Laborator Electricitate 4

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1 Problemă Seminar

O cantitate de sarcină Q este distribuită sub formă de strat subțire sferic a=raza sferei. De-a lungul unui diametr al acestei sfere, se plimbă un corp punctiform cu sarcină q.Calculați forța cu care acționează sarcina Q asupra corpului punctiform cu sarcina q.

1.1 Coordonate polare și aria de pe o sferă

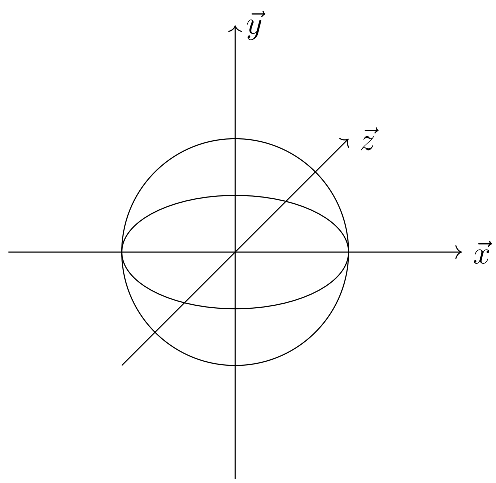

Figure 1: Cerc, elipsă și vectori

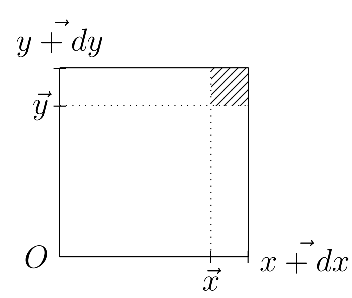

Figure 2: Coordonate Carteziene

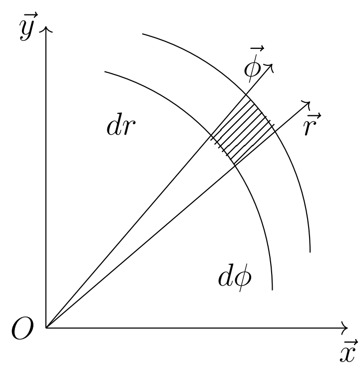

Figure 3: Coordonate Polare

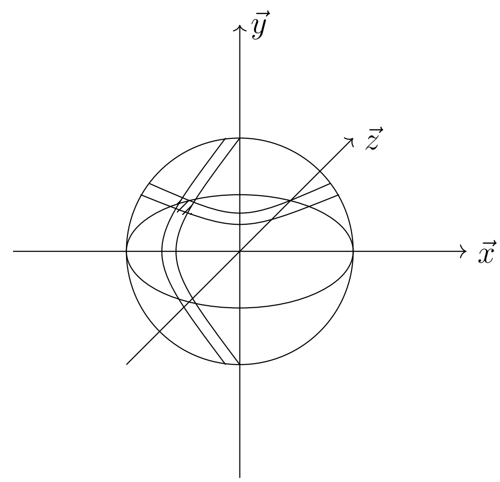

Figure 4: Arie de pe sferă

1.2 Formule Matematice

$$dS = rdrd\phi \qquad \phi \in [0, 2\pi] \tag{1}$$

$$(x, y, z) \longrightarrow (r, \theta, \phi) \qquad \theta \in [0, \pi] \tag{2}$$

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \qquad x, y, z \in [-r, r] \tag{3}$$

$$\begin{cases} L_1 = rsin(\theta)d\phi \\ L_2 = rd\theta \\ dS = r^2sin(\theta)d\theta d\phi \end{cases} \tag{4}$$

1.3 Simplificare Integrala dublă

$$\sum_{i=1}^{2}\sum_{j=1}^{3} a_ib_j = \sum_{i=1}^{2}(a_ib_1+a_ib_2+a_ib_3) = (a_1+a_2)(b_1+b_2+b_3) = (\sum_{i=1}^{2} a_i)(\sum_{j=1}^{3} b_j) = \sum_{i=1}^{2}\sum_{j=1}^{3} a_ib_j \tag{5}$$

$$\implies \int_a^b \int_a^b f(x)f(y) \ = \int_a^b f(x) \int_a^b f(y) \tag{6}$$

1.4 Rezolvare Problemă

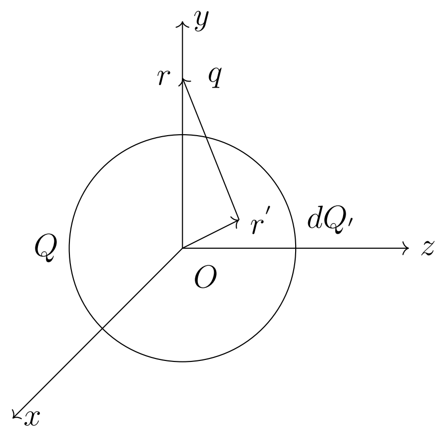

Figure 5: Schemă problemă

$$d\vec{F} = k\frac{qdQ_{\prime}}{\left|\vec{r}-\vec{r^{'}}\right|^{3}}(\vec{r}-\vec{r^{'}}) \tag{7}$$

$$\begin{cases} Q............4\pi a^2 \\ dQ^{'}............dS^{'} \\ \implies dQ^{'} = \frac{QdS^{'}}{4\pi a^2} \end{cases} \tag{8}$$

$$\begin{cases} \vec{r}-\vec{r^{'}}(-x^{'},-y^{'},z-z^{'}) \\ \vec{r}-\vec{r^{'}} = -x^{'}\vec{i}-y^{'}\vec{j}+(z-z^{'})\vec{k} \\ \left|\vec{r}-\vec{r^{'}}\right| = \sqrt{x'^2+y'^2+(z-z^{'})^2} \end{cases} \tag{9}$$

$$d\vec{F} = k\frac{Q\frac{dS^{'}}{4\pi a^2}q}{\sqrt{x'^2+y'^2+(z-z^{'})^2}^3}(-x^{'}\vec{i}-y^{'}\vec{j}+(z-z^{'})\vec{k}) \tag{10}$$

$$= \frac{kQq}{4\pi a^2}\frac{dS^{'}}{x'^2+y'^2+(z-z^{'})^{2\frac{3}{2}}}(-x^{'}\vec{i}-y^{'}\vec{j}+(z-z^{'})\vec{k}) \tag{11}$$

$$\begin{cases} dF_x = \frac{kQq}{4\pi a^2}\frac{dS^{'}(-x^{'})}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ dF_y = \frac{kQq}{4\pi a^2}\frac{dS^{'}(-y^{'})}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ dF_z = \frac{kQq}{4\pi a^2}\frac{dS^{'}(-z^{'})}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \end{cases} \tag{12}$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2}\int \frac{dS^{'}(-x^{'})}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ F_y = \frac{kQq}{4\pi a^2}\int \frac{dS^{'}(-y^{'})}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\ F_z = \frac{kQq}{4\pi a^2}\int \frac{dS^{'}(-z^{'})}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \end{cases} \tag{13}$$

$$\begin{cases} x^{'} = asin\theta cos\phi \\ y^{'} = asin\theta sin\phi \\ z^{'} = acos\theta \end{cases} \tag{14}$$

$$\implies (x'^2 + y'^2 + (z-z')^2)^{\frac{3}{2}} = (a^2sin^2\theta cos^2\phi + a^2sin^2\theta sin^2\phi + (z-acos\theta)^2)^{\frac{3}{2}}$$
$$= (a^2sin^2\theta + (z-acos\theta)^2)^{\frac{3}{2}} = (a^2sin^2\theta + z^2 + a^2cos^2\theta - 2a + cos\theta)^{\frac{3}{2}}$$
$$= (a^2 + z^2 - 2a + cos\theta)^{\frac{3}{2}} = a^3(1 + (\tfrac{z}{a})^2 - 2\tfrac{z}{a}cos\theta)^{\frac{3}{2}}$$
$$= a^3\left(\left(\tfrac{z}{a}\right)^2 - \cos\theta\tfrac{z}{a} + 1\right)^{\frac{3}{2}} \quad \text{notăm} \quad \tfrac{z}{a} = m$$
$$\implies a^3(m^2 - cos\theta m + 1)^{\frac{3}{2}}$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2}\int_0^{\pi}\int_0^{2\pi} \frac{a^2sin\theta d\theta d\phi(-asin\theta cos\phi)}{a^3(1+m^2-2mcos\theta)^{\frac{3}{2}}} \\ \\ F_y = \frac{kQq}{4\pi a^2}\int_0^{\pi}\int_0^{2\pi} \frac{a^2sin\theta d\theta d\phi(-asin\theta sin\phi)}{a^3(1+m^2-2mcos\theta)^{\frac{3}{2}}} \\ \\ F_z = \frac{kQq}{4\pi a^2}\int_0^{\pi}\int_0^{2\pi} \frac{a^2sin\theta d\theta d\phi(z-acos\theta)}{a^3(1+m^2-2mcos\theta)^{\frac{3}{2}}} \end{cases} \tag{15}$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2}\int_0^{\pi}\int_0^{2\pi} \frac{sin^2\theta d\theta(-cos\phi d\phi)}{(1+m^2-2mcos\theta)^{\frac{3}{2}}} \\ \\ F_y = \frac{kQq}{4\pi a^2}\int_0^{\pi}\int_0^{2\pi} \frac{sin^2\theta d\theta(-sin\phi d\phi)}{(1+m^2-2mcos\theta)^{\frac{3}{2}}} \\ \\ F_z = \frac{kQq}{4\pi a^2}\int_0^{\pi}\int_0^{2\pi} \frac{sin^2\theta d\theta d\phi(m-cos\theta)}{(1+m^2-2mcos\theta)^{\frac{3}{2}}} \end{cases} \tag{16}$$

$$\begin{cases} F_x = \frac{kQq}{4\pi a^2} \int_0^\pi \frac{sin^2\theta d\theta}{(1+m^2-2mcos\theta)^{\frac{3}{2}}} \int_0^{2\pi} (-cos\phi d\phi) = 0 \\ \\ F_y = \frac{kQq}{4\pi a^2} \int_0^\pi \frac{sin^2\theta d\theta}{(1+m^2-2mcos\theta)^{\frac{3}{2}}} \int_0^{2\pi} (-sin\phi d\phi) = 0 \\ \\ F_z = \frac{kQq}{4\pi a^2} \int_0^\pi \frac{sin^2\theta d\theta (m-cos\theta)}{(1+m^2-2mcos\theta)^{\frac{3}{2}}} \int_0^{2\pi} d\phi \end{cases} \tag{17}$$

$$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_z = \frac{kQq}{4\pi a^2} \int_0^\pi \frac{(m-cos\theta)sin\theta d\theta}{(1+m^2-2mcos\theta)^{\frac{3}{2}}} d\theta \end{cases} \tag{18}$$

$$\implies F_z = \frac{kQq}{4\pi a^2} \int_{-1}^{1} \frac{(m-x)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \tag{19}$$

$$F_z = \frac{kQq}{4\pi a^2} \int_{-1}^{1} \frac{(-m^2-2mx)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \tag{20}$$

$$F_z = \frac{kQq}{4\pi a^2} \int_{-1}^{1} \frac{(1+m^2-2mx+m^2-1)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \tag{21}$$

$$F_z = \frac{kQq}{4\pi a^2} \int_{-1}^{1} \frac{(m^2-2mx)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \int_{-1}^{1} \frac{(m^2-1)dx}{(1+m^2-2mx)^{\frac{3}{2}}} \tag{22}$$

$$F_z = \frac{kQq}{4\pi a^2} \int_{-1}^{1} (1+m^2-2mx)^{-\frac{1}{2}} \int_{-1}^{1} (1+m^2-2mx)^{-\frac{3}{2}} \tag{23}$$

$$F_z = -\frac{kQq}{4\pi a^2} \left(|m-1| - |m+1| - \left(m^2-1\right) \left(\frac{1}{|m-1|} - \frac{1}{|m+1|} \right) \right) \tag{24}$$

$m = \frac{z}{a}; m > 1 \implies z > adeasupramingii$

$$\implies F_z = -\frac{kQq}{4\pi a^2} \left(m-1-m+1 - \left(m^2-1\right) \left(\frac{1}{m-1} - \frac{1}{m+1} \right) \right) \tag{25}$$

$$F_z = -\frac{kQq}{4\pi a^2} (-4) \tag{26}$$

$$F_z = \frac{4kQq}{4\pi a^2} = \frac{kQq}{m^2 a^2} = \frac{kQq}{z^2} \tag{27}$$