In the name of God

**HW1**

Deadline: OCT 28th

Email: mlfall2023@gmail.com

---

About submission:

- File's naming format:
  - For Homeworks: "HWX _Student NO."
    Exe: HW2 _400613198.pdf
  - For MATLAB Homeworks: "PRJX _Student NO."
    Exe: PRJ1 _400613198.pdf
- About late submissions: %10 of daily reduction of the HW mark.

---

1) We are tasked with estimating the random varieable $y$ after observing the random variable $x$. If the loss function is defined as:

$$\ell(y, \hat{y}) = e^{|y-\hat{y}|}$$

   a. Find the optimal estimator and simplify it as much as possible.
   b. Assuming $p(y|x) = 2e^{-2y}u(y)$ (where $u(.)$ is Denoted as unit step function), what will be the optimal estimator's prediction for $y$?
   c. Having the same assumption as part b, find $\hat{y}_{MMSE}$ and $\hat{y}_{MAP}$, the output of MMSE and MAP estimator respectively, for y.

2) Check the eigenvalues and eigenvectors of $A, B$, $A^2$, $A^{-1}$ and $A + B$. What is the relation between them?

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

3) For what numbers $c$ and $d$ are $S$ and $T$ positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

4) When S and T are symmetric positive definite, ST might not even be symmetric. But its eigenvalues are still positive. Start from $STx = \lambda x$ and take dot products with $Tx$. Then prove$\lambda > 0$.

5) Find the eigenvalues and unit eigenvectors of $A^T A$ and $AA^T$ . Keep each $Av = \sigma u$. Then construct the singular value decomposition and verify that $A$ equals $U\Sigma V^T$ .

**Fibonacci matrix** $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

6) Under which condition $f(x) = x^T Ax$ is a convex function?

7) For $f(x) = h\big(g(x)\big)$, prove each of these statements:

a. $f$ is convex, if $h$ is convex and nondecreasing and $g$ is convex
b. $f$ is convex, if $h$ is convex and nonincreasing and $g$ is concave
c. $f$ is concave, if $h$ is concave and nondecreasing and $g$ is concave
d. $f$ is concave, if $h$ is concave and nonincreasing and $g$ is convex

8) Newton's method with fixed step size t = 1 can diverge if the initial point is not close to x ? . In this problem we consider two examples.

a. $f(x) = \log(e^x + e^{-x})$ has a unique minimizer $x^* = 0$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 1$and at $x^{(0)} = 1.1$.
b. $f(x) = -\log x + x$ has a unique minimizer $x^* = 1$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 3$.
c. Solve a and b using gradient descent.

*Good luck*