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Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making ☆

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ABSTRACT

For the real decision making problems, most criteria have inter-dependent or interactive characteristics so that it is not suitable for us to aggregate them by traditional aggregation operators based on additive measures. Thus, to approximate the human subjective decision making process, it would be more suitable to apply fuzzy measures, where it is not necessary to assume additivity and independence among decision making criteria. In this paper, an intuitionistic fuzzy Choquet integral is proposed for multiple criteria decision making, where interactions phenomena among the decision making criteria are considered. First, we introduced two operational laws on intuitionistic fuzzy values. Then, based on these operational laws, intuitionistic fuzzy Choquet integral operator is proposed. Moreover, some of its properties are investigated. It is shown that the intuitionistic fuzzy Choquet integral operator can be represented by some special t-norms and t-conorms, and it is also a generalization of the intuitionistic fuzzy OWA operator and intuitionistic fuzzy weighted averaging operator. Further, the procedure and algorithm of multi-criteria decision making based on intuitionistic fuzzy Choquet integral operator is given under uncertain environment. Finally, a practical example is provided to illustrate the developed approaches.

1. Introduction

Since Zadeh presented the theory of fuzzy sets in 1965 (Zadeh, 1965), fuzzy sets theory has been successfully used for handling fuzzy decision making problems. In 1986, Atanassov (1986) extended the concept of Zadeh's fuzzy sets, and introduced the intuitionistic fuzzy sets, whose a prominent characteristic is that it assigns to each element a membership degree and a non-membership degree. So it gives us a powerful tool to deal with uncertainty and vagueness in real applications. As a generalization of the fuzzy sets, the intuitionistic fuzzy set has received more and more attention since its appearance. In 1993, Gau and Buehrer (1993) presented the concept of vague set. However, Bustine and Burillo (1996) pointed out that vague sets are intuitionistic fuzzy sets. Recently, the intuitionistic fuzzy set has been widely applied to the decision making problems. Based on vague sets, Chen and Tan (1994), Hong and Choi (2000) utilized the minimum and maximum operations and different measure functions to develop some approximate technique for handling multi-attribute decision making problems under fuzzy environment. Szmidt and Kacprzyk (1996) used intuitionistic fuzzy sets to solve group decision making problems. Further, Szmidt and Kacprzyk (2002) proposed some solution concepts such as the intuitionistic fuzzy core and consensus winner in group decision making with intuitionistic fuzzy preference relations, and proposed a method to aggregate the individual intuitionistic fuzzy preference relations into a social fuzzy preference relation on the basis of fuzzy majority equated with a fuzzy linguistic quantifier. Gabriella, Yager, and Atanassov (2004) constructed a generalized net model of multi-person multi-criteria decision making process based on intuitionistic fuzzy graphs. Atanassov, Pasi, and Yager (2005) proposed an intuitionistic fuzzy interpretation of multi-person multi-criteria decision making. Li (2005) investigated multi-attribute decision making using intuitionistic fuzzy sets and constructed several linear programming models to generate optimal weights for criteria. Pankowska and Wygralak (2006) introduced a concept of general intuitionistic fuzzy sets with triangular norm-based hesitation degrees, and applied it to build a comprehensive family of general algorithms of group decision making with a majority defined via linguistic quantifiers. Xu and Yager (2006) developed some geometric aggregation operators based on intuitionistic fuzzy sets, and applied them to multiple attribute decision making. Liu and Wang (2007) introduced the intuitionistic fuzzy point operators, and defined a series of new score functions for the multi-attribute decision making problems based on intuitionistic fuzzy point operators and evaluation function. By linear programming model, Lin, Yuan, and Xia (2007) presented a new method for handling multi-criteria fuzzy decision making problems based on intuitionistic fuzzy sets. Xu (2007) develop some new intuitionistic fuzzy aggregation operators, such as the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, and intuitionistic

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fuzzy hybrid aggregation operator, for aggregating intuitionistic fuzzy information. However, this aggregation process is based on the assumption that the criteria (attribute) and preferences of decision makers are independent, which is characterized by an independence axiom (Keeney & Raiffa, 1976; Wakker, 1999), that is, these operators are based on the implicit assumption that criteria and preference of decision makers are independent of one another; their effects are viewed as additive. For real decision making problems, there is always some degree of inter-dependent characteristics between attributes. Usually, there is interaction among preference of decision makers. Thus this assumption is too strong to match decision behaviors in the real world. The independence axiom generally can not be satisfied. To overcome this limitation, motivated by the Choquet integral (Choquet, 1954; Denneberg, 1994), in this paper, we shall develop an intuitionistic fuzzy Choquet integral operator for aggregating intuitionistic fuzzy information in multi-criteria decision making, and investigate various properties of this operator.

In order to do this, the paper is organized as follows. In Section 2, we review the intuitionistic fuzzy sets, and introduce two operational laws on intuitionistic fuzzy values. In Section 3, we introduce fuzzy measure and Choquet integral. Moreover, based on operational laws and fuzzy measure, an intuitionistic fuzzy Choquet integral operator is proposed, and some of its properties are investigated in detail. In Section 4, the multi-criteria decision making procedure based on the intuitionistic fuzzy Choquet integral operator is presented under intuitionistic fuzzy environment. In Section 5, an example is given to illustrate the concrete application of the method and to demonstrate its feasibility and practicality. Conclusions are made in Section 6.

Throughout this paper, let us denote by $X = \{1, 2, \dots, n\}$ the set of criteria of a given decision problem, and by $P(X)$ the power set of X, i.e., the set of all subsets of X. To avoid a heavy notation we will often omit braces for singletons, e.g., writing $A(i), X \setminus i$ instead of $A(\{i\}), X \setminus \{i\}$. Moreover, cardinality of subsets $S, T, \dots$ will be denoted whenever possible by the corresponding lower case letters $s, t, \dots$, otherwise by the standard notation $|S|, |T|, \dots$

2. Intuitionistic fuzzy sets

The main characteristic of fuzzy sets is that: the membership function assigns to each element x in a universe of discourse X a membership degree in interval $[0,1]$ and the non-membership degree equals one minus the membership degree, i.e., this single membership degree combines the evidence for x and the evidence against x, without indicating how much there is of each. The single membership value tells us nothing about the lack of knowledge. In real applications, however, the information of an object corresponding to a fuzzy concept may be incomplete, i.e., the sum of the membership degree and the non-membership degree of an element in a universe corresponding to a fuzzy concept may be less than one. In fuzzy set theory, there is no means to incorporate the lack of knowledge with the membership degrees. In 1986, Atanassov (1986) generalized the concept of fuzzy set, and defined the concept of intuitionsitic fuzzy set as follows.

Let X be an ordinary finite non-empty set. An intuitionistic fuzzy set in X is an expression A given by

$$A = \{< x, t_A(x), f_A(x) > | x \in X\} \tag{1}$$

where $t_A : X \to [0, 1], f_A : X \to [0, 1]$ with the condition: $0 \leqslant t_A(x) + f_A(x) \leqslant 1$, for all x in X. The numbers $t_A(x)$ and $f_A(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set A.

Obviously, each fuzzy set may be represented by:

$$A' = \{< x, t_{A'}(x), 1 - t_{A'}(x) > | x \in X\},$$

i.e., if $t_A(x) + f_A(x) = 1$, then the intuitionistic fuzzy set reduces to a fuzzy set. For computational convenience, in this paper, we call $(t_A(x), f_A(x))$ an intuitionistic fuzzy value. For convenience, let Ω be the set of all intuitionistic fuzzy values on X.

For every two intuitionistic fuzzy values A and B the following operations and relations are valid:

$$(1)\ A = B \quad \text{if and only if} \quad t_A(x) = t_B(x) \quad \text{and} \quad f_A(x) = f_B(x) \text{for all} \quad x \in X; \tag{2}$$

$$(2)\ A \leqslant B \quad \text{if and only if} \quad t_A(x) \leqslant t_B(x) \quad \text{and} \quad f_A(x) \geqslant f_B(x) \quad \text{for all} \quad x \in X. \tag{3}$$

However, (3) is not satisfied in some situations. So it cannot be used to compare intuitionistic fuzzy values. In the following, we use a score function (Chen & Tan, 1994) and an accuracy function (Hong & Choi, 2000) of intuitionistic fuzzy values for the comparison between two intuitionistic fuzzy values (Xu & Yager, 2006).

Definition 1. Let $\tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$ and $\tilde{b} = (t_{\tilde{b}}, f_{\tilde{b}})$ be two intuitionistic fuzzy values, $S(\tilde{a}) = t_{\tilde{a}} - f_{\tilde{a}}$ and $S(\tilde{b}) = t_{\tilde{b}} - f_{\tilde{b}}$ be the score functions of $\tilde{a}$ and $\tilde{b}$, respectively, and let $H(\tilde{a}) = t_{\tilde{a}} + f_{\tilde{a}}$ and $H(\tilde{b}) = t_{\tilde{b}} + f_{\tilde{b}}$ be the accuracy functions of $\tilde{a}$ and $\tilde{b}$, respectively, then

If $S(\tilde{a}) < S(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$;
If $S(\tilde{a}) = S(\tilde{b})$, then

(1) If $H(\tilde{a}) < H(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$;
(2) If $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, denoted by $\tilde{a} = \tilde{b}$.

Motivated by the operations in (Atanassov, 1999; De, Biswas, & Roy, 2000), we define two operational laws of intuitionistic fuzzy values.

Definition 2. Let $\tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$ and $\tilde{b} = (t_{\tilde{b}}, f_{\tilde{b}})$ be two intuitionistic fuzzy values, then

(1) $\tilde{a} \oplus \tilde{b} = (t_{\tilde{a}} + t_{\tilde{b}} - t_{\tilde{a}} t_{\tilde{b}}, f_{\tilde{a}} f_{\tilde{b}})$,
(2) $\lambda \tilde{a} = (1 - (1 - t_{\tilde{a}})^{\lambda}, (f_{\tilde{a}})^{\lambda}), \lambda > 0$.

For two operational laws of Definition 2, it is the following properties (Xu, 2007).

Proposition 1. *Let $\tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$ and $\tilde{b} = (t_{\tilde{b}}, f_{\tilde{b}})$ be two intuitionistic fuzzy values, and let $\tilde{c} = \tilde{a} \oplus \tilde{b}$ and $\tilde{d} = \lambda \tilde{a}$, then both $\tilde{c}$ and $\tilde{d}$ are also intuitionistic fuzzy values.*

Proposition 2. *Let $\tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$ and $\tilde{b} = (t_{\tilde{b}}, f_{\tilde{b}})$ be two intuitionistic fuzzy values, $\forall \lambda_1, \lambda_2 > 0$. Then we have:*

(1) $\tilde{a} \oplus \tilde{b} = \tilde{b} \oplus \tilde{a}$;
(2) $\lambda_1 (\tilde{a} \oplus \tilde{b}) = \lambda_1 \tilde{b} \oplus \lambda_1 \tilde{a}$;
(3) $\lambda_1 \tilde{a} \oplus \lambda_2 \tilde{a} = (\lambda_1 + \lambda_2) \tilde{a}$.

According to the two operational laws of Definition 2, Xu (2007) extended the weighted averaging operator and the ordered weighted averaging operator to intuitionistic fuzzy sets, and proposed the intuitionistic fuzzy weighted averaging operator and intuitionistic fuzzy ordered weighted averaging operator, which are defined as follows, respectively.

Definition 3. Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i}) (i = 1, 2, \dots, n)$ be a collection of intuitionistic fuzzy values on X. An intuitionistic fuzzy weighted averaging (IFWA) operator of dimension n is a mapping IFWA: $\Omega^n \to \Omega$, and

$$\mathrm{IFWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = w_1 \tilde{a}_1 \oplus w_2 \tilde{a}_2 \oplus \cdots \oplus w_n \tilde{a}_n$$

where $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of $\tilde{a}_i (i = 1, 2, \dots, n)$, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 4. Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values on X. An intuitionistic fuzzy ordered weighted averaging (IFOWA) operator of dimension n is a mapping IFOWA: $\Omega^n \to \Omega$, that has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, and

$$\mathrm{IFOWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = w_1\tilde{a}_{\pi(1)} \oplus w_2\tilde{a}_{\pi(2)} \oplus \cdots \oplus w_n\tilde{a}_{\pi(n)}$$

where $\pi(1), \pi(2), \ldots, \pi(n)$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{a}_{\pi(i)} \leqslant \tilde{a}_{\pi(i+1)}$ for all $i, i = 1, 2, \ldots, n$.

Furthermore

$$\mathrm{IFOWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(1 - \prod_{i=1}^n \left(1 - t_{\tilde{a}_{\pi(i)}}\right)^{w_i}, \prod_{i=1}^n (f_{\tilde{a}_{\pi(i)}})^{w_i}\right).$$

3. Intuitionistic fuzzy Choquet integral operator and some of its properties

3.1. Fuzzy measure and Choquet integral

In 1974, Sugeno (1974) introduced the concept of fuzzy measure (non-additive measure), which only make a monotonicity instead of additivity property. For decision making problems, it does not need assumption that criteria or preferences are independent of one another, and was used as a powerful tool for modeling interaction phenomena in decision making (Grabisch, 1996; Ishii & Sugeno, 1985; Kojadinovic, 2002; Murofushi & Soneda, 1993; Roubens, 1996). As an aggregation operator, the Choquet integral has been proposed by many authors as an adequate substitute to the weighted arithmetic mean or OWA operator to aggregate interacting criteria (Grabisch, 1995; Grabisch, Murofushi, & Sugeno, 2000; Labreuche & Grabisch, 2003; Marichal, 2000). In the weighted arithmetic mean or OWA model, each criterion i is given a weight $w_i \in [0,1]$ representing the importance of this criterion in the decision. In the Choquet integral model, where criteria can be dependent, a fuzzy measure is used to define a weight on each combination of criteria, thus making it possible to model the interaction existing among criteria.

Definition 5. A fuzzy measure on X is a set function $\mu : P(X) \to [0,1]$, satisfying the following conditions:

(1) $\mu(\phi) = 0, \mu(X) = 1$ (boundary conditions)
(2) If $A, B \in P(X)$ and $A \subseteq B$ then $\mu(A) \leqslant \mu(B)$ (monotonicity)

If the universal set X is infinite, it is necessary to add an extra axiom of continuity (Denneberg, 1994; Wang & Klir, 1992). However, in actual practice, it is enough to consider the finite universal set. $\mu(S)$ can be viewed as the grade of subjective importance of decision criteria set S. Thus, in addition to the usual weights on criteria taken separately, weights on any combination of criteria are also defined. This makes possible the representation of interaction between criteria. Intuitively, we could say the following about any a pair of criteria sets $A, B \in P(X), A \cap B = \phi$:

(1) A and B are considered to be without interaction (or to be independent) if $\mu(A \cup B) = \mu(A) + \mu(B)$, which is called an additive measure.
(2) A and B exhibit a positive synergetic interaction between them (or are complementary) if $\mu(A \cup B) > \mu(A) + \mu(B)$, which is called a super-additive measure.
(3) A and B exhibit a negative synergetic interaction between them (or are redundant or substitutive) if $\mu(A \cup B) < \mu(A) + \mu(B)$, which is called a sub-additive measure.

Definition 6 (Grabisch et al., 2000). Let f be a positive real-valued function on X, and μ be a fuzzy measure on X. The discrete Choquet integral of f with respective to μ is defined by

$$C_\mu(f) = \sum_{i=1}^n f_{(i)}[\mu(A_{(i)}) - \mu(A_{(i+1)})] \tag{4}$$

where $(\cdot)$ indicates a permutation on X such that $f_{(1)} \leqslant f_{(2)} \leqslant \ldots \leqslant f_{(n)}$. Also $A_{(i)} = \{i, \ldots, n\}, A_{(n+1)} = \phi$.

It is seen that the discrete Choquet integral is a linear expression up to a reordering of the elements. Moreover, it identifies with the weighted mean (discrete Lebesgue integral) as soon as the fuzzy measure is additive. And in some condition, the Choquet integral operator coincides with the OWA operator (Grabisch, 1995; Marichal, 2002).

3.2. Intuitionistic fuzzy Choquet integral operator

We first give the definition of intuitionistic fuzzy Choquet integral operator as follows.

Definition 7. Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. The (discrete) intuitionistic fuzzy Choquet integral of $\tilde{a}_i$ with respective to μ is defined by

$$\begin{aligned}\mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) &= \tilde{a}_{(1)}(\mu(A_{(1)}) - \mu(A_{(2)})) \oplus \tilde{a}_{(2)}(\mu(A_{(2)}) \\ &\quad - \mu(A_{(3)})) \oplus \cdots \oplus \tilde{a}_{(n)}(\mu(A_{(n)}) - \mu(A_{(n+1)})) \\ &= \sum_{i=1}^{n}{}^{\oplus} \tilde{a}_{(i)}(\mu(A_{(i)}) - \mu(A_{(i+1)}))\end{aligned} \tag{5}$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{a}_{(1)} \leqslant \tilde{a}_{(2)} \leqslant \cdots \leqslant \tilde{a}_{(n)}$. And $A_{(i)} = (i, \ldots, n), A_{(n+1)} = \phi$.

Theorem 1. *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. then their aggregated value by using the IFC_μ operator is also an intuitionistic fuzzy value, and*

$$\mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) = \left(1 - \prod_{i=1}^n (1 - t_{\tilde{a}_{(i)}})^{\mu(A_{(i)}) - \mu(A_{(i+1)})}, \prod_{i=1}^n (f_{\tilde{a}_{(i)}})^{\mu(A_{(i)}) - \mu(A_{(i+1)})}\right) \tag{6}$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{a}_{(1)} \leqslant \tilde{a}_{(2)} \leqslant \cdots \leqslant \tilde{a}_{(n)}$. And $A_{(i)} = ((i), \ldots, (n)), A_{(n+1)} = \phi$.

Proof. The first result follows quickly from Definition 7 and Proposition 1. Below we prove Eq. (6) by using mathematical induction on n.

For $n = 2$, according to the operational laws of Definition 2, we have

$$(\mu(A_{(1)}) - \mu(A_{(2)})\tilde{a}_{(1)} = (1 - (1 - t_{\tilde{a}_{(1)}})^{\mu(A_{(1)}) - \mu(A_{(2)})}, (f_{\tilde{a}_{(1)}})^{\mu(A_{(1)}) - \mu(A_{(2)})}),$$
$$(\mu(A_{(2)}) - \mu(A_{(3)})\tilde{a}_{(2)} = (1 - (1 - t_{\tilde{a}_{(2)}})^{\mu(A_{(2)}) - \mu(A_{(3)})}, (f_{\tilde{a}_{(2)}})^{\mu(A_{(2)}) - \mu(A_{(3)})})$$

Since

$$\tilde{a}_1 \oplus \tilde{a}_2 = (t_{\tilde{a}_1} + t_{\tilde{a}_2} - t_{\tilde{a}_1}t_{\tilde{a}_2}, f_{\tilde{a}_1}f_{\tilde{a}_2}) = (1 - (1 - t_{\tilde{a}_1})(1 - t_{\tilde{a}_2}), f_{\tilde{a}_1}f_{\tilde{a}_2}),$$

then

$$\begin{aligned}\mathrm{IFC}_\mu(\tilde{a}_1, \tilde{a}_2) &= \tilde{a}_{(1)}(\mu(A_{(1)}) - \mu(A_{(2)})) \oplus \tilde{a}_{(2)}(\mu(A_{(2)}) - \mu(A_{(3)})) \\ &= (1 - (1 - t_{\tilde{a}_{(1)}})^{\mu(A_{(1)}) - \mu(A_{(2)})}(1 - t_{\tilde{a}_{(2)}})^{\mu(A_{(2)}) - \mu(A_{(3)})}, \\ &\quad (f_{\tilde{a}(1)})^{\mu(A_{(1)}) - \mu(A_{(2)})}(f_{\tilde{a}_{(2)}})^{\mu(A_{(2)}) - \mu(A_{(3)})})\end{aligned}$$

That is, for n = 2, the Eq. (6) holds.

Suppose that if for $n = k$, the Eq. (6) holds, i.e.,

$$\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_k) = \left(1 - \prod_{i=1}^{k}(1-t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{k}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)$$

Then, for $n = k+1$, according to Definition 7, we have

$$\begin{aligned}\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_{k+1}) &= \Bigg(1 - \prod_{i=1}^{k}(1-t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}(1-t_{\tilde{a}_{(k+1)}})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})},\\ &\qquad \prod_{i=1}^{k}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}(f_{\tilde{a}_{(k+1)}})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})}\Bigg)\\ &= \left(1 - \prod_{i=1}^{k+1}(1-t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{k+1}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right).\end{aligned}$$

That is, for $n = k+1$, the Eq. (6) still holds.

Therefore, for all n, the Eq. (6) always holds, which completes the proof of Theorem 1. □

Remark 1. Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ and $\tilde{b}_i = (t_{\tilde{b}_i}, f_{\tilde{b}_i})(i = 1,2,\ldots,n)$ be two collections of intuitionistic fuzzy values on X. Since $t_{\tilde{a}_i}$, $f_{\tilde{a}_i}, t_{\tilde{b}_i}, f_{\tilde{b}_i} \in [0,1]$ for any i, if we assume that $T_P(f_{\tilde{a}_i}, f_{\tilde{b}_i}) = f_{\tilde{a}_i}f_{\tilde{b}_i}$, $S_P(t_{\tilde{a}_i}, t_{\tilde{b}_i}) = t_{\tilde{a}_i} + t_{\tilde{b}_i} - t_{\tilde{a}_i}t_{\tilde{b}_i}$, then $T_P(t_{\tilde{a}_i}, t_{\tilde{b}_i})$ is one of the basic t-norms, called the product (Frank, 1979; Klement, Mesiar, & Pap, 2000), which is satisfying the following properties (Klement et al., 2000): $T_P(x,1) = x$ (boundary); $T_P(x,y) \leqslant T_P(x,z)$ whenever $y \leqslant z$ (monotonicity); $T_P(x,y) = T_P(y,x)$ (commutativity); $T_P(x, T_P(y,z)) = T_P(T_P(x,y)$ (associativity), where $x, y, z \in [0, 1]$. $S_P(f_{\tilde{a}_i}, f_{\tilde{b}_i})$ is one of the basic t-conorms, called the probabilistic sum (Frank, 1979; Klement et al., 2000), and S_P is also called the dual t-conorm of T_P, which is satisfying the boundary, i.e., $S_P(x,0) = x$, monotonicity, commutativity, and associativity (Klement et al., 2000). The associativity of t-norms and t-conorms allows us to extend the product T_P and probabilistic sum $S_P$in unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, \ldots, x_n) \in [0,1]^n$ and $(y_1, y_2, \ldots, y_n) \in [0,1]^n$, respectively:

$$T_P(x_1,x_2,\cdots,x_n) = \underset{i=1}{\overset{n}{T_P}}\, x_i = T_P\left(\underset{i=1}{\overset{n-1}{T_P}} x_i, x_n\right) = \prod_{i=1}^{n} x_i,$$

$$S_P(y_1,y_2,\cdots,y_n) = \underset{i=1}{\overset{n}{S_P}}\, y_i = S_P\left(\underset{i=1}{\overset{n-1}{S_P}} y_i, y_n\right) = 1 - \prod_{i=1}^{n}(1-y_i).$$

Assume that $y_i = 1-(1-t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, x_i = (f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, then from Theorem 1 we have

$$\mathrm{IFC}_\mu(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = (S_P(x_1,x_2,\cdots,x_n), T_P(y_1,y_2,\cdots,y_n)). \tag{7}$$

It is shown that the intuitionistic fuzzy Choquet integral operator can be represented by one of the basic t-norms T_P and t-conorms S_P.

Proposition 3. *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If all $\tilde{a}_i(i = 1,2,\ldots,n)$ are equal, that is, for all $i, \tilde{a}_i = \tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$, then*

$$\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_n) = \tilde{a}.$$

Proof. According to Theorem 1, if for all $i(i = 1,2,\ldots,n), \tilde{a}_i = \tilde{a}$, then

$$\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_n) = \left(1-(1-t_{\tilde{a}})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}, (f_{\tilde{a}})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}\right)$$

Since

$$\begin{aligned}\sum_{i=1}^{n}(\mu(A_{(i)})-\mu(A_{(i+1)})) &= (\mu(A_{(1)})-\mu(A_{(2)})) + (\mu(A_{(2)})-\mu(A_{(3)}))\\ &\quad + \cdots + (\mu(A_{(n)})-\mu(A_{(n+1)}))\\ &= \mu(A_{(1)})-\mu(A_{(n+1)}) = 1.\end{aligned}$$

So

$$\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_n) = (t_{\tilde{a}}, f_{\tilde{a}}) = \tilde{a}. \qquad \square$$

Proposition 4. *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ and $\tilde{b}_i = (t_{\tilde{b}_i}, f_{\tilde{b}_i})(i = 1,2,\ldots,n)$ be two collections of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. $(\cdot)$ indicates a permutation such that $\tilde{a}_{(1)} \leqslant \cdots \leqslant \tilde{a}_{(n)}$ and $\tilde{b}_{(1)} \leqslant \cdots \leqslant \tilde{b}_{(n)}$. If $\tilde{a}_{(i)} \leqslant \tilde{b}_{(i)}$ for all i, that is, $t_{\tilde{a}_{(i)}} \leqslant t_{\tilde{b}_{(i)}}$ and $f_{\tilde{a}_{(i)}} \geqslant f_{\tilde{b}_{(i)}}$, then*

$$\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_n) \leqslant \mathrm{IFC}_\mu(\tilde{b}_1,\cdots,\tilde{b}_n).$$

Proof. Since $A_{(i+1)} \subseteq A_{(i)}$, then $\mu(A_{(i)}) - \mu(A_{(i+1)}) \geqslant 0$. For all $i, t_{\tilde{a}_{(i)}} \leqslant t_{\tilde{b}_{(i)}}$ and $f_{\tilde{a}_{(i)}} \geqslant f_{\tilde{b}_{(i)}}$, we have

$$\begin{aligned}&1 - \prod_{i=1}^{n}(1-t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\\ &\quad \leqslant 1 - \prod_{i=1}^{n}(1-t_{\tilde{b}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\\ &\quad \geqslant \prod_{i=1}^{n}(f_{\tilde{b}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}.\end{aligned}$$

Since

$$\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_n) = \left(1 - \prod_{i=1}^{n}(1-t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right),$$

$$\mathrm{IFC}_\mu(\tilde{b}_1,\cdots,\tilde{b}_n) = \left(1 - \prod_{i=1}^{n}(1-t_{\tilde{b}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(f_{\tilde{b}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right).$$

According to (3), we have

$$\mathrm{IFC}_\mu(\tilde{a}_1,\cdots,\tilde{a}_n) \leqslant IFC_\mu(\tilde{b}_1,\cdots,\tilde{b}_n). \qquad \square$$

Remark 2. From the monotonicity and associativity of t-norms and t-conorms, we easily obtain that

$$T_P(x_1,x_2,\cdots,x_n) \leqslant T_P(x'_1,x'_2,\cdots,x'_n) \quad \text{whenever} \quad x_1 \leqslant x'_1, x_2 \leqslant x'_2, \cdots, x_n \leqslant x'_n, \tag{8}$$

$$S_P(y_1,y_2,\cdots,y_n) \leqslant S_P(y'_1,y'_2,\cdots,y'_n) \quad \text{whenever} \quad y_1 \leqslant y'_1, y_2 \leqslant y'_2, \cdots, y_n \leqslant y'_n. \tag{9}$$

Since the intuitionistic fuzzy Choquet integral operator can be represented by one of the basic t-norms T_P and t-conorms S_P. If $t_{\tilde{a}_{(i)}} \leqslant t_{\tilde{b}_{(i)}}$ and $f_{\tilde{a}_{(i)}} \geqslant f_{\tilde{b}_{(i)}}$, let

$$x_i = (t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \quad y_i = 1-(1-f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$
$$x'_i = (t_{\tilde{b}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \quad y'_i = 1-(1-f_{\tilde{b}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}.$$

Since $A_{(i+1)} \subset A_{(i)}\mu(A_{(i+1)}) \leqslant \mu(A_{(i)})(i = 1,.2,\ldots,n)$, then $x_i \leqslant x'_i, y_i \geqslant y'_i$. By Eqs. (7)–(9), the conclusion of Proposition 4 is obvious, too.

Proposition 5. *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. And*

$$\tilde{a}^- = (\min_i(t_{\tilde{a}_i}), \max_i(f_{\tilde{a}_i})), \quad \tilde{a}^+ = (\max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i})),$$

then

$$\tilde{a}^- \leqslant \mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) \leqslant \tilde{a}^+.$$

Proof. For any $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1, 2, \ldots, n)$, it is obvious that $\tilde{a}^- = (\min_i(t_{\tilde{a}_i}), \max_i(f_{\tilde{a}_i}))$, and $\tilde{a}^+ = (\max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i}))$ are intuitionistic fuzzy values.

Since $A_{(i+1)} \subseteq A_{(i)}$, then $\mu(A_{(i)}) - \mu(A_{(i+1)}) \geqslant 0$.

Let $(\cdot)$ indicates a permutation such that $\tilde{a}_{(1)} \leqslant \cdots \leqslant \tilde{a}_{(n)}$, we have

$$\min_i(t_{\tilde{a}_i}) \leqslant t_{\tilde{a}_{(i)}} \leqslant \max_i(t_{\tilde{a}_i}), \quad \min_i(f_{\tilde{a}_i}) \leqslant f_{\tilde{a}_{(i)}} \leqslant \max_i(f_{\tilde{a}_i}).$$

So,

$$\begin{aligned}
1 - \prod_{i=1}^{n}\left(1 - \min_i(t_{\tilde{a}_i})\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}
&\leqslant 1 - \prod_{i=1}^{n}\left(1 - t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \\
&\leqslant 1 - \prod_{i=1}^{n}\left(1 - \max_i(t_{\tilde{a}_i})\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}
\end{aligned}$$

and

$$\begin{aligned}
\prod_{i=1}^{n} \min_i(f_{\tilde{a}_i})^{\mu(A_{(i)})-\mu(A_{(i+1)})} &\leqslant \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \\
&\leqslant \prod_{i=1}^{n} \max_i(f_{a_i})^{\mu(A_{(i)})-\mu(A_{(i+1)})}
\end{aligned}$$

i.e.,

$$\begin{aligned}
1 - \left(1 - \min_i(t_{\tilde{a}_i})\right)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}
&\leqslant 1 - \prod_{i=1}^{n}\left(1 - t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \\
&\leqslant 1 - \left(1 - \max_i(t_{\tilde{a}_i})\right)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})},
\end{aligned}$$

and

$$\begin{aligned}
\min_i(f_{\tilde{a}_i})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} &\leqslant \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \\
&\leqslant \max_i(f_{\tilde{a}_i})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{i=1}^{n}(\mu(A_{(i)}) - \mu(A_{(i+1)})) &= (\mu(A_{(1)}) - \mu(A_{(2)})) + (\mu(A_{(2)}) - \mu(A_{(3)})) \\
&\quad + \cdots + (\mu(A_{(n)}) - \mu(A_{(n+1)})) \\
&= \mu(A_{(1)}) - \mu(A_{(n+1)}) = 1.
\end{aligned}$$

So we have

$$\min_i(t_{\tilde{a}_i}) \leqslant 1 - \prod_{i=1}^{n}(1 - t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \max_i(t_{\tilde{a}_i}),$$

$$\min_i(f_{\tilde{a}_i}) \leqslant \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \max_i(f_{\tilde{a}_i}).$$

According to (3), we have

$$\left(\min_i(t_{\tilde{a}_i}), \max_i(f_{\tilde{a}_i})\right) \leqslant \mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) \leqslant (\max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i})),$$

That is, $\tilde{a}^- \leqslant \mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) \leqslant \tilde{a}^+$. □

Remark 3. Proposition 3 can be immediately obtained from Proposition 5, too.

From Definition 7, the following property of the intuitionistic fuzzy Choquet integral operator can easily be obtained.

Proposition 6. *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, then*

$$\mathrm{IFC}_\mu(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \mathrm{IFC}_\mu(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n).$$

Proposition 7. *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $\tilde{s} = (t_{\tilde{s}}, f_{\tilde{s}})$is an intuitionistic fuzzy value on X, then*

$$\mathrm{IFC}_\mu(\tilde{a}_1 \oplus \tilde{s}, \cdots, \tilde{a}_n \oplus \tilde{s}) = \mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) \oplus \tilde{s}.$$

Proof. Since for any $i(i = 1, 2, \ldots, n)$

$$\tilde{a}_i \oplus \tilde{s} = (t_{\tilde{a}_i} + t_{\tilde{s}} - t_{\tilde{a}_i}t_{\tilde{s}}, f_{\tilde{a}_i}f_{\tilde{s}}) = \left(1 - (1 - t_{\tilde{a}_i})(1 - t_{\tilde{s}}), f_{\tilde{a}_i}f_{\tilde{s}}\right).$$

According to Theorem 1, we have

$$\begin{aligned}
&\mathrm{IFC}_\mu(\tilde{a}_1 \oplus \tilde{s}, \cdots, \tilde{a}_n \oplus \tilde{s}) \\
&= \left(1 - \prod_{i=1}^{n}((1 - t_{\tilde{a}_{(i)}})(1 - t_{\tilde{s}}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}}f_{\tilde{s}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right) \\
&= (1 - (1 - t_{\tilde{s}}))^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \prod_{i=1}^{n}(1 - t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)}}, \\
&\quad (f_{\tilde{s}})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}) \\
&= \left(1 - (1 - t_{\tilde{s}}) \prod_{i=1}^{n}(1 - t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, f_{\tilde{s}} \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right).
\end{aligned}$$

According to Definition 2, we have

$$\begin{aligned}
&\mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) \oplus \tilde{s} \\
&= \left(1 - (1 - t_{\tilde{s}}) \prod_{i=1}^{n}(1 - t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, f_{\tilde{s}} \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right).
\end{aligned}$$

Thus,

$$\mathrm{IFC}_\mu(\tilde{a}_1 \oplus \tilde{s}, \cdots, \tilde{a}_n \oplus \tilde{s}) = \mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n) \oplus \tilde{s}. \qquad \square$$

Proposition 8. *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})(i = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $r > 0$, then*

$$\mathrm{IFC}_\mu(r\tilde{a}_1, \cdots, r\tilde{a}_n) = r\mathrm{IFC}_\mu(\tilde{a}_1, \cdots, \tilde{a}_n).$$

Proof. According to Definition 2, for any $i(i = 1, 2, \ldots, n)$ and $r > 0$ we have

$$r\tilde{a}_i = \left(1 - (1 - t_{\tilde{a}_i})^r, (f_{\tilde{a}_i})^r\right).$$

According to Theorem 1, we have

$$\begin{aligned}
&\mathrm{IFC}_\mu(r\tilde{a}_1, \cdots, r\tilde{a}_n) \\
&= \left(1 - \prod_{i=1}^{n}((1 - t_{\tilde{a}_{(i)}})^r)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}((f_{\tilde{a}_{(i)}})^r)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right) \\
&= \left(1 - \prod_{i=1}^{n}(1 - t_{\tilde{a}_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \prod_{i=1}^{n}(f_{\tilde{a}_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\right).
\end{aligned}$$

Since

$$r\mathrm{IFC}_\mu(\tilde a_1,\ldots,\tilde a_n)=r\left(1-\prod_{i=1}^{n}(1-t_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\prod_{i=1}^{n}(f_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)$$
$$=\left(1-\left(\prod_{i=1}^{n}(1-t_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^r,\left(\prod_{i=1}^{n}(f_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^r\right)$$
$$=\left(1-\prod_{i=1}^{n}(1-t_{\tilde a_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))},\prod_{i=1}^{n}(f_{\tilde a_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\right).$$

Thus,

$$\mathrm{IFC}_\mu(r\tilde a_1,\ldots,r\tilde a_n)=r\mathrm{IFC}_\mu(\tilde a_1,\ldots,\tilde a_n).\quad\square$$

According to Propositions 7 and 8, we can obtain the following corollary.

Corollary 1. *Let $\tilde a_i=(t_{\tilde a_i},f_{\tilde a_i})(i=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $r>0$ and $\tilde s=(t_{\tilde s},f_{\tilde s})$is an intuitionistic fuzzy value on X, then*

$$\mathrm{IFC}_\mu(r\tilde a_1\oplus\tilde s,\cdots,r\tilde a_n\oplus\tilde s)=rIFC_\mu(\tilde a_1,\cdots,\tilde a_n)\oplus\tilde s.$$

According to Definition 2 and Theorem 1, it is easily obtained the following proposition.

Proposition 9. *Let $\tilde a_i=(t_{\tilde a_i},f_{\tilde a_i})(i=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X.*

(1) *If $\mu(A)=1$ for any $A\in P(X)$, then $\mathrm{IFC}_\mu(\tilde a_1,\cdots,\tilde a_n)=\max(\tilde a_1,\cdots,\tilde a_n)=\tilde a_{(n)}$*
(2) *If $\mu(A)=0$ for any $A\in P(X)$ and $A\neq X$, then $\mathrm{IFC}_\mu(\tilde a_1,\cdots,\tilde a_n)=\min(\tilde a_1,\cdots,\tilde a_n)=\tilde a_{(1)}$.*
(3) *For any $A,B\in P(X)$ such that $a=b$, if $\mu(A)=\mu(B)$ and $\mu\{(i),\ldots,(n)\}=\frac{n-i+1}{n},1\leqslant i\leqslant n$, then*

$$\mathrm{IFC}_\mu(\tilde a_1,\cdots,\tilde a_n)=\left(1-\prod_{i=1}^{n}(1-t_{\tilde a_{(i)}})^{\frac1n},\prod_{i=1}^{n}(f_{\tilde a_{(i)}})^{\frac1n}\right),$$

where $(\cdot)$ indicates a permutation on X such that $\tilde a_{(1)}\leqslant\tilde a_{(2)}\leqslant\cdots\leqslant\tilde a_{(n)}$.

Definition 8. Let $\tilde a_i=(t_{\tilde a_i},f_{\tilde a_i})$ and $\tilde b_i=(t_{\tilde b_i},f_{\tilde b_i})(i=1,2,\ldots,n)$ be two collections of intuitionistic fuzzy values on X. $\tilde a_i,\tilde b_i$are said to be comonotonic if

$$\tilde a_{(1)}\leqslant\tilde a_{(2)}\leqslant\cdots\leqslant\tilde a_{(n)}\Rightarrow\tilde b_{(1)}\leqslant\tilde b_{(2)}\leqslant\cdots\leqslant\tilde b_{(n)},$$

where $(\cdot)$ denotes a permutation.

Proposition 10. *Let μ be a fuzzy measure on X. $\tilde a_i=(t_{\tilde a_i},f_{\tilde a_i})$ and $\tilde b_i=(t_{\tilde b_i},f_{\tilde b_i})(i=1,2,\ldots,n)$ are two collections of intuitionistic fuzzy values on X. If $\tilde a_i$ and $\tilde b_i$ are comonotonic, then*

$$\mathrm{IFC}_\mu(\tilde a_1\oplus\tilde b_1,\cdots,\tilde a_n\oplus\tilde b_n)=\mathrm{IFC}_\mu(\tilde a_1,\cdots,\tilde a_n)\oplus\mathrm{IFC}_\mu(\tilde b_1,\cdots,\tilde b_n).$$

Proof. Let $(\cdot)$ indicate a permutation on X such that

$$\tilde a_{(1)}\leqslant\tilde a_{(2)}\leqslant\cdots\leqslant\tilde a_{(n)}\Rightarrow\tilde b_{(1)}\leqslant\tilde b_{(2)}\leqslant\cdots\leqslant\tilde b_{(n)}.$$

Then

$$\tilde a_{(i)}\oplus\tilde b_{(i)}=(t_{\tilde a_{(i)}}+t_{\tilde b_{(i)}}-t_{\tilde a_{(i)}}t_{\tilde b_{(i)}},f_{\tilde a_{(i)}}f_{\tilde b_{(i)}})$$
$$=(1-(1-t_{\tilde a_{(i)}})(1-t_{\tilde b_{(i)}}),f_{\tilde a_{(i)}}f_{\tilde b_{(i)}}).$$

For $\tilde a_{(i)}\leqslant\tilde a_{(i+1)}$, according to (3), we have

$$t_{\tilde a_{(i)}}\leqslant t_{\tilde a_{(i+1)}}\quad\text{and}\quad f_{\tilde a_{(i)}}\geqslant f_{\tilde a_{(i+1)}}.$$

Similarly, for $\tilde b_{(i)}\leqslant\tilde b_{(i+1)}$, we have

$$t_{\tilde b_{(i)}}\leqslant t_{\tilde b_{(i+1)}}\quad\text{and}\quad f_{\tilde b_{(i)}}\geqslant f_{\tilde b_{(i+1)}}.$$

So

$$(1-(1-t_{\tilde a_{(i)}})(1-t_{\tilde b_{(i)}})\leqslant(1-(1-t_{\tilde a_{(i+1)}})(1-t_{\tilde b_{(i+1)}}),\quad f_{\tilde a_{(i)}}f_{\tilde b_{(i)}}\geqslant f_{\tilde a_{(i+1)}}f_{\tilde b_{(i+1)}}.$$

That is,

$$\tilde a_{(i)}\oplus\tilde b_{(i)}\leqslant\tilde a_{(i+1)}\oplus\tilde b_{(i+1)}.$$

Then according to Theorem 1, we have

$$\mathrm{IFC}_\mu(\tilde a_1\oplus\tilde b_1,\cdots,\tilde a_n\oplus\tilde b_n)$$
$$=\left(1-\prod_{i=1}^{n}((1-t_{\tilde a_{(i)}})(1-t_{\tilde b_{(i)}}))^{\mu(A_{(i)})-\mu(A_{(i+1)})},\prod_{i=1}^{n}(f_{\tilde a_{(i)}}f_{\tilde b_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)$$
$$=(1-\prod_{i=1}^{n}(1-t_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\prod_{i-1}^{n}(1-t_{\tilde b_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$
$$\prod_{i=1}^{n}(f_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\prod_{i=1}^{n}(f_{\tilde b_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$$

Since

$$\mathrm{IFC}_\mu(\tilde a_1,\cdots,\tilde a_n)\oplus\mathrm{IFC}_\mu(\tilde b_1,\cdots,\tilde b_n)$$
$$=\left(1-\prod_{i=1}^{n}(1-t_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\prod_{i=1}^{n}(f_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)$$
$$\oplus\left(1-\prod_{i=1}^{n}(1-t_{\tilde b_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\prod_{i=1}^{n}(f_{\tilde b_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)$$
$$=\left(1-\prod_{i=1}^{n}(1-t_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\prod_{i-1}^{n}(1-t_{\tilde b_{(i)}})\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$
$$\prod_{i=1}^{n}(f_{\tilde a_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\prod_{i=1}^{n}(f_{\tilde b_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}.$$

Thus,

$$\mathrm{IFC}_\mu(\tilde a_1\oplus\tilde b_1,\ldots,\tilde a_n\oplus\tilde b_n)=\mathrm{IFC}_\mu(\tilde a_1,\ldots,\tilde a_n)\oplus\mathrm{IFC}_\mu(\tilde b_1,\ldots,\tilde b_n).\quad\square$$

According to Definition 3 and 7, it is easy to obtain the following proposition.

Proposition 11. *Let $\tilde a_i=(t_{\tilde a_i},f_{\tilde a_i})(i=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, if μ is an additive measure on X such that $\mu(A)=\sum_{i\in A}w_i(A\subseteq X)$, then*

$$\mathrm{IFC}_\mu(\tilde a_1,\cdots,\tilde a_n)=\mathrm{IFWA}_w(\tilde a_1,\tilde a_2,\cdots,\tilde a_n),$$

where IFWA is the intuitionistic fuzzy weighted averaging operator, and w_i is the weight index of $\tilde a_i(i=1,2,\ldots,n)$, with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$.

Proposition 12. *Let $\tilde a_i=(t_{\tilde a_i},f_{\tilde a_i})(i=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. Any intuitionistic fuzzy OWA operator with the weighted vector $w=(w_1,\ldots,w_n)^T$ is an intuitionistic fuzzy Choquet integral operator, whose fuzzy measure μ is defined by*

$$\mu(S)=\sum_{i=n-s+1}^{n}w_i(S\subseteq X,S\neq\emptyset).$$

Reciprocally, any commutative intuitionistic fuzzy Choquet integral operator is such that $\mu(S)$ depends only on s, where s denotes the cardinality of $S(S\subseteq X)$, and coincides with an intuitionistic fuzzy OWA operator, whose weights are $w_{n-s}=\mu(S\quad\cup\quad i)-\mu(S),i\in X,S\subseteq X\setminus i$.

Proof. According to Definitions 4 and 7, the proof of the proposition is similar to that of Corollary 4 in Grabisch (1995) or that of Proposition 1 in Marichal (2002). Here, we do not duplicate it.

From Propositions 11 and 12, it is easily known that the intuitionistic fuzzy Choquet integral operator generalizes both the intuitionistic fuzzy OWA operator and intuitionistic fuzzy weighted averaging operator. □

4. An approach to multi-criteria decision making with intuitionistic fuzzy Choquet integral operator

Multi-criteria decision making (MCDM) problem is the process of finding the best alternative from all of the feasible alternatives where all the alternatives can be evaluated according to a number of criteria or attribute. In general, multi-criteria decision making problem includes uncertain and imprecise data and information. We first describe the multi-criteria decision making problems under intuitionistic fuzzy environment in this paper:

For a multi-criteria decision making problem, let $A = (a_1, a_2, \ldots, a_m)$ be a set of alternatives, $C = (c_1, c_2, \ldots, c_n)$ be a set of criteria. Assume that with respect to criteria $c_j (i = 1, 2, \ldots, n)$ the alternative $a_i (i = 1, 2, \ldots, m)$ performance is measured by intuitionistic fuzzy values $\tilde{a}_{ij} = (t_{ij}, f_{ij})(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, where t_{ij} indicates the degree that the alternative a_i satisfies the criteria c_j, f_{ij} indicates the degree that the alternative a_i does not satisfy the criteria c_j, and $0 \leqslant t_{ij} \leqslant 1, 0 \leqslant f_{ij} \leqslant 1, t_{ij} + f_{ij} \leqslant 1$. The characteristics of the alternatives $a_i (i = 1, 2, \ldots, m)$ is represented by the intuitionistic fuzzy values:

$$a_i = \{(t_{i1}, f_{i1}), (t_{i2}, f_{i2}), \ldots, (t_{in}, f_{in}\}.$$

To get the best alternative, the multi-criteria decision making procedure is give by the intuitionistic fuzzy Choquet integral operator as follows.

Step 1. With respect to criteria $c_j (i = 1, 2, \ldots, n)$ the partial evaluation of the alternative $a_i (i = 1, 2, \ldots, m)$ is made by an intuitionistic fuzzy values $\tilde{a}_{ij} = (t_{ij}, f_{ij})(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Then we can obtain a decision making matrix as follow:

$$R = \begin{pmatrix} \tilde{a}_{11}, & \tilde{a}_{12}, & \cdots, & \tilde{a}_{1n} \\ \tilde{a}_{21}, & \tilde{a}_{22}, & \cdots, & \tilde{a}_{2n} \\ \cdots & \cdots & \cdots & \\ \tilde{a}_{m1}, & \tilde{a}_{m2}, & \cdots, & \tilde{a}_{mn} \end{pmatrix}.$$

Step 2. By score functions S of Definition 1, we calculate $S(\tilde{a}_{ij})$ of the partial evaluation $\tilde{a}_{ij}$ of the alternative $a_i (i = 1, 2, \ldots, m)$, and utilize $S(\tilde{a}_{ij})$ to rank the partial evaluation $\tilde{a}_{ij}$. If there is no difference between two score functions $S(\tilde{a}_{ij})$ and $S(\tilde{a}_{ik})$, then by accuracy function H of Definition 1, we calculate $H(\tilde{a}_{ij})$ and $H(\tilde{a}_{ik})$ of the partial evaluation $\tilde{a}_{ij}$ and $\tilde{a}_{ik}$, respectively, and rank the partial evaluation $\tilde{a}_{ij}$ and $\tilde{a}_{ik}$ in according with the accuracy degree $H(\tilde{a}_{ij})$ and $H(\tilde{a}_{ik})$. So the partial evaluation $\tilde{a}_{ij}$of the alternative a_i is reordered such that $\tilde{a}_{i(j)} \leqslant \tilde{a}_{i(j+1)}$.

Step 3. Confirm the fuzzy measures of criteria of C and criteria sets of C. There are several methods for the determination of the fuzzy measure. For instance, linear methods (Marichal & Roubens, 1998), quadratic methods (Grabisch, 1996; Grabisch & Nicolas, 1994), heuristic-based methods (Grabisch, 1995) and genetic algorithms (Wang, Wang, & Klir, 1998) and so on are available in the literature.

Step 4. Using the intuitionistic fuzzy Choquet integral operator:

$$\mathrm{IFC}_\mu(\tilde{a}_{i1}, \cdots, \tilde{a}_{in}) = \left(1 - \prod_{j=1}^{n} (1 - t_{\tilde{a}_{i(j)}})^{\mu(A_{(j)}) - \mu(A_{(j+1)})}, \prod_{j=1}^{n} (f_{\tilde{a}_{i(j)}})^{\mu(A_{(j)}) - \mu(A_{(j+1)})}\right)$$

aggregate all $\tilde{a}_{ij} = (t_{\tilde{a}_{ij}}, f_{\tilde{a}_{ij}})(j = 1, 2, \ldots, n)$ in the ith line of the intuitionistic fuzzy decision matrix into a overall values $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i}) = \mathrm{IFC}_\mu(\tilde{a}_{i1}, \cdots, \tilde{a}_{in})(i = 1, 2, \ldots, m)$ of the alternatives a_i, where $A_{(j)} = (j, \ldots, n), A_{(n+1)} = \phi$.

Step 5. According to the overall values $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ of the alternatives $a_i (i = 1, 2, \ldots, m)$, by Definition 1, we calculate the score function $S(\tilde{a}_i)$ or the accuracy degree $H(\tilde{a}_i)$ to rank the alternative $a_i (i = 1, 2, \ldots, m)$, which is similar to that of Step 2, then to select the best one,

Step 6. End.

5. A numerical example

In this section, a multi-criteria decision making problem is concerned with a manufacturing company which wants to select the best global supplier according to the core competencies of suppliers. Now suppose that there are four suppliers (a_1, a_2, a_3, a_4) whose core competencies are evaluated by means of the following four criteria (c_1, c_2, c_3, c_4):

(1) the level of technology innovation (c_1);
(2) the control ability of flow (c_2);
(3) the ability of management (c_3);
(4) the level of service (c_4).

Step 1. The intuitionistic fuzzy values decision matrix of suppliers is made up according to the five evaluating criteria. Now there are ten experts who are invited to evaluate the core competencies of four candidates. As for the c_1 of the candidate a_1, if five experts consider c_1 strong, three experts are against what c_1 is strong, that is, three experts consider c_1 low, two experts don't judge whether c_1 is strong or not, then the intuitionistic fuzzy evaluating value of c_1 of the candidate a_1 can express by intuitionistic fuzzy value (0.5, 0.3). Integrating the preference value results of ten experts to four candidate suppliers according to four criteria together, the intuitionistic fuzzy decision matrix of candidates can be gotten as Table 1.

Step 2. According to Table 1, by Definition 1, the partial evaluation $\tilde{a}_{ij}$of the candidate a_i is reordered such that $\tilde{a}_{i(j)} \leqslant \tilde{a}_{i(j+1)} (i = 1, 2, 3, 4)$ as follows:

$$\tilde{a}_{1(1)} = (0.3, 0.1), \quad \tilde{a}_{1(2)} = (0.5, 0.4),$$
$$\tilde{a}_{1(3)} = (0.5, 0.3), \quad \tilde{a}_{1(4)} = (0.7, 0.2),$$
$$\tilde{a}_{2(1)} = (0.3, 0.4), \quad \tilde{a}_{2(2)} = (0.4, 0.3),$$
$$\tilde{a}_{2(3)} = (0.5, 0.2), \quad \tilde{a}_{2(4)} = (0.9, 0.1),$$
$$\tilde{a}_{3(1)} = (0.5, 0.4), \quad \tilde{a}_{3(2)} = (0.5, 0.3),$$
$$\tilde{a}_{3(3)} = (0.4, 0.1), \quad \tilde{a}_{3(4)} = (0.6, 0.2),$$
$$\tilde{a}_{4(1)} = (0.2, 0.5), \quad \tilde{a}_{4(2)} = (0.4, 0.2),$$
$$\tilde{a}_{4(3)} = (0.6, 0.2), \quad \tilde{a}_{4(4)} = (0.7, 0.1)$$

Step 3. Suppose the fuzzy measures of criteria of C and criteria sets of C as follows:

Table 1
The intuitionistic fuzzy values decision matrix of candidate suppliers.

	c_1	c_2	c_3	c_4
a_1	(0.5, 0.3)	(0.5, 0.4)	(0.7, 0.2)	(0.3, 0.1)
a_2	(0.4, 0.3)	(0.3, 0.4)	(0.9, 0.1)	(0.5, 0.2)
a_3	(0.4, 0.1)	(0.5, 0.3)	(0.5, 0.4)	(0.6, 0.2)
a_4	(0.6, 0.2)	(0.2, 0.5)	(0.4, 0.2)	(0.7, 0.1)

$$\mu(c_1)=0.40,\quad \mu(c_2)=0.25,\quad \mu(c_3)=0.37,\quad \mu(c_4)=0.20,$$
$$\mu(c_1,c_2)=0.76,\quad \mu(c_1,c_3)=0.65,$$
$$\mu(c_1,c_4)=0.50,\quad \mu(c_2,c_3)=0.45,\quad \mu(c_2,c_4)=0.34,$$
$$\mu(c_3,c_4)=0.42,\quad \mu(c_1,c_2,c_3)=0.85,$$
$$\mu(c_1,c_2,c_4)=0.68,\quad \mu(c_1,c_3,c_4)=0.76,$$
$$\mu(c_2,c_3,c_4)=0.57,\quad \mu(c_1,c_2,c_3,c_4)=1.0.$$

Step 4. Utilizing the intuitionistic fuzzy Choquet integral operator

$$\tilde{a}_i = IFC_\mu(\tilde{a}_{i1},\cdots,\tilde{a}_{in}) = \left(1-\prod_{j=1}^{n}(1-t_{\tilde{a}_{i(j)}})^{\mu(A_{(j)})-\mu(A_{(j+1)})},\prod_{j=1}^{n}(f_{\tilde{a}_{i(j)}})^{\mu(A_{(j)})-\mu(A_{(j+1)})}\right)$$

aggregate $\tilde{a}_{ij}(j=1,2,3,4)$ corresponding to the supplier a_i into a overall value $\tilde{a}_i(i=1,2,3,4)$, where $A_{(j)}=((c_j),\ldots,(c_4)),\mu(A_{(4+1)})=0$,

$$\begin{aligned}\tilde{a}_1 &= \mathrm{IFC}_\mu(\tilde{a}_{11},\tilde{a}_{12},\tilde{a}_{13},\tilde{a}_{14}) \\ &= \left(1-\prod_{j=1}^{4}(1-t_{\tilde{a}_{1(j)}})^{\mu(A_{(j)})-\mu(A_{(j+1)})},\prod_{j=1}^{4}(f_{\tilde{a}_{1(j)}})^{\mu(A_{(j)})-\mu(A_{(j+1)})}\right) \\ &= (1-(1-0.3)^{1-0.85}(1-0.5)^{0.85-0.65}(1-0.5)^{0.65-0.37} \\ &\quad \times(1-0.7)^{0.37},(0.1)^{1-0.85}(0.4)^{0.85-0.65}(0.3)^{0.65-0.37}(0.2)^{0.37}) \\ &= (0.56,0.23).\end{aligned}$$

Similarly,

$$\tilde{a}_2=(0.68,0.21),\quad \tilde{a}_3=(0.49,0.22),\quad \tilde{a}_4=(0.51,0.22).$$

Step 5. According to the overall values $\tilde{a}_i$ of the supplier $a_i(i=1,2,3,4)$, by Definition 1, we can obtain that

$$\tilde{a}_2 > \tilde{a}_1 > \tilde{a}_4 > \tilde{a}_3.$$

Thus the order of the four suppliers is a_2, a_1, a_4, a_3. Hence, the best supplier is a_2.

6. Conclusion

Being a generalization of fuzzy sets, the intuitionistic fuzzy sets give us an additional possibility to represent imperfect knowledge. This allows us to use more flexible ways to simulate real decision situations. In this paper, based on two operational laws of intuitionistic fuzzy values, we have developed an intuitionistic fuzzy Choquet integral operator for multiple criteria decision making, where interactions phenomena among the decision making criteria are considered. It is shown that the intuitionistic fuzzy Choquet integral operator generalizes both the intuitionistic fuzzy OWA operator and intuitionistic fuzzy weighted averaging operator. Based on the intuitionistic fuzzy Choquet integral operator, we have presented a new method for handling multi-criteria decision making problems under intuitionistic fuzzy environment, where the characteristics of the alternatives are represented by intuitionistic fuzzy values. And an algorithm and procedure for multi-criteria decision making is proposed. Finally, an example is given to illustrate the multi-criteria decision making process. The proposed method differs from previous approaches for multi-criteria decision making not only due to the fact that the proposed method use intuitionistic fuzzy set theory rather than fuzzy set theory, but also due to the consideration the interactions phenomena among the decision making criteria, which makes it have more feasible and practical than other traditional aggregation operators for real decision making problems.

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