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Zero forcing parameters and minimum rank problems[☆]

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The zero forcing number $\mathrm{Z}(G)$, which is the minimum number of vertices in a zero forcing set of a graph G, is used to study the maximum nullity/minimum rank of the family of symmetric matrices described by G. It is shown that for a connected graph of order at least two, no vertex is in every zero forcing set. The positive semidefinite zero forcing number $\mathrm{Z}_+(G)$ is introduced, and shown to be equal to $|G| - \mathrm{OS}(G)$, where $\mathrm{OS}(G)$ is the recently defined ordered set number that is a lower bound for minimum positive semidefinite rank. The positive semidefinite zero forcing number is applied to the computation of positive semidefinite minimum rank of certain graphs. An example of a graph for which the real positive symmetric semidefinite minimum rank is greater than the complex Hermitian positive semidefinite minimum rank is presented.

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Positive semidefinite maximum nullity
Positive semidefinite minimum rank
Ordered set number

1. Introduction

The *minimum rank problem* for a (simple) graph asks for the determination of the minimum rank among all real symmetric matrices with the zero-nonzero pattern of off-diagonal entries described by a given graph (the diagonal of the matrix is free); the maximum nullity of the graph is the maximum nullity over the same set of matrices. This problem arose from the study of possible eigenvalues of real symmetric matrices described by a graph and has received considerable attention over the last ten years (see [7] and references therein). There has also been considerable interest in the related *positive semidefinite minimum rank problem*, where the minimum rank is taken over (real or complex Hermitian) positive semidefinite matrices described by a graph (see, for example, [4,6,10,12,13,15]).

Zero forcing sets and the zero forcing number were introduced in [1]. The zero forcing number is a useful tool for determining the minimum rank of structured families of graphs and small graphs, and is motivated by simple observations about null vectors of matrices. The zero forcing process is the same as graph infection used by physicists to study control of quantum systems [5], and the zero forcing number is becoming a graph parameter of interest in its own right.

A *graph* $G = (V_G, E_G)$ means a simple undirected graph (no loops, no multiple edges) with a finite nonempty set of vertices V_G and edge set E_G (an edge is a two-element subset of vertices). All matrices discussed are Hermitian; the set of real symmetric $n \times n$ matrices is denoted by S_n and the set of (possibly complex) Hermitian $n \times n$ matrices is denoted by H_n. For $A \in H_n$, the *graph* of A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i,j\} : a_{ij} \neq 0, 1 \leqslant i < j \leqslant n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The study of minimum rank has focused on real symmetric matrices (or in some cases, symmetric matrices over a field other than the real numbers), whereas much of the work on positive semidefinite minimum rank involves (possibly complex) Hermitian matrices. Whereas it is well known that using complex Hermitian matrices can result in a lower minimum rank than using real symmetric matrices, one of the issues in the study of minimum positive semidefinite rank has been whether or not using only real matrices or allowing complex matrices matters to minimum positive semidefinite rank. Example 4.1 below shows that complex Hermitian positive semidefinite minimum rank can be strictly lower than real symmetric positive semidefinite minimum rank.

Let G be a graph. The *set of real symmetric matrices described by G* is

$$\mathcal{S}(G) = \{A \in S_n : \mathcal{G}(A) = G\}.$$

The *minimum rank* of G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}$$

and the *maximum nullity of G* is

$$\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}.$$

Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices of G. The *set of real positive semidefinite matrices described by G* and the *set of Hermitian positive semidefinite matrices described by G* are, respectively,

$$\mathcal{S}_+(G) = \{A \in S_n : \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite}\}$$
$$\mathcal{H}_+(G) = \{A \in H_n : \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite}\}.$$

The *minimum positive semidefinite rank* of G and *minimum Hermitian positive semidefinite rank* of G are, respectively,

$$\mathrm{mr}_+^{\mathbb{R}}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}_+(G)\} \text{ and } \mathrm{mr}_+^{\mathbb{C}}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{H}_+(G)\}.$$

The *maximum positive semidefinite nullity of G* and the *maximum Hermitian positive semidefinite nullity of G* are, respectively,

$$\mathrm{M}_+^{\mathbb{R}}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}_+(G)\} \text{ and } \mathrm{M}_+^{\mathbb{C}}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{H}_+(G)\}.$$

Clearly $\mathrm{mr}_+^{\mathbb{R}}(G) + \mathrm{M}_+^{\mathbb{R}}(G) = |G|$ and $\mathrm{mr}_+^{\mathbb{C}}(G) + \mathrm{M}_+^{\mathbb{C}}(G) = |G|$. There are a variety of symbols in the literature (see, for example, [4,15]) for these parameters, including $\mathrm{msr}(G)$ and $\mathrm{hmr}_+(G)$ for what we denote by $\mathrm{mr}_+^{\mathbb{C}}(G)$. Clearly $\mathrm{M}_+^{\mathbb{R}}(G) \leqslant \mathrm{M}(G)$ and $\mathrm{mr}(G) \leqslant \mathrm{mr}_+^{\mathbb{R}}(G)$ for every graph G, and it is well known that these inequalities can be strict (for example, any tree T that is not a path has $\mathrm{mr}(T) < \mathrm{mr}_+^{\mathbb{R}}(T)$).

We need some additional graph terminology. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element sets from V that are not in E. We denote the complete graph on n vertices by K_n; a complete graph is also called a clique. The *degree* of vertex v in graph G is the number of edges incident with v, and the minimum degree of the vertices of G is denoted by $\delta(G)$. A set of subgraphs of G, each of which is a clique and such that every edge of G is contained in at least one of these cliques, is called a *clique covering* of G. The *clique covering number* of G, denoted by $\mathrm{cc}(G)$, is the smallest number of cliques in a clique covering of G.

Observation 1.1 ([7]). *For every graph G, $\mathrm{mr}_+^{\mathbb{R}}(G) \leqslant \mathrm{cc}(G)$, so $|G| - \mathrm{cc}(G) \leqslant \mathrm{M}_+^{\mathbb{R}}(G)$.*

For an $n \times n$ matrix A and $W \subseteq \{1, \ldots, n\}$, the principal submatrix $A[W]$ is the submatrix of A lying in the rows and columns that have indices in W. For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The induced subgraph $\mathcal{G}(A)[W]$ of the graph of A is naturally associated with the graph of the the principal submatrix for W, i.e., $\mathcal{G}(A[W])$. The subgraph induced by $\overline{W} = V_G \backslash W$ is usually denoted by $G - W$, or in the case W is a singleton $\{v\}$, by $G - v$.

The *path cover number* $\mathrm{P}(G)$ of G is the smallest positive integer m such that there are m vertex-disjoint induced paths $P_1, \ldots, P_m$ in G that cover all the vertices of G (i.e., $V_G = \dot{\bigcup}_{i=1}^{m} V_{P_i}$). A graph is *planar* if it can be drawn in the plane without crossing edges. A graph is *outerplanar* if it has such a drawing with a face that contains all vertices. Given two graphs G and H, the *Cartesian product* of G and H, denoted $G \,\square\, H$, is the graph whose vertex set is the Cartesian product of V_G and V_H, with an edge between two vertices exactly when they are identical in one coordinate and adjacent in the other.

Let $G = (V_G, E_G)$ be a graph. A subset $Z \subseteq V_G$ defines an initial set of black vertices (with all the vertices not in Z white), called a *coloring*. There are no constraints on permissible colorings; instead there are constraints on how new colorings can be derived. The *color change rule* (for the zero forcing number) is to change the color of a white vertex w to black if w is the unique white neighbor of a black vertex u; in this case we say *u forces w* and write $u \to w$. Given a coloring of G, the *derived set* is the set of black vertices obtained by applying the color change rule until no more changes are possible. A *zero forcing set* for G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived set is V_G. The *zero forcing number* $\mathrm{Z}(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V_G$.

Theorem 1.2 ([1, Proposition 2.4]). *For any graph G, $\mathrm{M}(G) \leqslant \mathrm{Z}(G)$.*

Suppose $S = (v_1, v_2, \ldots, v_m)$ is an ordered subset of vertices from a given graph G. For each k with $1 \leqslant k \leqslant m$, let G_k be the subgraph of G induced by $\{v_1, v_2, \ldots, v_k\}$, and let H_k be the connected component of G_k that contains v_k. If for each k, there exists a vertex w_k that satisfies: $w_k \neq v_l$ for $l \leqslant k$, $\{w_k, v_k\} \in E$, and $\{w_k, v_s\} \notin E$ for all v_s in H_k with $s \neq k$, then S is called an *ordered set of vertices* in G, or an OS-set. As defined in [10], the *OS number* of a graph G, denoted by $\mathrm{OS}(G)$, is the maximum of $|S|$ over all OS-sets S of G.

Theorem 1.3 ([10, Proposition 3.3]). *For any graph G, $\mathrm{OS}(G) \leqslant \mathrm{mr}_+^{\mathbb{C}}(G)$.*

In Section 2 we establish several properties of zero forcing, including the non-uniqueness of zero forcing sets. In Section 3 we introduce the positive semidefinite zero forcing number as an upper bound for maximum positive semidefinite nullity, show that the sum of the positive semidefinite zero forcing number and the OS number is the order of the graph, and apply the positive semidefinite zero forcing number to the computation of positive semidefinite minimum rank. Section 4 provides the first example showing that $\mathrm{mr}_+^{\mathbb{R}}(G)$ and $\mathrm{mr}_+^{\mathbb{C}}(G)$ need not be the same (described as unknown in [7]).

2. Properties of the zero forcing number

In this section, we establish several properties of zero forcing, including the non-uniqueness of zero forcing sets and the relationship of zero forcing number to path cover number. We need some additional definitions related to the zero forcing number.

Definition 2.1. A *minimum zero forcing set* is a zero forcing set Z such that $|Z| = \mathrm{Z}(G)$.

Zero forcing chains of digraphs were defined in [2]. We give an analogous definition for graphs.

Definition 2.2. Let Z be a zero forcing set of a graph G.

- Construct the derived set, recording the forces in the order in which they are performed. This is the *chronological list of forces*.
- A *forcing chain* (for this particular chronological list of forces) is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, $v_i \to v_{i+1}$.
- A *maximal forcing chain* is a forcing chain that is not a proper subsequence of another zero forcing chain.

Note that a zero forcing chain can consist of a single vertex (v_1), and such a chain is maximal if $v_1 \in Z$ and v_1 does not perform a force.

As noted in [1], the derived set of a given set of black vertices is unique; however, a chronological list of forces (of one particular zero forcing set) usually is not. At Rocky Mountain Discrete Mathematics Days held September 12–13, 2008 at the University of Wyoming, the following questions were raised.

Question 2.3. Is there a graph that has a unique minimum zero forcing set?

Question 2.4. Is there a graph G and a vertex $v \in V_G$ such that v is in every minimum zero forcing set?

We show the answers to both these questions are negative for nontrivial connected graphs.

Definition 2.5. Let Z be a zero forcing set of a graph G. A *reversal* of Z is the set of last vertices of the maximal zero forcing chains of a chronological list of forces.

Each vertex can force at most one other vertex and can be forced by at most one other vertex, so the maximal forcing chains are disjoint, and the elements of Z are the initial vertices of the maximal forcing chains. Thus the cardinality of a reversal of Z is the same as the cardinality of Z.

Theorem 2.6. *If Z is a zero forcing set of G then so is any reversal of Z.*

Proof. Write the chronological list of forces in reverse order, reversing each force (call this the reverse chronological list of forces) and let the reversal of Z for this list be denoted W. We show the reverse chronological list of forces is a valid list of forces for W. Consider the first "force" $u \to v$ on the reverse chronological list. All neighbors of u except v must be in W, since when the last force $v \to u$

of Z was done, each of them had the white neighbor u and thus did not force any vertex previously (in the original chronological list of forces). Thus $u \to v$ is a valid force for W. Continue in this manner or use induction on $|G|$. $\square$

Corollary 2.7. *No connected graph of order greater than one has a unique minimum zero forcing set.*

Lemma 2.8. *Let G be a connected graph of order greater than one and let Z be a minimum zero forcing set. Every $z \in Z$ has a neighbor $w \notin Z$.*

Proof. Suppose not. Then there is a vertex $z \in Z$ such that every neighbor of z is in Z (and z does have at least one neighbor v). Since z cannot perform a force, z is in the reversal W of Z. Using the reversed maximal forcing chains, no neighbor of z performs a force. So $W \backslash \{z\}$ is a zero forcing set of smaller cardinality, because after every vertex except z is black, v can force z. $\square$

Theorem 2.9. *If G is a connected graph of order greater than one, then*

$$\bigcap\nolimits_{Z \in ZFS(G)} Z = \emptyset,$$

where $ZFS(G)$ is the set of all minimum zero forcing sets of G.

Proof. Suppose not. Then there exists $v \in \cap_{Z \in ZFS(G)} Z$. In particular, for each Z and each reversal W of Z, v is in both Z and W. This means that there is a maximal forcing chain consisting of only v, or in other words v does not force any other vertex.

Let Z be a zero forcing set. If there is no chronological list of forces in which a neighbor of v performs a force, then replace Z by its reversal (since, by Lemma 2.8, v originally had a white neighbor u, in the reversal u performs a force). Let $u \to w$ be the first force in which the forcing vertex u is a neighbor of v. We claim that $Z \backslash \{v\} \cup \{w\}$is a zero forcing set for G. The forces can proceed until u is encountered as a forcing vertex. At that time, replace $u \to w$ by $u \to v$, and then continue as in the original chronological list of forces. $\square$

Next we show that for any graph the zero forcing number is an upper bound for the path cover number.

Proposition 2.10. *For any graph G, $\mathrm{P}(G) \leqslant \mathrm{Z}(G)$.*

Proof. Let Z be a zero forcing set. The vertices in a forcing chain induce a path in G because the forces in a forcing chain occur chronologically in the order of the chain (since only a black vertex can force). The maximal forcing chains are disjoint, contain all the vertices of G, and the elements of the set Z are the initial vertices of the maximal forcing chains. Thus $\mathrm{P}(G) \leqslant |Z|$. By choosing a minimum zero forcing set Z, $\mathrm{P}(G) \leqslant \mathrm{Z}(G)$. $\square$

In [14] it was shown that for a tree T, $\mathrm{P}(T) = \mathrm{M}(T)$, and in [1] it was shown that for a tree, $\mathrm{P}(T) = \mathrm{Z}(T)$ (and thus $\mathrm{M}(T) = \mathrm{Z}(T)$). In [3] it was shown that for graphs in general, $\mathrm{P}(G)$ and $\mathrm{M}(G)$ are not comparable. However, Sinkovic has established the following relationship for outerplanar graphs: If G is an outerplanar graph, then $\mathrm{M}(G) \leqslant \mathrm{P}(G)$ [16]. The next example shows that neither outerplanar graphs nor 2-trees require $\mathrm{M}(G) = \mathrm{Z}(G)$ or $\mathrm{P}(G) = \mathrm{Z}(G)$ (a 2-tree is constructed inductively by starting with a K_3 and connecting each new vertex to 2 adjacent existing vertices).

Example 2.11. Let G_{12} be the graph shown in Fig. 2.1, called the pinwheel on 12 vertices. Note that G_{12} is an outerplanar 2-tree. The set $\{1, 2, 6, 10\}$ is a zero forcing set for G_{12}, so $\mathrm{Z}(G_{12}) \leqslant 4$. We show that $\mathrm{Z}(G_{12}) \geqslant 4$, which implies $\mathrm{Z}(G_{12}) = 4$. Suppose to the contrary that Z is a zero forcing set for G_{12} and $|Z| = 3$. To start the forcing, at least two of the vertices must be in one of the sets $\{1, 2, 3\}, \{7, 8, 9\}, \{10, 11, 12\}$; without loss of generality, assume that two or three black vertices are in $\{1, 2, 3\}$. Then after several forces the vertices $\{1, 2, 3, 4, 5\}$ are black, and at most one additional vertex $v \notin \{1, 2, 3, 4, 5\}$ is in Z. To perform another force with only one more black vertex v,

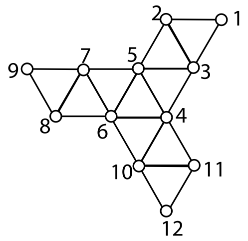

Fig. 2.1. The graph G_{12} for Example 2.11, the pinwheel on 12 vertices.

either 6 or 7 must be black, and 5 can force the other, but then no additional forces can be performed, so Z was not a zero forcing set for G_{12}. Clearly G_{12} can be covered by 9 triangles, so $\mathrm{cc}(G_{12}) \leqslant 9$ and $\mathrm{M}_+^{\mathbb{R}}(G_{12}) \geqslant 3$, by Observation 1.1. It is easy to find a path covering of three paths, so $\mathrm{M}_+^{\mathbb{R}}(G_{12}) = \mathrm{M}(G_{12}) = \mathrm{P}(G_{12}) = 3$ and $\mathrm{mr}_+^{\mathbb{R}}(G_{12}) = \mathrm{mr}(G_{12}) = \mathrm{cc}(G_{12}) = 9$. Since G_{12} is chordal, $\mathrm{mr}_+^{\mathbb{C}}(G_{12}) = \mathrm{cc}(G_{12})$ [4], and thus $\mathrm{M}_+^{\mathbb{C}}(G_{12}) = 3$.

3. The positive semidefinite zero forcing number

In this section, we introduce the positive definite zero forcing number, relate it to maximum positive semidefinite nullity and to the OS number, and apply it to compute maximum positive semidefinite nullity of several families of graphs. The definitions and terminology for zero forcing (coloring, derived set, etc.) are the same as for the zero forcing number $\mathrm{Z}(G)$, but the color change rule is different.

Definition 3.1

- The *positive semidefinite color change rule* is:
  Let B be the set consisting of all the black vertices. Let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$ (note that it is possible that $k = 1$). Let $w \in W_i$. If $u \in B$ and w is the only white neighbor of u in $G[W_i \cup B]$, then change the color of w to black.
- The *positive semidefinite zero forcing number of a graph* G, denoted by $\mathrm{Z}_+(G)$, is the minimum of $|X|$ over all positive semidefinite zero forcing sets $X \subseteq V_G$ (using the positive semidefinite color change rule).

Forcing using the positive semidefinite color change rule can be thought of as decomposing the graph into a union of certain induced subgraphs and using ordinary zero forcing on each of these induced subgraphs. The application of the positive semidefinite color change rule is illustrated in the next example.

Example 3.2. Let T be a tree. Then $\mathrm{Z}_+(T) = 1$, because any one vertex v is a positive semidefinite zero forcing set. Formally, this can be established by induction on $|T|$: If v is a leaf, it forces its neighbor; if not, a decomposition takes place. In either case smaller tree(s) are obtained. It has been known for a long time (see, for example, [7]) that $\mathrm{M}_+^{\mathbb{C}}(T) = 1$, but the use of Z_+ provides an easy proof of this result, because $\mathrm{M}_+^{\mathbb{C}}(T) = 1$ is an immediate consequence of $\mathrm{Z}_+(T) = 1$ by Theorem 3.5 below.

Observation 3.3. *Since any zero forcing set is a positive definite zero forcing set,*

$$\mathrm{Z}_+(G) \leqslant \mathrm{Z}(G).$$

Example 3.4. The pinwheel G_{12} shown in Figure 2.1 has $\mathrm{Z}_+(G_{12}) = 3 = \mathrm{M}_+^{\mathbb{R}}(G_{12})$ because $X = \{4, 5, 6\}$ is a positive semidefinite zero forcing set ($G_{12} - X$ is disconnected, and X is a zero forcing set for $G[\{1, 2, 3, 4, 5, 6\}]$, etc.).

For any graph G that is the disjoint union of connected components G_i, $i = 1, 2, \ldots, k$, $Z_+(G) = \sum_{i=1}^{k} Z_+(G_i)$ (the analogous results for M, $M_+^{\mathbb{R}}$, $M_+^{\mathbb{C}}$ and Z are all well known).

Theorem 3.5. *For any graph* G, $M_+^{\mathbb{C}}(G) \leqslant Z_+(G)$.

Proof. Let $A \in \mathcal{H}_+(G)$ with null $A = M_+^{\mathbb{C}}(G)$. Let $\mathbf{x} = [x_i]$ be a nonzero vector in ker A. Define B to be the set of indices u such that $x_u = 0$ and let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$. We claim that in $G[B \cup W_i]$, $w \in W_i$ cannot be the unique neighbor of any vertex $u \in B$. Once the claim is established, if X is a positive semidefinite zero forcing set for G, then the only vector in ker A with zeros in positions indexed by X is the zero vector, and thus $M_+^{\mathbb{C}}(G) \leqslant Z_+(G)$.

To establish the claim, renumber the vertices so that the vertices of B are last, the vertices of W_1 are first, followed by the vertices of W_2, etc. Then A has the block form

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 & C_1^* \\ 0 & A_2 & \cdots & 0 & C_2^* \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & A_k & C_k^* \\ C_1 & C_2 & \cdots & C_k & D \end{bmatrix}.$$

Partition $\mathbf{x}$ conformally as $\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_k^T, 0]^T$, and note that all entries of $\mathbf{x}_i$ are nonzero, $i = 1, \ldots, k$. Then $A\mathbf{x} = 0$ implies $A_i\mathbf{x}_i = 0$, $i = 1, \ldots, k$. Since A is positive semidefinite, each column in C_i^* is in the span of the columns of A_i by the column inclusion property of Hermitian positive semidefinite matrices [8]. That is, for $i = 1, \ldots, k$, there exists Y_i such that $C_i^* = A_iY_i$. Thus $C_i\mathbf{x}_i = Y_i^*A_i\mathbf{x}_i = 0$, and $w \in W_i$ cannot be the unique neighbor in W_i of any vertex $u \in B$. □

Theorem 3.5 is also a consequence of Theorem 3.6 below and Theorem 1.3 above, but using that as a justification obscures the motivation for the definition and the connection between zero forcing and null vectors that is given in the short direct proof.

In [15, Theorem 2.10] it is shown that $|G| - Z(G) \leqslant OS(G)$. A similar method can be used to show a more precise relationship between Z_+ and the OS number.

Theorem 3.6. *For any graph* $G = (V, E)$ *and any ordered set* S, $V \backslash S$ *is a positive semidefinite forcing set for* G, *and for any positive semidefinite forcing set* X *for* G, *there is an order that makes* $V \backslash X$ *an ordered set for* G. *Thus* $Z_+(G) + OS(G) = |G|$.

Proof. Let X be a positive semidefinite zero forcing set for G such that $|X| = Z_+(G)$. Let v_i be the vertex colored black by the ith application of the positive semidefinite color change rule. We show that $S = (v_t, v_{t-1}, \ldots, v_1)$ is an OS set for G, where $t = |G| - Z_+(G)$. Further define $X_0 = X$, and $X_{i+1} = X_i \cup \{v_{i+1}\}$, for $i = 0, 1, \ldots, t-1$. For each v_i, since it was initially white and then colored black on the ith application of the positive semidefinite color change rule, there exists a vertex $w_i \in X_i$ (the current black vertices) such that v_i is the only neighbor in the subgraph of G induced by $X_i \cup H_1$, where the subgraph $G \backslash X_i$ has components $H_1, H_2, \ldots H_p$ with $v_i \in H_1$. Since X is a positive semidefinite zero forcing set, no other vertex from the set $\{v_{i+1}, v_{i+2}, \ldots, v_t\}$ (the remaining white vertices) can be in H_1 and be a neighbor of w_i. Hence the set $(v_t, v_{t-1}, \ldots, v_1)$ is an OS-set. Therefore $t \leqslant OS(G)$. Thus

$$|G| - Z_+(G) \leqslant OS(G). \tag{3.1}$$

For the converse, we use the fact that if $S = (v_1, v_2, \ldots, v_m)$ is an OS set, then the set $S \backslash \{v_m\}$ is also an OS set. Suppose $S = (v_1, v_2, \ldots, v_m)$ is an OS set with $|S| = OS(G)$. Then we claim that $V \backslash S$ is a positive semidefinite zero forcing set. So color the vertices in $V \backslash S$ black, and suppose the subgraph G_m induced by the vertices of $\{v_1, \ldots, v_m\}$ has components induced by $U_1, U_2, \ldots, U_\ell$. Let $v_m \in U_1$. Since S is an OS-set there exists a vertex $w_m \in V \backslash S$ such that $\{w_m, v_m\} \in E$ and $\{w_m, v_s\} \notin E$ for all other $v_s \in U_1$. This implies that v_m can be colored black under the positive semidefinite color change rule.

Since $S\backslash\{v_m\}$ is also an OS-set for G, we may continue this argument and deduce that $V\backslash S$ is a positive semidefinite zero forcing set. Hence

$$|G| - \mathrm{OS}(G) = |V\backslash S| \geqslant Z_+(G), \tag{3.2}$$

as the positive semidefinite zero forcing number is defined as a minimum over all such zero forcing sets. From (3.1) and (3.2), $Z_+(G) + \mathrm{OS}(G) = |G|$. □

Corollary 3.7. *For every graph G,*

$$\delta(G) \leqslant \mathrm{Z}_+(G).$$

Proof. By [15, Corollary 2.19], $\mathrm{OS}(G) \leqslant |G| - \delta(G)$. Combining this with Theorem 3.6 gives the result. □

Another consequence of Theorem 3.6 is that there are examples of graphs for which Z_+ may not be equal to $\mathrm{M}_+^{\mathbb{C}}$. For example, in [15] it was shown that the Möbius Ladder on 8 vertices, sometimes denoted by ML_8 or V_8, satisfies $\mathrm{OS}(ML_8) = 4$ and $\mathrm{mr}_+^{\mathbb{C}}(ML_8) = 5$. In this case, by Theorem 3.6, it follows that $\mathrm{Z}_+(ML_8) = 4$, and hence $\mathrm{Z}_+(ML_8) > 3 = \mathrm{M}_+^{\mathbb{C}}(ML_8)$.

In [1], the zero forcing number was used to establish the minimum rank/maximum nullity of numerous families of graphs. The positive semidefinite zero forcing number is equally effective. Here we apply it to two families of graphs. The set of vertices associated with (the same) positive semidefinite zero forcing set in each copy of G is a positive semidefinite zero forcing set for $G \,\square\, H$.

Proposition 3.8. *For all graphs G and H,* $\mathrm{Z}_+(G \,\square\, H) \leqslant \min\{\mathrm{Z}_+(G)|H|, \mathrm{Z}_+(H)|G|\}$.

Corollary 3.9. *If T is a tree and G is a graph, then* $\mathrm{Z}_+(T \,\square\, G) \leqslant |G|$.

Theorem 3.10. *If T is a tree of order at least two, then* $\mathrm{M}_+^{\mathbb{R}}(T \,\square\, K_r) = \mathrm{M}_+^{\mathbb{C}}(T \,\square\, K_r) = \mathrm{Z}_+(T \,\square\, K_r) = r$.

Proof. Let T be a tree of order $n \geqslant 2$. By Corollary 3.9, $\mathrm{Z}_+(T \,\square\, K_r) \leqslant r$. We show $r \leqslant \mathrm{M}_+^{\mathbb{R}}(T \,\square\, K_r)$ by constructing a matrix $A \in \mathcal{S}_+(T \,\square\, K_r)$ of rank at most $(n-1)r$, and the result then follows from Theorem 3.5. The construction is by induction on n. Let P_2 denote the path on 2 vertices. To show that $\mathrm{mr}_+^{\mathbb{R}}(P_2 \,\square\, K_r) = r$, choose a nonsingular matrix $M \in \mathcal{S}_+(K_r)$ such that $M^{-1} \in \mathcal{S}_+(K_r)$ (for example, $M = I + J$, where I is the identity matrix and J is the all 1s matrix). Then $B = \begin{bmatrix} M & I \\ I & M^{-1} \end{bmatrix} \in \mathcal{S}_+(P_2 \,\square\, K_r)$ and rank B = rank $M = r$. Without loss of generality, in T vertex n is adjacent only to vertex $n-1$. We order the vertices (i,j) of $T \,\square\, K_r$ lexicographically. By the induction hypothesis, there is a matrix $C \in \mathcal{S}_+((T-n) \,\square\, K_r)$ such that rank $C = (n-2)r$; let $C' = C \oplus 0_{r\times r}$. Using $B \in \mathcal{S}_+(P_2 \,\square\, K_r)$ already constructed with rank r, let $B' = 0_{(n-2)r\times(n-2)r} \oplus B$. Then for $\alpha \in \mathbb{R}$ chosen to avoid cancellation, $A = C' + \alpha B' \in \mathcal{S}_+(T \,\square\, K_r)$ and rank $A \leqslant (n-2)r + r = (n-1)r$. □

A *book* with $m \geqslant 2$ pages, denoted B_m [9, p. 14], is m copies of a 4-cycle with one edge in common, or equivalently, $B_m = K_{1,m} \,\square\, P_2$, where $K_{1,m}$ is the complete bipartite graph with partite sets of 1 and m vertices. For $m \geqslant 2$, $t \geqslant 3$, we call m copies of a t-cycle with one edge in common a *generalized book*, denoted by B_m^t (obviously, $B_m = B_m^4$).

Proposition 3.11. *If B_m^t is a generalized book, then* $\mathrm{M}_+^{\mathbb{R}}(B_m^t) = \mathrm{M}_+^{\mathbb{C}}(B_m^t) = \mathrm{Z}_+(B_m^t) = 2$.

Proof. The two vertices in the common edge are a positive semidefinite zero forcing set, so $\mathrm{Z}_+(B_m^t) \leqslant 2$. Thus by Theorem 3.5, $\mathrm{M}_+^{\mathbb{C}}(B_m^t) \leqslant 2$. Since B_m^t is not a tree, $\mathrm{M}_+^{\mathbb{R}}(B_m^t) \geqslant 2$ [12]. □

4. Real versus complex minimum positive semidefinite rank

Clearly $\mathrm{mr}_+^{\mathbb{C}}(G) \leqslant \mathrm{mr}_+^{\mathbb{R}}(G)$ for every graph G. Previously it was not known whether $\mathrm{mr}_+^{\mathbb{C}}(G)$ could differ from $\mathrm{mr}_+^{\mathbb{R}}(G)$ [7, p. 578]. In this final section we provide an example of a graph for which these parameters are not identical.

Example 4.1. The "k-wheel with 4 hubs" (for k at least 3) is the graph on $4k+4$ vertices such that the outer cycle has $4k$ vertices, and each of the four hubs is attached to every 4th vertex of the cycle, and no others; this graph is denoted $H_4(k)$, and $H_4(3)$ is shown in Fig. 4.1. This family arose in Hall's investigation of graphs having minimum rank 3 [11]. We show $\mathrm{mr}_+^{\mathbb{C}}(\overline{H_4(3)}) = 3$ and $\mathrm{mr}_+^{\mathbb{R}}(\overline{H_4(3)}) = 4$. As numbered in Fig. 4.1, $H_4(3)$ is bipartite with partite sets consisting of the odd vertices and the even vertices. By [2, Theorem 3.1], $\mathrm{mr}_+^{\mathbb{R}}(\overline{H_4(3)}) = \mathrm{mr}^{\mathbb{R}}(Y_{\overline{H_4(3)}})$ where $Y_{\overline{H_4(3)}}$ is the biadjacency zero–nonzero pattern of $\overline{H_4(3)}$ and $\mathrm{mr}^{\mathbb{R}}(Y_{\overline{H_4(3)}})$ is the asymmetric minimum rank over the real numbers (Theorem 3.1 applies to $\overline{H_4(3)}$ because $H_4(3)$ is a bipartite graph). The same method used to prove Theorem 3.1 also shows that $\mathrm{mr}_+^{\mathbb{C}}(\overline{H_4(3)}) = \mathrm{mr}^{\mathbb{C}}(Y_{\overline{H_4(3)}})$ where $\mathrm{mr}^{\mathbb{C}}(Y_{\overline{H_4(3)}})$ is the asymmetric minimum rank over the complex numbers (in [2, Remark 3.2] it is noted that the method in Theorem 3.1 is valid for constructing a symmetric matrix over an infinite field, and the same reasoning applies to constructing a Hermitian matrix over $\mathbb{C}$ by using Hermitian adjoints in place of transposes). After scaling rows and columns, a minimum rank matrix having zero-nonzero pattern $Y_{\overline{H_4(3)}}$ has the form

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & a_{3,8} & a_{3,10} & a_{3,12} & a_{3,14} & 0 \\ 1 & 0 & 0 & a_{5,8} & a_{5,10} & a_{5,12} & 0 & a_{5,16} \\ 1 & a_{7,4} & 0 & 0 & a_{7,10} & a_{7,12} & a_{7,14} & 0 \\ 1 & a_{9,4} & a_{9,6} & 0 & 0 & a_{9,12} & 0 & a_{9,16} \\ 1 & a_{11,4} & a_{11,6} & a_{11,8} & 0 & 0 & a_{11,14} & 0 \\ 0 & 1 & 0 & a_{13,8} & 0 & a_{13,12} & a_{13,14} & a_{13,16} \\ 1 & 0 & a_{15,6} & 0 & a_{15,10} & 0 & a_{15,14} & a_{15,16} \end{bmatrix},$$

where the displayed entries a_{ij} are nonzero (real or complex) numbers. Since the principal submatrix in the first three rows and columns is nonsingular, rank $A = 3$ implies that rows 4 through 8 are linear combinations of rows 1 through 3. Computations show that the following assignments of variables are necessary:

$$\begin{gathered} a_{5,8} = (a_{3,8} - 1)\, a_{7,4}, \; a_{5,10} = (a_{3,10} - 1)\, a_{7,4} + a_{7,10}, \\ a_{5,12} = a_{3,12}a_{7,4} + a_{7,12}, \; a_{5,16} = -a_{7,4}, \\ a_{7,14} = -a_{3,14}a_{7,4}, \; a_{9,16} = a_{9,4} - a_{7,4}, \; a_{9,6} = a_{9,4}, \\ a_{9,12} = a_{3,12}a_{7,4} + a_{7,12} - a_{3,12}a_{9,4} + a_{3,12}a_{9,6}, \\ a_{7,10} = a_{7,4} - a_{3,10}a_{7,4} - a_{9,4}, \; a_{9,4} = (1 - a_{3,8})a_{7,4}, \\ a_{11,4} = a_{7,4}, \; a_{11,14} = a_{3,14}(a_{11,6} - a_{11,4}), \\ a_{7,12} = -a_{3,12}a_{11,6}, \; a_{11,8} = a_{3,8}a_{11,6}, \\ a_{3,8} = a_{3,10}(a_{7,4} - a_{11,6})/a_{7,4}, \; a_{13,16} = 1, \\ a_{13,14} = -a_{3,14}, a_{13,12} = -a_{3,12}, \; a_{3,10} = 1, \\ a_{13,8} = a_{11,6}/a_{7,4}, \; a_{15,16} = -a_{7,4}, \; a_{15,14} = a_{3,14}a_{15,6}, \\ a_{15,10} = -a_{11,6} + a_{15,6}, \; a_{11,6} = a_{7,4} + a_{15,6}. \end{gathered}$$

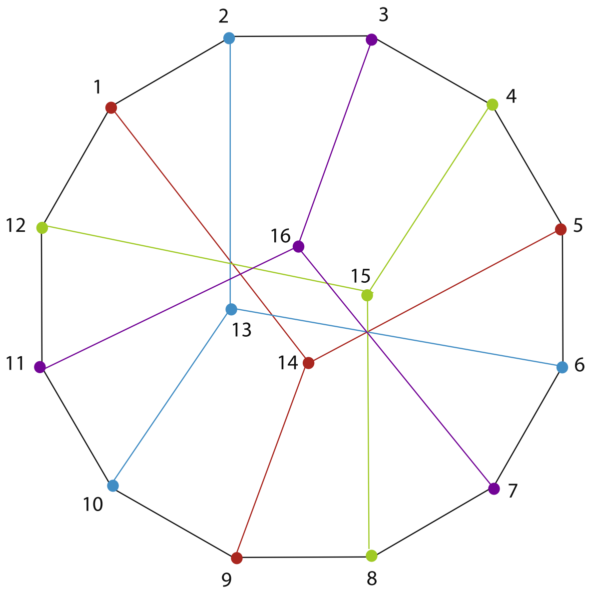

Fig. 4.1. The the 3-wheel on 4 hubs, $H_4(3)$, for Example 4.1.

After making these assignments, rows 4–7 are linear combinations of rows 1, 2, and 3, and in order for row 8 to be a linear combinations of rows 1, 2, and 3, it is necessary and sufficient that

$$1 + \frac{a_{7,4}}{a_{15,6}} + \left(\frac{a_{7,4}}{a_{15,6}}\right)^2 = 0. \tag{4.1}$$

Clearly (4.1) has a solution if and only if the field contains a primitive third root of unity. Thus $\mathrm{mr}^{\mathbb{C}}(Y_{\overline{H_4(3)}}) = 3$ whereas $\mathrm{mr}^{\mathbb{R}}(Y_{\overline{H_4(3)}}) = 4$, giving

$$\mathrm{mr}_+^{\mathbb{C}}(\overline{H_4(3)}) = 3 < 4 = \mathrm{mr}_+^{\mathbb{R}}(\overline{H_4(3)}).$$

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