Curs Analiză 11

Notație $T \in L(\mathbb{R}^n, \mathbb{R})$, $T(x_1, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$

$\pi_k : \mathbb{R}^n \to \mathbb{R}$, $\pi_k(x_1, \dots, x_n) = x_k$, $\pi_k \in L(\mathbb{R}^n, \mathbb{R})$

$T = \sum_{i=1}^{n} a_i \pi_i$ $\quad \pi_k \overset{not}{=} dx_k$

$f : D = \mathring{D} \subset \mathbb{R}^n \to \mathbb{R}$, $a \in D$

$$df'(a) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(a)\, dx_k$$

Ex $f : \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x, y, z) = x^4 y^2 + y^3 e^z$

$$df = 4x^3 y^2\, dx + (2x^4 y + 3y^2 e^z)\, dy + y^3 e^z\, dz$$

OBS $f : D = \mathring{D} \subset \mathbb{R}^n \to \mathbb{R}^m$, derivabilă pe D

$f' : D \to L(\mathbb{R}^n, \mathbb{R}^m) \approx \mathcal{M}_{m,n}(\mathbb{R}) \approx \mathbb{R}^{mn}$

$f'' : D \to L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m)) \approx \mathbb{R}^{n^2 m}$

$f''' : D \to L(\mathbb{R}^n, L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))) \approx \mathbb{R}^{n^3 m}$

1) $T(x_1 + x_2, y) = T(x_1, y) + T(x_2, y)$, $\forall x_1, x_2, y \in \mathbb{R}^n$

2) $T(x, y_1 + y_2) = T(x, y_1) + T(x, y_2)$ $\forall x, y_1, y_2 \in \mathbb{R}^n$

3) $aT(x, y) = T(ax, y) = T(x, ay)$ $\forall a \in \mathbb{R}$ $\forall x, y \in \mathbb{R}^n$

T se simetrică dacă $T(x, y) = T(y, x)$ $\forall x, y \in \mathbb{R}^n$

T s.n. antisimetrică dacă $T(x,y) = -T(y,x)$ $\forall x,y \in \mathbb{R}^n$

$L_2(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m)$

<u>Ex</u> $\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i \in L_2(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R})$

$A \in \mathcal{M}_{nm}(\mathbb{R})$ $\quad T_A(x,y) = \langle Ax, y\rangle$

$T \in L_2(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R})$ $\qquad \sum_{i=1}^{m} x_i e_i$

$$T(x,y) = T\left(x, \sum_{j=1}^{m} y_j e_j\right) = \sum_{j=1}^{m} y_j T(x, e_j) =$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \underset{\underset{a_{ij}}{\parallel}}{T(e_i, e_j)}\, x_i y_j \; ; \; A = (a_{ij})_{j,i = \overline{1,n}}$$

$$f'' \in L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m)) \simeq L_2(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m)$$

$$u : L_2(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m) \to L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$$

$T \in L_2(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m)$

$T_x : \mathbb{R}^n \to \mathbb{R}^m \quad T_x(y) = T(x,y)$

$T_x \in L(\mathbb{R}^n, \mathbb{R}^m)$

$x \to T_x$ liniară

$\hat{T}(x) = T_x \qquad \hat{T} : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$

$T_{x+y} = T_x + T_y$; $T_{x+y}(z) = T(x+y, z) = T(x, z) +$
$+ T(y, z) = T_x(z) + T_y(z)$

$u(T) = \hat{T}$ bijecție

$T \in L_2(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m)$; $T = \sum_{i=1}^{n} \sum_{j=1}^{n} T(e_i, e_j)\, \pi_i \cdot \pi_j$

$\pi_i \cdot \pi_j (x, y) = x_i \cdot y_j$

Fie $f : D = B \subset \mathbb{R}^n \to \mathbb{R}$

$$d^2 f = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d^2 f}{dx_i\, dx_j}\, dx_i\, dx_j$$

<u>T.T</u> Fie $D = \mathring{D} \subset \mathbb{R}^n$, x și $f : D \to \mathbb{R}$ a.î. f' pe D și $f''(a)$. Atunci

$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a, x-a) +$
$+ \|x-a\|^2 w(x)$ și $\lim_{x \to a} w(x) = 0$

<u>Ex</u> Fie $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = e^{ax + by}$

$\frac{\partial f}{\partial x} = a e^{ax+by} \qquad \frac{\partial^n f}{\partial x^n} = a^n e^{ax+by}$

$\frac{\partial^{n+1} f}{\partial y\, \partial x^n} = a^n \cdot b \cdot e^{ax+by} \qquad \frac{\partial^{n+m} f}{\partial y^m\, \partial x^n} = a^n b^m e^{ax+by} =$
$= \frac{\partial^{n+m} f}{\partial x^n\, \partial y^m}$

$$\frac{\partial^4 f}{\partial x \partial y^2 \partial x} = \frac{\partial^4 f}{\partial y \partial x \partial y \partial x} = \frac{\partial^4 f}{\partial y^2 \partial x^2} = a^2 b^2 e^{ax+by}$$

$$\partial f = a e^{ax+by} dx + b e^{ax+by} dy = e^{ax+by}(a dx + b dy)$$

$$\partial^2 f = a^2 e^{ax+by}(dx)^2 + 2ab e^{ax+by} \underset{dy\,dx}{\overset{\parallel}{dx\,dy}} + b^2 e^{ax+by}(dy)^2 =$$

forma asociată

$$= e^{ax+by}(a dx + b dy)^2$$

$$d^n f = \sum_{k=0}^{n} \frac{\partial^n f}{\partial y^{n-k} \partial x^k} C_n^k (dx)^k (dy)^n =$$ ← (polinom)

$$= \left[\sum_{k=0}^{n} C_n^k a^k b^{n-k} (dx)^k (dy)^{n-k}\right] e^{ax+by}$$

$\to d^2 f(a, u) \geq 0 \quad \forall u \geq 0$

$\to d^2 f(u, u) > 0 \quad \forall u \neq 0$

$x^2 + yx + y^2$

$= x^2 + xy + \frac{y^2}{4} + \frac{3}{4}y^2 = (x + \frac{y}{2})^2 + \frac{3}{4}y^2$

$x^2 + xy = (x + \frac{y}{2})^2 - \frac{y^2}{4}$

$1 > 0 \quad -\frac{1}{4} < 0$

Teorema multiplicatorilor lui Lagrange

Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f: D \to \mathbb{R}$ și $g: D \to \mathbb{R}^m$ $(m, 1 \le m < n)$. Presupunem că:

1) $f, g \in C^1$
2) rang $g'(a)$ este maxim
3) a este punct de extrem local pt f pe $A = \{g(x) = 0\}$

Atunci:

$\exists \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ a.î. $h_\lambda'(a) = 0$, unde

$\lambda_n \; = f + \lambda_1 g_1 + \dots + \lambda_m g_m$

Ex Să se determine extremele funcției $f: \mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z) = x + 2y + 3z$ pe mulțimea $A = \{x^2 + y^2 + z^2 = 1\}$

$g(x, y, z) = x^2 + y^2 + z^2 - 1$

1) $f, g \in C^\infty$
2) $g' = (2x \;\; 2y \;\; 2z)$
3) rang $g' = 1$

Pp rang $g' = 0 \Rightarrow$
$x = y = z = 0 \notin A$
$\Rightarrow$ rang $g' = 1$

3) Fie (x_0, y_0, z_0) un punct de extrem pentru f pe A

$\Rightarrow \exists \lambda \in \mathbb{R}$ a.î. $h_\lambda'(x_0, y_0, z_0) = 0$, unde

$h_\lambda = f + \lambda g = x + 2y + 3z + \lambda(x^2 + y^2 + z^2 - 1) = 0$

$\frac{\partial h_\lambda}{\partial x} = 1 + 2\lambda x = 0 \quad \frac{\partial h_\lambda}{\partial y} = 2 + 2\lambda y = 0 \quad \frac{\partial h_\lambda}{\partial z} = 3 + 2\lambda z = 0$

$$\begin{cases} x^2+y^2+z^2=1 & \\ 1+2\lambda x=0 & \Rightarrow x=-\dfrac{1}{2\lambda} \\ 2+2\lambda y=0 & \Rightarrow y=\dfrac{-2}{2\lambda} \\ 3+2\lambda z=0 & \Rightarrow z=\dfrac{-3}{2\lambda} \end{cases} \Rightarrow$$

$$\Rightarrow \frac{1}{4\lambda^2}+\frac{4}{4\lambda^2}+\frac{9}{4\lambda^2}=\frac{14}{4\lambda^2}=1$$

$$\frac{1}{2\lambda}=\pm\frac{1}{\sqrt{14}} \Rightarrow x=\mp\frac{1}{\sqrt{14}};\ y=\mp\frac{2}{\sqrt{14}}$$

$$z=\mp\frac{3}{\sqrt{14}}$$

<u>Metoda topologică</u> $A=\{x^2+y^2+z^2=1\} \Rightarrow |x|\le 1$, $|y|\le 1$, $|z|\le 1$

$\Rightarrow A$ mărginită

$A=g^{-1}(\{0\}) \Rightarrow A$ închisă

f continuă

$\exists (x_M, y_M, z_M)\in A$ a.î. $f(x_M,y_M,z_M)=\sup f(A) = \max$

$\exists (x_m, y_m, z_m)$ a.î.

$f(x_m, y_m, z_m)=\inf(A)$

$$\Rightarrow f\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)=\frac{14}{\sqrt{14}}=\sqrt{14} \text{ - maxim global}$$

$$f\left(-\frac{1}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{-3}{\sqrt{14}}\right)=\frac{-14}{\sqrt{14}}=-\sqrt{14} \text{ - minim global}$$

Metoda 2 Dacă h_λ are un extrem local în $(x_0, y_0, z_0) \Rightarrow f$ are un extrem local pe A în (x_0, y_0, z_0)

$\lambda m \, h_\lambda|_A = f|_A$; $h_\lambda = f + \lambda g|_{g=0}$

$$h_\lambda = \begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & 2\lambda & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} \quad \lambda = \pm \frac{\sqrt{14}}{2}$$

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \cdot \lambda = -\frac{\sqrt{14}}{2}$$

$$h_{-\frac{\sqrt{14}}{2}}\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) = \begin{pmatrix} -\sqrt{14} & 0 & 0 \\ 0 & -\sqrt{14} & 0 \\ 0 & 0 & -\sqrt{14} \end{pmatrix}$$

$\Delta_1 = -\sqrt{14}$
$\Delta_2 = 14$
$\Delta_3 = -14^{\frac{3}{2}}$

$- + -$

maxim

Metoda 3

$$d^2 h_\lambda = -\sqrt{14}\left((dx)^2 + (dy)^2 + (dz)^2\right) \leq 0$$

$x^2 + y^2 + z^2 = 1 \qquad 2x\,dx + 2y\,dy + 2z\,dz = 0$

$$\cancel{\frac{2}{\sqrt{14}}} \cdot (dx + 2dy + 3dz) = 0$$

$$dx = -2dy - 3dz$$