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(C3) -16. 17.10.2024

Spații vectoriale

$(\mathbb{K},+,\cdot)$ corp comutativ, V mulțime nevidă.

V are o structură de spațiu vectorial peste corpul $\mathbb{K}$

(3) dacă $\exists + : V\times V \longrightarrow V$

$\cdot : \mathbb{K}\times V \longrightarrow V$

aî 1) $(V,+)$ grup abelian

2) $a\cdot(b\cdot x) = (a\cdot b)\cdot x$

3) $(a+b)\cdot x = a\cdot x + b\cdot x$

4) $a\cdot(x+y) = a\cdot x + a y$

5) $1_{\mathbb{K}}\cdot x = x$, $\forall a,b\in\mathbb{K}$ (scalari)

$\forall x,y\in V$ (vectori)

Not $(V,+,\cdot)/\mathbb{K}$

OBS a) $0_{\mathbb{K}}\cdot x = 0_V$

b) $a\cdot 0_V = 0_V$

c) $(a-b)\cdot x = a\cdot x - b\cdot x$

d) $a\cdot(x-y) = a\cdot x - a\cdot y$.

Exemple

1) $(\mathbb{K},+,\cdot)$ corp com $\Rightarrow (\mathbb{K},+,\cdot)/\mathbb{K}$ sp vect.

$(\mathbb{R},+,\cdot)/\mathbb{R}$, $(\mathbb{C},+,\cdot)/\mathbb{C}$, $(\mathbb{Z}_p,+,\cdot)/\mathbb{Z}_p$ $p=$ prim.

2) $(\mathbb{K},+,\cdot)$ corp com $\Rightarrow (\mathbb{K},+,\cdot)/\mathbb{K}'$ sp vect.

$\mathbb{K}'\subset\mathbb{K}$ subcorp

Ex: $(\mathbb{R},+,\cdot)/\mathbb{Q}$; $(\mathbb{C},+,\cdot)/\mathbb{R}$; $(\mathbb{C},+,\cdot)/\mathbb{Q}$

3) Fie $(V_1, \oplus, \odot)/_{\mathbb{K}}, (V_2, \boxplus, \boxdot)/_{\mathbb{K}} \Rightarrow (V = V_1 \times V_2, +, \cdot)/_{\mathbb{K}}$
sp vect   sp. vect.

$+ : (V_1 \times V_2) \times (V_1 \times V_2) \longrightarrow V_1 \times V_2.$

$(x_1, x_2) + (y_1, y_2) := (x_1 \oplus y_1, x_2 \boxplus y_2)$

$\cdot : \mathbb{K} \times (V_1 \times V_2) \longrightarrow V_1 \times V_2$

$a \cdot (x_1, x_2) = (a \odot x_1, a \boxdot x_2),$

$\forall (x_1, x_2), (y_1, y_2) \in V_1 \times V_2$

$\forall a \in \mathbb{K}.$

<u>C. part</u> $(\mathbb{R}, +, \cdot)/_{\mathbb{R}}$ sp.v. $\Rightarrow (\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}, +, \cdot)/_{\mathbb{R}}$ sp.v.

$+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

$\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$a(x_1, \ldots, x_n) = (ax_1, \ldots, ax_n)$

4) $(V = \mathcal{M}_{m,n}(\mathbb{K}), +, \cdot)/_{\mathbb{K}}$ sp. vect.

$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \to (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn})$

$\cap$ $\cap$

$\mathcal{M}_{m,n}(\mathbb{R})$ $\mathbb{K}^{mn}$

5) $(\mathbb{K}[X], +, \cdot)/_{\mathbb{K}}$ sp. vect.

$P = a_0 + a_1 X + \ldots + a_n X^n \longrightarrow (a_0, a_1, \ldots, a_n) \in \mathbb{K}^{n+1}$

$\cap$

$\mathbb{K}_n[X] = \{P \in \mathbb{K}[X], \text{grad} P \leq n\}$

6) $I = [a,b]$, $a < b$

$(\mathcal{C}(I) = \{ f: I \to \mathbb{R} \mid f \text{ cont}\}, +, \cdot)|_{\mathbb{R}}$

$(\mathcal{D}(I) = \{ f: I \to \mathbb{R} \mid f \text{ deriv}\}, +, \cdot)|_{\mathbb{R}}$

$(\mathcal{P}(I) = \{ f: I \to \mathbb{R} \mid f \text{ admite primitive}\}, +, \cdot)|_{\mathbb{R}}$

$\mathcal{I}(I) = \{ f: I \to \mathbb{R} \mid f \text{ integrabile Riemann}\}, +, \cdot)|_{\mathbb{R}}$.

Subspații vectoriale

$(V, +, \cdot)|_{\mathbb{K}}$ sp. vect, $V' \subset V$ subm. nevidă

V' s.n. <u>subspațiu vectorial</u> $\Leftrightarrow$ 1) este închisă la „+" vect

2) —"— „$\cdot$" cu scalari

i.e. 1) $\forall v, w \in V' \Rightarrow v + w \in V'$

2) $\forall a \in \mathbb{K}, \forall v \in V' \Rightarrow av \in V'$

Prop (de caracterizare)

$(V, +, \cdot)|_{\mathbb{K}}$ sp. vect, $V' \subseteq V$ subm. nevidă

V' este subsp. vect $\Leftrightarrow \forall x, y \in V', \forall a, b \in \mathbb{K} : ax + by \in V'$

$\Leftrightarrow \forall x_1, \dots, x_n \in V', \forall a_1, \dots, a_n \in \mathbb{K} : a_1x_1 + \dots + a_nx_n \in V'$

Dem „$\Rightarrow$" Ip: V' subsp. vect.

$\forall x \in V', \forall a \in \mathbb{K} \Rightarrow ax \in V'$

$\forall y \in V', \forall b \in \mathbb{K} \Rightarrow by \in V'$

$\Rightarrow ax + by \in V'$

„$\Leftarrow$" Ip: $\forall x, y \in V', \forall a, b \in \mathbb{K} : ax + by \in V'$

- $b = 0_{\mathbb{K}} \Rightarrow ax \in V'$; $a = b = 1_{\mathbb{K}} \Rightarrow x + y \in V'$

Deci $V' \subset V$ subsp. vect.

OBS $(V,+,\cdot)/\mathbb{K}$ sp vect, $V' \subseteq V$ subsp vect $\Rightarrow (V',+,\cdot)/\mathbb{K}$ sp vec.

Exemple de subspații vect

1) $(V,+,\cdot)/\mathbb{K}$. $\{0\}, V$ ex. de ssp vect.

2) $n < m$ $(\mathbb{R}^m,+,\cdot)/\mathbb{R}$ sp vect și $\mathbb{R}^n$ subsp vect

3) $(\mathcal{M}_n(\mathbb{R}),+,\cdot)$

$V' = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = diag(a_1,\dots,a_n)\}$

$V'' = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = A^T\} = \mathcal{M}_n^s(\mathbb{R})$

$V''' = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = -A^T\} = \mathcal{M}_n^a(\mathbb{R})$

$W = \{A \in \mathcal{M}_n(\mathbb{R}) \mid Tr(A) = 0\}$

ssp vect

OBS $GL(n,\mathbb{R}), O(n), SO(n), SL(n,\mathbb{R}) \subset \mathcal{M}_n(\mathbb{R})$ NU sunt ssp vect.

4) $(\mathbb{R}^2,+,\cdot)/\mathbb{R}$ $W = \{(x,y) \in \mathbb{R}^2 \mid ax+by=0, a^2+b^2>0\}$
dreaptă în plan, care trece prin orig.

$(\mathbb{R}^3,+,\cdot)/\mathbb{R}$, $V = \{(x,y,z) \in \mathbb{R}^3 \mid ax+by+cz=0, a^2+b^2+c^2>0\}$
plan care trece prin orig.

$(\mathbb{R}^n,+,\cdot)/\mathbb{R}$ $U = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid a_1x_1+\dots+a_nx_n=0, a_1^2+\dots+a_n^2>0\}$
hiperplan care trece prin origine

(C3) - continuare AG-16

Spații vectoriale. Subspații vectoriale. Sistem liniar dependent / liniar independent. Sistem de generatori. Bază

• Subspațiul vectorial generat de o mulțime nevidă

$(V,+,\cdot)|_{\mathbb{K}}$ sp. vectorial, $S \subseteq V$ subm. nevidă

$$\langle S \rangle = \{x \in V \mid x = a_1x_1 + \dots + a_nx_n,\ x_i \in S, i=\overline{1,n},\ a_i \in \mathbb{K}, i=\overline{1,n}\}$$

Dacă $V = \langle S \rangle$, at S s.n. <u>sistem de generatori</u> (SG)

V s.n. spațiu vectorial <u>finit generat</u> $\Leftrightarrow$ $\exists S$ mulțime finită aî $V = \langle S \rangle$.

<u>OBS</u> a) $S \subset \langle S \rangle$ (din construcție)

b) $\langle S \rangle$ = cel mai mic subspațiu vect. al lui V, care conține S.

c) $\langle \phi \rangle = \{0_V\}$ (<u>Convenție</u>)

<u>Def</u> $(V,+,\cdot)|_{\mathbb{K}}$ sp. vect, $S \subseteq V$ subm. nevidă

1) S s.n. <u>sistem liniar independent</u> (SLI) $\Leftrightarrow$

$\forall x_1,\dots,x_n \in S$
$\forall a_1,\dots,a_n \in \mathbb{K}$ aî $a_1x_1 + \dots + a_nx_n = 0_V \Rightarrow a_1 = \dots = a_n = 0_{\mathbb{K}}$

($\forall$ combinație liniară nulă este trivială)

2) S s.n. <u>sistem liniar dependent</u> (SLD) $\Leftrightarrow$

$\exists x_1,\dots,x_n \in S$
$\exists a_1,\dots,a_n \in \mathbb{K}$, <u>nu</u> toți nuli aî $a_1x_1 + \dots + a_nx_n = 0_V$

Prop $\{x\}$, $x \neq 0_V$ este SLI

Dem Fie $a \in \mathbb{K}$ a.î. $a \cdot x = 0_V$ (*). Arătăm că $a = 0_{\mathbb{K}}$

Ip. abs $a \neq 0_{\mathbb{K}}$
dar $(\mathbb{K}, +, \cdot)$ corp $\Big| \Rightarrow \exists a^{-1}$

$(*) \Rightarrow \underbrace{a^{-1} \cdot a}_{1_{\mathbb{K}}} \cdot x = a^{-1} \cdot 0_V \xRightarrow{Ax5} x = 0_V$ Contrad.

Ip. este falsă și deci $a x = 0_V \Rightarrow a = 0_{\mathbb{K}}$
$\{x\}$, $x \neq 0_V$ SLI

Def $(V, +, \cdot)/\mathbb{K}$ sp. vect., $S \subseteq V$ subm. nevidă.
S s.n. bază $\Leftrightarrow \begin{cases} S \text{ este SLI} \\ S \text{ este SG.} \end{cases}$

Exemple

1) $(\mathbb{R}, +, \cdot)/\mathbb{R}$, $B_0 = \{1\}$ bază canonică

$1 \neq 0_V \Rightarrow \{1\}$ SLI
$\forall x \in \mathbb{R} \Rightarrow 1_{\mathbb{K}} \cdot x = x \Rightarrow \{1\}$ SG
$\mathbb{R} = \langle \{1\} \rangle$

OBS $B = \{a\}$, $a \neq 0_{\mathbb{R}}$ bază

2) $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$ $B_0 = \{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$ baza canonică

SLI: $\forall a_1, a_2 \in \mathbb{R}$ a.î. $a_1 e_1 + a_2 e_2 = 0_{\mathbb{R}^2} = (0,0)$
$a_1(1,0) + a_2(0,1) = (0,0) \Rightarrow (a_1, 0) + (0, a_2) = (0,0)$
$(a_1, a_2) = (0,0) \Rightarrow a_1 = a_2 = 0_{\mathbb{R}}$

SG: $\forall x = (x_1, x_2) \in \mathbb{R}^2$, $\Rightarrow x = (x_1, 0) + (0, x_2) =$

$= x_1(1,0) + x_2(0,1) = x_1 e_1 + x_2 e_2, \quad x_1, x_2 \in \mathbb{R}$

<u>OBS</u>

$B = \{ \underset{e_1'}{(1,1)}, \underset{e_2'}{(-1,2)} \}$ bază în $\mathbb{R}^2$

SLI $\forall a_1, a_2 \in \mathbb{R}$ aî $a_1 e_1' + a_2 e_2' = (0,0)$

$a_1(1,1) + a_2(-1,2) = (0,0) \Rightarrow (a_1 - a_2, a_1 + 2a_2) = (0,0)$

$(*) \begin{cases} a_1 - a_2 = 0 \\ a_1 + 2a_2 = 0 \end{cases} \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \Big| \begin{matrix} 0 \\ 0 \end{matrix} \quad \det A \neq 0$

sist $(*)$ SCD $\exists! (a_1, a_2) = (0,0)$

SG: $\forall x = (x_1, x_2) \in \mathbb{R}^2, \exists a_1, a_2 \in \mathbb{R}$ aî $x = a_1 e_1' + a_2 e_2'$

$(a_1 - a_2, a_1 + 2a_2) = (x_1, x_2)$

$(**) \begin{cases} a_1 - a_2 = x_1 \\ a_1 + 2a_2 = x_2 \end{cases} \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \Big| \begin{matrix} x_1 \\ x_2 \end{matrix} \quad \det A \neq 0$

sist $(**)$ SCD $\exists! (a_1, a_2)$

3) $(\mathcal{M}_{m,n}(\mathbb{R}), +, \cdot)/\mathbb{R}$ $\quad A \in \mathcal{M}_{m,n}(\mathbb{R})$
$\quad\quad = (a_{11}, \dots, a_{1n}, \dots, a_{m1}, \dots, a_{mn}) \in \mathbb{R}^{mn}$.

$$E_{ij} = \begin{array}{c|c} & j \\ \hline & \begin{pmatrix} 0 & & 0 \\ & & \\ i \cdots & 1 & \\ & & \\ 0 & & 0 \end{pmatrix} \end{array}$$

$B_0 = \{ E_{ij}, i = \overline{1,m}, j = \overline{1,n} \}$ baza canonică

$|B_0| = m \cdot n$.

4) $(\mathbb{R}[X], +, \cdot)/\mathbb{R}$. sp. vectorial care <u>nu</u> este finit generat

$B_0 = \{1, X, X^2, \dots\}$ bază $\quad P \in \mathbb{R}[X]$, $P = a_0 + a_1 X + \dots + a_n X^n$.

$\mathbb{R}_n[X] = \{ P \in \mathbb{R}[X] \mid \text{grad } P \leq n \}$ sp. vect. finit generat

$B_0 = \{1, X, \dots, X^n\}$ bază canonică

Prop $\forall$ subm. nevidă a unui SLI este un SLI.

Dem

$S = \{x_1, \dots, x_n\}$ SLI şi fie $S' = \{x_1, \dots, x_{n-1}\} \subset S$

Fie $a_1, \dots, a_{n-1} \in \mathbb{K}$ a.î. $\underbrace{a_1 x_1 + \dots + a_{n-1} x_{n-1}}_{\parallel} = 0_V$

$a_1 x_1 + \dots + a_{n-1} x_{n-1} + 0_{\mathbb{K}} \cdot x_n$

$\xrightarrow{S \text{ e SLI}} a_1 = \dots = a_{n-1} = 0_{\mathbb{K}} \Rightarrow S'$ e SLI

Prop $\forall$ supramulţime a unui SLD este un SLD.

Dem

$S = \{x_1, \dots, x_n\}$ SLD $\Rightarrow S' = S \cup \{x_{n+1}\}$ este SLD.

S SLD: $\exists\, a_1, \dots, a_n \in \mathbb{K}$, nu toţi nuli a.î.

$a_1 x_1 + \dots + a_n x_n = 0_V$

$\parallel$

$a_1 x_1 + \dots + a_n x_n + 0_{\mathbb{K}} \cdot x_{n+1} = 0_V$

$a_1, \dots, a_n, 0_{\mathbb{K}}$ nu sunt toţi nuli

$\Rightarrow S'$ este SLD.

Prop $\forall$ supramulţime a unui SG este un SG.

$x \in V = \langle S \rangle$

S e SG, $S = \{x_1, \dots, x_n\}$ Fie $S' = S \cup \{x_{n+1}\}$.

Dem că $V = \langle S' \rangle$

S este SG $\exists\, a_1, \dots, a_n \in \mathbb{K}$ a.î. $x = a_1 x_1 + \dots + a_n x_n$

$\parallel$

$a_1 x_1 + \dots + a_n x_n + 0_{\mathbb{K}} \cdot x_{n+1}$

$V = \langle S \cup \{x_{n+1}\} \rangle$

Teorema schimbului

$(V,+,\cdot)/\mathbb{K}$ sp vect

$\left.\begin{array}{l}\{x_1,\dots,x_n\} \text{ este SG} \\ \{y_1,\dots,y_n\} \text{ este SLI}\end{array}\right\} \Rightarrow \{y_1,\dots,y_n\} \text{ SG}$

Dem

(1) $\underset{\underset{y_1}{\Psi}}{V} = \langle\{x_1,\dots,x_n\}\rangle \Rightarrow \exists a_1,\dots,a_n \in \mathbb{K}$ aî $y_1 = a_1x_1+\dots+a_nx_n$

Ip. abs $a_1 = \dots = a_n = 0_{\mathbb{K}} \Rightarrow y_1 = 0_V$

! $\{0_V, y_2,\dots,y_n\}$ este SLI FALS

$a\cdot 0_V + 0_{\mathbb{K}}\cdot y_2 + \dots + 0_{\mathbb{K}}\cdot y_n = 0_V$, $a \neq 0_{\mathbb{K}}$

(comb. liniară nulă, netrivială).

Ip. este falsă $\Rightarrow \exists$ cel puțin un scalar nenul.

Eventual renumerotând, considerăm $a_1 \neq 0_{\mathbb{K}} \Rightarrow \exists a_1^{-1}$

$a_1x_1 = y_1 - a_2x_2 - \dots - a_nx_n \mid a_1^{-1}$

$x_1 = a_1^{-1}\cdot y_1 - a_1^{-1}\cdot a_2\cdot x_2 - \dots - a_1^{-1}a_n\cdot x_n$. (2)

Din (1), (2) $\underset{\underset{y_2}{\Psi}}{V} = \langle\{y_1, x_2,\dots,x_n\}\rangle$ (3)

$\exists b_1, a_2,\dots,a_n \in \mathbb{K}$ aî $y_2 = b_1y_1 + a_2x_2 + \dots + a_nx_n$.

Ip. abs $a_2 = \dots = a_n = 0_{\mathbb{K}} \Rightarrow y_2 = b_1y_1 \Rightarrow$

$b_1\cdot y_1 - \underset{\underset{0_{\mathbb{K}}}{\neq}}{1_{\mathbb{K}}}\cdot y_2 + 0_{\mathbb{K}}\cdot y_3 + \dots + 0_{\mathbb{K}}\cdot y_n = 0 \Rightarrow \underset{\text{Fals}}{\{y_1,\dots,y_n\} \text{ SLD}}$

Ip este falsă. Considerăm $a_2 \neq 0_{\mathbb{K}} \Rightarrow \exists a_2^{-1}$

$a_2x_2 = y_2 - b_1y_1 - a_3x_3 - \dots - a_nx_n \overset{(4)}{\Rightarrow} x_2 = a_2^{-1}y_2 - a_2^{-1}b_1y_1 - \dots - a_2^{-1}a_nx_n$

Dem (3), (4) $\Rightarrow V = \langle\{y_1, y_2, x_3, \dots, x_n\}\rangle$.
Repetăm raționamentul și după un nr finit de pași obț
$V = \langle\{y_1, \dots, y_n\}\rangle \Rightarrow \{y_1, \dots, y_n\}$ este SG.

<u>Prop</u> card. $\forall$ SG, finit $\geq$ card $\forall$ SLI

<u>Dem</u>

$\{x_1, \dots, x_n\}$ SG

Dem că $\underline{\{y_1, \dots, y_n}, y_{n+1}\} \subset V$ este SLD.

1) Dacă $\{y_1, \dots, y_n\}$ SLI, at. din Th. schimbului rez.
$\{y_1, \dots, y_n\}$ SG ie $\underset{\substack{\Psi \\ y_{n+1}}}{V} = \langle\{y_1, \dots, y_n\}\rangle \Rightarrow$

$\exists a_1, \dots, a_n \in \mathbb{K}$ ai $y_{n+1} = a_1 \cdot y_1 + \dots + a_n y_n \Rightarrow$

$\underbrace{a_1 \cdot y_1 + \dots + a_n y_n - 1_{\mathbb{K}} \cdot y_{n+1}}_{\text{(nu toți scalarii sunt 0)}} = 0_V \Rightarrow \{y_1, \dots, y_n, y_{n+1}\}$ SLD.

sau

2) Dc $\{y_1, \dots, y_n\}$ SLD, at $\underbrace{\{y_1, \dots, y_n\} \cup \{y_{n+1}\}}_{\text{(supramulțime)}}$ este SLD.

<u>Teoremă</u> $(V, +, \cdot)/\mathbb{K}$ sp. vect (finit generat)
$\forall B_1, B_2$ baze în V, at $|B_1| = |B_2| = n = \dim_{\mathbb{K}} V$
(dimensiunea lui V peste $\mathbb{K}$)

<u>Dem</u>

$\left.\begin{array}{l} B_1 \text{ SG} \\ B_2 \text{ SLI} \end{array}\right\} \xRightarrow{\text{prop preced}} |B_1| \geq |B_2|$; $\left.\begin{array}{l} B_2 \text{ SG} \\ B_1 \text{ SLI} \end{array}\right\} \xRightarrow{\text{prop}} |B_2| \geq |B_1|$

Deci $|B_1| = |B_2|$.

<u>Obs</u> $(V,+,\cdot)/_{\mathbb{K}}$, $\dim_{\mathbb{K}} V = n$

a) n = nr max de vect care form. SLI

b) n = nr min. de vect care form SG.

<u>Prop</u> $(V,+,\cdot)/_{\mathbb{K}}$, sp vect. finit generat, $\dim_{\mathbb{K}} V = n$.

$S = \{x_1, .., x_n\} \subset V$

UAE 1) S bază.

2) S este SLI.

3) S este SG.

<u>Exemple</u>

1) $(\mathbb{R}^3,+,\cdot)/_{\mathbb{R}}$ $B_0 = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ bază canonică

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3$

a) $S = \{(1,2,3), (-1,1,1), (0,2,3), (-1,0,0)\}$ este SLD.

M1 3 = nr. max. de vect care form. SLI, $|S| = 4$.

$\Rightarrow$ S este SLD.

M2 Fie $a_1, a_2, a_3, a_4 \in \mathbb{R}$ ai

$$a_1(1,2,3) + a_2(-1,1,1) + a_3(0,2,3) + a_4(-1,0,0) = (0,0,0)$$

$$(a_1 - a_2 - a_4,\ 2a_1 + a_2 + 2a_3,\ 3a_1 + a_2 + 3a_3) = (0,0,0)$$

$$\begin{cases} a_1 - a_2 - a_4 = 0 \\ 2a_1 + a_2 + 2a_3 = 0 \\ 3a_1 + a_2 + 3a_3 = 0 \end{cases} \quad A = \left(\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 3 & 1 & 3 & 0 & 0 \end{array}\right)$$

SCN $\exists$ și sol nenule $\Rightarrow$ SLD.

b) $S' = \{(1,2,-1), (\alpha,1,1), (0,2,1)\} \subset \mathbb{R}^3$, $\alpha \in \mathbb{R}$

$\alpha = ?$ aî S' este bază

$|S'| = 3$, $\dim_{\mathbb{R}} \mathbb{R}^3 = 3$.

Este suficient să considerăm S' SLI

$\forall a_1, a_2, a_3 \in \mathbb{R}$ aî $a_1(1,2,-1) + a_2(\alpha,1,1) + a_3(0,2,1) = (0,0,0)$

$\Rightarrow a_1 = a_2 = a_3 = 0_{\mathbb{R}}$

$\Leftrightarrow (a_1 + \alpha a_2, 2a_1 + a_2 + 2a_3, -a_1 + a_2 + a_3) = (0,0,0)$

$\Leftrightarrow$ (*) $\begin{cases} a_1 + \alpha a_2 = 0 \\ 2a_1 + a_2 + 2a_3 = 0 \\ -a_1 + a_2 + a_3 = 0 \end{cases}$ $A = \left(\begin{array}{ccc|c} 1 & \alpha & 0 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right)$

(*) SCD i.e sol unică nulă $(a_1, a_2, a_3) = (0,0,0) \Leftrightarrow \det A \neq 0$

$\begin{vmatrix} 1 & \alpha & 0 \\ 2 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix} \neq 0$ $\begin{vmatrix} 1 & \alpha & 0 \\ 4 & -1 & 0 \\ -1 & 1 & 1 \end{vmatrix} \neq 0$

$L_2 - 2L_3$ $-1 - 4\alpha \neq 0 \Rightarrow$

$\alpha \neq -\frac{1}{4}$

S' e SLI (adică bază) $\Leftrightarrow \alpha \neq -\frac{1}{4}$.