$Im(f)=\{y\in V_2 \mid \exists\, x_2\in V_1$ a.î. $f(x)=y\}$

Prop: $Ker(f)\subseteq V_1$ subsp vect

$Im(f)\subseteq V_2$ — // —

7.11.2024.

Seminarul 6: Aplicatii liniare

① $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x)=(x_1+2x_2+x_3,\ 2x_1+5x_2+3x_3,\ -3x_1-7x_2-4x_3)$

a) f liniară

b) $Ker f=?$; $Im f=?$

Precizați câte un reper din $Ker f$, $Im f$.

a) **Obs**: $(V,+,\cdot)_{/\mathbb{K}}$, $(W,+,\cdot)_{/\mathbb{K}}$

$f:V\to W$ aplicație liniară $\Leftrightarrow$ 1) $f(x+y)=f(x)+f(y)$

2) $f(ax)=af(x)$, $(\forall)\ x,y\in V$, $a\in\mathbb{R}$ $\Rightarrow$

$\Leftrightarrow f(ax+by)=af(x)+bf(y)$, $(\forall)\ x,y\in V$, $(\forall)\ a,b\in\mathbb{R}$

Obs: f liniară $\Rightarrow$

1) $f:(V,+)\to(V,\cdot)$ morfism de grupuri

$Ker(f)=\{x\in V \mid f(x)=0_W\}=f^{-1}(\{0_W\})$ nucleul lui f

$Im f=\{y\in W \mid (\exists)\ x\in V$ a.î. $f(x)=y\}$ imaginea lui f

$f(\underset{\sim}{ax+by}) = ax_1+by_1+2ax_2+2by_2+ax_3+by_3,$

$(ax_1+by_1, ax_2+by_2, ax_3+by_3)$ $\quad 2ax_1+2by_1+5ax_1+5by_2+3ax_3+3by_3, -3ax_1-3by_1-7ax_2-7by_2-4ax_3-4by_3)$

$f(ax+by) = a(x_1+2x_2+x_3, 2x_1+5x_2+3x_3, -3x_1-7x_2-4x_3)+$
$b(y_1+2y_2+y_3, 2y_1+5y_2+3y_3, -3y_1-7y_2-4y_3) =$
$= af(x)+bf(y) \Rightarrow f$ liniară

b) $\mathrm{Ker} f = \{x\in\mathbb{R}^3 \mid f(x)=0\} \subset \mathbb{R}^3$ (conform definiției)

$$\begin{cases} x_1+2x_2+x_3=0 \\ 2x_1+5x_2+3x_3=0 \\ -3x_1-7x_2-4x_3=0 \end{cases} \quad A=\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -3 & -7 & -4 & 0 \end{array}\right)$$ matricea asociată aplicației liniare

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{vmatrix} \xrightarrow{L_3'=L_3+L_2+L_1} \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \Rightarrow \mathrm{rg}\,A \neq 3 \,(\max) \Rightarrow \mathrm{rg}\,A = 2.$$

$\mathrm{Ker} f = S(A)$

$\dim(\mathrm{Ker} f) = 3 - \mathrm{rg}\,A = 3-2$

$x_3 = \alpha \in \mathbb{R}$

$$\begin{cases} x_1+2x_2 = -\alpha \mid \cdot(-2) \\ 2x_1+5x_2 = -3\alpha \end{cases} \Rightarrow \begin{cases} -2x_1-4x_2 = 2\alpha \\ 2x_1+5x_2 = -3\alpha \end{cases} (+)$$

$x_2 = -\alpha$

$\Rightarrow x_1 = \alpha$

$\Rightarrow \mathrm{Ker} f = \{(\alpha, -\alpha, \alpha) \mid \alpha\in\mathbb{R}\} = \langle\{\underset{R_1}{\underset{\parallel}{(1,-1,1)}}\}\rangle$

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 1 & y_1 \\ 2 & 5 & 3 & y_2 \\ -3 & -7 & -4 & y_3 \end{array}\right)$$

$Im f = \{ y \in \mathbb{R}^3 \mid (\exists) x \in \mathbb{R}^3 \text{ a.î. } f(x) = y \}$

$$(*) \begin{cases} x_1 + 2x_2 + x_3 = y_1 \\ 2x_1 + 5x_2 + 3x_3 = y_2 \\ -3x_1 + 7x_2 - 4x_3 = y_3 \end{cases}$$

$(*)$ este un sistem compatibil $\Leftrightarrow rg A = rg \bar{A}$

$$\Delta p = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \neq 0$$

Trebuie ca $\Delta c = \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} = 0$

$$\xrightarrow{L_1+L_2+L_3} \Delta c = \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ 0 & 0 & y_1+y_2+y_3 \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (y_1+y_2+y_3) \underbrace{\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}}_{1} = 0 \Rightarrow y_1+y_2+y_3 = 0$$

$Im f = \{ y \in \mathbb{R}^3 \mid y_1+y_2+y_3 = 0 \}$

$\dim \mathrm{Im} f = \underset{\underset{\mathrm{rg} A'}{\uparrow}}{3-1} = 2$

$A' = (1\ 1\ 1) | 0$

$y_1 + y_2 + y_3 = 0 \Rightarrow y_1 + y_2 = -y_3$

$\mathrm{Im} f = \{(y_1, y_2, -y_1 - y_2) \mid y_1, y_2 \in \mathbb{R}\} = \{y_1(1,0,-1) + y_2(0,1,-1)\} =$

$= \langle \{(1,0,-1), (0,1,-1)\} \rangle$

R_2 reper pt $\mathrm{Im} f$

II $\mathrm{Im} f$ reper pt imaginea lui f

$R_1 = \{(1,-1,1)\}$ reper

Extindem R_1 la un reper

$\mathrm{rg}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3$ (max)

$R_1 \cup \{e_1, e_3\} \to$ reper din $\mathbb{R}^3$

$\{f(e_1), f(e_3)\}$ reper in $\mathrm{Im} f$ (curs 6)

$f(e_1) = f((1,0,0)) = (1,2,-3)$

$f(e_3) = f((0,0,1)) = (1,3,-4)$

$\{(1,2,-3); (1,3,-4)\}$ reper in imagine

Obs: $\boxed{f(x) = y \Leftrightarrow Y = A \cdot X}$! (dacă se scrie așa e liniară)

$f: \mathbb{R}^3 \to \mathbb{R}^3$ $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

① $f : \mathbb{R}^2 \to \mathbb{R}^2$; $f(x) = (x_1 + x_2, -x_2)$

Dem $f \in Aut(\mathbb{R}^2)$ (f liniară + bij)

$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$f(x) = y \Leftrightarrow Y = A \cdot X \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$

$\Rightarrow f$ liniară
$\det A = -1 \neq 0$
$\Rightarrow f$ bij

$\Rightarrow f$ izomorfism între aceeași sp vect $\Rightarrow f$ automorfism

② $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x_1, x_2) = (3x_1 - 2x_2, 2x_1 - x_2, -x_1 + x_2)$

a) f liniară
b) f inj
c) $\operatorname{Im} f$

$f(x) = y \Leftrightarrow Y = AX \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow f$ liniară

Obs: $f : V \to W$ liniară

a) f inj $\Leftrightarrow \operatorname{Ker} f = \{0_V\} \Leftrightarrow \operatorname{rg} A = \dim V$
b) f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim W$

b) $\operatorname{Ker} f = \{x \in \mathbb{R}^2 \mid f(x) = 0_{\mathbb{R}^3}\}$

S(A)

$\Rightarrow \begin{cases} 3x_1 - 2x_2 = 0 \\ 2x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases}$

$A = \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right)$

$rg\, A = 2$ (maxim)

$rg\, \bar{A} = 2$

$\Rightarrow$ SCD

$\ker f = \{(0,0)\} \Rightarrow f$ inj

$rg\, A = 2 = \dim \mathbb{R}^2$

c) $\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid (\exists)\ x \in \mathbb{R}^2 \text{ a.î. } f(x) = y \}$

$$\begin{cases} 3x_1 - 2x_2 = y_1 \\ 2x_1 - x_2 = y_2 \\ -x_1 + x_2 = y_3 \end{cases} \qquad A = \left(\begin{array}{cc|c} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{array}\right)$$

$$\Delta c = 0 \Rightarrow \begin{vmatrix} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{vmatrix} = 0 \quad \underline{\underline{L_1 = L_1 - 2L_2}} \quad \begin{vmatrix} -1 & 0 & y_1 - 2y_2 \\ 2 & -1 & y_2 \\ 1 & 0 & y_2 + y_3 \end{vmatrix}$$

$\Delta c = y_1 - y_2 + y_3 = 0$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid y_1 - y_2 + y_3 = 0 \}$

!

<u>Teoremă</u>: $f: V \to W$ liniară

$\dim V = \dim \ker f + \dim \operatorname{Im} f$

f inj $\Rightarrow \dim V = \dim \operatorname{Im} f$

<u>Obs</u>: $f: V \to W$ $\dim V = n$, $\dim W = m$

$R_1 = \{e_1, \dots, e_n\} \xrightarrow{A} R_2 = \{\bar{e}_1, \dots, \bar{e}_m\}$

$A = [f]_{R_1, R_2}$

$$\boxed{f(e_i) = \sum_{j=1}^{m} a_{ji}\, \bar{e}_j \quad , i=\overline{1,n}}$$

<u>Document</u>: Seminar 6

(8) $f : \mathbb{R}[x] \to \mathbb{R}^3$

$f(ax+b) = (a, b, a+b)$

$R = \{2x-1, \ -x+1\}$

$R' = \{(1,1,1), (1,1,0), (1,0,0)\}$

a) f liniară

b) $[f]_{R,R'} = A = ?$

c) $\operatorname{Ker} f, \operatorname{Im} f = ?$

$$R = \{\underset{e_1}{2x-1}, \underset{e_2}{-x+1}\} \xrightarrow{A} R' = \{\underset{e_1'}{(1,1,1)}, \underset{e_2'}{(1,1,0)}, \underset{e_3'}{(1,0,0)}\}$$

$f(e_1) = (2x-1) = (2,-1,1) = a(1,1,1) + b(1,1,0) + c(1,0,0) =$

adunarea coef $(a+b)$

$= (a+b+c, a+b, a)$

$$\begin{cases} a+b+c = 2 \Rightarrow c = 2-a-b = 2-1+2 = 3 \\ a+b = -1 \\ a = 1 \end{cases} \Rightarrow b = -2$$

$$A = [f]_{R,R'} = \begin{pmatrix} a & a' \\ b & b' \\ c & c' \end{pmatrix}$$

$f(e_2) = (-x+1) = (-1, 1, 0) = a'(1,1,1) + b'(1,1,0) +$
$+ c'(1,0,0)$

$$\begin{cases} a'+b'+c' = -1 & \Rightarrow c' = -1-1 = -2 \\ a'+b' = 1 & \Rightarrow b' = 1 \\ a' = 0 \end{cases}$$

$$A = [f]_{R, R'} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 3 & -2 \end{pmatrix}$$

$[f]_{R_0, R}$

$R_0 = \{ e_1 = \underset{1}{(1,0)},\ e_2 = \underset{x}{(0,1)} \}$

$R = \{ \bar{e}_1 = (1,0,0),\ \bar{e}_2 = (0,1,0),\ \bar{e}_3 = (0,0,1) \}$

$$f(x) = y \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x_2, x_1, x_1 + x_2$

c) $\ker f = \{ P \in \mathbb{R}_1[X] \mid f(P) = 0_{\mathbb{R}^3} \}$

$f(ax+b) = 0_3 \Rightarrow a = 0,\ b = 0,\ a+b = 0$

Teorema dimensiunii

$$\dim \underset{\substack{\| \\ 2}}{\mathbb{R}_1[X]} = \dim \underset{\substack{\| \\ 0}}{\operatorname{Ker} f} + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$$

$$\operatorname{Im} f = \{ f \in \mathbb{R}^3 \mid \exists\, p \in \mathbb{R}_1[X] \text{ a.i. } f(p) = y \}$$

$$\begin{cases} a = y_1 \\ b = y_2 \\ a+b = y_3 \end{cases} \Rightarrow y_1 + y_2 = y_3 \Rightarrow y_1 + y_2 - y_3 = 0$$

$$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid y_1 + y_2 - y_3 = 0 \} = \{ (y_1, y_2, y_1 + y_2) \}$$

$$\operatorname{Im} f = \langle \{ (1,0,-1), (0,1,1) \} \rangle$$