C5 - AG 16 31.10.2024

- sp. vectoriale.
- SLI / SLD / SG / Bază / Reper

$(V,+,\cdot)/\mathbb{K}$ $\quad S=\{e_1,\dots,e_n\}\subset V$

- S este SLI $\iff [\forall a_1,\dots,a_n\in\mathbb{K}$ a.î. $a_1e_1+\dots+a_ne_n=0_V \Rightarrow a_1=\dots=a_n=0_{\mathbb{K}}.]$
- S este SLD $\iff [\exists a_1,\dots,a_n\in\mathbb{K}$ nu toți nuli a.î. $a_1e_1+\dots+a_ne_n=0_V]$
- S este SG pt $V \iff V=\langle S\rangle$

$\forall x\in V,\ \exists a_1,\dots,a_n\in\mathbb{K}$ a.î. $x=a_1e_1+\dots+a_ne_n$.

- S bază $\iff \begin{cases} S \text{ SLI}\\ S \text{ e SG.}\end{cases}$

V sp. vect. <u>finit generat</u> dacă $\exists S$ m. finită a.î. $V=\langle S\rangle$

V sp. v. finit generat, $B=\{e_1,\dots,e_n\}$ bază

$\dim_{\mathbb{K}} V=\operatorname{card} B$ (invariant)

$R=\{e_1,\dots,e_n\}$ reper $\iff$ bază ordonată.

$\forall x\in V,\ \exists!\ x_1,\dots,x_n\in\mathbb{K}$ a.î. $x=x_1e_1+\dots+x_ne_n$.

$(x_1,\dots,x_n)$ coordonatele lui x în raport cu reperul R

$R=\{e_1,\dots,e_n\}\xrightarrow{A} R'=\{e_1',\dots,e_n'\}$ repere

$$e_i'=\sum_{j=1}^{n} a_{ji}e_j,\ \forall i=\overline{1,n} \qquad X=AX'$$

$$\begin{cases} e_1'=a_{11}e_1+a_{21}e_2+\dots+a_{n1}e_n\\ \vdots\\ e_n'=a_{1n}e_1+a_{2n}e_2+\dots+a_{nn}e_n\end{cases} \qquad X=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix},\ X'=\begin{pmatrix}x_1'\\ \vdots\\ x_n'\end{pmatrix}$$

$$x=x_1'e_1'+\dots+x_n'e_n'$$

$\dim_{\mathbb{K}} V = n$

n = nr max de vect SLI

n = nr min de vect SG.

- $\left.\begin{matrix} S \text{ SLI} \\ S' \subset S \end{matrix}\right| \Rightarrow S' \text{ SLI}$

- S SLD $\Rightarrow$ $S \cup \{x\}$ SLD, $x \notin S$

- S SG $\Rightarrow$ $S \cup \{x\}$ SG, $x \notin S$

<u>OBS</u> V n-dim, $R = \{e_1, .., e_n\}$, $|R| = n$

UAE
1) R e bază
2) R e SLI
3) R e SG

<u>Crit LI</u> V n-dim, $S = \{v_1, .., v_m\} \subset V$, $m \leq n$.

S este SLI $\Leftrightarrow$ matricea comp. vect din S în rap cu $\forall$ reper are rg maxim $= m$.

- subsp. vect

$V' \subset V$ subsp $\Leftrightarrow \left[\begin{matrix} 1) \forall x, y \in V' \Rightarrow x + y \in V' \\ 2) \forall x \in V', \forall a \in \mathbb{K} \Rightarrow ax \in V' \end{matrix}\right] \Leftrightarrow$

$\forall x, y \in V', \forall a, b \in \mathbb{K} \Rightarrow ax + by \in V'$

$V_1, V_2 \subseteq V$ ssp vect $\Rightarrow$ $V_1 \cap V_2$ e ssp vect.

În general, $V_1 \cup V_2$ nu e ssp vect

$V_1 + V_2 = \langle V_1 \cup V_2 \rangle = \{x_1 + x_2 \mid x_1 \in V_1, x_2 \in V_2\}$

$V_1 \oplus V_2 \iff V_1 \cap V_2 = \{0_V\}$

$\iff \forall x \in V_1 + V_2, \exists!\ \begin{matrix} x_1 \in V_1 \\ x_2 \in V_2 \end{matrix}$ aî $x = x_1 + x_2$.

Dc $V = \underbrace{V_1}_{} \oplus \underbrace{V_2}_{}$, $\begin{matrix} V_2 \\ V_1 \end{matrix}$ $\begin{matrix} \text{ssp. complementar lui} \\ \text{—//—} \end{matrix}$ $\begin{matrix} V_1 \\ V_2. \end{matrix}$

V_1 dat $R_1 = \{e_1, \dots, e_m\}$ $\quad \dim V_1 = m$

$\dim V_2 = n - m$.

Extindem R_1 la un reper R în V.
(adaugăm $n-m$ vect aî R e SLI)

$R = R_1 \cup R_2$, $V_2 = \langle R_2 \rangle$

V_2 nu e unic

Teoremă $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$.

Aplicație

$(\mathbb{R}[X], +, \cdot)$ $\quad P = a_0 + a_1 X + a_2 X^2 + \dots$

$\mathbb{R}[X] = V_1 \oplus V_2$

$P = P_1 + P_2$ $\quad P = \overbrace{\frac{P(X) + P(-X)}{2}}^{P_1} + \overbrace{\frac{P(X) - P(-X)}{2}}^{P_2}$.

$P(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 + \dots$

$P(-X) = a_0 - a_1 X + a_2 X^2 - a_3 X^3 + a_4 X^4 + \dots$

$P(X) + P(-X) = 2(a_0 + a_2 X^2 + a_4 X^4 + \dots)$

$P(X) - P(-X) = 2(a_1 X + a_3 X^3 + \dots)$

$V_1 = \{a_0 + a_2 X^2 + \dots\}$ $\quad V_2 = \{a_1 X + a_3 X^3 + \dots\}$

$V_1 + V_2 \subseteq \mathbb{R}[X]$ (din def) $\Big| \Rightarrow \mathbb{R}[X] = V_1 + V_2$

$\mathbb{R}[X] \subseteq V_1 + V_2$

$V_1 \cap V_2 \ni P$ $\quad a_0 + a_2X^2 + \ldots = a_1X + a_3X^3 + \ldots$

$a_0 - a_1X + a_2X^2 - a_3X^3 + \ldots = 0 \Rightarrow a_k = 0$, $k \geq 0$

$\Rightarrow P = 0$

$\mathbb{R}[X] = V_1 \oplus V_2$.

Ex $(\mathbb{R}^3, +, \cdot)$ $\quad V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid \underbrace{x + 2y - z = 0}_{z = x+2y}\} = S(A_1)$

$A_1 = (1 \ 2 \ -1)$

a) $\mathcal{R}_1$ reper în V_1

b) $\mathbb{R}^3 = V_1 \oplus V_2$, $V_2 = ?$

c) $x = (-1, 2, 0)$ să se descompună în rap cu $\mathbb{R}^3 = V_1 \oplus V_2$

$\dim V_1 = 3 - rg A_1 = 3 - 1 = 2$

$V_1 = \{(x, y, x+2y) \mid x, y \in \mathbb{R}\}$

$\quad = x(1,0,1) + y(0,1,2)$

$\mathcal{R}_1 = \{(1,0,1), (0,1,2)\}$ SG

$\dim V_1 = |\mathcal{R}_1| = 2$

$\Longrightarrow \mathcal{R}_1$ reper în V_1

Extindem $\mathcal{R}_1$ la un reper în $\mathbb{R}^3$

$rg\begin{pmatrix} 1 & 0 & \boxed{1} \\ 0 & 1 & \boxed{0} \\ 1 & 2 & \boxed{0} \end{pmatrix} = 3$ (max)

$\mathcal{R}_2 = \{(1,0,0)\}$ reper în V_2

$V_2 = \langle \mathcal{R}_2 \rangle$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper în $\mathbb{R}^3$

c) $x = (-1, 2, 0) = \underset{V_1}{x_1} + \underset{V_2}{x_2}$ (with $x_1 \in V_1$, $x_2 \in V_2$)

$(-1, 2, 0) = \underbrace{a(\underline{1,0,1}) + b(\underline{0,1,2})}_{x_1} + \underbrace{c(\underline{1,0,0})}_{x_2} = (a+c, b, a+2b)$

$a + c = -1$ $\quad c = -1 + 4 = 3$

$b = 2$

$a + 2b = 0 \Rightarrow a = -4$

$\begin{cases} x_1 = -4(1,0,1) + 2(0,1,2) = (-4, 2, 0) \\ x_2 = (3, 0, 0) \end{cases}$

Teorema Grassmann

$(V,+,\cdot)/_{\mathbb{K}}$ sp. vect. finit generat și $V_1, V_2 \subseteq V$ subspații vect.

$\Rightarrow \dim(V_1+V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

Dem

Fie $B_0 = \{e_1, \dots, e_p\}$ bază în $V_1 \cap V_2$, $\dim(V_1 \cap V_2) = p$.

Fie $\dim V = n$, $\dim V_1 = n_1$, $\dim V_2 = n_2$.

Extindem B_0 la o bază în V_1

$B_1 = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}\}$.

Extindem B_0 la o bază în V_2

$B_2 = \{e_1, \dots, e_p, g_{p+1}, \dots, g_{n_2}\}$.

Arătăm că $B = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}, g_{p+1}, \dots, g_{n_2}\}$ este bază în $V_1 + V_2$.

1) B este SLI

$$\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k = 0_V$$

$$x = \underbrace{\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j}_{\in V_1} = \underbrace{-\sum_{k=p+1}^{n_2} c_k g_k}_{\in V_2} \in V_1 \cap V_2 = \langle B_0 \rangle$$

$$x \in V_1 \cap V_2 \Rightarrow x = \sum_{i=1}^{p} a_i' e_i$$

a) $\displaystyle\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j = \sum_{i=1}^{p} a_i' e_i \Rightarrow \sum_{i=1}^{p} (a_i - a_i') e_i + \sum_{j=p+1}^{n_1} b_j f_j = 0_V$

$$\xrightarrow{B_1 \text{ SLI}} \begin{cases} a_i - a_i' = 0, & \forall i = \overline{1,p} \\ b_j = 0 & \forall j = \overline{p+1, n_1} \end{cases} \quad (*)$$

b) $\displaystyle -\sum_{k=p+1}^{n_2} c_k g_k = \sum_{i=1}^{p} a_i' e_i \Rightarrow \sum_{i=1}^{p} a_i' e_i + \sum_{k=p+1}^{n_2} c_k g_k = 0_V \xRightarrow{B_2 \text{ SLI}}$

$\Rightarrow \begin{cases} a_i' = 0, & \forall i = \overline{1,p} \\ c_k = 0, & k = \overline{p+1, n_2} \end{cases} \quad (**)$ Din $(*)$, $(**)$ $\begin{cases} a_i = 0, & i = \overline{1,p} \\ b_j = 0 & j = \overline{p+1, n_1} \\ c_k = 0 & k = \overline{p+1, n_2} \end{cases}$

2) $\mathcal{B} = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}, g_{p+1}, \dots, g_{n_2}\}$ SG pt V_1+V_2

$\forall x \in V_1 + V_2$, $\exists \underset{\underset{\langle B_1 \rangle}{\overset{\shortparallel}{V_1}}}{x_1 \cap}, \underset{\underset{\langle B_2 \rangle}{\overset{\shortparallel}{V_2}}}{x_2 \cap}$ a.î. $x = x_1 + x_2$

$$x_1 = \sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j$$

$$x_2 = \sum_{i=1}^{p} a'_i e_i + \sum_{k=p+1}^{n_2} c_k g_k$$

$$x = \sum_{i=1}^{p} (a_i + a'_i) e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k \in \langle \mathcal{B} \rangle$$

$\Rightarrow \mathcal{B}$ SG

(1), (2) $\Rightarrow \mathcal{B}$ este bază în $V_1 + V_2$

$$\dim(V_1+V_2) = |\mathcal{B}| = \cancel{p} + n_1 - \cancel{p} + n_2 - p = n_1 + n_2 - p$$

$$= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

În particular

$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$.

Prop $(V, +, \cdot)/\mathbb{K}$, $A \in \mathcal{M}_{m,n}(\mathbb{K})$

$$S(A) = \{ (x_1, \dots, x_n) \in \mathbb{K}^n \mid \underset{(m,n)}{A} \underset{(n,1)}{X} = \underset{(m,1)}{0} \}$$

$S(A) \subset \mathbb{K}^n$ subsp vect

$\dim(S(A)) = n - \operatorname{rg} A$

Ex $(\mathbb{R}^4,+,\cdot)|\mathbb{R}$, $V' = \langle \{ \overset{u}{(1,1,0,0)}, \overset{v}{(1,0,1,-1)} \} \rangle$.

a) Să se descrie V' printr-un sistem de ec. liniare

b) Det V'' a.î. $\mathbb{R}^4 = V' \oplus V''$

c) Să se descompună $x = (1,1,2,1)$ în rap. cu $\mathbb{R}^4 = V' \oplus V''$

a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$ $rg A = 2 = \max$, $\mathcal{R}' = \{u, v\}$ SLI $\Big| \Rightarrow V' = \langle \mathcal{R}' \rangle$

$\mathcal{R}'$ este reper pt V'.

b) $\forall x \in V' \Rightarrow \exists a, b \in \mathbb{R}$ a.î. $x = au + bv$

$$(x_1, x_2, x_3, x_4) = a(1,1,0,0) + b(1,0,1,-1) = (a+b, a, b, -b)$$

$$(*)\begin{cases} a+b = x_1 \\ a = x_2 \\ b = x_3 \\ -b = x_4 \end{cases} \quad \bar{A'} = \left(\begin{array}{cc|c} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & -1 & x_4 \end{array}\right) \quad \Delta_p = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

$(*)$ SCD $\Leftrightarrow rg \bar{A'} = 2 \Leftrightarrow \begin{cases} \Delta_{c_1} = 0 \\ \Delta_{c_2} = 0 \end{cases}$

$$\Delta_{c_1} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \end{vmatrix} = 0 \quad \begin{vmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 - x_1 \\ 0 & 1 & x_3 \end{vmatrix} = 0 \Rightarrow -x_3 - x_2 + x_1 = 0$$

$$\Delta_{c_2} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & -1 & x_4 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 - x_1 \\ 0 & -1 & x_4 \end{vmatrix} = 0 \Rightarrow -x_4 + x_2 - x_1 = 0.$$

$$V' = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x_1 - x_2 - x_3 = 0 \\ x_1 - x_2 + x_4 = 0 \end{cases} \right\}$$

b) Extindem $\mathcal{R}' = \{u, v\}$ la un reper în $\mathbb{R}^4$

$$\det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \neq 0 \; (rg \text{ maxim} = 4) \quad V'' = \langle \underbrace{\{(0,0,0,1), (1,0,0,0)\}}_{\mathcal{R}''} \rangle$$

$\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$ reper în $\mathbb{R}^4$.

c) $x = (1,1,2,1) = \underset{V'}{\underset{\cap}{x'}} + \underset{V''}{\underset{\cap}{x''}}$ $\quad R = R' \cup R''$
$= \{(1,1,0,0), (1,0,1,-1), (0,0,0,1), (1,0,0,0)\}$.

$$(1,1,2,1) = \underbrace{a(1,1,0,0)+b(1,0,1,-1)}_{x'} + \underbrace{c(0,0,0,1)+d(1,0,0,0)}_{x''}$$

$(1,1,2,1) = (a+b+d, a, b, -b+c)$

$$\begin{cases} a+b+d = 1 \Rightarrow d = 1-3 = -2 \\ a = 1 \\ b = 2 \\ -b+c = 1 \Rightarrow c = 3 \end{cases}$$

$x' = (1,1,0,0) + 2(1,0,1,-1) = (3,1,2,-2)$

$x'' = 3(0,0,0,1) + (-2)(1,0,0,0) = (-2,0,0,3)$

Ex (temă) $(\mathbb{R}^4, +, \cdot)_{/\mathbb{R}}$, $V' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y-z-3t=0\}$
$V'' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y+z+2t=0\}$

Dem că $\mathbb{R}^4 = V' + V''$, dar nu $\oplus$.

OBS $\left.\begin{array}{l} V' \subset V \text{ subsp v.} \\ \dim V' = \dim V \end{array}\right| \Rightarrow V' = V$.

Prop $(V, +, \cdot)_{/\mathbb{K}}$, $V' \subset V$ subsp vect.
Coord. vect din V' în rap. cu $\forall$ reper reprezintă sol. unui sistem liniar și omogen.
$\exists A$ aî $V' = S(A)$

Morfisme de spații vectoriale (aplicații liniare)

Def $(V_1,+,\cdot)/\mathbb{K}_1$, $(V_2,+,\cdot)/\mathbb{K}_2$ sp. vectoriale

$f: V_1 \to V_2$ s.n. aplicație semiliniară $\Leftrightarrow$

1) $f(x+y) = f(x) + f(y)$, $\forall x, y \in V_1$

2) $\exists\ \theta: \mathbb{K}_1 \to \mathbb{K}_2$ izomorfism de corpuri

$f(\alpha x) = \theta(\alpha) \cdot f(x)$, $\forall \alpha \in \mathbb{K}_1$, $\forall x \in V_1$

Dacă $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$, $\theta: \mathbb{K} \to \mathbb{K}$, $\theta = id_{\mathbb{K}}$, at f s.n. aplicație liniară

$\boxed{\text{OBS}}$ 1) $(V_i,+,\cdot)/\mathbb{R}$, $i=\overline{1,2}$

$\theta: \mathbb{R} \to \mathbb{R}$, θ automorfism $\Rightarrow \theta(x) = x$, $\forall x \in \mathbb{R}$.

(θ - izomorfism de corpuri)

$f: V_1 \to V_2$ apl. semiliniară $\Rightarrow f$ liniară

2) $(\mathbb{C}^n,+,\cdot)/\mathbb{C}$, $\theta: \mathbb{C} \to \mathbb{C}$ automorfism.

$\theta(z) = \overline{z}$

$f: \mathbb{C}^n \to \mathbb{C}^n$, $f(z_1,..,z_n) = (\overline{z_1},..,\overline{z_n})$ f este semiliniară

a) $f(z+w) = f(z_1+w_1,..,z_n+w_n) = (\underset{\overline{z_1}+\overline{w_1}}{\overline{z_1+w_1}}, ..., \underset{\overline{z_n}+\overline{w_n}}{\overline{z_n+w_n}}) =$

$= (\overline{z_1},..,\overline{z_n}) + (\overline{w_1},..,\overline{w_n}) = f(z) + f(w)$

b) $f(\alpha z) = (\overline{\alpha z_1},..,\overline{\alpha z_n}) = \overline{\alpha}\,(\overline{z_1},..,\overline{z_n}) = \theta(\alpha) f(z)$

$(V_i,+,\cdot)/\mathbb{K}$, $i=\overline{1,2}$ sp vect

• $f: V_1 \to V_2$

apl. liniară $\Leftrightarrow$ 1) $f(x+y) = f(x) + f(y)$

2) $f(\alpha x) = \alpha f(x)$, $\forall x, y \in V_1$, $\forall \alpha \in \mathbb{K}$.

f s.n. izomorfism de sp vect $\Leftrightarrow$ 1) apl lin

2) f bij

f s.n. automorfism de sp vect $\Leftrightarrow$ $f: V \to V$ izom. de sp vect.

Not $\quad End(V) = \{ f : V \longrightarrow V \mid f \text{ liniară} \}$

$\quad Aut(V) = \{ f \in End(V) \mid f \text{ bij} \}$.

<u>OBS</u>

a) $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$, $g \circ f : V_1 \to V_3$ $\quad f, g$ liniare $\Rightarrow g \circ f$ liniară

b) $f : V_1 \longrightarrow V_2$ apl. liniară $\Rightarrow f : (V_1, +) \longrightarrow (V_2, +)$ morfism de grupuri

$f(0_{V_1}) = 0_{V_2}$.

<u>Exemple de apl. liniare</u>

1) $f : V \to V$, $\quad f(x) = x$ sau $f(x) = 0_V$

2) $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\quad f(x) = y$, unde $Y = AX$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

3) $f : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$, $\quad f(A) = Tr(A)$

$Tr(A+B) = Tr(A) + Tr(B)$

$Tr(\alpha A) = \alpha Tr(A)$, $\forall A, B \in \mathcal{M}_n(\mathbb{R})$, $\forall \alpha \in \mathbb{R}$

<u>OBS</u> $\quad f(A) = det(A)$ <u>NU</u> e apl. liniară.

4) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\quad f(x_1, x_2) = (x_1 + 2x_2, x_1 - x_2)$ ✓

$f(x_1, x_2) = (1 + x_1, 2 - x_2)$ NU

$f(x_1, x_2) = (x_1^2, x_1 x_2)$ NU

$f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ ✓

<u>Prop de caract. a apl. liniare</u>

$f : V_1 \to V_2$ apl lin $\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, $\forall x, y \in V_1$, $\forall \alpha, \beta \in K$

"⇒" I_p: f liniară

$f(x+y)=f(x)+f(y)$

$f(\alpha x)=\alpha f(x)$

$f(\alpha x+\beta y)=f(\alpha x)+f(\beta y)$
$=\alpha f(x)+\beta f(y)$

"⇐" I_p: $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$, $\forall x,y\in V_1$, $\forall \alpha,\beta\in \mathbb{K}$

1. $\alpha=\beta=1_{\mathbb{K}}$.
$f(1_{\mathbb{K}}\cdot x+1_{\mathbb{K}}\cdot y)=1_{\mathbb{K}}f(x)+1_{\mathbb{K}}f(y) \rightarrow f(x+y)=f(x)+f(y)$

2. $\beta=0_{\mathbb{K}}$
$f(\alpha x+0)=\alpha f(x)+0=\alpha f(x)$

<u>OBS</u> $f:V_1\longrightarrow V_2$ apl. liniară

Dacă $V'\subseteq V_1$ ssp vect, at $f(V')\subseteq V_2$ ssp. vect.

<u>Dem</u> $\forall y_1,y_2\in f(V') \Rightarrow ay_1+by_2\in f(V')$
$\forall a,b\in\mathbb{K}$

$\exists x_1,x_2\in V'$ aî $y_1=f(x_1)$, $y_2=f(x_2)$

$ay_1+by_2=af(x_1)+bf(x_2)=f(\underbrace{ax_1+bx_2}_{\in V' \text{ (ssp. vect)}})\in f(V')$

<u>Def</u> $f:V_1\rightarrow V_2$ apl. liniară

$Ker(f)=\{x\in V_1 \mid f(x)=0_{V_2}\}=f^{-1}(\{0_{V_2}\})$ nucleul lui V_1

$Im(f)=\{y\in V_2 \mid \exists x\in V_1 \text{ aî } f(x)=y\}$

<u>Prop</u> $Ker(f)\subseteq V_1$ subspvect

$Im(f)\subseteq V_2$ —"—