AN EXPLICIT IMPLIED VOLATILITY FORMULA

DAN STEFANICA AND RADOŠ RADOIČIĆ

Abstract. We show that an explicit approximate implied volatility formula can be obtained from a Black–Scholes formula approximation that is 2% accurate. The relative error of the approximate implied volatility is uniformly bounded for options with any moneyness and with arbitrary large or small option maturities and volatilities, including for long dated options and options on highly volatile underlying assets. For options within a large trading range, such as options with maturity less than five years and implied volatility less than 150%, the error of the approximate implied volatility relative to the Black–Scholes implied volatility is less than ten percentage points.

1. Introduction

The Black-Scholes-Merton theory of option pricing initiated in [Black and Scholes (1973), Merton 1973] is one of the most influential theories in finance. Its ultimate result is the celebrated Black-Scholes formula which is used daily by traders around the world. According to [Li and Lee (2007)], it is the most frequently used equation by human beings, ahead of both Newton's laws of motion in classical mechanics and Schrödinger's equation in quantum mechanics. Besides being used to price options on equities, it is also used to value options on futures, options on foreign currencies, and interest rates options.

In practice, the Black-Scholes formula is often used in the opposite direction: we observe the market price of the option and solve for a volatility value, called implied volatility that makes the Black–Scholes value of the option equal to its market price; see [Gatheral (2006)]. Originally suggested in [Latane and Rendleman (1976)], the implied volatility serves as a forward-looking measure of expectation about future market movements and its use is so pervasive in the financial markets that it is common practice for the implied volatility rather than the option price to be quoted. It is worth noting that only in extreme regimes (long/short expiry and fixed strike, or large/small strike and fixed expiry) the asymptotics of the implied volatility are well understood and determined to arbitrarily high order of precision; see [Benaim,Friz,Lee (2009), Gao and Lee (2014)].

Since the inversion of the Black-Scholes formula cannot be done in closed form, numerical solvers such as Newton-Raphson and Dekker-Brent are needed. While Newton-Raphson has quadratic convergence (when it does converge, see [Manaster and Koehler (1982)]), it has pitfalls such as divisions by the option vegas that can be extremely small for away-from-the-money options; see, e.g., [Jackel (2006)]. Most commercial software uses the considerably slower Dekker-Brent algorithm that is guaranteed to converge. Although they work well for single options, both solvers are slow for real-time applications when millions of options need to be simultaneously inverted.

To overcome these inefficiencies, analytical approximations and quasi-iterative methods can be used as alternatives to numerical solvers.

Most analytical approximations are based on the Taylor series expansion around 0 of the $N(\cdot)$ terms from the Black–Scholes formulas (5–6), where the second (or third) and higher order terms are ignored. This yields equations of degree at most 3 that are explicitly solvable for the

This work was supported by the Office of the Dean of the Weissman School of Arts and Sciences at Baruch College. The authors would like to thank their colleagues Jim Gatheral, Andrew Lesniewski, and Ivan Matić, as well as to Peter Carr and Travis Fisher for their comments and suggestions.

Baruch College, City University of New York, email: dan.stefanica@baruch.cuny.edu; corresponding author.
Baruch College, City University of New York, email: rados.radoicic@baruch.cuny.edu.

integrated volatility $\sigma\sqrt{T}$. The approximation formulas of [Brenner and Subrahmanyam (1988), Brenner and Subrahmanyam (1994)], [Feinstein (1988), Haug (2006)]), [Corrado and Miller Jr. (1996-1), Corrado and Miller Jr. (1996-2)], [Bharadia,Christofides,Salkin (1996)], [Hallerbach (2004)], [Li (2005)], and [Liang and Tahara (2009)] are all cast in this same mold. However, these formulas are accurate enough only in the moneyness[1] domain $0.9 < \eta < 1.1$ if the maturity is longer than 3 months, and in the domain $0.95 < \eta < 1.05$ if the maturity is longer than 1 month, with [Corrado and Miller Jr. (1996-1), Corrado and Miller Jr. (1996-2)] claiming the best accuracy in that moneyness range, closely followed in [Li (2005)]; see [Bridges,Curtis Jr.,Isengildina-Massa,Nian (2007)]. The square root terms in these formulas often yielding imaginary numbers, combined with the serious lack of precision for options that are more than 10% away from at-the-money, make these approximations impractical.

In contrast with the methods above, [Hofstetter and Selby (2001)] obtain an analytic approximation by replacing the $N(\cdot)$ terms by appropriate logistic distribution counterparts, which are, in turn, approximated by second-order Taylor series expansions around $\eta = 1$. However, the local nature of the Taylor expansions causes the performance of this method to deteriorate when options are more than 20% away from at-the-money.

The commercial software for computing Black-Scholes prices usually uses an economized rational approximation of the cumulative normal distribution, a method dating back to [Cody (1969)]; see [Acklam (2003), Jackel (2015)]. Hence, it was not long before this method was employed to invert the Black-Scholes formula, obtaining a better fit over a wider moneyness region, while losing some of the goodness of fit near at-the-money. [Li (2008)] used a fine mesh of about one million options to densely populate the 2-dimensional domain of approximation in log moneyness and integrated volatility and subsequently employed the downhill simplex search method to minimize an objective function, obtaining a rational function approximation with over 30 parameters. This approximation and a further improvement in [Li and Lee (2007)] are computationally intensive and devoid of intuition, and are still tailored to a specific fine mesh of options that does not cover options whose bid-ask spread is larger than the bid-ask midpoint.

A quasi-iterative method to invert the Black–Scholes formula was introduced in [Chance (1993), Chance (1996)] and later simplified in [Chambers and Nawalkha (2001)] and [Kelly (2006)]. The first step finds an initial guess σ_{ATM} of the volatility value. A significant shortcoming is already apparent, since the price of another at-the-money option must be known in order to provide a starting point. A second order Taylor series expansion is then performed around the at-the-money option leading to a quadratic equation in $\Delta\sigma = \sigma - \sigma_{ATM}$. These quasi-iterative methods have very limited effectiveness for options far from the money, with the greatest errors occurring for short term options with low vegas.

The above review of approaches to approximating Black-Scholes implied volatilities and their various setbacks articulates the need for a simple and computationally efficient explicit approximation formula.

Simple closed-form approximation formulas for the Black–Scholes implied volatility and the value of ATM-forward options, i.e., options with moneyness equal to 1, were introduced in [Pianca (2005)]. In [Stefanica,Radoicic (2016)], we showed that these formulas have relative errors that are uniformly bounded for options with arbitrary maturities and implied volatilities, and in [Matic,Radoicic,Stefanica (2017)] we further extended the ATM-forward approximation formulas from [Pianca (2005), Stefanica,Radoicic (2016)].

In this paper, we extend the approximation formula from [Stefanica,Radoicic (2016)] to options with arbitrary moneyness and find that the relative approximation errors are also uniformly bounded for any moneyness, for all maturities, and for arbitrarily large or small implied volatilities. Let

$$\begin{aligned} C_{approx} &= S_0 e^{-qT} A(d_1) \; - \; K e^{-rT} A(d_2); \\ P_{approx} &= K e^{-rT} A(-d_2) \; - \; S_0 e^{-qT} A(-d_1) \end{aligned}$$

[1] Here, $\eta = (Ke^{-rT})/(S_0 e^{-qT}) = K/F$ is the "quasi-debt ratio" introduced in [Merton 1973] and now usually referred to as the moneyness ratio.

be the approximation for the Black–Scholes formulas for European options obtained by replacing the $N(\cdot)$ terms by corresponding $A(\cdot)$ terms in (5–6), where $A(x)$ is the Pólya approximation function given by (8).

An approximate implied volatility $\sigma_{imp,approx,C}$ corresponding to a no–arbitrage call price C_m can be obtained by finding the unique value of σ such that $C_m = C_{approx}(\sigma)$. Similarly, an approximate implied volatility $\sigma_{imp,approx,P}$ corresponding to a no–arbitrage put price P_m can be obtained by finding the unique value of σ such that $P_m = P_{approx}(\sigma)$.

In Section 2, we solve explicitly $C_m = C_{approx}(\sigma)$ for $\sigma = \sigma_{imp,approx,C}$ and $P_m = P_{approx}(\sigma)$ for $\sigma = \sigma_{imp,approx,P}$, cf. Theorem 1. Pseudocodes for computing approximate implied volatilities are included in Table 1 and Table 2 for practical use.

TABLE 1. Pseudocode for the approximate implied volatility of call options

Input:
C_m = market price of call option
K = option strike
T = option maturity
F = forward price at T of underlying asset
r = constant interest rate
Output:
$\sigma_{imp,approx}$ = implied volatility approximation

$y = \ln\left(\frac{F}{K}\right)$; $\alpha_C = \frac{C_m}{Ke^{-rT}}$; $R = 2\alpha_C - e^y + 1$
Compute A, B, C from Table 3
$\beta = \frac{2C}{B+\sqrt{B^2+4AC}}$; $\gamma = -\frac{\pi}{2}\ln\beta$

if $y \geq 0$, $C_0 = Ke^{-rT}\left(e^y A(\sqrt{2y}) - \frac{1}{2}\right)$
  if $C_m \leq C_0$, $\sigma = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} - \sqrt{\gamma-y}\right)$
  if $C_m > C_0$, $\sigma = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} + \sqrt{\gamma-y}\right)$

if $y < 0$, $C_0 = Ke^{-rT}\left(\frac{e^y}{2} - A(-\sqrt{-2y})\right)$
  if $C_m \leq C_0$, $\sigma = \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma+y} + \sqrt{\gamma-y}\right)$
  if $C_m > C_0$, $\sigma = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} + \sqrt{\gamma-y}\right)$

$\sigma_{imp,approx} = \sigma$

TABLE 2. Pseudocode for the approximate implied volatility of put options

Input:
P_m = market price of put option
K = option strike
T = option maturity
F = forward price at T of underlying asset
r = constant interest rate
Output:
$\sigma_{imp,approx}$ = implied volatility approximation

$y = \ln\left(\frac{F}{K}\right)$; $\alpha_P = \frac{P_m}{Ke^{-rT}}$; $R = 2\alpha_P + e^y - 1$
Compute A, B, C from Table 3
$\beta = \frac{2C}{B+\sqrt{B^2+4AC}}$; $\gamma = -\frac{\pi}{2}\ln\beta$

if $y \geq 0$, $P_0 = Ke^{-rT}\left(\frac{1}{2} - e^y A(-\sqrt{2y})\right)$
  if $P_m \leq P_0$, $\sigma = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} - \sqrt{\gamma-y}\right)$
  if $P_m > P_0$, $\sigma = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} + \sqrt{\gamma-y}\right)$

if $y < 0$, $P_0 = Ke^{-rT}\left(A(\sqrt{-2y}) - \frac{e^y}{2}\right)$
  if $P_m \leq P_0$, $\sigma = \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma+y} + \sqrt{\gamma-y}\right)$
  if $P_m > P_0$, $\sigma = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} + \sqrt{\gamma-y}\right)$

$\sigma_{imp,approx} = \sigma$

TABLE 3. Values of A, B, and C for the codes from Table 1 and Table 2

$$A = \left(e^{\left(1-\frac{2}{\pi}\right)y} - e^{-\left(1-\frac{2}{\pi}\right)y}\right)^2$$
$$B = 4\left(e^{\frac{2}{\pi}y} + e^{-\frac{2}{\pi}y}\right) - 2e^{-y}\left(e^{\left(1-\frac{2}{\pi}\right)y} + e^{-\left(1-\frac{2}{\pi}\right)y}\right)\left(e^{2y} + 1 - R^2\right)$$
$$C = e^{-2y}\left(R^2 - (e^y-1)^2\right)\left((e^y+1)^2 - R^2\right)$$

In Section 3, we investigate the sharpness of the relative error of the approximate implied volatility for any moneyness. We obtain that the approximate implied volatility is within 12% of the corresponding Black–Scholes implied volatility, for options with any moneyness, and with arbitrary large or small option maturities and volatilities, i.e.,

$$-0.0418 \; < \; \frac{\sigma_{imp,BS,C} - \sigma_{imp,approx,C}}{\sigma_{imp,BS,C}} \; < \; 0.1138; \tag{1}$$

$$-0.0418 \; < \; \frac{\sigma_{imp,BS,P} - \sigma_{imp,approx,P}}{\sigma_{imp,BS,P}} \; < \; 0.1138. \tag{2}$$

While the relative error bounds (1) and (2) hold for all options, their precision is significantly better for options within trading range. For example, the absolute error $|\sigma_{imp,BS} - \sigma_{imp,approx}|$ is less than ten percentage points for integrated total volatility less than 4, i.e., for $\sigma_{imp,BS}\sqrt{T} < 4$. Moreover, it was established in [Gatheral,Matic,Radoicic,Stefanica (2017)] that $\sigma_{imp,approx,C}$ and $\sigma_{imp,approx,P}$ are sharp lower bounds for the Black–Scholes implied volatility which made them applicable for numerical computations of implied volatilities using the bisection method, as demonstrated on a set of SPX option closing prices of interest to options market makers; see also [Tehranchi (2016)]. As reported in [Le Floc'h (2017)], our approximation is more robust than the rational approximation of [Li (2008)] that applies only to a fixed range of strikes and maturities. Using our approximation as an initial guess for the adaptive SOR solver of [Li and Lee (2011)] results in a simple and efficient code, whose speed and accuracy (relative and absolute) improve upon the state-of-the-art solver of [Jackel (2015)]. Interested practitioners can find an open-source C++ implementation (by Spanderen) of this code in QuantLib [Spanderen 2017], where it was reported to be two to three times faster than their previously most efficient implementation on a large set of OTM and ITM options.

Note that (1) and (2) extend a result from [Stefanica,Radoicic (2016)], where we proved that

$$0 \ < \ \frac{\sigma_{imp,BS,ATM} - \sigma_{imp,approx,ATM}}{\sigma_{imp,BS,ATM}} \ < \ 1 - \frac{\sqrt{\pi}}{2} \approx 0.1138 \tag{3}$$

for any $\sigma_{imp,BS,ATM}$ and for any maturity T, for zero forward moneyness, i.e., for strike price K equal to the forward price F. The relative error bound (3) holding for any maturities and volatility levels was the first such bound derived in literature.

In Section 4, we prove that the option value approximations C_{approx} and P_{approx} are remarkably sharp approximations (within 2%) of the Black–Scholes values for in–the–money forward call and put options with arbitrary maturities and implied volatilities, including long dated options as well as options on highly volatile underlying assets. More precisely, for in–the–money forward call options,

$$-0.0067 \ < \ \frac{C_{approx} - C_{BS}}{C_{approx}} \ < \ 0.02, \tag{4}$$

see Theorem 2, and an identical result holds for in–the–money forward put options; cf. Theorem 3. This extends a similar result from [Stefanica,Radoicic (2016)] for ATM-forward options. Comments on why a similar result cannot hold for OTM-forward options are also included therein. The relative error bound (4) for arbitrary option moneyness is the first such bound established in the literature, and we note that [Hentschel (2003)] argues that minimizing the error in the normalized call price is more meaningful than minimizing the error in the integrated volatility.

Approximation formulas with similar properties can be introduced for Black's formula [Black (1976)] for futures options and the corresponding implied volatilities.

2. Explicit Approximation Formulas for Option Prices and Implied Volatility

The Black-Scholes values of European call and put options with maturity T and strike K on an asset with spot price S_0 following a lognormal distribution with volatility σ and paying dividends continuously at rate q, assuming a constant risk–free interest rate r, are, respectively,

$$C_{BS} \ = \ S_0 e^{-qT} N(d_1) \ - \ K e^{-rT} N(d_2); \tag{5}$$

$$P_{BS} \ = \ K e^{-rT} N(-d_2) \ - \ S_0 e^{-qT} N(-d_1), \tag{6}$$

where $N(x)$ is the cumulative distribution function of the standard normal and

$$d_1 \ = \ \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}; \quad d_2 \ = \ \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}. \tag{7}$$

To introduce an approximation formula for the Black–Scholes values of options with arbitrary moneyness, let $A(x)$ be the function given by

$$A(x) \;=\; \frac{1}{2} + \frac{\mathrm{sgn}(x)}{2}\sqrt{1-e^{-\frac{2x^2}{\pi}}} \;=\; \begin{cases} \frac{1}{2}+\frac{1}{2}\sqrt{1-e^{-\frac{2x^2}{\pi}}}, & \text{if } x \;\geq\; 0; \\ \frac{1}{2}-\frac{1}{2}\sqrt{1-e^{-\frac{2x^2}{\pi}}}, & \text{if } x \;<\; 0. \end{cases} \tag{8}$$

In Pólya [Polya (1949)], it is established that $A(x)$ is an accurate approximation for the cumulative density function $N(x)$, i.e.,

$$0 < A(x) - N(x) < 0.003, \; \forall \; x > 0; \quad 0 < N(x) - A(x) < 0.003, \; \forall \; x < 0. \tag{9}$$

Using the function $A(x)$ instead of $N(x)$ in (5–6),[2] we obtain the following approximation formulas for the Black–Scholes values of European options:

$$C_{approx} \;=\; S_0 e^{-qT} A(d_1) \;-\; K e^{-rT} A(d_2); \tag{10}$$

$$P_{approx} \;=\; K e^{-rT} A(-d_2) \;-\; S_0 e^{-qT} A(-d_1), \tag{11}$$

where d_1 and d_2 are given by (7). Note that C_{approx} and P_{approx} are consistent with the Put–Call parity; cf. Lemma 1.

If C_m is the market value of a plain vanilla European put option satisfying the model–independent no–arbitrage condition

$$\max(S_0 e^{-qT} - K e^{-rT}, 0) \;<\; C_m \;<\; S_0 e^{-qT}, \tag{12}$$

then the approximate implied volatility $\sigma_{imp,approx,C}$ corresponding to the no–arbitrage market price C_m can be obtained by finding the unique value $\sigma = \sigma_{imp,approx,C}$ such that

$$C_m \;=\; C_{approx}(\sigma); \tag{13}$$

cf. Lemma 3. Similarly, we show in Lemma 3 that, if P_m is the market value of a plain vanilla European put option satisfying the no–arbitrage condition

$$\max(K e^{-rT} - S_0 e^{-qT}, 0) \;<\; P_m \;<\; K e^{-rT}, \tag{14}$$

then the approximate implied volatility $\sigma_{imp,approx,P}$ corresponding to the no–arbitrage market price P_m can be obtained by finding the unique value $\sigma = \sigma_{imp,approx,P}$ such that

$$P_m \;=\; P_{approx}(\sigma). \tag{15}$$

The approximate implied volatilities $\sigma_{imp,approx,C}$ and $\sigma_{imp,approx,P}$ are equal for call and put options with the same strike and maturity on the same underlying asset, see Lemma 4 and will be denoted by $\sigma_{imp,approx}$ throughout the rest of the paper.

Unlike the implied volatility corresponding to Black–Scholes option values, for which explicit formulas do not exist, it is possible to solve (13) and (15) *exactly* for $\sigma = \sigma_{imp,approx}$.

Theorem 1. *Explicit solutions $\sigma = \sigma_{imp,approx}$ of (13) and of (15) exist and are as follows:*

Denote by $F = S_0 e^{(r-q)T}$ the forward price at time T of the underlying asset and let $y = \ln(F/K)$. Let $\alpha_C = C_m/(Ke^{-rT})$ and $\alpha_P = P_m/(Ke^{-rT})$, and let A, B, and C given by

$$A \;=\; \left(e^{\left(1-\frac{2}{\pi}\right)y} - e^{-\left(1-\frac{2}{\pi}\right)y}\right)^2; \tag{16}$$

$$B \;=\; 4\left(e^{\frac{2}{\pi}y} + e^{-\frac{2}{\pi}y}\right) \;-\; 2e^{-y}\left(e^{\left(1-\frac{2}{\pi}\right)y} + e^{-\left(1-\frac{2}{\pi}\right)y}\right)(e^{2y}+1-R^2); \tag{17}$$

$$C \;=\; e^{-2y}\left(R^2 - (e^y-1)^2\right)\left((e^y+1)^2 - R^2\right), \tag{18}$$

where

$$R \;=\; \begin{cases} 2\alpha_C - e^y + 1 & \textit{for a call option}; \\ 2\alpha_P + e^y - 1 & \textit{for a put option}. \end{cases}$$

[2] In [Matic,Radoicic,Stefanica (2016)], we introduced a more precise approximation of $N(x)$ for all values of x based on truncated Taylor series expansions of the function $A(x)$. However, that approximation does not lead to explicit formulas for the corresponding approximate implied volatility.

Let

$$\beta = \frac{2C}{B+\sqrt{B^2+4AC}}; \quad \gamma \;=\; -\frac{\pi}{2}\ln\beta.$$

If $y \geq 0$*, let* $C_0 = Ke^{-rT}\left(e^y A(\sqrt{2y}) - 1/2\right)$ *and* $P_0 = Ke^{-rT}\left(1/2 - e^y A(-\sqrt{2y})\right)$*. Then,*

$$\text{if } C_m > C_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y}+\sqrt{\gamma-y}\right); \tag{19}$$

$$\text{if } C_m \leq C_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y}-\sqrt{\gamma-y}\right); \tag{20}$$

$$\text{if } P_m > P_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y}+\sqrt{\gamma-y}\right); \tag{21}$$

$$\text{if } P_m \leq P_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y}-\sqrt{\gamma-y}\right). \tag{22}$$

If $y < 0$*, let* $C_0 = Ke^{-rT}\left(e^y/2 - A(-\sqrt{-2y})\right)$ *and* $P_0 = Ke^{-rT}\left(A(\sqrt{-2y}) - e^y/2\right)$*. Then,*

$$\text{if } C_m > C_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y}+\sqrt{\gamma-y}\right); \tag{23}$$

$$\text{if } C_m \leq C_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma+y}+\sqrt{\gamma-y}\right); \tag{24}$$

$$\text{if } P_m > P_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y}+\sqrt{\gamma-y}\right); \tag{25}$$

$$\text{if } P_m \leq P_0, \quad \text{then} \quad \sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma+y}+\sqrt{\gamma-y}\right). \tag{26}$$

Note that (19), (20), (23), and (24) are used in the pseudocode from Table 1 for computing the approximate implied volatility of call options, and (21), (22), (25), and (26) are used in the pseudocode from Table 2 for computing the approximate implied volatility of put options.

Proof. Note that $S_0e^{-qT} = Fe^{-rT}$ and the formulas (10, 11) can be written as

$$C_{approx} \;=\; Ke^{-rT}\left(\frac{F}{K}A(d_1) - A(d_2)\right); \tag{27}$$

$$P_{approx} \;=\; Ke^{-rT}\left(A(-d_2) - \frac{F}{K}A(-d_1)\right), \tag{28}$$

where d_1 and d_2, given by (7), can also be written as

$$d_1 \;=\; \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} \;+\; \frac{\sigma\sqrt{T}}{2}; \quad d_2 \;=\; \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} \;-\; \frac{\sigma\sqrt{T}}{2}. \tag{29}$$

Let x and y defined as follows:

$$x = \sigma\sqrt{T}; \quad y = \ln\left(\frac{F}{K}\right). \tag{30}$$

From (29), we obtain that

$$d_1 \;=\; \frac{y}{x}+\frac{x}{2}; \quad d_2 \;=\; \frac{y}{x}-\frac{x}{2}, \tag{31}$$

and the approximate formulas (27) and (28) can be written as

$$C_{approx} \;=\; Ke^{-rT}\left(e^y A\left(\frac{y}{x}+\frac{x}{2}\right) - A\left(\frac{y}{x}-\frac{x}{2}\right)\right); \tag{32}$$

$$P_{approx} \;=\; Ke^{-rT}\left(A\left(-\frac{y}{x}+\frac{x}{2}\right) - e^y A\left(-\frac{y}{x}-\frac{x}{2}\right)\right). \tag{33}$$

From (32), it follows that solving $C_m = C_{approx}(\sigma)$ is the same as solving

$$C_m \;=\; Ke^{-rT}\left(e^y A\left(\frac{y}{x}+\frac{x}{2}\right) - A\left(\frac{y}{x}-\frac{x}{2}\right)\right) \tag{34}$$

for $x = \sigma_{imp,approx}\sqrt{T}$, and, from (33), it follows that solving $P_m = P_{approx}(\sigma)$ is the same as solving

$$P_m \;=\; Ke^{-rT}\left(A\left(-\frac{y}{x}+\frac{x}{2}\right) - e^y A\left(-\frac{y}{x}-\frac{x}{2}\right)\right) \tag{35}$$

for $x = \sigma_{imp,approx}\sqrt{T}$.

Let $\alpha_C = C_m/(Ke^{-rT})$. From (34) and using the definition (8) of $A(\cdot)$, we find that

$$\alpha_C \;=\; e^y\left(\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(d_1)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}+\frac{x}{2}\right)^2}}\right) \;-\; \left(\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(d_2)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}-\frac{x}{2}\right)^2}}\right), \tag{36}$$

which is equivalent to

$$2\alpha_C - e^y + 1 \;=\; \mathrm{sgn}(d_1)e^y\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}+\frac{x}{2}\right)^2}} \;-\; \mathrm{sgn}(d_2)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}-\frac{x}{2}\right)^2}}. \tag{37}$$

Let

$$\beta \;=\; e^{-\frac{2}{\pi}\left(\frac{y^2}{x^2}+\frac{x^2}{4}\right)}. \tag{38}$$

Note that

$$e^{-\frac{2}{\pi}\left(\frac{y}{x}+\frac{x}{2}\right)^2} \;=\; e^{-\frac{2}{\pi}\left(\frac{y^2}{x^2}+\frac{x^2}{4}\right)\;-\;\frac{2y}{\pi}} \;=\; \beta\, e^{-\frac{2y}{\pi}}; \tag{39}$$

$$e^{-\frac{2}{\pi}\left(\frac{y}{x}-\frac{x}{2}\right)^2} \;=\; e^{-\frac{2}{\pi}\left(\frac{y^2}{x^2}+\frac{x^2}{4}\right)\;+\;\frac{2y}{\pi}} \;=\; \beta\, e^{\frac{2y}{\pi}}. \tag{40}$$

Then, the equation (37) can be written as

$$2\alpha_C - e^y + 1 \;=\; \mathrm{sgn}(d_1)e^y\sqrt{1-\beta e^{-\frac{2}{\pi}y}} \;-\; \mathrm{sgn}(d_2)\sqrt{1-\beta e^{\frac{2}{\pi}y}}. \tag{41}$$

Similarly, let $\alpha_P = P_m/(Ke^{-rT})$. From (35) and using the definition (8) of $A(\cdot)$, we find that

$$\alpha_P \;=\; \left(\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(-d_2)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}-\frac{x}{2}\right)^2}}\right) \;-\; e^y\left(\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(-d_1)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}+\frac{x}{2}\right)^2}}\right).$$

Since $\mathrm{sgn}(-d_1) = -\mathrm{sgn}(d_1)$ and $\mathrm{sgn}(-d_2) = -\mathrm{sgn}(d_2)$, we obtain that

$$2\alpha_P + e^y - 1 \;=\; \mathrm{sgn}(d_1)e^y\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}+\frac{x}{2}\right)^2}} \;-\; \mathrm{sgn}(d_2)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{y}{x}-\frac{x}{2}\right)^2}}. \tag{42}$$

Using (39) and (40), the equation (42) can be written as

$$2\alpha_P + e^y - 1 \;=\; \mathrm{sgn}(d_1)e^y\sqrt{1-\beta e^{-\frac{2}{\pi}y}} \;-\; \mathrm{sgn}(d_2)\sqrt{1-\beta e^{\frac{2}{\pi}y}}. \tag{43}$$

Note that both (41) and (43) can be written as

$$R \;=\; \mathrm{sgn}(d_1)e^y\sqrt{1-\beta e^{-\frac{2}{\pi}y}} \;-\; \mathrm{sgn}(d_2)\sqrt{1-\beta e^{\frac{2}{\pi}y}}, \tag{44}$$

where

$$R \;=\; \begin{cases} 2\alpha_C - e^y + 1 & \text{for a call option;} \\ 2\alpha_P + e^y - 1 & \text{for a put option.} \end{cases} \tag{45}$$

Thus, finding the implied volatility $\sigma_{imp,approx}$ for either a call option or for a put option requires solving (44) for $x = \sigma_{imp,approx}\sqrt{T}$.

We now show that if β is a solution of (44), then β is also a solution of the quadratic equation

$$A\beta^2 \;+\; B\beta \;-\; C = 0, \tag{46}$$

where A, B, and C are given by (16), (17), and (18), respectively.

By squaring (44) and using the facts that $(\mathrm{sgn}(t))^2 = 1$ for any $t \neq 0$ and $\mathrm{sgn}(d_1)\mathrm{sgn}(d_2) = \mathrm{sgn}(d_1 d_2)$, we obtain that

$$\begin{aligned}
& R \;=\; \mathrm{sgn}(d_1)e^y\sqrt{1-\beta e^{-\frac{2}{\pi}y}} \;-\; \mathrm{sgn}(d_2)\sqrt{1-\beta e^{\frac{2}{\pi}y}} \\
\Longleftrightarrow\;\; & R^2 \;=\; e^{2y}\left(1-\beta e^{-\frac{2}{\pi}y}\right) \;-\; 2\mathrm{sgn}(d_1d_2)e^y\sqrt{\left(1-\beta e^{-\frac{2}{\pi}y}\right)\left(1-\beta e^{\frac{2}{\pi}y}\right)} \;+\; \left(1-\beta e^{\frac{2}{\pi}y}\right) \\
\Longleftrightarrow\;\; & 2\mathrm{sgn}(d_1d_2)e^y\sqrt{\beta^2-\beta\left(e^{-\frac{2}{\pi}y}+e^{\frac{2}{\pi}y}\right)+1} \;=\; e^{2y}+1-R^2 \;-\; \beta e^y\left(e^{\left(1-\frac{2}{\pi}\right)y}+e^{-\left(1-\frac{2}{\pi}\right)y}\right) \\
\Longleftrightarrow\;\; & 2\mathrm{sgn}(d_1d_2)\sqrt{\beta^2-\beta\left(e^{-\frac{2}{\pi}y}+e^{\frac{2}{\pi}y}\right)+1} \;=\; e^{-y}\left(e^{2y}+1-R^2\right) \;-\; \beta\left(e^{\left(1-\frac{2}{\pi}\right)y}+e^{-\left(1-\frac{2}{\pi}\right)y}\right) \\
\Longleftrightarrow\;\; & 4\left(\beta^2-\beta\left(e^{-\frac{2}{\pi}y}+e^{\frac{2}{\pi}y}\right)+1\right) \;=\; e^{-2y}\left(e^{2y}+1-R^2\right)^2 \;+\; \beta^2\left(e^{\left(1-\frac{2}{\pi}\right)y}+e^{-\left(1-\frac{2}{\pi}\right)y}\right)^2 \\
& -\,2\beta e^{-y}\left(e^{2y}+1-R^2\right)\left(e^{\left(1-\frac{2}{\pi}\right)y}+e^{-\left(1-\frac{2}{\pi}\right)y}\right) \\
\Longleftrightarrow\;\; & A\beta^2 \;+\; B\beta \;-\; C = 0,
\end{aligned}$$

where A, B, and C are given by

$$\begin{aligned}
A &= \left(e^{\left(1-\frac{2}{\pi}\right)y}+e^{-\left(1-\frac{2}{\pi}\right)y}\right)^2-4 \\
&= e^{2\left(1-\frac{2}{\pi}\right)y}-2+e^{-2\left(1-\frac{2}{\pi}\right)y} \\
&= \left(e^{\left(1-\frac{2}{\pi}\right)y}-e^{-\left(1-\frac{2}{\pi}\right)y}\right)^2;
\end{aligned}$$

$$B \;=\; 4\left(e^{-\frac{2}{\pi}y}+e^{\frac{2}{\pi}y}+1\right)-2e^{-y}\left(e^{2y}+1-R^2\right)\left(e^{\left(1-\frac{2}{\pi}\right)y}+e^{-\left(1-\frac{2}{\pi}\right)y}\right);$$

$$\begin{aligned}
C &= 4-e^{-2y}\left(e^{2y}+1-R^2\right)^2 \\
&= e^{-2y}\left(4e^{2y}-\left(e^{2y}+1-R^2\right)^2\right) \\
&= e^{-2y}\left(2e^y-\left(e^{2y}+1-R^2\right)\right)\left(2e^y+\left(e^{2y}+1-R^2\right)\right) \\
&= e^{-2y}\left(R^2-(e^y-1)^2\right)\left((e^y+1)^2-R^2\right),
\end{aligned}$$

which are equivalent to (16), (17), and (18).

Note that, by dividing the no–arbitrage conditions (12) and (14) for the market values of plain vanilla options by Ke^{-rT} , we obtain that

$$\max\left(\frac{F}{K}-1,0\right) \;<\; \frac{C_m}{Ke^{-rT}} \;<\; \frac{F}{K}; \tag{47}$$

$$\max\left(1-\frac{F}{K},0\right) \;<\; \frac{P_m}{Ke^{-rT}} \;<\; 1, \tag{48}$$

where $F = S_0e^{(r-q)T}$. Using the facts that $\alpha_C = C_m/(Ke^{-rT})$, $\alpha_P = P_m/(Ke^{-rT})$, and $y = \ln(F/K)$, we find that (47) and (48) can be written as

$$\max(e^y-1,0) \;<\; \alpha_C \;<\; e^y; \tag{49}$$

$$\max(1-e^y,0) \;<\; \alpha_P \;<\; 1. \tag{50}$$

Since $R = 2\alpha_C - e^y + 1$ for call options, see (45), we find from (49) that

$$\begin{aligned}
& \max(e^y-1,0) \;<\; \alpha_C \;<\; e^y \\
\Longleftrightarrow\;\; & 2\max(e^y-1,0) \;-\; (e^y-1) \;<\; R \;<\; 2e^y-e^y+1 \\
\Longleftrightarrow\;\; & |e^y-1| \;<\; R \;<\; e^y+1,
\end{aligned} \tag{51}$$

where we used the fact that

$$2\max(t,0)-t \;=\; |t|. \tag{52}$$

Similarly, since $R = 2\alpha_P + e^y - 1$ for put options, see (45), we find from (50) and (52) that

$$\begin{aligned} & \max(1-e^y,0) \ < \ \alpha_P \ < \ 1 \\ \Longleftrightarrow \quad & 2\max(1-e^y,0) \ + \ (e^y-1) \ < \ R \ < \ 2+e^y-1 \\ \Longleftrightarrow \quad & |e^y-1| \ < \ R \ < \ e^y+1. \end{aligned} \tag{53}$$

From (51) and (53), we conclude that the no–arbitrage conditions for the market values of plain vanilla options can be written in terms of R as follows:

$$|e^y-1| \ < \ R \ < \ e^y+1. \tag{54}$$

From (18) and (54), we obtain that $C > 0$. Since $A > 0$, see (16), it follows that $B^2+4AC > 0$. Therefore, the quadratic equation (46) has exactly one positive root, equal to

$$\frac{-B+\sqrt{B^2+4AC}}{2A} \ = \ \frac{2C}{B+\sqrt{B^2+4AC}},$$

and, since $\beta = e^{-\frac{2}{\pi}\left(\frac{y^2}{x^2}+\frac{x^2}{4}\right)} > 0$ is a positive root of (46), we conclude that

$$\beta \ = \ \frac{2C}{B+\sqrt{B^2+4AC}}, \tag{55}$$

where A, B and C are given by (16–18).

From (38), we find that

$$-\frac{2}{\pi}\left(\frac{y^2}{x^2}+\frac{x^2}{4}\right) \ = \ \ln\beta. \tag{56}$$

Using the notation $\gamma = -(\pi/2)\ln\beta$, formula (56) becomes

$$\frac{y^2}{x^2}+\frac{x^2}{4} \ = \ \gamma. \tag{57}$$

From (57), it follows that

$$\left(\frac{y}{x}+\frac{x}{2}\right)^2 \ = \ \gamma+y \quad \text{and} \quad \left(\frac{y}{x}-\frac{x}{2}\right)^2 \ = \ \gamma-y$$

and therefore

$$\left|\frac{y}{x}+\frac{x}{2}\right| \ = \ \sqrt{\gamma+y}; \quad \left|\frac{y}{x}-\frac{x}{2}\right| \ = \ \sqrt{\gamma-y}. \tag{58}$$

Since $d_1 = y/x + x/2$ and $d_2 = y/x - x/2$, see (31), we obtain from (58) that

$$|d_1| \ = \ \sqrt{\gamma+y}; \quad |d_2| \ = \ \sqrt{\gamma-y}. \tag{59}$$

Then,

$$\frac{y}{x}+\frac{x}{2} \ = \ d_1 \ = \ \mathrm{sgn}(d_1)|d_1| \ = \ \mathrm{sgn}(d_1)\sqrt{\gamma+y}; \tag{60}$$

$$\frac{y}{x}-\frac{x}{2} \ = \ d_2 \ = \ \mathrm{sgn}(d_2)|d_2| \ = \ \mathrm{sgn}(d_2)\sqrt{\gamma-y}. \tag{61}$$

By subtracting (61) from (60), and recalling that $x = \sigma_{imp,approx}\sqrt{T}$, we conclude that

$$\sigma_{imp,approx} \ = \ \frac{1}{\sqrt{T}}\left(\mathrm{sgn}(d_1)\sqrt{\gamma+y} - \mathrm{sgn}(d_2)\sqrt{\gamma-y}\right). \tag{62}$$

The explicit formulas (19–26) for $\sigma_{imp,approx}$ are obtained by identifying the signs of d_1 and d_2 in (62) in terms of the sign of $y = \ln(F/K)$.

Case 1: If $y \geq 0$, then $d_1 = y/x + x/2 > 0$, since $x > 0$. Thus, $\mathrm{sgn}(d_1) = 1$ and (62) becomes

$$\sigma_{imp,approx} \ = \ \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} - \mathrm{sgn}(d_2)\sqrt{\gamma-y}\right). \tag{63}$$

Since $d_2 = y/x - x/2$ and $x = \sigma_{imp,approx}\sqrt{T}$, it follows that

$$d_2 > 0 \iff \frac{y}{x} > \frac{x}{2} \iff x < \sqrt{2y} \iff \sigma_{imp,approx} < \frac{\sqrt{2y}}{\sqrt{T}} \iff \sigma_{imp,approx} < \sigma_0, \tag{64}$$

where $\sigma_0 = \sqrt{2y}/\sqrt{T}$.

Using the facts that $C_{approx}(\sigma)$ is a strictly increasing function of σ, see Lemma 2, and that $C_{approx}(\sigma_{imp,approx}) = C_m$ by the definition of $\sigma_{imp,approx}$, see (13), we obtain that

$$\sigma_{imp,approx} < \sigma_0 \iff C_{approx}(\sigma_{imp,approx}) < C_{approx}(\sigma_0) \iff C_m < C_0, \tag{65}$$

where $C_0 = C_{approx}(\sigma_0)$ is the value of the approximation formula (32) corresponding to $\sigma_0 = \sqrt{2y}/\sqrt{T}$, and therefore to $x_0 = \sigma_0\sqrt{T} = \sqrt{2y}$, i.e.,

$$\begin{aligned} C_0 &= Ke^{-rT}\left(e^y A\left(\frac{y}{x_0}+\frac{x_0}{2}\right) - A\left(\frac{y}{x_0}-\frac{x_0}{2}\right)\right) = Ke^{-rT}\left(e^y A(\sqrt{2y}) - A(0)\right) \\ &= Ke^{-rT}\left(e^y A(\sqrt{2y}) - \frac{1}{2}\right). \end{aligned} \tag{66}$$

For put options, since $P_{approx}(\sigma)$ is a strictly increasing function of σ, see Lemma 2, and $P_{approx}(\sigma_{imp,approx}) = P_m$ by the definition of $\sigma_{imp,approx}$, see (15), we obtain that

$$\sigma_{imp,approx} < \sigma_0 \iff P_{approx}(\sigma_{imp,approx}) < P_{approx}(\sigma_0) \iff P_m < P_0, \tag{67}$$

where $P_0 = P_{approx}(\sigma_0)$ is the value of the approximation formula (33) corresponding to $\sigma_0 = \sqrt{2y}/\sqrt{T}$, and therefore to $x_0 = \sigma_0\sqrt{T} = \sqrt{2y}$, i.e.,

$$\begin{aligned} P_0 &= Ke^{-rT}\left(A\left(-\frac{y}{x_0}+\frac{x_0}{2}\right) - e^y A\left(-\frac{y}{x_0}-\frac{x_0}{2}\right)\right) = Ke^{-rT}\left(A(0) - e^y A(-\sqrt{2y})\right) \\ &= Ke^{-rT}\left(\frac{1}{2} - e^y A(-\sqrt{2y})\right). \end{aligned} \tag{68}$$

From (64) and (65), we obtain that $d_2 > 0 \iff C_m < C_0$, where C_0 is given by (66). From (64) and (67), we obtain that $d_2 > 0 \iff P_m < P_0$, where P_0 is given by (68). Then, in both cases, $\operatorname{sgn}(d_2) = 1$, and, from (63), we conclude that

$$\text{if } y \geq 0 \text{ and } C_m < C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} - \sqrt{\gamma-y}\right); \tag{69}$$

$$\text{if } y \geq 0 \text{ and } P_m < P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} - \sqrt{\gamma-y}\right). \tag{70}$$

Similarly, $d_2 < 0 \iff C_m > C_0$ and $d_2 < 0 \iff P_m > P_0$, in which case $\operatorname{sgn}(d_2) = -1$. From (63), it follows that

$$\text{if } y \geq 0 \text{ and } C_m > C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} + \sqrt{\gamma-y}\right);$$

$$\text{if } y \geq 0 \text{ and } P_m > P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} + \sqrt{\gamma-y}\right).$$

Moreover, $d_2 = 0 \iff C_m = C_0$ and $d_2 = 0 \iff P_m = P_0$, in which case $\operatorname{sgn}(d_2) = 0$. From (63), it follows that

$$\text{if } y \geq 0 \text{ and } C_m = C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\sqrt{\gamma+y}; \tag{71}$$

$$\text{if } y \geq 0 \text{ and } P_m = P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\sqrt{\gamma+y}. \tag{72}$$

Note that, if $d_2 = 0$, then $\gamma - y = 0$, see (59). Thus, (69) and (71), and (70) and (72), respectively, can be written compactly as

$$\text{if } y \geq 0 \text{ and } C_m \leq C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} - \sqrt{\gamma-y}\right);$$

$$\text{if } y \geq 0 \text{ and } P_m \leq P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma+y} - \sqrt{\gamma-y}\right).$$

Case 2: If $y < 0$, then $d_2 = y/x - x/2 < 0$ since $x > 0$. Thus, $\mathrm{sgn}(d_2) = -1$, and (62) becomes

$$\sigma_{imp,approx} \;=\; \frac{1}{\sqrt{T}}\left(\mathrm{sgn}(d_1)\sqrt{\gamma+y}+\sqrt{\gamma-y}\right). \tag{73}$$

Since $d_1 = y/x + x/2$ and $x = \sigma_{imp,approx}\sqrt{T}$, it follows that

$$d_1 > 0 \iff \frac{x}{2} > -\frac{y}{x} \iff x > \sqrt{-2y} \iff \sigma_{imp,approx} > \frac{\sqrt{-2y}}{\sqrt{T}} \iff \sigma_{imp,approx} > \sigma_0, \tag{74}$$

where $\sigma_0 = \sqrt{-2y}/\sqrt{T}$.

For call options, we use the facts that $C_{approx}(\sigma)$ is a strictly increasing function of σ, see Lemma 2, and that $C_{approx}(\sigma_{imp,approx}) = C_m$, see (13), to obtain that

$$\sigma_{imp,approx} > \sigma_0 \iff C_{approx}(\sigma_{imp,approx}) > C_{approx}(\sigma_0) \iff C_m > C_0, \tag{75}$$

where $C_0 = C_{approx}(\sigma_0)$ is the value of the approximation formula (32) corresponding to $\sigma_0 = \sqrt{-2y}/\sqrt{T}$, and therefore to $x_0 = \sigma_0\sqrt{T} = \sqrt{-2y}$, i.e.,

$$\begin{aligned} C_0 \;&=\; Ke^{-rT}\left(e^y A\left(\frac{y}{x_0}+\frac{x_0}{2}\right) - A\left(\frac{y}{x_0}-\frac{x_0}{2}\right)\right) \\ &=\; Ke^{-rT}\left(e^y A\left(\frac{y}{\sqrt{-2y}}+\frac{\sqrt{-2y}}{2}\right) - A\left(\frac{y}{\sqrt{-2y}}-\frac{\sqrt{-2y}}{2}\right)\right). \end{aligned} \tag{76}$$

Since $y < 0$, note that $y = -|y| = -\sqrt{y^2}$ and therefore

$$\frac{y}{\sqrt{-2y}} \;=\; \frac{-\sqrt{y^2}}{\sqrt{-2y}} \;=\; -\sqrt{\frac{y^2}{-2y}} \;=\; -\sqrt{\frac{-y}{2}}. \tag{77}$$

Then, from (77), we find that

$$\frac{y}{\sqrt{-2y}}+\frac{\sqrt{-2y}}{2} \;=\; -\sqrt{\frac{-y}{2}}+\sqrt{\frac{-y}{2}} \;=\; 0; \tag{78}$$

$$\frac{y}{\sqrt{-2y}}-\frac{\sqrt{-2y}}{2} \;=\; -\sqrt{\frac{-y}{2}}-\sqrt{\frac{-y}{2}} \;=\; -2\sqrt{\frac{-y}{2}} \;=\; -\sqrt{-2y}. \tag{79}$$

From (76), (78), and (79), and using the fact that $A(0) = 1/2$, we obtain that

$$C_0 \;=\; Ke^{-rT}\left(e^y A(0) - A(-\sqrt{-2y})\right) \;=\; Ke^{-rT}\left(\frac{e^y}{2} - A(-\sqrt{-2y})\right). \tag{80}$$

For put options, since $P_{approx}(\sigma)$ is a strictly increasing function of σ, see Lemma 2, and $P_{approx}(\sigma_{imp,approx}) = P_m$, see (15), we obtain that

$$\sigma_{imp,approx} > \sigma_0 \iff P_{approx}(\sigma_{imp,approx}) > P_{approx}(\sigma_0) \iff P_m > P_0, \tag{81}$$

where $P_0 = P_{approx}(\sigma_0)$ is the value of the approximation formula (33) corresponding to $\sigma_0 = \frac{\sqrt{-2y}}{\sqrt{T}}$, and therefore to $x_0 = \sigma_0\sqrt{T} = \sqrt{-2y}$, i.e.,

$$\begin{aligned} P_0 \;&=\; Ke^{-rT}\left(A\left(-\frac{y}{x_0}+\frac{x_0}{2}\right) - e^y A\left(-\frac{y}{x_0}-\frac{x_0}{2}\right)\right) \\ &=\; Ke^{-rT}\left(A\left(-\frac{y}{\sqrt{-2y}}+\frac{\sqrt{-2y}}{2}\right) - e^y A\left(-\frac{y}{\sqrt{-2y}}-\frac{\sqrt{-2y}}{2}\right)\right) \\ &=\; Ke^{-rT}\left(A\left(\sqrt{-2y}\right) - e^y A(0)\right) && (82) \\ &=\; Ke^{-rT}\left(A\left(\sqrt{-2y}\right) - \frac{e^y}{2}\right), && (83) \end{aligned}$$

where for (82) we used the facts that

$$-\frac{y}{\sqrt{-2y}} + \frac{\sqrt{-2y}}{2} = -\left(\frac{y}{\sqrt{-2y}} - \frac{\sqrt{-2y}}{2}\right) = -\left(-\sqrt{-2y}\right) = \sqrt{-2y};$$

$$-\frac{y}{\sqrt{-2y}} - \frac{\sqrt{-2y}}{2} = -\left(\frac{y}{\sqrt{-2y}} + \frac{\sqrt{-2y}}{2}\right) = 0;$$

see (79) and (78), respectively.

From (74) and (75), we obtain that $d_1 > 0 \Longleftrightarrow C_m > C_0$, where C_0 is given by (80). From (74) and (81), we obtain that $d_1 > 0 \Longleftrightarrow P_m > P_0$, where P_0 is given by (83). Then, in both cases, $\mathrm{sgn}(d_1) = 1$, and, from (73), we conclude that

$$\text{if } y < 0 \text{ and } C_m > C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma + y} + \sqrt{\gamma - y}\right);$$

$$\text{if } y < 0 \text{ and } P_m > P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(\sqrt{\gamma + y} + \sqrt{\gamma - y}\right).$$

Similarly, $d_1 < 0 \Longleftrightarrow C_m < C_0$ and $d_1 < 0 \Longleftrightarrow P_m < P_0$, in which case $\mathrm{sgn}(d_1) = -1$. From (73), it follows that

$$\text{if } y < 0 \text{ and } C_m < C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma + y} + \sqrt{\gamma - y}\right); \tag{84}$$

$$\text{if } y < 0 \text{ and } P_m < P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma + y} + \sqrt{\gamma - y}\right). \tag{85}$$

Moreover, $d_1 = 0 \Longleftrightarrow C_m = C_0$ and $d_1 = 0 \Longleftrightarrow P_m = P_0$, in which case $\mathrm{sgn}(d_1) = 0$. From (73), it follows that

$$\text{if } y < 0 \text{ and } C_m = C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\sqrt{\gamma - y}; \tag{86}$$

$$\text{if } y < 0 \text{ and } P_m = P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\sqrt{\gamma - y}. \tag{87}$$

Note that, if $d_1 = 0$, then $\gamma + y = 0$, see (59). Thus, (84) and (86), and (85) and (87), respectively, can be written compactly as

$$\text{if } y < 0 \text{ and } C_m \leq C_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma + y} + \sqrt{\gamma - y}\right);$$

$$\text{if } y < 0 \text{ and } P_m \leq P_0, \quad \text{then} \quad \sigma_{imp,approx} = \frac{1}{\sqrt{T}}\left(-\sqrt{\gamma + y} + \sqrt{\gamma - y}\right).$$

□

3. The sharpness of the implied volatility approximation

Recall from [Stefanica,Radoicic (2016)] that for at–the–money forward options the implied volatility approximation $\sigma_{imp,approx}$ introduced above is an accurate approximation of the Black–Scholes implied volatility $\sigma_{imp,BS}$ for any integrated volatility, i.e.,

$$0 < \frac{\sigma_{imp,BS,ATM} - \sigma_{imp,approx,ATM}}{\sigma_{imp,BS,ATM}} < 1 - \frac{\sqrt{\pi}}{2} \approx 0.1138, \tag{88}$$

for any $\sigma_{imp,BS,ATM}$ and for any maturity T, where $\sigma_{imp,approx,ATM}$ is the same as $\sigma_{imp,approx}$ for $K = F$.

Similarly, the approximate implied volatility $\sigma_{imp,approx}$ for both call and put options whose explicit form is given in Section 2 and implemented in Theorem 1 and 2 is within 12% of the corresponding Black–Scholes implied volatility for options with any moneyness and with arbitrary large or small option maturities and volatilities, i.e.,

$$-0.0418 < \frac{\sigma_{imp,BS} - \sigma_{imp,approx}}{\sigma_{imp,BS}} < 0.1138; \tag{89}$$

see Figure 1 for a plot of the absolute relative error as a function of the Black–Scholes integrated total volatility $\sigma_{imp,BS}\sqrt{T}$ and log forward moneyness $\ln(F/K)$.

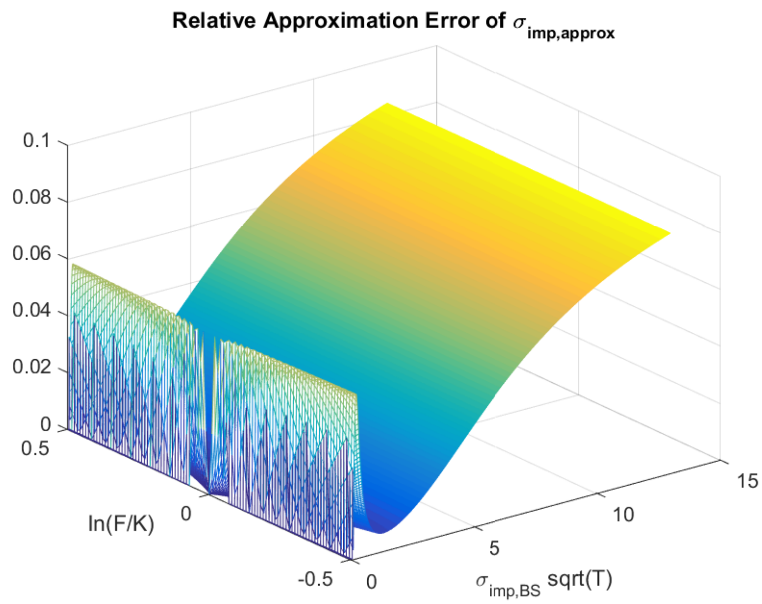

FIGURE 1. Relative approximation errors for the Black–Scholes implied volatility $\frac{|\sigma_{imp,BS}-\sigma_{imp,approx}|}{\sigma_{imp,BS}}$ for log–monenyness $\ln(F/K)$ between -0.5 and 0.5

The bound (89) holds for the implied volatility approximations of both call and put options. More precisely,

$$-0.0418 \;<\; \frac{\sigma_{imp,BS,C} - \sigma_{imp,approx,C}}{\sigma_{imp,BS,C}} \;<\; 0.1138;$$

$$-0.0418 \;<\; \frac{\sigma_{imp,BS,P} - \sigma_{imp,approx,P}}{\sigma_{imp,BS,P}} \;<\; 0.1138,$$

where $\sigma_{imp,BS,C}$ and $\sigma_{imp,approx,C}$ denote the Black–Scholes implied volatility and the approximate implied volatility corresponding to the same call option, and $\sigma_{imp,BS,P}$ and $\sigma_{imp,approx,P}$ denote the Black–Scholes implied volatility and the approximate implied volatility corresponding to the same put option. Note that the approximate implied volatilities for a call and put option with the same strike on the same underlying asset are equal, as was the case for the Black–Scholes implied volatilities; see Lemma 4 for details.

Note that the precision of the relative error bound (89) is significantly better for options within trading range. For example, the absolute error $|\sigma_{imp,BS} - \sigma_{imp,approx}|$ is less than ten percentage points for integrated total volatility less than 4, i.e., for $\sigma_{imp,BS}\sqrt{T} < 4$; see Figure 2.

The approximate values $\sigma_{imp,approx}\sqrt{T}$ corresponding to $\sigma_{imp,BS}\sqrt{T}$ equal to 0.10, 0.15, 0.20, and 0.30, which corresponds to a Black–Scholes implied volatility of about 30% and one month, three months, six months, and one year options, respectively, can be found in Figure 3.

4. THE SHARPNESS OF THE APPROXIMATION FORMULA FOR ITM-FORWARD OPTIONS

In [Stefanica,Radoicic (2016)], we showed that, for ATM-forward options with strike equal to the forward price at maturity of the underlying asset, the approximate values C_{approx} and P_{approx} of ATM-forward call and put options given by (10) and (11) are within 2% of the Black–Scholes values C_{BS} and P_{BS}. In this section, we show that a similar result holds for in–the–money forward call and put options.

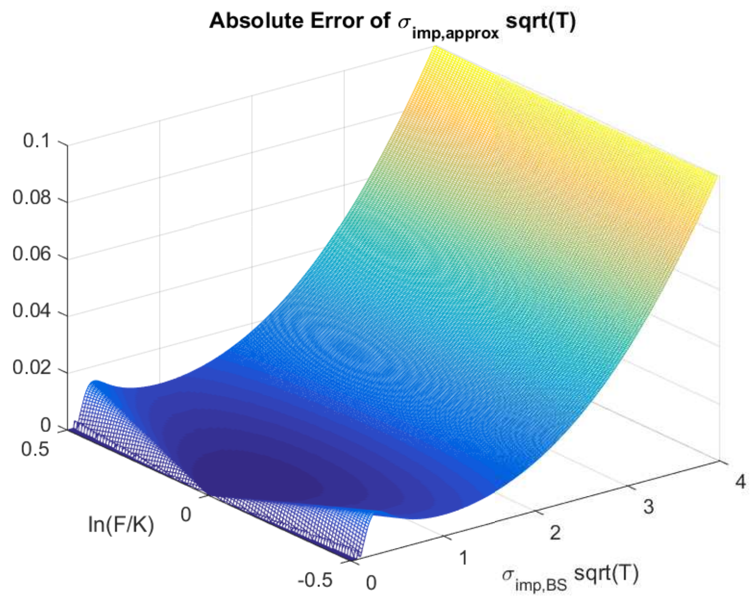

FIGURE 2. Absolute approximation errors for the Black–Scholes implied volatility $|\sigma_{imp,BS}\sqrt{T} - \sigma_{imp,approx}\sqrt{T}|$ for log–monenyness $\ln(F/K)$ between -0.5 and 0.5

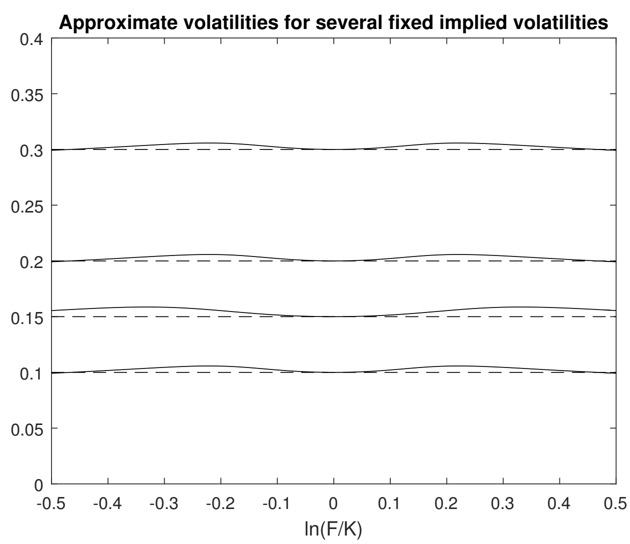

FIGURE 3. Implied volatility approximation $\sigma_{imp,approx}\sqrt{T}$ (solid lines) for $\sigma_{imp,BS}\sqrt{T}$ equal to 0.10, 0.15, 0.20, 0.30 (dotted lines)

Recall from (27) that the approximate value C_{approx} of a call option can be expressed as

$$C_{approx} \;=\; Ke^{-rT}\left(\frac{F}{K}A(d_1) - A(d_2)\right). \tag{90}$$

Similarly, the Black–Scholes value of a call option given by (5) can be expressed as

$$C_{BS} \;=\; Ke^{-rT}\left(\frac{F}{K}N(d_1) - N(d_2)\right). \tag{91}$$

By using the same notation as in (30), i.e., $y = \ln(F/K)$ and $x = \sigma\sqrt{T}$, the formulas (90) and (91) can be written as

$$C_{approx} \;=\; Ke^{-rT}\left(e^y A(d_1) - A(d_2)\right); \tag{92}$$

$$C_{BS} \;=\; Ke^{-rT}\left(e^y N(d_1) - N(d_2)\right), \tag{93}$$

where d_1 and d_2 are given by (29) and can be expressed as

$$d_1 = \frac{y}{x} + \frac{x}{2}; \quad d_2 = \frac{y}{x} - \frac{x}{2}; \tag{94}$$

see also (31). From (94), it follows that

$$x \;=\; d_1 - d_2; \quad y \;=\; \frac{d_1^2 - d_2^2}{2}. \tag{95}$$

In–the–money forward (ITM-forward) call options are call options with $F > K$, where $F = S_0 e^{(r-q)T}$ is the forward price at time T of the underlying asset. Note that $y = \ln(F/K) > 0$ for ITM-forward options and $x = \sigma\sqrt{T} > 0$. Then, from (95), we obtain that

$$d_1 > d_2 \text{ and } d_1 > -d_2. \tag{96}$$

In Theorem 2, we prove that C_{approx} is a remarkably sharp approximation (within 2% accurate) of the Black–Scholes value for ITM-forward call options, for any integrated volatilities $\sigma\sqrt{T}$, including long dated options and options on highly volatile underlying assets.

Theorem 2. *The approximate value C_{approx} of an ITM-forward call option is within 2% of the Black–Scholes option value C_{BS}, i.e.,*

$$-\bar{c} \;<\; \frac{C_{approx} - C_{BS}}{C_{approx}} \;<\; c, \tag{97}$$

for any volatility parameter σ of the underlying asset and for any maturity T, where

$$c = 1 - \left(\frac{4-\pi}{\pi-2}\right)^{\frac{\pi-3}{2}} \approx 0.019982 < 0.02 \quad \textit{and} \quad \bar{c} = 0.0067.$$

Proof. Note that (97) is equivalent to

$$(1-c)\; C_{approx} \;<\; C_{BS} \;<\; (1+\bar{c})\; C_{approx}, \;\forall\; \sigma \geq 0,\; T > 0. \tag{98}$$

From (92) and (93), we obtain that establishing the left hand side inequality from (98) is equivalent to proving that

$$\begin{aligned}
& (1-c)\; C_{approx} \;<\; C_{BS} & (99)\\
\Longleftrightarrow\quad & (1-c)\;\left(e^y A(d_1) - A(d_2)\right) \;<\; e^y N(d_1) - N(d_2) & \\
\Longleftrightarrow\quad & N(d_2) - (1-c)A(d_2) \;<\; e^y\left(N(d_1) - (1-c)A(d_1)\right) & \\
\Longleftrightarrow\quad & e^{\frac{d_2^2}{2}}\left(N(d_2) - (1-c)A(d_2)\right) \;<\; e^{\frac{d_1^2}{2}}\left(N(d_1) - (1-c)A(d_1)\right) & (100)\\
\Longleftrightarrow\quad & f_1(d_1) \;>\; f_1(d_2), & (101)
\end{aligned}$$

where for (100), we used the fact that $y = (d_1^2 - d_2^2)/2$, see (95), and the function $f_1(t)$ is given by

$$f_1(t) \;=\; e^{\frac{t^2}{2}}\left(N(t) - (1-c)A(t)\right).$$

We will use the facts that $f_1(t)$ is a strictly increasing function for $t \geq 0$ and that $f_1(t) > f_1(-t)$ for all $t > 0$, see Lemma 5, to prove (101).

Recall from (96) that, if $y = \ln(F/K) > 0$, then $d_1 > d_2$ and $d_1 > -d_2$.

If $d_2 \geq 0$, then $f_1(d_1) > f_1(d_2)$ since $d_1 > d_2$ and $f_1(t)$ is strictly increasing for $t \geq 0$.

If $d_2 < 0$, then $-d_2 > 0$ and $f_1(-d_2) > f_1(d_2)$ since $f_1(t) > f_1(-t)$ for all $t > 0$. Since $d_1 > -d_2 > 0$, and since $f_1(t)$ is strictly increasing for $t \geq 0$, we find that $f_1(d_1) > f_1(-d_2)$. Thus, $f_1(d_1) > f_1(-d_2) > f_1(d_2)$.

We conclude that $f_1(d_1) > f_1(d_2)$ regardless of the sign of d_2. From (99) and (101), we obtain that the left hand side inequality from (98) is established.

The proof of the right hand side inequality from (98) follows along the same lines. From (92) and (93), we obtain that

$$\begin{aligned}
& C_{BS} \ < \ (1+\bar{c})\ C_{approx} && (102)\\
\Longleftrightarrow\quad & e^y N(d_1) - N(d_2) \ < \ (1+\bar{c})\ \left(e^y A(d_1) - A(d_2)\right) \\
\Longleftrightarrow\quad & (1+\bar{c})A(d_2) - N(d_2) \ < \ e^y\left((1+\bar{c})A(d_1) - N(d_1)\right) \\
\Longleftrightarrow\quad & e^{\frac{d_2^2}{2}}\ \left((1+\bar{c})A(d_2) - N(d_2)\right) \ < \ e^{\frac{d_1^2}{2}}\ \left((1+\bar{c})A(d_1) - N(d_1)\right) && (103)\\
\Longleftrightarrow\quad & f_2(d_1) \ > \ f_2(d_2), && (104)
\end{aligned}$$

where for (103), we used the fact that $y = (d_1^2 - d_2^2)/2$, see (95), and the function $f_2(t)$ is given by

$$f_2(t) \ = \ e^{\frac{t^2}{2}}\ \left((1+\bar{c})A(t) - N(t)\right).$$

In Lemma 6, we show that $f_2(t)$ is a strictly increasing function for $t \geq 0$ and that $f_2(t) > f_2(-t)$, for all $t > 0$, which are the same properties of function $f_1(t)$ which allowed us to show that $f_1(d_1) > f_1(d_2)$; see (101). Thus, following the same proof as above, we obtain that $f_2(d_1) > f_2(d_2)$, which is the same as (104). From (102) and (104), we conclude that the right hand side inequality from (98) holds true. $\square$

A result similar to that of Theorem 2 holds for ITM-forward put options, i.e., for put options with $F < K$; see Theorem 3.

From (28), we obtain that P_{approx} can be expressed as

$$P_{approx} \ = \ Ke^{-rT}A(-d_2) - S_0 e^{-qT}A(-d_1) \ = \ S_0 e^{-qT}\left(\frac{K}{F}A(-d_2) - A(-d_1)\right). \qquad (105)$$

Similarly, the Black–Scholes value of a put option given by (6) can be expressed as

$$P_{BS} \ = \ S_0 e^{-qT}\left(\frac{K}{F}N(-d_2) - N(-d_1)\right). \qquad (106)$$

Let

$$\widehat{y} \ = \ \ln\left(\frac{K}{F}\right) \quad \text{and} \quad x \ = \ \sigma\sqrt{T}.$$

Note that $\widehat{y} > 0$, since $F < K$.

Then, d_1 and d_2, which are given by (29), can be written as follows:

$$\begin{aligned}
d_1 \ &= \ \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \ = \ -\left(\frac{\ln\left(\frac{K}{F}\right)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right) \ = \ -\left(\frac{\widehat{y}}{x} - \frac{x}{2}\right); && (107)\\
d_2 \ &= \ \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} \ = \ -\left(\frac{\ln\left(\frac{K}{F}\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) \ = \ -\left(\frac{\widehat{y}}{x} + \frac{x}{2}\right). && (108)
\end{aligned}$$

Using (107) and (108), the formulas (105) and (106) can be written as

$$\begin{aligned}
P_{approx} \ &= \ S_0 e^{-qT}\left(e^{\widehat{y}}A\left(\frac{\widehat{y}}{x} + \frac{x}{2}\right) \ - \ A\left(\frac{\widehat{y}}{x} - \frac{x}{2}\right)\right); && (109)\\
P_{BS} \ &= \ S_0 e^{-qT}\left(e^{\widehat{y}}N\left(\frac{\widehat{y}}{x} + \frac{x}{2}\right) \ - \ N\left(\frac{\widehat{y}}{x} - \frac{x}{2}\right)\right). && (110)
\end{aligned}$$

Theorem 3. *The approximate value P_{approx} of an ITM-forward put option is within 2% of the Black–Scholes option value P_{BS}, i.e.,*

$$-\bar{c} \;<\; \frac{P_{approx} - P_{BS}}{P_{approx}} \;<\; c, \tag{111}$$

for any volatility parameter σ of the underlying asset and for any maturity T, where

$$c = 1 - \left(\frac{4-\pi}{\pi-2}\right)^{\frac{\pi-3}{2}} \approx 0.019982 < 0.02 \quad \textit{and} \quad \bar{c} = 0.0067.$$

Proof. The inequality (97) established in Theorem 2 can be written using (92), (93), and (94) as follows:

$$-\bar{c} \;<\; 1 - \frac{C_{BS}}{C_{approx}} \;=\; 1 - \frac{e^y N\left(\frac{y}{x} + \frac{x}{2}\right) - N\left(\frac{y}{x} - \frac{x}{2}\right)}{e^y A\left(\frac{y}{x} + \frac{x}{2}\right) - A\left(\frac{y}{x} - \frac{x}{2}\right)} \;<\; c, \;\forall\, y > 0,\; x > 0. \tag{112}$$

Using (109) and (110), the inequality (111) becomes

$$-\bar{c} \;<\; 1 - \frac{e^{\widehat{y}} N\left(\frac{\widehat{y}}{x} + \frac{x}{2}\right) - N\left(\frac{\widehat{y}}{x} - \frac{x}{2}\right)}{e^{\widehat{y}} A\left(\frac{\widehat{y}}{x} + \frac{x}{2}\right) - A\left(\frac{\widehat{y}}{x} - \frac{x}{2}\right)} \;<\; c, \;\forall\, \widehat{y} > 0,\; x > 0, \tag{113}$$

which is the same as (112) for $y = \widehat{y}$, and is therefore established. □

Note that similar relative approximation results cannot hold true for out–of–the–money (OTM) forward options, since, from the Put–Call parity which is satisfied by both the Black–Scholes option values and the approximate option values, see Lemma 1, it follows that

$$C_{BS} - P_{BS} \;=\; S_0 e^{-qT} - K e^{-rT} \;=\; C_{approx} - P_{approx}.$$

Thus, $C_{approx} - C_{BS} = P_{approx} - P_{BS}$ and therefore

$$\frac{C_{approx} - C_{BS}}{C_{BS}} \;=\; \frac{P_{approx} - P_{BS}}{C_{BS}} \;=\; \frac{P_{approx} - P_{BS}}{P_{BS}} \cdot \frac{P_{BS}}{C_{BS}}. \tag{114}$$

An OTM-forward call is a call option with strike higher than the forward price of the underlying asset, i.e., with $K > F$. Then, if $K > F$, the term $(P_{approx} - P_{BS})/P_{BS}$ from (114) corresponds to the relative approximation error of an ITM-forward put and is bounded uniformly by 0.02 as well as bounded away from 0 for any fixed integrated total volatility $\sigma_{imp,BS}\sqrt{T}$. However, the term P_{BS}/C_{BS} from (114) is unbounded since, for large strike K, the value of the OTM-forward call C_{BS} goes to zero, while the value of the ITM-forward put P_{BS} goes to infinity. Thus, the relative approximation error $(C_{approx} - C_{BS})/C_{BS}$ of an OTM-forward call is unbounded.

However, this does not impact the implied volatility results from the previous section, since the implied volatilities for OTM-forward calls are equal to the implied volatilities for ITM-forward puts with the same strike, and, reciprocally, the implied volatilities for OTM-forward puts are equal to the implied volatilities for ITM-forward calls with the same strike.

Appendix A. General Properties of the Approximation Formula

In this section, we show that the approximate option values given by (10–11) satisfy the Put–Call parity and are increasing functions of the volatility parameter (i.e., have positive vega), we establish the existence and uniqueness of an approximate implied volatility for no–arbitrage market values of call and put options, and we show that approximate implied volatilities for calls and puts with the same strike and maturity are equal.

Lemma 1. *The approximate option values C_{approx} and P_{approx} given by (10) and (11), respectively, satisfy the Put–Call parity, i.e.,*

$$C_{approx} - P_{approx} \;=\; S_0 e^{-qT} - Ke^{-rT}. \tag{115}$$

Proof. From (8), we obtain that $A(x) + A(-x) = 1$ for all x and therefore

$$\begin{aligned} C_{approx} - P_{approx} &= (S_0 e^{-qT} A(d_1) - Ke^{-rT} A(d_2)) \;-\; (Ke^{-rT} A(-d_2) - S_0 e^{-qT} A(-d_1)) \\ &= S_0 e^{-qT}(A(d_1) + A(-d_1)) \;-\; Ke^{-rT}(A(d_2) + A(-d_2)) \\ &= S_0 e^{-qT} - Ke^{-rT}. \end{aligned}$$

□

Lemma 2. *The approximate option values C_{approx} and P_{approx} given by (10) and (11), respectively, are both strictly increasing functions of volatility.*

Proof. Since $S_0 e^{-qT}$ and Ke^{-rT} are independent of σ, it follows from the Put–Call parity (115) that P_{approx} and C_{approx} have the same monotonicity as functions of σ. It is thus enough to show that C_{approx} is a strictly increasing function of the volatility σ.

From (32), we find that

$$\frac{C_{approx}}{Ke^{-rT}} \;=\; e^y A\left(\frac{y}{x} + \frac{x}{2}\right) - A\left(\frac{y}{x} - \frac{x}{2}\right), \tag{116}$$

where $y = \ln(F/K)$ and $x = \sigma\sqrt{T}$. From (116), and using the definition (8) of $A(\cdot)$, it follows that showing that C_{approx} is a strictly increasing function of σ is equivalent to showing that the function

$$f(x) \;=\; e^y \left(\frac{1}{2} + \frac{1}{2}\mathrm{sgn}\left(\frac{y}{x} + \frac{x}{2}\right)\sqrt{1 - e^{-\frac{2}{\pi}\left(\frac{y}{x}+\frac{x}{2}\right)^2}}\right) \tag{117}$$

$$-\left(\frac{1}{2} + \frac{1}{2}\mathrm{sgn}\left(\frac{y}{x} - \frac{x}{2}\right)\sqrt{1 - e^{-\frac{2}{\pi}\left(\frac{y}{x}-\frac{x}{2}\right)^2}}\right) \tag{118}$$

is a strictly increasing function of $x = \sigma\sqrt{T}$, for all $x > 0$.

Let

$$z_1 \;=\; e^{-\frac{2}{\pi}\left(\frac{y}{x}+\frac{x}{2}\right)^2}; \quad z_2 \;=\; e^{-\frac{2}{\pi}\left(\frac{y}{x}-\frac{x}{2}\right)^2}. \tag{119}$$

Since $x > 0$, note that

$$\mathrm{sgn}\left(\frac{y}{x} + \frac{x}{2}\right) \;=\; \mathrm{sgn}\left(\frac{2y + x^2}{2x}\right) \;=\; \mathrm{sgn}\left(2y + x^2\right); \tag{120}$$

$$\mathrm{sgn}\left(\frac{y}{x} - \frac{x}{2}\right) \;=\; \mathrm{sgn}\left(\frac{2y - x^2}{2x}\right) \;=\; \mathrm{sgn}\left(2y - x^2\right). \tag{121}$$

Using (120), (121), and the notation (119), the function $f(x)$ given by (117–118) can be expressed as

$$f(x) \;=\; \frac{e^y - 1}{2} \;+\; \mathrm{sgn}\left(2y + x^2\right)\frac{e^y}{2}\sqrt{1 - z_1} \;-\; \mathrm{sgn}\left(2y - x^2\right)\frac{1}{2}\sqrt{1 - z_2}. \tag{122}$$

Note from (119) that $0 < z_1 < z_2 < 1$ and that

$$\frac{z_1}{z_2} \;=\; e^{-\frac{2}{\pi}\left(\left(\frac{y}{x}+\frac{x}{2}\right)^2 - \left(\frac{y}{x}-\frac{x}{2}\right)^2\right)} \;=\; e^{-\frac{4}{\pi}y}. \tag{123}$$

From (123) it follows that

$$e^y \;=\; \left(\frac{z_1}{z_2}\right)^{-\frac{\pi}{4}} \;=\; \left(\frac{z_2}{z_1}\right)^{\frac{\pi}{4}}. \tag{124}$$

Note that

$$\frac{dz_1}{dx} = -\frac{4}{\pi} z_1 \left(\frac{y}{x} + \frac{x}{2}\right)\left(-\frac{y}{x^2} + \frac{1}{2}\right) = z_1 \frac{(2y+x^2)(2y-x^2)}{\pi x^3}; \tag{125}$$

$$\frac{dz_2}{dx} = -\frac{4}{\pi} z_2 \left(\frac{y}{x} - \frac{x}{2}\right)\left(-\frac{y}{x^2} - \frac{1}{2}\right) = z_2 \frac{(2y+x^2)(2y-x^2)}{\pi x^3}. \tag{126}$$

From (122), and using (124), (125), and (126), we find that

$$\begin{aligned} f'(x) &= -\operatorname{sgn}\left(2y+x^2\right) \frac{e^y}{4\sqrt{1-z_1}} \cdot \frac{dz_1}{dx} + \operatorname{sgn}\left(2y-x^2\right) \frac{1}{4\sqrt{1-z_2}} \cdot \frac{dz_2}{dx} \\ &= \frac{1}{4\pi x^3} (2y+x^2)(2y-x^2) \left(-\operatorname{sgn}\left(2y+x^2\right) e^y \frac{z_1}{\sqrt{1-z_1}} + \operatorname{sgn}\left(2y-x^2\right) \frac{z_2}{\sqrt{1-z_2}}\right) \\ &= \frac{1}{4\pi x^3} (2y+x^2)(2y-x^2) \left(-\operatorname{sgn}\left(2y+x^2\right) \left(\frac{z_2}{z_1}\right)^{\frac{\pi}{4}} \frac{z_1}{\sqrt{1-z_1}} + \operatorname{sgn}\left(2y-x^2\right) \frac{z_2}{\sqrt{1-z_2}}\right) \\ &= \frac{z_2^{\frac{\pi}{4}}}{4\pi x^3} (2y+x^2)(2y-x^2) \left(-\operatorname{sgn}\left(2y+x^2\right) \frac{z_1^{1-\frac{\pi}{4}}}{\sqrt{1-z_1}} + \operatorname{sgn}\left(2y-x^2\right) \frac{z_2^{1-\frac{\pi}{4}}}{\sqrt{1-z_2}}\right). \end{aligned} \tag{127}$$

The following cases are considered separately:

Case 1: If $y = 0$, then $z_1 = z_2 = e^{-x^2/(2\pi)}$, see (119), and, from (127), we obtain that

$$\begin{aligned} f'(x) &= \frac{z_2^{\frac{\pi}{4}}}{4\pi x^3} (-x^4) \left(-\frac{z_1^{1-\frac{\pi}{4}}}{\sqrt{1-z_1}} - \frac{z_2^{1-\frac{\pi}{4}}}{\sqrt{1-z_2}}\right) \\ &= -\frac{z_2^{\frac{\pi}{4}}}{4\pi} x \cdot \left(-2\frac{z_2^{1-\frac{\pi}{4}}}{\sqrt{1-z_2}}\right) = x \frac{z_2}{4\pi\sqrt{1-z_2}} \\ &> 0, \ \forall\, x > 0, \end{aligned}$$

and therefore $f(x)$ is a strictly increasing function for $x > 0$.

Case 2: If $y > 0$ and $x > \sqrt{2y}$, then $2y + x^2 > 0$, $2y - x^2 < 0$, $\operatorname{sgn}\left(2y+x^2\right) = 1$, $\operatorname{sgn}\left(2y-x^2\right) = -1$, and we find from (127) that

$$f'(x) = \frac{z_2^{\frac{\pi}{4}}}{4\pi x^3} (2y+x^2)(2y-x^2) \left(-\frac{z_1^{1-\frac{\pi}{4}}}{\sqrt{1-z_1}} - \frac{z_2^{1-\frac{\pi}{4}}}{\sqrt{1-z_2}}\right) > 0.$$

If $y > 0$ and $x < \sqrt{2y}$, then $2y + x^2 > 0$, $2y - x^2 > 0$, $\operatorname{sgn}\left(2y+x^2\right) = 1$, $\operatorname{sgn}\left(2y-x^2\right) = 1$, and we find from (127) that

$$f'(x) = \frac{z_2^{\frac{\pi}{4}}}{4\pi x^3} (2y+x^2)(2y-x^2) \left(\frac{z_2^{1-\frac{\pi}{4}}}{\sqrt{1-z_2}} - \frac{z_1^{1-\frac{\pi}{4}}}{\sqrt{1-z_1}}\right). \tag{128}$$

From (119) and (123), it is easy to see that $0 < z_1 < z_2 < 1$, if $y > 0$. Note that the function $z^{1-\pi/4}/\sqrt{1-z}$ is increasing for $0 < z < 1$, since both $z^{1-\pi/4}$ and $1/\sqrt{1-z}$ are increasing for $0 < z < 1$. Then, from (128), we find that $f'(x) > 0$ if $x < \sqrt{2y}$.

We conclude that, if $y > 0$, then $f'(x) > 0$ and therefore $f(x)$ is a strictly increasing function for $x > 0$.

Case 3: If $y < 0$ and $x > \sqrt{-2y}$, then $2y + x^2 > 0$, $2y - x^2 < 0$, $\operatorname{sgn}\left(2y+x^2\right) = 1$, $\operatorname{sgn}\left(2y-x^2\right) = -1$, and we find from (127) that

$$f'(x) = \frac{z_2^{\frac{\pi}{4}}}{4\pi x^3} (2y+x^2)(2y-x^2) \left(-\frac{z_1^{1-\frac{\pi}{4}}}{\sqrt{1-z_1}} - \frac{z_2^{1-\frac{\pi}{4}}}{\sqrt{1-z_2}}\right) > 0.$$

If $y < 0$ and $x < \sqrt{-2y}$, then $2y + x^2 < 0$, $2y - x^2 < 0$, $\operatorname{sgn}\left(2y + x^2\right) = -1$, $\operatorname{sgn}\left(2y - x^2\right) = -1$, and we find from (127) that

$$f'(x) \;=\; \frac{z_2^{\frac{\pi}{4}}}{4\pi x^3}\,(2y+x^2)(2y-x^2)\,\left(\frac{z_1^{1-\frac{\pi}{4}}}{\sqrt{1-z_1}} \;-\; \frac{z_2^{1-\frac{\pi}{4}}}{\sqrt{1-z_2}}\right). \tag{129}$$

Recall from (119) that $0 < z_1, z_2 < 1$, and, from (123), that $z_1/z_2 = e^{-4y/\pi}$. If $y < 0$, then $z_1 > z_2$, and therefore we conclude that $0 < z_2 < z_1 < 1$. Since the function $z^{1-\pi/4}/\sqrt{1-z}$ is increasing for $0 < z < 1$, we conclude from (129) that $f'(x) > 0$.

We conclude that, if $y < 0$, then $f'(x) > 0$ and therefore $f(x)$ is a strictly increasing function for $x > 0$. □

Lemma 3. *If C_m is the market value of a European call option satisfying the no–arbitrage condition*

$$\max(S_0e^{-qT} - Ke^{-rT}, 0) \;<\; C_m \;<\; S_0e^{-qT}, \tag{130}$$

a unique value for the volatility parameter σ (which is, by definition, the approximate implied volatility $\sigma_{imp,approx,C}$) exists such that the approximate value of the option given by (10) is equal to the market price of the option, i.e., such that $C_{approx}(\sigma) = C_m$.

If P_m is the market value of a European put option satisfying the no–arbitrage condition

$$\max(Ke^{-rT} - S_0e^{-qT}, 0) \;<\; P_m \;<\; Ke^{-rT}, \tag{131}$$

a unique value for the volatility parameter σ (which is, by definition, the approximate implied volatility $\sigma_{imp,approx,P}$) exists such that the approximate value of the option given by (11) is equal to the market price of the option, i.e., such that $P_m = P_{approx}(\sigma)$.

Proof. The approximate value $C_{approx}(\sigma)$ is a strictly increasing function of σ, see Lemma 2, and therefore $C_{approx}(\sigma) = C_m$ has a unique solution $\sigma = \sigma_{imp,approx,C}$ if and only if

$$\lim_{\sigma\searrow 0} C_{approx}(\sigma) \;<\; C_m \;<\; \lim_{\sigma\to\infty} C_{approx}(\sigma). \tag{132}$$

Based on the fact that $C_{approx}(\sigma)$ is a strictly increasing function of σ, see Lemma 2, the following limits can be obtained as in the Black–Scholes framework:

$$\lim_{\sigma\searrow 0} C_{approx}(\sigma) \;=\; \max(S_0e^{-qT} - Ke^{-rT}, 0); \tag{133}$$

$$\lim_{\sigma\to\infty} C_{approx}(\sigma) \;=\; S_0e^{-qT}, \tag{134}$$

which allows us to conclude that the no arbitrage bounds (130) and (132) are the same.

The bounds (131) follow from (130) by using the Put–Call parity for the market prices of the options, i.e., $P_m = C_m - S_0e^{-qT} + Ke^{-rT}$. □

Lemma 4. *The approximate implied volatilities for a call and put option with the same strike on the same underlying asset are equal.*

Using the facts that the approximate option values satisfy the Put–Call parity and the approximate implied volatilities are strictly increasing functions of the implied volatility parameter, see Lemma 1 and Lemma 2, a proof of this result can be given following step by step the proof for Black–Scholes implied volatilities; see, e.g., section 3.7 of [Stefanica 2011].

Appendix B. Technical Results

This section contains a series of technical results used for the proof of Theorem 2.

Lemma 5. *Define the function $f_1(t)$ as follows:*

$$f_1(t) \;=\; e^{\frac{t^2}{2}}\,\left(N(t) - (1-c)A(t)\right) \tag{135}$$

where

$$c = 1 - \left(\frac{4-\pi}{\pi-2}\right)^{\frac{\pi-3}{2}}. \tag{136}$$

(i) The function $f_1(t)$ is strictly increasing for $t \geq 0$;
(ii) $f_1(t) > f_1(-t)$, for all $t > 0$.

Proof. (i) We will show that the function

$$f(t) \;=\; N(t) - (1-c)A(t) \tag{137}$$

is positive and strictly increasing for $t \geq 0$, which allows us to conclude that the function $f_1(t)$ is a strictly increasing function for $t \geq 0$, since it is the product of two positive and strictly increasing functions.

The derivative of the function $f(t)$ from (137) is

$$f'(t) \;=\; \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \;-\; \frac{(1-c)t}{\pi\sqrt{1-e^{-\frac{2t^2}{\pi}}}} e^{-\frac{2t^2}{\pi}}.$$

Then,

$$\begin{aligned}
& f'(t) > 0,\ \forall\, t > 0 && (138)\\
\Longleftrightarrow\quad & e^{\frac{2t^2}{\pi}-\frac{t^2}{2}}\ \sqrt{1-e^{-\frac{2t^2}{\pi}}} \;>\; (1-c)\sqrt{\frac{2}{\pi}}\, t,\ \forall\, t > 0 &&\\
\Longleftrightarrow\quad & e^{\frac{4t^2}{\pi}-t^2}\ \left(1-e^{-\frac{2t^2}{\pi}}\right) \;>\; (1-c)^2\frac{2t^2}{\pi},\ \forall\, t > 0 &&\\
\Longleftrightarrow\quad & e^{\frac{4-\pi}{2}\tau}\ \left(1-e^{-\tau}\right) \;>\; (1-c)^2\tau,\ \forall\, \tau > 0 && (139)\\
\Longleftrightarrow\quad & g(\tau) \;>\; 0,\ \forall\, \tau > 0, && (140)
\end{aligned}$$

where $\tau = 2t^2/\pi$ in (139) and therefore

$$\frac{4t^2}{\pi} - t^2 \;=\; \frac{4-\pi}{2}\tau,$$

and the function $g(\tau)$ from (140) is given by

$$g(\tau) \;=\; e^{\frac{4-\pi}{2}\tau}\left(1-e^{-\tau}\right) \;-\; (1-c)^2\tau.$$

Note that

$$\begin{aligned}
g'(\tau) \;&=\; e^{\frac{4-\pi}{2}\tau}\left(\frac{\pi}{2}-1\right)\left(\frac{4-\pi}{\pi-2}+e^{-\tau}\right) \;-\; (1-c)^2;\\
g''(\tau) \;&=\; e^{\frac{4-\pi}{2}\tau}\left(\frac{\pi}{2}-1\right)^2\left(\left(\frac{4-\pi}{\pi-2}\right)^2 - e^{-\tau}\right).
\end{aligned}$$

Let $\tau_0 = 2\ln\left((\pi-2)/(4-\pi)\right) > 0$ be the only solution of $g''(\tau) = 0$. A direct computation[3] shows that, if c is given by (136), then $g'(\tau_0) = 0$. Note that $g''(\tau) < 0$ if $0 < \tau < \tau_0$ and $g''(\tau) > 0$ if $\tau > \tau_0$. In other words, the function $g'(\tau)$ is strictly decreasing when $0 < \tau < \tau_0$ and strictly increasing when $\tau > \tau_0$, and therefore $g'(\tau) > g'(\tau_0) = 0$ for all $\tau > 0$ with $\tau \neq \tau_0$. Thus, the function $g(\tau)$ is strictly increasing for $\tau > 0$, and, since $g(0) = 0$, it follows that $g(\tau) > 0$ for all $\tau > 0$.

[3] Recall that $e^{-\tau_0} = \left(\frac{4-\pi}{\pi-2}\right)^2$ and $c = 1 - \left(\frac{4-\pi}{\pi-2}\right)^{\frac{\pi-3}{2}}$. Then, $\left(\frac{4-\pi}{\pi-2}\right)^{\pi-3} = (1-c)^2$ and

$$\begin{aligned}
g'(\tau_0) \;&=\; e^{\frac{4-\pi}{2}\tau_0}\left(\frac{\pi}{2}-1\right)\left(\frac{4-\pi}{\pi-2}+e^{-\tau_0}\right) \;-\; (1-c)^2\\
&=\; \left(\frac{4-\pi}{\pi-2}\right)^{-4+\pi}\cdot\frac{\pi-2}{2}\cdot\left(\frac{4-\pi}{\pi-2}+\left(\frac{4-\pi}{\pi-2}\right)^2\right) \;-\; (1-c)^2\\
&=\; \left(\frac{4-\pi}{\pi-2}\right)^{\pi-3} \;-\; (1-c)^2\\
&=\; 0.
\end{aligned}$$

From (138) and (140), we obtain that $f'(t) > 0$ for all $t > 0$, and conclude that the function $f(t)$ is strictly increasing for $t \geq 0$. Since $f(0) = \frac{c}{2}$, it follows that $f(t) > 0$ for all $t > 0$.

(ii) From (135), it follows that $f_1(t) > f_1(-t)$ for all $t > 0$ is equivalent to

$$N(t) - (1-c)A(t) \; > \; N(-t) - (1-c)A(-t), \; \forall \; t > 0. \tag{141}$$

Since $N(-t) = 1 - N(t)$ and $A(-t) = 1 - A(t)$, the inequality (141) can be written as

$$2N(t) - 1 \; \geq \; (1-c)(2A(t) - 1), \; \forall \; t \geq 0. \tag{142}$$

In [Stefanica,Radoicic (2016)], we showed that

$$A(t) - \frac{1}{2} \; < \; \left(\frac{\pi - 2}{4 - \pi}\right)^{\frac{\pi - 3}{2}} \left(N(t) - \frac{1}{2}\right) \; = \; \frac{1}{1-c}\left(N(t) - \frac{1}{2}\right), \; \forall \; t > 0; \tag{143}$$

see (47) and (49) from the proof of Theorem 4.1 therein. From (143), we find that

$$2N(t) - 1 \; > \; (1-c)(2A(t) - 1), \; \forall \; t > 0,$$

and therefore the inequality (142) is established. □

Lemma 6. *Let $\bar{c} = 0.0067$ and define the function $f_2(t)$ as follows:*

$$f_2(t) \; = \; e^{\frac{t^2}{2}} \left((1+\bar{c})A(t) - N(t)\right). \tag{144}$$

(i) $f_2(t)$ is a strictly increasing function for $t \geq 0$;

(ii) $f_2(t) > f_2(-t)$, for all $t > 0$.

Proof. (i) We will show that $f_2'(t) > 0$ for all $t \geq 0$ and conclude that $f_2(t)$ is an increasing function for $t \geq 0$. Note that

$$\begin{aligned} f_2'(t) \; &= \; e^{\frac{t^2}{2}} t((1+\bar{c})A(t) - N(t)) \; + \; e^{\frac{t^2}{2}}((1+\bar{c})A'(t) - N'(t)) \\ &= \; e^{\frac{t^2}{2}} t(A(t) - N(t)) \; + \; e^{\frac{t^2}{2}} \bar{c}(tA(t) + A'(t)) \; + \; e^{\frac{t^2}{2}}(A'(t) - N'(t)) \\ &> \; e^{\frac{t^2}{2}} \bar{c}(tA(t) + A'(t)) \; + \; e^{\frac{t^2}{2}}(A'(t) - N'(t)), \; \forall \; t > 0, \end{aligned} \tag{145}$$

since $A(t) > N(t)$ for $t > 0$; cf. (9).

From (145), it follows that, if

$$\bar{c}(tA(t) + A'(t)) \; + \; A'(t) - N'(t) \; > \; 0, \; \forall \; t > 0, \tag{146}$$

then $f_2'(t) > 0$ for all $t > 0$.

After elementary yet elaborate computations that we omit here, it can be shown that the function $tA(t) + A'(t)$ is increasing for $t \geq 0$; see also Figure 4. Therefore, $tA(t) + A'(t) \geq A'(0)$, and, since $A'(0) = 1/\sqrt{2\pi}$, we obtain that

$$tA(t) + A'(t) \; \geq \; \frac{1}{\sqrt{2\pi}}, \; \forall \; t \geq 0. \tag{147}$$

Similarly, it can be shown that the function $A'(t) - N'(t)$ increases from the value 0 at $t = 0$ to a global maximum $M = 0.003451$ achieved at the point $t_M = 0.967728$, then decreases to a global minimum $m = -0.002648$ corresponding to the point $t_m = 2.321564$, and increases when $t \geq t_m$, with $\lim_{t\to\infty}(A'(t) - N'(t)) = 0$; see Figure 4 for more details. In other words,

$$M \; \geq \; A'(t) - N'(t) \; \geq \; m, \; \forall \; t \geq 0. \tag{148}$$

From (145–148), we conclude that, for any $t \geq 0$,

$$\bar{c}(tA(t) + A'(t)) \; + \; A'(t) - N'(t) \; \geq \; \frac{\bar{c}}{\sqrt{2\pi}} + m \; = \; 0.0000245 \; > \; 0,$$

which is what we wanted to show; cf. (146).

(ii) From (144), it follows that $f_2(t) > f_2(-t)$ for all $t > 0$ is equivalent to

$$(1+\bar{c})A(t) - N(t) \; > \; (1+\bar{c})A(-t) - N(-t) \; \forall \; t > 0. \tag{149}$$

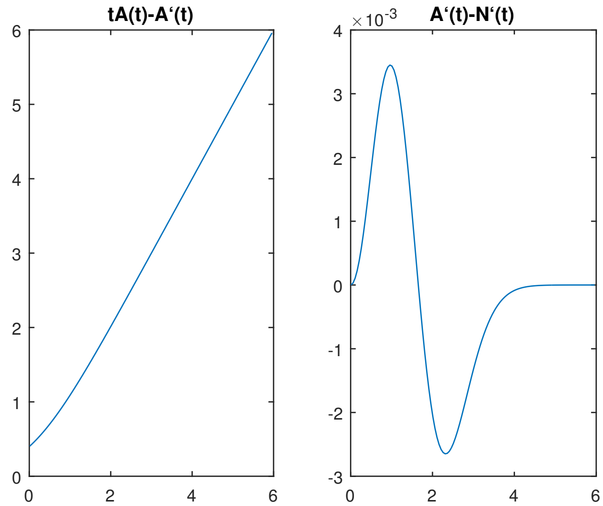

FIGURE 4. Functions $tA(t) + A'(t)$ and $A'(t) - N'(t)$ for $t > 0$; used in Lemma 6

Since $N(-t) = 1 - N(t)$ and $A(-t) = 1 - A(t)$, the inequality (149) can be written as

$$(1 + \bar{c})(2A(t) - 1) \ = \ \bar{c}(2A(t) - 1) \ + \ 2A(t) - 1 \ > \ 2N(t) - 1, \ \forall \ t > 0. \tag{150}$$

The inequality (150) holds true since $A(t) > 1/2$ if $t > 0$ and $A(t) > N(t)$ for all $t > 0$; cf. (9). $\square$

REFERENCES

[Acklam (2003)] P. J. Acklam. An algorithm for computing the inverse normal cumulative distribution function. *online*: home.online.no/∼pjacklam/notes/invnorm/index.html, 2003.

[Bridges,Curtis Jr.,Isengildina-Massa,Nian (2007)] W. C. Bridges Jr., C. E. Curtis Jr., O. Isengildina-Massa, and M. Nian. Accuracy of implied volatility approximations using "nearest-to-the-money" option premiums. 2007 Annual Meeting, February 4-7, 2007, Mobile, Alabama, *Southern Agricultural Economics Association*.

[Bharadia,Christofides,Salkin (1996)] M. A. Bharadia, N. Christofides, and G. R. Salkin. Computing the Black-Scholes implied volatility. *Advances in Futures and Options Research*, 8:15–29, 1996.

[Benaim,Friz,Lee (2009)] S. Benaim, P. Friz, and R. Lee. On Black-Scholes implied volatility at extreme strikes. In Cont, R., editor, *Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling*, pages 19–45. John Wiley and Sons, Hoboken, NJ, 2009.

[Black (1976)] F. Black. The pricing of commodity contracts. *Journal of Financial Economics*, 3:167–179, 1976.

[Black and Scholes (1973)] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

[Brenner and Subrahmanyam (1988)] M. Brenner and M. Subrahmanyam. A simple formula to compute the implied standard deviation. *Financial Analyst Journal*, 44:80–83, 1988.

[Brenner and Subrahmanyam (1994)] M. Brenner and M. Subrahmanyam. A simple approach to option valuation and hedging in the Black-Scholes model. *Financial Analyst Journal*, 50:25–28, 1994.

[Cody (1969)] W. J. Cody. Rational Chebyshev approximations for the error function. *Mathematics of Computation*, 23:631–638, 1969.

[Chance (1993)] D. M. Chance. Leap into the unknown. *Risk*, 6:60–66, 1993.

[Chance (1996)] D. M. Chance. A generalized simple formula to compute the implied volatility. *The Financial Review*, 31:859–867, 1996.

[Corrado and Miller Jr. (1996-1)] C. J. Corrado and T. W. Miller Jr. A note on a simple, accurate formula to compute implied standard deviations. *Journal of Banking and Finance*, 20:595–603, 1996.

[Corrado and Miller Jr. (1996-2)] C. J. Corrado and T. W. Miller Jr. Volatility without tears. *Risk*, 7:49–52, 1996.

[Corrado and Miller Jr. (2004)] C. J. Corrado and T. W. Miller Jr. Tweaking Implied Volatility. *online*: papers.ssrn.com/sol3/papers.cfm?abstract_id=584982, 2004.

[Chambers and Nawalkha (2001)] D. R. Chambers and S. J. Nawalkha. An improved approach to computing implied volatility. *The Financial Review*, 36:89–100, 2001.

[Feinstein (1988)] S. Feinstein. A source of unbiased implied volatility forecasts. *Working paper*, 88-9, Federal Reserve Bank of Atlanta, 1988.

[Le Floc'h (2017)] F. Le Floc'h. Implied Volatility from Black-Scholes price. *online*: chasethedevil.github.io/post/implied-volatility-from-black-scholes-price, 2017.

[Gatheral (2006)] J. Gatheral. *The Volatility Surface: A Practitioner's Guide*. John Wiley and Sons, Hoboken, NJ, 2006.

[Gatheral,Matic,Radoicic,Stefanica (2017)] J.Gatheral, I.Matić, R. Radoičić, and D. Stefanica. Tighter bounds for implied volatility. *International Journal of Theoretical and Applied Finance*, 20:5, 2017.

[Gao and Lee (2014)] K. Gao and R. Lee. Asymptotics of implied volatility to arbitrary order. *Finance and Stochastics*, 18:349–392, 2014.

[Hentschel (2003)] L. Hentschel. Errors in implied volatility estimation. *The Journal of Financial and Quantitative Analysis*, 38:779–810, 2003.

[Hallerbach (2004)] W. G. P. M. Hallerbach. An improved estimator for Black-Scholes-Merton implied volatility. *Erasmus Research Institute of Management* Research Paper ERS-2004-054, 2004.

[Haug (2006)] E. G. Haug. *The Complete Guide to Option Pricing Formulas*. McGraw-Hill, 2nd edition, 2006.

[Hofstetter and Selby (2001)] E. Hofstetter and M. J. P. Selby. The logistic function and implied volatility: quadratic approximation and beyond. *online*: www2.warwick.ac.uk/fac/soc/wbs/research/wfri/wpaperseries/pp_04.141.pdf, 2001.

[Jackel (2006)] P. Jäckel. By implication. *Wilmott*, November: 60–66, 2006.

[Jackel (2015)] P. Jäckel. Let's be rational. *Wilmott*, January:40–53, 2015.

[Kelly (2006)] M. A. Kelly. Faster implied volatilities via the implicit function theorem. *Financial Review*, 41:589–597, 2006.

[Li (2005)] S. Li. A new formula on computing the implied volatility. *Applied Mathematics and Computation*, 170:611–625, 2005.

[Li (2008)] M. Li. Approximate inversion of the Black-Scholes formula using rational functions. *European Journal of Operational Research*, 185:743–759, 2008.

[Latane and Rendleman (1976)] H. A. Latané and R. J. Rendleman. Standard deviation of stock price ratios implied by option premia. *Journal of Finance*, 31:369–382, 1976.

[Li and Lee (2007)] M. Li and K. Lee. An adaptive successive over-relaxation method for computing the Black-Scholes implied volatility. *Quantitative Finance*, 11:1245–1269, 2011.

[Li and Lee (2011)] M. Li and K. Lee. An adaptive successive over-relaxation method for computing the Black-Scholes implied volatility. *Quantitative Finance*, 11:1245–1269, 2011.

[Liang and Tahara (2009)] S. Liang and Y. J. Tahara. A formula to compute implied volatility, with error estimate. *Interdisciplinary Information Sciences*, 15:267–272, 2009.

[Matic,Radoicic,Stefanica (2017)] I.Matić, R. Radoičić, and D. Stefanica. Pólya-Based Approximation for the ATM-Forward Implied Volatility. To appear, *Int. J. Finan. Eng.*, 2017.

[Matic,Radoicic,Stefanica (2016)] I.Matić, R. Radoičić, and D. Stefanica. A Sharp Pólya-Based Approximation to the Normal CDF. *online*: ssrn.com/abstract=2842681, 2016.

[Merton 1973] R. C. Merton. The theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

[Manaster and Koehler (1982)] S. Manaster and G. Koehler. The calculation of implied variances from the Black-Scholes model. *The Journal of Finance*, 38:227–230, 1982.

[Pianca (2005)] P. Pianca. Simple formulas to option pricing and hedging in the Black-Scholes model. *Finance*, 0511005, EconWPA, 2005.

[Polya (1949)] G. Pólya. Remarks on computing the probability integral in one and two dimensions. In Neyman, J., editor, *Proceedings of the First Berkeley Symposium on Mathematical Statistics and Probability*, pages 63 –78. University of California Press, Berkeley, Calif., 1949.

[Spanderen 2017] K. Spanderen. Adaptive SOR Method for Implied Volatility Calculation. *online*: hpcquantlib.wordpress.com/2017/08/06/adaptive-sor-method-for-implied-volatility-calculation, 2017.

[Stefanica 2011] D. Stefanica. *A Primer for the Mathematics of Financial Engineering*. Financial Engineering Press, New York, 2nd edition, 2011.

[Stefanica,Radoicic (2016)] D. Stefanica and R. Radoičić. A sharp approximation for ATM-forward option prices and implied volatilities. *Int. J. Finan. Eng.*, 3:1, 2016.

[Tehranchi (2016)] M. R. Tehranchi (2016) Uniform bounds for Black–Scholes implied volatility. *SIAM Journal on Financial Mathematics*, 7(1):893–916.