Curs 9 Analiză

Def Fie $f:(a,b)\to\mathbb{R}^m$, $f=(f_1,\dots,f_m)$ şi $c\in(a,b) \Rightarrow$

$$f'(c) = \frac{f(x)-f(c)}{x-c}$$

Ex $f:\mathbb{R}\to\mathbb{R}^3$ $\quad f(x)=(x^4+x^2,\ \sin 2x,\ a^{2x})$

$$f'(x)=(4x^3+2x \quad 2\cos 2x \quad 2a^{2x})$$

OBS f este derivabilă în $c \Leftrightarrow$ funcţiile f_i $i=\overline{1,m}$ sunt derivabile în c.

Ex $f:\mathbb{R}^3\to\mathbb{R}$; $f(x,y,z)=x^2y^3z^4$

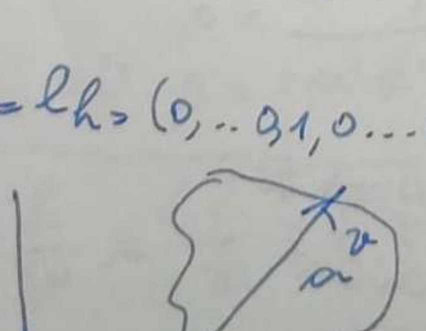
$$f'=\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z}\right)=(2xy^3z^4 \quad 3x^2y^2z^4 \quad 4x^2y^3z^3)$$

$$\frac{\partial f}{\partial x}=2xy^3z^4 \quad \frac{\partial f}{\partial y}=3x^2y^2z^4 \quad \frac{\partial f}{\partial z}=4x^2y^3z^3$$

Def Fie $A=\mathring{A}\subset\mathbb{R}^n$, $f:A\to\mathbb{R}^m$, $a\in A$ şi $v\in\mathbb{R}^n\setminus\{0\}$

$$\frac{\partial f}{\partial v}(a)=\lim_{t\to 0}\frac{f(a+tv)-f(a)}{t}$$

$v=e_k=(0,..0,1,0...0)$ $\quad \frac{\partial f}{\partial e_k}=\frac{\partial f}{\partial x_k}$



$\{a+tv \mid t\in\mathbb{R}\}$ - dreaptă

Ex $f:\mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2y^3$

$$\frac{\partial f}{\partial x} = 2xy^3; \quad \frac{\partial f}{\partial y} = 3x^2y^2 \quad v = (m,n) \in \mathbb{R}^2 \quad m^2+n^2 \neq 0$$
$$c = (a,b)$$

$$\frac{\partial f}{\partial v}(c) = \lim_{t\to 0} \frac{f(c+tv) - f(c)}{t} =$$

$$= \lim_{t\to 0} \frac{f(a+tm, b+tn) - f(a,b)}{t} = \lim_{t\to 0} \frac{(a+tm)^2(b+tn)^3 - a^2b^3}{t}$$

$$\overset{L'H}{=} \lim_{t\to 0} 2m(a+tm)(b+tn)^3 + 3n(a+tm)^2(b+tn)^2 = m\, 2ab^3 +$$

$$+ n\cdot 3a^2b^2 = m\cdot\frac{\partial f}{\partial x}(a,b) + n\frac{\partial f}{\partial y}(a,b) =$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = T(v)$$

$$T = f'(a)$$

<u>OBS</u> $f:(a,b) \to \mathbb{R}$ $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \iff$

$$\iff \lim_{x\to c} \frac{f(x) - f(c) - f'(c)(x-c)}{x-c} = 0$$

$$\iff \lim_{x\to c} \frac{f(x) - f(c) - f'(c)(x-c)}{|x-c|} = 0$$

Def Fie $D = \mathring{D} \subseteq \mathbb{R}^n$, $a \in D$, $f: D \to \mathbb{R}^m$

Spunem că f este derivabilă în $a \iff$

$\iff T \in L(\mathbb{R}^n, \mathbb{R}^m)$ a.î.

$$\lim_{x \to a} \frac{f(x) - f(a) - T(x-a)}{\|x-a\|_2} = 0$$

T - diferențiala lui f în a (diferențială)

matricea asociată derivata lui f în a (derivată)

"sunt practic același lucru"

<u>Def</u> Fie $D = \overset{\circ}{D} \subset \mathbb{R}^n$, $a \in D$, $f: D \to \mathbb{R}^m$

Spunem că f este derivabilă dacă $\exists\, T \in L(\mathbb{R}^n, \mathbb{R}^m)$

și $\omega : D \to \mathbb{R}^m$ a.î.

1) $f(x) = f(a) + T(x-a) + \|x-a\|\omega(x)$

2) $\lim_{x \to a} \omega(x) = 0$

<u>Ex</u> $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2y^3$ $c = (a,b)$

f este derivabilă în $c \Leftrightarrow T \in L(\mathbb{R}^2, \mathbb{R})$, a.î.

$$\lim_{\substack{x \to a \\ y \to b}} \frac{f(x,y) - f(a,b) - T(x-a, y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

$T(x,y) = \alpha x + \beta y$ $\alpha = \frac{\partial f}{\partial x}(a,b)$ și $\beta = \frac{\partial f}{\partial y}(a,b) \Rightarrow$

$\Rightarrow \alpha = 2ab^3 \cdot \beta = 3a^2b^2$

$$\boxed{\frac{\partial f}{\partial x} = \frac{df}{dx} = f'_x}$$

$$\left|\frac{xy^3 - x^2b^3 - 3a^2b^2(y-b) + x^2b^3 - a^2b^3 - 2ab^3(x-a)}{\sqrt{(x-a)^2+(y-b)^2}}\right| \le$$

$$\le \frac{|y-b|\,|x^2(y^2+yb+b^2) - 3a^2b^2|}{\sqrt{(x-a)^2+(y-b)^2}} + \frac{|x-a|}{\sqrt{(x-a)^2+(y-b)^2}}|b^3(x+a)-2ab^3|$$

$$\le |x(y^2+yb+b^2) - 3a^2b^2| + |b^3||x-a| \xrightarrow[x\to a]{} 0$$

Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f : D \to \mathbb{R}^m$, $v \in \mathbb{R}^n / \{a\}$

<u>Obs 1</u> Dacă $\exists f'(a) \Rightarrow f$ este continuă în a

$$\lim_{x\to a} f(x) = \lim_{x\to a} f(a) + f'(a)(x-a) + \|x-a\|\, w(x) \| = f(a)$$
$$f(a)\cdot 0 = 0$$

<u>OBS 2</u> <u>Derivata este unică</u>

Dacă $\exists T_1, T_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$, a.î. $\lim\limits_{x\to a} \frac{f(x)-f(a)-T_1(x-a)}{\|x-a\|} = 0$ (2)

$$\lim_{x\to a} \frac{f(x)-f(a)-T_2(x-a)}{\|x-a\|} = 0 \text{ (1)}$$

(1) și (2) $\Rightarrow T_1 = T_2$

$$\lim_{x\to a} \frac{(T_1-T_2)(x-a)}{\|x-a\|} = 0 \Rightarrow \lim_{t\to 0} \frac{(T_1-T_2)(tv)}{\|tv\|} = 0 \Rightarrow$$

$$x = a + tv \qquad x \to a \Leftrightarrow t \to 0 \quad \Big| \Rightarrow \lim_{t\to 0} \frac{t(T_1 - T_2)(v)}{t\|v\|} = 0 \Rightarrow$$

$$(T_1 - T_2)(v) = 0 \quad \forall v \neq 0$$

OBS 3 Dacă $\exists f'(a) \Rightarrow \frac{\partial f}{\partial v}(a) = f'(a)(v)$

$$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{\|x-a\|} = 0 \qquad x = a + tv \Rightarrow$$

$$\lim_{t\to 0} \frac{f(a+tv) - f(a) - f'(a)(tv)}{\|tv\|} = 0 \quad \Rightarrow$$

$$\Rightarrow \lim_{t\to 0} \frac{f(a+tv) - f(a)}{t} - f'(a)(v) = 0$$

$$\frac{\partial f}{\partial x}(a) = \frac{\partial f}{\partial e_1}(a) = \lim_{x_1 \to a_1} \frac{f(x_1, a_2, \dots, a_n) - f(a_1, \dots, a_n)}{x_1 - a_1}$$

$$T(x) = \sum_{k=1}^{n} x_k T(e_k) = \sum_{k=1}^{n} x \begin{pmatrix} \frac{\partial f_1}{\partial x_k} \\ \vdots \\ \frac{\partial f_m}{\partial x_k} \end{pmatrix}$$

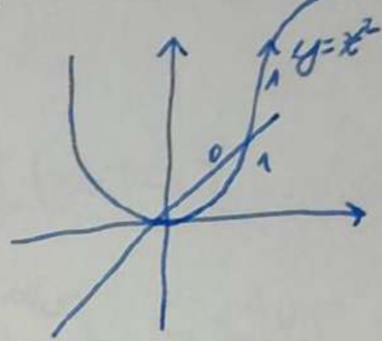
$$x = \sum_{k=1}^{n} x_k e_k$$

$$f' = \begin{pmatrix} \frac{\partial f_1}{x_1} & \dots & \frac{\partial f_1}{\partial_n} \\ \vdots & & \\ \frac{\partial f_m}{x_1} & \dots & \frac{\partial f_m}{\partial_n} \end{pmatrix}$$

<u>Ex</u> Fie $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} 1 & , y = x^2 \neq 0 \\ 0 & , \text{rest} \end{cases}$

$\frac{\partial f}{\partial v}(0,0) = 0 \quad \forall v \quad$ f deriv. $(0,0)$

$f\left(\frac{1}{n}, \frac{1}{n^2}\right) = 1 \to 1 \neq 0$



<u>OBS.</u> Fie $f: B(a,r) \to \mathbb{R}^m$ $(B(a,r) \subset \mathbb{R}^n)$, a.î. $\exists \frac{\partial f}{\partial x_i}$ pe $B(a,r)$ $\forall i = \overline{1,n}$. Atunci

1) Dacă $\exists M$, a.î. $\left|\frac{\partial f}{\partial x_i}(x)\right| \leq M \quad \forall i = \overline{1,n} \quad \forall x \in B(a,r) \Rightarrow$

$\Rightarrow f$ este continuă în a

2) dacă $\frac{\partial f}{\partial x_i}$ sunt continue în a $\Rightarrow \exists f'(a)$

<u>Prop</u> Fie $D = \overset{\circ}{D} \subset \mathbb{R}^n$, $f, g: D \to \mathbb{R}^m$, $\varphi: D \to \mathbb{R}$, $a \in D$ și $v \in \mathbb{R}^n \setminus \{0\}$

Dacă $\exists \frac{\partial f}{\partial v}(a), \frac{\partial g}{\partial v}(a)$ și $\frac{\partial \varphi}{\partial v}(a)$ atunci $\Rightarrow$

1) $\exists \frac{\partial (f+g)}{\partial v}(a) = \frac{\partial f}{\partial v}(a) + \frac{\partial g}{\partial v}(a)$

2) $\exists \frac{\partial (\alpha f)}{\partial v}(a) = \alpha \cdot \frac{\partial f}{\partial v}(a)$

3) $\exists \frac{\partial \varphi f}{\partial v}(a) = \frac{\partial \varphi}{\partial v}(a) f(a) + \varphi(a) \frac{\partial f}{\partial v}(a)$

4) $\frac{\partial}{\partial v}\langle f, g\rangle(a) = \langle \frac{\partial f}{\partial v}(a), g(a)\rangle + \langle f(a), \frac{\partial g}{\partial v}(a)\rangle$

Dacă $\exists f'(a)$ și $\exists g'(a)$ atunci

1) $\exists (f+g)'(a) = f'(a) + g'(a)$ și $\exists (\alpha f)'(a) = \alpha f'(a)$

Ex $f : \mathbb{R}^3 \to \mathbb{R}^2$ $f(x,y,z) = (x^3y^2 + y^3z, x^2 + y^3 + z^4)$

$f_1(x,y,z) = x^3y^2 + y^3z$

$f_2(x,y,z) = x^2 + y^3 + z^4$

$$f' = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$$

Prop Fie $D = \mathring{D} \subset \mathbb{R}^n$, $G = \mathring{G} \subset \mathbb{R}^m$, $a \in D$, $f : D \to G$, $g : G \to \mathbb{R}^p$

Dacă $\exists f'(a)$ și $\exists g'(f(a))$

atunci: $\exists (g \circ f)'(a) = g'(f(a)) \circ f'(a)$

$$\begin{tikzcd} \mathbb{R}^n & \mathbb{R}^m & \\ D \arrow[u, phantom, "\cup"] \arrow[r, "f"] & G \arrow[u, phantom, "\cup"] \arrow[r, "g"] & \mathbb{R}^p \\ a \arrow[u, phantom, "\in"] \arrow[rr, bend right, "g \circ f"'] & f(a) \arrow[u, phantom, "\in"] & g \circ f(a) \end{tikzcd}$$

Prop. Fie $D = \mathring{D} \subset \mathbb{R}^n$, $G = \mathring{G} \subset \mathbb{R}^n$, $f: D \to G$ bijectivă și $a \in D$ a.î.

1) $\exists f'(a)$.

2) $\exists (f'(a))^{-1}$ și

3) f^{-1} este continuă în $f(a)$

Atunci $\exists (f^{-1})'(f(a)) = (f'(a))^{-1}$

$f(a) = b \Rightarrow (f^{-1})'(b) = (f'(f^{-1}(b)))^{-1}$

Teorema de inversare locală

Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$ și $\varphi: D \to \mathbb{R}^m$ a.î.

1) $\varphi \in C^1(D)$ ($\exists \varphi'$ pe D și φ' este continuă pe D)

2) $\det \varphi'(a) \neq 0 \Rightarrow \exists D_1 = \mathring{D}_1 \subset \mathbb{R}^3$ și $D_2 = \mathring{D}_2 \subset \mathbb{R}^n$ a.î.

$a \in D_1 \subset D$, $f(a) \in D_2$ și $\psi: D_1 \to D_2$

$\psi(x) = \varphi(x)$ să ~~fie~~ fie bijectivă și ψ^{-1} să fie de clasă C^1

$(\psi^{-1}(\varphi(a)) = (\varphi'(a))^{-1})$