Curs (10) Analiză

Def Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f : D \to \mathbb{R}^m$ și $v \in \mathbb{R}^n \setminus \{0\}$

1) $\frac{\partial f}{\partial v}(a) = \lim\limits_{t \to 0} \frac{f(a+tv) - f(a)}{t}$,

$\frac{\partial f}{\partial e_k}(a) = \frac{\partial f}{\partial x_k}(a) = f'_{x_k}(a)$

2) f este derivabilă în $a \Leftrightarrow \exists T \in L(\mathbb{R}^n, \mathbb{R}^m)$ a.î.

$\lim\limits_{x \to a} \frac{f(x) - f(a) - T(x-a)}{\|x - a\|_2} = 0 \quad T = f'(a)$

OBS $f'(a)(v) = \frac{\partial f}{\partial v}(a) \quad f'(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$

Ex Fie $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{x^8 y^4}{x^{10} + y^{10}}, & x^2 + y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$

a) cont f

b) $\exists$ cont $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

c) derivabil

a) f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{\substack{x\to 0\\ y=ax}} f(x,y) = \lim_{x\to 0} f(x,ax) = \lim_{x\to 0} \frac{x^8 a^4 x^4}{x^{10}(1+a^{10})} = \lim_{x\to 0} \frac{x^2 a^4}{1+a^{10}} = 0$$

nu e suficient

$x_n = y_n = \frac{1}{n}$ trebuie să majorăm

$$|f(x,y)| = \frac{x^8 y^4}{x^{10}+y^{10}} \le \underbrace{\left(\frac{x^{10}}{x^{10}+y^{10}}\right)^{\frac{8}{10}}}_{\le 1} + \left(\frac{y^{10}}{x^{10}+y^{10}}\right)^{\frac{4}{10}} \cdot$$

$$\cdot \left(x^{10}+y^{10}\right)^{\frac{8}{10}+\frac{4}{10}-1=\frac{1}{5}} \le \left(x^{10}+y^{10}\right)^{\frac{1}{5}} \xrightarrow[\substack{x\to 0\\ y\to 0}]{} 0$$

b) $$\frac{\partial f}{\partial x} = \frac{8x^7 y^4 (x^{10}+y^{10}) - x^8 y^4 10 x^9}{(x^{10}+y^{10})^2} =$$

$$= \frac{-2y^4 x^{17} + 8x^7 y^{14}}{(x^{10}+y^{10})^2} \Rightarrow$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\substack{x\to 0\\ y=0}} \frac{f(x,0)-f(0,0)}{x} = 0$$

$$\lim_{\substack{x\to 0\\ y=ax}} \frac{\partial f}{\partial x} = \lim_{x\to 0} \frac{-2x^{17}a^4x^4 + 8x^7a^{14}x^{14}}{x^{20}(1+a^{10})^2} = 0$$

$$\frac{y^4 |x^{17}|}{(x^{10}+y^{10})^2} = \underbrace{\left(\frac{x^{10}}{x^{10}+y^{10}}\right)^{\frac{17}{10}}}_{\le 1} \left(\frac{y^{10}}{x^{10}+y^{10}}\right)^{\frac{4}{10}} \cdot$$

$$\cdot (x^{10}+y^{10})^{\frac{21}{10}-2} \le (x^{10}+y^{10})^{\frac{1}{10}} \xrightarrow[\substack{x\to 0\\ y\to 0}]{} 0$$

c) f este dif in $(0,0) \Leftrightarrow \exists\, T \in L(\mathbb{R}^2;\mathbb{R})$ ai

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-T(x,y)}{\sqrt{x^2+y^2}} = 0$$

$T(x,y) = ax + by$ $\quad a = \frac{\partial f}{\partial x}(0,0) = 0 \quad b = \frac{\partial f}{\partial y}(0,0) =$

$$= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0$$

Ex $f:\mathbb{R}^3\to\mathbb{R}$ $f(x,y,z) = x^2y^3z^4$ $\quad f' = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \frac{\partial f}{\partial z}\right.$

$= \left(2xy^3z^4 \quad 3x^2y^2z^4 \quad 4x^2y^3z^3\right) = g(g_1\ g_2\ g_3)$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right) = 6xy^2z^4$$

$$\frac{\partial^2 f}{\partial z \partial x} = 8xy^3z^3 \qquad \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial^2 f}{\partial x^2} = 2y^3z^4$$

$$\frac{\partial^3 f}{\partial z \partial y \partial x} = 24xy^2z^3$$

$$g' = \begin{pmatrix} \frac{\partial g^1}{\partial x} & \frac{\partial g^1}{\partial y} & \frac{\partial g^1}{\partial z} \\ \frac{\partial g^2}{\partial x} & \frac{\partial g^2}{\partial y} & \frac{\partial g^2}{\partial z} \\ \frac{\partial g^3}{\partial x} & \frac{\partial g^2}{\partial y} & \frac{\partial g^3}{\partial z} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} =$$

$$= \begin{pmatrix} 2y^3z^4 & 6xy^2z^4 & 8xy^2z^3 \\ 6xy^2z^4 & 6x^2yz^4 & 12x^2y^2z^3 \\ 8xy^2z^3 & 12x^2y^2z^3 & 12x^2y^2z^2 \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial x \partial y}$$ (majoritatea cazurilor sunt egale)

<u>Def</u> Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f : D \to \mathbb{R}^m$ și $v_1, \dots, v_k \in \mathbb{R}^n \setminus \{0\}$

$$\frac{\partial^k f}{\partial v_k \partial v_{k-1} \dots \partial v_1}(a) = \frac{\partial}{\partial v_k}\left(\frac{\partial^{k-1} f}{\partial v_{k-1} \dots \partial v_1}\right)(a)$$

<u>Def</u> Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f : D \to \mathbb{R}$ a.î. $\exists f'$ pe D

$$f''(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

T. Young Fie $D=\mathring{D}\subset\mathbb{R}^n$, $a\in D$, $f:D\to\mathbb{R}^m$ și $u,v\in\mathbb{R}^n\setminus\{0\}$

Dacă $\exists$ f' pe D și $f''(a)$ atunci

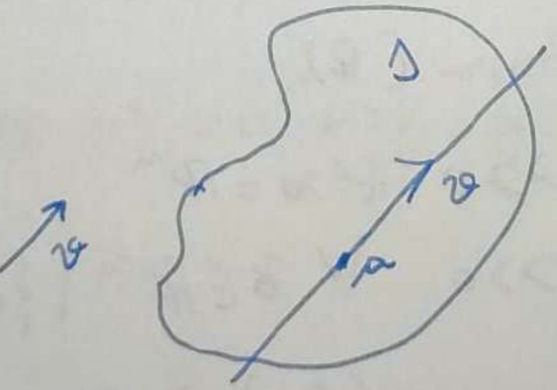
$$\frac{\partial^2 f}{\partial v\,\partial u}(a)=\frac{\partial^2 f}{\partial u\,\partial v}(a)=f''(a)(u,v)=f''(a)(v,u)$$

T. Schwarz Fie $D=\mathring{D}\subset\mathbb{R}^n$, $f:D\to\mathbb{R}^m$, $a\in D$ și $u,v\in\mathbb{R}^n\setminus\{0\}$ și $\exists$ $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial u\,\partial v}$ pe D și $\frac{\partial^2 f}{\partial u\,\partial v}$ să fie continuă în a.

Atunci $\exists$ $\frac{\partial^2 f}{\partial v\,\partial u}(a)=\frac{\partial^2 f}{\partial u\,\partial v}(a)$

T. Fermat Fie $D=\mathring{D}\subset\mathbb{R}^n$, $a\in D$, $f:D\to\mathbb{R}$.

Dacă $\exists$ $f'(a)$ și a este punct de extrem local pentru f în a $\Rightarrow f'(a)=0$



$$\frac{\partial f}{\partial v}(a)=f'(a)(v)=0 \quad \forall v$$

KYSSS

T. Taylor $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f : D \to \mathbb{R}$ a.î. $\exists f'$ pe D și $f''(a)$. Atunci $\exists\, \omega : D \to \mathbb{R}$ a.î.

1) $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a, x-a) + \|x-a\|^2 \omega(x)$ și 2) $\lim\limits_{x \to a} \omega(x) = 0$

$\boxed{T}$ Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f : D \to \mathbb{R}$ a.î. $\exists f'$ pe D și $f''(a)$.

Atunci

(1) Dacă a este punct de maxim local pt $f \Rightarrow f'(a) = 0$ și $f''(a) \geq 0$

(2) Dacă $f'(a) = 0$ și $f''(a) > 0 \Rightarrow a$ este punctul de minim local pt f

(3) Dacă a este punct de maxim local $\Rightarrow f'(a) = 0$ și $f'(a) \leq 0$

(4) Dacă $f'(a) = 0$ și $f''(a) < 0 \Rightarrow a$ este punct de maxim local pentru f

Def Fie $A = A^t \in \mathcal{M}_{n,n}(\mathbb{R})$

1) $A \geq 0 \Leftrightarrow \langle Ax, x \rangle \geq 0 \quad \forall x \in \mathbb{R}^n$

2) $A > 0 \Leftrightarrow \langle Ax, x \rangle > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

$\Leftrightarrow \Delta_1, \ldots, \Delta_n > 0 \Leftrightarrow A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} =$

$= (a_{ij})_{i,j=\overline{1,n}}$, $\Delta_k = \det (a_{ij})_{i,j=\overline{1,k}} > 0$

Ex 1 $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2 - xy + y^2$

$$\frac{\partial f}{\partial x} = 2x - y = 0 \qquad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \neq 0$$

$$\Rightarrow x = y = 0$$

$$\frac{\partial f}{\partial y} = -x + 2y = 0$$

$$f''(0,0) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \begin{aligned} \Delta_1 &= 2 \quad + \\ \Delta_2 &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \quad + \end{aligned}$$

punct de minim local

$$\det(\lambda I - A) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 1$$

$$2 - \lambda = \pm 1$$

$$\lambda_1 = 3 > 0$$

$$\lambda_2 = 1 > 0$$

punct de minim local

$$z(x,y) \Rightarrow \langle Az, z\rangle = 2x^2 - 2xy + 2y^2 =$$

$$= 2(x^2 - xy + y^2) = 2\left(x^2 - xy + \frac{y^2}{4} + \frac{3}{4}y^2\right) = 2\left(\left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2\right) > 0$$

$$z \neq 0$$

Ex 2 $g : \mathbb{R}^2 \to \mathbb{R}$ $g(x,y) = -x^2 - y^2$

$\frac{\partial g}{\partial x} = -2x = 0$

$\frac{\partial g}{\partial y} = -2y = 0$ $\Rightarrow x = y = 0$

$g'' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$\lambda_1 = \lambda_2 = -2$ max local

$\Delta_1 = -2 < 0$

$\Delta_2 = 4 > 0$

$- + - + - +$

maxim

Ex 3 $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, $f'(x) = 3x^2 = 0$ $f'(0) = 0$

Ex $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = x^2 - y^2$

$\frac{\partial h}{\partial x} = 2x = 0$

$\frac{\partial h}{\partial y} = -2y = 0$ $\Rightarrow x = y = 0$

$h'' = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$\lambda_1 = 2 > 0$ punct șa

$\lambda_2 = -2 < 0$

Ex $f:\mathbb{R}\to\mathbb{R}$ $f(x)=e^x$, $a=0$

$\lim_{x\to 0} e^x = 1$

$\lim_{x\to 0} \frac{e^x-1}{x} = 1$

$\lim_{x\to 0} \frac{e^x-1-x}{x} = 0$

$\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{e^x-1}{2x} = \frac{1}{2}$

$\lim_{x\to 0} \frac{e^x-\boxed{1-x-\frac{x^2}{2}}}{x^2} = 0$

$\lim_{x\to 0} \frac{e^x-1-x-\frac{x^2}{2}}{x^3} = \frac{1}{6}$

$= \lim_{x\to 0} \frac{e^x-1-x}{3x^2} = \frac{1}{6}$; $\lim_{x\to 0} \frac{e^x-1-x-\frac{x^2}{2}-\dots-\frac{x^n}{n!}}{x^{n+1}} = \frac{1}{(n+1)!}$

Def Fie $f:(a,b)\to\mathbb{R}$, $c\in(a,b)$ a.î. $\exists f^{(n-1)}$ pe (a,b) și $\exists f^{(n)}(c)$

Polinomul $T_{f,n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ s.n. polinomul Taylor asociat funcției f de ordin n în a

Ex $f(x)=e^x$ $f^{(n)}(x)=e^x$ $f^{(n)}(0)=1$

$T_{f,n,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

Prima teoremă a lui Taylor Fie $f:(a,b)\to\mathbb{R}$ și $c\in(a,b)$ a.î. $\exists\, f^{(n-1)}$ pe (a,b) și $\exists\, f^{(n)}(c)$

Atunci $\exists\, \omega:(a,b)\to\mathbb{R}$ a.î.

$$f(x)=T_{f,n,c}(x)+(x-c)^n\,\omega(x) \text{ și } \lim_{x\to c}\omega(x)=0$$

A doua teoremă a lui Taylor Fie $f:(a,b)\to\mathbb{R}$ și $c\in(a,b)$ a.î. $\exists\, f^{(n+1)}$ pe (a,b). Atunci $\exists\, \alpha$ între x și c a.î.

$$f(x)=T_{f,n,c}(x)+\frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-c)^{n+1}=R_{f,n,c}(x)$$