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Orlicz projection bodies ☆

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Abstract

Minkowski's projection bodies have evolved into L_p projection bodies and their asymmetric analogs. These all turn out to be part of a far larger class of Orlicz projection bodies. The analog of the classical Petty projection inequality is established for the new Orlicz projection bodies.

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As Schneider [58] observes, the classical Brunn–Minkowski theory emerged at the turn of the 19th into the 20th century, when Minkowski began his study of the volume of (what would become known as) the Minkowski sum of convex bodies. In addition to mixed volumes, one of the core concepts that Minkowski introduced within the Brunn–Minkowski theory is that of a projection body (precise definitions to follow). Four decades ago, in a highly influential paper, Bolker [5] illustrated how Minkowski's projection operator, its range, and its polar were in fact objects of independent investigation in a number of disciplines.

Within the Brunn–Minkowski theory, the two classical inequalities which connect the volume of a convex body with that of its polar projection body are the Petty and Zhang projection inequalities. In retrospect, it is interesting to observe that these inequalities were not discovered by Minkowski. Nor did they emerge out of Blaschke's school. In fact, it took seven decades from Minkowski's discovery of projection bodies for the Petty projection inequality to appear

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(see e.g., the books by Gardner [17], Leichtweiss [27], Schneider [58], and Thompson [61] for reference). It took yet another two decades for the Zhang projection inequality to be discovered. Establishing the analogs of the Petty and Zhang projection inequalities for the projection operator (as opposed to the polar projection operator) are today major open problems within the field of convex geometric analysis.

Unlike the classical isoperimetric inequality, the Petty and Zhang projection inequalities are affine isoperimetric inequalities in that they are inequalities between a pair of geometric functionals whose ratio is invariant under affine transformations. The Petty projection inequality is not only stronger than (i.e., directly implies) the classical isoperimetric inequality, but it can be viewed as an optimal isoperimetric inequality. In the same manner that the classical isoperimetric inequality leads to (in fact is equivalent to) the Sobolev inequality, the Petty projection inequality has led to the affine Sobolev inequality [66] that is stronger than the classical Sobolev inequality and yet is independent of any underlying Euclidean structure.

In the early 1960s, Firey (see e.g. Schneider [58, p. 383]) introduced an L_p extension of Minkowski's addition (now known as Firey–Minkowski L_p-addition) of convex bodies. In the mid 1990s, it was shown in [36,37], that a study of the volume of these Firey–Minkowski L_p-combinations leads to an embryonic L_p Brunn–Minkowski theory. This theory has expanded rapidly. (See e.g. [2,3,6–12,16,19–25,28–34,36–47,50,51,57,59,60,62–65].)

An early achievement of the new L_p Brunn–Minkowski theory was the discovery of L_p analogs of projection bodies and of the Petty projection inequality [40], with an alternate approach to establishing this inequality presented by Campi and Gronchi in [6]. These new inequalities have found applications in the field of analytic inequalities where they led to affine L_p Sobolev inequalities [43] and ultimately to affine Moser–Trudinger and affine Morrey–Sobolev inequalities [11].

Work of Ludwig [31] (see also [28]) showed that the known L_p extension of the projection operator considered in [40] is only one of a family of natural L_p extensions of their classical counterpart. Using this insight, Haberl and Schuster [23] (see also [24]) considered so-called "asymmetric" L_p analogs and obtained "asymmetric" L_p analogs of the Petty projection inequality. For bodies that are not symmetric about the origin, the inequalities of Haberl and Schuster are stronger than the original L_p Petty projection inequality. The operators considered by Haberl and Schuster appear to be ideally suited for non-symmetric bodies. This can be seen most clearly by looking at the L_p analog of the classical Blaschke–Santaló inequality presented in [39]. For origin symmetric bodies, this L_p extension does recover the original Blaschke–Santaló inequality as $p \to \infty$. However, for arbitrary bodies only the Haberl–Schuster version does so.

The above cited work of Haberl and Schuster and the recent work of Ludwig and Reitzner [34], as well as Ludwig [33], makes it apparent that the time is ripe for the next step in the evolution of the Brunn–Minkowski theory towards what we call an Orlicz–Brunn–Minkowski theory. It is the aim of this paper to define Orlicz projection bodies and establish an Orlicz analog of the classical Petty projection inequality — an inequality that we call the Orlicz Petty projection inequality. The new inequality has all its predecessors (including the Haberl–Schuster version) as special cases.

Another classical affine isoperimetric inequality is the Busemann–Petty centroid inequality [56]. This is an inequality between the volume of a convex body and that of its centroid body. The centroid body is a concept that goes back at least to Dupin. Another early achievement within the L_p Brunn–Minkowski theory (and its dual) was the establishment of the L_p analog of the Busemann–Petty centroid inequality [6,40] for the natural L_p extension of centroid bodies. The L_p extensions of the centroid operator quickly became an object of interest in asymptotic geometric analysis (see e.g. [13,15,18,26,52–55]) and even the theory of stable distributions (see [51]).

It was shown in [35] that once the Busemann–Petty centroid inequality has been established, the Petty projection inequality can be obtained as an almost effortless consequence. In addition, it was shown in [35] that also the reverse is the case: the Busemann–Petty centroid inequality can be obtained easily from the Petty projection inequality. As shown in [40], a similar relationship holds between the L_p Petty projection inequality and the L_p Busemann–Petty centroid inequalities: only one of these two inequalities needs to be established and then the other can be quickly derived. It appears that this might not be the case for the Orlicz analogues of these classical inequalities. (See the discussion in Section 4.) Neither the Orlicz Petty projection inequality nor the Orlicz Busemann–Petty centroid inequality appears to lead to the other in some manner discernable to the authors. Therefore, the topic of Orlicz centroid bodies is treated in a separate work [48].

We consider convex $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$. This means that ϕ must be decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. We will assume throughout that one of these is happening strictly so; i.e., ϕ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$.

Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and that has volume $|K|$. The *Orlicz projection body* $\Pi_\phi K$ of K is defined as the body whose support function (see Section 1 for definitions) is given by

$$h_{\Pi_\phi K}(x) = \inf\left\{\lambda > 0\colon \int_{\partial K} \phi\left(\frac{x \cdot \nu(y)}{\lambda y \cdot \nu(y)}\right) y \cdot \nu(y)\, d\mathcal{H}^{n-1}(y) \leqslant n|K|\right\},$$

where $\nu(y)$ is the outer unit normal of ∂K at $y \in \partial K$, where $x \cdot \nu(y)$ denotes the inner product of x and $\nu(y)$, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. Recall that $\nu(y)$ exists for $\mathcal{H}^{n-1}$-almost all $y \in \partial K$. For the polar (see Section 1 for definitions) of $\Pi_\phi K$ we will write $\Pi_\phi^* K$.

With $\phi_1(t) = |t|$, it turns out that for $u \in S^{n-1}$,

$$h_{\Pi_{\phi_1} K}(u) = \frac{c_n}{|K|}|K_u|,$$

where $|K_u|$ denotes the $(n-1)$-dimensional volume of K_u, the image of the orthogonal projection of K onto the subspace $u^\perp$. Thus

$$\Pi_{\phi_1} K = \frac{c_n}{|K|}\Pi K,$$

where ΠK is the classical projection body of K introduced by Minkowski. With $\phi_p(t) = |t|^p$, and $p \geqslant 1$,

$$\Pi_{\phi_p} K = \frac{c_{n,p}}{|K|^{\frac{1}{p}}}\Pi_p K,$$

where $\Pi_p K$ is the L_p projection body of K, defined as the convex body whose support function is given by

$$h_{\Pi_p K}(x) = \left\{ \int_{\partial K} \left|x \cdot \nu(y)\right|^p \left|y \cdot \nu(y)\right|^{1-p} \, d\mathcal{H}^{n-1}(y) \right\}^{1/p}.$$

We will prove the following volume ratio inequality.

Orlicz Petty projection inequality. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the volume ratio*

$$\left|\Pi_\phi^* K\right| / |K|$$

is maximized when K is an ellipsoid centered at the origin. If ϕ is strictly convex, then ellipsoids centered at the origin are the only maximizers.

When $\phi(t) = |t|$, the Orlicz Petty projection inequality is the classical Petty projection inequality. When $\phi(t) = |t|^p$, and $p > 1$, the Orlicz Petty projection inequality is the L_p Petty projection inequality (established in [40], with an alternate proof given by Campi and Gronchi in [6]). Haberl and Schuster's recent extension [23] of the L_p Petty projection inequality is the case $\phi(t) = (|t| + \alpha t)^p$, for $-1 \leqslant \alpha \leqslant 1$ of the Orlicz Petty projection inequality.

In Section 1, we establish notation and list some basic facts regarding convex functions and convex bodies. In Section 2 some of the basic properties of Orlicz projection bodies are established. Section 3 contains the proof of the Orlicz Petty projection inequality. In Section 4 some questions are posed.

1. Basics regarding convex bodies

The setting will be Euclidean n-space $\mathbb{R}^n$. We write $e_1, \dots, e_n$ for the standard orthonormal basis of $\mathbb{R}^n$ and when we write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ we always assume that e_n is associated with the last factor.

We will attempt to use x, y for vectors in $\mathbb{R}^n$ and x', y' for vectors in $\mathbb{R}^{n-1}$, and $u, v \in S^{n-1}$ for unit vectors. We will use a, b, s, t, α for numbers in $\mathbb{R}$ and c, λ for strictly positive reals. If Q is a Borel subset of $\mathbb{R}^n$ and Q is contained in an i-dimensional affine subspace of $\mathbb{R}^n$ but in no affine subspace of lower dimension, then $|Q|$ will denote the i-dimensional Lebesgue measure of Q. If $x \in \mathbb{R}^n$ then by abuse of notation we will write $|x| = \sqrt{x \cdot x}$.

For $A \in \mathrm{GL}(n)$ write A^t for the transpose of A and A^{-t} for the inverse of the transpose (contragradient) of A. Write $|A|$ for the absolute value of the determinant of A.

We shall write c_n for a constant depending only on n and $c_{n,p}$ for a constant depending only on n and p. For $a \in \mathbb{R}$ define

$$(a)_+ = \max\{a, 0\} \quad \text{and} \quad (a)_- = \min\{a, 0\}.$$

Let $\mathcal{C}$ be the class of convex functions $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$ and such that ϕ is either strictly decreasing on $(-\infty, 0]$ or ϕ is strictly increasing on $[0, \infty)$. We say that the sequence $\{\phi_i\}$, where the $\phi_i \in \mathcal{C}$, is such that $\phi_i \to \phi_o \in \mathcal{C}$ provided

$$|\phi_i - \phi_o|_I := \max_{t\in I}\bigl|\phi_i(t) - \phi_o(t)\bigr| \to 0,$$

for every compact interval $I \subset \mathbb{R}$. The subclass of $\mathcal{C}$ consisting of those $\phi \in \mathcal{C}$ that are strictly convex will be denoted by $\mathcal{C}_s$.

We shall make use of the fact that for $\phi \in \mathcal{C}$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$ with $a_3, a_4 > 0$,

$$(a_3 + a_4)\phi\left(\frac{a_1 + a_2}{a_3 + a_4}\right) \leqslant a_3\phi\left(\frac{a_1}{a_3}\right) + a_4\phi\left(\frac{a_2}{a_4}\right). \tag{1.1}$$

This is a trivial consequence of the convexity of ϕ. If $\phi \in \mathcal{C}_s$, then observe that there is equality in (1.1) if and only if $a_1/a_3 = a_2/a_4$.

Define c_ϕ by

$$c_\phi = \min\bigl\{c > 0\colon\ \max\bigl\{\phi(c), \phi(-c)\bigr\} \leqslant 1\bigr\}. \tag{1.2}$$

We write $\mathcal{K}^n$ for the set of convex bodies (compact convex subsets) of $\mathbb{R}^n$. We write $\mathcal{K}^n_o$ for the set of convex bodies that contain the origin in their interiors.

For $K \in \mathcal{K}^n$, let $h(K;\cdot) = h_K : \mathbb{R}^n \to \mathbb{R}$ denote the *support function* of K; i.e., $h(K;x) = \max\{x \cdot y\colon\ y \in K\}$. Thus, if $y \in \partial K$, then

$$h_K\bigl(\nu_K(y)\bigr) = \nu_K(y)\cdot y,$$

where $\nu_K(y)$ denotes an outer unit normal to ∂K at y. Obviously, when $c > 0$, for the support function of the convex body $cK = \{cx\colon\ x \in K\}$ we have

$$h_{cK} = ch_K. \tag{1.3}$$

Observe that from the definition of the support function it follows immediately that for $A \in \mathrm{GL}(n)$ the support function of the image $AK = \{Ay\colon\ y \in K\}$ is given by

$$h_{AK}(x) = h_K\bigl(A^t x\bigr). \tag{1.4}$$

If $K_i \in \mathcal{K}^n$, we say that $K_i \to K_o \in \mathcal{K}^n$ provided

$$|h_{K_i} - h_{K_o}|_\infty := \max_{u\in S^{n-1}}\bigl|h_{K_i}(u) - h_{K_o}(u)\bigr| \to 0.$$

If $K \in \mathcal{K}^n_o$, then the *polar body* K^* is defined by

$$K^* = \bigl\{x \in \mathbb{R}^n\colon\ x\cdot y \leqslant 1 \text{ for all } y \in K\bigr\}.$$

It is easy to see that for $c > 0$,

$$(cK)^* = \frac{1}{c}K^*, \tag{1.5}$$

and more generally that for $A \in \mathrm{GL}(n)$

$$(AK)^* = A^{-t}K^*.$$

It is easy to verify that

$$K^{**} = K.$$

We require the easily established continuity of the polar operator $^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$ that maps $K \mapsto K^*$.

Let $\rho(K;\cdot) = \rho_K : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$ denote *radial function* of $K \in \mathcal{K}_o^n$; i.e. $\rho_K(x) = \max\{\lambda > 0\colon\ \lambda x \in K\}$. From the definition, it follows immediately that

$$x \in K \setminus \partial K \quad \text{if and only if} \quad \rho_K(x) > 1. \tag{1.6}$$

It is easily verified that

$$h_{K^*} = 1/\rho_K \quad \text{and} \quad \rho_{K^*} = 1/h_K. \tag{1.7}$$

Observe that from (1.6) and (1.7) it follows immediately that for $x \in \mathbb{R}^n$

$$h(K;x) = 1 \quad \text{if and only if} \quad x \in \partial K^*. \tag{1.8}$$

The classical Aleksandrov–Fenchel–Jessen *surface area measure*, S_K, of the convex body K can be defined as the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u)\,dS_K(u) = \int_{\partial K} f\big(\nu_K(y)\big)\,d\mathcal{H}^{n-1}(y), \tag{1.9}$$

for each continuous $f : S^{n-1} \to \mathbb{R}$. We shall require the trivial observation that for the surface area measure of cK we have

$$S_{cK} = c^{n-1}S_K, \tag{1.10}$$

and the fact that for $K \in \mathcal{K}_o^n$ the measure S_K cannot be concentrated on a hemisphere of S^{n-1}. Less trivial, but much needed is the fact that S_K is weakly continuous in K (see e.g. Schneider [58, p. 201]); i.e., if $K_i \in \mathcal{K}_o^n$, then

$$K_i \to K_o \in \mathcal{K}_o^n \quad \Longrightarrow \quad S_{K_i} \to S_{K_o}, \ \text{weakly}.$$

That

$$\frac{1}{n}\int_{S^{n-1}} h_K(u)\,dS_K(u) = \frac{1}{n}\int_{\partial K} h_K\big(\nu_K(y)\big)\,d\mathcal{H}^{n-1}(y) = \frac{1}{n}\int_{\partial K} y\cdot\nu_K(y)\,d\mathcal{H}^{n-1}(y)$$

is equal to $|K|$ can be easily seen by considering a polytope $P \in \mathcal{K}_o^n$ whose faces have areas (i.e., $(n-1)$-dimensional volumes) $a_1, \dots, a_m$, corresponding outer unit normals $u_1, \dots, u_m$, with *facial cones* (i.e., the cones whose bases are the faces of P and whose vertex is the origin) of volume $V_i = \frac{1}{n} a_i h_P(u_i)$. Here

$$\frac{1}{n} \int_{\partial P} h_P\big(\nu_P(y)\big)\, d\mathcal{H}^{n-1}(y) = \frac{1}{n} \sum_{i=1}^{m} a_i h_P(u_i) = \sum_{i=1}^{m} V_i = |P|.$$

For $K \in \mathcal{K}_o^n$, it will be convenient to use the volume-normalized *conical measure* V_K defined by

$$|K|\, dV_K = \frac{1}{n} h_K\, dS_K.$$

Observe that

$$V_K \text{ is a probability measure on } S^{n-1}. \tag{1.11}$$

For $K \in \mathcal{K}_o^n$, define $R_K, r_K \in (0, \infty)$ by

$$R_K = \max_{u \in S^{n-1}} h_K(u) \quad \text{and} \quad r_K = \min_{u \in S^{n-1}} h_K(u). \tag{1.12}$$

From definition (1.9) we have Cauchy's projection formula,

$$\int_{S^{n-1}} (u \cdot v)_+\, dS_K(v) = |K_u| \quad \text{and} \quad \int_{S^{n-1}} (u \cdot v)_-\, dS_K(v) = -|K_u|,$$

for $u \in S^{n-1}$. From definition (1.12) we see that the diameter,

$$D_K = \max_{u \in S^{n-1}} \big[h_K(u) + h_K(-u)\big],$$

of a body K is at most $2R_K$, and since K is obviously contained in the right cylinder whose base is K_u and whose height is D_K we have the crude estimates

$$\int_{S^{n-1}} (u \cdot v)_+ \frac{dS_K(v)}{|K|} \geqslant \frac{1}{2R_K} \quad \text{and} \quad \int_{S^{n-1}} (u \cdot v)_- \frac{dS_K(v)}{|K|} \leqslant -\frac{1}{2R_K}. \tag{1.13}$$

For a convex body $K' \in \mathcal{K}_o^{n-1}$ and a function $g : K' \to \mathbb{R}$ whose gradient exists a.e., define $\langle g \rangle : K' \to \mathbb{R}$ by

$$\langle g \rangle(x') = g(x') - x' \cdot \nabla g(x').$$

We shall often make use of the fact that $\langle\cdot\rangle$ is a linear operator; i.e., for $g_1, g_2 : K' \to \mathbb{R}$ whose gradient exists a.e., and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\langle\alpha_1 g_1 + \alpha_2 g_2\rangle = \alpha_1\langle g_1\rangle + \alpha_2\langle g_2\rangle. \tag{1.14}$$

For a convex body K and a direction $u \in S^{n-1}$, let $\underline{\ell}_u(K;\cdot) : K_u \to \mathbb{R}$ and $\bar{\ell}_u(K;\cdot) : K_u \to \mathbb{R}$ denote the *undergraph* and *overgraph functions* of K with respect to u; i.e.

$$K = \big\{y' + tu\colon\ -\underline{\ell}_u(K;y') \leqslant t \leqslant \bar{\ell}_u(K;y') \text{ for } y' \in K_u\big\}.$$

Thus, for the *Steiner symmetral*, $\mathrm{S}_u K$, of K in direction u, we see that the image of the orthogonal projections onto $u^\perp$ of both K and $\mathrm{S}_u K$ are identical, and that

$$\underline{\ell}_u(\mathrm{S}_u K;y') = \frac{1}{2}\big(\underline{\ell}_u(K;y') + \bar{\ell}_u(K;y')\big) = \bar{\ell}_u(\mathrm{S}_u K;y'), \tag{1.15}$$

for all $y' \in K_u$. Both K and u will be suppressed when clear from the context, and thus we will often denote the undergraph and overgraph functions of K with respect to u simply by $\underline{\ell} : K_u \to \mathbb{R}$ and $\bar{\ell} : K_u \to \mathbb{R}$.

When considering the convex body $K \in \mathcal{K}_o^n$ as $K \subset \mathbb{R}^{n-1}\times\mathbb{R}$, then for $(x',t) \in \mathbb{R}^{n-1}\times\mathbb{R}$ we will usually write $h(K;x',t)$ rather than $h(K;(x',t))$. Note that the Steiner symmetral, $\mathrm{S}_{e_n}K$, of K in the direction e_n can be given by

$$\mathrm{S}_{e_n}K = \left\{\left(x', \frac{1}{2}t + \frac{1}{2}s\right) \in \mathbb{R}^{n-1}\times\mathbb{R}\colon\ (x',t), (x',-s) \in K \text{ with } t \neq -s\right\}, \tag{1.16}$$

and its boundary can be given by

$$\partial\mathrm{S}_{e_n}K = \left\{\left(x', \frac{1}{2}t + \frac{1}{2}s\right) \in \mathbb{R}^{n-1}\times\mathbb{R}\colon\ (x',t), (x',-s) \in \partial K \text{ with } t \neq -s\right\}. \tag{1.17}$$

We shall make use of the fact that if $u_i \in S^{n-1}$, then

$$u_i \to u \quad\Longrightarrow\quad \mathrm{S}_{u_i}L \to \mathrm{S}_u L, \tag{1.18}$$

for each $L \in \mathcal{K}_o^n$.

Finally, we shall make critical use of the following fact that is inspired by the work of Ball [1] and Meyer and Pajor [49] and that follows directly from (1.16), (1.17), and (1.8).

Lemma 1.1. *Suppose $K, L \in \mathcal{K}_o^n$ and consider $K, L \subset \mathbb{R}^{n-1}\times\mathbb{R}$. Then*

$$\mathrm{S}_{e_n}K^* \subset L^*,$$

if and only if

$$h(K;x',t) = 1 = h(K;x',-s), \quad \textit{with } t \neq -s \quad\Longrightarrow\quad h\left(L;x', \frac{1}{2}t + \frac{1}{2}s\right) \leqslant 1.$$

In addition, if $\mathrm{S}_{e_n}K^* = L^*$*, then* $h(K; x', t) = 1 = h(K; x', -s)$*, with* $t \neq -s$*, implies that* $h(L; x', \frac{1}{2}t + \frac{1}{2}s) = 1$.

We say that ∂K is *line free* in direction $u \in S^{n-1}$ if $\partial K \cap (x + \mathbb{R}u)$ consists of no more than two points, for each $x \in \partial K$. Note that if ∂K is line free in direction u then $\partial \mathrm{S}_u K$ is line free in direction u. It follows from the Ewald–Larman–Rogers theorem [14] (or see [58, p. 80]) that for each convex body K

$$\mathcal{H}^{n-1}\big(\big\{u \in S^{n-1}\colon\ \partial K \text{ is not line free in direction } u\big\}\big) = 0. \tag{1.19}$$

We will make use of the well-known and easily established fact that if ∂K is line free in direction e_n then for a continuous $g : \partial K \to \mathbb{R}$,

$$\begin{aligned}&\int\limits_{\partial K} g(x)\, d\mathcal{H}^{n-1}(x)\\ &\quad = \int\limits_{K'} g\big(x', \bar{\ell}(x')\big)\sqrt{1 + \big|\nabla \bar{\ell}(x')\big|^2}\, dx' + \int\limits_{K'} g\big(x', -\underline{\ell}(x')\big)\sqrt{1 + \big|\nabla \underline{\ell}(x')\big|^2}\, dx',\end{aligned} \tag{1.20}$$

where $K' = K_{e_n}$.

2. Definition and basic properties of Orlicz projection bodies

The Orlicz projection body $\Pi_\phi K$ of $K \in \mathcal{K}_o^n$ is defined as the body whose support function is given by

$$h_{\Pi_\phi K}(x) = \inf\left\{\lambda > 0\colon \int\limits_{\partial K} \phi\left(\frac{x \cdot \nu(y)}{\lambda y \cdot \nu(y)}\right) y \cdot \nu(y)\, d\mathcal{H}^{n-1}(y) \leqslant n|K|\right\}, \tag{2.1}$$

where $\nu(y) = \nu_K(y)$ is the outer unit normal of ∂K at $y \in \partial K$, or equivalently, using (1.9),

$$h_{\Pi_\phi K}(x) = \inf\left\{\lambda > 0\colon \int\limits_{S^{n-1}} \phi\left(\frac{x \cdot u}{\lambda h_K(u)}\right) h_K(u)\, dS_K(u) \leqslant n|K|\right\}. \tag{2.2}$$

It will be easier to see the affine nature (Lemma 2.6) of the Orlicz projection body if we use (1.7) to rewrite (2.2) as

$$h_{\Pi_\phi K}(x) = \inf\left\{\lambda > 0\colon \int\limits_{S^{n-1}} \phi\left(\frac{1}{\lambda}(x \cdot u)\rho_{K^*}(u)\right) dV_K(u) \leqslant 1\right\}. \tag{2.3}$$

The polar body of $\Pi_\phi K$ will be denoted by $\Pi_\phi^* K$, rather than $(\Pi_\phi K)^*$.

Since the area measure S_K cannot be concentrated on a closed hemisphere of S^{n-1}, and since we assume that ϕ is strictly increasing on $[0, \infty)$ or strictly decreasing on $(-\infty, 0]$ it follows that the function

$$\lambda \longmapsto \int\limits_{S^{n-1}} \phi\left(\frac{1}{\lambda}(x \cdot u)\rho_{K^*}(u)\right) dV_K(u)$$

is strictly decreasing in $(0, \infty)$. Thus, we have,

Lemma 2.1. *Suppose $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$. If $x_o \in \mathbb{R}^n \setminus \{0\}$, then*

$$\int\limits_{S^{n-1}} \phi\left(\frac{x_o \cdot u}{\lambda_o h_K(u)}\right) dV_K(u) = 1$$

if and only if

$$h_{\Pi_\phi K}(x_o) = \lambda_o.$$

We first show that $h_{\Pi_\phi K}$ is a support function.

Lemma 2.2. *Suppose $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$. Then the function $h_{\Pi_\phi K}$ is the support function of a convex body, $\Pi_\phi K$, that contains the origin in its interior.*

Proof. It follows immediately from definition (2.1) that for all $x \in \mathbb{R}^n$, and for $c > 0$

$$h_{\Pi_\phi K}(cx) = c h_{\Pi_\phi K}(x).$$

We now show that for $x_1, x_2 \in \mathbb{R}^n$,

$$h_{\Pi_\phi K}(x_1 + x_2) \leqslant h_{\Pi_\phi K}(x_1) + h_{\Pi_\phi K}(x_2).$$

To that end let

$$h_{\Pi_\phi K}(x_i) = \lambda_i.$$

By Lemma 2.1 this means that

$$\int\limits_{S^{n-1}} \phi\left(\frac{x_1 \cdot u}{\lambda_1}\rho_{K^*}(u)\right) dV_K(u) = 1 \quad \text{and} \quad \int\limits_{S^{n-1}} \phi\left(\frac{x_2 \cdot u}{\lambda_2}\rho_{K^*}(u)\right) dV_K(u) = 1.$$

The convexity of the function $s \mapsto \phi(s\rho_{K^*}(u))$ shows that

$$\phi\left(\frac{x_1 \cdot u + x_2 \cdot u}{\lambda_1 + \lambda_2}\rho_{K^*}(u)\right) \leqslant \frac{\lambda_1}{\lambda_1 + \lambda_2}\phi\left(\frac{x_1 \cdot u}{\lambda_1}\rho_{K^*}(u)\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2}\phi\left(\frac{x_2 \cdot u}{\lambda_2}\rho_{K^*}(u)\right).$$

Integrating both sides of this inequality with respect to the measure V_K gives us

$$\int\limits_{S^{n-1}} \phi\left(\frac{(x_1 + x_2) \cdot u}{\lambda_1 + \lambda_2}\rho_{K^*}(u)\right) dV_K(u) \leqslant 1, \tag{2.4}$$

from $h_{\Pi_\phi K}(x_i) = \lambda_i$. But using definition (2.3), inequality (2.4) gives the desired result that

$$h_{\Pi_\phi K}(x_1 + x_2) \leqslant \lambda_1 + \lambda_2.$$

Thus $h_{\Pi_\phi K}$ is indeed the support function of a compact convex set. That this set has the origin in its interior (i.e., that $h_{\Pi_\phi K}(x) > 0$ whenever $x \neq 0$) follows easily from the fact that we always have either $\lim_{s\to\infty} \phi(s) = \infty$ or $\lim_{s\to-\infty} \phi(s) = \infty$. $\square$

We shall require more than the fact that $\Pi_\phi K$ contains the origin in its interior.

Lemma 2.3. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$, then*

$$\frac{1}{2c_\phi R_K} \leqslant h_{\Pi_\phi K}(u) \leqslant \frac{1}{c_\phi r_K},$$

for all $u \in S^{n-1}$.

Proof. Suppose $u_o \in S^{n-1}$ and $h_{\Pi_\phi K}(u_o) = \lambda_o$; i.e.

$$\frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o h_K(u)}\right)\frac{h_K(u)\,dS_K(u)}{|K|} = 1 = \int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o h_K(u)}\right) dV_K(u).$$

To obtain the lower estimate we proceed as follows. From the definition (1.2), either $\phi(c_\phi) = 1$ or $\phi(-c_\phi) = 1$. Suppose $\phi(-c_\phi) = 1$. Hence from the fact that ϕ is non-negative and $\phi(0) = 0$, Jensen's inequality, and (1.13) together with the fact that ϕ is monotone decreasing on $(-\infty, 0]$,

$$\begin{aligned}
\phi(-c_\phi) &= 1 \\
&= \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o h_K(u)}\right)\frac{h_K(u)\,dS_K(u)}{|K|} \\
&\geqslant \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{(u_o \cdot u)_-}{\lambda_o h_K(u)}\right)\frac{h_K(u)\,dS_K(u)}{|K|} \\
&\geqslant \phi\left(\frac{1}{n}\int_{S^{n-1}} \frac{(u_o \cdot u)_-}{\lambda_o h_K(u)}\frac{h_K(u)\,dS_K(u)}{|K|}\right) \\
&\geqslant \phi\left(-\frac{1}{2\lambda_o R_K}\right).
\end{aligned}$$

Since ϕ is monotone decreasing on $(-\infty, 0]$, from this we obtain the lower bound for $h_{\Pi_\phi K}$,

$$\frac{1}{2c_\phi R_K} \leqslant \lambda_o.$$

The case where $\phi(c_\phi) = 1$ is handled the same way and gives the same result.

To obtain the upper estimate, observe that from the definition (1.2), together with the fact that the function $t \mapsto \max\{\phi(t), \phi(-t)\}$ is monotone increasing on $[0, \infty)$ and definition (1.12), and (1.11) it follows that

$$\begin{aligned}
\max\{\phi(c_\phi), \phi(-c_\phi)\} &= 1 \\
&= \int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o h_K(u)}\right) dV_K(u) \\
&\leqslant \int_{S^{n-1}} \max\left\{\phi\left(\frac{|u_o \cdot u|}{\lambda_o h_K(u)}\right), \phi\left(\frac{-|u_o \cdot u|}{\lambda_o h_K(u)}\right)\right\} dV_K(u) \\
&\leqslant \int_{S^{n-1}} \max\left\{\phi\left(\frac{1}{\lambda_o h_K(u)}\right), \phi\left(\frac{-1}{\lambda_o h_K(u)}\right)\right\} dV_K(u) \\
&\leqslant \int_{S^{n-1}} \max\left\{\phi\left(\frac{1}{\lambda_o r_K}\right), \phi\left(\frac{-1}{\lambda_o r_K}\right)\right\} dV_K(u) \\
&= \max\left\{\phi\left(\frac{1}{\lambda_o r_K}\right), \phi\left(\frac{-1}{\lambda_o r_K}\right)\right\}.
\end{aligned}$$

But the even function $t \mapsto \max\{\phi(t), \phi(-t)\}$ is monotone increasing on $[0, \infty)$ so we conclude

$$\lambda_o \leqslant \frac{1}{c_\phi r_K}. \qquad \square$$

For $c > 0$, from (1.3), (1.10) and definition (2.2) we have

$$\Pi_\phi cK = \frac{1}{c}\Pi_\phi K, \tag{2.5}$$

and taking the polars, and using (1.5),

$$\Pi_\phi^* cK = c\Pi_\phi^* K. \tag{2.6}$$

The Orlicz projection operator $\Pi_\phi : \mathcal{K}_o^n \to \mathcal{K}_o^n$ is continuous.

Lemma 2.4. *If $K_i \in \mathcal{K}_o^n$, then*

$$K_i \to K \in \mathcal{K}_o^n \quad \Longrightarrow \quad \Pi_\phi K_i \to \Pi_\phi K,$$

for each $\phi \in \mathcal{C}$.

Proof. Suppose $u_o \in S^{n-1}$. We will show that

$$h_{\Pi_\phi K_i}(u_o) \to h_{\Pi_\phi K}(u_o).$$

Let

$$h_{\Pi_\phi K_i}(u_o) = \lambda_i,$$

and note that Lemma 2.3 gives

$$\frac{1}{2c_\phi R_{K_i}} \leqslant \lambda_i \leqslant \frac{1}{c_\phi r_{K_i}}.$$

Since $K_i \to K \in \mathcal{K}_o^n$, we have $r_{K_i} \to r_K > 0$ and $R_{K_i} \to R_K < \infty$, and thus there exist a, b such that $0 < a \leqslant \lambda_i \leqslant b < \infty$, for all i. To show that the bounded sequence $\{\lambda_i\}$ converges to $h_{\Pi_\phi K}(u_o)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $h_{\Pi_\phi K}(u_o)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence we have

$$\lambda_i \to \lambda_*.$$

Obviously, $0 < a \leqslant \lambda_* \leqslant b$. Let $\bar{K}_i = \lambda_i K_i$. Since $\lambda_i \to \lambda_*$ and $K_i \to K$, we have

$$\bar{K}_i \to \lambda_* K.$$

The fact that $h_{\Pi_\phi K_i}(u_o) = \lambda_i$, together with (2.5) and (1.3), shows that $h_{\Pi_\phi \bar{K}_i}(u_o) = 1$; i.e.

$$\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{h_{\bar{K}_i}(u)}\right) dV_{\bar{K}_i}(u) = 1,$$

for all i. But $\bar{K}_i \to \lambda_* K$ implies that the functions $h_{\bar{K}_i} \to h_{\lambda_* K}$, uniformly, and the measures $S_{\bar{K}_i} \to S_{\lambda_* K}$, weakly. This in turn implies that the measures $V_{\bar{K}_i} \to V_{\lambda_* K}$, weakly, and hence using the continuity of ϕ we have

$$\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{h_{\lambda_* K}(u)}\right) dV_{\lambda_* K}(u) = 1,$$

which by Lemma 2.1 gives

$$h_{\Pi_\phi \lambda_* K}(u_o) = 1.$$

This, together with (2.5) and (1.3), yields the desired

$$h_{\Pi_\phi K}(u_o) = \lambda_*,$$

and shows that $h_{\Pi_\phi K_i}(u_o) \to h_{\Pi_\phi K}(u_o)$.

Since the support functions $h_{\Pi_\phi K_i} \to h_{\Pi_\phi K}$ pointwise (on S^{n-1}) they converge uniformly (see e.g., Schneider [58, p. 54]) completing the proof. $\square$

The continuity of the Orlicz projection operator $\Pi_\phi : \mathcal{K}_o^n \to \mathcal{K}_o^n$, together with the continuity of the projection operator, now yields the continuity of the polar Orlicz projection operator $\Pi_\phi^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$.

It turns out that the Orlicz projection body $\Pi_\phi K$ is continuous in ϕ as well as in K.

Lemma 2.5. *If $\phi_i \in \mathcal{C}$, then*

$$\phi_i \to \phi \in \mathcal{C} \quad \Longrightarrow \quad \Pi_{\phi_i} K \to \Pi_\phi K,$$

for each $K \in \mathcal{K}_o^n$.

Proof. Suppose $K \in \mathcal{K}_o^n$ and $u_o \in S^{n-1}$. We will first show that

$$h_{\Pi_{\phi_i} K}(u_o) \to h_{\Pi_\phi K}(u_o).$$

Let

$$h_{\Pi_{\phi_i} K}(u_o) = \lambda_i,$$

and note that Lemma 2.3 gives

$$\frac{1}{2c_{\phi_i} R_K} \leqslant \lambda_i \leqslant \frac{1}{c_{\phi_i} r_K}.$$

Since $\phi_i \to \phi \in \mathcal{C}$, we have $c_{\phi_i} \to c_\phi \in (0, \infty)$ and thus there exist a, b such that $0 < a \leqslant \lambda_i \leqslant b < \infty$, for all i. To show that the bounded sequence $\{\lambda_i\}$ converges to $h_{\Pi_\phi K}(u_o)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $h_{\Pi_\phi K}(u_o)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence we have

$$\lambda_i \to \lambda_*.$$

Obviously, $0 < a \leqslant \lambda_* \leqslant b$. Since $h_{\Pi_{\phi_i} K}(u_o) = \lambda_i$, Lemma 2.1 gives

$$1 = \int_{S^{n-1}} \phi_i\left(\frac{u_o \cdot u}{\lambda_i h_K(u)}\right) dV_K(u).$$

This, together with the facts that $\phi_i \to \phi \in \mathcal{C}$ and $\lambda_i \to \lambda_* \in (0, \infty)$, gives

$$1 = \int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_* h_K(u)}\right) dV_K(u).$$

When combined with Lemma 2.1, this gives the desired

$$h_{\Pi_\phi K}(u_o) = \lambda_*,$$

and completes the argument showing that $h_{\Pi_{\phi_i} K}(u_o) \to h_{\Pi_\phi K}(u_o)$.

Since the support functions $h_{\Pi_{\phi_i} K} \to h_{\Pi_\phi K}$ pointwise (on S^{n-1}) they converge uniformly and hence

$$\Pi_{\phi_i} K \to \Pi_\phi K. \qquad \square$$

We now demonstrate the affine nature of the Orlicz projection operator.

Lemma 2.6. *If $K \in \mathcal{K}_o^n$ and $A \in \mathrm{SL}(n)$, then*

$$\Pi_\phi AK = A^{-t} \Pi_\phi K.$$

Proof. Suppose P is a polytope whose $(n-1)$-dimensional faces are $F_1, \dots, F_m$. Let $u_1, \dots, u_m$ be the outer unit normals to the faces, and let $h_1, \dots, h_m$ denote the *support numbers* of the faces of P; i.e., $h(P; u_i) = h_i$. Let $V_1, \dots, V_m$ denote the volumes of the facial cones, so that, $V_i = \frac{1}{n} h_i |F_i|$. Finally, let V denote the volume of the polytope P.

For $A \in \mathrm{SL}(n)$, let $P^\diamond = AP = \{Ax\colon\ x \in P\}$. Let $F_1^\diamond, \dots, F_m^\diamond$ denote the faces of $P^\diamond$, let $u_1^\diamond, \dots, u_m^\diamond$ be the outer unit normals of the faces of $P^\diamond$ and let $h_1^\diamond, \dots, h_m^\diamond$ denote the corresponding support numbers of $P^\diamond$. Since $A \in \mathrm{SL}(n)$, obviously the volumes $V_1^\diamond, \dots, V_m^\diamond$ of the facial cones of $P^\diamond$ are such that $V_i^\diamond = V_i$.

The face F_i parallel to the subspace $u_i^\perp$ is transformed by A into the face $F_i^\diamond = AF_i$ parallel to $(A^{-t} u_i)^\perp$ and thus

$$u_i^\diamond = A^{-t} u_i / \left|A^{-t} u_i\right|. \tag{2.7}$$

Now $h_i^\diamond = h(P^\diamond, u_i^\diamond) = h(AP, u_i^\diamond) = h(P, A^t u_i^\diamond)$, by (1.4). Thus, from (2.7) we have

$$h_i^\diamond = h\big(P, A^t u_i^\diamond\big) = h\big(P, u_i / \left|A^{-t} u_i\right|\big) = h(P, u_i) / \left|A^{-t} u_i\right| = h_i / \left|A^{-t} u_i\right|. \tag{2.8}$$

Now from definition (2.3), the fact that $V^\diamond = V$ and $V_i^\diamond = V_i$ together with (2.7) and (2.8), definition (2.3) again, and finally (1.4), we have

$$\begin{aligned}
h_{\Pi_\phi AP}(x) &= h_{\Pi_\phi P^\diamond}(x) \\
&= \inf\left\{\lambda > 0\colon\ \sum_{i=1}^m \phi\left(\frac{x \cdot u_i^\diamond}{\lambda h_i^\diamond}\right) \frac{V_i^\diamond}{V^\diamond} \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0\colon\ \sum_{i=1}^m \phi\left(\frac{x \cdot A^{-t} u_i}{\lambda h_i}\right) \frac{V_i}{V} \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0\colon\ \sum_{i=1}^m \phi\left(\frac{A^{-1} x \cdot u_i}{\lambda h_i}\right) \frac{V_i}{V} \leqslant 1\right\} \\
&= h_{\Pi_\phi P}\big(A^{-1} x\big) \\
&= h_{A^{-t} \Pi_\phi P}(x),
\end{aligned}$$

showing that $\Pi_\phi AP = A^{-t} \Pi_\phi P$. This along with Lemma 2.4 completes the proof. $\square$

3. Proof of the Orlicz Petty projection inequality

We shall require the following.

Lemma 3.1. *If $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is a convex body that contains the origin in its interior, and if ∂K is line free in direction e_n, then for $(y', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$,*

$$\int_{\partial K} \phi\left(\frac{(y', t) \cdot \nu_K(x)}{x \cdot \nu_K(x)}\right) x \cdot \nu_K(x)\, d\mathcal{H}^{n-1}(x)$$
$$= \int_{K'} \phi\left(\frac{t - y' \cdot \nabla \bar{\ell}(x')}{\langle \bar{\ell}\rangle(x')}\right) \langle \bar{\ell}\rangle(x')\, dx' + \int_{K'} \phi\left(\frac{-t - y' \cdot \nabla \underline{\ell}(x')}{\langle \underline{\ell}\rangle(x')}\right) \langle \underline{\ell}\rangle(x')\, dx',$$

where $K' = K_{e_n}$ denotes the image of the orthogonal projection of K onto the subspace $e_n^{\perp} = \mathbb{R}^{n-1}$.

Proof. Note that we abbreviated the overgraph and undergraph functions of K in the direction e_n by $\bar{\ell} = \bar{\ell}_{e_n}(K; \cdot) : K' \to \mathbb{R}$ and $\underline{\ell} = \underline{\ell}_{e_n}(K; \cdot) : K' \to \mathbb{R}$; i.e.,

$$K = \left\{x' + s e_n \colon x' \in K',\ -\underline{\ell}(x') \leqslant s \leqslant \bar{\ell}(x')\right\}.$$

For $x' \in K'$, denote the outer unit normal of the overgraph of K at $(x', \bar{\ell}(x'))$ by $\bar{\nu}(x')$. Thus,

$$\bar{\nu}(x') = \frac{(-\nabla \bar{\ell}(x'), 1)}{(1 + |\nabla \bar{\ell}(x')|^2)^{\frac{1}{2}}}.$$

Denote the outer unit normal of the undergraph of K at $(x', -\underline{\ell}(x'))$ by $\underline{\nu}(x')$, so that

$$\underline{\nu}(x') = \frac{(-\nabla \underline{\ell}(x'), -1)}{(1 + |\nabla \underline{\ell}(x')|^2)^{\frac{1}{2}}}.$$

When x is on the overgraph of ∂K, i.e., $x = (x', \bar{\ell}(x'))$, we have

$$x \cdot \nu_K(x) = \left(x', \bar{\ell}(x')\right) \cdot \bar{\nu}(x') = \frac{\langle \bar{\ell}\rangle(x')}{(1 + |\nabla \bar{\ell}(x')|^2)^{\frac{1}{2}}}. \tag{3.1}$$

When x is on the undergraph of ∂K, i.e., $x = (x', -\underline{\ell}(x'))$, we have

$$x \cdot \nu_K(x) = \left(x', -\underline{\ell}(x')\right) \cdot \underline{\nu}(x') = \frac{\langle \underline{\ell}\rangle(x')}{(1 + |\nabla \underline{\ell}(x')|^2)^{\frac{1}{2}}}. \tag{3.2}$$

To complete the proof we now appeal to (1.20). □

We now establish the main ingredient in the proof of the Orlicz Petty projection inequality.

Proposition 3.1. *Suppose $\phi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$ and ∂K is line free in direction u, then*

$$\mathrm{S}_u \Pi_\phi^* K \subseteq \Pi_\phi^*(\mathrm{S}_u K). \tag{3.3}$$

If $\phi \in \mathcal{C}_s$ and $\mathrm{S}_u \Pi_\phi^* K = \Pi_\phi^*(\mathrm{S}_u K)$, then all of the midpoints of the chords of K parallel to u lie on a hyperplane that passes through the origin.

Proof. Without loss of generality, assume that $u = e_n$. We will be appealing to Lemma 1.1 and thus we begin by supposing that

$$h(\Pi_\phi K; y', t) = 1 \quad \text{and} \quad h(\Pi_\phi K; y', -s) = 1,$$

with $t \neq -s$, or equivalently, by (1.8), that

$$(y', t) \in \partial \Pi_\phi^* K \quad \text{and} \quad (y', -s) \in \partial \Pi_\phi^* K.$$

By Lemma 2.1 this means that

$$\frac{1}{n|K|} \int_{\partial K} \phi\left(\frac{(y', t) \cdot \nu_K(x)}{x \cdot \nu_K(x)}\right) x \cdot \nu_K(x)\, d\mathcal{H}^{n-1}(x) = 1 \tag{3.4a}$$

and

$$\frac{1}{n|K|} \int_{\partial K} \phi\left(\frac{(y', -s) \cdot \nu_K(x)}{x \cdot \nu_K(x)}\right) x \cdot \nu_K(x)\, d\mathcal{H}^{n-1}(x) = 1. \tag{3.4b}$$

By Lemma 1.1, the desired inclusion (3.3) will have been established if we can show that

$$h\left(\Pi_\phi \mathrm{S}_u K; y', \frac{1}{2}t + \frac{1}{2}s\right) \leqslant 1. \tag{3.5}$$

By Lemma 3.1 and (1.15), (1.14), (1.1), and Lemma 3.1 once again, we have

$$\begin{aligned}
&\int_{\partial(\mathrm{S}_u K)} \phi\left(\frac{(y', \frac{1}{2}t + \frac{1}{2}s) \cdot \nu_{\mathrm{S}_u K}(x)}{x \cdot \nu_{\mathrm{S}_u K}(x)}\right) x \cdot \nu_{\mathrm{S}_u K}(x)\, d\mathcal{H}^{n-1}(x) \\
&\quad = \int_{K'} \phi\left(\frac{\frac{1}{2}t + \frac{1}{2}s - y' \cdot \nabla(\frac{1}{2}\underline{\ell} + \frac{1}{2}\overline{\ell})(x')}{\langle \frac{1}{2}\underline{\ell} + \frac{1}{2}\overline{\ell}\rangle(x')}\right) \left\langle \frac{1}{2}\underline{\ell} + \frac{1}{2}\overline{\ell}\right\rangle(x')\, dx' \\
&\qquad + \int_{K'} \phi\left(\frac{-\frac{1}{2}t - \frac{1}{2}s - y' \cdot \nabla(\frac{1}{2}\underline{\ell} + \frac{1}{2}\overline{\ell})(x')}{\langle \frac{1}{2}\underline{\ell} + \frac{1}{2}\overline{\ell}\rangle(x')}\right) \left\langle \frac{1}{2}\underline{\ell} + \frac{1}{2}\overline{\ell}\right\rangle(x')\, dx'
\end{aligned}$$

$$\leqslant \frac{1}{2}\int_{K'} \phi\left(\frac{t - y'\cdot\nabla\bar{\ell}(x')}{\langle\bar{\ell}\rangle(x')}\right)\langle\bar{\ell}\rangle(x')\,dx' + \frac{1}{2}\int_{K'} \phi\left(\frac{s - y'\cdot\nabla\underline{\ell}(x')}{\langle\underline{\ell}\rangle(x')}\right)\langle\underline{\ell}\rangle(x')\,dx'$$
$$+ \frac{1}{2}\int_{K'} \phi\left(\frac{-t - y'\cdot\nabla\underline{\ell}(x')}{\langle\underline{\ell}\rangle(x')}\right)\langle\underline{\ell}\rangle(x')\,dx' + \frac{1}{2}\int_{K'} \phi\left(\frac{-s - y'\cdot\nabla\bar{\ell}(x')}{\langle\bar{\ell}\rangle(x')}\right)\langle\bar{\ell}\rangle(x')\,dx'$$
$$= \frac{1}{2}\int_{\partial K} \phi\left(\frac{(y',t)\cdot\nu_K(x)}{x\cdot\nu_K(x)}\right)x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)$$
$$+ \frac{1}{2}\int_{\partial K} \phi\left(\frac{(y',-s)\cdot\nu_K(x)}{x\cdot\nu_K(x)}\right)x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x). \tag{3.6}$$

As an aside, observe that if ϕ is strictly convex, then (1.1) tells us that equality in the inequality of (3.6) would imply that

$$\frac{t - y'\cdot\nabla\bar{\ell}(x')}{\langle\bar{\ell}\rangle(x')} = \frac{s - y'\cdot\nabla\underline{\ell}(x')}{\langle\underline{\ell}\rangle(x')} \quad\text{and}\quad \frac{-s - y'\cdot\nabla\bar{\ell}(x')}{\langle\bar{\ell}\rangle(x')} = \frac{-t - y'\cdot\nabla\underline{\ell}(x')}{\langle\underline{\ell}\rangle(x')},$$

for almost all $x' \in K'$.

Since $|\mathrm{S}_u K| = |K|$, it follows from (3.4) and (3.6), that

$$\frac{1}{n|\mathrm{S}_u K|}\int_{\partial(\mathrm{S}_u K)} \phi\left(\frac{(y', \frac{1}{2}t + \frac{1}{2}s)\cdot\nu_{\mathrm{S}_u K}(x)}{x\cdot\nu_{\mathrm{S}_u K}(x)}\right)x\cdot\nu_{\mathrm{S}_u K}(x)\,d\mathcal{H}^{n-1}(x) \leqslant 1.$$

This and a glance at definition (2.1), gives (3.5), and thus (3.3) is proved.

Suppose that ϕ is strictly convex and

$$\mathrm{S}_u \Pi_\phi^* K = \Pi_\phi^*(\mathrm{S}_u K). \tag{3.7}$$

For each $y' \in K'$ that is sufficiently close to the origin, there exist real $t_{y'}$ and $s_{y'}$, with $t_{y'} \neq -s_{y'}$, such that

$$h(\Pi_\phi K; y', t_{y'}) = 1 \quad\text{and}\quad h(\Pi_\phi K; y', -s_{y'}) = 1; \tag{3.8}$$

or equivalently by (1.8)

$$(y', t_{y'}) \in \partial\Pi_\phi^* K \quad\text{and}\quad (y', -s_{y'}) \in \partial\Pi_\phi^* K. \tag{3.9}$$

By Lemma 1.1, (3.7) and (3.8) forces

$$h\left(\Pi_\phi \mathrm{S}_u K; y', \frac{1}{2}t_{y'} + \frac{1}{2}s_{y'}\right) = 1. \tag{3.10}$$

Now (3.10) forces equality in (3.6). The strict convexity of ϕ and the equality conditions of (1.1) now show that

$$\frac{t_{y'} - y' \cdot \nabla\bar{\ell}(x')}{\langle\bar{\ell}\rangle(x')} = \frac{s_{y'} - y' \cdot \nabla\underline{\ell}(x')}{\langle\underline{\ell}\rangle(x')}, \tag{3.11a}$$

$$\frac{-s_{y'} - y' \cdot \nabla\bar{\ell}(x')}{\langle\bar{\ell}\rangle(x')} = \frac{-t_{y'} - y' \cdot \nabla\underline{\ell}(x')}{\langle\underline{\ell}\rangle(x')}, \tag{3.11b}$$

for almost all $x' \in K'$. Let $y' = 0$ and note from (3.9) that $s_0 \neq 0$ and $t_0 \neq 0$. Observe that the denominators in (3.11) are strictly positive for almost all $x' \in K'$, and by solving (3.11) we see that

$$\langle\bar{\ell}\rangle(x') = \langle\underline{\ell}\rangle(x'),$$

for almost all $x' \in K'$; i.e.,

$$(\bar{\ell} - \underline{\ell})(x') - x' \cdot \nabla(\bar{\ell} - \underline{\ell})(x') = 0 \tag{3.12}$$

for almost all $x' \in K'$. But the fact that $\langle\bar{\ell}\ \rangle(x') = \langle\underline{\ell}\rangle(x')$, for almost all $x' \in K'$, together with (3.11) shows that

$$(t_{y'} - s_{y'}) - y' \cdot \nabla(\bar{\ell} - \underline{\ell})(x') = 0, \tag{3.13}$$

for almost all $x' \in K'$. However (3.13) says that, for some null set $N' \subset K'$

$$\left\{\nabla(\bar{\ell} - \underline{\ell})(x'):\ x' \in K' \setminus N'\right\}$$

is a set of points that must lie in a plane of $\mathbb{R}^{n-1}$ with normal vector y'. But y' can be chosen in any direction in $\mathbb{R}^{n-1}$, so there exists an $x'_o \in \mathbb{R}^{n-1}$ with

$$\left\{\nabla(\bar{\ell} - \underline{\ell})(x'):\ x' \in K' \setminus N'\right\} = \left\{x'_o\right\}.$$

Substituting this into (3.12) shows that

$$(\bar{\ell} - \underline{\ell})(x') = x' \cdot x'_o$$

for almost all $x' \in K'$ and hence for all $x' \in K'$. This shows that the midpoints of the chords of K parallel to e_n,

$$\left\{\left(x', \frac{1}{2}\bar{\ell}(x') - \frac{1}{2}\underline{\ell}(x')\right):\ x' \in K'\right\},$$

lie in the subspace $\{(x', \frac{1}{2}x'_o \cdot x'):\ x' \in \mathbb{R}^{n-1}\}$ of $\mathbb{R}^n$. □

The observation (1.18), the continuity of $\Pi^*_\phi : \mathcal{K}^n_o \to \mathcal{K}^n_o$, and the fact that every convex body has a boundary that is line free in almost all directions (1.19), allows us to conclude from Proposition 3.1:

Corollary 3.1. *If $\phi \in \mathcal{C}$, and $K \in \mathcal{K}_o^n$, then*

$$\mathrm{S}_u \Pi_\phi^* K \subseteq \Pi_\phi^*(\mathrm{S}_u K),$$

for all $u \in S^{n-1}$.

To establish the equality conditions of our theorem we will make use of the following classical characterization of ellipsoids centered at the origin: A convex body $K \in \mathcal{K}_o^n$ is an ellipsoid centered at the origin if and only if there exists a dense set D of directions in S^{n-1} such that for each $u \in D$, the midpoints of the chords of K parallel to u lie in a subspace of $\mathbb{R}^n$.

Theorem 3.1. *Suppose $\phi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$, then the volume ratio*

$$\left|\Pi_\phi^* K\right| / |K|$$

is maximized when K is an ellipsoid centered at the origin. If $\phi \in \mathcal{C}_s$, then ellipsoids centered at the origin are the only maximizers.

Proof. Suppose $\phi \in \mathcal{C}_s$ and K is not an ellipsoid centered at the origin. Choose a direction u in which ∂K is line free and for which the chords of K (in direction u) have midpoints which do not lie in a subspace of $\mathbb{R}^n$. Let $K_1 = \mathrm{S}_u K$. From Proposition 3.1 and the fact that Steiner symmetrization leaves volume unchanged, it follows that

$$\left|\Pi_\phi^* K\right| < \left|\Pi_\phi^* K_1\right| \quad \text{with } |K| = |K_1|.$$

Choose a sequence of line-free directions $\{u_i\}$ such that the sequence defined by $K_{i+1} = \mathrm{S}_{u_i} K_i$, converges to cB, where $c > 0$. Here we are using (1.19) together with the fact (see [4]) that a dense set of directions is sufficient to Steiner transform any convex body into a ball. But Proposition 3.1 and the continuity of $\Pi_\phi^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$ can now be used to conclude that

$$\left|\Pi_\phi^* K\right| < \left|\Pi_\phi^* K_1\right| \leqslant \cdots \leqslant \left|\Pi_\phi^* K_i\right| \to \left|\Pi_\phi^* cB\right|,$$

with $|K_{i+1}| = |K_i| = \cdots = |K_1| = |K|$ and thus $|K| = |cB|$, since $K_i \to cB$. From (2.6) we know that $\Pi_\phi^* cB = c\, \Pi_\phi^* B$ and from this we have

$$\frac{|\Pi_\phi^* K|}{|K|} < \frac{|\Pi_\phi^* B|}{|B|}.$$

For $\phi \in \mathcal{C}$ that is not necessarily strictly convex, use the same argument — but now with a first step that results in an inequality that is not necessarily strict. □

4. Open problems

The equality conditions in Theorem 3.1 were only established under the assumption that ϕ is strictly convex. Was this restriction necessitated by our methods?

Conjecture 4.1. *Suppose $\phi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$, then the volume ratio*

$$\left|\Pi_\phi^* K\right| / |K|$$

is maximized only when K is an ellipsoid.

For $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$, the *Orlicz centroid body $\Gamma_\phi K$* of K is defined in [48] as the convex body whose support function at $x \in \mathbb{R}^n$ is given by

$$h(\Gamma_\phi K; x) = \inf\left\{\lambda > 0\colon \frac{1}{|K|} \int_K \phi\left(\frac{1}{\lambda} x \cdot y\right) dy \leqslant 1\right\},$$

where the integration is with respect to Lebesgue measure in $\mathbb{R}^n$.

In [48] the following Orlicz version of the Busemann–Petty centroid inequality is established.

Orlicz Busemann–Petty centroid inequality. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$ then the volume ratio*

$$|\Gamma_\phi K| / |K|$$

is minimized if and only if K is an ellipsoid centered at the origin.

A technique introduced in [35] shows that once the Petty projection inequality has been established one can easily derive the Busemann–Petty centroid inequality as a consequence. Is there a simple path from the Orlicz Petty projection inequality to the Orlicz Busemann–Petty centroid inequality?

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