Curs 2 Analiză

Def 1) O funcție $d : X \times X \to [0, +\infty)$ s.n. distanța dacă

i) $d(x, y) = 0 \Leftrightarrow x = y$

ii) $d(x, y) = d(y, x) \ \forall x, y \in X$ și

iii) $d(x, y) + d(y, z) \geq d(x, z)$; $\forall (x, y, z \in X$

2) $x_n \xrightarrow{d} \Leftrightarrow \forall \varepsilon > 0 \ \exists n_\varepsilon$ a.î. $\forall n > n_\varepsilon \Rightarrow d(x_n, a) < \varepsilon$

OBS 1 $x_n \xrightarrow{d} a \Leftrightarrow d(x_n, a) \to 0$

$d(x_n, a) < \varepsilon \Leftrightarrow x_n \in B(a, \varepsilon) = \{x \in X \mid d(a, x) < \varepsilon\}$

3) $A \subset X$ s.n. mărginită $\Leftrightarrow \exists B(a, r) \subset X$ a.î. $A \subset B(a, r)$

4) Șirul $(x_n)_n$ s.n. Cauchy $\Leftrightarrow \forall \varepsilon > 0 \ \exists n_\varepsilon$ a.î. $\forall n, m \geq n_\varepsilon \Rightarrow$

$\Rightarrow d(x_n, x_m) < \varepsilon$

T (x_n este conv $\overset{\not\Leftarrow}{\Rightarrow}$ este cauchy $\Rightarrow$ este mărginit

$$\left.\begin{array}{l} (x_n)_n \text{ Cauchy} \\ \exists x_{n_k} \to a \end{array}\right| \Rightarrow x_n \to a$$

$\mathbb{R}^n = \overset{n}{\underset{i=1}{\times}} \mathbb{R} \ni x = (x_1, x_2, \dots, x_n)$

$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \qquad x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$\cdot : \mathbb{R} \times \mathbb{R}^n = \qquad a x = (a x_1, a x_2, \dots, a x_n)$

1) $(\mathbb{R}^n, +)$ grup comutativ 2) $a(x + y) = ax + ay$

3) $(a+b)x = ax + bx$; 4) $(ab)x = a(bx)$

$\forall x, y \in \mathbb{R}^n$ și $\forall a, b \in \mathbb{R}$

Def O mulțime X, împreună cu $+ : X \times X \to X$ și

și $\cdot : \mathbb{R} \times X \to X$ care verifică 1)-5) s.n. spațiu vectorial

Exemple 1) $\mathbb{R}[x]$, $P+Q$, aP

2) $\mathbb{R}_n[X] = \{P \in \mathbb{R}[X], \text{grad} P \leq n\}$

3) $\mathcal{M}_{m,n}(\mathbb{R})$; 4) $\mathcal{F}(A, \mathbb{R}) = \{f : A \to \mathbb{R}\}$ $f+g$, $a \cdot f$

$\mathbb{C}[X]$

$\mathbb{R}^n$ $d_1, d_2, d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

$$d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

$$d_\infty(x,y) = \max_{i=1}^{n} |x_i - y_i|$$

Verificăm 1) $d_1(x,y) = 0 \iff x = y$; $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = 0$

$\iff x_i = y_i \; \forall i = \overline{1,n} \iff x = y$

2) $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d_1(y,x)$

3) $d_1(x,y) + d_1(y,z) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| =$

$= \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) \geq \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| = d_1(x,z)$

4) $d_1(x+z, y+z) = \sum_{i=1}^{n} |x_i + \cancel{z_i} - (y_i + \cancel{z_i})| = d_1(x, z)$

5) $d_1(ax, ay) = \sum_{i=1}^{n} |ax_i - ay_i| = |a| \cdot \sum_{i=1}^{n} |x_i - y_i| = |a| \, d_1(x, y)$

---

1) $d_2(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}} = 0 \Leftrightarrow x = y \; (x_i = y_i, \; i = \overline{1, n})$

2) $d_2(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} (y_i - x_i)^2 \right)^{\frac{1}{2}} = d_2(y, x)$

3) $d(x, y) + d_2(y, z) \geq d_2(x, z)$

$\left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} + \left(\sum_{i=1}^{n} (y_i - z_i) \right)^{\frac{1}{2}} \geq \left(\sum_{i=1}^{n} (x_i - z_i)^2 \right)^{\frac{1}{2}}$ | $x_i - y_i = a_i$, $y_i - z_i = b_i$, $x_i - z_i = a_i + b_i$

$\left(\sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} b_i^2 \right)^{\frac{1}{2}} \geq \sqrt{\sum_{i=1}^{n} (a_i + b_i)^2}$

$\cancel{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2} + 2\sqrt{\left(\sum_{i=1}^{n} a_i^2 \right)\left(\sum_{i=1}^{n} b_i^2 \right)} \geq$

$\geq \sum_{i=1}^{n} (a_i + b_i)^2 = \cancel{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i} + \cancel{2} \sum_{i=1}^{n} a_i b_i$

$\Leftrightarrow \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2$

$\mathbb{R}^2$ $m=2$ $z_n = (x_n, y_n) \xrightarrow{d_1} z = (x,y) \Leftrightarrow$

$d_1(z_n, z) \to 0 \Leftrightarrow \underline{|x_n - x| + |y_n - y|} \to 0 \Leftrightarrow \begin{matrix} x_n \to x \\ y_n \to y \end{matrix}$

$d_2(z_n, z) \to 0 \Leftrightarrow \sqrt{(x_n-x)^2 + (y_n-y)^2} \to 0 \Rightarrow \begin{matrix} x_n \to 0 \\ y_n \to 0 \end{matrix}$

$\begin{matrix} \geq & & \geq \\ |x_n - x| & & |y_n - y| \end{matrix}$

$\underbrace{|\cancel{x} - x_n| + |y - y_n|}_{\downarrow 0} \geq \underbrace{d_2(z_n, z)}_{\downarrow 0} \geq 0$

$\left(\frac{2n+1}{3n+1}, \left(1+\frac{1}{n}\right)^{2n}, \sqrt{n+1} - \sqrt{n} \right) \to \left(\frac{2}{3}, e^2, 0\right)$

$\frac{2n+1}{3n+1} \to \frac{2}{3}$, $\left(1+\frac{1}{n}\right)^{2n} \to e^{-2}$, $\sqrt{n+1} - \sqrt{n} \to 0$

în $d_1 (d_2, d_\infty)$

$d_\infty(x,y) \leq d_2(x,y) \leq d_1(x,y) \leq n\, d_\infty(x,y)$ $(*)$

$\max_{i=1}^{n} |x_i - y_i| \leq \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \leq \sum_{i=1}^{n} |x_i - y_i| \leq$

$\leq n \cdot \max_{i=1}^{n} |x_i - y_i|$

$0 \leq d_\infty(x_n, x) = d_1(x_n, x)$

$\downarrow \qquad \downarrow 0$

$x_n \xrightarrow{d_1} x \Rightarrow x_n \xrightarrow{d_\infty} x$

$d_1(x,y) = d_1(x-y, 0)$ $\qquad$ $d_1(x,0) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$

$d_2(x,0) = \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

__Def__ o aplicatie $\| \ \| : \mathbb{R}^n \to [0,\infty)$ s.n __norma__ dacă

1) $\|x\| = 0 \Leftrightarrow x = 0$

2) $\|x+y\| \le \|x\| + \|y\| \ \forall x,y \in \mathbb{R}^3$ și 3) $\|\alpha x\| = |\alpha| \cdot \|x\|$

$\| \ \| \to d_{\| \ \|}$ $\qquad$ $d_{\| \ \|}(x,y) = \|x-y\|$

norma e tie de distanta tier

__Def__ Două norme $\|\cdot\|, |||\cdot|||$, s.n echivalente dacă

$\exists \ 0 < \alpha \le \beta$ a.î

$\alpha\|x\| \le |||x||| \le \beta\|x\| \quad \forall x \in \mathbb{R}^n$ $\qquad$ $\|x\| = |||x|||$

(*) $(\| \ \|_\infty \le \| \ \|_2 \le \| \ \|_1 \le n \| \ \|_\infty)$

"echiv" cu $||| \ \ |||$

__Teorema__ În $\mathbb{R}^n$ orice două norme sunt echivalente

$v = (2,1) \in \mathbb{R}^2 \quad n = 2 \quad \dim \mathbb{R}^2 = n = 2$

$d_1(v,0) = \|v\|_1 = |2-0| + |1-0| = 3 = \|v\|_1$

$d(2,0) + d(1,0) = 3$

$a + A = \{a + x / x \in A\}$

$a \cdot A = \{a \cdot x \,/\, x \in A\}$

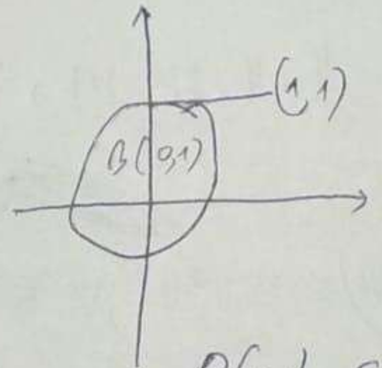
$\|v\|_2 = d_2(v,0) = \sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5}$

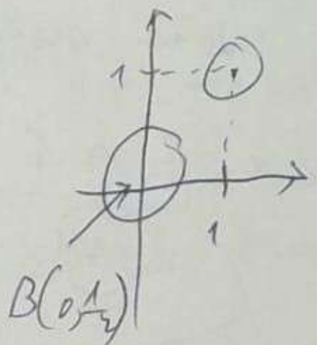
$\|v\|_\infty = d_\infty(v,0) = \max(|2-0|, |1-0|) = 2$ inf

$B(a,r) = a + B(0,r)$ $\qquad$ $B(a,r) = a + r \cdot B(0,1)$

$B(0,r) = r\, B(0,1)$

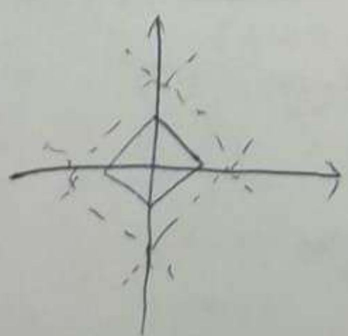
$d_2 \quad B((1,1), \frac{1}{2}) = \{ (x,y) \mid \sqrt{(x-1)^2 + (y-1)^2} < \frac{1}{2} \}$





$B(0,1) = \{x^2 + y^2 < 1\}$

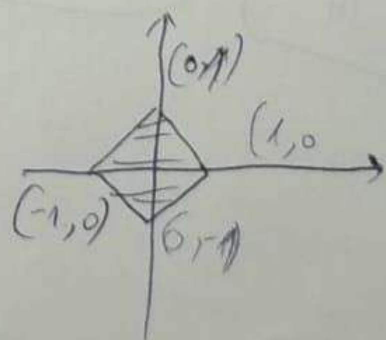
$B_1(0,1) = \{(x,y) \mid \|(x,y)\|_1 < 1 \Leftrightarrow |x| + |y| < 1\}$



$x > 0 \quad x + y = 1$

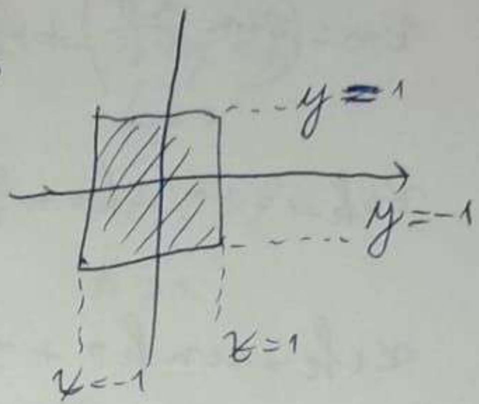
$y > 0$

$x - y = 1$



$B_\infty(0,1) = \{ \max(|x|, |y|) < 1 \} \Leftrightarrow$

$\Leftrightarrow \|(x,y)\|_\infty \}$



Teoremă Un șir $x_n =$

$= (x_1^n, x_2^n, \dots, x_m^n) \in \mathbb{R}^m$ este Cauchy în $d_2 \Leftrightarrow$ șirurile $(x_k^n)_{n \geq 1}$ sunt Cauchy $\forall k \in \overline{1,m}$

Teoremă Un șir mărginit din $(\mathbb{R}^m, d_2)$ are un subșir convergent

Teoremă Orice șir Cauchy din $\mathbb{R}^m$ este convergent

$x_n = (-1)^n \frac{2n+1}{3n+1} + \frac{n}{n+1}$ (cu $\frac{2n+1}{3n+1} \to \frac{2}{3}$, $\frac{n}{n+1} \to 1$)

$\overline{l}$ = limita superioară

$\overline{l} > \underline{l}$

$x_{2n} = \frac{4n+1}{6n+1} + \frac{2n}{2n+1} \to \frac{2}{3} + 1 = \frac{5}{3}$

$x_{2n+1} = -\frac{4n+3}{6n+4} + \frac{2n+1}{2n+2} \to -\frac{2}{3} + 1 = \frac{1}{3}$

$L = \left\{ \frac{1}{3}, \frac{5}{3} \right\}$

$\frac{5}{3} = \lim_{n \geq 1} \sup x_n = \overline{\lim_{n \to \infty}} x_n = \overline{l}$

$\underline{l} = \frac{1}{3}$ = limita inferioară

Ex 2 $x_n = \left(\sin \frac{n\pi}{2}\right) + \frac{1}{n}$

$x_{4k+1} = \sin\left(2k\pi + \frac{\pi}{2}\right) + \frac{1}{4k+1} \to 1$

$x_{2k} = \sin k\pi + \frac{1}{2k} \to 0$

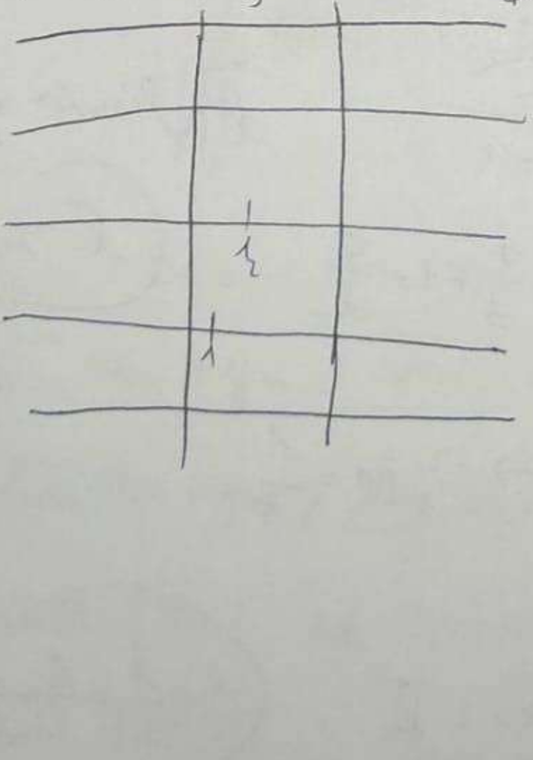
$x_{4k+3} = \underbrace{\sin\left(2k\pi + \frac{3\pi}{2}\right)}_{= -1} + \frac{1}{4k+3} \to -1$

$\mathcal{L} = \{-1, 0, 1\}$ $\overline{\lim}\, x_n = 1$

$\underline{\lim}\, x_n = -1$

$\frac{n(n+1)}{2}$

$\underbrace{0}_{1}, \underbrace{0, 1}_{2}, \underbrace{0, \frac{1}{2}, 1}_{3}, \underbrace{0, \frac{1}{3}, \frac{2}{3}, 1}_{4} \dots \quad \Big| \ 0 \ \frac{1}{n} \ \frac{2}{n} \dots \frac{n}{n} = 1$



$\mathcal{L} = [0, 1]$

$x_n \in [0,1] \Rightarrow \mathcal{L} \subset [0,1]$

$x \in [0,1]\ \exists K$ a.i $\left|x - \underbrace{\frac{k}{n}}_{(x_n)_n}\right| < \frac{1}{n}$

$k \in \{0, \dots, n\}$

Fie $(x_n)_n \subseteq \mathbb{R}$ şi notăm $u_n = \sup\limits_{k \geq n} x_k$ şi

$A \subset B \Rightarrow \sup A \leq \sup B$ $\qquad$ $v_n = \inf\limits_{k \geq n} x_k$

$\Rightarrow u_{n+1} = \sup\limits_{k \geq n+1} x_k \leq u_n$ $\qquad$ $u_n \geq v_n$

$v_n \leq v_{n+1} \leq u_{n+1} \leq u_n$

$v_n \nearrow$ $\quad \lim\limits_{n \to \infty} v_n = \underline{\lim}\, x_n$

$v_n \searrow$ $\quad \lim\limits_{n \to \infty} u_n = \inf\limits_{n \geq 1} u_n = \overline{\lim}\, x_n$

<u>Proprietăţi</u>

1) $\overline{\lim}\, -x_n = -\underline{\lim}\, x_n$

2) $\overline{\lim}\, x_n + y_n \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$

3) $\overline{\lim}\, x_n + y_n \geq \underline{\lim}\, x_n + \overline{\lim}\, y_n$

4) dacă $(x_n)_n$ este convergent $\Rightarrow \overline{\lim}\, x_n + y_n = \lim x_n + \overline{\lim}\, y_n$

2) + 3) $\Rightarrow$ 4)

5) $\underline{\lim}\, x_n + y_n \geq \underline{\lim}\, x_n + \underline{\lim}\, y_n$

6) $\underline{\lim}\, x_n + y_n \le \overline{\lim}\, x_n + \underline{\lim}\, y_n$

7) $x_n > 0$ $\quad \overline{\lim} \frac{1}{x_n} = \frac{1}{\underline{\lim}\, x_n}$

Obs ① $\sup(-A) = -\inf A \qquad A = (1,2) \sup A = 2 \quad -A = (-2,-1) \quad \inf(-A) = -2$

$-A = \{-x \mid x \in A\}$

1) $\overline{\lim}(-x_n) = \inf_{n\ge 1} \sup_{k \ge n} -x_k = \inf_{n \ge 1}\left(-\inf_{k\ge n} x_k\right) = -\sup_{n\ge 1} \inf_{k \ge n} x_k =$

$= -\underline{\lim}\, x_n$

② $\overline{\lim}\, x_n + y_n \le \overline{\lim}\, x_n + \overline{\lim}\, y_n$

$\lim_{n\to\infty} \sup_{k\ge n} x_k + y_k \le \lim_{n\to\infty} \sup_{k \ge 2} x_k + \lim_{n\to\infty} \sup_{k\ge n} y_k$

$\xrightarrow{} \lim_{n\to\infty}$

$\sup_{k\ge n} x_n + y_n \le \sup_{k\ge n} x_k + \sup_{k \ge n} y_k$

$l \ge n$ $\qquad x_l + y_l \le \sup_{k\ge n} x_k + \sup_{k\ge n} y_k$