem Q la o formă canonică (met Gauss/Jacobi)

$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2$

$\tau: X' = CX''$, $C \in GL(2, \mathbb{R})$

Deci: $\underbrace{\tau \circ \theta(\Gamma)}_{\Gamma'}: X = CX'' + X_0$ (transf. afină)

$\Gamma \sim \Gamma'$ afin echivalente

$\tau \circ \theta(\Gamma): \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \dfrac{\Delta}{\delta} = 0$.

(2) $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$ sp. punctual euclidian.

a) Analog cu cazul (1)

b) $Q: \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = X'^T A X'$

$\exists$ un reper ortonormat format din versori proprii ai A diag.

$P(\lambda) = \det(A - \lambda I_2) = 0. \Rightarrow \lambda_1, \lambda_2 \Rightarrow e_1', e_2'$ versori proprii ortogonali

$e_k' = (\ell_k, m_k)$

$R = \begin{pmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{pmatrix}$ Pt $\det R = 1 \Rightarrow$ reper pozitiv orientat

$\tau: X' = RX''$ izometrie ($\det R = 1 \Rightarrow$ spetă 1)

$\tau \circ \theta(\Gamma) = \Gamma' : \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0$

$\Gamma \equiv \Gamma'$ congruente metric.

$\tau \circ \theta : \{ X = RX'' + X_0$, $R \in SO(2)$, ($\det R = 1$)

$\mathcal{R} \xrightarrow{\theta} \mathcal{R}' \xrightarrow{\tau} \mathcal{R}''$ repere ortonormate

translație rotație

EX5 $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$

Fie conica:

$\Gamma: f(x_1, x_2) = 5x_1^2 + 8x_1x_2 + 5x_2^2 - 18x_1 - 18x_2 + 9 = 0$

Să se aducă la o formă canonică, efectuând izometrii.

Reprezentare grafică

EX6 Fie conica:

$\Gamma: f(x_1, x_2) = 3x_1^2 - 8x_1x_2 + 3x_2^2 + 2x_1 + 2x_2 + 2 = 0$

Să se aducă la o formă canonică, ef. izometrii.

Reprez. grafică

EX7 $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$ Fie conica Γ:

a) $f(x_1, x_2) = 3x_1^2 - 4x_1x_2 - 2x_1 + 4x_2 - 3 = 0$

b) $f(x_1, x_2) = 4x_1x_2 - 3x_2^2 + 4x_1 - 14x_2 - 7 = 0$

c) $f(x_1, x_2) = 3x_1^2 - 4x_1x_2 + 3x_2^2 - 4x_1 + 6x_2 - 4 = 0$

d) $f(x_1, x_2) = 16x_1^2 + 4x_1x_2 + 19x_2^2 + 80x_1 + 10x_2 + 40 = 0$

1) Să se det centrul conicei

2) Să se aducă conica Γ la o formă canonică, efectuând o izometrie de speța 1. Precizați schimbările de repere ortonormate

3) Reprez. grafică