C_9 - AG

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Spații vectoriale euclidiene
Procedeul Gram - Schmidt

Def $(V,+,\cdot)|_{\mathbb{R}}$ sp. vect. real

$g: V\times V \longrightarrow \mathbb{R}$ s.n. produs scalar $\Leftrightarrow$

1) $g \in L^s(V,V;\mathbb{R})$ (formă biliniară, simetrică)

i.e. $g(x,y) = g(y,x)$

și

$g(ax+by, z) = a\,g(x,z) + b\,g(y,z)$, $\forall x,y,z\in V$, $a,b\in\mathbb{R}$.

2) g pozitiv definită.

i.e $\begin{cases} g(x,x) > 0, \ \forall x\in V\setminus\{0_V\} \\ g(x,x) = 0 \Leftrightarrow x = 0_V. \end{cases}$

OBS V este înzestrat cu structura euclidiană g.

(E,g), $(E,\langle\cdot,\cdot\rangle)$, $(E,(\cdot,\cdot))$ sp. euclidian.

$g = \langle\cdot,\cdot\rangle ; (\cdot,\cdot)$ $\qquad g(x,y) = \langle x,y\rangle = (x,y)$

Def (E,g) s.v.e.r

$\mathcal{R} = \{e_1,\dots,e_n\}$ reper

a) $\mathcal{R}$ s.n. reper ortogonal $\Leftrightarrow g(e_i,e_j) = 0, \forall i\neq j$

b) $\mathcal{R}$ s.n. reper ortonormat $\Leftrightarrow g(e_i,e_j) = \delta_{ij} = \begin{cases} 0, & i\neq j \\ 1, & i=j \end{cases}$

$\|x\| = \sqrt{g(x,x)} = \sqrt{Q(x)}$

$g(e_i,e_i) = 1 \Rightarrow \|e_i\|^2 = 1 \Rightarrow e_i$ versor $\forall i=\overline{1,n}$

Prop

(E, g) s.v.e.r.

$\mathcal{R} = \{e_1, .., e_n\} \xrightarrow{C} \mathcal{R}' = \{e'_1, .., e'_n\}$ repere ortonormate.

$\Rightarrow C \in O(n)$

Dem

$$\delta_{rs} = g(e'_r, e'_s) = g\left(\sum_{i=1}^{n} c_{ir} e_i, \sum_{j=1}^{n} c_{js} e_j\right)$$
$$= \sum_{i,j=1}^{n} c_{ir} c_{js} \underbrace{g(e_i, e_j)}_{\delta_{ij}} = \sum_{i=1}^{n} c_{ir} c_{is}$$

$I_n = C^T \cdot C \Rightarrow C \in O(n)$

OBS A da un produs scalar $\Leftrightarrow$ a preciza un reper ortonormat

$g : V \times V \to \mathbb{R}$ produs scalar. Considerăm $\mathcal{R} = \{e_1, .., e_n\}$ reper a.î. $g(e_i, e_j) = \delta_{ij}$ $\forall i,j = \overline{1,n}$.

Reciproc $\mathcal{R} = \{e_1, .., e_n\}$ reper ortonormat

$g(e_i, e_j) = \delta_{ij}$

Prelungim g prin biliniaritate

$$g(x, y) = g\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \sum_{i,j=1}^{n} x_i y_j \overbrace{g(e_i, e_j)}^{\delta_{ij}} =$$
$$= \sum_{i=1}^{n} x_i y_i$$

$G = I_n$ matricea asociată lui g în rap. cu reperul ortonormat $\mathcal{R} = \{e_1, .., e_n\}$

g este biliniară (din construcție)

$G = G^T \Rightarrow g$ simetrică

$Q(x) = g(x, x) = \sum_{i=1}^{n} (x_i)^2$ are signatura $(n, 0) \Rightarrow$ poz. def

$$\left(\begin{array}{l} g(x,x) > 0, \ \forall x \in V \setminus \{0_V\} \\ g(x,x) = 0 \Rightarrow x_i = 0, \ \forall i = \overline{1,n} \Rightarrow x = 0_V \end{array} \right)$$

Exemplu $g_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g_0(x,y) = x_1y_1 + \dots + x_ny_n$

$\mathcal{R}_0 = \{e_1, \dots, e_n\}$ (reperul canonic) → reper ortonormat

$G = I_n$.

g_0 produs scalar canonic (str. euclidiană canonică).

Produs vectorial

$(\mathbb{R}^3, g_0)$ str. euclidiană canonică. Fie $x, y \in \mathbb{R}^3$

$S = \{x, y\}$.

Definim $z = x \times y$ (produs vectorial) astfel

I) S este SLD, at $z = 0_{\mathbb{R}^3}$

II S este SLI, atunci

a) $\|z\|^2 = \|x \times y\|^2 = g_0(z, z) = \begin{vmatrix} g_0(x,x) & g_0(x,y) \\ g_0(y,x) & g_0(y,y) \end{vmatrix}$

b) $g_0(x, z) = 0$, $g_0(y, z) = 0$.

c) reperul $\mathcal{R} = \{x, y, z = x \times y\}$ reper pozitiv orientat (la fel orientat cu reperul canonic $\mathcal{R}_0$)

OBS $x \times y$ este un determinant „formal"

$x = x_1e_1 + x_2e_2 + x_3e_3$

$y = y_1e_1 + y_2e_2 + y_3e_3$

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$x \times y = e_1 \cdot \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$x \times y = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)$$

Prop $(\mathbb{R}^3, g_0)$

a) $x \times y = - y \times x$ („$\times$" este anticomutativ)

b) $(x \times y) \times z = \langle z, x \rangle y - \langle z, y \rangle x$ (temă)

c) (identitatea Jacobi) $(x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$

$$\sum_{x,y,z}^{c} (x \times y) \times z = (x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$$

Def (produs mixt)

$(\mathbb{R}^3, g_0)$ $\quad x, y, z \in \mathbb{R}^3$

$$z \wedge x \wedge y \overset{def}{=} g_0(z, x \times y) = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Obs

$$z \wedge x \wedge y = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \underset{\displaystyle g_0(x, y \times z)}{\underset{\shortparallel}{x \wedge y \wedge z}}$$

Exemplu

$(\mathbb{R}^3, g_0)$ $\quad g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$

$u = (1, -1, 2)$, $v = (0, 1, 3)$, $w = (1, 1, 0)$

a) $u \times v$, b) $w \wedge u \wedge v$

SOL a)
$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = e_1 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + e_3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$= (-5, -3, 1)$

$$rg \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} = 2 \ (\max) \Rightarrow$$

$S = \{u, v\}$ e SLI

b)
$$w \wedge u \wedge v = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = g_0(w, u \times v)$$
$$= g_0((1,1,0), (-5,-3,1))$$
$$= -5 - 3 = -8$$

Problemă

$$\mathcal{R} \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R}''$$

reper $\forall$ reper ortogonal   reper ortonormat

Procedeul de ortogonalizare Gram - Schmidt

(E, g) s.v.e.r. , $\dim E = n$.

$\mathcal{R} = \{f_1, \dots, f_n\}$ reper arbitrar

$\Rightarrow \exists\ \mathcal{R}' = \{e_1, \dots, e_n\}$ reper ortogonal ai $Sp\{f_1, \dots, f_i\} = Sp\{e_1 \dots e_i\}$, $i = \overline{1,n}$ (sp. generat)

Dem

Dem este inductivă

$f_1 \neq 0_V \qquad e_1 = f_1$

Fie $e_2 = f_2 + \alpha_{21} f_1 = \underline{f_2 + \alpha_{21} e_1}$

$0 = \langle e_2, e_1 \rangle = \langle f_2 + \underline{\alpha_{21}} e_1, e_1 \rangle = \langle f_2, e_1 \rangle + \alpha_{21} \langle e_1, e_1 \rangle$

$$\alpha_{21} = -\frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \Rightarrow e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 .$$

$$\begin{cases} f_1 = e_1 . \\ f_2 = \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 . \end{cases} \Rightarrow Sp\{f_1, f_2\} = Sp\{e_1, e_2\}.$$

Pp. adev P_{k-1}. $\{e_1, \dots, e_{k-1}\}$ vect. ortog , $Sp\{f_1, \dots, f_i\} = Sp\{e_1 \dots e_i\}$, $i = \overline{1, k-1}$

Dem P_k.

$$e_k = f_k + \sum_{i=1}^{k-1} \alpha_{ki} e_i$$

$$0 = \langle e_k, e_1 \rangle = \langle f_k + \sum_{i=1}^{k-1} \alpha_{ki} e_i, e_1 \rangle = \langle f_k, e_1 \rangle + \sum_{i=1}^{k-1} \alpha_{ki} \cdot \langle e_i, e_1 \rangle$$

$$0 = \langle f_k, e_1 \rangle + \alpha_{k1} \langle e_1, e_1 \rangle \Rightarrow \alpha_{k1} = -\frac{\langle f_k, e_1 \rangle}{\langle e_1, e_1 \rangle}$$

$\vdots$

$$0 = \langle e_k, e_{k-1} \rangle \Rightarrow \alpha_{k\,k-1} = -\frac{\langle f_k, e_{k-1} \rangle}{\langle e_{k-1}, e_{k-1} \rangle}$$

$$e_k = f_k - \sum_{j=1}^{k-1} \frac{\langle f_k, e_j \rangle}{\langle e_j, e_j \rangle} \cdot e_j$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_1, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 + e_2 \\ \vdots \\ f_k = \frac{\langle f_k, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 + \dots + \frac{\langle f_k, e_{k-1} \rangle}{\langle e_{k-1}, e_{k-1} \rangle} e_{k-1} + e_k \end{cases} \Rightarrow$$

$$Sp\{f_1, \dots, f_i\} = Sp\{e_1, \dots, e_i\}, \ \forall i = \overline{1,k}$$

După n pasi construim

$R' = \{e_1, \dots, e_n\}$ sist. de vectori ortogonali (nenuli)

Lemă (E, g)

$S = \{v_1, \dots, v_k\}$ $\underbrace{\text{vect. ortog}}_{\text{nenuli}} \Rightarrow S$ este SLI

$k \leq n$, $n = \dim E$.

Dem (lemă)

$$\sum_{i=1}^{k} a_i v_i = 0 \Rightarrow a_1 v_1 + \dots + a_k v_k = 0$$

$$\left.\begin{array}{l} \langle a_1 v_1 + \dots + a_k v_k, v_1 \rangle = 0 \Rightarrow a_1 = 0 \\ \qquad \shortparallel \\ \quad a_1 \langle v_1, v_1 \rangle \\ \vdots \\ \langle a_1 v_1 + \dots + a_k v_k, v_k \rangle = 0 \Rightarrow a_k = 0 \\ \qquad \shortparallel \\ \quad a_k \langle v_k, v_k \rangle \end{array}\right| \Rightarrow S \text{ este SLI}$$

$$\left.\begin{array}{l} \text{Cf lemei } R' \text{ este SLI} \\ |R'| = \dim E = n \end{array}\right\} \Rightarrow R' \text{ este un reper ortogonal.}$$

Obs

$$\underset{\text{reper arb.}}{R = \{f_1, \dots, f_n\}} \underset{A}{\overset{A^{-1}}{\rightleftarrows}} \underset{\text{reper ortogonal}}{R' = \{e_1, \dots, e_n\}} \xrightarrow{B} \underset{\text{reper ortonormat}}{R'' = \left\{\frac{e_1}{\|e_1\|}, \dots, \frac{e_n}{\|e_n\|}\right\}}$$

OBS

a) $v \neq 0 \Rightarrow \left\| \frac{v}{\|v\|} \right\| = 1$.

$$\|x\| = \sqrt{g(x,x)} = \sqrt{\langle x, x\rangle}.$$

$$\left\| \frac{v}{\|v\|} \right\| = \sqrt{\left\langle \frac{v}{\|v\|}, \frac{v}{\|v\|} \right\rangle} = \sqrt{\frac{1}{\|v\|^2} \underbrace{\langle v, v\rangle}_{\|v\|^2}} = 1.$$

b) $\|\alpha v\| = |\alpha| \cdot \|v\|$

OBS $\mathcal{R}, \mathcal{R}', \mathcal{R}''$ sunt repere la fel orientate.

$$\mathcal{R}' \xrightarrow{A} \mathcal{R}$$

$$A = \begin{pmatrix} 1 & \frac{\langle f_1, e_1\rangle}{\langle e_1, e_1\rangle} & \dots & \frac{\langle f_n, e_1\rangle}{\langle e_1, e_1\rangle} \\ 0 & 1 & & \vdots \\ 0 & 0 & & \\ \vdots & \vdots & & \frac{\langle f_n, e_{n-1}\rangle}{\langle e_{n-1}, e_{n-1}\rangle} \\ 0 & 0 & & 1 \end{pmatrix} \Rightarrow \det A = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det A} = 1$$

$$\mathcal{R}' \xrightarrow{B} \mathcal{R}'' \qquad B = \begin{pmatrix} \frac{1}{\|e_1\|} & 0 & \dots & 0 \\ 0 & \frac{1}{\|e_2\|} & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & & \frac{1}{\|e_n\|} \end{pmatrix}$$

$$\det B = \frac{1}{\|e_1\|} \cdots \frac{1}{\|e_n\|} > 0$$

$\mathcal{R}, \mathcal{R}', \mathcal{R}''$ sunt la fel orientate.

OBS $\mathcal{R} = \{f_1, \dots, f_n\} \longrightarrow \mathcal{R}' = \{e_1, \dots, e_n\}$

$$\begin{cases} e_1 = f_1 \\ e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 \\ e_3 = f_3 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} \cdot e_2 \\ \vdots \\ e_n = f_n - \frac{\langle f_n, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \dots - \frac{\langle f_n, e_{n-1}\rangle}{\langle e_{n-1}, e_{n-1}\rangle} e_{n-1}. \end{cases}$$

Def (E, g) s.v.e.r., $x \in E$, $x \neq 0_E$.

a) $<\{x\}>^{\perp} = \{x\}^{\perp} = \{y \in E \mid g(x, y) = 0\} \subseteq E$ subsp. vect

b) $U \subseteq E$ subsp. vect

$U^{\perp} = \{y \in E \mid g(x, y) = 0, x \in U\} \subseteq E$ subsp. vect

Exemplu

$(\mathbb{R}^3, g_0)$, $u = (1, 2, -1)$

a) $<\{u\}>^{\perp}$; b) Det. un reper ortonormat în $<\{u\}>^{\perp}$

SOL

a) $<\{u\}>^{\perp} = \{x \in \mathbb{R}^3 \mid g_0(x, u) = 0\} = \{(x_1, x_2, x_1 + 2x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$x = (x_1, x_2, x_3)$, $u = (1, 2, -1)$, $g_0(x, u) = x_1 + 2x_2 - x_3$

$(x_1, x_2, x_1 + 2x_2) = (x_1, 0, x_1) + (0, x_2, 2x_2)$

$x_3 = x_1 + 2x_2$

$U = <\{u\}>^{\perp} = <\{(1, 0, 1), (0, 1, 2)\}>$, $\dim U = 3 - 1 = 2$

$\mathcal{R} = \{f_1 = (1, 0, 1), f_2 = (0, 1, 2)\}$ reper $\forall$ în U.

Aplicăm Gram-Schmidt

$e_1 = f_1 = (1, 0, 1)$

$$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1, e_1>} \cdot e_1 = (0, 1, 2) - \frac{2}{2}(1, 0, 1)$$

$$= (0, 1, 2) - (1, 0, 1) = (-1, 1, 1)$$

$\mathcal{R}' = \{e_1 = (1, 0, 1), e_2 = (-1, 1, 1)\}$ reper ortogonal

$$\|x\| = \sqrt{g_0(x, x)} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$\mathcal{R}'' = \left\{ \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}(1, 0, 1), \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{3}}(-1, 1, 1) \right\}$ reper ortonormat.

<u>Teoremă</u> (E, g) s.v.e.r., $U \subseteq E$ subsp vect.

$\Rightarrow E = U \oplus U^{\perp}$ (scrierea este unică)

<u>Dem</u>

Dem că $\oplus$.

Fie $x \in U \cap U^{\perp} \Rightarrow g(x,x) = 0 \xrightarrow{g \text{ poz. def}} x = 0$ ✓

$E \supseteq U \oplus U^{\perp}$ (din construcție)

Dem că $E \subseteq U \oplus U^{\perp}$

Fie $R = \{e_1, .., e_k\}$ reper $\overset{\text{ortonormat}}{\text{în}}$ U.

Fie $v \in E$ și $\quad v' = v - \underbrace{\sum_{i=1}^{k} \langle v, e_i \rangle \cdot e_i}_{v'' \in U}$

$v = \underset{U}{\underset{\cap}{v''}} + v'$?

Dem că $v' \in U^{\perp}$

$\langle v', e_1 \rangle = \langle v - \sum_{i=1}^{k} \langle v, e_i \rangle \cdot e_i, e_1 \rangle$

$= \langle v, e_1 \rangle - \underbrace{\sum_{i=1}^{k} \langle v, e_i \rangle \langle e_i, e_1 \rangle}$

$= \langle v, e_1 \rangle - \langle v, e_1 \rangle \underset{1}{\underset{\parallel}{\langle e_1, e_1 \rangle}} = 0$

Analog $\langle v', e_j \rangle = 0, \forall j = \overline{2, k}$

$\Rightarrow \langle v', x \rangle = 0, \forall x \in U \Rightarrow v' \in U^{\perp}$

$v = \underset{U}{\underset{\cap}{v''}} + \underset{U^{\perp}}{\underset{\cap}{v'}}$

<u>Exemplul 1</u> $(\mathbb{R}^4, g_0)$

$U = \left\{ x \in \mathbb{R}^4 \,\middle|\, \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \right\}$ $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$

a) $U^{\perp}$; b) Să se det un reper ortonormat $R = R_1 \cup R_2$ în $\mathbb{R}^4$, aî R_1 reper ortonormat în U

R_2 — // — $U^{\perp}$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 = -x_3 \\ x_1 + x_2 = x_4 \end{cases} \qquad \begin{cases} x_1 = -\frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_2 = \frac{1}{2}x_3 + \frac{1}{2}x_4 \end{cases}$$

$$2x_1 \ / \ = -x_3 + x_4$$

$$U = \{ (-\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4, \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4, x_3, x_4) \ , \ x_3, x_4 \in \mathbb{R} \}$$

$$\tfrac{1}{2}x_3(-1, 1, 2, 0) + \tfrac{1}{2}x_4(1, 1, 0, 2)$$

$$\dim U = 4 - 2 = 2$$

$$U = \langle \{ \underbrace{(-1,1,2,0)}_{f_1}, \underbrace{(1,1,0,2)}_{f_2} \} \qquad R_1' = \{f_1, f_2\} \text{ reper în } U \text{ ortogonal}$$

$$g_0(f_1, f_2) = 0 \ ..$$

$$U^\perp = \left\{ x \in \mathbb{R}^4 \ \middle| \ \begin{cases} g_0(x, f_1) = 0 \\ g_0(x, f_2) = 0 \end{cases} \right\}$$

$$= \left\{ x \in \mathbb{R}^4 \ \middle| \ \begin{cases} -x_1 + x_2 + 2x_3 = 0 \\ x_1 + x_2 + 2x_4 = 0 \end{cases} \right\} \qquad \begin{pmatrix} \boxed{\begin{matrix} -1 & 1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} \end{pmatrix}$$

$$\mathbb{R}^4 = U \oplus U^\perp$$

$$\begin{cases} -x_1 + x_2 = -2x_3 \\ x_1 + x_2 = -2x_4 \end{cases} \qquad \begin{cases} x_2 = -x_3 - x_4 \\ x_1 = x_3 - x_4 \end{cases}$$

$$U^\perp = \{ (x_3 - x_4, -x_3 - x_4, x_3, x_4) \mid x_3, x_4 \in \mathbb{R} \}$$

$$= \langle \{ \underbrace{(1,-1,1,0)}_{f_3}, \underbrace{(-1,-1,0,1)}_{f_4} \} \rangle \quad R_2' = \{f_3, f_4\} \text{ reper ortogonal în } U^\perp$$

$$R_1 = \left\{ \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{6}}(-1,1,2,0), \ \frac{f_2}{\|f_2\|} = \frac{1}{\sqrt{6}}(1,1,0,2) \right\} \text{ reper orton în } U$$

$$R_2 = \left\{ \frac{f_3}{\|f_3\|} = \frac{1}{\sqrt{3}}(1,-1,1,0), \ \frac{f_4}{\|f_4\|} = \frac{1}{\sqrt{3}}(-1,-1,0,1) \right\} \ \text{—"—} \ U^\perp$$

$$R = R_1 \cup R_2 .$$