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A modification of Newton's method for nondifferentiable equations

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Abstract

In this paper, we study the problem of approximating a solution of a nonlinear equation where the operator involved is nondifferentiable. Having the Newton method as origin we construct a uniparametric family of iterative processes to approximate a solution of the equation. To finish we consider several test problems.

MSC: 47H17; 65J15

Keywords: The Secant method; Nondifferentiable operator

1. Introduction

In this paper, we study the problem to approximate a solution of a nonlinear equation

$$H(x) = 0, \tag{1}$$

where $H : \Omega \subseteq X \to Y$ is a continuous but nondifferentiable operator defined on a convex nonempty subset Ω of a Banach space X with values in a Banach space Y. Newton's method [5,11] is the most used iteration to solve (1), as a consequence of its computational efficiency, and is given by

$$x_{n+1} = x_n - (H'(x_n))^{-1}H(x_n), \quad n \geqslant 0, \; x_0 \in \Omega \text{ given}, \tag{2}$$

but this method needs the existence of H'. For this reason Newton's method cannot be applied. In this situation, the well-known Secant method has been considered. An important feature of this method is that it uses divided differences instead of the first derivative of the operator involved. We shall use, as in [12], the known definition for divided differences of an operator. Let us denote

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by $\mathscr{L}(X,Y)$ the space of bounded linear operators from X to Y. An operator $[x,y;H]\in\mathscr{L}(X,Y)$ is called a first order divided difference for the operator H on the points x and y $(x\neq y)$ if the following equality holds:

$$[x,y;H](x-y)=H(x)-H(y). \tag{3}$$

Using this definition, the Secant method is described by the following algorithm:

$$x_{n+1}=x_n-[x_{n-1},x_n;H]^{-1}H(x_n),\quad n\geqslant 0,\ x_{-1},x_0\in\Omega \text{ given.} \tag{4}$$

The order of convergence of this method is superlinear under certain conditions for the divided differences (see [6,12]). This iteration has been also used to solve equations with nondifferentiable operators (see [7,8]). Other types of approximations to the derivative of the operator has been also considered and Newton-like methods are obtained

$$x_{n+1}=x_n-(A(x_n))^{-1}H(x_n),\quad n\geqslant 0,\ x_0\in\Omega. \tag{5}$$

The convergence analysis has been given by several authors [1,3,12–14].

In order to improve the last situations we have worked in two ways. Firstly, taking into account the loss of convergence speed produced when (4) is used instead of (2), a uniparametric family of iterative processes has been defined by

$$\begin{aligned}&x_{-1},x_0\in\Omega,\\&y_n=\lambda x_n+(1-\lambda)x_{n-1},\quad \lambda\in[0,1),\\&x_{n+1}=x_n-[y_n,x_n;H]^{-1}H(x_n),\quad n\geqslant 0.\end{aligned} \tag{6}$$

Observe that (4) is obtained if $\lambda=0$ in (6). This family allows us to improve the convergence speed of Secant method and obtain interesting convergence results [9,10].

Secondly, we have considered the case in which the operator H is such that

$$H(x)=F(x)+G(x),$$

where $F,G:\Omega\subseteq X\to Y$, are nonlinear operators, F is differentiable and G is continuous but nondifferentiable [2,4]. So in [2], Catinas considers

$$A(x_n)=F'(x_n)+[x_{n-1},x_n;G] \tag{7}$$

and obtains superlinear convergence under certain conditions on the operators involved. This choice (7) improves the convergence speed of method

$$x_{n+1}=x_n-(F'(x_n))^{-1}H(x_n),\quad n\geqslant 0,\ x_0\in\Omega$$

and of the Secant method (4).

In the present paper, we combine both arguments and then consider (5) with $A(x_n)=F'(x_n)+[y_n,x_n;G]$; i.e.

$$\begin{aligned}&x_{-1},x_0\in\Omega,\\&y_n=\lambda x_n+(1-\lambda)x_{n-1},\quad \lambda\in[0,1),\\&x_{n+1}=x_n-(F'(x_n)+[y_n,x_n;G])^{-1}H(x_n),\quad n\geqslant 0.\end{aligned} \tag{8}$$

Two are the advantages: the first, the differentiable part of the operator is considered in the optimal situation, namely $F'(x_n)$; and the second, for the nondifferentiable part, the class of iterations (6) is considered, which improves the results given by the Secant method. Then, a more suitable situation for $A(x_n)$ is considered than the known ones until now. Moreover, notice that (8) generalizes the last situations without increasing the operational cost since if $F=0$ and $\lambda=0$ are taken, iteration (4) is obtained; if $F=0$ and $\lambda\in[0,1)$ whatever, (6) is obtained; if $\lambda=0$, the method considered by Catinas and other authors is obtained; and if H is differentiable ($G=0$), Newton's method (2) is obtained.

So, in this paper, a semilocal convergence result is given when mild conditions are required. This generalizes the convergence results obtained by other authors for (4), (6) and (7). Finally, some numerical test are presented where the use of method (8) is justified.

2. Convergence study

In this section, we are going to analyze the semilocal convergence of the uniparametric family (8). For this, we consider $x_{-1},x_0\in\Omega$ and assume

(I) $\|x_{-1}-x_0\|=\alpha$,
(II) there exists $L_0^{-1}=(F'(x_0)+[y_0,x_0;G]t)^{-1}$, such that $\|L_0^{-1}\|\leqslant\beta$,
(III) $\|L_0^{-1}H(x_0)\|\leqslant\eta$,
(IV) $\|F'(x)-F'(y)\|\leqslant\omega_1(\|x-y\|)$; $x,y\in\Omega$, where $\omega_1:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous nondecreasing function,
(V) $\|[x,u;G]-[y,v;G]\|\leqslant\omega_2(\|x-y\|,\|u-v\|)$; $x,y,u,v\in\Omega$, where $\omega_2:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a continuous nondecreasing function in its two arguments,
(VI) there exist a continuous and nondecreasing function $h:[0,1]\to\mathbb{R}_+$, such that $\omega_1(tz)\leqslant h(t)\omega_1(z)$, with $t\in[0,1]$ and $z\in[0,\infty)$. We denote $T=\int_0^1 h(t)\,\mathrm{d}t$.

Note that condition (VI) does not involve any restriction, since h always exists, such that $h(t)=1$, as a consequence of ω_1 being a nondecreasing function. We can even consider $h(t)=\sup_{z\geqslant 0}\omega_1(tz)/\omega_1(z)$. We use it to sharpen the bounds that we obtain for particular expressions, as we will see later.

It is interesting to note that usually it is considered Lipschitz-like conditions on the operators involved. Particularly, in [2]. Catinas assumes that the Fréchet-derivative F' is Lipschitz continuous. We give a semilocal convergence result under rather general situation. Besides, we use a new proof technique, that is, we fix the radius R of the domain of existence and we try to calculate it, so that, the sequence $\{x_n\}$ is contained in the ball whose center is the starting point x_0 and radius R.

Theorem 2.1. *We assume that F is once Fréchet differentiable and for every pair of distinct points $x,y\in\Omega$, there exists a first order divided difference $[x,y;G]\in\mathscr{L}(X,Y)$. Under conditions* (I)–(VI), *we denote by* $m=\max\{\beta(\omega_2((1-\lambda)\alpha,\eta)+T\omega_1(\eta)),\beta(\omega_2((1-\lambda)\eta,\eta)+T\omega_1(\eta))\}$ *and assume that the equation*

$$r\left(1-\frac{m}{1-\beta(\omega_1(r)+\omega_2(r+(1-\lambda)\alpha,r))}\right)-\eta=0, \tag{9}$$

has at least one positive zero, let R be the smallest positive one. We denote by $d = \beta(\omega_1(R) + \omega_2(R + (1-\lambda)\alpha, R))$, if $\overline{B(x_0,R)} \subset \Omega$ and $m + d < 1$, then, the sequence $\{x_n\}$ given by (8) is well defined, remain in $\overline{B(x_0,R)}$ and converges to the unique solution x^* of equation $H(x)=0$ in $\overline{B(x_0,R)}$.

Proof. To simplify the notation, we denote $F'(x_n) + [y_n,x_n;G] = L_n$. Firstly, we prove, by mathematical induction, that the sequence given in (8) is well defined, namely iterative procedure (8) makes sense if, at each step, the operator L_n is invertible and the point x_{n+1} lies in Ω.

From the initial hypotheses, it follows that x_1 is well defined and it is easy to check that $\|x_1 - x_0\| \leqslant \eta < R$. Therefore, $x_1 \in B(x_0,R) \subseteq \Omega$.

Now, using (IV)–(VI) and as ω_1 and ω_2 are nondecreasing, we obtain

$$\begin{aligned}\|I - L_0^{-1}L_1\| &\leqslant \|L_0^{-1}\|\,\|L_0 - L_1\| \leqslant \|L_0^{-1}\|(\|F'(x_0) - F'(x_1)\| + \|[y_0,x_0;G] - [y_1,x_1;G]\|)\\ &\leqslant \|L_0^{-1}\|(\omega_1(\|x_1 - x_0\|) + \omega_2(\|y_1 - y_0\|, \|x_1 - x_0\|))\\ &\leqslant \|L_0^{-1}\|(\omega_1(\|x_1 - x_0\|) + \omega_2(\lambda\|x_1 - x_0\| + (1-\lambda)\|x_0 - x_{-1}\|, \|x_1 - x_0\|))\\ &\leqslant \beta(\omega_1(\eta) + \omega_2(\lambda\eta + (1-\lambda)\alpha, \eta)) \leqslant \beta(\omega_1(R) + \omega_2(R + (1-\lambda)\alpha, R)) < 1\end{aligned}$$

and, by the Banach lemma, L_1^{-1} exists and

$$\|L_1^{-1}\| \leqslant \frac{\beta}{1 - \beta(\omega_1(R) + \omega_2(R + (1-\lambda)\alpha, R))} = \frac{\beta}{1-d}.$$

As F is a differentiable operator, from the Taylor's formula it follows that

$$\begin{aligned}F(x_1) &= F(x_0) + F'(x_0)(x_1 - x_0) + \int_{x_0}^{x_1} (F'(x) - F'(x_0))\,\mathrm{d}x\\ &= F(x_0) + F'(x_0)(x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)\,\mathrm{d}t.\end{aligned}$$

On the other hand, using (3)

$$G(x_1) = G(x_0) - [x_0,x_1;G](x_0 - x_1),$$

and, therefore

$$\begin{aligned}H(x_1) = F(x_1) + G(x_1) &= H(x_0) + F'(x_0)(x_1 - x_0) + [x_0,x_1;G](x_1 - x_0)\\ &\quad + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)\,\mathrm{d}t.\end{aligned}$$

Then, from (8)

$$\begin{aligned}H(x_1) &= -[F'(x_0) + [y_0,x_0;G]](x_1 - x_0) + F'(x_0)(x_1 - x_0) + [x_0,x_1;G](x_1 - x_0)\\ &\quad + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)\,\mathrm{d}t\\ &= ([x_0,x_1;G] - [y_0,x_0;G))(x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)\,\mathrm{d}t,\end{aligned}$$

and

$$\|H(x_1)\| \leqslant \left(\|[x_0,x_1;G]-[y_0,x_0;G)\| + \int_0^1 \|F'(x_0+t(x_1-x_0))-F'(x_0)\|\,\mathrm{d}t \right) \|x_1-x_0\|.$$

Now, by (IV)–(VI), we have

$$\|H(x_1)\| \leqslant \left(\omega_2(\|x_0-y_0\|,\|x_1-x_0\|) + \int_0^1 \omega_1(\|t(x_1-x_0)\|)\,\mathrm{d}t \right) \|x_1-x_0\|.$$

$$\leqslant \left(\omega_2(\|x_0-y_0\|,\|x_1-x_0\|) + \omega_1(\|(x_1-x_0)\|) \int_0^1 h(t)\,\mathrm{d}t \right) \|x_1-x_0\|.$$

$$\leqslant (\omega_2((1-\lambda)\alpha,\eta)+T\omega_1(\eta))\|x_1-x_0\|$$

and consequently, iterate x_2 is well defined since $H(x_1)$ and L_1^{-1} are. Moreover,

$$\|x_2-x_1\| \leqslant \|L_1^{-1}\|\,\|H(x_1)\| \leqslant \frac{m}{1-d}\|x_1-x_0\| = M\|x_1-x_0\| < \eta,$$

where $M=m/(1-d)$.

On the other hand, if we take into account that R is a solution of (9), then

$$\|x_2-x_0\| \leqslant \|x_2-x_1\|+\|x_1-x_0\| \leqslant (M+1)\|x_1-x_0\| \leqslant (M+1)\eta < R$$

and $x_2 \in B(x_0,R)$.

Then, by induction on n, from the previous reasoning, it is easy to prove the following items for $n \geqslant 2$:

(i$_n$) $\exists L_n^{-1}=(F'(x_n)+[y_n,x_n;G])^{-1}$ such that $\|L_n^{-1}\| \leqslant \beta/(1-d)$,
(ii$_n$) $\|x_{n+1}-x_n\| \leqslant M\|x_n-x_{n-1}\| \leqslant M^n\|x_1-x_0\| \leqslant \eta$ and $x_{n+1}\in B(x_0,R)$.

Secondly, we prove that $\{x_n\}$ is a Cauchy sequence. For $k \geqslant 1$ we obtain

$$\|x_{n+k}-x_n\| \leqslant \|x_{n+k}-x_{n+k-1}\|+\|x_{n+k-1}-x_{n+k-2}\|+\cdots+\|x_{n+1}-x_n\|$$

$$\leqslant [M^{k-1}+M^{k-2}+\cdots+1]\|x_{n+1}-x_n\|$$

$$\leqslant \frac{1-M^k}{1-M}\|x_{n+1}-x_n\| < \frac{1}{1-M}M^n\|x_1-x_0\|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in \overline{B(x_0,R)}$.

Finally, we see that x^* is a zero of H. Since

$$\|H(x_n)\| \leqslant (\omega_2((1-\lambda)\eta,\eta)+T\omega_1(\eta))\|x_n-x_{n-1}\|$$

and $\|x_n-x_{n-1}\| \to 0$ as $n\to\infty$, we obtain $H(x^*)=0$.

To show uniqueness, we assume that there exists a second solution $y^* \in \overline{B(x_0,R)}$ and consider the operator $P=\int_0^1 F'(x^*+t(y^*-x^*))\,\mathrm{d}t+[y^*,x^*;G]$. Since $P(y^*-x^*)=H(y^*)-H(x^*)$, if operator P is invertible then $x^*=y^*$. Indeed,

$$\|L_0^{-1}P-I\| \leqslant \|L_0^{-1}\|\,\|P-L_0\|$$

$$\leqslant \|L_0^{-1}\| \left[\int_0^1 \|F'(x^*+t(y^*-x^*))-F'(x_0)\|\,\mathrm{d}t + \|[y^*,x^*;G]-[y_0,x_0;G]\| \right]$$

$$\leqslant \beta\left[\int_0^1 \omega_1(\|(1-t)(x^*-x_0)+t(y^*-x_0)\|)\,\mathrm{d}t+\omega_2(\|y^*-y_0\|,\|x^*-x_0\|)\right]$$

$$\leqslant \beta\left[\int_0^1 \omega_1((1-t)\|x^*-x_0\|+t\|y^*-x_0\|)\mathrm{d}t+\omega_2(\|y^*-x_0\|+\|x_0-y_0\|,\|x^*-x_0\|)\right]$$

$$\leqslant \beta\left[\int_0^1 \omega_1(R)\,\mathrm{d}t+\omega_2(R+(1-\lambda)\alpha,R)\right]$$

$$=\beta(\omega_1(R)+\omega_2(R+(1-\lambda)\alpha,R))<1$$

and the operator P^{-1} exists. □

Remark. As we said before, in the introduction, this result is true for differentiable operators. So, if H is differentiable ($H=F$), we consider $A(x_n)=F'(x_n)$ and we obtain a semilocal convergence result for the Newton method.

On the other hand, if H does not have differentiable part ($H=G$), taking $A(x_n)=[y_n,x_n;G]$, we obtain a semilocal convergence result for the family (6).

3. Applications

In this section, we present two types of applications. The first one is theoretical, where it is proved the semilocal convergence when the operator F' is not Lipschitz continuous. The second one is practical, to show how the convergence speed for (8) varies according to λ and we compare the methods presented in the paper with the method (7).

3.1. Example 1

Now we apply the semilocal convergence result given above to the following nonlinear system:

$$x^{3/2}-y-\tfrac{3}{4}+\tfrac{1}{9}|x-1|=0,$$

$$y^{3/2}+\tfrac{2}{9}x-\tfrac{3}{8}+\tfrac{1}{9}|y|=0. \tag{10}$$

We therefore have an operator $H:\mathbb{R}^2\to\mathbb{R}^2$ such that $F=(F_1,F_2)$ and $G=(G_1,G_2)$. For $x=(x_1,x_2)\in\mathbb{R}^2$ we take $F_1(x_1,x_2)=x_1^{3/2}-x_2-\frac{3}{4}$, $F_2(x_1,x_2)=x_2^{3/2}+\frac{2}{9}x_1-\frac{3}{8}$, $G_1(x_1,x_2)=\frac{1}{9}|x_1-1|$, $G_2(x_1,x_2)=\frac{1}{9}|x_2|$.

Let $x=(x_1,x_2)\in\mathbb{R}^2$ then our norm will be $\|x\|=\|x\|_\infty=\max_{1\leqslant i\leqslant 2}|x_i|$. The corresponding norm on $A\in\mathbb{R}^2\times\mathbb{R}^2$ is

$$\|A\|=\max_{1\leqslant i\leqslant 2}\sum_{j=1}^{2}|a_{ij}|.$$

For $u,v\in\mathbb{R}^2$, we shall take $[u,v;G]\in\mathscr{L}(\mathbb{R}^2,\mathbb{R}^2)$ as

$$[u,v;G]_{i1}=\frac{G_i(u_1,v_2)-G_i(v_1,v_2)}{u_1-v_1},\quad [u,v;G]_{i2}=\frac{G_i(u_1,u_2)-G_i(u_1,v_2)}{u_2-v_2},\quad i=1,2. \tag{11}$$

Then,

$$F'(x) = \begin{pmatrix} \frac{3}{2}x_1^{1/2} & -1 \\ \frac{2}{9} & \frac{3}{2}x_2^{1/2} \end{pmatrix},$$

$$[u,v;G] = \frac{1}{9}\begin{pmatrix} |u_1-1|-|v_1-1|/u_1-v_1 & 0 \\ 0 & |u_2|-|v_2|/u_2-v_2 \end{pmatrix}.$$

It follows

$$\begin{aligned} \|F'(x)-F'(y)\| &= |\mathrm{diag}\left\{\tfrac{3}{2}(x_i^{1/2}-y_i^{1/2})\right\}| \\ &= \max_{1\leqslant i\leqslant 2}|\tfrac{3}{2}(x_i^{1/2}-y_i^{1/2})| \leqslant \tfrac{3}{2}\max_{1\leqslant i\leqslant 2}|x_i^{1/2}-y_i^{1/2}| \\ &\leqslant \tfrac{3}{2}\left[\max_{1\leqslant i\leqslant 2}|x_i-y_i|\right]^{1/2} = \tfrac{3}{2}\|x-y\|^{1/2} \end{aligned}$$

and

$$\|[x,y;G]-[u,v;G]\| \leqslant \tfrac{2}{9}.$$

Therefore, we consider the functions

$$\omega_1(z) = \tfrac{3}{2}z^{1/2}, \quad \omega_2(s_1,s_2) = \tfrac{2}{9}, \quad h(t) = t^{1/2}.$$

Now, we apply iteration (8) for $\lambda=0$ to approximate the solution of $H(x)=0$. We choose $z_{-1}=(5,5)$ and $z_0=(1,0)$. After three iterations we obtain

$$z_2 = (1.06157, 0.329438) \quad \text{and} \quad z_3 = (1.00309, 0.253723).$$

Then we take $x_{-1}=z_2$ and $x_0=z_3$. With the notation of Theorem 2.1 we can easily obtain the following results:

$$\alpha = 0.0757149, \quad \beta = 1.15189, \quad \eta = 0.00371354, \quad T = \tfrac{2}{3}, \quad m = 0.380907, \quad R = 0.0117634,$$

$$d = 0.443373, \quad m+d < 1.$$

Therefore, the hypotheses of Theorem 2.1 are fulfilled, what ensures that a unique solution x^* of equation $H(x)=0$ exists in $\overline{B(x_0,R)}$. We obtain the vector $x^*=(1,0.25)$ as the solution of system (10).

Note that, the convergence conditions that have been required in [2] is not satisfied in this example. Therefore the result given in [2] cannot be applied.

3.2. Example 2

We will complete this work with an example that shows how the convergence speed for (8) varies along with λ. Consider the system,

$$\begin{aligned} &3x^2y + y^2 - 1 + |x-1|^{3/2} = 0, \\ &x^4 + xy^3 - 1 + |y|^{3/2} = 0. \end{aligned} \tag{12}$$

Table 1
Method (7) with $x_{-1}=(5,5)$ and $x_0=(1,0)$

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x^*-x_n\|$
1	0.7029611634677623	0.5313592243548252	$2.3538\cdot 10^{-1}$
2	1.2870582391737477	0.0509764025163120	$3.48717\cdot 10^{-1}$
3	1.0688539033796350	0.0183797527165918	$1.47537\cdot 10^{-1}$
4	0.9727120859005791	0.3132618441588043	$3.4371\cdot 10^{-2}$
5	0.9397905155513006	0.3343285084854082	$3.08399\cdot 10^{-3}$
6	0.9383204784322101	0.3312908801696560	$4.63665\cdot 10^{-5}$
7	0.9383410858073490	0.3312444807518935	$4.05776\cdot 10^{-8}$
8	0.9383410452295753	0.3312445136372174	$2.96929\cdot 10^{-13}$
9	0.9383410452297656	0.3312445136375143	10^{-17}

Table 2
Method (8) with $\lambda=0.5$ and $x_{-1}=(5,5)$, $x_0=(1,0)$

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x^*-x_n\|$
1	0.8380633103097624	0.4096710209352457	$1.00278\cdot 10^{-1}$
2	0.9671504284898262	0.3088816498691156	$2.88094\cdot 10^{-2}$
3	0.9401498155763583	0.3293393381113563	$1.90518\cdot 10^{-3}$
4	0.9383494363047030	0.3312501668841831	$8.39107\cdot 10^{-6}$
5	0.9383410426310081	0.3312445184176730	$4.78016\cdot 10^{-9}$
6	0.9383410452297709	0.3312445136375069	$7.38298\cdot 10^{-15}$
7	0.9383410452297656	0.3312445136375143	10^{-17}

Table 3
Method (8) for $\lambda=0.99$ and $x_{-1}=(5,5)$, $x_0=(1,0)$

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x^*-x_n\|$
1	0.9812963959707791	0.3345802402686147	$4.29554\cdot 10^{-2}$
2	0.9408773086450006	0.3318555241033399	$2.53626\cdot 10^{-3}$
3	0.9383501073096226	0.3312485113552779	$9.06208\cdot 10^{-6}$
4	0.9383410453055872	0.3312445138367638	$1.9925\cdot 10^{-10}$
5	0.9383410452297656	0.3312445136375143	10^{-17}

We will consider $H:\mathbb{R}^2\to\mathbb{R}^2$, $F=(F_1,F_2)$, $G=(G_1,G_2)$. For $x=(x_1,x_2)\in\mathbb{R}^2$ we take $F_1(x_1,x_2)=3x_1^2x_2+x_2^2-1$, $F_2(x_1,x_2)=x_1^4+x_1x_2^3-1$, $G_1(x_1,x_2)=|x_1-1|^{3/2}$, $G_2(x_1,x_2)=|x_2|^{3/2}$.

For $u,v\in\mathbb{R}^2$, we shall take $[u,v;G]\in\mathscr{L}(\mathbb{R}^2,\mathbb{R}^2)$ as (11) and we consider the max-norm.

Now, we apply several methods to solve (12). See Table 1 for method (7) with $x_{-1}=(5,5)$ and $x_0=(1,0)$. Note that the approximated solution used is

$$x^*=(0.9383410452297656,0.3312445136375143).$$

For method (8) with $\lambda=0.5$, $x_{-1}=(5,5)$ and $x_0=(1,0)$, see Table 2 and for method (8) with $\lambda=0.99$, $x_{-1}=(5,5)$ and $x_0=(1,0)$, see Table 3.

The numerical results indicate that method (7) is not optimal for approximating the solution x^* of $H(x)=0$. Moreover, iteration (8) converges faster to x^* for increasing values of the parameter λ.

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