Supplemental Material – Towards Quantum Advantage via Topological Data Analysis

I.   LLSD IS DQC1-HARD

Following the definition of [1], for any problem $L \in \mathsf{DQC1}$ and every $x \in L$, there exists a quantum circuit U of depth $T \in \mathcal{O}\left(\text{poly}(|x|)\right)$ that operates on $n \in \mathcal{O}\left(\text{poly}(|x|)\right)$ qubits such that

- $x \in L_{yes} \implies p_0 \geq \frac{1}{2} + \frac{1}{\text{poly}(|x|)}$,

- $x \in L_{no} \implies p_0 \leq \frac{1}{2} - \frac{1}{\text{poly}(|x|)}$,

where $p_0 = \text{Tr}\left[(|0\rangle\langle 0| \otimes I)U\rho U^\dagger\right]$ and $\rho = |0\rangle\langle 0| \otimes I/2^{n-1}$. From this it can be gathered that if we can estimate p_0 to within $1/\text{poly}(|x|)$ additive precision, then we can solve L.

For a positive semidefinite matrix $H \in \mathbb{C}^{2^n \times 2^n}$ and a threshold $b \in \mathbb{R}_{\geq 0}$, we define the *normalized subtrace* of H up to b as

$$\overline{\text{Tr}_b}(H) = \frac{1}{2^n} \sum_{0 \leq \lambda_k \leq b} \lambda_k,$$

where $\lambda_0 \leq \cdots \leq \lambda_{2^n-1}$ denote the eigenvalues of H. The following result by Brandão shows that if we can estimate the normalized subtrace $\overline{\text{Tr}_b}$ of log-local Hamiltonians up to additive inverse polynomial precision, then we can solve any problem in $\mathsf{DQC1}$. In other words, estimating $\overline{\text{Tr}_b}$ of log-local Hamiltonians up to additive inverse polynomial precision is $\mathsf{DQC1}$-hard.

Proposition 1 (Brandão [2]). *Given as input a description of an n-qubit quantum circuit U of depth $T \in \mathcal{O}\left(poly(n)\right)$ together with a polynomial $r(n)$, one can efficiently construct a log-local Hamiltonian $H \in \mathbb{C}^{T2^n \times T2^n}$ and a threshold $b \in \mathcal{O}\left(poly(n)\right)$ such that*

$$\left|\overline{\text{Tr}_b}(H) - p_0\right| \leq \frac{1}{r(n)}, \tag{1}$$

where $p_0 = \text{Tr}\left[(|0\rangle\langle 0| \otimes I)U\rho U^\dagger\right]$ and $\rho = |0\rangle\langle 0| \otimes I/2^{n-1}$. Moreover, H also satisfies:

(i) H is positive semidefinite.

(ii) There exists a $\delta \in \Omega\left(1/poly(n)\right)$ such that H has no eigenvalues in the interval $[b, b+\delta]$.

Remark. *The Hamiltonian in the above proposition is obtained by applying Kitaev's circuit-to-Hamiltonian construction directly to the circuit U, but only constraining the input and output of the clean qubit while leaving the other qubits unconstrained (emulating the maximally mixed state).*

We will show that we can efficiently estimate the normalized subtrace $\overline{\text{Tr}_b}$ in Equation 1 to within additive inverse polynomial precision using an oracle for LLSD. To be precise, we show that we can estimate this normalized subtrace to within additive inverse polynomial precision using a polynomial amount of nonadaptive queries to an oracle for LLSD (whose input is restricted to log-local Hamiltonians), together with polynomial-time classical preprocessing of the inputs and postprocessing of the outputs. In other words, we provide a polynomial-time truth-table reduction from the problem of estimating $\overline{\text{Tr}_b}$ to LLSD. We gather this in Lemma 2, which together with Proposition 1 shows that LLSD with the input restricted to log-local Hamiltonians is $\mathsf{DQC1}$-hard under polynomial-time truth-table reductions.

Lemma 2. *Given as input $H \in \mathbb{C}^{T2^n \times T2^n}$ and $b \in \mathcal{O}\left(poly(n)\right)$ as described in Proposition 1, together with a polynomial $q(n)$, one can compute a quantity Λ that satisfies*

$$|\Lambda - \overline{\text{Tr}_b}(H)| \leq \frac{1}{q(n)},$$

using a polynomial number of queries to an oracle for LLSD, together with polynomial-time classical preprocessing of the inputs and postprocessing of the outputs.

Proof. Define $\Delta = (3q(n))^{-1}$, $M = b/\Delta$, $\epsilon = (6Mbq(n))^{-1}$ and let $\delta < \Delta/3$ be such that H has no eigenvalues in the interval $[b, b+\delta]$. Also, define the thresholds $x_j = (j+1)\Delta$, for $j = 0, \ldots, M-1$. Next, denote by $\hat{\chi}_j$ the outcome of LLSD with threshold $b = x_j$ and precision parameters δ, ϵ as defined above. That is, $\hat{\chi}_j$ is an estimate of $\hat{y}_j$ to within additive accuracy ϵ, where

$$\hat{y}_j = N_H(0, x_j) + \hat{\gamma}_j, \text{ with } 0 \le \hat{\gamma}_j \le N_H(x_j, x_j + \delta).$$

Subsequently, define $\chi_0 = \hat{\chi}_0$, $y_0 = \hat{y}_0$ and

$$y_j = \hat{y}_j - \hat{y}_{j-1}, \tag{2}$$
$$\chi_j = \hat{\chi}_j - \hat{\chi}_{j-1}, \tag{3}$$

for $1 \le j \le M-1$. Finally, define the estimate

$$\Lambda = \sum_{j=0}^{M-1} \chi_j x_j. \tag{4}$$

We will show that Λ is indeed an estimate of $\overline{\mathrm{Tr}_b}(H)$ to within additive precision $\pm 1/q(n)$. To do so, we define $\gamma_0 = \hat{\gamma}_0$ and $\gamma_j = \hat{\gamma}_j - \hat{\gamma}_{j-1}$ for $1 \le j \le M-1$, and we define and expand

$$\Gamma = \sum_{j=0}^{M-1} y_j x_j = \sum_{j=0}^{M-1} (N_H(x_{j-1}, x_j) + \gamma_j)\, x_j = \underbrace{\sum_{j=0}^{M-1} N_H(x_{j-1}, x_j)x_j}_{\mathcal{B}:=} + \underbrace{\sum_{j=0}^{M-1} \gamma_j x_j}_{\mathcal{E}_{bin}:=}.$$

We start by upper-bounding the magnitude of the $\mathcal{E}_{bin}$ term. To do so, we rewrite

$$
\begin{aligned}
\mathcal{E}_{bin} &= \sum_{j=0}^{M-1} \gamma_j x_j = \hat{\gamma}_0 x_0 + \sum_{j=1}^{M-1} (\hat{\gamma}_j - \hat{\gamma}_{j-1}) x_j \\
&= \sum_{j=0}^{M-1} \hat{\gamma}_j x_j - \sum_{j=1}^{M-1} \hat{\gamma}_{j-1} x_j \\
&= \sum_{j=0}^{M-1} \hat{\gamma}_j x_j - \sum_{j=1}^{M-1} \hat{\gamma}_{j-1}(x_{j-1} + \Delta) \\
&= \sum_{j=0}^{M-1} \hat{\gamma}_j x_j - \sum_{j=1}^{M-1} \hat{\gamma}_{j-1} x_{j-1} - \Delta \sum_{j=1}^{M-1} x_{j-1} \\
&= \underbrace{\hat{\gamma}_{M-1} x_{M-1}}_{=0} - \Delta \underbrace{\sum_{j=1}^{M-1} \hat{\gamma}_{j-1}}_{\le 1},
\end{aligned}
$$

and we conclude that $|\mathcal{E}_{bin}| \le \Delta$. Next, we upper-bound the absolute difference of $\mathcal{B}$ and $\overline{\mathrm{Tr}_b}(H)$.

$$\left|\mathcal{B} - \overline{\mathrm{Tr}_b}(H)\right| = \left| \sum_{j=0}^{M-1} N_H(x_{j-1}, x_j)x_j - \overline{\mathrm{Tr}_b}(H) \right| \le \sum_{j=0}^{M-1} \Delta \cdot N_H(x_{j-1}, x_j) \le \Delta.$$

Finally, we upper-bound the absolute difference between Λ and Γ.

$$|\Lambda - \Gamma| = \left| \sum_{j=0}^{M-1} (\chi_j - y_j) x_j \right| \le \left| \sum_{j=0}^{M-1} 2\epsilon x_j \right| \le M \cdot 2\epsilon \cdot b = \frac{1}{3q(n)}$$

Combining all of the above we find that

$$|\Lambda - \overline{\mathrm{Tr}_b}(H)| \le |\Lambda - \Gamma| + |\Gamma - \overline{\mathrm{Tr}_b}(H)| \le |\Lambda - \Gamma| + |\mathcal{B} - \overline{\mathrm{Tr}_b}(H)| + |\mathcal{E}| \le \frac{1}{3q(n)} + \Delta + \Delta = \frac{1}{q(n)}.$$

$\square$

II. QUANTUM ALGORITHMS FOR SUES AND LLSD

In this section we give a quantum algorithm for SUES and a quantum algorithm for LLSD. Moreover, if the input is a log-local Hamiltonian, then the quantum algorithms we give in this section turn out to be a DQC1 algorithm in the case of LLSD, and a $\mathsf{DQC1}_{\log n}$ algorithm in the case of SUES. That is, if the input is a log-local Hamiltonian, then these algorithms can be implemented in the one clean qubit model, where in the case of SUES we need to measure logarithmically many qubits (as opposed to just one), in order to read out the entire encoding of the eigenvalue.

By scaling the input $H' = H/\Lambda$, where $\Lambda \in \mathcal{O}(\mathrm{poly}(n))$ is an upper bound on the largest eigenvalue of H, we can assume without loss of generality that $||H|| < 1$. Moreover, we will use that allowing up to $\mathcal{O}(\log(n))$ clean qubits does not change the class DQC1 [1]. That is, the class of problems that can be solved in polynomial time using the one clean qubit model of computation is the same as the class of problems that can be solved in polynomial time using the k-clean qubit model of computation, for $k \in \mathcal{O}(\log n)$. We use this result since the quantum algorithms we describe need additional ancilla qubits, which have to be initialized in the all-zeros state and hence be 'clean'.

A. Quantum algorithm for SUES

In this section we describe a quantum algorithm for SUES, which when the input is restricted to log-local Hamiltonians turns out to be a $\mathsf{DQC1}_{\log n}$ algorithm. That is, if the input is a log-local Hamiltonian, then this algorithm can be implemented using the one clean qubit model of computation where we are allowed to measure logarithmically many of the qubits at the end, in order to read out the encoding of the eigenvalue.

The quantum algorithm for SUES implements an approximation of the unitary e^{iH} using Hamiltonian simulation, to which it applies quantum phase estimation with the eigenvector register starting out in the maximally mixed state. In the remainder of this section we will show that we can control the errors such that quantum phase estimation applied to the approximation of e^{iH} outputs the corresponding eigenvalue of H up to precision $\delta \in \Omega(1/\mathrm{poly}(n))$, with error probability $\mu \in \Omega(1/\mathrm{poly}(n))$. Because the maximally mixed state is in a given eigenstate with uniform probabilities over all eigenstates, this shows that this quantum algorithm is able to output a sample from a (δ, μ)-approximation of the uniform distribution over the eigenvalues of H.

Errors can arise in two places, namely due to the imprecisions of the unitary implemented by the Hamiltonian simulation and due to the imprecisions of estimating eigenvalues using quantum phase estimation. First, we discuss the errors of the Hamiltonian simulation step. Given sparse access to H, we can implement a unitary V such that

$$||V - e^{iH}|| < \gamma, \tag{5}$$

in time $\mathcal{O}(\mathrm{poly}(n, \log(1/\gamma)))$ [3]. The algorithms for Hamiltonian simulation of matrices specified by an oracle unfortunately require more than $\mathcal{O}(\log n)$ ancilla qubits, which implies that they can not be implemented using the one clean qubit model. On the other hand, if H is a log-local Hamiltonian, then Hamiltonian simulation techniques based on the Trotter-Suzuki formula can implement a unitary V that satisfies Equation 5 in time $\mathcal{O}(\mathrm{poly}(n, 1/\gamma))$ [4], while only using a constant number of ancilla qubits [5]. Therefore, if H is a log-local Hamiltonian, then using the one clean qubit model we can implement a unitary V that satisfies Equation 5 in time $\mathcal{O}(\mathrm{poly}(n, 1/\gamma))$.

Denote by λ_j and ζ_j the output of the quantum phase estimation routine (where for now we assume that it works perfectly, i.e., introduces no error) when run using e^{iH} and V, respectively. Then, by Equation 5 we have

$$|e^{i\lambda_j} - e^{i\zeta_j}| \leq \gamma,$$

where we assume that $|\lambda_j - \zeta_j| \leq \pi$ by adding multiples of 2π to λ_j if necessary. With some algebra [5], we can show that this implies that

$$|\lambda_j - \zeta_j| \leq \pi\gamma/2.$$

Choosing the accuracy of the Hamiltonian simulation to be $\gamma = \delta/\pi \in \Omega(1/\mathrm{poly}(n))$, we get that

$$|\lambda_j - \zeta_j| < \delta/2. \tag{6}$$

Next, we will consider the errors that arise from using the quantum phase estimation routine to estimate the eigenvalues ζ_j of the unitary V. The quantum phase estimation routine requires a register of t ancilla qubits (also called the eigenvalue register), onto which the eigenvalue will be loaded. If we take

$$t = \log(2/\delta) + \lceil \log(2 + 1/2\mu) \rceil \in \mathcal{O}(\log n)$$

qubits in the eigenvalue register, then quantum phase estimation outputs an estimate $\overline{\zeta_j}$ that satisfies

$$|\overline{\zeta_j} - \zeta_j| \leq \delta/2,$$

with probability at least $(1 - \mu)$ [6]. In particular, with probability at least $(1 - \mu)$ this estimate satisfies

$$|\overline{\zeta_j} - \lambda_j| \leq |\overline{\zeta_j} - \zeta| + |\zeta_j - \lambda_j| \leq \delta.$$

This requires $\widetilde{\mathcal{O}}(2^t) = \widetilde{\mathcal{O}}(\text{poly}(n))$ applications of the unitary V, each of which can be implemented in $\mathcal{O}(\text{poly}(n))$ time as discussed above. In addition, this quantum phase estimation step requires only $\mathcal{O}(\log n)$ ancilla qubits, making it possible to be implemented using the one clean qubit model.

In conclusion, both the Hamiltonian simulation and the quantum phase estimation can be implemented up to the required precision in time $\mathcal{O}(\text{poly}(n))$. Moreover, if H is a log-local Hamiltonian, then this can be done using the one clean qubit model. Finally, to read out the encoding of the eigenvalue, we need to measure the $t \in \mathcal{O}(\log(n))$ qubits in the eigenvalue register, resulting in a $\mathsf{DQC1}_{\log n}$ algorithm for SUES if the input is a log-local Hamiltonian.

B. Quantum algorithm for LLSD

In this section, we will describe two quantum algorithms for LLSD, both of which turn into $\mathsf{DQC1}$ algorithms when the input is restricted to log-local Hamiltonians. That is, if the input is a log-local Hamiltonian, then these algorithms can be implemented using the one clean qubit model of computation.

1. Counting eigenvalues below the threshold

A straightforward approach is to solving LLSD is to repeatedly sample from the output of SUES and then compute the fraction of samples that lie below the given threshold. The downside of this is that it requires one to measure the entire eigenvalue register consisting of logarithmically many qubits, which is prohibitive as we are only allowed to measure a single qubit in the one clean qubit model. This can be circumvented by simply adding an extra clean qubit and flipping this qubit conditioned on the state in the eigenvalue register being smaller than the given threshold. This extra qubit will be flipped with probability close to the low-lying spectral density, allowing us to obtain a solution to LLSD by only measuring this single qubit. Moreover, if H is a log-local Hamiltonian, then this 'fully quantum' algorithm can be implemented using the one clean qubit model, as it requires only a few more additional clean qubits on top of those required for the quantum algorithm for SUES discussed in Section II A.

Note that the outcome probabilities of this 'fully quantum' algorithm are identical to those obtained by measuring the entire eigenvalue register, followed by classical counting of the number of samples below the given threshold. Consequently, the same error analysis applies in both cases. In the rest of this section we will discuss the error analysis of classically counting the number of samples below the given threshold.

Let $m = \epsilon^{-2} \in \mathcal{O}(\text{poly}(n))$ and draw for $j = 1, \ldots, m$ a sample $\overline{\lambda}_{k_j}$ from SUES with $\delta/2$ as the precision parameter. Next, compute

$$\chi_j = \begin{cases} 1 \text{ if } \overline{\lambda}_{k_j} \in (a - \delta/2, b + \delta/2), \\ 0 \text{ otherwise.} \end{cases}$$

For now we assume that all samples $\overline{\lambda}_{k_j}$ were *correctly sampled*, i.e., each k_j is drawn uniformly at random from the set $\{0, \ldots, 2^n - 1\}$ and $|\lambda_{k_j} - \overline{\lambda}_{k_j}| \leq \delta/2$, where λ_{k_j} denotes the eigenvalue of which $\overline{\lambda}_{k_j}$ is an estimate. We now show that under this assumption the quantity

$$\chi := \frac{1}{m} \sum_{j=1}^{m} \chi_j \tag{7}$$

is, with high probability, a correct solution to LLSD. By the Chernoff-Hoeffding inequality χ is, with high probability, an estimate to within additive precision ϵ of

$$y := \Pr_{\overline{\lambda} \sim \text{SUES}} \left[\overline{\lambda} \in (a - \delta/2, b + \delta/2) \right],$$

where the probability is taken over the $\overline{\lambda}$ being correctly sampled from SUES. Because we assume that the $\overline{\lambda}$ are correctly samples from SUES, we know that they satisfy $|\lambda - \overline{\lambda}| \leq \delta/2$, where λ denotes the eigenvalue of which $\overline{\lambda}$ is an estimate. This implies that

(i) $y \le \Pr_{\lambda \sim_U \{\lambda_j\}_{j=1}^{2^n}} \left[\lambda \in (a - \delta, b + \delta) \right] = N_H(a - \delta, b + \delta)$,

(ii) $y \ge \Pr_{\lambda \sim_U \{\lambda_j\}_{j=1}^{2^n}} \left[\lambda \in (a, b) \right] = N_H(a, b)$,

where the probabilities are taken over the λ being sampled uniformly from the set of all eigenvalues of H. Combining this with the Chernoff-Hoeffding inequality, we find that χ is, with high probability, an estimate of y up to additive precision ϵ, where y satisfies

$$N_H(a, b) \le y \le N_H(a - \delta, b + \delta).$$

That is, if all $\overline{\lambda}_{k_j}$ were sampled correctly from SUES, then χ is with high probability a correct solution to LLSD.

Finally, we consider the probability that all samples $\overline{\lambda}_{k_j}$ were indeed sampled correctly. By the union bound this probability is at least $1 - m\mu$, where μ denotes the sampling error probability of SUES. Because $m \in \mathcal{O}(\text{poly}(n))$, we can choose $\mu \in \Omega\left(1/\text{poly}(\epsilon^{-2}, n)\right) = \Omega(1/\text{poly}(n))$ such that all our samples are sampled correctly with probability close to 1. Therefore, we conclude that the χ defined in Equation 7 is a correct solution to LLSD, with probability close to 1. Moreover, χ can be obtained from a polynomial number of samples from SUES, and can therefore be computed in time $\mathcal{O}(\text{poly}(n))$.

2. Using trace estimation of eigenvalue transform

In our paper, we use a result of Cade & Montanaro [5] to argue that the complexity of estimating the spectral entropy of a Hermitian matrix is closely related to DQC1. In their work, Cade & Montanaro describe a DQC1 algorithm can estimate traces of general functions of Hermitian matrices (i.e., beyond spectral entropies). This algorithm could also be used to extract other interesting properties encoded in the spectrum of the combinatorial Laplacian. To illustrate this and connect even further to this line of work, we provide an alternative algorithm for LLSD based on this algorithm. The main result we will utilize is the following Lemma.

Lemma 3 (Cade & Montanaro [5]). *For a log-local Hamiltonian $H \in \mathbb{C}^{2^n \times 2^n}$, and any log-space polynomial-time computable function $f : I \to [-1, 1]$ (where I contains the spectrum of H) that is Lipschitz continuous with constant K (i.e., $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in I$), there exists a DQC1 algorithm to estimate $\text{Tr}(f(H))/2^n = \sum_j f(\lambda_j)/2^n$ up to additive accuracy $\epsilon(K + 1)$, where λ_j denote the eigenvalues of H, and $\epsilon \in \Omega(1/poly(n))$.*

It is clear that if the function f is the step-function with threshold $b + \delta/2$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \le b + \delta/2, \\ 0 & \text{otherwise,} \end{cases}$$

then the quantity estimated by the algorithm of Lemma 3 is a correct solution to LLSD. However, as this function is not Lipschitz continuous, we will use a smooth approximation based on the following lemma.

Lemma 4 (Smooth approximation of the sign function). *Let $\delta > 0$, $\epsilon \in (0, 1)$ and $\gamma = \frac{\delta\sqrt{2\epsilon - \epsilon^2}}{1 - \epsilon}$. Then, the function $g_\gamma(x) = \frac{x}{\sqrt{x^2 + \gamma^2}}$ satisfies*

(i) *for all $x \in [-2, 2]$: $-1 \le g_\gamma(x) \le 1$,*

(ii) *for all $x \in [-2, 2] \backslash (-\delta, \delta)$: $|g_\gamma(x) - sgn(x)| \le \epsilon$, and*

(iii) *$\sup_{x \in [-2,2]} |g'_\gamma(x)| \le \frac{1}{\gamma}$.*

Proof. (i) It is clear that for all $x \in [-2, 2]$ we have: $-1 \le g_\gamma(-2) \le g_\gamma(x) \le g_\gamma(2) \le 1$.

(ii) Let $x \in (\delta, 2]$, then

$$|g_\gamma(x) - \text{sgn}(x)| = |g_\gamma(x) - 1| \le |g_\gamma(\delta) - 1| = \epsilon.$$

For $x \in [-2, -\delta)$ we note that

$$|g_\gamma(x) - \text{sgn}(x)| = |g_\gamma(x) + 1| \le |g_\gamma(-\delta) + 1| = |-(g_\gamma(\delta) - 1)| = \epsilon.$$

(iii) It is clear that: $\sup_{x\in[-2,2]}|g'_\gamma(x)| = |g'_\gamma(0)| = \frac{1}{\gamma}$.

$\square$

Let $\gamma = \frac{(\delta/2)\sqrt{2\epsilon-\epsilon^2}}{1-\epsilon}$ and define $g = g_\gamma$ as in Lemma 4. We define our smooth approximation of the step-function by

$$\hat{f}(x) = \frac{g(-x+b')+1}{2},$$

where $b' = b + \delta/2$. By Lemma 4 we know that $\hat{f}$ is Lipschitz continuous on $[0,1]$ with constant $1/\gamma \in \mathcal{O}\left(\text{poly}(n)\right)$, and that it satisfies

- for all $x \in [0,1] : 0 \le \hat{f}(x) \le 1$, and

- for all $x \in [0,1]\backslash(b,b+\delta) : |\hat{f}(x) - f(x)| \le \epsilon/2$.

Subsequently, we define our estimation objective

$$y = \frac{1}{2^n}\left(\sum_{j \,:\, \lambda_j\in[0,b]} f(\lambda_j) + \sum_{j \,:\, \lambda_j\in[b,b+\delta]} \hat{f}(\lambda_j)\right),$$

and we note that y indeed satisfies $N_H(0,b) \le y \le N_H(0,b+\delta)$, since

$$y = \frac{1}{2^n}\left(\sum_{j \,:\, \lambda_j\in[0,b]} f(\lambda_j) + \sum_{j \,:\, \lambda_j\in[b,b+\delta]} \hat{f}(\lambda_j)\right)$$

$$= N_H(0,b) + \underbrace{\frac{1}{2^n}\sum_{j \,:\, \lambda_j\in[b,b+\delta]} \underbrace{\hat{f}(\lambda_j)}_{\in[0,1]}}_{\in[0,N_H(b,b+\delta)]}.$$

Now our goal is to use Lemma 3 to obtain an ϵ-approximation of y. To this end, we first define

$$\Lambda = \frac{1}{2^n}\sum_{j=1}^{2^n} \hat{f}(\lambda_j),$$

and we upper-bound the absolute difference between y and Λ as follows

$$\left|\Lambda - y\right| \le \frac{1}{2^n}\left|\sum_{j \,:\, \lambda_j\in[0,b]}\left(\hat{f}(\lambda_j) - f(\lambda_j)\right) + \sum_{j \,:\, \lambda_j\in[b+\delta,1]} \hat{f}(\lambda_j)\right|$$

$$\le \frac{1}{2^n}\left(\sum_{j \,:\, \lambda_j\in[0,b]}\left|\hat{f}(\lambda_j) - f(\lambda_j)\right| + \sum_{j \,:\, \lambda_j\in[b+\delta,1]}\left|\hat{f}(\lambda_j)\right|\right)$$

$$\le \frac{1}{2^n}\left(\sum_{j \,:\, \lambda_j\in[0,b]} \epsilon/2 + \sum_{j \,:\, \lambda_j\in[b+\delta,1]} \epsilon/2\right) \le \epsilon/2.$$

Finally, let χ be the output of the algorithm of Lemma 3 applied to our function $\hat{f}$ with precision parameter $\hat{\epsilon} = \epsilon/(2(K+1)) \in \Omega(1/\text{poly}(n))$. In particular, χ satisfies $|\chi - \Lambda| \le \hat{\epsilon}(K+1) = \epsilon/2$. We conclude that χ is a correct solution to LLSD since

$$|\chi - y| \le |\chi - \Lambda| + |\Lambda - y| \le \epsilon/2 + \epsilon/2 = \epsilon,$$

and y indeed satisfies $N_H(0,b) \le y \le N_H(0,b+\delta)$ as discussed earlier.

III.   SWES IS DQC1-HARD

In this section, we will show that SWES is DQC1-hard. We will do so by showing that we estimate the DQC1-hard normalized subtrace $\overline{\mathrm{Tr}}_b(H)$ from Proposition 1 up to additive polynomial precision $\epsilon \in \Omega(1/\mathrm{poly})$ using a polynomial number of queries to an oracle for SWES, together with polynomial-time classical preprocessing of the input and postprocessing of the output.

First, by considering how H is constructed in [2], we note that $\mathrm{Tr}(H)$ is known and that $\mathrm{Tr}(H)/2^n \in \mathcal{O}\left(\mathrm{poly}(n)\right)$. Next, we define $\hat{\epsilon} = (\epsilon/(\mathrm{Tr}(H)/2^n))$ and $m = 1/\hat{\epsilon}^2$. Subsequently, let $\overline{\lambda}_{k_1}, \ldots, \overline{\lambda}_{k_m}$ denote samples drawn from SWES with estimation precision $\delta/2$, where δ is such that H has no eigenvalues in $[b, b+\delta]$. For now we assume that all samples were *correctly sampled*, i.e., $|\overline{\lambda}_{k_j} - \lambda_{k_j}| \leq \delta/2$, where λ_{k_j} denotes the eigenvalue of which $\overline{\lambda}_{k_j}$ is an estimate. Afterwards, we estimate the normalized subtrace $\overline{\mathrm{Tr}}_b(H)$ by computing the ratio of samples that is below $b + \delta/2$

$$\chi = \frac{1}{m} \sum_{j \,:\, \overline{\lambda}_{k_j} \leq b + \delta/2} 1$$

By the Chernoff-Hoeffding inequality (together with the fact that H has no eigenvalues in $[b, b+\delta]$), this ratio χ is, with high probability, an estimate of

$$\Lambda = \sum_{j \,:\, \lambda_j \leq b} \lambda_j / \mathrm{Tr}(H),$$

up to additive precision $\hat{\epsilon}$. Therefore, $(\mathrm{Tr}(H)/2^n) \cdot \chi$ is, with high probability, an ϵ estimate of $(\mathrm{Tr}(H)/2^n) \cdot \Lambda = \overline{\mathrm{Tr}}_b(H)$.

Finally, we consider the probability that all samples $\overline{\lambda}_{k_j}$ were indeed sampled correctly. By the union bound this probability is $M \cdot \mu$, where μ denotes the sampling error probability of SWES. Because $m \in \mathcal{O}\left(\mathrm{poly}(n)\right)$, we can choose $\mu \in \Omega(1/m) = \mathcal{O}\left(\mathrm{poly}(n)\right)$ such that all our samples are sampled correctly with probability close to 1.

---

[1] P. W. Shor and S. P. Jordan, Quantum Information & Computation **8**, 681 (2008), `arXiv:0707.2831`.

[2] F. G. Brandão, *Entanglement theory and the quantum simulation of many-body physics*, Ph.D. thesis, University of London (2008), `arXiv:0810.0026`.

[3] G. H. Low and I. L. Chuang, Physical review letters **118**, 010501 (2017), `arXiv:1606.02685`.

[4] S. Lloyd, Science , 1073 (1996).

[5] C. Cade and A. Montanaro, in *13th Conference on the Theory of Quantum Computation, Communication and Cryptography* (2018) `arXiv:1706.09279`.

[6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, 10th ed. (Cambridge University Press, 2011).