Seminar 8 Algebră și Geometrie

Vectori proprii. Valori proprii. Diagonalizare

$\boxed{Ex\ 8}$ $f \in End(\mathbb{R}^3)$; $R_0 = \{e_1, e_2, e_3\}$

$f(e_1) = e_3$? $\exists$ R reper în $\mathbb{R}^3$ ai $\overset{A'}{\overset{\parallel}{[f]}}_{R,R}$ este diagonală

$f(e_2) = e_2$

$f(e_3) = e_1$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} ; A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad f(x_1, x_2, x_3) = (x_3, x_2, x_1)$$

I $P_f(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda+1 & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (-\lambda+1) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} =$

$= (-\lambda+1)(+\lambda^2 - 1) = (-\lambda+1)(\lambda-1)(\lambda+1) = (-\lambda+1)(\lambda-1)^2$

$\lambda_1 = 1, m_1 = 2$

$\lambda_1 = -1, n_2 = 1$

II $V_{\lambda_1} = \{ X \in \mathbb{R}^3 \mid AX = \lambda_1 X \}$

$$(A - \lambda_1 I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim V_{\lambda_1} = \dim \mathbb{R}^3 - rg\, A = 3 - 1 = 2 = m_1$

$\mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_n}$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid -x_1 + x_3 = 0 \} \Rightarrow x_1 = x_3 \Rightarrow (x_1, x_2, x_1)$

$= x_1(1, 0, 1) + x_2(0, 1, 0)$

$\Rightarrow \mathcal{R}_1 = \langle\{(1,0,1),(0,1,0)\}\rangle$ SG reper în V_{λ_1}

și e SLI (ca teorema demonstrăm)

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid AX = -X\} = \left\{x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases}\right\}$

$(A + I_3) \cdot X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A + I_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\dim V_{\lambda_2} = 3 - \operatorname{rg}(A + I_3) = 3 - 2 = 1 = m_2$

$\mathcal{R}_2 = \langle\{(1,0,-1)\}\rangle$ SG reper în V_{λ_2}

și e SLI (—— ıı ——)

$\Rightarrow$ A este diagonalizabilă

Pentru că $\mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2}$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 = \{(1,0,1),(0,1,0),(1,0,-1)\}$ reper în $\mathbb{R}^3$

$[f]_{\mathcal{R},\mathcal{R}_0} = A' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(12) $f \in \operatorname{End}(\mathbb{R}^3)$, $\lambda_1 = 3$; $\lambda_2 = -2$, $\lambda_3 = 1$ valori proprii

$v_1 = (-3,2,1)$, $v_2 = (-2,1,0)$, $v_3 = (-6,3,1)$ vectori proprii

$[f]_{\mathcal{R}_0\mathcal{R}_0} = A$ — vectori proprii corespunzători la valori proprii distincte

$\mathcal{R} = \{v_1, v_2, v_3\}$ reper, SLI și dim $\mathbb{R}^3$

$\mathcal{R}_0 = \{(1,0,0),(0,1,0),(0,0,1)\}$

$f(v_1) = \lambda_1 v_1$

$f(v_2) = \lambda_2 v_2$

$f(v_3) = \lambda_3 v_3$

$$[f]_{\mathcal{R},\mathcal{R}} = A' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$\mathcal{R}_0 \xrightarrow{C} \mathcal{R} = \{v_1 = -3e_1 + 2e_2 + e_3, v_2 = -2e_1 + e_2, v_3 = -6e_1 + 3e_2 + e_3\}$

$$C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$A' = C^{-1} A C \mid \cdot C;\ A = ?$

$CA' = AC \mid \cdot C^{-1}$

$CA'C^{-1} = A$

$$\left(\begin{array}{ccc|ccc} -3 & -2 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$ facem calculul

$$A^n = (CA'\underbrace{C^{-1})\ (C}_{I_n}A'C^{-1}) \dots (CA'C^{-1}) = CA'^n C^{-1}$$

$$A'^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & (-2)^n & 0 \\ 0 & 0 & 1^n \end{pmatrix}$$

T3C $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 - x_2 + x_3, x_1 - x_2 + x_3, x_3)$

a) $[f]_{\mathcal{R}_0\mathcal{R}_0} = ?$

b) $\operatorname{Ker} f$, $\operatorname{Im} f$ şi repere

c) $\mathbb{R}^3 = \operatorname{Ker} f \oplus W$

$W = ?$

$p: \operatorname{Ker} f \oplus W \to \operatorname{Ker} f$ $\qquad p(1,0,3) = ?$

$s: \mathbb{R}^3 \to \mathbb{R}^3$ $\qquad s(1,0,3) = ?$

ⓐ $f(x) = y \Rightarrow Y = AX \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{[f]_{R_0 R_0}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

ⓑ $\mathrm{Ker}\, f = \left\{ x \in \mathbb{R}^3 \,/\, AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = x_2 \\ x_3 = 0 \end{cases}$

$\Rightarrow \mathrm{Ker}\, f = \langle \{(1,1,0)\} \rangle$; $R_1 = \{(1,1,0)\}$ reper în Ker f

Extindem R_1 la un reper în $\mathbb{R}^3$

$\mathrm{rg} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$ (max)

$\qquad\quad e_2 \quad e_3$

$\Rightarrow R_1 \cup \{e_2, e_3\}$ - reper în $\mathbb{R}^3$

$\{f(e_2), f(e_3)\}$ - reper în Im f

$f(0,1,0) = (-1,-1,0) \Rightarrow R_2 = \{(-1,-1,0), (1,1,1)\}$ - reper în Im f

$f(e_3) = (1,1,1)$

ⓓ $p': \mathrm{Ker}\, f \oplus W \longrightarrow W$ proiecția pe W de-a lungul lui Ker f

$s': \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ simetria față de W de-a lungul lui $\mathbb{R}^3$