Seminar 6 - A. G

2) $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = (x_1 + 2x_2 + x_3, 2x_1 + 5x_2 + 3x_3, -3x_1 - 7x_2 - 4x_3)$$

ⓐ f liniară?

ⓑ Ker (f) = ?, Im (f) = ?

ⓒ Precizați câte un reper la fiecare

<u>OBS</u> $(V, +, \cdot)/\mathbb{K}$ $(W, +, \cdot)/\mathbb{K}$

$f : V \to W$ aplicație liniară $\Leftrightarrow$

$$1)\ \left.\begin{array}{l} f(x+y) = f(x) + f(y) \\ f(ax) = a f(x),\ x, y \in V \\ \qquad a \in \mathbb{K} \end{array}\right| \Leftrightarrow \begin{array}{l} f(ax + by) = a f(x) + b f(y) \\ \forall x, y \in V \\ \forall a, b \in \mathbb{K} \end{array}$$

<u>OBS</u> f liniară $\Leftrightarrow$ $f : (V, +) \to (W, +)$ morfism de grupuri

$$\mathrm{Ker}(f) = \{x \in V \mid f(x) = 0_W\} = f^{-1}(\{0_W\}) \text{ nucleul lui } f$$

$$\mathrm{Im}(f) = \{y \in W \mid \exists x \in V \text{ a.î. } f(x) = y\} \text{ imaginea lui } f$$

ⓐ $\underbrace{f(ax+by)}$

$(ax_1+by_1, ax_2+by_2, ax_3+by_3) = \cancel{(ax_1+by_1+2ax+2by_2+ax_3+by_3)}$

$= (ax_1 + by_1 + 2ax_2 + 2by_2 + ax_3 + by_3;$

$2ax_1 + 2by_1 + 5ax_2 + 5by_2 + 3ax_3 + 2by_3;$

$-3ax_1 - 3by_1 - 7ax_2 - 7by_2 - 4ax_3 - 4by_3)$

$f(ax+by) = a(x_1+2x_2+x_3, 2x_1+5x_2+3x_3, -3x_1-7x_2-4x_2) +$
$b(y_1+2y_2+y_3, 2y_1+5y_2+3y_3, -3y_1-7y_2-4y_3) =$
$= a f(x) + b f(y) \Rightarrow f$ liniară

ⓑ $\mathrm{Ker}(f) = \{x \in \mathbb{R}^3 \mid f(x) = 0\} \subset \mathbb{R}^3$

$$\begin{cases} x_1+2x_2+x_3=0 \\ 2x_1+5x_2+3x_3=0 \\ -3x_1-7x_2-4x_3=0 \end{cases} ; A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}$$

sunt pe rânduri, nu pe coloane

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{vmatrix} = -20 - 14 - 18 + 15 + 21 + 16 = 1 + 31 - 32 = 0$$

$\mathrm{Ker}(f) = S(A)$ $\quad \dim(\mathrm{Ker}(f)) = 3 - \mathrm{rg}\, A = 3 - 2 = 1$

nucleul are dimensiune

$$\begin{cases} x_1 + 2x_2 = -\alpha \\ 2x_1 + 5x_2 = -3\alpha \\ \quad x_2 = -\alpha \Rightarrow x_1 = -\alpha + 2\alpha = \alpha \end{cases}$$

$$\Rightarrow Ker(f) = \{ (\alpha, -\alpha, \alpha) / \alpha \in \mathbb{R} \} = < \underset{\overset{\shortparallel}{R_1}}{\{(1,-1,1)\}} >$$

$$Im(f) = \{ y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3 \text{ a.î. } f(x) = y \}$$

$$\begin{cases} x_1 + x_2 + x_3 = y_1 \\ 2x_1 + 5x_2 + 3x_3 = y_2 \\ -3x_1 - 7x_2 - 4x_3 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -1 & -4 \end{array}\right) \left| \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right.$$

$$\det(A) = 0 \Rightarrow \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \neq 0$$

$$\Rightarrow \boxed{rg\, A = rg\, \bar{A}} \qquad \Delta_C = \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} = 0$$

$$\Delta_C = \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ 0 & 0 & y_1+y_2+y_3 \end{vmatrix} = (y_1 + y_2 + y_3) \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \Rightarrow y_1 + y_2 + y_3 = 0$$

$$Im(f) = \{ y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = 0 \}$$

$$\dim Im(f) = 3 - 1 = 2$$

$$\boxed{\dim V = \dim[Ker(f)] + \dim[Im(f)]}$$

Teoremă

în cazul nostru $\dim \mathbb{R}^3 = 1 + 2 = 3$ ④

M_2 Im(f), reper pentru imaginea lui f

$\mathcal{R}_1 = \{(1,-1,1)\}$ reper în Ker(f)

Extindem $\mathcal{R}_1$ la un reper cu dimensiunea 3

$$rg\begin{pmatrix} & e_1 & e_3 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = (\text{maxim})\, 3 \; ; \; \mathcal{R}_1 \cup \{e_1, e_3\} \text{- reper în } \mathbb{R}^3$$

$\{f(e_1), f(e_3)\}$ reper în Im(f) (C 6)

$$f(e_1) = f(1,0,0) = (1,2,-3) = (1+2\cdot 0+3\cdot 0, 2+5\cdot 0+3\cdot 0, -3-7\cdot 0-4$$

$$f(e_3) = f(0,0,1) = (1,3,-4)$$

$$= (0+0+1, 0+0+1, -3\cdot 0-7\cdot 0-4)$$

$\Rightarrow \{(1,2,-3),(1,3,-4)\}$ reper în Im(f)

OBS $f(X) = y \iff Y = AX$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

matricea se scrie pe linie

SKIBIDI TOILET

① $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, -x_2)$

$f \in Aut(\mathbb{R}^2)$ (f liniară + bijectivă)

$f(x) = y \Leftrightarrow Y = AX \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$

$\Rightarrow \left.\begin{matrix} \text{f liniară} \\ \det A = 1 \neq 0 \end{matrix} \Rightarrow \text{f bij} \right\} \Rightarrow$ f izomorfism intre aceleași

$\Rightarrow$ f automorfism

③ $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x_1, x_2) = (3x_1 - 2x_2, 2x_1 - x_2, -x_1 + x_2)$

ⓐ liniară $\quad f(x) = y \Leftrightarrow Y = AX$

ⓑ bijectivă

ⓒ $Im(f) = ?$ $\quad \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$ f liniară

$f: V \to W$ liniară

① f inj $\Leftrightarrow Ker(f) = \{0_V\} \Leftrightarrow rg A = \dim V$

② f surj $\Leftrightarrow \dim(Im(f)) = \dim W$

ⓑ $\mathrm{Ker}(f) = \{ x \in \mathbb{R}^2 \mid f(x) = 0_{\mathbb{R}^3} \} \Rightarrow \begin{cases} 3x_1 - 2x_2 = 0 \\ 2x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases}$

$$A = \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \quad \begin{array}{l} \mathrm{rg}\, A = 2 \text{ (maxim)} \\ \mathrm{rg}\, \bar{A} = 2 \\ \Rightarrow SCD \end{array} \quad = \{(0,0)\} \Rightarrow f \text{ inj}$$

ⓒ $\mathrm{Im}(f) = \{ y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^2 \text{ a.î } f(x) = y \}$

$$\begin{cases} 3x_1 - 2x_2 = y_1 \\ 2x_1 - x_2 = y_2 \\ -x_1 + x_2 = y_3 \end{cases} \qquad A = \left(\begin{array}{cc|c} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{array}\right)$$

$$\Delta c = 0 = \begin{vmatrix} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{vmatrix}$$

$= -3y_3 - 4y_1 + 2y_2 + y_1 - 3y_2 + 4y_3$

$= y_3 - 3y_1 - y_2 = 0$

$\Rightarrow \mathrm{Im}(f) = \{ y \in \mathbb{R}^3 \mid$ ~~[illegible]~~ $y_1 - y_2 + y_3 = 0 \}$

Teoremă $f: V \to W$ liniară

$\dim V = \dim \operatorname{Ker}(f) + \dim \operatorname{Im}(f)$

f inj $\Rightarrow \dim V = \dim \operatorname{Im}(f)$

OBS: $f: V \to W$ $\quad \dim W = m$; $\dim V = n$

$\mathcal{R}_1 = \{e_1, e_2, \dots, e_n\} \xrightarrow{A} \mathcal{R}_2 = \{\bar{e}_1, \dots, \bar{e}_m\}$

$A = [f]_{\mathcal{R}_1, \mathcal{R}_2}$; A matricea asociată lui $\mathcal{R}_1, \mathcal{R}_2$

$$f(e_i) = \sum_{j=1}^{m} a_{ji} \bar{e}_j, \quad i = \overline{1,n}$$

⑧ $f: \overset{\overset{\mathbb{R}^2}{\parallel}}{\mathbb{R}_1[X]} \to \mathbb{R}^3$, $f(aX + b) = (a, b, a + b)$

$\mathcal{R} = \{2X - 1, -X + 1\}$, $\mathcal{R}' = \{(1,1,1), (1,1,0), (1,0,0)\}$

repere în $\mathbb{R}_1[X]$, respectiv $\mathbb{R}^3$

ⓐ f liniară

ⓑ $[f]_{\mathcal{R}, \mathcal{R}'} = A = ?$

ⓒ $\operatorname{Ker}(f)$, $\operatorname{Im}(f)$

$f(ax+b) = (a, b, a+b)$

$\Rightarrow \{2x-1, -x+1\}$, $\mathcal{R}' = \{(1,1,1), (1,1,0), (1,0,0)\}$

baze in $\mathbb{R}_1[X]$,

$$\mathcal{R} = \{\underset{e_1''}{2x-1}, \underset{e_2''}{-x+1}\} \xrightarrow[\mathcal{R}']{A} \{(1,1,1),(1,1,0),(1,0,0)\}$$

$f(e_1) = f(2x-1) = (2,-1,1) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$$A = [f]_{\mathcal{R}\mathcal{R}'} = \begin{pmatrix} a & a' \\ b & b' \\ c & c' \end{pmatrix}$$

$$\begin{cases} a+b+c=2 \\ a+b=-1 & \Rightarrow b=-2 \\ a=1 & -1+c=2 \Rightarrow c=3 \end{cases}$$

$f(e_2) = f(-x+1) = (-1,1,0) = a'(1,1,1) + b'(1,1,0) + c'(1,0,0)$

$$\begin{cases} a'+b'+c'=-1 \\ a'+b'=1 \\ \boxed{a'=0} \end{cases} \Rightarrow \boxed{b'=1} \quad \boxed{c'=-2} \Rightarrow A = [f]_{\mathcal{R}\mathcal{R}'} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 3 & 2 \end{pmatrix}$$

$\mathcal{R}_0 = \{e_1 = \overset{\overset{1}{\parallel}}{(1,0)}, e_2 = \overset{X}{(0,1)}\}$

$\mathcal{R}_0' = \{\bar{e_1} = (1,0,0), \bar{e_2} = (0,1,0), \bar{e_3} = (0,0,1)\}$

$$f(x) = y \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

ⓒ $\mathrm{Ker}(f) = ?$; $\mathrm{Im}(f) = ?$

$\mathrm{Ker}(f) = \{P \in \mathbb{R}_1[X] / f(P) = 0_{\mathbb{R}^3}\} = \{0\}$

$f(aX + b) = 0_{\mathbb{R}^3} \Rightarrow a = 0, b = 0, c = 0$
$\Rightarrow P = 0$

$\mathrm{Im}(f) = \{y \in \mathbb{R}^3 / \exists P \in \mathbb{R}_1[X]$ a.î $f(P) = y\}$

$$\begin{cases} a = y_1 \\ b = y_2 \\ a + b = y_3 \Rightarrow y_1 + y_2 = y_3 \quad y_1 + y_2 - y_3 = 0 \end{cases}$$

$\mathrm{Im}(f) = \{y \in \mathbb{R}^3 \mid y_1 + y_2 - y_3 = 0\} =$

$= \langle\{(y_1, y_2, y_1 + y_2)\}\rangle$

$\Rightarrow \mathrm{Im}(f) = \langle\{(1,0,1), (0,1,1)\}\rangle$