

Weakly group-theoretical and solvable fusion categories

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To Izrail Moiseevich Gelfand on his 95th birthday with admiration

Abstract

We introduce two new classes of fusion categories which are obtained by a certain procedure from finite groups – weakly group-theoretical categories and solvable categories. These are fusion categories that are Morita equivalent to iterated extensions (in the world of fusion categories) of arbitrary, respectively solvable finite groups. Weakly group-theoretical categories have integer dimension, and all known fusion categories of integer dimension are weakly group-theoretical. Our main results are that a weakly group-theoretical category \mathcal{C} has the strong Frobenius property (i.e., the dimension of any simple object in an indecomposable \mathcal{C} -module category divides the dimension of \mathcal{C}), and that any fusion category whose dimension has at most two prime divisors is solvable (a categorical analog of Burnside’s theorem for finite groups). This has powerful applications to classification of fusion categories and semisimple Hopf algebras of a given dimension. In particular, we show that any fusion category of integer dimension < 84 is weakly group-theoretical (i.e. comes from finite group theory), and give a full classification of semisimple Hopf algebras of dimensions pqr and pq^2 , where p, q, r are distinct primes.

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1. Introduction and main results

The goal of this paper is to introduce and study two classes of fusion categories (over \mathbb{C}): weakly group-theoretical categories and solvable categories.

Namely, recall [20] that a fusion category \mathcal{C} is said to be *nilpotent* if there is a sequence of fusion categories $\mathcal{C}_0 = \text{Vec}$, $\mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C}$ and a sequence G_1, \dots, G_n of finite groups such that \mathcal{C}_i is obtained from \mathcal{C}_{i-1} by a G_i -extension (i.e., \mathcal{C}_i is faithfully graded by G_i with trivial component \mathcal{C}_{i-1}). Let us say that \mathcal{C} is *cyclically nilpotent* if the groups G_i can be chosen to be cyclic (or, equivalently, cyclic of prime order).

Definition 1.1. A fusion category \mathcal{C} is *weakly group-theoretical* if it is Morita equivalent to a nilpotent fusion category.¹

Here the notion of *Morita equivalence* means the same as *weak monoidal Morita equivalence* introduced by M. Müger in [27]. This is a categorical analogue of the familiar notion of Morita equivalence for rings.

Definition 1.2. A fusion category \mathcal{C} is *solvable*² if any of the following two equivalent conditions is satisfied³:

- (i) \mathcal{C} is Morita equivalent to a cyclically nilpotent fusion category;
- (ii) there is a sequence of fusion categories $\mathcal{C}_0 = \text{Vec}$, $\mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C}$ and a sequence G_1, \dots, G_n of cyclic groups of prime order such that \mathcal{C}_i is obtained from \mathcal{C}_{i-1} either by a G_i -equivariantization or as a G_i -extension.

Thus, the class of weakly group-theoretical categories contains the classes of solvable and group-theoretical categories (i.e. those Morita equivalent to pointed categories, see [14] and Section 2.5 below). In fact, it contains all fusion categories we know which are weakly integral, i.e., have integer Frobenius–Perron dimension.

Our first main result is the following characterization of fusion categories Morita equivalent to group extensions of a given fusion category.

Theorem 1.3. *Let \mathcal{D} be a fusion category and let G be a finite group. A fusion category \mathcal{C} is Morita equivalent to a G -extension of \mathcal{D} if and only if its Drinfeld center $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\mathcal{E} = \text{Rep}(G)$ such that the de-equivariantization of \mathcal{E}' by \mathcal{E} is equivalent to $\mathcal{Z}(\mathcal{D})$ as a braided tensor category.*

The precise definition of de-equivariantization can be found in Section 2.6 below and in [9, Section 4]. In the context of modular categories it is the *modularization* construction introduced by A. Bruguières [3] and M. Müger [28].

¹ It will be shown in a subsequent paper that a weakly group-theoretical category is in fact Morita equivalent to a nilpotent category of nilpotency class $n = 2$, i.e., to a group extension of a pointed category. This should allow one to describe weakly group-theoretical categories fairly explicitly in group-theoretical terms.

² This definition is motivated by the fact the category $\text{Rep}(G)$ of representations of a finite group G is solvable if and only if G is a solvable group.

³ The equivalence of these two conditions is proved in Proposition 4.4.

In the special case when G is the trivial group, Theorem 1.3 simply says that two fusion categories are Morita equivalent if and only if their centers are equivalent as braided categories (Theorem 3.1). This important result, which answers a question of V. Drinfeld, has been announced by A. Kitaev and M. Müger, and is used in the proof of the more general Theorem 1.3. Since, as far as we know, a proof of this result is unavailable in the literature, we give such a proof in the beginning of Section 3. That Morita equivalent fusion categories have braided equivalent centers was shown by Müger in [27]; we prove the opposite implication.

Our second main result may be viewed as a strong form of Kaplansky's 6-th conjecture (stating that the dimension of an irreducible representation of a semisimple Hopf algebra divides the dimension of the Hopf algebra) for weakly group-theoretical fusion categories. To state it, we need the following definition.

Definition 1.4. We will say that a fusion category \mathcal{C} has the *strong Frobenius* property if for every indecomposable \mathcal{C} -module category \mathcal{M} and any simple object X in \mathcal{M} the number $\frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(X)}$ is an algebraic integer, where the Frobenius–Perron dimensions in \mathcal{M} are normalized in such a way that $\text{FPdim}(\mathcal{M}) = \text{FPdim}(\mathcal{C})$.

Obviously, the strong Frobenius property of a fusion category implies the usual Frobenius property, i.e. that the Frobenius–Perron dimension of any simple object divides the Frobenius–Perron dimension of the category (indeed, it suffices to take $\mathcal{M} = \mathcal{C}$).

Theorem 1.5. *Any weakly group-theoretical fusion category has the strong Frobenius property.*

Finally, our third main result is an analog of Burnside's theorem for fusion categories.

Theorem 1.6. *Any fusion category of Frobenius–Perron dimension $p^r q^s$, where p and q are primes, and r, s are nonnegative integers, is solvable.*

Theorems 1.5 and 1.6 and the intermediate results used in their proofs provide powerful methods for studying weakly group-theoretical and solvable fusion categories, in particular those of dimension $p^r q^s$, and of (quasi-)Hopf algebras associated to them. As an illustration, we show that any non-pointed simple weakly group-theoretical fusion category is equivalent to $\text{Rep}(G)$ for a finite non-abelian simple group G , and that any fusion category of dimension 60 is weakly group-theoretical (in particular, if it is simple, it is equivalent to $\text{Rep}(A_5)$). We also show that any fusion category of dimension pqr (where p, q, r are distinct primes) and any semisimple Hopf algebra of dimension pqr or pq^2 is group-theoretical, and classify such Hopf algebras. However, most of such applications will be discussed in future publications. In particular, the classification of fusion categories of dimension pq^2 , where q and p are primes, is given in the paper [22] (as pointed out in [14], not all such categories are group-theoretical, already for $p = 2$).

The structure of the paper is as follows. In Section 2, we discuss preliminaries. In Section 3 we prove Theorem 1.3. In Section 4, we state and prove the basic properties of weakly group-theoretical and solvable fusion categories. In Section 5, we describe module categories over equivariantizations; this description needed for the proof of Theorem 1.5. In Section 6, we prove Theorem 1.5. In Section 7, we prove some important properties of non-degenerate and slightly degenerate categories containing a simple object of prime power dimension, which are needed for the proof of Theorem 1.6. In particular, Corollary 7.2 is similar to the classical Burnside Lemma in the theory of group representations. In Section 8, we prove Theorem 1.6. In Section 9,

we discuss applications of our results to concrete problems in the theory of fusion categories and Hopf algebras. In Section 10 we briefly discuss the relation between our results and classification results on semisimple Hopf algebras and fusion categories available in the literature. Finally, in Section 11 we formulate some open questions.

2. Preliminaries

In this paper, we will freely use the basic theory of fusion categories, module categories over them, Frobenius–Perron dimensions, and modular categories. For basics on these topics, we refer the reader to [2,41,14,9].⁴ However, for reader’s convenience, we recall some of the most important definitions and facts that are used below.

2.1. Graded categories [14,20]

Let \mathcal{C} be a fusion category and let G be a finite group. We say that \mathcal{C} is *graded* by G if $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, and for any $g, h \in G$, one has $\otimes : \mathcal{C}_g \times \mathcal{C}_h \rightarrow \mathcal{C}_{gh}$, $*$: $\mathcal{C}_g \rightarrow \mathcal{C}_{g^{-1}}$. The fusion category \mathcal{C}_e corresponding to the neutral element $e \in G$ is called the *trivial component* of the G -graded category \mathcal{C} . A grading is *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$. If \mathcal{C} is faithfully graded by G , one says that \mathcal{C} is a G -*extension* of \mathcal{C}_e . The *adjoint category* \mathcal{C}_{ad} is the smallest fusion subcategory of \mathcal{C} containing all objects $X \otimes X^*$, where $X \in \mathcal{C}$ is simple.

There exists a unique faithful grading of \mathcal{C} for which $\mathcal{C}_e = \mathcal{C}_{\text{ad}}$. It is called the *universal grading* of \mathcal{C} . The corresponding group is called the *universal grading group* of \mathcal{C} , and denoted by $U_{\mathcal{C}}$. All faithful gradings of \mathcal{C} are induced by the universal grading, in the sense that for any faithful grading $U_{\mathcal{C}}$ canonically projects onto the grading group G , and \mathcal{C}_e contains \mathcal{C}_{ad} .

A fusion category \mathcal{C} is said to be *nilpotent* if it can be reduced to the category of vector spaces by iterating the operation of taking the adjoint category. This is equivalent to the condition that \mathcal{C} can be included into a chain $\text{Vec} = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_n = \mathcal{C}$, where each \mathcal{C}_i is faithfully graded by a finite group G_i , and has trivial component \mathcal{C}_{i-1} .

The simplest example of a nilpotent category is a *pointed* category, i.e. a fusion category where all simple objects are invertible. Such a category is the category of vector spaces graded by some finite group G with associativity defined by a cohomology class $\omega \in H^3(G, \mathbb{C}^*)$, denoted by $\text{Vec}_{G, \omega}$. If $\omega = 1$, we denote this category by Vec_G .

2.2. Frobenius–Perron dimensions in a module category

Let \mathcal{C} be a fusion category, and \mathcal{M} an indecomposable module category over \mathcal{C} . Let M_i , $i \in I$, be the simple objects of \mathcal{M} . Then it follows from the Frobenius–Perron theorem that there exists a unique, up to a common factor, collection of positive numbers d_i , $i \in I$, such that whenever $X \in \mathcal{C}$, and $X \otimes M_i = \bigoplus_{j \in I} N_{ij}(X) M_j$, one has $\text{FPdim}(X) d_i = \sum_{j \in I} N_{ij}(X) d_j$. We will normalize d_i in such a way that $\sum_{i \in I} d_i^2 = \text{FPdim}(\mathcal{C})$. The numbers d_i normalized in such a way are called the Frobenius–Perron dimensions of M_i . By additivity, this defines the Frobenius–Perron dimension of any object of \mathcal{M} .

⁴ All fusion categories and (quasi-)Hopf algebras in this paper will be over \mathbb{C} (which can be replaced by any algebraically closed field of characteristic zero). All module categories will be left module categories. For a fusion category \mathcal{C} we use notation $\mathcal{Z}(\mathcal{C})$ for its Drinfeld center (see e.g. [9, §2.9]).

2.3. Weakly integral and integral categories

A fusion category is said to be *weakly integral*, if its Frobenius–Perron dimension is an integer. Recall [14, Proposition 8.27] that in such a category, the Frobenius–Perron dimension of any simple object is the square root of an integer.

A fusion category is called *integral* if the Frobenius–Perron dimension of every (simple) object is an integer. A weakly integral fusion category is automatically pseudounitary and has a canonical spherical structure with respect to which categorical dimensions coincide with the Frobenius–Perron dimensions [14, Propositions 8.23, 8.24].

2.4. Tannakian categories

Recall [5] that a symmetric fusion category \mathcal{C} is *Tannakian* if it is equivalent to the representation category of a finite group as a symmetric fusion category. More generally, let us say that \mathcal{C} is *super-Tannakian* if there exists a finite group G and a central element $u \in G$ of order 2, such that \mathcal{C} , as a symmetric category, is equivalent to the category of representations of G on super vector spaces, on which u acts by the parity operator.

Theorem 2.1. (See Deligne’s theorem, [6].) Any symmetric fusion category is super-Tannakian.

In particular, if \mathcal{C} has Frobenius–Perron dimension bigger than 2, then it contains a nontrivial Tannakian subcategory (the category of representations of $G/(u)$).

2.5. Morita equivalence [27]; see also [14, 41]

One says that two fusion categories \mathcal{C} and \mathcal{D} are *Morita equivalent* if $\mathcal{D} \cong (\mathcal{C}_{\mathcal{M}}^*)^{op}$ for some indecomposable \mathcal{C} -module category \mathcal{M} (the category of \mathcal{C} -module endofunctors of \mathcal{M} , with opposite composition). Equivalently, there is an algebra A in \mathcal{C} such that \mathcal{D} is equivalent to the category of A -bimodules in \mathcal{C} .

The above is an equivalence relation on fusion categories of a given Frobenius–Perron dimension. A fusion category is said to be *group-theoretical* if it is Morita equivalent to a pointed category. A (quasi-)Hopf algebra is group-theoretical if its representation category is group-theoretical.

2.6. Equivariantization and de-equivariantization [3, 28] and [9, Section 4]

Let \mathcal{C} be a fusion category with an action of a finite group G . In this case one can define the fusion category \mathcal{C}^G of G -equivariant objects in \mathcal{C} . Objects of this category are objects X of \mathcal{C} equipped with an isomorphism $u_g : g(X) \rightarrow X$ for all $g \in G$, such that

$$u_{gh} \circ \gamma_{g,h} = u_g \circ g(u_h),$$

where $\gamma_{g,h} : g(h(X)) \rightarrow gh(X)$ is the natural isomorphism associated to the action. Morphisms and tensor product of equivariant objects are defined in an obvious way. This category is called the *G -equivariantization* of \mathcal{C} . One has $\text{FPdim}(\mathcal{C}^G) = |G| \text{FPdim}(\mathcal{C})$.

For example, $\text{Vec}^G = \text{Rep}(G)$ (for the trivial action of G on Vec). A more interesting example is the following. Let K be a normal subgroup of G . Then we have a natural action of G/K on $\text{Rep}(K)$, and $\text{Rep}(K)^{G/K} = \text{Rep}(G)$.

There is a procedure opposite to equivariantization, called the *de-equivariantization*. In the context of modular categories it is the *modularization* construction introduced by A. Bruguières and M. Müger, see Remark 2.3 below. It is also closely related to the dynamical extensions of monoidal categories of J. Donin and A. Mudrov [7].

Namely, let \mathcal{C} be a fusion category and let $\mathcal{E} = \text{Rep}(G) \subset \mathcal{Z}(\mathcal{C})$ be a Tannakian subcategory which embeds into \mathcal{C} via the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Let A be the algebra in $\mathcal{Z}(\mathcal{C})$ corresponding to the algebra $\text{Fun}(G)$ of functions on G under the above embedding. It is a commutative algebra in $\mathcal{Z}(\mathcal{C})$ and so the category \mathcal{C}_G of left A -modules in \mathcal{C} is a fusion category, called *de-equivariantization* of \mathcal{C} by \mathcal{E} . The free module functor $\mathcal{C} \rightarrow \mathcal{C}_G : X \mapsto A \otimes X$ is a surjective tensor functor. One has $\text{FPdim}(\mathcal{C}_G) = \text{FPdim}(\mathcal{C})/|G|$.

The above constructions are canonically inverse to each other, i.e., there are canonical equivalences $(\mathcal{C}_G)^G \cong \mathcal{C}$ and $(\mathcal{C}^G)_G \cong \mathcal{C}$. See [9, Proposition 4.19].

2.7. The crossed product fusion category

Let \mathcal{C} be a fusion category, and G a finite group acting on \mathcal{C} . Then the *crossed product category* $\mathcal{C} \rtimes G$ is defined as follows [43].

For a pair of Abelian categories $\mathcal{A}_1, \mathcal{A}_2$, let $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ denote their Deligne's tensor product [5].⁵ We set $\mathcal{C} \rtimes G = \mathcal{C} \boxtimes \text{Vec}_G$ as an Abelian category, and define a tensor product by

$$(X \boxtimes g) \otimes (Y \boxtimes h) := (X \otimes g(Y)) \boxtimes gh, \quad X, Y \in \mathcal{C}, \quad g, h \in G. \quad (1)$$

Then $\mathcal{M} = \mathcal{C}$ is naturally a module category over $\mathcal{C} \rtimes G$ and we have the following result.

Proposition 2.2. (See [40, Proposition 3.2].) $\mathcal{C}^G \cong (\mathcal{C} \rtimes G)_{\mathcal{M}}^{*op}$.

In other words, the crossed product category $\mathcal{C} \rtimes G$ is dual to the equivariantization \mathcal{C}^G with respect to the \mathcal{C}^G -module category \mathcal{C} .

2.8. The Müger centralizer [29]

Let \mathcal{C} be a braided fusion category, and $\mathcal{D} \subset \mathcal{C}$ a full subcategory. The *Müger centralizer* \mathcal{D}' of \mathcal{D} in \mathcal{C} is the category of all objects $Y \in \mathcal{C}$ such that for any $X \in \mathcal{D}$ the squared braiding on $X \otimes Y$ is the identity. The *Müger center* of \mathcal{C} is the Müger centralizer \mathcal{C}' of the entire category \mathcal{C} . The category \mathcal{C} is *non-degenerate* (in the sense of Müger) if $\mathcal{C}' = \text{Vec}$. If \mathcal{C} is a non-degenerate braided fusion category then one has $\text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C})$ (see [29, Theorem 3.2] and [9, Theorem 3.14]). If \mathcal{C} is non-degenerate then $\mathcal{D} \subset \mathcal{C}$ is called *Lagrangian* if $\mathcal{D} = \mathcal{D}'$.

Remark 2.3. (See [3, 28].) Let \mathcal{C} be a braided fusion category and let $\mathcal{E} \subset \mathcal{C}'$ be a Tannakian subcategory. Then the de-equivariantization of \mathcal{C} by \mathcal{E} is a braided fusion category. It is non-degenerate if and only if $\mathcal{E} = \mathcal{C}'$.

⁵ For semisimple categories \mathcal{A}_i , the Deligne tensor product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is just the category whose simple objects are $X_1 \otimes X_2$, where $X_i \in \mathcal{A}_i$ are simple.

Note that if \mathcal{C} is a weakly integral non-degenerate category, then by a result of [14], it is pseudounitary, which implies that it is canonically a modular category. This fact will be used throughout the paper.

2.9. Müger's theorem [29, Theorem 4.2], [9, Theorem 3.13]

Theorem 2.4. *Let \mathcal{C} be a braided category, and \mathcal{D} a non-degenerate subcategory in \mathcal{C} . Then \mathcal{C} is naturally equivalent, as a braided category, to $\mathcal{D} \boxtimes \mathcal{D}'$.*

2.10. Slightly degenerate categories

Definition 2.5. A braided fusion category \mathcal{C} is called *slightly degenerate* if its Müger center \mathcal{C}' is equivalent, as a symmetric category, to the category SuperVec of super vector spaces.

Proposition 2.6.

- (i) (cf. [26, Lemma 5.4]) *Let \mathcal{C} be a braided fusion category such that its Müger center \mathcal{C}' contains SuperVec (for example, a slightly degenerate category). Let χ be the invertible object generating $\text{SuperVec} \subseteq \mathcal{C}'$, and let Y be any simple object of \mathcal{C} . Then $\chi \otimes Y$ is not isomorphic to Y .*
- (ii) *Let \mathcal{C} be slightly degenerate and pointed. Then $\mathcal{C} = \text{SuperVec} \boxtimes \mathcal{C}_0$, where \mathcal{C}_0 is a non-degenerate pointed category.*

Proof. (i) Assume the contrary, i.e., $\chi \otimes Y = Y$. Since χ centralizes Y , we have from this identity that the trace T of the Drinfeld isomorphism $u : Y \rightarrow Y^{**}$ is equal to $-T$ (as $u|_{\chi} = -1$). This is a contradiction, as $T \neq 0$.

(ii) This statement is proved in [9, Corollary A.19]. We provide a proof here for the reader's convenience. Our job is to show that $\chi \neq \xi^{\otimes 2}$ for any ξ (this is the condition for the group of invertible objects of \mathcal{C} to be the direct product of the $\mathbb{Z}/2\mathbb{Z}$ generated by χ with another subgroup). Assume the contrary, and let Q be the quadratic form defining the braiding. Then we have $Q(\xi)^4 = Q(\chi) = -1$, $Q(\chi \otimes \xi) = Q(\xi^{\otimes 3}) = Q(\xi)^9 = Q(\xi)$, so the squared braiding of ξ and χ is

$$\beta_{\chi, \xi} = Q(\chi \otimes \xi) / Q(\chi)Q(\xi) = -1,$$

which is a contradiction with the centrality of χ . \square

Note that if \mathcal{C} is a weakly integral braided category, then it is pseudounitary, and hence is canonically a ribbon category [14]. By the S -matrix of a ribbon category \mathcal{C} we understand a square matrix $S := \{s_{X,Y}\}$ whose columns and rows are labeled by simple objects of \mathcal{C} and the entry $s_{X,Y}$ is equal to the quantum trace of $c_{Y,X}c_{X,Y} : X \otimes Y \rightarrow X \otimes Y$, where c denotes the braiding of \mathcal{C} .

Corollary 2.7. *If \mathcal{C} is a weakly integral slightly degenerate braided category, then the S -matrix of \mathcal{C} is $S = I \otimes S'$, where S' is a non-degenerate matrix with orthogonal rows and columns, and I is the 2 by 2 matrix consisting of ones.*

Proof. This is an easy consequence of Proposition 2.6(i) since the rows and columns of S corresponding to $\mathbf{1}$ and χ coincide. \square

Corollary 2.8. *Let \mathcal{C} be a slightly degenerate integral category of dimension > 2 . Then \mathcal{C} contains an odd-dimensional simple object outside of the Müger center \mathcal{C}' of \mathcal{C} .*

Proof. Let χ be the invertible object generating \mathcal{C}' . Let X be any simple object outside of \mathcal{C}' . By Proposition 2.6(i), $\chi \otimes X \neq X$, which implies that $X \otimes X^*$ does not contain χ . Thus, either X itself is odd-dimensional, or $X \otimes X^*/\mathbf{1}$ is odd-dimensional, and is a direct sum of simple objects not contained in \mathcal{C}' . In this case one of the summands has to be odd-dimensional. \square

2.11. Interpretation of extensions and equivariantizations in terms of the center [3,26,28,9,19]

Proposition 2.9. *Let \mathcal{C} be a fusion category.*

- (i) *If $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\text{Rep}(G)$ which maps to Vec under the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ then \mathcal{C} is a G -extension of some fusion category \mathcal{D} .*
- (ii) *Let \mathcal{C} be a G -extension of \mathcal{D} . Then $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\mathcal{E} = \text{Rep}(G)$ mapping to Vec in \mathcal{C} , such that the de-equivariantization of \mathcal{E}' by \mathcal{E} is equivalent to $\mathcal{Z}(\mathcal{D})$ as a braided tensor category.*

Proof. (i) Suppose there is a Tannakian subcategory $\mathcal{E} = \text{Rep}(G) \subset \mathcal{Z}(\mathcal{C})$ such that the restriction of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ maps \mathcal{E} to Vec . Then every simple object X of \mathcal{C} determines a tensor automorphism of $F|_{\mathcal{E}}$ as follows. Given an object Y in \mathcal{E} , the permutation isomorphism $\eta_{X,Y} : X \otimes F(Y) \xrightarrow{\sim} F(Y) \otimes X$ defining the central structure of Y yields an automorphism $\eta_{X,Y} \circ \delta$ of $F(Y) \otimes X$, where $\delta : F(Y) \otimes X \rightarrow X \otimes F(Y)$ is the “trivial” isomorphism, coming from the fact that $F(Y) \in \text{Vec}$. Since $\text{End}_{\mathcal{C}}(F(Y) \otimes X) = \text{End}_{\mathcal{C}} F(Y)$, we obtain a linear automorphism $i_X : F(Y) \rightarrow F(Y)$. Clearly, i_X gives rise to a tensor automorphism of $F|_{\mathcal{E}}$. Since the group of tensor automorphisms of $F|_{\mathcal{E}}$ is isomorphic to G , we have a canonical assignment $X \mapsto i_X \in G$. It is multiplicative in X (in the sense that $i_Z = i_X i_Y$ for any simple $Z \subset X \otimes Y$), and thus defines a grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$.

Now note that every simple object of the center $\mathcal{Z}(\mathcal{C})$ of a graded category \mathcal{C} is either supported on its trivial component or is disjoint from it. By construction, \mathcal{E}' coincides with the category $\mathcal{Z}(\mathcal{C})_e$ of objects of $\mathcal{Z}(\mathcal{C})$ supported on \mathcal{C}_e (indeed, X is in \mathcal{E}' iff i_X is identity). Therefore, F restricts to a surjective functor $\mathcal{E}' \rightarrow \mathcal{C}_e$. Using the identity in [14, proof of Corollary 8.11] we obtain

$$\text{FPdim}(\mathcal{C}_e) = \frac{\text{FPdim}(\mathcal{E}')}{\text{FPdim}(\mathcal{C})} = \frac{\text{FPdim}(\mathcal{C})}{|G|},$$

which means that the above grading of \mathcal{C} is faithful.

(ii) This statement is proved in [19], we include its proof for the reader’s convenience. Suppose $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ with $\mathcal{C}_e = \mathcal{D}$. We construct a subcategory $\mathcal{E} \subset \mathcal{Z}(\mathcal{C})$ as follows. For any representation $\pi : G \rightarrow GL(V)$ of G consider an object Y_π in $\mathcal{Z}(\mathcal{C})$ where $Y_\pi = V \otimes \mathbf{1}$ as an object of \mathcal{C} with the permutation isomorphism

$$c_{Y_\pi, X} := \pi(g) \otimes \text{id}_X : Y_\pi \otimes X \cong X \otimes Y_\pi, \quad \text{when } X \in \mathcal{C}_g, \quad (2)$$

where we identified $Y_\pi \otimes X$ and $X \otimes Y_\pi$ with $V \otimes X$. Let \mathcal{E} be the fusion subcategory of $\mathcal{Z}(\mathcal{C})$ consisting of objects Y_π , where π runs through all finite-dimensional representations of G . Clearly, \mathcal{E} is equivalent to $\text{Rep}(G)$ with its standard braiding.

By construction, the forgetful functor maps \mathcal{E} to Vec and \mathcal{E}' consists of all objects in $\mathcal{Z}(\mathcal{C})$ whose forgetful image is in \mathcal{C}_e . Consider the surjective braided functor $H : \mathcal{E}' \rightarrow \mathcal{Z}(\mathcal{C}_e)$ obtained by restricting the braiding of $X \in \mathcal{E}'$ from \mathcal{C} to \mathcal{C}_e . One can check that H can be factored through the de-equivariantization functor $\mathcal{E}' \rightarrow \mathcal{E}'_G$ (see [3, Theorem 3.1]). This yields a braided equivalence between \mathcal{E}'_G and $\mathcal{Z}(\mathcal{C}_e)$, since the two categories have equal Frobenius–Perron dimension. \square

The following Proposition 2.10 can be derived from [3,26,27]. Again, we include the proof for the reader's convenience.

Proposition 2.10.

- (i) If $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\text{Rep}(G)$ which embeds to \mathcal{C} under the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ then \mathcal{C} is a G -equivariantization of some fusion category \mathcal{D} .
- (ii) Let \mathcal{C} be a G -equivariantization of \mathcal{D} . Then $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\mathcal{E} = \text{Rep}(G)$ such that the de-equivariantization of \mathcal{E}' by \mathcal{E} is equivalent to $\mathcal{Z}(\mathcal{D})$ as a braided tensor category.

Proof. (i) Let $A = \text{Fun}(G)$ be the algebra of functions on G . It is a commutative algebra in $\mathcal{Z}(\mathcal{C})$. Therefore, the category $\mathcal{D} := \mathcal{C}_G$ of A -modules in \mathcal{C} is a fusion category. The action of G on A via right translations gives rise to an action of G on \mathcal{D} . It is straightforward to check that the corresponding equivariantization of \mathcal{D} is equivalent to \mathcal{C} (see Section 2.6 and [9, Section 4.2]).

(ii) By Proposition 2.2 \mathcal{D}^G is Morita equivalent to a G -graded fusion category $\mathcal{D} \rtimes G$ whose trivial component is \mathcal{D} . Hence $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{D} \rtimes G)$ and the required statement follows immediately from Proposition 2.9(ii). \square

2.12. The divisibility theorems

Theorem 2.11. (See [10, Lemma 1.2] and [14, Propositions 8.23, 8.24, 8.27].)

- (i) Let \mathcal{C} be a weakly integral non-degenerate braided category. Then for any simple object $X \in \mathcal{C}$, the ratio $\text{FPdim}(\mathcal{C})/\text{FPdim}(X)^2$ is an integer.
- (ii) Let \mathcal{C} be a weakly integral braided fusion category. Then for any simple object $X \in \mathcal{C}$, the ratio $\text{FPdim}(\mathcal{C})/\text{FPdim}(X)$ is the square root of an integer.

Theorem 2.12. (See [14, Corollary 8.11, Proposition 8.15].) The Frobenius–Perron dimension of a full fusion subcategory or a component in a quotient category of a fusion category \mathcal{C} divides the Frobenius–Perron dimension of \mathcal{C} in the ring of algebraic integers.

2.13. Kac algebras (Abelian extensions) [32]

Let G be a finite group, and $G = KL$ be an exact factorization of G into a product of two subgroups K, L (exactness means that $K \cap L = 1$). Let $\omega \in Z^3(G, \mathbb{C}^*)$ be a 3-cocycle, and $\psi : C^2(K, \mathbb{C}^*), \phi \in C^2(L, \mathbb{C}^*)$ be 2-cochains such that $d\psi = \omega|_K, d\phi = \omega|_L$. Consider

the fusion category $\mathcal{C} = \text{Vec}_{G,\omega,K,\psi}$ of (K, ψ) -biequivariant (G, ω) -graded vector spaces (see [41,42]). This category has a module category \mathcal{M} of (G, ω) -graded vector spaces which are equivariant under (K, ψ) on the left, and under (L, ϕ) on the right. This module category has only one simple object, so it defines a fiber functor on \mathcal{C} , hence a group-theoretical Hopf algebra $H = H(G, K, L, \omega, \psi, \phi)$ with $\text{Rep}(H) = \mathcal{C}$. This Hopf algebra is an *Abelian extension*

$$\mathbb{C} \rightarrow \text{Fun}(L) \rightarrow H \rightarrow \mathbb{C}[K] \rightarrow \mathbb{C},$$

and is sometimes called a *Kac algebra*.

3. Proof of Theorem 1.3

We start with the following characterization of Morita equivalence of fusion categories in terms of their centers.

Theorem 3.1. *Two fusion categories \mathcal{C} and \mathcal{D} are Morita equivalent if and only if $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{D})$ are equivalent as braided tensor categories.*

Proof. By the result of Müger [27, Remark 3.18] Morita equivalent fusion categories have braided equivalent centers. Thus we need to prove the opposite implication.

Given an algebra A in a fusion category \mathcal{C} let $A\text{-mod}_{\mathcal{C}}$ and $A\text{-bimod}_{\mathcal{C}}$ denote, respectively, the categories of right A -modules and A -bimodules in \mathcal{C} . In the case when the category \mathcal{C} is braided and the algebra A is commutative, the category $A\text{-mod}_{\mathcal{C}}$ has a natural structure of tensor category, see e.g. [3]. Namely any $M \in A\text{-mod}_{\mathcal{C}}$ can be turned into A -bimodule using the morphism $A \otimes M \xrightarrow{c_{A,M}} M \otimes A \rightarrow M$ and the tensor product on $A\text{-mod}_{\mathcal{C}}$ is defined to be tensor product \otimes_A over A , see e.g. [41].

For a fusion category \mathcal{C} let $F_{\mathcal{C}} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ and $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ denote the forgetful functor and its right adjoint. The following lemma is a special case of a more general result obtained in [4].

Lemma 3.2.

- (i) *The object $A = I_{\mathcal{C}}(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ has a natural structure of commutative algebra; moreover for any $X \in \mathcal{C}$ the object $I_{\mathcal{C}}(X)$ has a natural structure of right A -module.*
- (ii) *The functor $I_{\mathcal{C}}$ induces an equivalence of tensor categories $\mathcal{C} \simeq A\text{-mod}_{\mathcal{Z}(\mathcal{C})}$.*

Proof. Consider the category \mathcal{C} as a module category over $\mathcal{Z}(\mathcal{C})$ via the functor $F_{\mathcal{C}}$. Then [15, Lemma 3.38] says that $I_{\mathcal{C}}(X) = \underline{\text{Hom}}(\mathbf{1}, X)$ for any $X \in \mathcal{C}$. Thus $A = I_{\mathcal{C}}(\mathbf{1}) = \underline{\text{Hom}}(\mathbf{1}, \mathbf{1})$ has a natural structure of algebra in $\mathcal{Z}(\mathcal{C})$; for any $X \in \mathcal{C}$ the object $I_{\mathcal{C}}(X) = \underline{\text{Hom}}(\mathbf{1}, X)$ is naturally right A -module and the functor $I_{\mathcal{C}}(?) = \underline{\text{Hom}}(\mathbf{1}, ?)$ induces an equivalence of categories $\mathcal{C} \simeq A\text{-mod}_{\mathcal{Z}(\mathcal{C})}$, see [15, Theorem 3.17]. It remains to explain that A is a commutative algebra and that the functor $I_{\mathcal{C}}$ has a structure of tensor functor.

It follows from definitions (see [15,41]) that the multiplication on the algebra A can be described as follows. Let $\mu \in \text{Hom}(F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})), \mathbf{1})$ be the image of id under the canonical isomorphism $\text{Hom}(I_{\mathcal{C}}(\mathbf{1}), I_{\mathcal{C}}(\mathbf{1})) \simeq \text{Hom}(F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})), \mathbf{1})$. The multiplication morphism $m : A \otimes A \rightarrow A$ is the image of $\mu \otimes \mu$ under the isomorphism $\text{Hom}(F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) \otimes F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})), \mathbf{1}) \simeq \text{Hom}(I_{\mathcal{C}}(\mathbf{1}) \otimes I_{\mathcal{C}}(\mathbf{1}), I_{\mathcal{C}}(\mathbf{1}))$. By definition the commutativity of A means that $m \in \text{Hom}(I_{\mathcal{C}}(\mathbf{1}) \otimes I_{\mathcal{C}}(\mathbf{1}), I_{\mathcal{C}}(\mathbf{1}))$

is invariant under the action of the braiding permuting two copies of $I_{\mathcal{C}}(\mathbf{1})$. Using the definition of m we see that this is equivalent to the invariance of $\mu \otimes \mu \in \text{Hom}(F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) \otimes F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})), \mathbf{1})$ under the braiding $c_{A,A}$ permuting the two copies of $F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) \in \mathcal{C}$ (note that $F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1}))$ has a canonical lift to $\mathcal{Z}(\mathcal{C})$, namely $A = I_{\mathcal{C}}(\mathbf{1})$, so we can talk about the braiding). The naturality of the braiding with a central object implies the commutativity of the following diagram:

$$\begin{array}{ccccc}
 F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) \otimes F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) & \xrightarrow{c_{A,A}} & F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) \otimes F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) & & \\
 \downarrow \text{id} \otimes \mu & & \downarrow \mu \otimes \text{id} & & \\
 F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) \otimes \mathbf{1} & \xrightarrow{\text{id}} & F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) & \xrightarrow{\text{id}} & \mathbf{1} \otimes F_{\mathcal{C}}(I_{\mathcal{C}}(\mathbf{1})) \\
 \searrow \mu \otimes \text{id} & & & & \swarrow \text{id} \otimes \mu \\
 & \mathbf{1} \otimes \mathbf{1} & & &
 \end{array} \quad (3)$$

Applying the functor $\text{Hom}(\cdot, \mathbf{1})$ to this diagram we obtain the desired invariance of $\mu \otimes \mu$.

For any $X \in \mathcal{C}$ let $\mu_X : F_{\mathcal{C}}(I_{\mathcal{C}}(X)) \rightarrow X$ be the image of id under the canonical isomorphism $\text{Hom}(I_{\mathcal{C}}(X), I_{\mathcal{C}}(X)) \simeq \text{Hom}(F_{\mathcal{C}}(I_{\mathcal{C}}(X)), X)$ (so we have $\mu_{\mathbf{1}} = \mu$ in the notation used above) and for $X, Y \in \mathcal{C}$ let $\mu_{X,Y} : I_{\mathcal{C}}(X) \otimes I_{\mathcal{C}}(Y) \rightarrow I_{\mathcal{C}}(X \otimes Y)$ be the image of $\mu_X \otimes \mu_Y$ under the canonical isomorphism $\text{Hom}(F_{\mathcal{C}}(I_{\mathcal{C}}(X)) \otimes F_{\mathcal{C}}(I_{\mathcal{C}}(Y)), X \otimes Y) \simeq \text{Hom}(I_{\mathcal{C}}(X) \otimes I_{\mathcal{C}}(Y), I_{\mathcal{C}}(X \otimes Y))$ (in the notation above $\mu_{\mathbf{1},\mathbf{1}} = m$ is the multiplication morphism on $A = I_{\mathcal{C}}(\mathbf{1})$ and $\mu_{X,\mathbf{1}}$ is the morphism making $I_{\mathcal{C}}(X)$ into right A -module). It is straightforward to verify that $\mu_{X,Y}$ satisfies all the axioms of a tensor functor except for being an isomorphism. In particular, the morphism $\mu_{\mathbf{1},X}$ makes $I_{\mathcal{C}}(X)$ into left A -module; moreover $\mu_{\mathbf{1},X}$ and $\mu_{X,\mathbf{1}}$ make $I_{\mathcal{C}}(X)$ into A -bimodule. The diagram similar to (3) shows that $\mu_{\mathbf{1},X}$ can be described as a composition $A \otimes I_{\mathcal{C}}(X) \xrightarrow{c_{A,I_{\mathcal{C}}(X)}} I_{\mathcal{C}}(X) \otimes A \xrightarrow{\mu_{X,\mathbf{1}}} I_{\mathcal{C}}(X)$, so the structure of $I_{\mathcal{C}}(X)$ as A -bimodule is the same as the structure used in the definition of tensor structure on $A\text{-mod}_{\mathcal{Z}(\mathcal{C})}$.

It is immediate to check that $\mu_{X,Y}$ factorizes through the canonical map $I_{\mathcal{C}}(X) \otimes I_{\mathcal{C}}(Y) \rightarrow I_{\mathcal{C}}(X)$ as $I_{\mathcal{C}}(X) \otimes I_{\mathcal{C}}(Y) \rightarrow I_{\mathcal{C}}(X) \otimes_A I_{\mathcal{C}}(Y) \xrightarrow{\tilde{\mu}_{X,Y}} I_{\mathcal{C}}(X \otimes Y)$ and that $\tilde{\mu}_{X,Y}$ satisfies all the axioms of a tensor functor with a possible exception of being an isomorphism. Finally one verifies that for $X = F_{\mathcal{C}}(Z)$ with $Z \in \mathcal{Z}(\mathcal{C})$ we have $I_{\mathcal{C}}(X) = Z \otimes A$ (as A -modules) and under this isomorphism $\tilde{\mu}_{X,Y}$ goes to the canonical isomorphism $I_{\mathcal{C}}(X) \otimes_A I_{\mathcal{C}}(Y) = Z \otimes I_{\mathcal{C}}(Y) \simeq I_{\mathcal{C}}(F_{\mathcal{C}}(Z) \otimes Y)$ from [15, Proposition 3.39(iii)]. Since the functor $F_{\mathcal{C}}$ is surjective (see [15, Proposition 3.39(i)] we get that $\tilde{\mu}_{X,Y}$ is always an isomorphism. Thus the isomorphisms $\tilde{\mu}_{X,Y}$ define a tensor structure on the functor $I_{\mathcal{C}}$ and lemma is proved. \square

Let \mathcal{C}, \mathcal{D} be fusion categories such that there is a braided tensor equivalence $a : \mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{D})$. Since $I_{\mathcal{D}}(\mathbf{1})$ is a commutative algebra in $\mathcal{Z}(\mathcal{D})$ and a is a *braided* equivalence, we have that $L := a^{-1}(I_{\mathcal{D}}(\mathbf{1}))$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$. Furthermore,

$$\mathcal{D} \cong L\text{-mod}_{\mathcal{Z}(\mathcal{C})} \quad (4)$$

as a fusion category.

Note that L is indecomposable in $\mathcal{Z}(\mathcal{C})$ but might be decomposable as an algebra in \mathcal{C} , i.e.,

$$L = \bigoplus_{i \in J} L_i,$$

where $L_i, i \in J$, are indecomposable algebras in \mathcal{C} such that the multiplication of L is zero on $L_i \otimes L_j, i \neq j$ (e.g., if $\mathcal{C} = \text{Rep}(G)$ then $L = \text{Fun}(G, k)$ with the adjoint action of G and J is the set of conjugacy classes of G).⁶

We would like to show that for each $i \in J$:

$$L_i\text{-bimod}_{\mathcal{C}} \cong L\text{-mod}_{\mathcal{Z}(\mathcal{C})}. \quad (5)$$

In view of (4) this will mean that \mathcal{D} is dual to \mathcal{C} with respect to the \mathcal{C} -module category $L_i\text{-mod}_{\mathcal{C}}$ for any $i \in J$.

Consider the following commutative diagram of tensor functors:

$$\begin{array}{ccccc} \mathcal{Z}(\mathcal{C}) & \xrightarrow{Z \mapsto Z \otimes L_i} & & \mathcal{Z}(L_i\text{-bimod}_{\mathcal{C}}) & \\ \downarrow Z \mapsto Z \otimes L & & & \downarrow F_{L_i\text{-bimod}_{\mathcal{C}}} & \\ L\text{-mod}_{\mathcal{Z}(\mathcal{C})} & \xrightarrow{F_{\mathcal{C}}} & \bigoplus L_i\text{-bimod}_{\mathcal{C}} \subset L\text{-bimod}_{\mathcal{C}} & \xrightarrow{\pi_i} & L_i\text{-bimod}_{\mathcal{C}} \end{array}$$

Here π_i is a projection from $L\text{-bimod}_{\mathcal{C}} = \bigoplus_{i,j} (L_i - L_j)\text{-bimod}_{\mathcal{C}}$ to its (i, i) component. We have $\pi_i(X \otimes L) = X \otimes L_i$ for all $X \in \mathcal{C}$. The top arrow is an equivalence by [27, Remark 3.18] (see also [15, Corollary 3.35]) and the forgetful functor $\mathcal{Z}(L_i\text{-bimod}_{\mathcal{C}}) \rightarrow L_i\text{-bimod}_{\mathcal{C}}$ (the right down arrow) is surjective. Hence, the composition $F_i := \pi_i F_{\mathcal{C}}$ of the functors in the bottom row is surjective. But F_i is a tensor functor between fusion categories of equal Frobenius–Perron dimension and hence it is an equivalence by [15, Proposition 2.20]. \square

Remark 3.3.

- (1) The above characterization of Morita equivalence was announced earlier and independently by A. Kitaev and M. Müger.
- (2) For group-theoretical categories Theorem 3.1 was proved in [31].
- (3) A crucial idea of the proof of Theorem 3.1 (which is to consider a commutative algebra $L \in \mathcal{Z}(\mathcal{C})$ as an algebra in \mathcal{C}) also appears in [24, Theorem 3.22].

Lemma 3.4. *Let G be a finite group, let \mathcal{D} be a G -extension of a fusion category \mathcal{D}_0 , and let $\tilde{\mathcal{D}}_0$ be a fusion category Morita equivalent to \mathcal{D}_0 . There exists a G -extension $\tilde{\mathcal{D}}$ of $\tilde{\mathcal{D}}_0$ which is Morita equivalent to \mathcal{D} .*

Proof. Let A be an indecomposable algebra in \mathcal{D}_0 such that $\tilde{\mathcal{D}}_0$ is equivalent to the category of A -bimodules in \mathcal{D}_0 . Observe that the tensor category $\tilde{\mathcal{D}} = A\text{-bimod}_{\mathcal{D}}$ of A -bimodules in \mathcal{D}

⁶ Here and below we abuse notation and write L for an object of $\mathcal{Z}(\mathcal{C})$ and its forgetful image in \mathcal{C} .

inherits the G -grading (since A belongs to the trivial component of \mathcal{D}). Since the category of A -modules in \mathcal{D} is indecomposable, $\tilde{\mathcal{D}}$ is a fusion category. \square

We are now ready to prove Theorem 1.3.

Suppose \mathcal{C} is Morita equivalent to a G -extension $\tilde{\mathcal{C}}$ of \mathcal{D} . By [27, Remark 3.18] there is a braided tensor equivalence $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\tilde{\mathcal{C}})$. By Proposition 2.9(ii) $\mathcal{Z}(\tilde{\mathcal{C}})$ contains a Tannakian subcategory $\mathcal{E} = \text{Rep}(G)$ with the specified property, as desired.

Conversely, suppose that $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\mathcal{E} = \text{Rep}(G)$ such that the de-equivariantization \mathcal{Z} of \mathcal{E}' by \mathcal{E} is equivalent to $\mathcal{Z}(\mathcal{D})$ as a braided tensor category. Let $I : \mathcal{D} \rightarrow \mathcal{Z}(\mathcal{D}) \cong \mathcal{Z}$ be the composition of the left adjoint of the forgetful functor $\mathcal{Z}(\mathcal{D}) \rightarrow \mathcal{D}$ with the equivalence $\mathcal{Z}(\mathcal{D}) \cong \mathcal{Z}$. Then $A_1 := I(\mathbf{1})$ is a commutative algebra in \mathcal{Z} . Let $J : \mathcal{Z} = A_1\text{-mod}_{\mathcal{E}'} \rightarrow \mathcal{E}'$ be the functor forgetting the A_1 -module structure then $A := J(A_1)$ is a commutative algebra in \mathcal{E}' and, hence, in $\mathcal{Z}(\mathcal{C})$.

It was explained in [9] that for every $Z \in \mathcal{Z}(\mathcal{C})$ the object $Z \otimes A$ has a structure of an object in the center of $A\text{-mod}_{\mathcal{C}}$ and that the functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(A\text{-mod}_{\mathcal{C}}) : Z \mapsto A \otimes Z$ is a braided tensor equivalence. By Theorem 3.1 \mathcal{C} and $A\text{-mod}_{\mathcal{C}}$ are Morita equivalent.

The composition $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(A\text{-mod}_{\mathcal{C}}) \rightarrow A\text{-mod}_{\mathcal{C}}$ identifies with the free A -module functor. This functor takes $\mathcal{E} = \text{Rep}(G) \subset \mathcal{Z}(A\text{-mod}_{\mathcal{C}})$ to Vec . By Proposition 2.9 $A\text{-mod}_{\mathcal{C}}$ is a G -extension of some fusion category $\tilde{\mathcal{D}}$ and there is a braided tensor equivalence $\mathcal{Z}(\mathcal{D}) \cong \mathcal{Z}(\tilde{\mathcal{D}})$. By Theorem 3.1 \mathcal{D} and $\tilde{\mathcal{D}}$ are Morita equivalent. So \mathcal{D} is Morita equivalent to a G -extension of \mathcal{D} by Lemma 3.4, as required.

4. Properties of weakly group-theoretical and solvable fusion categories

4.1. Properties of weakly group-theoretical categories

The basic properties of weakly group-theoretical fusion categories (see Definition 1.1) are summarized in the following two Propositions.

Proposition 4.1. *The class of weakly group-theoretical categories is closed under taking extensions, equivariantizations, Morita equivalent categories, tensor products, the center, subcategories and component categories of quotient categories.*

Proof. The invariance under taking Morita equivalent categories and tensor products is obvious. The invariance under taking extensions follows from Lemma 3.4 and the invariance under equivariantizations follows from Proposition 2.2. The invariance under taking the center then follows from Morita invariance, as $\mathcal{Z}(\mathcal{C})$ is Morita equivalent to $\mathcal{C} \boxtimes \mathcal{C}^{op}$. The rest of the proof is similar to the proof of Proposition 8.44 in [14]. Namely, to prove the invariance under taking subcategories, let \mathcal{C} be a weakly group-theoretical category, and $\mathcal{D} \subset \mathcal{C}$ a fusion subcategory. Let \mathcal{M} be an indecomposable \mathcal{C} -module category such that $\mathcal{C}_{\mathcal{M}}^*$ is nilpotent. Then every component category of $\mathcal{D}_{\mathcal{M}}^*$ is nilpotent, since it is easy to see that every component category in a quotient of a nilpotent category is nilpotent. The case of a component in a quotient category reduces to the case of a subcategory by taking duals. \square

Proposition 4.2. *A fusion category \mathcal{C} is weakly group-theoretical if and only if there exists a sequence of non-degenerate braided categories*

$$\text{Vec} = \mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n = \mathcal{Z}(\mathcal{C})$$

and finite groups G_1, \dots, G_n , such that for all $1 \leq i \leq n$, the Tannakian category $\text{Rep}(G_i)$ is contained in \mathcal{D}_i as an isotropic subcategory, and \mathcal{D}_{i-1} is the de-equivariantization of the Müger centralizer $\text{Rep}(G_i)'$ of $\text{Rep}(G_i)$ in \mathcal{D}_i .

Proof. To prove the “only if” part it suffices to assume that \mathcal{C} is nilpotent. Suppose first that \mathcal{C} is a G -extension of another category \mathcal{D} . By Proposition 2.9 $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\text{Rep}(G)$ such that the de-equivariantization of $\text{Rep}(G)'$ by $\text{Rep}(G)$ is $\mathcal{Z}(\mathcal{D})$. Since every nilpotent category is obtained from Vec by a sequence of extensions, this implies the desired statement.

To prove the “if” part, we argue by induction in n . For $n = 1$ we must have a Lagrangian subcategory $\text{Rep}(G_1) \subset \mathcal{Z}(\mathcal{C})$ and so \mathcal{C} is group-theoretical by [8, Corollary 4.14]. Suppose the statement is true for $n = l$ and let $\text{Vec} = \mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{l+1} = \mathcal{Z}(\mathcal{C})$ be a sequence as in the statement of the proposition. By Theorem 1.3 \mathcal{C} is Morita equivalent to a G_1 -extension of a fusion category $\tilde{\mathcal{C}}$ such that $\mathcal{Z}(\tilde{\mathcal{C}})$ is braided equivalent to the de-equivariantization of $\text{Rep}(G_1)'$ by $\text{Rep}(G_1)$ in \mathcal{D}_1 . By induction, $\tilde{\mathcal{C}}$ is weakly group-theoretical, hence \mathcal{C} is weakly group-theoretical by Proposition 4.1. \square

Remark 4.3. Note that the class of group-theoretical categories is *not* closed under taking extensions and equivariantizations, see [40] and [14, Remark 8.48].

4.2. Properties of solvable fusion categories

Let \mathcal{C} be a fusion category.

Proposition 4.4. *The following conditions are equivalent:*

- (1) \mathcal{C} is solvable in the sense of Definition 1.2(i).
- (2) $\mathcal{Z}(\mathcal{C})$ admits a chain as in Proposition 4.2, where all the groups G_i are cyclic of prime order.
- (3) There is a sequence of fusion categories $\mathcal{C}_0 = \text{Vec}, \mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C}$ and a sequence G_1, \dots, G_n of cyclic groups of prime order such that \mathcal{C}_i is obtained from \mathcal{C}_{i-1} either by a G_i -equivariantization or as a G_i -extension.

Proof. (1) \Rightarrow (2) This follows by iterating Proposition 2.9.

(2) \Rightarrow (3) We argue by induction in n . Consider the image of $\text{Rep}(G_n)$ in \mathcal{C} under the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Since G_n is cyclic of prime order, either $\text{Rep}(G_n)$ maps to Vec , in which case \mathcal{C} is a G_n -extension of some category \mathcal{D} by Proposition 2.9(i), or $\text{Rep}(G_n)$ embeds into \mathcal{C} , in which case \mathcal{C} is a G_n -equivariantization of some category \mathcal{D} by Proposition 2.10(i). In both cases, $\mathcal{Z}(\mathcal{D}) = \mathcal{D}_{n-1}$, and by the induction assumption \mathcal{D} satisfies (3), so we are done.

(3) \Rightarrow (1) By Proposition 2.2 \mathcal{C}_i is Morita equivalent to a G_i -extension of \mathcal{C}_{i-1} , $i = 1, \dots, n$. Combining induction with Lemma 3.4 we see that \mathcal{C} is Morita equivalent to a cyclically nilpotent fusion category, i.e., \mathcal{C} is solvable. \square

Proposition 4.5.

- (i) *The class of solvable categories is closed under taking extensions and equivariantizations by solvable groups, Morita equivalent categories, tensor products, center, subcategories and component categories of quotient categories.*
- (ii) *The categories $\text{Vec}_{G,\omega}$ and $\text{Rep}(G)$ are solvable if and only if G is a solvable group.*

- (iii) A braided nilpotent fusion category is solvable.
- (iv) A solvable fusion category $\mathcal{C} \neq \text{Vec}$ contains a nontrivial invertible object.

Proof. (i) As in the proof of Proposition 4.1, everything follows from the easy fact that a component category in a quotient of a cyclically nilpotent category is cyclically nilpotent.

(ii) One direction is obvious, since if G is solvable, $\text{Vec}_{G,\omega}$ is cyclically nilpotent. Since $\text{Rep}(G)$ is Morita equivalent to Vec_G it is also solvable by (i).

To prove the converse implication it suffices to show that if $\text{Rep}(G)$ is solvable then so is G . Indeed, $\mathcal{Z}(\text{Vec}_{G,\omega})$ contains $\text{Rep}(G)$ as a fusion subcategory, so the solvability of $\text{Vec}_{G,\omega}$ implies solvability of $\text{Rep}(G)$ by (i). We have two possibilities: either $\text{Rep}(G)$ is an H -extension or $\text{Rep}(G) = \mathcal{C}^H$ for some fusion category \mathcal{C} , where H is a cyclic group of prime order. In the former situation G must have a nontrivial center Z and we can pass to the fusion subcategory $\text{Rep}(G/Z) \subset \text{Rep}(G)$ which is again solvable by (i). In the latter situation $\text{Rep}(G)$ contains a fusion subcategory of prime order by Proposition 2.9(i), therefore, G contains a normal subgroup G_1 of prime index and we can pass to the solvable quotient category $\text{Rep}(G_1)$. So the required statement follows by induction.

(iii) Follows from [8, Theorem 6.12] combined with [14, Theorem 8.28].

(iv) The proof is by induction in the dimension of \mathcal{C} . The base of induction is clear, and only the induction step needs to be justified. If \mathcal{C} is an extension of a smaller solvable category \mathcal{D} , then either $\mathcal{D} \neq \text{Vec}$ and the statement follows from the induction assumption, or $\mathcal{D} = \text{Vec}$ and \mathcal{C} is pointed, so the statement is obvious. On the other hand, if \mathcal{C} is a \mathbb{Z}/p -equivariantization of a smaller solvable category \mathcal{D} , then $\text{Rep}(\mathbb{Z}/p)$ sits inside \mathcal{C} , so we are done. \square

Remark 4.6.

- (1) Note that a non-braided nilpotent fusion category need not be solvable (e.g., Vec_G for a non-solvable group G).
- (2) The notion of a solvable fusion category is close in spirit to the notions of upper and lower solvable and semisolvable Hopf algebras introduced by Montgomery and Witherspoon [30]. However, we would like to note that a semisimple Hopf algebra H such that $\text{Rep}(H)$ is solvable in our sense is not necessarily upper or lower semisolvable in the sense of [30]. For example, Galindo and Natale constructed in [17] self-dual Hopf algebras without nontrivial normal Hopf subalgebras as twisting deformations of solvable groups. Clearly, the representation category of any such Hopf algebra is solvable. It is also easy to construct an example of an upper and lower solvable semisimple Hopf algebra H , such that $\text{Rep}(H)$ is not solvable. For this, it suffices to take the Kac algebra associated to the exact factorization of groups $A_5 = A_4 \cdot \mathbb{Z}/5\mathbb{Z}$.

5. Module categories over equivariantized categories

Let \mathcal{C} be a fusion category and let G be a finite group acting on \mathcal{C} . In this section we obtain a description of module categories over the equivariantization \mathcal{C}^G .

Let \mathcal{M} be a \mathcal{C} -module category, and let t be a tensor autoequivalence of \mathcal{C} . Define a twisted \mathcal{C} -module category \mathcal{M}^t by setting $\mathcal{M}^t = \mathcal{M}$ as an Abelian category and defining a new action of \mathcal{C} :

$$X \otimes^t M := t(X) \otimes M,$$

for all objects M in \mathcal{M} and X in \mathcal{C} , cf. [40]. Given a \mathcal{C} -module functor $F : \mathcal{M} \rightarrow \mathcal{N}$, we define a \mathcal{C} -module functor $F^t : \mathcal{M}^t \rightarrow \mathcal{N}^t$ in an obvious way. Given a natural transformation $\nu : F \rightarrow G$ between \mathcal{C} -module functors $F, G : \mathcal{M} \rightarrow \mathcal{N}$ we define a natural transformation $\nu^t : F^t \rightarrow G^t$.

Remark 5.1. If A is an algebra in \mathcal{C} such that \mathcal{M} is equivalent to the category of A -modules in \mathcal{C} then \mathcal{M}^t is equivalent to the category of $t(A)$ -modules in \mathcal{C} .

An action of a group G on \mathcal{C} gives rise to \mathcal{C} -module equivalences

$$\Gamma_{g,h} : (\mathcal{M}^h)^g \cong \mathcal{M}^{gh}, \quad g, h \in G.$$

Definition 5.2. A G -equivariant \mathcal{C} -module category is a pair consisting of a \mathcal{C} -module category \mathcal{M} along with \mathcal{C} -module equivalences $U_g : \mathcal{M}^g \xrightarrow{\sim} \mathcal{M}$, $g \in G$, and natural isomorphisms of tensor functors $\mu_{g,h} : U_{gh} \Gamma_{g,h} \xrightarrow{\sim} U_g (U_h)^g$, $g, h \in G$, satisfying the following compatibility conditions:

$$(\mu_{f,g} (U_h)^{fg}) \circ (\mu_{fg,h} \Gamma_{f,g}) = (U_f (\mu_{g,h})^f) \circ (\mu_{f,gh} \Gamma_{g,h}), \quad f, g, h \in G. \quad (6)$$

Definition 5.3. Let \mathcal{M} be a G -equivariant \mathcal{C} -module category. An *equivariant object* in \mathcal{M} is a pair consisting of an object M of \mathcal{M} along with isomorphisms $v_g : U_g(M) \xrightarrow{\sim} M$, $g \in G$, such that the diagrams

$$\begin{array}{ccc} U_g(U_h(M)) & \xrightarrow{U_g(v_h)} & U_g(M) \\ \mu_{g,h}(M) \downarrow & & \downarrow v_g \\ U_{gh}(M) & \xrightarrow{v_{gh}} & M \end{array}$$

commute for all $g, h \in G$.

Let H be a subgroup of G and let \mathcal{M} be an H -equivariant \mathcal{C} -module category. Let \mathcal{M}^H denote the category of equivariant objects in \mathcal{M} . Then \mathcal{M}^H is a \mathcal{C}^G -module category. Namely, the equivariant structure on $X \otimes M$, where X is an object of \mathcal{C}^G and M is an object of \mathcal{M}^H , is given by the product of equivariant structures of X and M .

Proposition 5.4. Every indecomposable \mathcal{C}^G -module category is equivalent to one of the form \mathcal{M}^H , where H is a subgroup of G , and \mathcal{M} is an H -equivariant indecomposable \mathcal{C} -module category.

Proof. Consider the crossed product category $\mathcal{C} \rtimes G$ (see Section 2.7). Indecomposable $(\mathcal{C} \rtimes G)$ -module categories were studied in [40]. Every such module category \mathcal{N} decomposes into a direct sum of \mathcal{C} -module categories $\mathcal{N} = \bigoplus_{s \in S} \mathcal{N}_s$, where S is a homogeneous G -set. Let H be the stabilizer of $s \in S$ so that $S \cong G/H$. It follows that $\mathcal{M} := \mathcal{N}_s$ is an H -equivariant \mathcal{C} -module category which completely determines \mathcal{N} . By Proposition 2.2, any indecomposable \mathcal{C}^G -module category is equivalent to the category of $(\mathcal{C} \rtimes G)$ -module functors from \mathcal{C} to \mathcal{N} for some \mathcal{N} as above. It is easy to see that such functors correspond to equivariant objects in \mathcal{M} . \square

Example 5.5. Let $\mathcal{C} = \text{Vec}$. We have $\text{Vec} \rtimes G = \text{Vec}_G$ and $\text{Vec}^G = \text{Rep}(G)$. An H -equivariant Vec -module category is nothing but a 2-cocycle $\mu \in Z^2(H, \mathbb{C}^*)$. An equivariant object in this category is the same thing as a projective representation of H with the Schur multiplier μ . Thus, our description agrees with that of [42].

6. Proof of Theorem 1.5

Let \mathcal{C} be a fusion category, and let \mathcal{M} be an indecomposable module category over \mathcal{C} . We will denote the Frobenius–Perron dimension of $X \in \mathcal{M}$ normalized as in Definition 1.4 by $\text{FPdim}_{\mathcal{M}}(X)$.

Definition 6.1. Let $m \neq 0$ be an algebraic integer. Let us say that a fusion category \mathcal{C} has the *strong m -Frobenius property* if for any indecomposable \mathcal{C} -module category \mathcal{M} and any simple $X \in \mathcal{M}$, the ratio $\frac{\text{FPdim}(\mathcal{C})}{m \text{FPdim}_{\mathcal{M}}(X)}$ is an algebraic integer.

Proposition 6.2. Let \mathcal{C} be a fusion category having the strong m -Frobenius property, and let G be a finite group. Then:

- (1) a G -equivariantization of \mathcal{C} has the strong m -Frobenius property, and
- (2) a G -extension of \mathcal{C} has the strong $m\sqrt{|G|}$ -Frobenius property.

Proof. (1) By Proposition 5.4 every indecomposable \mathcal{C}^G -module category \mathcal{M} is equivalent to \mathcal{N}^H , where H is a subgroup of G and \mathcal{N} is an H -equivariant indecomposable \mathcal{C} -module category. For any simple object X in $\mathcal{M} = \mathcal{N}^H$ choose and fix its simple constituent Y in \mathcal{N} . Let $\text{Stab}(Y)$ denote the stabilizer of Y in H . Then X corresponds to an irreducible representation π of $\text{Stab}(Y)$ and

$$\text{FPdim}_{\mathcal{N}}(X) = \deg(\pi)[H : \text{Stab}(Y)] \text{FPdim}_{\mathcal{N}}(Y). \quad (7)$$

We claim that

$$\frac{\text{FPdim}_{\mathcal{M}}(X)}{\text{FPdim}_{\mathcal{N}}(X)} = \sqrt{[G : H]}. \quad (8)$$

Indeed, we have

$$\begin{aligned} \sum_X \text{FPdim}_{\mathcal{N}}(X)^2 &= \sum_Y \left(\sum_{\pi} \deg(\pi)^2 \right) [H : \text{Stab}(Y)]^2 \text{FPdim}_{\mathcal{N}}(Y)^2 \\ &= |H| \sum_Y [H : \text{Stab}(Y)] \text{FPdim}_{\mathcal{N}}(Y)^2 \\ &= |H| \text{FPdim}(\mathcal{C}). \end{aligned}$$

On the other hand, $\sum_X \text{FPdim}_{\mathcal{M}}(X)^2 = \text{FPdim}(\mathcal{C}^G) = |G| \text{FPdim}(\mathcal{C})$. Combining these two equations we obtain (8). Comparing with (7) we see that

$$\frac{\text{FPdim}(\mathcal{C}^G)}{\text{FPdim}_{\mathcal{M}}(X)} = \frac{\sqrt{|G:H|} |\text{Stab}(Y)|}{\deg(\pi)} \times \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}_{\mathcal{N}}(Y)},$$

and so \mathcal{C}^G has the strong m -Frobenius property.

(2) Let $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$, $\mathcal{D}_e = \mathcal{C}$, be a G -extension of \mathcal{C} , and let \mathcal{M} be an indecomposable \mathcal{D} -module category. Let $\mathcal{M} = \bigoplus_{s \in S} \mathcal{M}_s$ be its decomposition as a \mathcal{C} -module category. It was shown in [20] that S is a homogeneous G -set and $\text{FPdim}(\mathcal{M}_s)/\text{FPdim}(\mathcal{M}_t) = 1$ for all $s, t \in S$. Let $H \subset G$ be a subgroup such that $S = G/H$. Then for any simple object X in \mathcal{M}_s we have $\frac{\text{FPdim}_{\mathcal{M}}(X)}{\text{FPdim}_{\mathcal{M}_s}(X)} = \sqrt{|H|}$, and therefore,

$$\frac{\text{FPdim}(\mathcal{D})}{\text{FPdim}_{\mathcal{M}}(X)} = \frac{|G|}{\sqrt{|H|}} \times \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}_{\mathcal{M}_s}(X)},$$

and so \mathcal{D} has the strong $m\sqrt{|G|}$ -Frobenius property. \square

Lemma 6.3. *Let \mathcal{D} be a non-degenerate braided fusion category containing a Tannakian subcategory $\mathcal{E} = \text{Rep}(G)$. Let \mathcal{Z} be the de-equivariantization of \mathcal{E}' by \mathcal{E} . Then \mathcal{D} is equivalent, as a fusion category, to a G -equivariantization of a G -extension of \mathcal{Z} .*

Proof. Let $A = \text{Fun}(G)$ be the algebra of functions on G . The category \mathcal{D}_G of A -modules in \mathcal{C} is faithfully G -graded with the trivial component \mathcal{Z} , and \mathcal{D} is a G -equivariantization of \mathcal{D}_G , see [23,28]. \square

Corollary 6.4. *Let \mathcal{C} be a weakly group-theoretical fusion category. Then its center $\mathcal{Z}(\mathcal{C})$ has the strong $\sqrt{\text{FPdim}(\mathcal{C})}$ -Frobenius property.*

Proof. It follows from Proposition 4.2 and Lemma 6.3 that there exists a sequence of finite groups G_1, \dots, G_n such that $\mathcal{Z}(\mathcal{C})$ can be obtained from Vec by the following sequence of $2n$ operations: G_1 -extension, G_1 -equivariantization, \dots , G_n -extension, G_n -equivariantization. Therefore, the result follows from Proposition 6.2. \square

Now we are in a position to prove Theorem 1.5. Let \mathcal{C} be a weakly group-theoretical fusion category, and let \mathcal{M} be an indecomposable module category over \mathcal{C} . Let $\widetilde{\mathcal{M}}$ be the pullback of \mathcal{M} under the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Then it is obvious that for any $X \in \mathcal{M}$, one has $\text{FPdim}_{\widetilde{\mathcal{M}}}(X) = \text{FPdim}_{\mathcal{M}}(X)\sqrt{\text{FPdim}(\mathcal{C})}$. On the other hand, Corollary 6.4 implies that $\text{FPdim}(\mathcal{C})^2/\text{FPdim}_{\widetilde{\mathcal{M}}}(X)$ is an algebraic integer divisible by $\sqrt{\text{FPdim}(\mathcal{C})}$. This implies that $\text{FPdim}(\mathcal{C})/\text{FPdim}_{\mathcal{M}}(X)$ is an algebraic integer, i.e., \mathcal{C} has the strong Frobenius property.

7. Non-degenerate and slightly degenerate categories with a simple object of prime power dimension

In this section we will prove several results on non-degenerate and slightly degenerate braided categories containing a simple object of prime power dimension. These results will be of central importance for the proof of Theorem 1.6 and further results of the paper, and are parallel to the character-theoretic lemmas used in the classical proof of Burnside's theorem in group theory.

Lemma 7.1. *Let X and Y be two simple objects of an integral braided category with coprime dimensions⁷ d_X, d_Y . Then one of two possibilities hold:*

- (i) X and Y projectively centralize each other (i.e. the square of the braiding on $X \otimes Y$ is a scalar);
- (ii) $s_{X,Y} = 0$ (where s is the S -matrix).

Proof. It suffices to consider the case when the category is non-degenerate, since any braided category can be embedded into a non-degenerate one (its center). In this case, by the Verlinde formula, $\frac{s_{X,Y}}{d_X}$ and $\frac{s_{X,Y}}{d_Y}$ are algebraic integers. Since d_X and d_Y are coprime, $\frac{s_{X,Y}}{d_X d_Y}$ is also an algebraic integer. Since $s_{X,Y}$ is a sum of $d_X d_Y$ roots of unity, we see that $\frac{s_{X,Y}}{d_X d_Y}$ is either a root of unity (in which case the square of the braiding must be a scalar, option (i)), or 0 (option (ii)). \square

Corollary 7.2. *Let \mathcal{E} be an integral non-degenerate braided category which contains a simple object X with dimension $d_X = p^r$, $r > 0$, where p is a prime. Then \mathcal{E} contains a nontrivial symmetric subcategory.*

Proof. We first show that \mathcal{E} contains a nontrivial proper subcategory. Assume not. Take any simple $Y \neq \mathbf{1}$ with d_Y coprime to d_X . We claim that $s_{X,Y} = 0$. Indeed, otherwise X and Y projectively centralize each other, hence Y centralizes $X \otimes X^*$, so the Müger centralizer of the category generated by Y is nontrivial, and we get a nontrivial proper subcategory, a contradiction.

Now let us use the orthogonality of columns ($s_{X,Y}$) and (d_Y) of the S -matrix:

$$\sum_{Y \in \text{Irr } \mathcal{E}} \frac{s_{X,Y}}{d_X} d_Y = 0.$$

As we have shown, all the nonzero summands in this sum, except the one for $Y = \mathbf{1}$, come from objects Y of dimension divisible by p . Therefore, all the summands in this sum except for the one for $Y = \mathbf{1}$ (which equals 1) are divisible by p . This is a contradiction.

Now we prove the corollary by induction in $\text{FPdim}(\mathcal{E})$. Let \mathcal{D} be a nontrivial proper subcategory of \mathcal{E} . If \mathcal{D} is degenerate, then its Müger center is a nontrivial proper symmetric subcategory of \mathcal{E} , so we are done. Otherwise, \mathcal{D} is non-degenerate, and by Theorem 2.4, $\mathcal{E} = \mathcal{D} \boxtimes \mathcal{D}'$. Thus $X = X_1 \otimes X_2$, where $X_1 \in \mathcal{D}$, $X_2 \in \mathcal{D}'$ are simple. Since the dimension of X_1 or X_2 is a positive power of p , we get the desired statement from the induction assumption applied to \mathcal{D} or \mathcal{D}' (which are non-degenerate braided categories of smaller dimension). \square

Remark 7.3. The proof above shows that for a simple object X of an integral non-degenerate braided category \mathcal{E} with prime power dimension we can find another nontrivial simple object Y such that X and Y projectively centralize each other. This can be used to give a proof of Burnside's theorem that a finite group G with a conjugacy class C of prime power size cannot be simple (together with Sylow theorem this implies immediately the solvability of groups of order $p^a q^b$) as follows. Assume that G is simple (it is also non-abelian since it contains conjugacy class C of size > 1). Then G is generated by any of its nontrivial conjugacy classes and $\text{Rep}(G)$ has no nontrivial fusion subcategories. This implies that the category $\mathcal{Z}(\text{Rep}(G))$ has a unique proper

⁷ Here and below, we use shortened notation d_X for the Frobenius–Perron dimension $\text{FPdim}(X)$.

fusion subcategory, namely $\text{Rep}(G)$ (we recall that the category $\mathcal{Z}(\text{Rep}(G))$ is identified with the category of sheaves on G which are G -equivariant with respect to the adjoint action). Now let X be the simple object of $\mathcal{Z}(\text{Rep}(G))$ which corresponds to a trivial sheaf supported on C ; then $d_X = |C|$ is a prime power, hence X projectively centralizes some nontrivial object Y . The object Y is not invertible since the group G is simple non-abelian; hence X generates a nontrivial fusion subcategory of $\mathcal{Z}(\text{Rep}(G))$ contained in Müger centralizer of $Y \otimes Y^*$. This is a contradiction since X is not contained in $\text{Rep}(G) \subset \mathcal{Z}(\text{Rep}(G))$ (recall that the subcategory $\text{Rep}(G)$ consists of objects supported on the unit element $e \in G$).

Proposition 7.4. *Let \mathcal{C} be a slightly degenerate integral braided category, which contains a simple object X of dimension p^r for some prime $p > 2$. Then \mathcal{C} contains a nontrivial Tannakian subcategory.*

Proof. The proof is by induction on the dimension of \mathcal{C} . Let \mathcal{B} be the category spanned by the invertible objects of \mathcal{C} . Then the Müger center of \mathcal{B} contains the category SuperVec .

If the Müger center of \mathcal{B} is bigger than SuperVec , then it contains a nontrivial Tannakian subcategory, and we are done. Otherwise, by Proposition 2.6(ii), $\mathcal{B} = \text{SuperVec} \boxtimes \mathcal{B}_0$, where \mathcal{B}_0 is a pointed non-degenerate braided category. If \mathcal{B}_0 is nontrivial, then $\mathcal{C} = \mathcal{B}_0 \boxtimes \mathcal{B}'_0$, and \mathcal{B}'_0 is slightly degenerate, so we are done by the induction assumption. Thus, it suffices to consider the case $\mathcal{B} = \text{SuperVec}$, which we do from now on.

Let $\mathbf{1}$ and χ be the simple objects of $\text{SuperVec} \subset \mathcal{C}$ (which are the only invertible objects of \mathcal{C}). Let Y be a non-invertible simple object of \mathcal{C} of dimension not divisible by p .

Assume that X and Y projectively centralize each other. In this case the category generated by Y and χ centralizes $X \otimes X^*$, so it is a proper subcategory of \mathcal{C} . If it is not slightly degenerate, its Müger center contains more than two simple objects, hence contains a nontrivial Tannakian subcategory. So we may assume that this subcategory is slightly degenerate, in which case we are done by the induction assumption.

Thus, we may assume that X and Y do not projectively centralize each other. In this case Lemma 7.1 tells us that $s_{X,Y} = 0$.

Now, since \mathcal{C} is slightly degenerate, by Corollary 2.7, we have

$$\sum_Y \frac{s_{X,Y}}{d_X} d_Y = 0,$$

and all the nonzero terms in this sum correspond to either $\dim Y = 1$ (there are two such terms, both equal to 1), or $\dim Y$ divisible by p , which gives terms divisible by p . So we get that 2 is divisible by p , a contradiction. \square

8. Proof of Theorem 1.6

We are going to prove the following theorem, which will easily imply Theorem 1.6.

Theorem 8.1. *Let \mathcal{E} be an integral non-degenerate braided category of dimension $p^a q^b$, where $p < q$ are primes, and a, b nonnegative integers. If \mathcal{E} is not pointed, then it contains a Tannakian subcategory of the form $\text{Rep}(G)$, where G is a cyclic group of prime order.*

Let us explain how Theorem 8.1 implies Theorem 1.6. Let \mathcal{C} be a fusion category of dimension $p^r q^s$, where $p < q$ are primes, and $r, s \geq 0$ are nonnegative integers. We prove that \mathcal{C} is solvable by induction in $r + s$. We can assume that \mathcal{C} is integral, because if not, then \mathcal{C} is $\mathbb{Z}/2\mathbb{Z}$ -graded, so we are done by the induction assumption. Also, we can clearly assume that \mathcal{C} is not pointed. Clearly, the center $\mathcal{E} := \mathcal{Z}(\mathcal{C})$ is not pointed. So the result follows by using Theorem 8.1 and Proposition 4.4.

The rest of the section is devoted to the proof of Theorem 8.1.

Proposition 8.2. *Let \mathcal{E} be an integral non-degenerate braided category of dimension $p^a q^b$, $a + b > 0$. Then \mathcal{E} contains a nontrivial invertible object.*

Proof. By [14] a fusion category of a prime power Frobenius–Perron dimension is nilpotent and hence contains a nontrivial invertible object. So we may assume that $a, b > 0$. Assume the contrary, i.e. that \mathcal{E} does not contain nontrivial invertible objects. By Theorem 2.11, the squared dimensions of simple objects of \mathcal{E} divide $p^a q^b$. Therefore, \mathcal{E} must contain a simple object of dimension p^r , $r > 0$. Hence by Corollary 7.2, it contains a nontrivial symmetric subcategory \mathcal{D} . By Theorem 2.1 \mathcal{D} is super-Tannakian, and therefore by the usual Burnside’s theorem for finite groups (saying that a group of order $p^a q^b$ is solvable), it must contain nontrivial invertible objects, which is a contradiction. \square

Consider now the subcategory \mathcal{B} spanned by all invertible objects of \mathcal{E} . Proposition 8.2 implies that this subcategory is nontrivial. If \mathcal{B} is non-degenerate, then by Theorem 2.4, $\mathcal{E} = \mathcal{B} \boxtimes \mathcal{B}'$, where \mathcal{B}' is another non-degenerate braided category, which is nontrivial (as \mathcal{E} is not pointed), but has no nontrivial invertible objects. Thus, by Proposition 8.2, this case is impossible.

Therefore, \mathcal{B} is degenerate. Consider the Müger center \mathcal{Z} of \mathcal{B} . It is a nontrivial pointed symmetric subcategory in \mathcal{E} . So if $\text{FPdim}(\mathcal{Z}) > 2$, we are done (as \mathcal{Z} necessarily contains a Tannakian subcategory $\text{Rep}(G)$, where G is a cyclic group of prime order).

It remains to consider the case $\text{FPdim}(\mathcal{Z}) = 2$. In this case, we must consider the additional possibility that \mathcal{Z} is the symmetric category SuperVec of super vector spaces (in which case $p = 2$). In this situation, by Proposition 2.6(ii), $\mathcal{B} = \mathcal{Z} \boxtimes \mathcal{D}$, where \mathcal{D} is non-degenerate, so if \mathcal{D} is nontrivial, by Theorem 2.4 $\mathcal{E} = \mathcal{D} \boxtimes \mathcal{D}'$, where \mathcal{D}' is another non-degenerate braided category, whose subcategory of invertible objects is \mathcal{Z} . Thus, it is sufficient to consider the case $\mathcal{B} = \mathcal{Z} = \text{SuperVec}$. In this case, let $\mathcal{C} \supset \mathcal{Z}$ be the Müger centralizer of \mathcal{Z} . This category has dimension is $2^{a-1} q^b > 2$, contains only two invertible objects, and its Müger center is $\mathcal{Z} = \text{SuperVec}$, i.e. it is slightly degenerate. Therefore, Theorem 8.1 follows from the following proposition.

Proposition 8.3. *Let \mathcal{C} be a slightly degenerate integral braided category of dimension $2^r q^s > 2$, where $q > 2$ is a prime, and r, s are nonnegative integers. Suppose that \mathcal{C} contains only two invertible objects. Then \mathcal{C} contains a nontrivial Tannakian subcategory.*

Proof. It follows from Theorem 2.11 that there exists a non-invertible simple object Y of \mathcal{C} whose dimension is a power of 2. Also, by Corollary 2.8, \mathcal{C} contains a simple object X of dimension q^m , $m > 0$. Now the statement follows from Proposition 7.4. \square

9. Applications

9.1. Fusion categories of dimension pqr

Proposition 9.1. *A weakly group-theoretical integral fusion category of square-free dimension is group-theoretical.*

Proof. It follows from [20, Corollary 5.3] that any nilpotent integral fusion category of square-free dimension is automatically pointed. \square

Theorem 9.2. *Let $p < q < r$ be a triple of distinct primes. Then any integral fusion category \mathcal{C} of dimension pqr is group-theoretical.⁸*

Proof. By Proposition 9.1, it suffices to show that \mathcal{C} is Morita equivalent to a nilpotent category, i.e., is weakly group-theoretical. It suffices to show that the category $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory; then the result will follow from Proposition 4.2 and Theorem 1.6.

Lemma 9.3. *$\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory.*

Proof. By Corollary 7.2, if $\mathcal{Z}(\mathcal{C})$ contains a simple object of prime power dimension, then it contains a nontrivial symmetric subcategory and we are done. So it suffices to consider the case when $\mathcal{Z}(\mathcal{C})$ does not contain simple objects of prime power dimension. In this case, by Theorem 2.11, the dimensions of simple objects of $\mathcal{Z}(\mathcal{C})$ can be 1, pq , pr , and qr .

Consider first the case when $\mathcal{Z}(\mathcal{C})$ contains nontrivial invertible objects. In this case, let \mathcal{B} be the category spanned by the invertible objects of $\mathcal{Z}(\mathcal{C})$. If \mathcal{B} is degenerate, its Müger center is a nontrivial symmetric category, and we are done. If \mathcal{B} is non-degenerate, then by Theorem 2.4, $\mathcal{Z}(\mathcal{C}) = \mathcal{B} \boxtimes \mathcal{B}'$, where \mathcal{B}' has no nontrivial invertible objects. It is clear that $\text{FPdim}(\mathcal{B}')$ is not divisible by one of the numbers p^2, q^2, r^2 . Say it is p^2 . Then by Theorem 2.11, all nontrivial simple objects of \mathcal{B}' have dimension qr , which is a contradiction.

Now consider the remaining case, i.e., when $\mathcal{Z}(\mathcal{C})$ has no nontrivial invertible objects. Then the dimensions of nontrivial simple objects in $\mathcal{Z}(\mathcal{C})$ are pq, pr, qr . Let X be a simple object of $\mathcal{Z}(\mathcal{C})$ of dimension qr (it is easy to see that it exists). We have the orthogonality relation

$$\sum_{Y \in \text{Irr } \mathcal{Z}(\mathcal{C})} \frac{s_{X,Y}}{d_X} d_Y = 0.$$

Hence there exists $Y_0 \in \text{Irr } \mathcal{Z}(\mathcal{C})$ of dimension pq such that $s_{X,Y_0} \neq 0$ (otherwise the left-hand side will be equal to 1 modulo r). Since $\frac{s_{X,Y_0}}{d_X}$ and $\frac{s_{X,Y_0}}{d_{Y_0}}$ are algebraic integers, we have that $\frac{s_{X,Y_0}}{pqr}$ is an algebraic integer; thus $\frac{s_{X,Y_0}}{d_X}$ is divisible by p . Now we have

$$\sum_{Y \in \text{Irr } \mathcal{Z}(\mathcal{C})} \left| \frac{s_{X,Y}}{d_X} \right|^2 = \frac{\text{FPdim}(\mathcal{C})^2}{d_X^2} = p^2. \quad (9)$$

⁸ It is easy to see that any weakly integral but not integral fusion category of dimension pqr is solvable, because in this case $p = 2$, and the category has a $\mathbb{Z}/2\mathbb{Z}$ -grading with trivial component of dimension qr . Such categories are not hard to classify, but we won't do it here.

Notice that since $s_{X,Y}$ is a sum of roots of unity, every summand on the left-hand side is a totally positive algebraic integer; the summand corresponding to $Y = \mathbf{1}$ is 1 and the summand s corresponding to $Y = Y_0$ is an algebraic integer divisible by p^2 . Thus there exists a Galois automorphism g such that $g(s) \geq p^2$. Applying g to both sides of (9), we get a contradiction, as the left-hand side is $\geq 1 + p^2$. \square

Now let us finish the proof of Theorem 9.2. We are done in the case when pqr is odd since a symmetric category of odd dimension is automatically Tannakian. So let us assume that $p = 2$. Let us prove that $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory. Assume not. Then a maximal symmetric subcategory of $\mathcal{Z}(\mathcal{C})$ is the category of super vector spaces; let $\mathcal{Z} \subset \mathcal{Z}(\mathcal{C})$ be its Müger centralizer. Clearly, \mathcal{Z} is slightly degenerate, and $\text{FPdim}(\mathcal{Z}) = 2q^2r^2$.

Assume first that \mathcal{Z} has no invertible objects outside of $\mathcal{Z}' = \text{SuperVec}$. In this case by Proposition 2.8 \mathcal{Z} contains a non-invertible simple object X of odd dimension. We must have $d_X = q$ or r , since if $d_X = qr$ then $\chi \otimes X \neq X$ would also have dimension qr . But \mathcal{Z} cannot contain two simple qr -dimensional objects since then $\text{FPdim}(\mathcal{Z}) \geq 1 + 2q^2r^2$. Thus we are done by Proposition 7.4.

Now assume that \mathcal{Z} does contain invertible objects outside of $\mathcal{Z}' = \text{SuperVec}$. In this case, consider the category \mathcal{B} spanned by the invertible objects of \mathcal{Z} of odd order. If \mathcal{B} is degenerate, it contains a Tannakian subcategory and we get a contradiction. If \mathcal{B} is non-degenerate, then by Theorem 2.4, $\mathcal{Z} = \mathcal{B} \boxtimes \mathcal{B}'$, and \mathcal{B}' is a slightly degenerate category of dimension dividing $2q^2r^2$ with no invertible objects outside of SuperVec . By the above argument, either this category must contain a simple object of dimension q or r , in which case we are done by Proposition 7.4, or $\mathcal{B}' = \text{SuperVec}$. In the latter case, $\mathcal{Z}(\mathcal{C}) = \mathcal{B} \boxtimes \hat{\mathcal{B}}$, where $\hat{\mathcal{B}}$ is a 4-dimensional integral non-degenerate braided category, hence $\mathcal{Z}(\mathcal{C})$ is pointed and there is nothing to prove. The theorem is proved. \square

Corollary 9.4. *Let H be a semisimple Hopf algebra of dimension pqr , where $p < q < r$ are primes. Then there exists a finite group G of order pqr and an exact factorization $G = KL$ of G into a product of subgroups, such that H is the split Abelian extension $H(G, K, L, 1, 1, 1) = \mathbb{C}[K] \rtimes \text{Fun}(L)$ associated to this factorization.*

Proof. By Theorem 9.2, H is group-theoretical. Thus the result follows from the following lemma.

Lemma 9.5. *Let H be a group-theoretical semisimple Hopf algebra of square-free dimension. Then H is a split Abelian extension of the form $H(G, K, L, 1, 1, 1)$.*

Proof. Since H is group theoretical, there exist a group G and a cocycle $\omega \in Z^3(G, \mathbb{C}^*)$ such that $\text{Rep}(H)$ is the group-theoretical category $\text{Vec}_{G, \omega, K, \psi}$, of (K, ψ) -biequivariant (G, ω) -graded vector spaces (here ψ is a 2-cochain on K such that $d\psi = \omega|_K$). The fiber functor on $\text{Rep}(H)$ corresponds to a module category \mathcal{M} over $\text{Vec}_{G, \omega, K, \psi}$ with one simple object. It is the category of (G, ω) -graded vector spaces which are left-equivariant under (K, ψ) and right equivariant under (L, ϕ) , for another subgroup $L \subset G$ and 2-cochain ϕ on L such that $d\phi = \omega|_L$. The condition of having one simple object implies that $KL = G$. Moreover, \mathcal{M} is the category of projective representations of the group $K \cap L$ with a certain 2-cocycle. But the group $K \cap L$ has square free order, so its Sylow subgroups are cyclic, and thus this 2-cocycle must be trivial. So the one simple object condition implies that $K \cap L = 1$, so $G = KL$ is an exact factorization.

Also, since $[\omega]_K = [\omega]_L = 1$, we find that ω represents the trivial cohomology class. Finally, if we choose $\omega = 1$, then ψ and ϕ are coboundaries, so we can choose $\psi = 1, \phi = 1$.

Thus, we have shown that both the category $\text{Rep}(H)$ and the fiber functor on it attached to H are the same as those for $H(G, K, L, 1, 1, 1)$. This implies that $H = H(G, K, L, 1, 1, 1)$, as desired. \square

9.2. Classification of semisimple Hopf algebras of dimension pq^2

In this section we classify semisimple Hopf algebras of dimension pq^2 , generalizing the results of [18,35,36].

Let p, q be distinct primes.

Proposition 9.6. (See [22].) *Every semisimple Hopf algebra H of dimension pq^2 is group-theoretical.*

Proof. By Theorem 1.6 $\text{Rep}(H)$ is either an extension or an equivariantization of a fusion category of smaller dimension.

Suppose $\text{Rep}(H)$ is an extension. Then H contains a central Hopf subalgebra K of prime dimension, and therefore it is an extension of the form

$$\mathbb{C} \rightarrow K \rightarrow H \rightarrow L \rightarrow \mathbb{C},$$

where L is a Hopf algebra with $\dim L$ being a product of two primes. If L is cocommutative then H is a Kac algebra, hence group-theoretical (see [32]). Otherwise, $\dim L = pq$, and L must be commutative by [11], so the trivial component of $\text{Rep}(H)$ is pointed of dimension pq , hence $\text{Rep}(H)$ must be pointed.

Suppose now that $\text{Rep}(H)$ is an equivariantization, i.e., $\text{Rep}(H) = \mathcal{C}^G$ for a cyclic group G of prime order. By [40, Corollary 3.6] \mathcal{C}^G is group-theoretical if and only if there is a G -invariant indecomposable \mathcal{C} -module category \mathcal{M} such that the dual category $\mathcal{C}_{\mathcal{M}}^*$ is pointed. Clearly, such a category always exists if \mathcal{C} itself is pointed (take $\mathcal{M} = \mathcal{C}$). By [13], the only non-pointed possibility for \mathcal{C} is the representation category of a non-commutative group algebra of dimension pq . But this category has a unique (and hence G -invariant) fiber functor. The dual with respect to this functor is pointed, and so $\text{Rep}(H)$ is group-theoretical. \square

Corollary 9.7. *A semisimple Hopf algebra of dimension pq^2 is either a Kac algebra, or a twisted group algebra (by a twist corresponding to the subgroup $(\mathbb{Z}/q\mathbb{Z})^2$), or the dual of a twisted group algebra.*

Proof. The situation is the same as in the proof of Lemma 9.5, except that now the group $K \cap L$ does not have to be trivial. The condition on this group is that it must have a non-degenerate 2-cocycle. The only case when this group can be nontrivial is when $K = G, L = (\mathbb{Z}/q\mathbb{Z})^2$, or $L = G, K = (\mathbb{Z}/q\mathbb{Z})^2$, which implies the statement. \square

Remark 9.8. There exist integral fusion categories of dimension pq^2 which are *not* group-theoretical, e.g., certain Tambara–Yamagami categories [44,14]. By Proposition 9.6 they are not equivalent to representation categories of semisimple Hopf algebras. A classification of such categories and another proof of Proposition 9.6 based on this classification will appear in [22].

9.3. Semisimple quasi-Hopf algebras of dimension $p^r q^s$

Proposition 9.9. *Any semisimple quasi-Hopf (in particular, Hopf) algebra of dimension $p^r q^s > 1$ (where p, q are primes) has a nontrivial 1-dimensional representation. Therefore, any semisimple Hopf algebra of dimension $p^r q^s > 1$ has a nontrivial group-like element.*

Proof. The first statement follows from Theorem 1.6 and Proposition 4.5. The second statement follows from the first one by taking the dual. \square

9.4. Simple fusion categories

Definition 9.10. A fusion category is called *simple* if it has no nontrivial proper fusion subcategories.

Clearly, a pointed fusion category is simple iff it is equivalent to $\text{Vec}_{G,\omega}$ for a cyclic group G of prime order.

Proposition 9.11. *If \mathcal{C} is a weakly group-theoretical simple fusion category which is not pointed, then $\mathcal{C} = \text{Rep}(G)$, where G is a non-abelian finite simple group.*

Proof. Consider the center $\mathcal{Z}(\mathcal{C})$. It contains a nontrivial Tannakian subcategory $\text{Rep}(G)$ with $|G| \leq \text{FPdim}(\mathcal{C})$ that maps to \mathcal{C} . If it maps to Vec , we get a G -grading on \mathcal{C} , and we are done. Otherwise, the image of $\text{Rep}(G)$ in \mathcal{C} is a nontrivial fusion subcategory. So it must be the whole \mathcal{C} , and by dimension argument the functor $\text{Rep}(G) \rightarrow \mathcal{C}$ is an equivalence, so we are done. \square

9.5. Simple categories of dimension 60

By Theorems 1.6 and 9.2, all weakly integral fusion categories of dimension < 60 are solvable. Thus, the only simple weakly integral fusion categories of dimension < 60 are the categories $\text{Vec}_{G,\omega}$ for cyclic groups G of prime order.

The goal of this subsection is to prove the following result:

Theorem 9.12. *Let \mathcal{C} be a simple fusion category of dimension 60. Then $\mathcal{C} \cong \text{Rep}(A_5)$, where A_5 is the alternating group of order 60.*

Proof. It is clear that the category \mathcal{C} is integral, since otherwise it would contain a nontrivial subcategory \mathcal{C}_{ad} , see [14, Proposition 8.27].

Consider first the case when $\mathcal{Z}(\mathcal{C})$ is not simple. In this case $\mathcal{Z}(\mathcal{C})$ contains a nontrivial subcategory \mathcal{D} of dimension ≤ 60 (if the dimension of \mathcal{D} is > 60 , we will replace \mathcal{D} with its Müger centralizer \mathcal{D}').

Let $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ be the forgetful functor. The fusion subcategory $F(\mathcal{D}) \subset \mathcal{C}$ is nontrivial, since \mathcal{C} has no nontrivial gradings. Since \mathcal{C} is simple, this means that \mathcal{D} has dimension exactly 60 and $F : \mathcal{C} \cong \mathcal{D}$ is an equivalence, so \mathcal{C} is a braided category. Clearly, \mathcal{C} contains objects of prime power dimension, since for any representation $60 = 1 + \sum n_i^2$, $n_i > 1$, some n_i has to be a prime power. Thus, \mathcal{C} cannot be non-degenerate by Corollary 7.2. Therefore, \mathcal{C} must be symmetric, i.e. $\mathcal{C} \cong \text{Rep}(G)$ for a simple group G . Since A_5 is the unique simple group of order 60, we obtain $G \cong A_5$.

Now let us assume that $\mathcal{Z}(\mathcal{C})$ is simple.

Lemma 9.13. *The dimensions of nontrivial simple objects in $\mathcal{Z}(\mathcal{C})$ are among the numbers 6, 10, 15, 30.*

Proof. Theorem 2.11(i) and Corollary 7.2 show that possible dimensions of nontrivial simple objects of $\mathcal{Z}(\mathcal{C})$ are 6, 10, 12, 15, 20, 30. Thus we just need to exclude the dimensions $60/p$, where $p = 3$ or 5 . In both cases the argument is parallel to the proof of Lemma 9.3. Namely, let X be a simple object of $\mathcal{Z}(\mathcal{C})$ of dimension $60/p$. We have the orthogonality relation

$$\sum_{Y \in \text{Irr } \mathcal{Z}(\mathcal{C})} \frac{s_{X,Y}}{d_X} d_Y = 0.$$

Hence there exists $Y_0 \in \text{Irr } \mathcal{Z}(\mathcal{C})$ of dimension divisible by p such that $s_{X,Y_0} \neq 0$ (otherwise the left-hand side will be equal to 1 modulo $15/p$). Since $\frac{s_{X,Y_0}}{d_X}$ and $\frac{s_{X,Y_0}}{d_{Y_0}}$ are algebraic integers, we have that $\frac{s_{X,Y_0}}{60}$ is an algebraic integer; thus $\frac{s_{X,Y_0}}{d_X}$ is divisible by p . The rest of the argument is word for word as in the proof of Lemma 9.3. \square

There is only one decomposition of $60 - 1 = 59$ into the sum of numbers 6, 10, 15, 30, namely $59 = 15 + 10 + 10 + 6 + 6 + 6 + 6$. It follows that the object $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ (where $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ is the induction functor) has precisely 8 simple direct summands, hence $\dim \text{Hom}(I(\mathbf{1}), I(\mathbf{1})) \geq 8$. Then [14, Proposition 5.6] implies that the category \mathcal{C} contains at least 8 simple objects. Hence \mathcal{C} contains a nontrivial simple object with the square of dimension less or equal to $\frac{59}{7} < 9$; thus this object is of dimension 2. But it is proved in [38] that an integral simple fusion category cannot contain a simple object of dimension 2.⁹ The theorem is proved. \square

Corollary 9.14. *Up to isomorphism, the only semisimple Hopf algebras of dimension 60 without nontrivial Hopf algebra quotients are the group algebra $\mathbb{C}[A_5]$ and its twisting deformation $\mathbb{C}[A_5]_J$ constructed in [39].*

Proof. Let H be such a Hopf algebra. Then $\text{Rep}(H)$ is simple and hence $\text{Rep}(H) \cong \text{Rep}(A_5)$ by Theorem 9.12. Therefore, H is a twisting deformation of $\mathbb{C}[A_5]$. By [12] twisting deformations of a group algebra $\mathbb{C}[G]$ correspond to non-degenerate 2-cocycles on subgroups of G . Each Sylow 2-subgroup of A_5 admits a unique (up to cohomological equivalence) non-degenerate 2-cocycle. All such subgroups are conjugate and so the corresponding twisting deformations are gauge equivalent and yield the example of [39]. \square

Remark 9.15. The property of a Hopf algebra having no nontrivial quotients is *stronger* than that of having no nontrivial normal Hopf subalgebras. In particular, there exist other semisimple Hopf algebras of dimension 60 without nontrivial normal Hopf subalgebras. Such are, e.g., $\mathbb{C}[A_5]_J^*$ and the example constructed in [17, 4.2].

⁹ To be more precise, the argument in [38] is for comodule categories over finite-dimensional cosemisimple Hopf algebras, but it uses only the Grothendieck ring arguments, and therefore applies verbatim to the case of fusion categories.

9.6. Non-simple fusion categories of dimension 60

Theorem 9.16. *Any fusion category \mathcal{C} of dimension 60 is weakly group-theoretical.*

Proof. If \mathcal{C} is simple, then the result follows from the previous subsection. So let us consider the case when \mathcal{C} is not simple. In this case, we may assume that \mathcal{C} is integral (otherwise \mathcal{C} is $\mathbb{Z}/2\mathbb{Z}$ graded, hence solvable).

Let $\mathcal{Z}(\mathcal{C})$ be the center of \mathcal{C} . It suffices to show that $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory.

Lemma 9.17. *$\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory.*

Proof. Recall that in $\mathcal{Z}(\mathcal{C})$ we have a standard commutative algebra $A = I(\mathbf{1})$, whose category of modules is \mathcal{C} . We may assume that A does not contain nontrivial invertible objects; otherwise \mathcal{C} is faithfully graded by a nontrivial group, and we are done.

Let $\mathcal{D} \subset \mathcal{C}$ be a nontrivial proper fusion subcategory, of codimension $1 < d < 60$ (i.e., dimension $60/d$). We claim that there exists an algebra $B \subset A$ in $\mathcal{Z}(\mathcal{C})$ of dimension d . Indeed, consider the category \mathcal{E} of pairs (X, η) , where X is an object of \mathcal{C} , and $\eta : \otimes_{\mathcal{C}, \mathcal{D}} \rightarrow \otimes_{\mathcal{D}, \mathcal{C}}$ is a functorial isomorphism satisfying the hexagon relation (where $\otimes_{\mathcal{C}, \mathcal{D}}, \otimes_{\mathcal{D}, \mathcal{C}}$ are the tensor product functors $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C}, \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{C}$). In other words, \mathcal{E} is the dual category to $\mathcal{C} \boxtimes \mathcal{D}^{op}$ with respect to the module category \mathcal{C} . Thus, we have a diagram of tensor functors $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{E} \rightarrow \mathcal{C}$, whose composition is the standard forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Denote the functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{E}$ by F . This functor is surjective, as it is dual to the inclusion $\mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{D}^{op}$. Let F^\vee be the adjoint functor to F . Then $B = F^\vee(\mathbf{1}) \subset A$ is the desired subalgebra.

The existence of the algebra B implies that $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ contains a simple object of prime power dimension. Indeed, if not, then the dimensions of simple objects can be 1, 6, 10, 12, 15, 20, 30. On the other hand, the dimension of B is some divisor d of 60, $1 < d < 60$. Thus, we have $d - 1 = \sum n_i$, where $n_i = 6, 10, 12, 15, 20, 30$. It is checked by inspection that this is impossible. \square

Consider the nontrivial symmetric category contained in $\mathcal{Z}(\mathcal{C})$. If its dimension is bigger than 2, it contains a nontrivial Tannakian subcategory, and we are done. So it remains to consider the case when the only nontrivial symmetric subcategory of $\mathcal{Z}(\mathcal{C})$ is SuperVec. We make this assumption in the remainder of the section.

Let \mathcal{Z} be the Müger centralizer of the subcategory SuperVec. Proposition 2.6(i) implies that if \mathcal{Z} contains a simple object of odd prime power dimension, then it contains a nontrivial Tannakian subcategory. Let us now consider the case of even prime power dimension.

Lemma 9.18.

- (i) *If \mathcal{Z} contains a 2-dimensional simple object X , then it contains a nontrivial Tannakian subcategory.*
- (ii) *If \mathcal{Z} contains a 4-dimensional simple object X , then it contains a nontrivial Tannakian subcategory.*

Proof. (i) Let \mathcal{B} be the category spanned by the invertible objects of \mathcal{Z} . By Proposition 2.6(ii), $\mathcal{B} = \text{SuperVec} \boxtimes \mathcal{B}_0$, where \mathcal{B}_0 is a non-degenerate pointed category. Then $\mathcal{Z} = \mathcal{B}_0 \boxtimes \mathcal{B}'_0$, and \mathcal{B}'_0

contains only two invertible objects, $\mathbf{1}$ and χ (the generator of SuperVec). Clearly, $X \otimes X^* \in \mathcal{B}'_0$, and $\chi \otimes X \neq X$ by Proposition 2.6(i), so $X \otimes X^* = \mathbf{1} \oplus Y$, where Y is 3-dimensional. Thus we are done by Proposition 7.4.

(ii) Arguing as in (i), we see that $X \otimes X^* = \mathbf{1} + \cdots$, where \cdots is a direct sum of simple objects of \mathcal{Z} of dimension > 1 . Moreover, at least one of these dimensions must be odd, since the total is 15. If there is an object of dimension 3 or 5, then we are done by Proposition 7.4. Otherwise, 15 is the smallest odd dimension that can occur (by Theorem 2.11), so we must have $X \otimes X^* = \mathbf{1} \oplus Y$, where Y has dimension 15. Then we would have $s_{XX^*} = \lambda + 15\mu$, where λ, μ are roots of unity. On the other hand, s_{XX^*} is divisible by $d_X = 4$. Thus $\lambda - \mu$ is divisible by 4. So $\lambda - \mu = 0$, and X projectively centralizes its dual, hence itself. Thus Y centralizes itself, so it generates a symmetric category, which must contain a nontrivial Tannakian subcategory. \square

Lemma 9.18 shows that it suffices to prove that \mathcal{Z} contains a simple object of prime power dimension; then we will be done by Proposition 7.4.

To show this, let $A_+ \subset A$ be $\chi \otimes K$, where K is the kernel of $c^2 - 1$ (squared braiding minus one) on $\chi \otimes A$. Then A_+ is a subalgebra of A contained in \mathcal{Z} , and by the argument at the end of the proof of Lemma 9.17, A_+ contains a simple object of prime power dimension. The theorem is proved. \square

10. Relation with previous results on classification of semisimple Hopf algebras and fusion categories

1. Theorem 1.6 for $s = 0$, i.e. for fusion categories of prime power dimension, follows from [14], where it is shown that any such category is cyclically nilpotent. For $r = s = 1$ (i.e. for fusion categories of dimension pq), Theorem 1.6 follows from the paper [13], where such categories are classified (we note that the main results of [11] and [13] are trivial consequences of Theorem 1.6).

2. Semisimple Hopf algebras of small dimension were studied extensively in the literature, see [33, Table 1] for the list of references. In particular, in the monograph [33] it is shown that semisimple Hopf algebras of dimension < 60 are either upper or lower semisolvable up to a cocycle twist, and in [37] it is shown that any semisimple Hopf algebra of dimension < 36 is group-theoretical (this is not true for dimension 36, see [40]). Our Theorems 1.6 and 9.2 along with [8] further describe the structure (of representation categories) of such Hopf algebras in group-theoretical terms.

Numerous classification results and nontrivial examples of Hopf algebras of dimension pq^n are obtained in [25, 11, 16, 18, 33–36, 21]. Some of these results use an assumption of Hopf algebras involved being of Frobenius type. Our Theorem 1.5 shows that this assumption is always satisfied.

Semisimple Hopf algebras of dimension pqr we studied in [1, 34]. In particular, Hopf algebras of dimension 30 and 42 were classified as Abelian extensions (Kac algebras). Our Corollary 9.4 is a generalization of these results. It implies that [34, Theorem 4.6] can be used to obtain a complete classification of such Hopf algebras.

11. Questions

We would like to conclude the paper with two questions.

Question 1. Does there exist a fusion category that does not have the strong Frobenius property?

Question 2. Does there exist a weakly integral fusion category which is not weakly group-theoretical?

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