Seminar 7 Algebră și Geometrie

Aplicații liniare Vectori proprii

L6 5) $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, -x_1 - 2x_2 - x_3, x_1 + x_2 + x_3)$

$$V' = x \in \mathbb{R}^3 \Big/ \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}$$

c) $f(V') = ?$

<u>OBS</u> $f/V' : V' \to \mathbb{R}^3$

Teorema dimensiunii

$\dim V' = \dim \operatorname{Ker}(f/V') + \dim [f(V')] \Rightarrow \dim f(V') \leq \dim V'$

$$A = \left(\begin{array}{cc|c|c} 1 & -1 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 - x_2 = -\alpha \\ x_1 + 2x_2 = \alpha \end{array}$$

$3x_2 = 2\alpha \Rightarrow \boxed{x_2 = \frac{2}{3}\alpha}$

$\boxed{x_1 = -\frac{1}{3}\alpha}$

$V' = \left\{ \left(-\frac{1}{3}\alpha, \frac{2}{3}\alpha, \alpha\right) \middle| \alpha \in \mathbb{R} \right\}$

$\left(-\frac{1}{3} \cdot \alpha, \frac{2}{3} \cdot \alpha, \alpha\right) = \frac{\alpha}{3}(-1, 2, 3)$

$\Rightarrow V' = \langle \{(-1, 2, 3)\} \rangle$ $\dim V' = 1 \Rightarrow$ dreaptă

$f(v') = (-1+4, +3, 1-4-3, -1+5) = (6, -6, 4) = 2(3, -3, 2)$

$f(V') = <\{(3, -3, 2)\}>$ și unidimensional

L6

(6) $(\mathbb{R}^2, +, \cdot)|\mathbb{R}$ $\quad \mathcal{B}_e = \{e_1 = (1,0), e_2 = (0,1)\} \xrightarrow{C}$

$\mathcal{B}' = \{e_1' = e_1 - e_2, e_2' = e_1 + 2e_2\}$

$((\mathbb{R}^2)^*, +, \cdot)|\mathbb{R}$ $\quad (\mathbb{R}^2)^* = \{f: \mathbb{R}^2 \to \mathbb{R} \mid f \text{ liniară}\}$

spațiul dual lui $V = \mathbb{R}^2$

$\mathcal{B}^* = \{e_1^*, e_2^*\} \xrightarrow{D} \mathcal{B}'^* = \{e_1'^*, e_2'^*\}$

$e_i^*: \mathbb{R}^2 \to \mathbb{R}$

$\boxed{e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}}$ $\quad e_i'^*: \mathbb{R}^2 \to \mathbb{R}$, $\quad e_i'^*(e_j') = \delta_{ij}$

$C, D = ?$ $\boxed{\text{matricea } I_n}$

$e_1' = e_1 - e_2$

$e_2' = e_1 + 2e_2$

$C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$1 = e_1'^*(e_1') = (ae_1^* + be_2^*)(e_1 - e_2)$

$= \underbrace{ae_1^*(e_1)}_{1} + \underbrace{be_2^*(e_1)}_{0} - \underbrace{ae_1^*(e_2)}_{0} - \underbrace{be_2^*(e_2)}_{1} =$

$= a - b$

$0 = e_1'^*(e_2') = (a e_1^* + b e_2^*)(e_1 + 2e_2)$

$= \underbrace{a e_1^*(e_1)}_{1} + \underbrace{b e_2^*(e_1)}_{0} + \underbrace{a e_1^*(2e_2)}_{0} + \underbrace{b e_2^*(2e_2)}_{1}$

$= a + 2b$

$\begin{cases} a - b = 1 \\ a + 2b = 0 \end{cases} \Rightarrow 3b = -1 \Rightarrow b = \frac{-1}{3}$, $a = \frac{2}{3}$

$\Rightarrow \boxed{a = \frac{2}{3},\ b = -\frac{1}{3}}$ prima coloană din D

$\cancel{e_2'^* = a e_1^* + b e_2^*}$

$e_2'^* = c e_1^* + d e_2^*$

$0 = e_2'^*(e_1') = (c e_1^* + d e_2^*)(e_1 - e_2) =$

$= \underbrace{c e_1^*(e_1)}_{1} + \underbrace{c e_1^*(-e_2)}_{0} + \underbrace{d e_2^*(e_1)}_{0} - \underbrace{d e_2^*(e_2)}_{1}$

$= c - d = 0$

$1 = e_2'^*(e_2') = (c e_1^* + d e_2^*)(e_1 + 2e_2)$

$= \underbrace{c e_1^*(e_1)}_{1} + \underbrace{d e_2^*(e_1)}_{0} + \underbrace{c e_1^*(2e_2)}_{0} + \underbrace{d e_2^*(2e_2)}_{1}$

$= c + 0 + 0 + 2d$

$\begin{cases} c - d = 0 \Rightarrow 3d = 1 \\ c + 2d = 1 \quad d = \frac{1}{3} \\ \quad c = d = \frac{1}{3} \end{cases} \Rightarrow D = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\det(C) = 2+1 = 3$

$C^T = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$; $C^* = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow C^{-1} = \frac{C^*}{3} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

$(C^{-1})^T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \Rightarrow \boxed{(C^{-1})^T = D}$

L6

(9) $g : \mathbb{R}^3 \to \mathbb{R}^3$ liniară, $g(v_i) = u_i$, $i = \overline{1,3}$

$R = \{v_1 = (-1,1,1),\ v_2 = (1,1,1),\ v_3 = (0,2,1)\}$

$R' = \{u_1 = 2v_1 + 3v_2 - v_3,\ u_2 = v_1 + 3v_2 + v_3,\ u_3 = v_3\}$

(a) $g = ?$ (a.2) $[f]_{R,R'}$

~~(b) $[g]_{R,R'}$~~ (b) $[g]_{R_0 R_0}$

(c) $\operatorname{Ker}(g)$, $\operatorname{Im}(g)$

$f(v_1) = u_1 = 2v_1 + 3v_2 - v_3$

$f(v_2) = u_2 = v_1 + 3v_2 + v_3$

$f(v_3) = u_3 = v_3$

$$[f]_{R,R'} = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$R_0 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$f(v_1) = f(-e_1+e_2+e_3) = -f(e_1)+f(e_2)+f(e_3) =$$

$$\overset{\parallel}{u_1} = 2(-1,1,1)+3(1,1,1)-(0,2,1)$$

$$= (1,3,4) = e_1+3e_2+4e_3$$

$$\Rightarrow -f(e_1)+f(e_2)+f(e_3) = e_1+3e_2+4e_3$$

$$f(v_2) = u_2 = f(e_1+e_2+e_3) = (-1,1,1)+3(1,1,1)+(0,2,1)$$

$$\Rightarrow f(e_1)+f(e_2)+f(e_3) = (2,6,5) = 2e_1+6e_2+5e_3$$

$$f(e_1)+f(e_2)+f(e_3) = 2e_1+6e_2+5e_3$$

$$f(v_3) = u_3 \Rightarrow f(2e_2+e_3) = (0,2,1) = 2f(e_2)+f(e_3) = 2e_2+e_3$$

$$\Rightarrow \begin{cases} -f(e_1)+f(e_2)+f(e_3) = e_1+3e_2+4e_3 & \Big| \oplus \\ f(e_1)+f(e_2)+f(e_3) = 2e_1+6e_2+5e_3 & \Big| \\ 0 + 2f(e_2)+f(e_3) = 0+2e_2+e_3 & \big| \cdot(-1) \end{cases}$$

$$0+2f(e_2)+2f(e_3) = 3e_1+9e_2+9e_3 \quad \Big| \oplus$$

$$\Rightarrow 0-2f(e_2)-f(e_3) = 0-(+2e_2+e_3) \quad \Big|$$

$$f(e_3) = 3e_1+9e_2-2e_2+9e_3-e_3$$

$$\cancel{f(e_3) = e_1+\frac{7}{3}e_2+\frac{6}{3}e_3}$$

$$f(e_3) = e_1+7e_2+8e_3$$

$$\Rightarrow f(e_2) = \frac{2e_2 + e_3 - 3e_1 - 7e_2 - 8e_3}{2}$$

$$\Rightarrow \boxed{f(e_2) = -\frac{3}{2}e_1 - \frac{5}{2}e_2 - \frac{7}{2}e_3}$$

$$f(e_1) = 2e_1 + 6e_2 + 5e_1 - 3e_1 - 7e_2 - 8e_3 + \frac{3}{2}e_1 + \frac{5}{2}e_2 + \frac{7}{2}e_3$$

$$\boxed{f(e_1) = \frac{1}{2}e_1 + \frac{3}{2}e_2 + \frac{1}{2}e_3}$$

matricea asociată lui $[f]_{R_0 R_0} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & 1 \\ \frac{3}{2} & -\frac{5}{2} & 7 \\ \frac{1}{2} & -\frac{7}{2} & 8 \end{pmatrix}$

~~[illegible] e_1, e_2, e_3~~

$e_1, e_2, e_3 \in R_0$

și faci sistem cu funcțiile

$$A' = [f]_{R, R'}$$

$$A = D A' C^{-1}$$

$$\begin{array}{ccc} R_0 = \{e_1, e_2, e_3\} & \xrightarrow{C} & R = \{v_1, v_2, v_3\} \\ \Big\downarrow A = [f]_{R_0 R_0} & & \Big\downarrow A' = [f]_{R, R'} \\ R_0 = \{e_1, e_2, e_3\} & \xrightarrow{D} & R' = \{u_1, u_2, u_3\} \end{array}$$

$$A C = D A'$$

$f: \mathbb{R}_1[X] \to \mathbb{R}_1[X]$ liniară

Să se afle expresia analitică pentru f dată

ⓐ $[f]_{R_0,R_0} = A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

ⓑ $[f]_{R,R} = A' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$, $R = \{X-1,\ 2X+2\}$

$R_0 = \{1, X\}$ $\quad f(1) = 1\cdot 1 + 3\cdot X$

$\qquad\qquad\qquad f(X) = 2\cdot 1 + 4\cdot X$

$$f(a+bX) = a f(1) + b f(X) = a(1+3X) + b(2+4X) =$$
$$= a + 2b + (3a+4b)X$$

$f(x_1, x_2) = (x_1 + 2x_2,\ 3x_1 + 4x_2)$

$f(x_1 + x_2 X) = x_1 + 2x_2 + (3x_1 + 4x_1)X$

$$\begin{tikzcd} R_0 \arrow[r, "{[f]_{R_0,R_0} = A}"] \arrow[d, "C"'] & R_0 \arrow[d, "C"] \\ R \arrow[r, "{[f]_{R,R} = A'}"'] & R \end{tikzcd}$$

$R_0 = \{1, X\} = \{(1,0), (0,1)\}$

$R = \{(-1,1), (2,2)\} = \{-1+X,\ 2+2X\}$

$C = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$

$A = CA'C^{-1}$ $\quad$ Analog pt a)

(m II) $f(x+1) = f(-1,1) = 1(x-1) + 0(2x-2) = x-1$

$f(2x+2) = 2(x-1) + (-1)(2x+2) = -4$

$$\begin{cases} -f(1) + f(x) = -1 + x \quad \Big| \cdot 2 \\ 2f(1) + 2f(x) = -4 \end{cases} \quad \oplus$$

$4f(x) = -6 + 2x \Rightarrow f(x) = \frac{-3}{2} + \frac{1}{2}x \Rightarrow$

$\Rightarrow f(1) = \frac{-3}{2} + \frac{1}{2}x + 1 - x = \boxed{-\frac{1}{2} - \frac{1}{2}x = f(1)}$

$f(a+bx) = a f(1) + b f(x) = a\left(-\frac{1}{2} - \frac{1}{2}x\right) + b\left(-\frac{3}{2} + \frac{1}{2}x\right)$

$\boxed{= -\frac{a}{2} - \frac{3b}{2} - \left(\frac{a}{2} - \frac{b}{2}\right)x}$

rang (matrice) = maxim $\Rightarrow$ matricea este bijectivă

$f: \mathbb{R}^3 \to \mathbb{R}^3$ ~~[illegible]~~

$f(x) = (x_1 - 2x_2 + 5x_3,\ m x_1 + 3x_2 - x_3,\ x_2 - 3x_3)$

(a) $m = ?$ ai f inj

(b) $m = ?$ ai f surj

$$[f]_{R_0,R_0} = \begin{pmatrix} 1 & -2 & 5 \\ m & 3 & -1 \\ 0 & 1 & -3 \end{pmatrix}$$

$$[f]_{R_0 R_0} = \begin{pmatrix} 1 & -2 & 5 \\ m & 3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + 5x_3 \\ m x_1 + 3x_2 - x_3 \\ 0 + x_2 - 3x_3 \end{pmatrix}$$

OBS $f : V \to W$

(a) f inj $\Leftrightarrow$ $\dim V = \operatorname{rg} A$

(b) f surj $\Leftrightarrow$ $\dim W = \operatorname{rg} A$

$$\begin{vmatrix} 1 & -2 & 5 \\ m & 3 & -1 \\ l & 1 & 3 \end{vmatrix} \underset{C_3+3C_2}{=}$$

$$\begin{vmatrix} 1 & -2 & -1 \\ m & 3 & 8 \\ 0 & 1 & 0 \end{vmatrix} = (-1)^5 \begin{vmatrix} 1 & -1 \\ m & 8 \end{vmatrix} = -(8+m) = -8-m$$

$m \neq -8$

$\Rightarrow \operatorname{rg} A = 3 \Rightarrow f$ inj şi f surj $\Leftrightarrow \operatorname{rg} A = 3 \Leftrightarrow m \neq -8$