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Seminar 5 A.G

Repere. Coordonate. Funcții în subspații vectoriale

$(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

$\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reper canonic

$\mathcal{R}' = \{e_1' = e_1 + 2e_2 + e_3,\ e_2' = e_1 + 7e_2 + e_3,\ e_3' = -e_1 + e_2 + e_3\}$

a) $\mathcal{R}'$ reper în $\mathbb{R}^3$, $\mathcal{R}_0 \xrightarrow{A} \mathcal{R}'$, $A = ?$, matricea de trecere

b) $x = (3,2,1)$ coord lui x, în raport cu $\mathcal{R}'$

a) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\mathcal{R}'$ SLI $\Leftrightarrow$ rang(A) maxim

$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \Rightarrow \mathcal{R}'$ SLI

$\dim \mathbb{R}^3 = 3;\ |\mathcal{R}'| = 3 \Rightarrow \mathcal{R}'$ SLI $\Leftrightarrow \mathcal{R}'$ S.G.

$e_i' = \sum_{j=1}^{n} a_{ji} e_j$ (coloana i)

b) $x = (3,2,1) = x_1' \cdot e_1' + x_2' e_2' + x_3' e_3'$

$= x_1'(1,2,1) + x_2'(1,7,1) + x_3'(-1,1,1)$

$(x_1' + x_2' - x_3',\ 2x_1' + 7x_2' + x_3',\ x_1' + x_2' + x_3') = (3,2,1)$

$$\begin{cases} x_1'+x_2'-x_3'=3 \\ 2x_1'+7x_2'+x_3'=2 \\ x_1'+x_2'+x_3'=1 \end{cases} \qquad \begin{vmatrix} 1 & 1 & -1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 7-2+1+7-1-2 = 14-5 = 9$$

$$E_3+E_1 = \begin{cases} x_1'+x_2'=2 \quad | -2 \\ 2x_1'+7x_2'=3 \end{cases} \qquad x_1' = \frac{11}{5}$$

$$\oplus \quad 5x_2' = -1 \rightarrow x_2' = -\frac{1}{5}$$

$$(x_1', x_2', x_3') = \left(\frac{11}{5}, -\frac{1}{5}, -1\right)$$

OBS $X = A \cdot X'$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

A matricea de trecere

$$X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

② $(\mathbb{R}_2[X], +, \cdot)|_{\mathbb{R}}$, $\mathcal{R}_0 = \{e_1 = 1, e_2 = X, e_3 = X^2\}$ reperul canonic

$\mathcal{R}' = \{-1+2x+7x^2, x-x^2, x-2x^2\}$

a) $\mathcal{R}'$ reper? ; $\mathcal{R}_0 \xrightarrow{A} \mathcal{R}'$, $A = ?$

b) coordonatele $P = 3 - x + x^2$ în $\mathcal{R}'$

(a) $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & -2 \end{pmatrix}$ $\det A = 1 \neq 0 \Rightarrow \operatorname{rg} A = 3$ (maxim) $\Rightarrow$ SLI $\Rightarrow$

$\mathcal{R}'$ SG

$|\mathcal{R}'| = \dim \mathbb{R}_2(x) = 3$

$\Rightarrow \mathcal{R}'$ reper

b) $3 - x + x^2 = a_1' e_1' + a_2' e_2' + a_3' e_3'$

$3 - x + x^2 = a_1'(-1 + 2x + 3x^2) + a_2'(x - x^2) + a_3'(x - 2x^2)$

$$\begin{cases} 3 = -a_1' \Rightarrow a_1' = -3 \\ -1 = 2a_1' + a_2' + a_3' \Rightarrow a_2' + a_3' = -5 \\ 1 = 3a_1' - a_2' - 2a_3' \Rightarrow -a_2' - 2a_3' = 10 \end{cases} \quad (+)$$

$-a_3' = 5 \Rightarrow a_3' = -5$

$\Rightarrow a_2' = 0$

$\Rightarrow$ coordonatele lui p în raport cu $\mathcal{R}'$ $(-3, 0, -5)$

$(V^3, +, \cdot)|_{\mathbb{R}}$, 3-dim

$\mathcal{R} = \{v_1, v_2, v_3\}$ reper în V^3

$\mathcal{R}' = \{v_1' = v_1, v_2' = v_1 + v_2, v_3' = v_1 + v_2 + v_3\}$

a) $\mathcal{R}'$ reper, $\mathcal{R} \xrightarrow{A} \mathcal{R}'$, $A = ?$

b) $v \in V$ coord (x_1, x_2, x_3) în raport cu $\mathcal{R}$

(x_1', x_2', x_3') în raport cu $\mathcal{R}'$

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; rang $A = 3$ (max) $\Rightarrow \mathcal{R}'$ SLI

$|\mathcal{R}'| = 3$; $\dim V^3 = 3 \Rightarrow \mathcal{R}'$ SLI $\Leftrightarrow \mathcal{R}'$ SG

$\Rightarrow \mathcal{R}'$ reper

b) $x = x_1 v_1 + x_2 v_2 + x_3 v_3 = x_1' v_1' + x_2' v_2' + x_3' v_3'$

$= x_1'(v_1) + x_2'(v_1 + v_2) + x_3'(v_1 + v_2 + v_3)$

$= v_1(x_1' + x_2' + x_3') + v_2(x_2' + x_3') + v_3 \cdot x_3'$

$$\begin{cases} x_1' + x_2' + x_3' = x_1 \\ x_2' + x_3' = x_2 \\ x_3' = x_3 \end{cases} \Rightarrow x_2' = x_2 - x_3$$

$x_1' + x_2 - \cancel{x_3} + \cancel{x_3} = x_1$

$x_1' = x_1 - x_2$

$(x_1', x_2', x_3') = (x_1 - x_2, x_2 - x_3, x_3)$

$\{^1$ ③ $V_2 = < \{(1,-1,4),(3,1,0)\}>$

$V^1 = \{(x,y,z) \in \mathbb{R}^3 \mid 2x - y + z = 0$

ⓐ ~~$x_1(1,-1,2) + x_2(3,1,0) = (0,0,0)$~~

fie $x \in V_2 \Rightarrow$

$\exists\ a_1, a_2$ scalari $\in \mathbb{R}$

a.i. $x = (x_1, x_2, x_3) = a_1(1,-1,2) + a_2(3,1,0)$

$$\begin{cases} a_1 + 3a_2 = x_1 \\ -a_1 + a_2 = x_2 \\ 2a_1 + 0 = x_3 \end{cases}$$

$a_1 = \frac{x_3}{2}$

$a_2 = x_2 + \frac{x_3}{2}$

$\frac{x_3}{2} + 3\left(x_2 + \frac{x_3}{2}\right) = x_1$

$\frac{x_3}{2} + 3x_2 + \frac{3x_3}{2} = x_1$

$2x_3 + 3x_2 = x_1$

$\circledast$ SCD $\Leftrightarrow$ rg A = rg $\bar{A}$

$$A\Delta = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 1 + 3 = 4$$

$$\Delta c = \begin{vmatrix} 1 & 3 & x_1 \\ -1 & 1 & x_2 \\ 2 & 0 & x_3 \end{vmatrix} = 0$$

$$= x_1 \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} - x_2 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + x_3 \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} =$$

$$-2x_1 + 6x_2 + 4x_3 = 0$$

$\Rightarrow x_1 - 3x_2 - 2x_3 = 0 \Rightarrow V_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x_1 - 3x_2 - 2x_2 = 0\}$

(b) $V_1: y = 2x + z$ $\qquad$ $S(A_1) \Rightarrow A_1(2-1+1) = 1$

$V_1 = \{(x, 2x+z, z) \mid x, z \in \mathbb{R}\}$

$= x(1,2,0) + z(0,1,1)$.

$\mathcal{R}_1 = \{(1,2,0), (0,1,1)\}$ SG pt V_1 (1)

$\dim V_1 = 3 - 1 = 2$ (2)

(1), (2) $\Rightarrow \mathcal{R}_1$, reper în V_1

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

$\mathcal{R}_2 = \{(1,-3,2), (3,1,0)\}$ reper în V_2

$V_1 \cap V_2 = \{x, y, z \in \mathbb{R}^3$ $\begin{cases} 2x - y + z = 0 \\ x - 3y - 2z = 0 \end{cases}$

$\begin{cases} 2x - y = -z \\ x - 3y = 2z \mid (-2) \end{cases}$ $\quad z = \alpha$

$0 - 5y = -5z \Rightarrow y = -z$

$x = 2z - 3z \Rightarrow x = -z$

$V_1 \cap V_2 = \{(-z, -z, z), z \in \mathbb{R}\}$

$= z(-1,-1,1)$

reper $< \{(-1,-1,1)\} >$ $\mathbb{R}^3$

$\{(-1,-1,1), (0,1,1), (2,0,1)\}$ reper în $V_1 + V_2$

$\in V_1 \cap V_2$, $\in V_1$, $\in V_2$

din o teoremă Grassmann?

⑥ $(\mathbb{R}^8, +, \cdot)|\mathbb{R}$

$\dim U = 3$, $\dim W = 5$

$\dim(U+W) = 8$

$\dim(U+W) = \dim U + \dim W - \dim U \cap W$

$\Rightarrow 8 = 5 + 3 - \dim U \cap W$

$\Rightarrow \dim U \cap W = 0$

$\Rightarrow U + W$ este sumă directă

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⑧ $(\mathbb{R}^4, +, \cdot)|\mathbb{R}$, $V = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_4 = 0 \end{cases} \right.$

ⓐ $\overline{A} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right)$ rang $(A) = 2 \Rightarrow \dim V = 4 - 2 = 2$

$\begin{cases} x_1 + x_2 = -x_3 - x_4 \\ x_1 = -x_4 \end{cases} \Rightarrow x_2 = -x_3$

$\Rightarrow V = \{(-x_4, -x_3, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\}$

$\quad\quad \shortparallel$

$x_4(-1, 0, 0, 1) + x_3(0, -1, 1, 0)$

$\Rightarrow V = \langle\{(-1, 0, 0, 1), (0, -1, 1, 0)\}\rangle$; $R' =$

ⓑ R' reper în V

$\det \begin{pmatrix} -1 & 0 & \boxed{0} & \boxed{1} \\ 0 & -1 & \boxed{0} & \boxed{0} \\ 0 & 1 & \boxed{1} & \boxed{0} \\ 1 & 0 & \boxed{0} & \boxed{0} \end{pmatrix} \neq 0$

$V' = \langle\{(0, 0, 1, 0), (1, 0, 0, 0)\}\rangle$

R'' reper în V'

(c) $\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$, reper in $\mathbb{R}^4$ a.î. $\mathbb{R}^2$ reper

$x = v + v'$ $\quad \mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$ reper în $\mathbb{R}^4$

$\quad \cap \quad \cap$

$\quad V \quad V'$

$$x = (1,2,-1,3) = \underbrace{a(-1,0,0,1) + b(0,-1,1,0)}_{v} + \underbrace{c(0,0,1,0) + d(1,0,0,0)}_{v'}$$

$$\begin{cases} -a + d = 1 \Rightarrow d = 4 \\ -b = 2 \Rightarrow b = -2 \\ b + c = -1 \Rightarrow c = 1 \\ a = 3 \end{cases} \Rightarrow v = (-3,0,0,3) + (0,2,-2,0)$$

$$v = (-3,2,-2,3)$$

$$v' = (0,0,1,0) + (4,0,0,0)$$

$$\Rightarrow v' = (4,0,1,0)$$

$$\Rightarrow x = \underset{v}{(-3,2,-2,3)} + \underset{v'}{(4,0,1,0)} = (1,2,-1,3) = x$$

(d) ?