Curs 3 Analiză

Fie $A \subset \mathbb{R}$ și $a \in \mathbb{R}$ (A mărginită)

$a = \sup A \iff$ 1) $\forall x \in A \Rightarrow x \leq a$

2) $\forall b$ a.î. $\forall x \in A \Rightarrow x \leq b \Rightarrow a \leq b$

$M_A = \{b\}\ b \geq x\ \forall\ x \in A = [\sup A, +\infty)$

$\iff$ 1) $x \in A \Rightarrow x \leq a$

2) $\forall \varepsilon > 0\ \exists x_\varepsilon \in A$ a.î. $x_\varepsilon > a - \varepsilon$

$\left(\varepsilon = \frac{1}{n},\ y_n = x_{\frac{1}{n}} \in A;\ a - \frac{1}{n} \leq y_n \leq a \Rightarrow y_n \to a\right)$

$A, B \subset \mathbb{R},\ x \in \mathbb{R}$

$x + A = \{x + a \mid a \in A\}$

$x \cdot A = \{x a \mid a \in A\}$

$A + B = \{a + b \mid a \in A, b \in B\}$

Prop 1) $A \subset B \Rightarrow \sup A \leq \sup B$ și $\inf A \geq \inf B$

2) $\sup(x + A) = x + \sup A \quad \inf(x + A) = x + \inf A$

3) $\sup(-A) = -\inf A$

4) $\sup(A + B) = \sup A + \sup B$

$\inf(A + B) = \inf A + \inf B$

5) $\sup(x \cdot A) = x \sup A \quad , x > 0$

$A+B=\{z \mid z=x+y \quad x\in A, y\in B\}$

$a=\sup A$ și $b=\sup B$

? $a+b \overset{?}{\underset{\geq}{=}} \sup(A+B)$

$z\in A+B \Rightarrow z=x+y$ cu $x\in A$ și $y\in B \Rightarrow z=x+y\leq a+b$

$\Rightarrow \sup(A+B)\leq a+b$

$a=\sup A \Rightarrow \exists (x_n)_n \subset A$ a.î. $x_n\to a$

$b=\sup B \Rightarrow \exists (y_n)_n \subset B$ a.î. $y_n\to b$

$z_n = x_n+y_n \to a+b$ $\qquad a+b=\sup(A+B)$

$\in A+B$

Ex: $A=(0,1)$; $B=(1,2)$ $\qquad \sup A=1$

$\inf A=0$

$x=1$; $1+A=(1,2)$ $\sup(1+A)=2$

$-B=(-2,-1)$; $\inf(-B)=-2$

$A+B=(1,3)$

$\sup(A+B)=3=\underset{\sup A}{\underset{\shortparallel}{1}}+\underset{\sup B}{\underset{\shortparallel}{2}}$

Fie $(x_n)_n \subset \mathbb{R}$ un şir mărginit $\mathcal{L} = \{a \in \mathbb{R} \mid \exists x_{n_k} \to a\}$

$\boxed{\sup \mathcal{L} = \overline{\lim_{n\to\infty}} x_n = \overline{l}}$; $\inf \mathcal{L} = \underline{\lim} x_n = \underline{l}$

$M = \{v \in \mathbb{R} \mid \exists n_v \text{ a.î. } v \geq x_n \ \forall n \geq n_v\}$ $l^* = \inf M$

$u_n = \sup_{k \geq n} x_k \in M$ $\quad u_n \geq u_{n+1}$ $\quad \widetilde{l} = \lim_{n\to\infty} u_n =$

$= \inf_{n \geq 1} u_n$

$\left(\overline{\lim}(x_n + y_n) \leq \overline{\lim} x_n + \overline{\lim} y_n\right.$

OBS $u_n \in M$

$\Rightarrow \widetilde{l} \geq l^* = \inf M$

$v_n = l^{\sim}$

$v \in M \Rightarrow \exists n_v \text{ a.î. } \forall n \geq n_v \Rightarrow x_n \leq v \Rightarrow \underbrace{\sup_{n \geq n_v} x_n}_{=u_{n_v}} \leq v$

$\Rightarrow v \geq \widetilde{l}$

$\Rightarrow \underbrace{\inf M}_{= l^*} \geq \widetilde{l}$

$a \in \mathcal{L}$ $\quad x_{n_k} \to a \Rightarrow a \leq \sup_{k \geq 1} x_{n_k} \leq \sup_{n \geq 1} y_n = u_1$

$\Rightarrow a \leq \underset{\downarrow \widetilde{l}}{u_n} \ \forall n \geq 1 \Rightarrow a \leq \widetilde{l} \Rightarrow \overline{l} \leq \widetilde{l}$

$\overline{l} \leq \widetilde{l} = l^*$

Propoziție Fie $(x_n)_n$ un șir mărginit. Atunci

$\exists\, x_{n_k} \to \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \sup\limits_{k\geq n} x_k$

1) $\bar{\ell} = \tilde{\ell} = \ell^*$

2) $\overline{\lim}\, x_n \in \mathcal{L}$

3) Orice șir mărginit are un subșir convergent

Propoziție Fie $(x_n)_{n\geq 1}$ un șir mărginit și $a \in \mathbb{R}$

A.U.A.S.E.

1) $a = \sup \mathcal{L}$

2) $a = \lim\limits_{n\to\infty} \sup\limits_{k\geq n} x_k = \inf\limits_{n\geq 1} \sup\limits_{k\geq n} x_k$

3) $a = \inf \{ v \mid \exists n_v \text{ a.î. } \forall n \geq n_v \Rightarrow x_n \leq v \}$

4) $\forall \varepsilon > 0 \Rightarrow$ i) $\exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon \Rightarrow x_n < a + \varepsilon$

ii) $\exists x_{n_k}$ a.î. $x_{n_k} > a - \varepsilon$

i) se aplică în exerciții mai des

$a_n > 0 \quad \lim \frac{a_{n+1}}{a_n} = \ell \qquad \ell > 1 \Rightarrow a_n \to +\infty$

$\ell < 1 \Rightarrow a_n \to 0$

Dacă $\overline{\lim} \frac{a_{n+1}}{a_n} < 1 \Rightarrow a_n \to 0$ și $\underline{\lim} \frac{a_{n+1}}{a_n} > 1 \Rightarrow a_n \to +\infty$

$\bar{l} = \overline{\lim} \frac{a_{n+1}}{a_n} < 1 \quad \bar{l} < \frac{\bar{l}+1}{2} < 1; \quad \frac{\bar{l}+1}{2} - \bar{l} = \frac{1-\bar{l}}{2} > 0$

i) $\forall \varepsilon > 0 \ \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon \Rightarrow \frac{a_{n+1}}{a_n} < \bar{l} + \varepsilon$

ii)

$\varepsilon = \frac{1-\bar{l}}{2} > 0 \quad \forall n \geq n_\varepsilon \quad \frac{a_{n+1}}{a_n} < \bar{l} + \varepsilon = \frac{\bar{l}+1}{2} < 1$

$\frac{a_{n+2}}{a_n} = \frac{a_{n+2}}{a_{n+1}} \cdot \frac{a_{n+1}}{a_n} \leq \left(\frac{\bar{l}+1}{2}\right)^2$

$\frac{a_{n+p}}{a_n} \leq \left(\frac{\bar{l}+1}{2}\right)^p$

$a_{n+p} < a_n \left(\frac{\bar{l}+1}{2}\right)^p \to 0, \ p \to \infty$

$0 \leq \overline{\lim_{n\to\infty}} a_n = \lim_{p\to\infty} a_{n+p} \leq a_n \cdot 0 = 0$

$\Rightarrow \boxed{a_n \to 0}$

Serii

Def. O pereche de șiruri $((x_n)_{n\geq p}, (y_n)_{n\geq p})$; unde $y_n = \sum_{k=p}^{n} x_k$

s.n. serie și se notează cu $\sum_{n\geq p} x_n = \sum_{n=p}^{\infty} x_n$

x_n - termenii seriei

y_n - sumele parțiale

dacă $\exists \lim\limits_{n\to\infty} y_n$ spunem că seria are limită și notăm

$\lim\limits_{n\to\infty} y_n = \sum\limits_{n=1}^{\infty} x_n$

dacă $\sum\limits_{n=1}^{\infty} x_n$ este finită spunem că seria e convergentă

Ex $\sum\limits_{n\geq 1} \frac{1}{n(n+1)}$ $\quad x_n = \frac{1}{n(n+1)}$; $y_n = \sum\limits_{k=1}^{n} \frac{1}{k(k+1)} =$

$= \sum\limits_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1} \to 1$

$1 = \sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$

Ex$_2$ $\sum\limits_{n=0}^{\infty} a^n$ $\quad (0^0 = 1)$ $\quad x_n = a^n \Rightarrow$

$|a| < 1$

$\shortparallel$

$1 + a + a^2 + \dots + a^n + \dots$

$\Rightarrow y_n = \sum\limits_{k=0}^{n} a^k = 1 + a + \dots + a^n = \frac{1-a^{n+1}}{1-a} \to \frac{1}{1-a}$

$(a < 1 \Rightarrow |a^n| \to 0$

$a=1 \quad y_n=\sum_{k=0}^{n} 1 = n+1 \to \infty \qquad \sum_{n\geq 0} 1 = +\infty$

$a>1 \quad y_n = \frac{a^{n+1}-1}{a-1} \to \infty = \sum_{n\geq 1} a^n$

$a=-1 \quad y_n = \sum_{k=0}^{n} (-1)^n \quad y_{2n+1}=0$

$y_{2n}=1$

seria nu are limită

Ex 3 $\sum_{n\geq 1} \frac{1}{(n+1)^2}$; $x_n = \frac{1}{(n+1)^2}$; $y_n = \sum_{k=1}^{n} \frac{1}{(k+1)^2}$

$y_n - y_{n-1} = x_n = \frac{1}{(n+1)^2} > 0 \Rightarrow y_n \nearrow$ (1)

$\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} \qquad \Rightarrow y_n = \sum_{k=1}^{n} \frac{1}{k+1} \leq \sum_{k=1}^{n} \frac{1}{k(k+1)} < 1$

$\Rightarrow y_n$ mărginit (2)

Din (1) și (2) $\Rightarrow y_n$ convergent

OBS (1) $\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$,

(2) $\sum_{n=1}^{\infty} (\alpha \cdot x_n) = \alpha \cdot \sum_{n=1}^{\infty} x_n$

dacă seriile $\sum_{n=1}^{\infty} x_n$ și $\sum_{n\geq 1} y_n$ sunt convergente

OBS 2 Dacă seria $\sum_{n=1}^{\infty} x_n$ este convergentă $\Rightarrow x_n \to 0$

$a = \lim_{n\to\infty} y_n \Rightarrow x_n = y_n - y_{n-1} \to a - a = 0$

$x_n \not\to 0 \Rightarrow$ seria este divergentă

Ex $\sum_{n\geq 1} \frac{1}{n} = +\infty$; $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n} = 1$

$\frac{1}{n} \to 0$

Serii cu termeni pozitivi

OBS $\sum_{n=1}^{\infty} x_n$ cu $x_n \geq 0$. Atunci $y_{n+1} - y_n = x_{n+1} \geq 0 \Rightarrow$ șirul este monoton $\Rightarrow$ seria este convergentă $\Leftrightarrow$ este mărginită

Criteriul 1 Criteriul comparației

Fie seriile $\sum_{n\geq 1} a_n$ și $\sum_{n=1}^{\infty} b_n$ cu $a_n, b_n \geq 0 \ \forall n \geq 1$

Dacă $M \geq 0$ și $\exists n_0$, a.î. $\forall n \geq n_0 \Rightarrow$

$a_n \leq M b_n$ ($\Leftrightarrow \lim \frac{a_n}{b_n} < +\infty$) atunci

1. dacă $\sum_{n=1}^{\infty} b_n < +\infty \Rightarrow \sum_{n=1}^{\infty} a_n < +\infty$
2. dacă $\sum_{n=1}^{\infty} a_n = +\infty \Rightarrow \sum_{n=1}^{\infty} b_n = +\infty$

Dem $a_n < M b_n \; \forall n > n_0 \Rightarrow \sum_{n=n_0}^{\infty} a_n \leq M \sum_{n=n_0}^{\infty} b_n \leq M + \sum_{n\geq 1} b_n$

$\sum_{n\geq 1} a_n \leq \sum_{h=1}^{n_0} a_h + M \sum_{n=1}^{\infty} b_n$

Dacă $\lim_{n\to\infty} \frac{a_n}{b_n} = l \in (0,+\infty) \Rightarrow \boxed{\sum_{n\geq 1} a_n \sim \sum_{n\geq 1} b_n}$

(seriile $\sum_{n=1}^{\infty} a_n$ și $\sum_{n=1}^{\infty} b_n$ sunt ambele convergente sau divergente)

Ex

$0 < \lim \frac{a_n}{b_n} < +\infty \Rightarrow$

$\Rightarrow \lim \frac{a_n}{b_n} < +\infty$ și $\lim \frac{b_n}{a_n} < +\infty$

$a_n \leq M \cdot b_n$ $\qquad b_n \leq a_n$

$$\sum_{n=1}^{\infty} \underbrace{\frac{n}{2n^2+n+1}}_{b_n} \sim \sum \underset{a_n}{\underset{\|}{\frac{1}{n}}}$$

$\frac{b_n}{a_n} = \frac{n}{2n^2+n+1} \; ; n \to \frac{1}{2} \in (0,+\infty)$

$$\sum_{n\geq 1} \underset{a_n}{\underset{\|}{\frac{n}{n^3+1}}} \sim \sum_{n=1}^{\infty} \underset{b_n}{\underset{\|}{\frac{1}{n^2}}} \text{ convergentă}$$

$\frac{a_n}{b_n} \to 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \text{ conv} \Leftrightarrow \alpha > 1$$

$$\sum_{n\geq 1} \frac{2\sqrt{n}}{n^2+1} \sim \sum_{n\geq 1} \frac{1}{n^{2-\frac{1}{2}}} = \frac{3}{2} \quad \frac{3}{2} > 1 \text{ e convergent}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n+1}}{n+2} \sim \sum_{n=1}^{\infty} \frac{1}{n^{1-\frac{1}{3}=\frac{2}{3}}}$$

Criteriul 2 Criteriul condensării

Dacă $a_n \searrow 0 \Rightarrow \sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} 2^n \cdot a_{2^n}$

Ex $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$; $\lim_{n\to\infty} \frac{1}{n^{\alpha}} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 0 \\ +\infty, & \alpha < 0 \end{cases} \Rightarrow$ s. divergent

$\alpha > 0 \quad \frac{1}{n^{\alpha}} \to 0$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}}_{a_n} \sim \sum_{n\geq 1} 2^n \cdot \frac{1}{(2^n)^{\alpha}} = \sum_{n\geq 1} \frac{2^n}{2^{n\alpha}} = \sum_{n=1}^{\infty} \underbrace{\left(2^{1-\alpha}\right)^n}_{a}$$

seria este conv $\Leftrightarrow a < 1 \Leftrightarrow 2^{1-\alpha} < 2^0 \Leftrightarrow 1 < \alpha$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \searrow 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2} \sim \sum_{n=2}^{\infty} 2^n \frac{1}{2^n (\ln 2^n)^2} =$$

$$= \sum_{n=2}^{\infty} \frac{1}{n(\ln 2)^2} = \frac{1}{(\ln 2)^2} \sum_{n=2}^{\infty} \frac{1}{n^2}$$

convergent

$$\sum_{n\geq 1}\frac{1}{n} = 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq \frac{1}{4}+\frac{1}{4}=\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq \frac{1}{8}+\frac{1}{8}+\dots+\frac{1}{8}=\frac{1}{2}}+\dots+\underbrace{\frac{1}{2^{n+1}}+\dots+\frac{1}{2^n}}_{\geq \frac{1}{2^{n+1}}+\dots+\frac{1}{2^{n+1}}=\frac{1}{2}}$$

2 termeni   4 termeni 2^n termeni

$\Rightarrow$ seria tinde la infinit

<u>Criteriul 3</u> <u>Criteriul raportului</u>

Seria $\sum_{n=1}^{\infty} x_n$ și $l = \lim_{n\to\infty}\frac{x_{n+1}}{x_n}$. Dacă $l>1 \Rightarrow$ s. este divergentă și dacă $l<1 \Rightarrow$ s. este convergentă

$$\sum_{n=1}^{\infty} x^n \frac{\sqrt{n}}{n^2+1} \qquad a_n = x^n\frac{\sqrt{n}}{n^2+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n}\cdot\underbrace{\frac{\sqrt{n+1}}{\sqrt{n}}}_{\to 1}\cdot\underbrace{\frac{n^2+1}{(n+1)^2+1}}_{\to 1} \to x$$

$x>1 \Rightarrow$ divergentă

$x<1 \Rightarrow$ convergentă