Seminar 2 Analiză

$d_\infty(x,y) = \max_{i=1}^{n} |x_i - y_i|$; $d_\infty : \mathbb{R}^n \times \mathbb{R}^n [0, +\infty)$

1) $d_\infty(x,y) = 0 \Leftrightarrow x = y$

$\max_{i=1}^{n} |x_i - y_i| = 0 \Leftrightarrow x_i = y_i, i = \overline{1,n} \Leftrightarrow x = y$

2) $d_\infty(x,y) = d_\infty(y,x) \Leftrightarrow \max_{i=1}^{n} |x_i - y_i| = \max_{i=1}^{n} |y_i - x_i| =$

$= \max_{i=1}^{n} |x_i - y_i| \Rightarrow$ (A)

3) $d_\infty(x,y) + d_\infty(y,z) \geq d_\infty(x,z)$

$\max_{i=1}^{n} |x_i - y_i| + \max_{i=1}^{n} |y_i - z_i| \geq \max_{i=1}^{n} |x_i - z_i|$ (A)

$(|x_i - y_i| + |y_i - z_i|)$

II

1) $d_\varphi(x,y) = 0 \Leftrightarrow |\varphi(x) - \varphi(y)| = 0 \Leftrightarrow \varphi_x = \varphi_y$

$\varphi : \mathbb{R} \to \mathbb{R}$ $\varphi(x) = x^3$ $d_\varphi : \mathbb{R}^2 \to [0, \infty)$ $d_\varphi(x,y)$

2) $d_\varphi(x,y) = d_\varphi(y,x)$

$\Leftrightarrow |\varphi(x) - \varphi(y)| = |\varphi(y) - \varphi(x)|$

3) $d\varphi(x,y) + d\varphi(y,z) \geq \varphi(x,z)$

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \text{ continuă}\}$$

$$d_1(f,g) = \int_a^b |f(t) - g(t)|\, dt$$

$$d_2(f,g) = \left(\int_a^b (f(t) \cdot g(t))^2\, dt\right)^{\frac{1}{2}}$$

$$d_\infty(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

① $d_1(f,g) = 0 \Rightarrow \left.\begin{array}{l} \int_a^b |f(t) - g(t)|\, dt = 0 \\ f, g \text{ continue} \end{array}\right\} \Rightarrow \begin{array}{l} f(t) - g(t) = 0 \\ \Rightarrow f(t) = g(t) \end{array}$

② $d_1(f,g) = d_1(g,f)$???

$$d_1(f,g) = \int_a^b |f(t) - g(t)|\, dt = \int_a^b |g(t) - f(t)|\, dt = d_1(g,f)$$

③ $d_1(f,g) + d_1(g,h) \geq d_1(f,h)$

$$\int_a^b |f(t) - g(t)|\, dt + \int_a^b |g(t) - h(t)|\, dt > \int_a^b |f(t) - h(t)|\, dt$$

④ $d_1(f+h, g+h) = d_1(f,g)$

$$\int_a^b |f(t) + h(t) - g(t) + h(t)|\, dt$$

④ d_∞

d_2: ① $d_2(f, g) = 0 \Leftrightarrow f = g$

$$\left(\int_a^b |f(t) - g(t)|^2 dt\right)^{\frac{1}{2}} = 0$$

f, g cont $\Leftrightarrow f = g$

② $d_2(f, g) = d_2(g, f)$

$$\left[\int_a^b [f(t) - g(t)|^2 dt\right]^{\frac{1}{2}} = \left[\int_a^b [g(t) - f(t)]^2 dt\right]^{\frac{1}{2}} =$$

③ $d_2(f, g) + d_2(g, h) \geq d_2(f, h)$

$$\left[\int_a^b (\underbrace{f(t) - g(t)}_{\alpha(t)})^2\right]^{\frac{1}{2}} + \left[\int_a^b (\underbrace{g(t) - h(t)}_{\beta(t)})^2\right]^{\frac{1}{2}} \geq$$

$$\geq \left[\int_a^b (f(t) - h(t))^2\right]^{\frac{1}{2}}$$

$$\left[\int_a^b \alpha^2(t)\, dt\right]^{\frac{1}{2}} + \left[\int_a^b \beta^2(t)\, dt\right]^{\frac{1}{2}} \geq \left[\int_a^b (\alpha(t) + \beta(t))^2 dt\right]^{\frac{1}{2}} \Big|^2$$

$$\int_a^b \alpha^2(t)\, dt + \int_a^b \beta^2(t)\, dt + 2\int_a^b \alpha(t)\beta(t)\, dt \geq \int_a^b (\alpha(t) + \beta(t))^2 dt$$

$$\int_a^b \alpha^2(t)\, dt \cdot \int_a^b \beta^2(t)\, dt \geq \int_a^b \alpha(t)\beta(t)^2\, dt$$

se simplifică chestii

$$\underbrace{\int_a^b \alpha^2(t)}_{u^2} \cdot \underbrace{\int_a^b \beta^2(t)}_{v^2} \geqslant \left[\int_a^b \alpha(t)\cdot\beta(t)\right]^2$$

$$\left(\frac{\alpha(t)}{u}\right)^2 + \left(\frac{\beta(t)}{v}\right)^2 \geqslant 2\,\frac{\alpha(t)\cdot\beta(t)}{u\cdot v}$$

$$\cancel{d_\infty(f,g) = \sup|f(t)-g(t)|}$$

$$\cancel{d_\infty(f,g) = \sup|f(t)-g(t)| = 0 \Leftrightarrow f(t)=g(t)}$$

$$f_n(t) = \begin{cases} 0; & t\in\left[\frac{1}{n},1\right] \\ \sqrt{n}; & t\in\left[0,\frac{1}{n}\right) \end{cases}$$

$$d_1(f_n,f) = \int_0^1 |f_n(t)-f(t)|\,dt$$

$$\Rightarrow = \int_0^{\frac{1}{n}} |\sqrt{n}-0|\,dt + \int_{\frac{1}{n}}^1 |0-0|\,dt = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \longrightarrow 0$$

$$d_\infty(f_n,f) = \sup_{x\in[0,1]} |f_n(x)-0| = \sqrt{n} \to \infty$$

$$d_2 = \left(\int_0^1 f_n^2(t)\right)^{\frac{1}{2}} = \sqrt{\int_0^{\frac{1}{n}} n\,dt + \int_{\frac{1}{n}}^1 0\,dt} = 1$$

$$g_n(t) = \begin{cases} 0, & t \in [\frac{1}{n}, 1] \\ -n\sqrt{n}\,(x - \frac{1}{n}), & x \in [0, \frac{1}{n}) \end{cases}$$

$$d_\infty(g_n, 0) = \sup_{x \in [0,1]} |g_n(t)| = \sqrt{n} - x_0 \quad g_n$$

$$d_1(g_n, 0) = \int_0^1 |g_n(t)|\,dt = \int_0^{1/n} -n\sqrt{n}\,(t - \frac{1}{n})\,dt = \frac{\sqrt{n}\cdot\frac{1}{n}}{2} = \frac{\sqrt{n}}{2n} =$$

$$= \frac{1}{2\sqrt{n}} \to 0$$

$$d_2(g_n, 0) = \sqrt{\int_0^{\frac{1}{n}} (-n\sqrt{n}\cdot(t - \frac{1}{n}))^2} =$$

$$= \left(n^3 \left(\frac{t - \frac{1}{n}}{3}\right)^3 \Bigg|_0^{\frac{1}{n}}\right)^{1/2} = n^{\frac{3}{2}} \cdot \frac{1}{\sqrt{3}\cdot n^3} = \frac{1}{\sqrt{3}} \Rightarrow g_n \overset{d_2}{\not\to} 0$$

$$\|f\|_\infty = d_\infty(f, 0) = \sup_{t \in [a,b]} f(t)$$

$$\|f\|_1 = \int_a^b |f(t)|\,dt \le \int_a^b \|f\|_\infty\,dt = (b-a)\|f_n\|_\infty$$

$$\|f\|_2 = \left[\int_a^b f^2(t)\,dt\right]^{\frac{1}{2}} \le \sqrt{\int_a^b \|f\|_\infty^2\,dt} = \|f\|_\infty \sqrt{b-a}$$

$$\int_a^b f^2(t)\,dt \cdot \int_a^b 1^2\,dt \ge \left(\int_a^b f(t)\,dt\right)^2$$

$$\|f\|_2 \sqrt{b-a} \ge \|f\|_1$$

$$x_n = \frac{3n + (-1)^n n}{n+1} + \frac{1}{n}$$

$$x_{2n} = \frac{6n + (-1)^{2n} \cdot 2n}{2n+1} + \frac{1}{2n} \to 4 \qquad L = \{2;4\}$$

$$x_{2n+1} = \frac{6n+3-(2n+1)}{2n+2} + \frac{1}{2n+1} \to 2 \qquad \underline{l} = 2$$

$2n+1 \to \infty$

$$y_n = (-1)^n \frac{n}{2n+1} + \frac{2n + (-1)^{\frac{n(n+1)}{2}} n}{n+1}$$

$$y_{4n} = \frac{4n}{8n+1} + \frac{8n+4n}{4n+1} \to \frac{1}{2} + 3 \to \frac{7}{2}$$

$$y_{4n+1} = -\frac{4n+1}{8n+1} + \frac{8n+2+(-)4n-1}{4n+2} \to \frac{4}{8} + 1 \to \frac{3}{2}$$

$$\cancel{\Rightarrow L = \left\{\frac{1}{2}, \frac{3}{2}\right\}, \text{ai}}$$

$$\cancel{\Rightarrow \underline{l} = \frac{3}{2};\ \bar{l} = \frac{7}{2}}$$

$$y_{4n+2} = \frac{4n+2}{8n+5} + \frac{8n+4-4n-2}{4n+3} \to \frac{5}{2}$$

$$y_{4n+3} \to \frac{1}{2}$$

$L = \left\{\frac{7}{2}, \frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right\}$ și mai multe

$$\underline{l} = \frac{1}{2};\ \bar{l} = \frac{7}{2}$$

$$z_n = \frac{n}{n+1}\cos\frac{n\pi}{2} + \frac{\sqrt{n}}{n+1}$$

$$z_{4n} = \frac{4n}{4n+2}\cos 2n\pi + \frac{\sqrt{4n}}{4n+1} \to 1$$

$$z_{4n+1} = \frac{4n+1}{4n+2}\cos\frac{(4n+1)\cdot\pi}{2} + \frac{\sqrt{4n+1}}{4n+2} \to 0$$

$$z_{4n+3} = \frac{4n+3}{4n+4}\cos\frac{(4n+3)\pi}{2} + \frac{\sqrt{4n+3}}{4n+4} \to 0$$

$$z_{4n+2} = \frac{4n+2}{4n+3}\underbrace{\cos 2n\pi+\pi}_{-1} + \frac{\sqrt{4n+2}}{4n+3} \to -1$$

$$\Rightarrow \mathcal{L} = \{-1, 0, 1\}$$

$$\Rightarrow \underline{l} = -1 \quad \bar{l} = 1$$

$$x_n = \underbrace{\frac{2n+(-1)^n n}{n+3}}_{y_n} + \underbrace{\left\{\frac{n}{3}\right\}}_{z_n} \frac{\sqrt{n}}{2\sqrt{n}+1}$$

$$\frac{\sqrt{n}}{2\sqrt{n}+1} \to \frac{1}{2}$$

$y_{2n} \to 3$

$y_{2n+1} \to 1$

$z_{3n} = 0$

$z_{3n+1} = \frac{1}{3}$

$z_{3n+2} = \frac{2}{3}$

	x	$y \cdot \frac{1}{2}$	$x y \frac{1}{2}$
$6n$	3	0	3
$6n+1$	1	$\frac{1}{6}$	$\frac{7}{6}$
$6n+2$	3	$\frac{1}{3}$	$\frac{10}{3}$
$6n+3$	1	0	1
$6n+4$	3	$\frac{1}{6}$	$\frac{19}{6}$
$6n+5$	1	$\frac{1}{3}$	$\frac{4}{3}$

$\underline{\lim} (x_n + y_n) \geq \underline{\lim}\, x_n + \underline{\lim}\, y_n$

$\lim\limits_{n\to\infty} \inf\limits_{k\geq n} (x_k + y_k) \geq \lim\limits_{n\to\infty} \inf\limits_{k\geq n} x_k + \lim\limits_{n\to\infty} \inf\limits_{k\geq n} y_k$

$\inf\limits_{k\geq n} (x_k + y_k) \geq \inf\limits_{k\geq n} x_k + \inf\limits_{k\geq n} y_k \quad ; l \geq n \Rightarrow$

$\left.\begin{array}{l} x_l \geq \inf\limits_{k\geq n} x_k \\ y_l \geq \inf\limits_{k\geq n} y_k \end{array}\right\} \Rightarrow$

$\Rightarrow x_l + y_l \geq \inf\limits_{n\geq k}(x_k) + \inf\limits_{n\geq k}(y_k)$

$\forall k \geq n$

$x_n = (-1)^n \quad y_n = (-1)^{n+1}$

$\left.\begin{array}{l} \underline{\lim}\, x_n = -1 \\ \underline{\lim}\, y_n = -1 \end{array}\right\} \Rightarrow \underline{\lim}\, x_n + y_n \overset{\lim}{=} (-1)^n + (-1)^{n+1} = 0$

$0 \geq -1 + (-1) \Rightarrow 0 \geq -2$ (A) did not get this

Ex $\sup_{k\geq n} x_k + y_k \geq \inf_{k\geq n} x_k + \sup_{k\geq n} y_k$

$\sup_{k\geq n} (x_k+y_k) \geq \sup_{k\geq n} (\inf x_l + y_k) = \inf_{l\geq n} x_l + \sup_{k\geq n} y_k$

$\overline{\lim}\, x_n + y_n \geq \underline{\lim}\, y_n + \overline{\lim}\, y_n$

$\overline{\lim}\,(x_n+y_n) \geq \underline{\lim}\, x_n + \overline{\lim}\, y_n$

$\lim_{n\to\infty} \sup_{k\geq 1} (x_k+y_k) \geq \lim_{n\to\infty} \inf_{k\geq n} x_k + \lim_{n\to\infty} \sup_{k\geq n} y_k$

Ex $\underline{\lim}\, x_n+y_n \leq \overline{\lim}\, x_n + \underline{\lim}\, y_n$

$\lim_{n\to\infty} \inf_{k\geq n} x_k + y_k \leq \lim_{n\to\infty} \sup_{k\geq n} x_k + \lim_{n\to\infty} \inf y_k$

$\inf_{k\geq n} x_k + y_k \leq \sup_{k\geq n} x_k + \inf_{k\geq n} y_k$

$x_k + y_k \leq \sup_{l\geq n} (x_l + y_k)$

$\inf_{k\geq n} (x_k + y_k) \leq \inf_{k\geq n} \left(\sup_{l\geq n} x_l + y_k \right) =$

$= \sup_{k\geq n} (x_l) + \inf_{k\geq n} y_k$