Seminar Analiză

① $\sqrt{n^2+n}+n$

$$\sqrt{n^2+n}-n=\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}\to\frac{1}{2}$$

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② $$\sqrt[3]{3n^3+n}-\sqrt{2n^2+n}=n\sqrt[3]{3+\frac{1}{n^2}}-n\sqrt{2+\frac{1}{n}}=n\left(\sqrt[3]{3+\frac{1}{n^2}}-\sqrt{2+\frac{1}{n}}\right)$$

$$\lim_{n\to\infty}n\left(\sqrt[3]{3+\frac{1}{n^2}}-\sqrt{2+\frac{1}{n}}\right)=\infty\left(\sqrt[3]{3+0}-\sqrt{2+0}\right)=\infty\left(\sqrt[3]{3}-\sqrt{2}\right)=$$

$$=\infty$$

$3^2<2^3$

$3^2 \gtrless 8$

$9>8$

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③ $$\sqrt[3]{n^3+n^2}-\sqrt[3]{n^3+n^2}=\frac{n^3+n^2-n^3+n^2}{\sqrt[3]{(n^3+n^2)^2}+\sqrt[3]{(n^3+n^2)(n^3-n^2)}+\sqrt[3]{(n^3-n^2)^2}}=$$

$$=\frac{2n^2}{n^2\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+n^2\sqrt[3]{\left(1-\frac{1}{n^2}\right)}+n^3\sqrt[3]{\left(1-\frac{1}{n}\right)^2}}\to\frac{2}{3}$$

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④ $$\frac{1+\sqrt{2}+\dots+\sqrt{n}}{n\sqrt{n}}=\sum_{k=1}^{n}\frac{\sqrt{k}}{n\sqrt{n}}=\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}=\frac{1}{2}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)$$

$$f(x)=\sqrt{x}$$

$$\int_0^1\sqrt{x}\,dx=\int_0^1 x^{\frac{1}{2}}\,dx=\frac{2x^{\frac{3}{2}}}{3}\Big|_0^1=\frac{2}{3}(1-0)=\frac{2}{3}$$

⑤ $\left(1+\sqrt{n+1}-\sqrt{n}\right)^{\sqrt{n}} = \left[\left(1+\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\sqrt{n+1}-\sqrt{n}}\right]^{\frac{\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}} =$

$= e^{\frac{\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}} \Rightarrow e^{\frac{1}{2}}$

$$\boxed{\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}}$$

⑥ $\frac{\sqrt[n]{n!}}{n} = \sqrt[n]{\frac{n!}{n^n}} \Rightarrow \lim_{n\to\infty}$

$\Rightarrow \lim_{x\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{x\to\infty} \frac{\cancel{(n+1)!}^{\,n+1} \cdot n^n}{\cancel{n!}\,(n+1)^{\cancel{n+1}}} = \lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n =$

$= \lim_{n\to\infty}\left[\left(1-\frac{1}{n+1}\right)^{n+1}\right]^{\frac{n}{n+1}} = e^{\frac{n}{n+1}} = e^{1} = e$

⑦ $1 < 1+\frac{1}{2}+\dots+\frac{1}{n} = \sum_{k=1}^{n}\frac{1}{k} < n$

$\frac{1}{n}\to\infty$ $\boxed{\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}}$

$\Rightarrow \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\ln(n+1)-\ln n} = \lim_{n\to\infty}\frac{1}{(n+1)\ln\frac{n+1}{n}} = \lim_{n\to\infty}\frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}} =$

$= \frac{1}{\ln e} = 1$

$$= \lim_{n\to\infty} \frac{n^{\cancel{\alpha}}\left(1+\frac{1}{n}\right)^{\alpha}}{\cancel{n}^{\alpha+1}\left(\left(1+\frac{1}{n}\right)^{\alpha+1}-1\right)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\left(1+\frac{1}{n}\right)^{\alpha+1}-1} \quad \boxed{\lim_{x\to 0} \frac{(1+x)^{\alpha}-1}{x} = \alpha}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\left(1+\frac{1}{n}\right)^{\alpha+1}} = \frac{1}{\alpha+1}$$

$$n\sqrt{n}\left(\sqrt{n+1}+\sqrt{n-1}-2\sqrt{n}\right)$$

$$= n\sqrt{n^2+n} + n\sqrt{n^2-n} - 2n = \lim_{n\to\infty} n\left(\sqrt{n^2+n}+\sqrt{n^2-n}-2\cancel{n}\right) =$$

$$\cancel{= \lim_{n\to\infty} n\left(n\sqrt{1+\frac{1}{n}} + n\sqrt{1-\frac{1}{n}} - 2\right)} = n\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}+\sqrt{n-1}-\sqrt{n}\right) =$$

$$= n\sqrt{n}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} + \frac{n-1-n}{\sqrt{n-1}+\sqrt{n}}\right) = n\sqrt{n}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}} - \frac{1}{\sqrt{n-1}+\sqrt{n}}\right) =$$

$$= n\sqrt{n}\left(\frac{\sqrt{n-1}+\sqrt{n}-\sqrt{n+1}-\sqrt{n}}{\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n-1}+\sqrt{n}\right)}\right) = n\sqrt{n}\left(\frac{\sqrt{n-1}-\sqrt{n+1}}{\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n-1}+\sqrt{n}\right)}\right) =$$

$$\cancel{= n\sqrt{n}\left(\frac{n-1-n-1}{\sqrt{n-1}+\sqrt{n+1}} \quad \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}\right)}$$

(8) $$\lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}(n+1-n)} = 2$$

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(9) $$\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^n = \left(\frac{2+a^{\frac{1}{n}}-1+b^{\frac{1}{n}}-1}{2}\right)^n = \left[\left(1+\frac{a^{\frac{1}{n}}-1+b^{\frac{1}{n}}-1}{2}\right)^{\frac{2}{a^{\frac{1}{n}}+b^{\frac{1}{n}}}}\right]$$

$$\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}-2}{2}\cdot n = e^{\frac{1}{2}\left(\lim_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}+\lim_{n\to\infty}\frac{b^{\frac{1}{n}}-1}{\frac{1}{n}}\right)} = e^{\frac{1}{2}\ln a\cdot b} =$$

$$= e^{\ln\sqrt{ab}}$$

$\sqrt[n]{a^n+b^n+c^n}$, $a,b,c>0$ i.f. $a>b>c$

$$\sqrt[n]{a^n\left(1+\frac{b^n}{a^n}+\frac{c^n}{a^n}\right)} = a\sqrt[n]{1+\frac{b^n}{a^n}+\frac{c^n}{a^n}} = a$$

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(10) $$\frac{1^r+2^r+\dots+n^r}{n^{r+1}} = \frac{1}{n}\sum_{k=1}^{n}\frac{k^r}{n^r} = \frac{1}{n}\left(\frac{k}{n}\right)^r \Rightarrow$$

$$\Rightarrow \int_0^1 x^r\,dx = \left(\frac{x^{r+1}}{r+1}\right)\Big|_0^1 = \frac{1}{r+1}$$

$$\Rightarrow \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{\cancel{1^r+\dots+n^r+(n+1)^r-1^r-\dots-n^r}}{(n+1)^{r+1}-n^{r+1}}$$

$$\cancel{= \lim_{n\to\infty}\frac{(n+1)^r}{(n+1)^r(n+1)-}}$$

$$\lim_{n\to\infty} \sqrt[n]{C_{2n}^{n}} \Rightarrow \sqrt[n]{\frac{(2n)!}{n!(2n-n)!}} = \sqrt[n]{\underbrace{\frac{(n+1)(n+2)\cdots(2n)}{n!}}_{a}} =$$

$$\cancel{a = \frac{(2(n+1))!}{n!(n+1)} = \sqrt[n]{\frac{(n+1)(n+2)\cdots(2n)}{n!}} =}$$

$$\cancel{\frac{x_{n+1}}{x_n}} =$$

$$x_{n+1} = x_n - x_n^2 \in (0,1)$$

$$x_n = x_n(1-x_n) \in (0,1)$$

$$x_n \searrow l;\ l = l - l^2$$

$$\Rightarrow \text{I } l = 0$$

$$\text{II } l \neq 0 \Rightarrow 1 = 1 - l \Rightarrow l = 0$$

$$x_n \searrow 0$$

$$n \cdot x_n = \frac{x_n}{\frac{1}{n}} = \frac{x_{n+1} - x}{\frac{1}{n+1} - \frac{1}{n}} = \frac{-x^2}{\frac{n-n+1}{n(n+1)}} = x_n^2 (n+1) n$$

$$= \frac{n}{\frac{1}{x_n}} = \frac{n+1-n}{\frac{1}{x_{n+1}} - \frac{1}{x_2}} = \frac{x_{n+1} \cdot x_n}{x_n \cdot x_{n+1}} = \frac{(x_n - x_n^2)\, x_n}{x_n^2} =$$

$$= 1 - x_n \to 1$$

$$\lim_{n\to\infty}\left(1+\frac{a^n}{n^n}\right)^{n!} = \lim_{n\to\infty}\left[\left(1+\frac{a^n}{n^n}\right)^{\frac{n^n}{a^n}}\right]^{\frac{a^n}{n^n}\cdot n!} =$$

$$= e^{\lim\limits_{n\to\infty}\frac{a^n}{n^n}\cdot n!}$$

$$a_n>0,\ \frac{a_{n+1}}{a_n}\to l \quad \left.\begin{matrix} l>0 \Rightarrow a_n\to\infty \\ l<0 \Rightarrow a_n\to 0\end{matrix}\right\} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{a^{n+1}\cancel{(n+1)}}{(n+1)^{n+1}} =$$

$$= \frac{a\cdot n^n}{(n+1)^n} = a\left(\frac{n}{n+1}\right)^n = \frac{a}{\left(1+\frac{1}{n}\right)^n} \to \frac{a}{e}$$

$$\Rightarrow e^{\infty}=\infty,\ a>e$$

$$e^0 = 1,\ a<e$$

$$\sqrt[n]{a^n+b^n+c^n} \Rightarrow a<b<c$$

$$c^n \le a^n+b^n+c^n \le 3c^n$$

$$c \le \sqrt[n]{a^n+b^n+c^n} \le c\sqrt[n]{3}$$

$$\sqrt[n]{3}\to 1$$

$$\to c$$

$$(f^g)' = \left(e^{g\ln f}\right)'$$