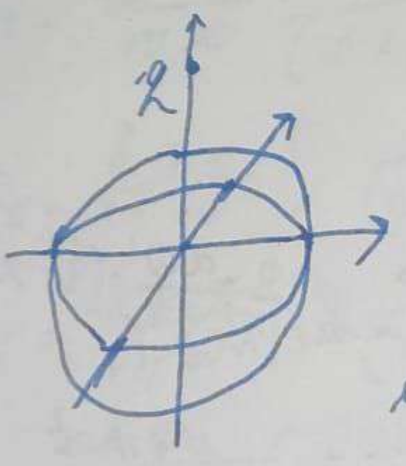
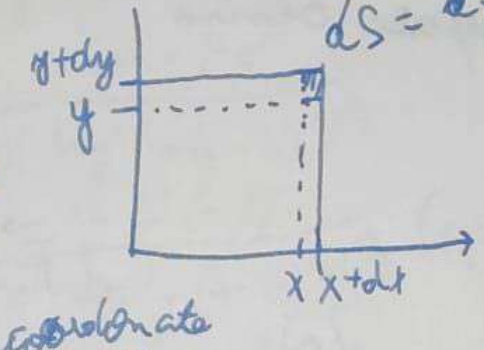
Laborator 4 Electricitate

① O cantitate de sarcină Q este distribuită sub formă de strat subțire sferic a = raza sferei. De-a lungul unui diametru al acestei sfere, se plimbă un corp punctiform cu sarcina q. Calculați forța cu care acționează sarcina Q asupra corpului punctiform cu sarcina q.

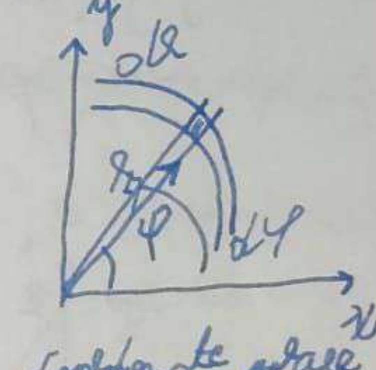
2.





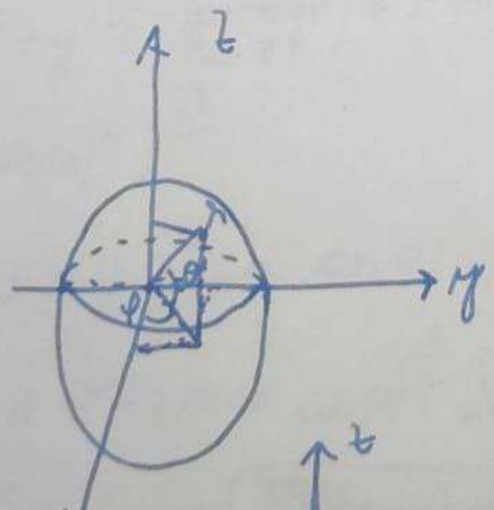
$dS = dx\,dy$

coordonate carteziene



coordonate polare

$dS = dr \cdot r\,d\varphi = r\,dr\,d\varphi$



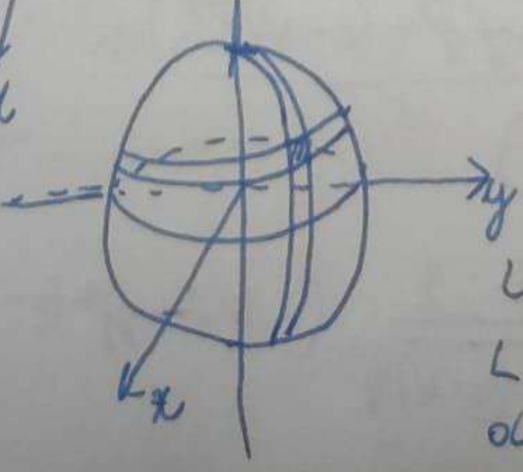
$(x, y, z) \longrightarrow (r, \theta, \varphi)$

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$

$x, y, z \in [-r, r]$



$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi]$

$L_1 = r \sin\theta\, d\varphi$

$L_2 = r\, d\theta$

$dS = r^2 \sin\theta\, d\theta\, d\varphi$

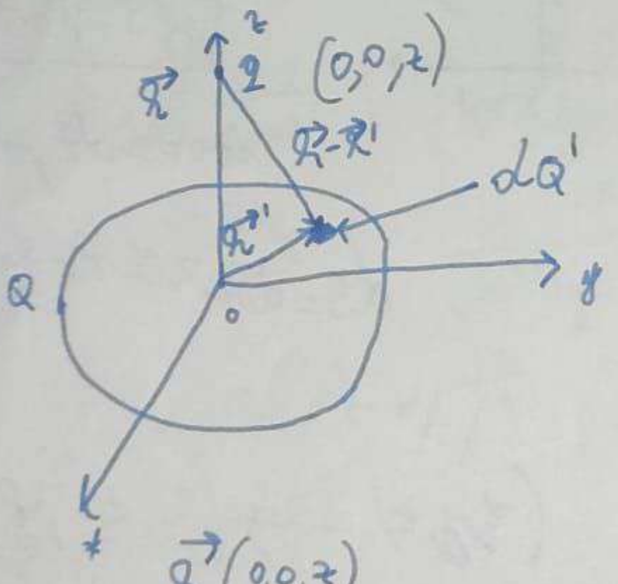
$$\sum_{i=1}^{2}\sum_{j=1}^{3} a_i b_j = \sum_{i=1}^{2}\left(a_i b_1 + a_i b_2 + a_i b_3\right) = (a_1 + a_2)(b_1 + b_2 + b_3)$$

$$= \left(\sum_{i=1}^{2} a_i\right)\left(\sum_{j=1}^{3} b_j\right) = \sum_{i=1}^{2}\sum_{j=1}^{3} a_i b_j$$

$$\Rightarrow \int\cdot\int(\text{ceva}) = \left(\int\right)\cdot\left(\int\right)$$

dacă poți separa termenii

$$\vec{r}' + \vec{t} = \vec{r}$$
$$\vec{t} = \vec{r} - \vec{r}'$$



$$d\vec{F} = k\cdot\frac{q\cdot dQ'}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

Q .... $4\pi a^2$

dQ' .... dS'

$$dQ' = \frac{Q\cdot dS'}{4\pi a^2}$$

$\vec{r}\,(0,0,z)$

$\vec{r}'\,(x',y',z')$

$\vec{r}-\vec{r}'\,(-x',-y',z-z')$

$$\boxed{\vec{r}-\vec{r}' = -x'\vec{i} - y'\vec{j} + (z-z')\vec{k}}$$

$$|\vec{r}-\vec{r}'| = \sqrt{x'^2+y'^2+(z-z')^2}$$

$$d\vec{F} = k\cdot\frac{Q\cdot\frac{dS'}{4\pi a^2}\cdot q}{\left(\sqrt{x'^2+y'^2+(z-z')^2}\right)^3}\left(-x'\vec{i} + y'\vec{j} + (z-z')\vec{k}\right)$$

$$= \frac{kQq}{4\pi a^2} \cdot \frac{dS'}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}}\left(-x'\vec{i} - y'\vec{j} + (z-z')\vec{k}\right)$$

$$\begin{cases} dF_x = \dfrac{k\cdot Qq}{4\pi a^2} \cdot \dfrac{dS'(-x')}{(x'^2+y'^2+(z-z')^2)^{3/2}} \\[2ex] dF_y = \dfrac{k\cdot Qq}{4\pi a^2} \cdot \dfrac{dS'\cdot(-y')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\[2ex] dF_z = \dfrac{k\cdot Qq}{4\pi a^2} \cdot \dfrac{dS'(z-z')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \end{cases}$$

integrala pe suprafață

$$\begin{cases} F_x = \dfrac{kQq}{4\pi a^2} \displaystyle\int \dfrac{dS'(-x')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\[2ex] F_y = \dfrac{kQq}{4\pi a^2} \displaystyle\int \dfrac{dS'(-y')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \\[2ex] F_z = \dfrac{kQq}{4\pi a^2} \displaystyle\int \dfrac{dS'(z-z')}{(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}}} \end{cases} \qquad \begin{cases} x' = a\sin\theta\cos\varphi \\ y' = a\sin\theta\sin\varphi \\ z' = a\cos\theta \end{cases}$$

$$(x'^2+y'^2+(z-z')^2)^{\frac{3}{2}} = (a^2\sin^2\theta\cos^2\varphi + a^2\sin^2\theta\sin^2\varphi +$$

$$+ (z - a\cos\theta)^2)^{\frac{3}{2}}$$

$$= (a^2\sin^2\theta + (z - a\cos\theta)^2)^{\frac{3}{2}} = (a^2\sin^2\theta + z^2 + a^2\cos^2\theta - 2az\cos\theta)^{\frac{3}{2}}$$

$$= (a^2+z^2-2az\cos\theta)^{\frac{3}{2}} = a^3\left(1+\left(\frac{z}{a}\right)^2 - 2\frac{z}{a}\cos\theta\right)^{\frac{3}{2}}$$

$$= a^3\left(\left(\frac{z}{a}\right)^2 - \cos\theta\,\frac{z}{a} + 1\right)^{\frac{3}{2}} \quad ; \text{ notăm } \frac{z}{a} = m$$

$$= a^3(m^2 - \cos\theta\, m + 1)^{\frac{3}{2}}$$

$$\begin{cases} F_x = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi}\int_0^{2\pi} \dfrac{a^2 \sin\theta \, d\theta \, d\varphi \, (-a \sin\theta \cos\varphi)}{a^3 (1+m^2-2m\cos\theta)^{\frac{3}{2}}} \\ F_y = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi}\int_0^{2\pi} \dfrac{a^2 \sin\theta \, d\theta \, d\varphi \, (-a \sin\theta \sin\varphi)}{a^3 (1+m^2-2m\cos\theta)^{\frac{3}{2}}} \\ F_z = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi}\int_0^{2\pi} \dfrac{a^2 \sin\theta \, d\theta \, d\varphi \, (z - a\cos\theta)}{a^3 (1+m^2-2m\cos\theta)^{\frac{3}{2}}} \end{cases}$$

$$\begin{cases} F_x = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi}\int_0^{2\pi} \dfrac{\sin^2\theta \, d\theta \, (-\cos\varphi \, d\varphi)}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} \\ F_y = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi}\int_0^{2\pi} \dfrac{\sin^2\theta \, d\theta \, (-\sin\varphi \, d\varphi)}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} \\ F_z = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi}\int_0^{2\pi} \dfrac{\sin\theta \, d\theta \, d\varphi \, (m-\cos\theta)}{(1+m^2-2m\cos(\theta))^{\frac{3}{2}}} \end{cases}$$

$$\begin{aligned} \Delta &= 4\cos^2\theta - 4 \\ &= 4(\cos^2\theta - 1) \\ &= 4(-\sin^2\theta) \end{aligned}$$

$$\begin{cases} F_x = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi} \dfrac{\sin^2\theta \, d\theta}{(m^2-2m\cos\theta+1)^{\frac{3}{2}}} \cdot \int_0^{2\pi} -\cos\varphi \, d\varphi \to 0 \\ F_y = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi} \dfrac{\sin^2\theta \, d\theta}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} \cdot \int_0^{2\pi} -\sin\varphi \, d\varphi \to 0 \\ F_z = \dfrac{kQq}{4\pi a^2} \displaystyle\int_0^{\pi} \dfrac{\sin\theta \, d\theta \, (m-\cos\theta)}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}} \cdot \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \end{cases}$$

$F_x = 0$

$F_y = 0$

$$F_z = \frac{kQq}{2a^2} \int_0^{\pi} \frac{(m-\cos\theta) \sin\theta \, d\theta}{(1+m^2-2m\cos\theta)^{\frac{3}{2}}}$$

$\cos\theta = x$, $dx = -\sin\theta \, d\theta$

$\theta = 0 \Rightarrow x = 1$

$\theta = \pi \Rightarrow x = -1$

$$\frac{m+1-m+1}{(m^2-1)} \ (m^2-1$$

$$\Rightarrow F_z = \frac{kQq}{2a^2}\int_{-1}^{1} \frac{(m-x) \, dx}{(1+m^2-2xm)^{\frac{3}{2}}}$$

$$F_z = \frac{kQq}{4ma^2} \int_{-1}^{1} \frac{-m^2-2mx}{(1-m^2-2mx)^{\frac{3}{2}}} dx$$

$$= \frac{kQq}{4ma^2} \int_{-1}^{1} \frac{1+m^2-2mx+m^2-1}{(1+m^2-2mx)^{\frac{3}{2}}} dx$$

$$= \frac{kQq}{4ma^2}\left[\int_{-1}^{1} \frac{1+m^2-2mx}{(1+m^2-2mx)^{3/2}} dx + \int_{-1}^{1} \frac{m^2-1}{(1+m^2-2mx)^{\frac{3}{2}}} dx\right]$$

$$= \frac{kQq}{4ma^2}\left[\int_{-1}^{1} (1+m^2-2mx)^{-\frac{1}{2}} dx + (m^2-1)\int_{-1}^{1} (1+m^2-2mx)^{-\frac{3}{2}} dx\right]$$

$$F_z = \frac{kQq}{4ma^2}\left[\frac{(1+m^2-2mx)^{\frac{1}{2}}}{\frac{1}{2}(-2m)}\Bigg|_{-1}^{1} + (m^2-1)\frac{(1+m^2-2mx)^{-\frac{1}{2}}}{-\frac{1}{2}(-2m)}\Bigg|_{-1}^{1}\right]$$

$$F_z = -\frac{kQq}{4m^2a^2}\left[(1+m^2-2mx)^{\frac{1}{2}}\Big|_{-1}^{1} - (m^2-1)(1+m^2-2mx)^{-\frac{1}{2}}\Big|_{-1}^{1}\right]$$

$$F_z = -\frac{kQq}{4m^2a^2}\left[|m-1| - |m+1| - (m^2-1)\left(\frac{1}{|m-1|} - \frac{1}{|m+1|}\right)\right]$$

$m = \frac{z}{a}$ $m > 1 \Rightarrow z > a$ deasupra mingii

$$\Rightarrow F_z = -\frac{kQq}{4m^2a^2}\left(\cancel{m} - 1 - \cancel{m} - 1 - (m^2-1)\left(\frac{1}{m-1} - \frac{1}{m+1}\right)\right)$$

$$F_z = -\frac{kQq}{4m^2a^2}\left(-2 - \cancel{m} - 1 + \cancel{m} - 1\right) \Rightarrow F_z = \frac{4kQq}{4m^2a^2} = \frac{kQq}{m^2a^2} = \frac{kQq}{z^2}$$