Seminar 10 A-G

Forme pătratice. Formă canonică

Spații vectoriale cu produs scalar. Procedeul Gram-Schmidt

Ex 6 $Q: \mathbb{R}^3 \to \mathbb{R}, Q(x) = 2x_1^2 + 5x_2^2 + 2x_3^2 - 4x_1x_2 - 2x_1x_3 + 4x_2x_3$

ⓐ $G = ?$ în raport cu R_0

ⓑ $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, formă polară asociată

ⓓ Forma canonică ⓒ este g nedegenerată!

ⓐ $G = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{pmatrix}$

ⓑ $g(x, y) = 2^{-1}[Q(x+y) - Q(x) - Q(y)]$

$\sum_{i,j=1}^{3} g_{ij} x_i y_j = 2x_1y_1 - 2x_1y_2 - x_1y_3 - 2x_2y_1 + 5x_2y_2 +$

$+ 2x_2y_3 - x_3y_1 + 2x_3y_2 + 2x_3y_3$

ⓒ $g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$ $\text{Ker } g = \{x \in \mathbb{R}^3 \mid g(x,y) = 0, x, y \in \mathbb{R}^3\}$

g nedegenerată $\Leftrightarrow \text{Ker } g = \{0_{\mathbb{R}^3}\}$

$\begin{vmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & 2 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -(-4-3) = 7 \neq 0$

$C_1 + 2C_3$ $\Rightarrow G$ nedegenerată

$C_2 - 2C_3$ $\Rightarrow g$ nedegenerată

d) Jacobi

$\Delta_1 = |2| = 2$

$\Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 5 \end{vmatrix} = 6$

$\Delta_3 = \begin{vmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{vmatrix} = +7$

(*) $\begin{cases} x_1' = (x_1 - x_2 - \frac{1}{2}x_3) \\ x_2' = (x_2 + \frac{1}{3}x_3) \\ x_3' = x_3 \end{cases}$

$-\frac{6}{2} + \frac{3}{2} \quad -3$

$Q(x) = \frac{1}{\Delta_1}x_1^2 + \frac{\Delta_1}{\Delta_2}x_2^2 + \frac{\Delta_2}{\Delta_3}x_3^2$

$Q(x) = \frac{1}{2}x_1^2 + \frac{1}{3}x_2^2 + \frac{6}{7}x_3^2$

$\Rightarrow$ signatura $(3,0)$; Q pozitiv definit

$Q(x) = \underline{2x_1^2} + 5x_2^2 + 2x_3^2 - \underline{4x_1x_2} - \underline{2x_1x_3} + 4x_2x_3$

$Q(x) = 2(x_1^2 - 2x_1x_2 - x_1x_3) + 5x_2^2 + 2x_3^2 + 4x_2x_3 =$

$= 2(x_1 - x_2 - \frac{1}{2}x_3)^2 - \frac{1}{2}x_3^2 + 2x_3x_2 + 5x_2^2 + 2x_3^2 + 4x_2x_3 =$

$= -2x_2^2 = 2(x_1 - x_2 - \frac{1}{2}x_3)^2 + \underline{3x_2^2} + \frac{3}{2}x_3^2 + \underline{6x_2x_3}$

$= 2(x_1 - x_2 - \frac{1}{2}x_3)^2 + 3(x_2^2 + 2x_2x_3 + x_3^2) - 3x_3^2 + \frac{3}{2}x_3^2$

$= 2(x_1 - x_2 - \frac{1}{2}x_3)^2 + 3(x_2 + x_3)^2 - \frac{3}{2}x_3^2$

$= 3(x_2^2 + \frac{2}{3}x_2x_3 + \frac{1}{9}x_3^2) - \frac{x_3^2}{3} + \frac{3}{2}x_3^2$

$Q(x) = 2(x_1 - x_2 - \frac{1}{3}x_3)^2 + 3(x_2 + \frac{1}{3}x_3)^2 + \frac{7}{3}x_3^2$

(*) $\Rightarrow Q(x) = 2x_1'^2 + 3x_2'^2 + \frac{7}{3}x_3'$ $\Rightarrow$ signatura $(3,0)$

Q pozitiv definit; coeficienții diferiți

Ex 2 (59) $G = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$

$Q(x) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$ când diagonala are zerouri

$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_1 - x_2 \\ x_3' = x_3 \end{cases}$ $\begin{cases} x_1 = \frac{1}{2}(x_1' + x_2') \\ x_2 = \frac{1}{2}(x_1' - x_2') \\ x_3 = x_3' \end{cases}$

$Q(x) = \frac{1}{2}(x_1'^2 - x_2'^2) - 6x_3' \cdot (x_1')$

$= \frac{1}{2}(x_1'^2 - 12x_1'x_3' + 36x_3'^2) - \frac{36}{2}x_3'^2 - \frac{1}{2}x_2'^2$

$= \frac{1}{2}(x_1' + 6x_3')^2 - 18x_3'^2 - \frac{1}{2}x_2'^2$

$\Rightarrow \begin{cases} x_1'' = x_1' + 6x_3' \\ x_2'' = x_2' \\ x_3'' = x_3' \end{cases} \Rightarrow Q(x) = \frac{1}{2}x_1''^2 - \frac{1}{2}x_2''^2 - 18x_3''^2$

$\Rightarrow$ signatura $(1,2)$

Q nu e pozitiv definit

Listă Seminarul 10

② $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ asociată în raport cu R_0

Este $(\mathbb{R}^3, g)$ spațiu vectorial euclidian real?

$\Leftrightarrow$ g este produs scalar $\Leftrightarrow$ g formă biliniară
g simetrică 1°
g pozitiv definită

$G = G^T \Rightarrow$ g simetrică
dar g biliniară $\Rightarrow g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

$|3| = 3 \quad \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 6 - 4 = 2$

$\begin{vmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 6 + 0 + 0 - 0 - 4 \cdot 3 - 4 = 6 - 12 - 4 = -10$

Metoda Jacobi

$Q(x) = \frac{1}{3} x_1^2 + \frac{2}{3} x_2^2 - \frac{2}{10} x_3^2$

$S(3\ 1) \Rightarrow$

g nu e pozitiv definită

$\Rightarrow$ g nu e produs scalar

$p_1 \circ s_1 (u_1 + u_2) = p_2 (u_1 - u_2) = -u_2$

$s_1 = 2p^1 - id$

$s_1 (u_1 + u_2) = 2u_1 - (u_1 + u_2) = u_1 - u_2$

$p_1 \circ s_2 (u_1 + u_2) = p_1 (-u_1 + u_2) = -u_1$

③ $(\mathbb{R}^3, g_0)$ spațiu vectorial euclidian real

$g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3)$

$U = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0 \}$

ⓐ $U^\perp$ ortogonal? $\perp \Leftrightarrow$ ortogonal

ⓑ Să se afle un reper ortonormat $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ în $\mathbb{R}^3$ unde $\mathcal{R}_1 =$ reper ortonormat în U
$\mathcal{R}_2$ ——— " ——— $U^\perp$

$U^\perp = \{ x \in \mathbb{R}^3 \mid g_0(x, y) = 0 \ \forall y \in U \}$

$1 x_1 + 1 \cdot x_2 + (-1) x_3 = g_0((1, 1, -1), (x_1, x_2, x_3))$

ⓐ

$U^\perp = \langle \{ (1, 1, -1) \} \rangle$

ⓑ $U = \{ (x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R} \} \Rightarrow U = \langle \{ \overset{f_1}{(1,0,1)}, \overset{f_2}{(0,1,1)} \} \rangle$

Aplicăm procedeul de ortogonalizare Gram-Schmidt

$e_1 = f_1$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (0,1,1) - \frac{1}{2}(1,0,1) = \frac{1}{2}(-1,2,1)$

$\langle x, y \rangle = (x_1 y_1, x_2 y_2, x_3 y_3)$ sau n

$\|x\| = \sqrt{g_0(x,x)} = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{x_1 x_1 + x_2 x_2 + x_3 x_3}$

$$\begin{cases} e_1' = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}(1,0,1) \\ e_2' = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{6}}(-1,2,1) \\ e_3' = \frac{e_3}{\|e_3\|} = \frac{1}{\sqrt{3}}(1,1,-1) \end{cases}$$

$e_3 = (1,1,-1)$

$\mathcal{R}_1 = \{e_1', e_2'\}$

reper ortonormat în U

versori, perpendiculari

$\mathcal{R} = \{e_1', e_2', \frac{1}{\sqrt{3}}(1,1,-1)\} \Rightarrow \mathcal{R}_2 = \{\frac{1}{\sqrt{3}}(1,1,-1)\}$

reper ortonormat în $U^\perp$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$

$\mathbb{R}^3 = U \oplus U^\perp$

<u>OBS</u> $U^\perp = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} g_0(x, f_1) = 0 \\ g_0(x, f_2) = 0 \end{array} \right\} =$

$= \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{array} \right\}$

$= \{(-x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} \Rightarrow = \langle \{(-1,-1,1)\} \rangle$

$$\det A = \frac{1}{6}\begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{vmatrix} \overset{L_3 - L_1}{=\!=} \frac{1}{6}\begin{vmatrix} 0 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & -2 \end{vmatrix} = \frac{1}{6}(-4-2) = -1$$

$$\mathcal{R}_0 \xrightarrow{A} \mathcal{R}$$
inversabilă, ortogonală

$\mathcal{R}$ nu e pozitiv orientat ca și $\mathcal{R}_0$

⑤ $(\mathbb{R}^3, g_0)$, $\mathcal{R} = \{f_1 = (1,2,3),\ f_2 = (0,1,1),\ f_3 = (1,1,5)\}$ — spațiu euclidian real

ⓐ $\mathcal{R}$ reper în $\mathbb{R}^3$. Să se ortonormeze G-S

ⓑ $f_1 \times f_2$

ⓒ $f_1 \wedge f_2 \wedge f_3$

$\det A > 0 \Rightarrow \mathcal{R}_0, \mathcal{R}$ la fel orientate

$\wedge$ = wedge

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{vmatrix} = 5 + 2 + 0 - 3 - 2 - 0 = 7 - 5 = 2 \neq 0$$
$\Rightarrow \mathcal{R}$ SLI

ⓐ

$e_1 = f_1 \Rightarrow e_1' = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{14}}(1,2,3)$

$e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 =$

$= (0,1,1) - \frac{5}{14}(1,2,3) = \frac{1}{14}(-5,4,-1)$

$e_2' = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{42}}(-5,4,-1)$

e_3

(b) e_3 e pe a lung ob calculat (se poate de fapt)

facem la $f_1 \times f_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = \left(\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \right) =$

$= (-1, -1, 1)$, $e_3' = \frac{1}{\sqrt{3}}(-1, -1, 1)$ $\quad f_1 \times f_2 = e_3$

$$f_1 \wedge f_2 \wedge f_3 = \begin{vmatrix} 1 & 2 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = g_0(f_3, f_1 \times f_2) = -1 - 2 + 5 = 2$$