Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 225 (2010) 2828–2839
ADVANCES IN
Mathematics
www.elsevier.com/locate/aim

Algebraic methods in discrete analogs of the Kakeya problem

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Received 9 December 2008; accepted 12 May 2010

Available online 11 June 2010

Communicated by Charles Fefferman

Abstract

We prove the joints conjecture, showing that for any N lines in $\mathbb{R}^3$, there are at most $O(N^{\frac{3}{2}})$ points at which 3 lines intersect non-coplanarly. We also prove a conjecture of Bourgain showing that given N^2 lines in $\mathbb{R}^3$ so that no N lines lie in the same plane and so that each line intersects a set P of points in at least N points then the cardinality of the set of points is $\Omega(N^3)$. Both our proofs are adaptations of Dvir's argument for the finite field Kakeya problem.

Keywords: Kakeya problem; Joints; Incidence problem; Bezout's theorem; Dvir argument

1. Introduction

Various authors have considered the joints problem. It asks, given N lines in space, how many "joints" can the lines form, where a joint is defined as a point where three lines with linearly independent directions intersect. Obviously given a $\sqrt{\frac{N}{3}} \times \sqrt{\frac{N}{3}} \times \sqrt{\frac{N}{3}}$ cube in the integer lattice, we get N lines with $\frac{N^{\frac{3}{2}}}{3\sqrt{3}}$ joints by simply taking all lines in coordinate directions which intersect the cube and the lattice. The joints problem is to prove:

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0001-8708/$ – see front matter © 2010 Published by Elsevier Inc.
doi:10.1016/j.aim.2010.05.015

Theorem 1.1. *Any set of N lines in $\mathbb{R}^3$ form at most $O(N^{\frac{3}{2}})$ joints.*

The previous best bound in the joints problem is due to Feldman and Sharir [6], who proved that the number of joints is $O(N^{1.6232})$. Earlier bounds were reported in [11,10,2]. Bennett, Carbery, and Tao obtained a result conditioned on the angles at the joints in [1].

At AIM in 2004, Bourgain conjectured the following:

Theorem 1.2. *Let L be a set of N^2 lines in $\mathbb{R}^3$ and let P be a set of points in $\mathbb{R}^3$. Suppose no more than N lines of L lie in the same plane and suppose the each line of L contains at least N points of P. Then $|P| = \Omega(N^3)$.*

The previous best bound in Bourgain's incidence problem is due to Solymosi and Tóth [12], who proved that the number of points is $\Omega(N^{11/4})$.

What both conjectures have in common is that they are discrete models of the Kakeya problem. Work of Sharir on joints helped inspire Schlag's program on Kakeya, see [9]. Bourgain's conjecture was posed with the analogy to Kakeya in mind.

In 2008, Dvir [5] solved the finite field version of the Kakeya problem. His technique was to study the properties of a polynomial which vanishes on the Kakeya set. We adapt this technique, proving the above theorems through a study of a polynomial which vanishes on the point sets in question.

The main idea of both proofs is as follows. We find a polynomial of as low degree as possible which vanishes on the set of joints (*resp*. points) in question. We factorize the polynomial to irreducibles and find an irreducible factor vanishing on many joints (points). That irreducible factor will also vanish on many lines. In the event that the gradient, too, vanishes on many lines, a variant of Bezout's theorem leads to a contradiction by reducing our irreducible. Otherwise at each point where such lines intersect, all the lines lie in the tangent plane. In the joints problem this leads to an immediate contradiction, since intersections must be non-coplanar. In the Bourgain problem, this leads to many flat points of the zero set of the irreducible, which force that zero set to be a plane. This contradicts the hypothesis of fewer than N lines in a plane. The idea of planiness, that in Kakeya problems, lines at a given point of intersection lie mainly in a plane, seems first to have arisen in the work of the second author with Laba and Tao [8]. The idea that this plane is the tangent space to a polynomial vanishing on the set comes from the work of the first author on the endpoint multilinear Kakeya problem [7].

We have tried to minimize the amount of algebra background needed for the paper. The small amount of algebra we use is summarized in the next section with references. The main ingredient is Bezout's theorem. In order to minimize the algebra, we focus on irreducible polynomials, and we use pigeonhole estimates to locate an irreducible polynomial that vanishes on many joints or points. It is also possible to give a proof using reducible polynomials. Such a proof would need less pigeonhole estimates, but it would require more algebra.

2. Algebraic preliminaries

In this section, we bring together various algebraic facts that we shall need. Good references are the books of Cox, Little, and O'Shea [3,4].

We recall a fundamental object, the *resultant* of two polynomials. Given f and g elements of $\mathbb{C}[x]$ having degree l and m respectively, and given as

$$f(x) = a_l x^l + a_{l-1} x^{l-1} + \cdots + a_0$$

and

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0,$$

we define the resultant of f and g, namely $Res(f, g)$ as the determinant of the $l + m \times l + m$ matrix whose coefficients c_{ij} satisfy $c_{ij} = a_{j-i}$ if $1 \leqslant i \leqslant m$ and $i \leqslant j \leqslant i + l$, satisfy $c_{ij} = b_{j-i+m}$ if $m + 1 \leqslant i \leqslant m + l$ and $i - m \leqslant j \leqslant i - m + l$ and are equal to zero otherwise.

The columns of the matrix c_{ij} represent coefficients of the polynomial f multiplied by x^j where j runs from 0 to $m - 1$ and the coefficients of the polynomial g multiplied by x^k where k runs from 0 to $l - 1$. The resultant tests whether this set of polynomials is linearly independent. Linear independence fails exactly when f and g have a common factor. (The resultant was first defined in this way by Sylvester.) We bring together some basic properties of the resultant following [4].

Now suppose instead that we work with polynomials $f, g \in \mathbb{C}[x_1, \ldots, x_n]$. We may view them as polynomials in x_1 with coefficients that are polynomials in $x_2, \ldots, x_n$. Then we denote the resultant, a polynomial in $x_2, \ldots, x_n$ as $Res(f, g; x_1)$. In fact, computing resultants is all we need to do in order to determine whether polynomials in several variables have a common factor.

Theorem 2.1. *Let $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ and suppose that both f and g have positive degree when viewed as polynomials in x_1 then f and g have a common factor if and only if $Res(f, g; x_1)$ is identically zero.*

Theorem 2.1 is §3.6 Proposition 1(ii) in [4].

Proposition 2.2. *Let f and g be elements of $\mathbb{C}[x_1, x_2]$ and suppose that f and g have degrees l and m respectively. Furthermore, assume that f has degree l in x_1 and g has degree m in x_1. Then $Res(f, g; x_1)$ is a polynomial of x_2 of degree at most lm.*

Proof. Given two polynomials of one variable,

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_l)$$

and

$$g(x) = (x - s_1)(x - s_2) \cdots (x - s_m),$$

we have that

$$Res(f, g) = \prod_{j=1}^{l} \prod_{k=1}^{m} r_j - s_k.$$

The coefficient a_i of f is a symmetric polynomial in the roots of f which is homogeneous of degree $l - i$. Similarly, b_i is a homogeneous polynomial of degree $m - i$ in the roots of g. We therefore assign the variable a_i a degree $l-i$ and the variable b_i the degree $m-i$. With respect to these degrees, the resultant $Res(f,g)$ is a homogeneous polynomial of degree lm. On the other hand, the coefficient a_i is a polynomial in x_2 of degree at most $l-i$ and the coefficient b_i is a polynomial in x_2 of degree at most $m-i$. Therefore, $Res(f,g;x_1)$ is a polynomial of x_2 of degree at most lm. $\square$

Proposition 2.2 is the main point in the proof of the celebrated Bezout's theorem.

Corollary 2.3 *(Bezout's theorem). Let f and g be elements of $\mathbb{C}[x_1,x_2]$ and suppose that f and g have positive degrees l and m respectively. Suppose there are more than lm points of $\mathbb{C}^2$ where f and g both vanish. Then f and g have a common factor.*

Proof. We begin by changing coordinates. We may choose x_1 and x_2 so that f has degree l in x_1 and g has degree m in x_1. We may also guarantee that there are more than lm distinct values of x_2 among the points where f and g both vanish. A generic choice of coordinates accomplishes these goals. By Proposition 2.2, we see that $Res(f,g;x_1)$ is a polynomial of x_2 with degree at most lm. Since it vanishes at more than lm points, it must vanish identically. An application of Theorem 2.1 completes the proof. $\square$

We shall need a small generalization of Corollary 2.3 that works in $\mathbb{C}^3$ when two polynomials vanish simultaneously on many lines.

Corollary 2.4. *Let f and g be elements of $\mathbb{C}[x_1,x_2,x_3]$ and suppose that f and g have positive degrees l and m respectively. Suppose there are more than lm lines on which f and g simultaneously vanish identically. Then f and g have a common factor.*

Proof. Without loss of generality, we may choose x_1 so that f and g have positive degree in x_1 and x_3 so that the $x_3=0$ plane is transverse to at least $lm+1$ of the lines of vanishing. Then fixing x_3 and apply Bezout's theorem and Theorem 2.1, we get that $Res(f,g;x_1)$ vanishes identically as a function of x_2. Since this happens for all values of x_3, we have that $Res(f,g;x_1)$ vanishes identically and therefore applying Theorem 2.1, we get the desired result. $\square$

Finally we prove the real analog of Corollary 2.4. The result below is the one we apply in the proof of our theorems.

Corollary 2.5. *Let f and g be elements of $\mathbb{R}[x_1,x_2,x_3]$, and suppose that f and g have positive degrees l and m respectively. Suppose that there are more than lm lines on which f and g simultaneously vanish identically. Then f and g have a common factor.*

Proof. We can think of f and g as elements of $\mathbb{C}[x_1,x_2,x_3]$, and they must vanish on more than lm complex lines in $\mathbb{C}^3$. By Corollary 2.4, f and g must have a common factor h in $\mathbb{C}[x_1,x_2,x_3]$. We can assume h is irreducible. A priori, the polynomial h may or may not be real. But, if h is non-real, then the irreducible factorization of f must contain both h and $\bar{h}$. Hence f is divisible by the real polynomial $h\bar{h}$. Similarly, g is divisible by $h\bar{h}$. $\square$

We take this moment to state an additional algebraic proposition which we will use in what follows.

Proposition 2.6. *Let X be a set of N points in $\mathbb{R}^3$. Then there is a nontrivial polynomial in $\mathbb{R}[x_1, x_2, x_3]$ (*a fortiori* in $\mathbb{C}[x_1, x_2, x_3]$) which vanishes on every point of X of degree less than $CN^{\frac{1}{3}}$, with a universal constant C.*

Proof. A polynomial of three variables and degree d has $\frac{(d+3)(d+2)(d+1)}{6}$ coefficients. Requiring that a polynomial vanish at a point gives a homogeneous linear equation for the coefficients. Underdetermined systems of homogeneous linear equations always have nontrivial solutions. $\square$

3. Geometric preliminaries

In this section, we will recall some basic facts of the extrinsic geometry of irreducible algebraic varieties in $\mathbb{R}^3$.

We let p be a nontrivial irreducible polynomial on $\mathbb{R}^3$ of degree $d > 0$. We consider

$$S = \{(x, y, z):\ p(x, y, z) = 0\}.$$

We say that a point $a \in S$ is *critical* if $\nabla p(a) = 0$. Otherwise, we say that a is *regular*. (By the implicit function theorem, S is locally a manifold in a neighborhood of a regular point.) We say a line l is *critical* if $l \subset S$ and every point of l is critical.

Proposition 3.1. *The set S contains no more than $d(d-1)$ critical lines.*

Proof. Suppose not. We apply Corollary 2.5 to p and a nontrivial component of ∇p. This contradicts the irreducibility of p. $\square$

Next, we begin to investigate regular points of S. We immediately get the following.

Proposition 3.2. *Let a be a regular point of S. Let l be a line contained in S which passes through a. Then $l \subset T_a S$, where $T_a S$ is the tangent plane to S at a.*

Let a be a regular point of S. We would like to investigate the extrinsic curvature of S at a. That is, we want to understand infinitesimally how the direction of ∇p is changing in a neighborhood of a in S. We define

$$\mathbf{II}(p)(a) = \{\nabla_{\nabla p \times e_j} \nabla p \times \nabla p\}_{j=1,2,3},$$

where $\times$ is the cross product of vectors and e_1, e_2, e_3 are the standard basis vectors in $\mathbb{R}^3$. Clearly $\mathbf{II}(p)$ is a set of three vectors. Thus it has nine components. Each of the components is a polynomial of degree no more than $3d - 4$.

We will refer to $\mathbf{II}(p)$ as the algebraic second fundamental form of S. (The geometric second fundamental form is a quadratic form on $T_p S$ obtained by differentiating the unit normal vector to S along S.) However since the algebraic fundamental form measures the normal component of the change of ∇p along three directions which span $T_p S$, it is easy to see that for any regular point a, all the components of $\mathbf{II}(p)(a)$ vanish if and only if the second fundamental form of S vanishes.

We say a regular point a of S is *flat* if all the components of $\mathbf{II}(p)(a)$ vanish. We give a sufficient condition for a regular point a to be flat.

Lemma 3.3. *Let a be a regular point of S. Suppose that S contains three distinct lines all of which intersect at a, then a is a flat point.*

Proof. The quadratic surface which most closely approximates S at a contains the three lines. (This is because the Taylor series of p along the lines vanishes.) However so does $T_a S$. Thus since $3 > 2$ by Corollary 2.5 the quadratic surface most closely approximating S must contain $T_a S$. Therefore, the second fundamental form of S vanishes at a. □

We say that a line l is flat if it is contained in S, it is not critical, and every regular point of l is flat.

Corollary 3.4. *Let p be an irreducible polynomial of degree $d > 0$. Let S be the zero set of p. Suppose that S contains more than $3d^2 - 4d$ flat lines, then S is a plane.*

Proof. By Corollary 2.5, each component of $\mathbf{II}(p)$ has p as a factor. Therefore the direction of the normal to S at regular points of S does not vary. Therefore S contains a plane. But p is irreducible so S is a plane. □

4. Analytic preliminaries

In the following two sections we will prove "big oh" results by contradiction. Thus we will make an assumption involving a large constant K. We keep track of anything that depends on K. However, we ignore constants which are independent of K. Thus we write

$$A \lesssim B$$

if A and B are quantities and C is a universal constant.

There are two variants of the pigeonhole principle which we will use frequently. The first is often referred to as the popularity lemma.

Lemma 4.1. *Let (X, Y, E) be a bipartite graph with edges E and two sets of vertices X and Y. Suppose that $|E| > \rho|Y|$. Let Y' be the set of vertices of Y having degree at least μ and let E' be the set of edges in E between Y' and X. Then*

$$\left|E'\right| > (\rho - \mu)|Y|.$$

Proof. The vertices in $Y \backslash Y'$ are incident to at most $\mu|Y|$ edges. □

We now give the second which we'll refer to freely as the pigeonhole principle.

Lemma 4.2. *Let $x_1, \ldots, x_m$ be positive quantities and $y_1, \ldots, y_m$ positive quantities, then there is an integer $1 \leqslant k \leqslant m$ so that*

$$\frac{x_k}{y_k} \geqslant \frac{\sum_{j=1}^m x_j}{\sum_{j=1}^m y_j}.$$

Proof. Suppose not. Let $\rho = \frac{\sum_{j=1}^m x_j}{\sum_{j=1}^m y_j}$. Then $x_k < \rho y_k$ for all k. Thus

$$\sum_{j=1}^m x_j < \rho \sum_{j=1}^m y_j,$$

which is a contradiction. □

5. Joints problem

In this section, we prove Theorem 1.1.

We suppose that we are given a set of lines L of cardinality N. Let J be the set of joints determined by L. We suppose

$$|J| \geqslant K N^{\frac{3}{2}},$$

with a large, but universal, constant K.

We create a bipartite, three-colored graph (L, J, R, G, B) between the set of lines and the set of joints. Each joint is incident to at least three non-coplanar lines. For each joint, we arbitrarily color one of the incidences red, one green, and one blue. The sets R, G, and B consist of, respectively, the red, green, and blue incidences.

We will now refine the sets slightly. We let L_R be the set of all lines which have at least $\frac{K}{1000} N^{\frac{1}{2}}$ incidences in R. We let J_R be the set of joints having a red incidence with a line of L_R. By Lemma 4.1,

$$|J_R| \geqslant \frac{999}{1000} |J|.$$

Now we let L_G and L_B be those lines having respectively at least $\frac{K}{1000} N^{\frac{1}{2}}$ green or blue incidences with joints in J_R. We let J' denote that set of joints which has red, green, and blue incidences with lines in L_G and L_B and by Lemma 4.1, it is easy to show that

$$\left|J'\right| \geqslant \frac{99}{100} |J|.$$

Our goal now is to produce a polynomial of relatively low degree vanishing on all the points of J'. (Any degree which is substantially lower than $N^{\frac{1}{2}}$ will suffice.) We say a line l of L_G or L_B meets a line l' of L_R if $l \cap l'$ is a joint of J_R. Each line of L_G and each line of L_B meets at least $\frac{K}{1000} N^{\frac{1}{2}}$ lines of L_R. We now take a random subset L'_R of the lines of L_R, picking each line with probability $\frac{1}{K}$.

By the law of large numbers, with positive probability, the following events occur: Each line of L_G and L_B meets at least $\frac{1}{2000} N^{\frac{1}{2}}$ lines of L'_R and

$$\left|L'_R\right| \leqslant \frac{2N}{K}.$$

We make a set of points S by selecting $\frac{1}{2}N^{\frac{1}{2}}$ points on each line of L'_R. Then

$$|S| \leqslant \frac{N^{\frac{3}{2}}}{K}.$$

We find a polynomial p which vanishes on all points of S. By the estimate on the size of S, we may select p with degree d which is $\lesssim \frac{N^{\frac{1}{2}}}{K^{\frac{1}{3}}}$. With sufficiently large K, this means that p must vanish on each line of L'_R and because of the number of lines of L'_R that each line of L_G and L_B meet, it means that p must vanish identically on each line of L_G and L_B. Therefore, the polynomial p must vanish on the entire set J'.

Now, it is not necessarily the case that p is irreducible. Thus we factor p into irreducibles

$$p = \prod_{j=1}^{m} p_j.$$

We denote the degree of the polynomial p_j by d_j and observe that

$$\sum_{j=1}^{m} d_j \lesssim \frac{N^{\frac{1}{2}}}{K^{\frac{1}{3}}}.$$

We let J_j be the subset of J' on which p_j vanishes, and we have

$$\sum_{j=1}^{m} |J_j| \gtrsim K N^{\frac{3}{2}}.$$

Thus by Lemma 4.2, we find j for which

$$|J_j| \gtrsim K^{\frac{4}{3}} N d_j.$$

From now on, we restrict our attention to J_j.

We denote by $L_{R,j}$, $L_{G,j}$, and $L_{B,j}$ those lines in L_R, L_B, and L_G which are incident to at least $d_j + 1$ elements of J_j, and we let J'_j be those element of J_j incident to a line each from $L_{R,j}$, $L_{G,j}$, and $L_{B,j}$. With K sufficiently large, we have

$$|J'_j| \geqslant \frac{999}{1000}|J_j|.$$

We now define $L'_{R,j}$, $L'_{G,j}$, and $L'_{B,j}$ as the set of lines which are incident to more than $d_j + 1$ points of J'_j. We define J''_j to be the set of joints defined by these lines. We have

$$|J''_j| \geqslant \frac{99}{100}|J_j|.$$

We now break into two cases. In the first case, there are fewer than d_j^2 lines in each of $L'_{R,j}$, $L'_{G,j}$, and $L'_{B,j}$. In this case, we start over again, having a joints problem with fewer lines and more favorable exponents than the original.

In the second case, we may assume without loss of generality that $L'_{R,j}$ contains at least d_j^2 lines. By the definition of $L_{R,j}$, $L_{G,j}$, and $L_{B,j}$, the polynomial p_j vanishes identically on each line in these sets. However, this implies that each point of J'_j is a critical point of p_j, because otherwise it would be impossible for p_j to vanish on three, intersecting, non-coplanar lines. But this implies that each component of the gradient of p_j vanishes at each point of J'_j. Let q be one of the components of the gradient which does not vanish identically. Then q has degree at most $d_j - 1$. Thus, it must vanish on every line of $L'_{R,j}$. But this is a contradiction by Proposition 3.1.

6. Bourgain's incidence problem

In this section, we prove Theorem 1.2. We suppose we have a set of points $X \subset \mathbb{R}^3$ of cardinality $\frac{N^3}{K}$, with K large to be specified later and a set L of N^2 lines so that no N lines lie in the same plane and so that each line $l \in L$ is incident to at least N points of X. We may assume in what follows that each line is incident to exactly N points by coloring N of its incidences black and only counting black incidences below.

We say that a point $x \in X$ is *valuable* if it is incident to at least $\frac{K}{1000}$ lines. We define $v(x)$ the value of x to be the number of lines it is incident to. We let X_v be the set of valuable points. Clearly

$$|X_v| \leqslant \frac{1000N^3}{K},$$

and by Lemma 4.1

$$\sum_{x \in X_v} v(x) \geqslant \frac{999N^3}{1000}.$$

We now perform some dyadic pigeonholing to uniformize the quantity $v(x)$. We define X_j to be the set of those $x \in X_v$ so that

$$\frac{2^{j-1}K}{1000} \leqslant v(x) < \frac{2^j K}{1000}.$$

We define

$$V_j = \sum_{x \in X_j} v(x).$$

Then note that

$$\sum_{j=1}^{\infty} V_j = \sum_{x \in X_v} v(x) \geqslant \frac{999N^3}{1000}.$$

Note also that

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} < 2.$$

Hence, by the pigeonhole principle, we can fix a positive number j so that

$$V_j \geqslant \frac{999N^3}{2000j^2}.$$

(In the argument below, the reader is encouraged to assume that $j = 1$, since this is indeed the worst case.)

From this we see that

$$\frac{N^3}{K2^j} \gtrsim |X_j| \gtrsim \frac{N^3}{K2^j j^2}.$$

Next we find a polynomial p which vanishes on every point of X_j. We may choose p to have degree d satisfying

$$d \lesssim \frac{N}{K^{\frac{1}{3}} 2^{\frac{j}{3}}}.$$

The polynomial p needs not be irreducible. Thus we factor it

$$p = p_1 p_2 \dots p_m,$$

with p_k having degree d_k. We let $X_{j,k}$ be the set of points of X_j on which p_k vanishes. Clearly, we have

$$d_1 + d_2 + \cdots + d_m \lesssim \frac{N}{K^{\frac{1}{3}} 2^{\frac{j}{3}}},$$

while

$$|X_{j,1}| + |X_{j,2}| + \cdots + |X_{j,m}| \gtrsim |X_j| \gtrsim \frac{N^3}{K2^j j^2}.$$

Thus by the pigeonhole principle, we can fix a k with

$$|X_{j,k}| \gtrsim \frac{N^2 d_k}{K^{\frac{2}{3}} 2^{\frac{2j}{3}} j^2}.$$

We let $Y = X_{j,k}$ and by the definition of X_j, if I is the number of incidences between L and Y, we have

$$I \gtrsim N^2 d_k K^{\frac{1}{3}} 2^{\frac{j}{3}} j^{-2} \gg N^2 d_k.$$

We let L' be the set of lines incident to more than $100d_k$ points of Y and let I' be the number of incidences between L' and Y. Then clearly

$$I' \gtrsim I.$$

Note that each line of L' is in the zero set of p_k. Now let Y' be the set of points of Y incident to more than 3 lines of L'. Then each point of Y' is either a critical point of $p_k = 0$ or else by Lemma 3.3, it must be a flat point of p_k.

We let L'' be the set of lines in L' incident to at least $10d_k$ points of Y'. If I'' is the number of incidences between lines of L'' and points of Y', we still have

$$I'' \gtrsim N^2 d_k K^{\frac{1}{3}} 2^{\frac{j}{3}} j^{-2}.$$

We let I_{flat} and I_{crit} be the number of those incidences with flat points and critical points respectively. Note that

$$I_{crit} + I_{flat} \geqslant I''.$$

There are two cases. In the first case

$$I_{crit} \gtrsim N^2 d_k K^{\frac{1}{3}} 2^{\frac{j}{3}} j^{-2},$$

which means that there are at least $N d_k K^{\frac{1}{3}} 2^{\frac{j}{3}} j^{-2} \gg d_k^2$ lines in the surface $p_k = 0$ on which are critical. But this is a contradiction in light of Proposition 3.1.

In the second case

$$I_{flat} \gtrsim N^2 d_k K^{\frac{1}{3}} 2^{\frac{j}{3}} j^{-2},$$

which means that there are at least $N d_k K^{\frac{1}{3}} 2^{\frac{j}{3}} j^{-2} \gg 3d_k^2$ flat lines in the surface $p_k = 0$. In light of Corollary 3.4 the surface $p_k = 0$ is in fact a plane. But now we have more than N lines of L lying in a plane which is also a contradiction.

Acknowledgments

The second author would like to thank the University of Toronto and the Fields Institute for their hospitality which allowed this collaboration to take place. He would also like to thank his colleague Michael Larsen for a useful conversation about resultants. He is partially supported by NSF grant DMS 0653763.

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