Transformări Ortogonale
Endomorfisme simetrice
Seminar 12 A-G

1. $(\mathbb{R}^3, g_0)$; $U = (1,1,0)$;

ⓐ $U^\perp = ?$, Reper ortonormat?

ⓑ $\varphi = \frac{\pi}{2}$ $f \in O(\mathbb{R}^3)$, rotație

axa $= \langle u \rangle$ $f = ?$

$$U^\perp = \{x \in \mathbb{R}^3 \mid g_0(x,y) = 0\} = \{x \in \mathbb{R} \mid (-x_2, x_2, x_3)\} = \langle \{\underset{f_1}{(-1,1,0)}, \underset{f_2}{(0,0,1)}\} \rangle$$

$x_1 \cdot u_1 + x_2 \cdot u_2 + x_3 \cdot u_3 = 0$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$$\left.\begin{matrix} \mathbb{R}^3 = \{U\} \oplus U^\perp \quad \dim U^\perp = 2 \\ U^\perp \text{ SG} \end{matrix}\right\} \Rightarrow \text{SLI} \Rightarrow \text{Reper}$$

Aplicăm Gram-Schmidt

$e_2 = f_1 = (-1,1,0)$

$$e_3 = f_3 - \underbrace{\frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle}}_{0} e_2 = f_3$$

$$\left.\begin{matrix} e_2' = \frac{(-1,1,0)}{\sqrt{2}} \\ e_3' = (0,0,1) \end{matrix}\right. \Rightarrow \{e_2', e_3'\} \text{ Reper ortonormat în } U^\perp$$

$$e_1 = \frac{(1,1,0)}{\sqrt{2}}$$

$\mathcal{R}' = \{e_1', e_2', e_3'\}$ $\quad \varphi = \frac{\pi}{2}$

$$A' = [f]_{\mathcal{R}'\mathcal{R}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{R}_0 \xrightarrow{C} \mathcal{R}'$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A' = C^T A C$$

$$A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = C A' C^T$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}.$$

$f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = \frac{1}{2}\left(x_1 + x_2 + \sqrt{2}x_3,\ x_1 + x_2 - \sqrt{2}x_3,\ -\sqrt{2}x_1 + \sqrt{2}x_2\right)$$

$Q(x) = \langle x, f(x)\rangle$

2. $(\mathbb{R}^3, g_0)$ $f \in \mathrm{End}(\mathbb{R}^3)$ a) $f \in \mathrm{Sim}_2(\mathbb{R}^3)$, $f = ?$

$A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ b) $Q : \mathbb{R}^3 \to \mathbb{R}$ forma pătratică (?)

c) Q la o formă canonică, efectuând o transformare ortogonală pe h

(schimb de repere)

(a) $A = A^T \Rightarrow f \in \mathcal{L}in(\mathbb{R}^3)$

$f : \mathbb{R}^3 \to \mathbb{R}^3 \; ; f(x) = (x_1 + x_3, x_2, x_1 + x_3)$

(b) $\Phi(x) = \langle x, f(x) \rangle = g_0(x, f(x))$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1 + 0 + 0 - 1 - 0 - 0 = 0$$

$\Phi(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_3$

(c) $P_A(\lambda) \underset{det}{=} (A - \lambda I_3) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^3 + 0 + 0 - (1-\lambda) - 0 - 0$$

$\Rightarrow (1-\lambda)^3 - (1-\lambda) = 0$

$(1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$

$\lambda_1 = 0$ $\quad m_1 = 1$

$A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$(1-\lambda)^2 - 1 = 0$

$1 - 2\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda_2 = 1$ $\quad m_2 = 1$

$\lambda_3 = 2$ $\quad m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 / f(x) = \lambda x\} = \{(-x_3, 0, x_3) / x_3 \in \mathbb{R}\} = \langle \{(-1, 0, 1)\} \rangle$

$\Rightarrow (A - 0 I_3)X = 0$

$AX = 0 \cdot X$

$AX = 0$

$e_1 = \frac{1}{\sqrt{2}}(-1, 0, 1)$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases}$$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid Ax = 1 \cdot x \}$

$$(A - I_3)x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_3 = 0 \\ x_1 = 0 \end{cases} \Rightarrow V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid (0, x_2, 0) \} = \langle \{ (0,1,0) \} \rangle$$

$$e_2 = \frac{1}{\sqrt{1}} (0,1,0) = (0,1,0)$$

$V_{\lambda_3} = \{ x \in \mathbb{R}^3 \mid Ax = 2x \}$

$$(A - 2 \cdot I_3)x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = 0 \end{cases}$$

$$\Rightarrow V_{\lambda_3} = \{ (x_1, 0, x_1) \mid x_1 \in \mathbb{R} \} = \langle \{ (1,0,1) \} \rangle$$

$$e_3 = \frac{1}{\sqrt{2}} (1,0,1)$$

$\Rightarrow$ baza ortonormat $\mathcal{R}' = \{ \overset{e_1}{(-1,0,1)}, \overset{e_2}{(0,1,0)}, \overset{e_3}{(1,0,1)} \}$

$$\mathcal{R}_0 = \{ e_1^0, e_2^0, e_3^0 \} \xrightarrow{C} \mathcal{R} = \{ e_1 = \frac{1}{\sqrt{2}}(-1,0,1), e_2 = (0,1,0), e_3 = \frac{1}{\sqrt{2}}(1,0,1) \}$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$h \in O(\mathbb{R}^3)$

$$h(x) = \frac{1}{\sqrt{2}} (-x_1 + x_3, \sqrt{2} x_2, x_1 + x_3)$$

sau

$h \in O(\mathbb{R}^3)$, $h(e_i^0) = e_i$, $i = \overline{1,3}$

$A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = [f]_{\mathcal{R}, \mathcal{R}}$

$Q(x) = x_2'^2 + 2x_3'^2 \quad S(3,0)$

④ $(\mathbb{R}^3, g_0)$, $f : \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = g_0(x, u) \cdot u$, $u = (1, -1, 2)$

a) $f \in \operatorname{Sim}(\mathbb{R}^3)$, $f = ?$

b) $Q = ?$ (met. val. proprii)

ⓐ $g_0(x, u) = x_1 - x_2 + 2x_3$

$f(x) = (x_1 - x_2 + 2x_3) \cdot (1, -2, 2) = (x_1 - x_2 + 2x_3, -x_1 + x_2 - 2x_3, 2x_1 - 2x_2 + 4x_3)$

$[f]_{\mathcal{R}_0 \mathcal{R}_0} = A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = A^T \Rightarrow f \in \operatorname{Sim}(\mathbb{R}^3)$

ⓑ $Q : \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$

Se face cu valori proprii ca la exercițiul precedent

Ex) $(\mathbb{R}^3, g_0)$, $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = u \times x$, $u = (1, 2, 3)$

$$u \times x = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 3 \\ x_1 & x_2 & x_3 \end{vmatrix} = e_1(2x_3 - 3x_2) - e_2(x_3 - 3x_1) + e_3(x_2 - 2x_1)$$

$\Rightarrow f(x) = (-3x_2 + 2x_3, 3x_1 - x_3, -2x_1 + x_2)$

$$A = [f]_{R_0, R_0} = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

f nu se poate diagonaliza

Se calculează valorile proprii

Geometrie analitică euclidiană

$(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$ spațiu afin euclidian canonic, cu $\mathcal{A} = \mathbb{R}^3$, $V = \mathbb{R}^3$

$g_0: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$g_0(x, y) = x_1y_1 + x_2y_2 + x_3y_3$ produs scalar canonic

$\varphi: \underset{\mathcal{A}}{\mathbb{R}^3} \times \underset{\mathcal{A}}{\mathbb{R}^3} \to \underset{V}{\mathbb{R}^3}$, $\varphi(u, v) = v - u$ structură afină canonică

Ex $(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$

$A(3,-1,3), B(\ ,1,-1), u = (-3,5,-6)$

a) ec dreptei d ai $A \in d$, $V_d = \langle \{u\} \rangle$

b) ec dreptei AB

c) Să se afle punctele de intersecție ale dreptei d cu planele de coordonate

a) d A u $M(x_1,x_2,x_3)$

$\forall M \in d\ \exists t \in \mathbb{R}$ ai $\overrightarrow{AM} = tu =$

$= t(-3,5,-6)$

A B

$\overrightarrow{AM} = (x_1-3, x_2+1, x_3-3)$

$d: \begin{cases} x_1 - 3 = -3t \\ x_2+1 = 5t \\ x_3 - 3 = -6t \end{cases}$ (ecuație parametrică)

$d: \frac{x_1-3}{-3} = \frac{x_2+1}{5} = \frac{x_3-3}{-6} = t$

$u_{AB} = \overrightarrow{AB} = (5-3, 1+1, -1-3) = (2,2,-4)$

b) $AB: \begin{cases} x_1 - 3 = 2t \\ x_2+1 = 2t \\ x_3-3 = -4t \end{cases}, t \in \mathbb{R}$

$AB: \frac{x_1-3}{2} = \frac{x_2+1}{2} = \frac{x_3-3}{-4} = t$

(c) $\alpha \cap (Ox_1x_2)$

$x_3 = 0$

$\pi: ax_1 + bx_2 + cx_3 + d = 0$

$\pi \cap \alpha$

α cu ecuatii parametrice

$\alpha: \begin{cases} x_1 = 3 - 3t \\ x_2 = -1 + 5t \\ x_3 = 3 - 6t \end{cases} \Rightarrow 3 - 6t = 0 \Rightarrow t = \frac{1}{2} \Rightarrow$

$\Rightarrow P\left(3 - \frac{3}{2}, -1 + \frac{5}{2}, 0\right)$

$P\left(\frac{3}{2}, \frac{3}{2}, 0\right)$

$\alpha \cap (Ox_1x_3)$

$x_2 = 0 \Rightarrow -1 + 3t = 0 \Rightarrow t = \frac{1}{5}$

$Q\left(3 - \frac{3}{5}, 0, 3 - \frac{6}{5}\right)$

$Q\left(\frac{12}{5}, 0, \frac{9}{5}\right)$

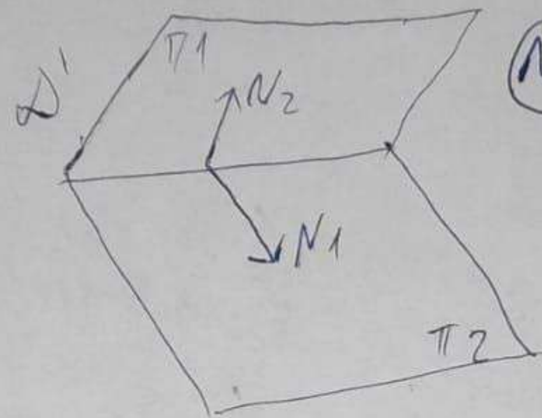
$\alpha \cap (Ox_2x_3)$

$x_1 = 0 \Rightarrow 3 - 3t = 0 \Rightarrow t = 1$

$S(0, -1 + 5, 3 - 6)$; $S(0, 4, -3)$

(8) $\Delta = !$ ai $A(2,-5,3) \in \Delta$

și $\Delta \parallel \Delta'$, unde Δ': $\begin{cases} 2x_1 - x_2 + 3x_3 + 1 = 0 & (1) \\ 5x_1 + 4x_2 - x_3 + 1 = 0 & (2) \end{cases}$



(M1) (1) $N_1 = (2, -1, 3)$

(2) $N_2 = (5, 4, -1)$

$$u_{\Delta'} = N_1 \times N_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & -1 & 3 \\ 5 & 4 & -1 \end{vmatrix}$$

$$= e_1 \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} - e_2 \begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} + e_3 \begin{vmatrix} 2 & -1 \\ 5 & 4 \end{vmatrix} =$$

$$= -11e_1 + 17e_2 + 13e_3 = (-11, 17, 13)$$

$$\frac{x_1 - 2}{-11} = \frac{x_2 + 5}{17} = \frac{x_3 - 3}{13} = t$$

(M2) matricea sistemului Δ' $\Rightarrow A = \left(\boxed{\begin{matrix} 2 & -1 \\ 5 & 4 \end{matrix}} \begin{matrix} 3 \\ -1 \end{matrix} \middle| \begin{matrix} -1 \\ -1 \end{matrix}\right)$

$$x_3 = t \Rightarrow \begin{aligned} 2x_1 - x_2 &= -1 - 3t \quad |\cdot 4 \\ 5x_1 - 4x_2 &= -1 + t \\ \hline 13x_1 \quad / &= -5 - 11t \end{aligned}$$

$$x_1 = \frac{-5 - 11t}{13}$$

$$\Rightarrow 2\left(-\frac{5}{13}-\frac{11}{3}t\right)+1+3t=x_2$$

$$\Rightarrow x_2 = t\left(\frac{-22+39}{13}\right)+\frac{-10+13}{13} =$$

$$= \frac{17}{13}t + \frac{2}{13}$$

$$D': \begin{cases} x_1 = -\frac{5}{13} - \frac{11}{13}t \\ x_2 = \frac{3}{13} + \frac{17}{13}t \\ x_3 = t \end{cases}$$

$$\left(-\frac{11}{13}, \frac{17}{13}, 1\right)$$

$$= \frac{1}{13}(-11, 17, 13)$$

am gășit din nou directia