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Advances in Mathematics 223 (2010) 1521–1544

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On the noncommutative Donaldson–Thomas invariants arising from brane tilings

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Received 1 April 2009; accepted 2 October 2009

Available online 11 November 2009

Communicated by Hiraku Nakajima

Abstract

Given a brane tiling, that is a bipartite graph on a torus, we can associate with it a quiver potential and a quiver potential algebra. Under certain consistency conditions on a brane tiling, we prove a formula for the Donaldson–Thomas type invariants of the moduli space of framed cyclic modules over the corresponding quiver potential algebra. We relate this formula with the counting of perfect matchings of the periodic plane tiling corresponding to the brane tiling. We prove that the same consistency conditions imply that the quiver potential algebra is a 3-Calabi–Yau algebra. We also formulate a rationality conjecture for the generating functions of the Donaldson–Thomas type invariants.

Keywords: Hilbert scheme; Brane tiling; Quiver potential algebra; Donaldson–Thomas invariant

Contents

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2009.10.001

1. Introduction

The main objective of this paper is to generalize the results of Szendrői [18] on the noncommutative Donaldson–Thomas theory in the case of the conifold to the case of quiver potentials arising from arbitrary brane tilings (see Section 3). That is, we compute the Donaldson–Thomas type invariants [1] of the moduli spaces of framed cyclic modules over the quiver potential algebra.

The Donaldson–Thomas type invariants are the weighted Euler numbers and they can therefore be computed using the localization technique, whenever we can find an appropriate torus action on the moduli space. In this paper we show that the quiver potential algebra arising from the brane tiling has a canonical grading. This grading provides us with a torus action on the moduli space, so that there is a finite number of fixed points, and we get a purely combinatorial formula that counts ideals in some poset of paths in the quiver (see Proposition 4.14).

This nice state of affairs will not be for free. We need to impose certain conditions on the brane tiling (see Conditions 4.12, 5.3), which we call the consistency conditions. These conditions should be compared with the consistency conditions arising in physics [9]. It follows from the results of [8,3,4] that our consistency conditions are satisfied under the physical consistency conditions (see also Remark 4.13).

In the presence of the consistency conditions, we construct a bijection between the set of ideals in the poset of paths mentioned above and the set of perfect matchings (having certain prescribed behavior at infinity) of the periodic plane tiling induced by the brane tiling. This bijection generalizes the folklore result on the correspondence between the 3-dimensional Young diagrams and the so-called honeycomb dimers (see Section 5). Earlier Szendrői [18] observed such correspondence in the conifold case, where it also has a visual interpretation like for the 3-dimensional Young diagrams.

Furthermore, we prove that, under the consistency conditions, the quiver potential algebra is always a 3-Calabi–Yau algebra (and actually a graded 3-Calabi–Yau algebra, when provided with a canonical grading). This does not contradict the results of Bocklandt [2, Theorem 3.1], who proved that a graded quiver potential algebra can be a graded 3-Calabi–Yau algebra only under some rather restrictive conditions (for example, the algebra $\mathbb{C}[x, y, z]$ does not satisfy those conditions) because the grading group in [2] can only be $\mathbb{Z}$ and all arrows have degree one there. We learned that Nathan Broomhead [3] recently proved the 3-Calabi–Yau property under the condition that there exists an R-charge on the brane tiling (see Remark 4.13).

Our formulas give a way to compute the noncommutative DT invariants with a computer. In the cases of the orbifolds $\mathbb{C}^3/\mathbb{Z}_n$, $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and in the case of the conifold there exist nice compact formulas due to Benjamin Young [19]. Using the results of Young and computer evidence in other cases, we formulate a rationality conjecture on the noncommutative DT invariants in Section 8. We learned from Nagao Kentaro about joint work with Hiraku Nakajima [17,16] on the proof of Young's formulas in the conifold case using wall-crossing formulas. Their technique could possibly provide a formula for general brane tilings.

The paper is organized as follows: In Section 2 we construct the moduli spaces of framed cyclic modules and describe the general localization technique to reduce the problem of computation of the Euler number of the moduli space to some combinatorial problem. In Section 3 we consider the quiver potentials associated with brane tilings and study the canonical grading of the quiver potential algebra. In Section 4 we study different equivalence relations on paths induced by a potential. In Section 5 we prove a bijection between the set of finite ideals in the poset of paths and the set of perfect matchings of the periodic plane tiling having some prescribed behavior at infinity. In Section 6 we prove that the quiver potential algebra associated to a brane tiling satisfying the consistency conditions is a 3-Calabi–Yau algebra. In Section 7 we relate the Donaldson–Thomas type invariants of moduli spaces of framed cyclic modules with their Euler numbers. In Section 8 we formulate the rationality conjecture.

We would like to thank Raf Bocklandt and Alastair King for many helpful discussions and useful comments. We would like to thank Tom Bridgeland for pointing out a mistake in Section 8 in the first version of the paper.

2. Hilbert schemes

Let $Q = (Q_0, Q_1)$ be a quiver. Here Q_0 is the set of vertices of Q, Q_1 is the set of arrows of Q, and there are two maps $s, t : Q_1 \to Q_0$ called the source and the target maps respectively. A path in Q is a sequence of arrows $a_1 \dots a_n$, such that $s(a_i) = t(a_{i+1})$ for $1 \leqslant i \leqslant n-1$. We consider also the trivial paths e_i, $i \in Q_0$. We define the path algebra $\mathbb{C}Q$ to be generated as a vector space by all paths in Q. Multiplication in $\mathbb{C}Q$ is given by

$$(a_1 \dots a_m) \cdot (b_1 \dots b_n) = \begin{cases} a_1 \dots a_m b_1 \dots b_n, & s(a_m) = t(b_1), \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $e_i \in \mathbb{C}Q$, $i \in Q_0$ are idempotents of $\mathbb{C}Q$ and that $1 = \sum_{i \in Q_0} e_i$ is a unit of $\mathbb{C}Q$. Let $I \subset \mathbb{C}Q$ be an ideal of the path algebra, and let $A = \mathbb{C}Q/I$ be the factor algebra. The A-module $P_i := Ae_i$ is a projective A-module. Let M be some A-module. For any vertex $i \in Q_0$, we define a vector space $M_i = e_i M$. For any arrow $a : i \to j$, there is a linear map $M_a : M_i \to M_j$ induced by the action of a on M. There is an isomorphism of vector spaces $M = \bigoplus_{i \in Q_0} M_i$. We define $\dim M := (\dim M_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$.

Definition 2.1. For any $i \in Q_0$, we define an i-cyclic A-module to be a pair (M, m) where M is a finite-dimensional left A-module, $m \in M_i$, and $Am = M$. The i-cyclic A-modules form a category, where morphisms $f : (M, m) \to (N, n)$ are A-module homomorphisms $f : M \to N$ such that $f(m) = n$.

It is clear that the i-cyclic A-modules correspond to the finite-dimensional quotients of P_i. We want to construct the moduli spaces of isomorphism classes of i-cyclic A-modules, which we will call the Hilbert schemes.

Let $\widehat{Q}$ be a new quiver with

$$\widehat{Q}_0 = Q_0 \cup \{*\}, \qquad \widehat{Q}_1 = Q_1 \cup \{a_* : * \to i\}.$$

Let $\widehat{I} \subset \mathbb{C}\widehat{Q}$ be the ideal generated by I. Then a module over $\widehat{A} = \mathbb{C}\widehat{Q}/\widehat{I}$ can be identified with a triple (M, V, f), where M is an A-module, V is a vector space, and $f : V \to M_i$ is a linear map.

Definition 2.2. Given an element $\theta \in \mathbb{N}^{Q_0}$, called a stability, we define a slope function

$$\mu : \mathbb{N}^{Q_0} \backslash \{0\} \to \mathbb{Q}, \qquad \alpha \mapsto \frac{\theta \cdot \alpha}{\sum_{i \in Q_0} \alpha_i}.$$

For any finite-dimensional nonzero A-module M, we define $\mu(M) := \mu(\dim M)$. The module M is called θ-semistable (resp. θ-stable) if for any its proper nonzero submodule $N \subset M$ we have $\mu(N) \leqslant \mu(M)$ (resp. $\mu(N) < \mu(M)$). The stability condition for $\widehat{A}$-modules is defined in the same way.

Given a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, we define a new dimension vector $\widehat{\alpha} = (\alpha, 1) \in \mathbb{N}^{Q_0} \times \mathbb{N} = \mathbb{N}^{\widehat{Q}_0}$. Define a stability $\widehat{\theta} = (0, \ldots, 0, 1) \in \mathbb{N}^{Q_0} \times \mathbb{N} = \mathbb{N}^{\widehat{Q}_0}$.

Lemma 2.3. *Let (M, V, f) be an $\widehat{A}$-module of dimension $\widehat{\alpha}$. Then the following conditions are equivalent*

(1) *(M, V, f) is $\widehat{\theta}$-semistable.*
(2) *(M, V, f) is $\widehat{\theta}$-stable.*
(3) *$f(V)$ generates M as an A-module.*

This lemma implies that the moduli space

$$\mathrm{Hilb}_i^{\alpha}(A) := M_{\theta}^{ss}(\widehat{A}, \widehat{\alpha})$$

of semistable $\widehat{A}$-modules [13] parametrizes the i-cyclic modules, or equivalently, the quotients of P_i of dimension α.

The goal of this paper is to study the Donaldson–Thomas type invariant of $\mathrm{Hilb}_i^{\alpha}(A)$. It is defined as a certain weighted Euler number (see e.g. [1]). We will first develop certain techniques to compute the usual Euler number of this Hilbert scheme and then prove in Section 7 that its DT-invariant differs from the Euler number just by sign (in the case of quiver potentials induced by brane tilings). This should be compared with the dimension zero MNOP conjecture [1, Theorem 4.12], [14]. We define a partition function

$$Z^i(A) = \sum_{\alpha \in \mathbb{N}^{Q_0}} \chi_c\big(\mathrm{Hilb}_i^{\alpha}(A)\big) x^{\alpha} \in \mathbb{Q}[[x_j \mid j \in Q_0]],$$

where χ_c means the Euler number of cohomology with compact support.

Let $\mathrm{wt} : Q_1 \to \Lambda$ be a map, called a weight function, to a free abelian finitely generated group Λ. The path algebra $\mathbb{C}Q$ is then automatically a Λ-graded algebra. We assume that $I \subset \mathbb{C}Q$ is a Λ-homogeneous ideal. The quotient algebra $A = \mathbb{C}Q/I$ is again a Λ-graded algebra.

We define the action of the torus $T = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$ on the Hilbert scheme $\mathrm{Hilb}_i^{\alpha}(A)$ of i-cyclic A-modules as follows. For any $t \in T$ and any i-cyclic A-module (M, m), we define $(M', m') = t(M, m)$ by $M'_i = M_i$ for $i \in Q_0$, $m' = m$ and

$$M'_a = t_a M_a, \quad \text{for } a \in Q_1,$$

where $t_a := t(\mathrm{wt}(a))$. By localization, we have

$$\chi_c\big(\mathrm{Hilb}_i^\alpha(A)\big) = \chi_c\big(\mathrm{Hilb}_i^\alpha(A)^T\big),$$

where $\mathrm{Hilb}_i^\alpha(A)^T$ is the subvariety of T-fixed points.

Theorem 2.4. *An i-cyclic A-module (M,m) in $\mathrm{Hilb}_i^\alpha(A)$ is T-fixed if and only if M possesses a Λ-grading as an A-module such that m has degree zero (such a grading is unique as m generates M).*

Proof. Let (M,m) be some T-fixed point of $\mathrm{Hilb}_{i_0}^\alpha(A)$, $i_0 \in Q_0$. For any $t \in T$ there exists $g = (g_i)_{i \in Q_0} \in \mathrm{GL}_\alpha$ such that for any arrow $a : i \to j$, we have (recall that $t_a = t(\mathrm{wt}(a))$)

$$t_a M_a = g_j M_a g_i^{-1}$$

and $g_{i_0} m = m$. Consider the subgroup $H \subset T \times \mathrm{GL}_\alpha$ of all pairs (t,g) that satisfy this condition. Then $p_1 : H \to T$ is surjective. But its kernel is trivial. It consists of pairs $(1,g)$, where g is an automorphism of M that fixes m. It follows that g acts trivially on M, as m generates M. Consider the composition

$$\psi = p_2 \circ p_1^{-1} : T \to \mathrm{GL}_\alpha$$

and split it to components $\psi_i : T \to \mathrm{GL}(M_i)$, $i \in Q_0$. Then for any arrow $a : i \to j$, we have

$$t_a M_a = \psi_j(t_a) M_a \psi_i(t_a)^{-1}$$

and $\psi_{i_0}(t)m = m$ for any $t \in T$. Using the action of T on M_i defined by ψ_i, we can decompose M_i with respect to the character group $X(T) \simeq \Lambda$ of T

$$M_i = \bigoplus_{\lambda \in \Lambda} M_{i,\lambda}.$$

Then the above condition implies that for any arrow $a : i \to j$

$$M_a(M_{i,\lambda}) \subset M_{j,\lambda+\mathrm{wt}(a)}$$

and $m \in M_{i_0,0}$. This means that M is a Λ-graded A-module and m has degree zero. The converse statement is easy. $\square$

An i-cyclic A-module (M,m) as in the above theorem, will be called a Λ-graded i-cyclic A-module. Given a path u in Q, let us define its weight $\mathrm{wt}(u) \in \Lambda$ to be the sum of the weights of arrows from u. We assume that:

(1) Any two paths in Q having the same startpoint and the same weight are proportional in A.
(2) The weight of any nontrivial path is nonzero.

The first assumption implies that the Λ-graded module P_i has dimension at most one at every degree. It follows that any Λ-graded quotient of P_i is determined by the set of weights from its support. We define $\Delta = \Delta_i$ to be the set of paths starting at i modulo an equivalence relation

$u \sim v$ if $\mathrm{wt}(u) = \mathrm{wt}(v)$. It follows from our assumptions that there is a poset structure on Δ, given by the rule $u \leqslant v$ if there exists some path w with $wu \sim v$. There is a bijection between Δ and the set of weights $\lambda \in \Lambda$ such that $P_{i,\lambda} \neq 0$.

Definition 2.5. Let Δ be a poset. A subset $\Omega \subset \Delta$ is called an ideal of Δ if whenever $x \leqslant y$ in Δ and $y \in \Omega$, we have $x \in \Omega$.

Lemma 2.6. *There is a bijection between the set of isomorphism classes of Λ-graded i-cyclic A-modules and the set of finite ideals of Δ_i given by the rule*

$$(M, m) \mapsto \{u \in \Delta_i \mid M_{\mathrm{wt}(u)} \neq 0\}.$$

In the next sections we will construct the weight functions for the quivers induced by brane tilings and investigate when the above assumptions are satisfied.

3. Brane tilings and quiver potentials

Definition 3.1. A bipartite graph $G = (G_0^+, G_0^-, G_1)$ consists of two sets of vertices G_0^+, G_0^-, called the sets of white and black vertices respectively, a set of edges G_1 and a map $G_1 \to G_0^+ \times G_0^-$. The corresponding CW-complex is also denoted by G. We denote the set of all vertices of G by $G_0 = G_0^+ \cup G_0^-$.

Definition 3.2. A brane tiling is a bipartite graph G together with an embedding of the corresponding CW-complex into the real two-dimensional torus T so that the complement $T \backslash G$ consists of simply-connected components. We identify any two homotopy equivalent embeddings. The set of connected components of $T \backslash G$ is denoted by G_2 and is called the set of faces of G.

We will always assume that the connected components of $T \backslash G$ are convex polygons. We define a quiver $Q = (Q_0, Q_1)$ dual to the brane tiling Gas follows. The set of vertices Q_0 is G_2, the set of arrows Q_1 is G_1. For any arrow $a \in Q_1$ we define its endpoints to be the polygons in G_2 adjacent to a. The direction of a is chosen in such a way that the white vertex is on the right of a. The CW-complex corresponding to Q is automatically embedded in T. The set of connected components of the complement, called the set of faces of Q, will be denoted by Q_2. It can be identified with G_0. There is a decomposition $Q_2 = Q_2^+ \cup Q_2^-$ corresponding to the decomposition $G_0 = G_0^+ \cup G_0^-$. It follows from our definition that the arrows of the face from Q_2^+ go clockwise and the arrows of the face from Q_2^- go anti-clockwise.

For any face $F \in Q_2$, we will denote by w_F the necklace (equivalence class of cycles in Q up to shift) obtained by going along the arrows of F. We define the potential of Q (see e.g. [7,2] for the relevant definitions) by

$$W = \sum_{F \in Q_2^+} w_F - \sum_{F \in Q_2^-} w_F.$$

We want to apply the results of Section 2 to the algebra $\mathbb{C}Q/(\partial W)$, which we call a quiver potential algebra.

Consider the complex

$$\mathbb{Z}^{Q_2} \xrightarrow{d_2} \mathbb{Z}^{Q_1} \xrightarrow{d_1} \mathbb{Z}^{Q_0},$$

where $d_2(F) = \sum_{a \in F} a$ for $F \in Q_2$ and $d_1(a) = s(a) - t(a)$ for $a \in Q_1$. Its homology groups are isomorphic to $H_*(T, \mathbb{Z})$. We define the group Λ as

$$\Lambda = \mathbb{Z}^{Q_1} / \langle d_2(F) - d_2(F') \mid F, F' \in Q_2 \rangle$$

and define the weight function $\mathrm{wt} : \mathbb{Z}^{Q_1} \to \Lambda$ to be the projection. It is clear that the ideal $(\partial W) \subset \mathbb{C}Q$ is automatically Λ-homogeneous. For any path u in Q, we define its content $|u| \in \mathbb{Z}^{Q_1}$ by counting the multiplicities of arrows in u. We define the weight of the path u by $\mathrm{wt}(u) := \mathrm{wt}(|u|)$. Let $\varpi := \mathrm{wt}(d_2(F))$ for some (any) $F \in Q_2$. The weight of an arrow $a \in Q_1$ will also usually be denoted by a.

Lemma 3.3. *Assume that one of the following conditions is satisfied:*

(1) *There exists at least one perfect matching of G.*
(2) *All faces of Q contain the same number of arrows.*

Then the group Λ is free.

Proof. Let us show first that $\operatorname{coker} d_2 = \mathbb{Z}^{Q_1} / \operatorname{im} d_2$ is free. Let $\lambda \in \mathbb{Z}^{Q_1}$ be such that $k\lambda \in \operatorname{im} d_2$ for some $k \geqslant 1$. We have to show that $\lambda \in \operatorname{im} d_2$. We have $d_1(k\lambda) = 0$ and therefore $d_1(\lambda) = 0$. Let $\pi : \widetilde{T} \to T$ be the universal covering of the torus T and let $\widetilde{Q}$ be the corresponding periodic quiver, which is the inverse image of Q in $\widetilde{T}$. We will consider paths in Q and $\widetilde{Q}$, consisting of arrows and their inverses (we call them weak paths). The condition $d_1(\lambda) = 0$ implies that we can construct a weak cycle (i.e. a weak path with equal source and target) u in Q with content λ (every arrow is counted with a plus sign and its inverse with a minus sign). We can lift u to some weak path $\widetilde{u}$ in $\widetilde{Q}$. The condition $k\lambda \in \operatorname{im} d_2$ implies that $\widetilde{u}$ is actually a weak cycle. Its content λ can be represented as a sum of $\pm d_2(F)$ for faces F contained inside the cycle. This means that $\lambda \in \operatorname{im} d_2$.

We note that there is an exact sequence

$$\mathbb{Z} \to \Lambda \to \operatorname{coker} d_2 \to 0,$$

where $\mathbb{Z} \to \Lambda$ is given by $1 \mapsto \varpi = \mathrm{wt}(d_2(F))$ for some $F \in Q_2$. To show that Λ is free we need to show that the first map is injective. Assume that $k\varpi = 0$ in Λ for some $k \geqslant 1$. Then $kd_2(F_0) = \sum_{F \in Q_2} y_F d_2(F)$ for some $F_0 \in Q_2$ and integers y_F, $F \in Q_2$, with $\sum_{F \in Q_2} y_F = 0$. Let $kd_2(F) = \sum_{a \in Q_1} x_a a \in \mathbb{Z}^{Q_1}$. Then all x_a, $a \in Q_1$, are nonnegative.

If there exists some perfect matching I of G then $\sum_{a \in I} x_a = \sum_{F \in Q_2} y_F$. For $a \in F_0 \cap I$, we have $x_a > 0$ and therefore $\sum_{F \in Q_2} y_F > 0$, contradicting our assumption.

If all faces of Q contain the same number of arrows, say r, then $r \sum_{a \in Q_1} x_a = \sum_{F \in Q_2} y_F$. For any $a \in F_0$, we have $x_a > 0$ and therefore $\sum_{F \in Q_2} y_F > 0$, contradicting our assumption. $\square$

Definition 3.4. A bipartite graph (dimer model) is called nondegenerate if all of its edges belong to some perfect matching.

The following result ensures that the second assumption of Section 2 is satisfied.

Lemma 3.5. *Assume that one of the following conditions is satisfied:*

(1) *The bipartite graph G is nondegenerate.*
(2) *All faces of Q contain the same number of arrows.*

Then for any $(x_a)_{a\in Q_1}\in\mathbb{N}^{Q_1}\backslash\{0\}$, we have $\sum_{a\in Q_1}x_a a\neq 0$ in Λ.

Proof. Assume that $\sum_{a\in Q_1}x_a a\in\mathbb{Z}^{Q_1}$ is zero in Λ, where all $x_a\geqslant 0$ and some of them are nonzero. This means

$$\sum x_a a=\sum_{F\in Q_2}\lambda_F d_2(F),$$

for some $\lambda_F\in\mathbb{Z}$ with $\sum\lambda_F=0$.

If G is nondegenerate, then for any $b\in Q_1$ with $x_b>0$ we can find a perfect matching I of G containing b. Then $\sum_{a\in I}x_a=\sum_{F\in Q_2}\lambda_F$ and the left-hand side is strictly positive, as $x_b>0$ and $b\in I$. This contradicts the assumption $\sum_{F\in Q_2}\lambda_F=0$.

If all faces of Q contain the same number of arrows, say r, then $r\sum_{a\in Q_1}x_a=\sum_{F\in Q_2}\lambda_F$. But the left-hand side of this equation is nonzero and this contradicts the assumption $\sum_{F\in Q_2}\lambda_F=0$. $\square$

It follows that the arrows of Q generate a strongly convex cone in $\Lambda_{\mathbb{R}}=\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$. From now on we will always assume that the bipartite graph is nondegenerate.

Remark 3.6. The main result of Ishii and Ueda [11] relies on the existence of some map $R:Q_1\to\mathbb{R}_{>0}$, such that for any face $F\in Q_2$, we have $\sum_{a\in F}R(a)=2$. If the bipartite graph G is nondegenerate, then the above lemma allows us to construct a linear map $R:\Lambda_{\mathbb{R}}\to\mathbb{R}$ such that $R(a)>0$ for every arrow a and $R(\overline{\omega})=2$. This gives a map required in [11].

4. Path groupoid of a quiver potential

By a weak path in a quiver Q we will mean a path consisting of arrows of a quiver and their inverses (for any arrow a we identify aa^{-1} and $a^{-1}a$ with trivial paths). The usual paths will be sometimes called strict paths. We define a cycle (resp. a weak cycle) to be a path (resp. a weak path) with equal endpoints. We will show that the potential W from the previous section defines an equivalence relation on the set of strict paths (it will also be called the strict equivalence relation) and on the set of weak paths (it will also be called the weak equivalence relation). It is possible that strict paths are weakly equivalent but not strictly equivalent. The goal of this section is to prove Proposition 4.8 stating that two paths with the same startpoints are weakly equivalent if and only if their weights are equal. In Lemma 4.11 we investigate when the weak equivalence of paths implies the strict equivalence. This is needed in order to satisfy the first assumption of Section 2.

Let $\mathcal{P}Q$ be the category of paths of a quiver Q (objects are vertices of Q and morphisms are paths in Q) and let $\mathcal{P}_{\mathrm{w}}Q$ be the groupoid of weak paths of Q.

For any arrow $a \in Q_1$ we have $\partial W/\partial a = u - v$ for some paths u, v. We denote the pair (u, v) also by $\partial W/\partial a$ and consider it as a relation on the set of paths. We denote by ∂W the set of all relations obtained in this way.

Let $\mathcal{P}Q/(\partial W)$ (resp. $\mathcal{P}_{\mathrm{w}}Q/(\partial W)$) be the factor category (resp. factor groupoid) with respect to the equivalence relation (∂W) generated by ∂W (this is the minimal equivalence relation containing ∂W and such that $(u, v) \in (\partial W)$ implies $(xuy, xvy) \in (\partial W)$ for any morphisms x, y of $\mathcal{P}Q$ (resp. $\mathcal{P}_{\mathrm{w}}Q$) with $t(y) = s(u) = s(v)$, $s(x) = t(u) = t(v)$).

Remark 4.1 *(Quotients of groupoids).* We can formalize the operation of taking the quotient by an equivalence relation in a groupoid with the help of quotients by normal subgroupoids. Let G be a groupoid. We call the set of groups $H = (H_i \subset G(i,i))_{i \in \mathrm{Ob}\, G}$ a normal subgroupoid of G if for any $h \in H_i$ and $x \in G(i, j)$, the element xhx^{-1} belongs to H_j. One can then construct the quotient groupoid G/H in the usual way, namely $x, y \in G(i, j)$ are equivalent if and only if $x^{-1}y \in H_i$. In the above example, for any arrow $a \in Q_1$, there is a pair of cycles (au, av) in W, and we take the minimal normal subgroupoid generated by the elements $u^{-1}v = (au)^{-1}(av)$.

Lemma 4.2. *Any two cycles along the faces of Q with the same endpoints are equal in $\mathcal{P}Q/(\partial W)$ and $\mathcal{P}_{\mathrm{w}}Q/(\partial W)$.*

Proof. Two cycles as above starting at $i \in Q_0$ correspond to some faces incident to i. We may assume that these two faces have a common arrow incident to i. This means that the corresponding cycles have either equal first arrows or equal last arrows. Let us assume that they have the same first arrow a. Then they are of the form ua and va. The set of relations ∂W contains the pair (u, v). Therefore $(ua, va) \in (\partial W)$. $\square$

Remark 4.3. The groupoid $\mathcal{P}_{\mathrm{w}}Q/(\partial W)$ can be also constructed as a quotient of $\mathcal{P}_{\mathrm{w}}Q$ modulo the relation $u \sim v$ for any two cycles u, v of W with the same endpoints.

Let $\pi : \widetilde{T} \to T$ be the universal covering of a torus and let $\widetilde{Q}$ be the inverse image of Q, called a periodic quiver. For any point $i \in \widetilde{Q}_0$ and any path u in Q with $s(u) = \pi(i)$ there is a unique lifting u' of u in $\widetilde{Q}$ with $s(u') = i$. Analogously to the weak equivalence in Q, we introduce an equivalence relation on the set of weak paths in $\widetilde{Q}$. It is generated by the pairs (u, v), where u, v are the cycles along the faces of $\widetilde{Q}$ having the same endpoints. An equivalence class of the cycles along the faces with the endpoint i will be denoted by ω_i. We will write just ω if we do not want to specify the endpoint of the cycle.

Lemma 4.4. *For any weak path $u : i \to j$ we have $u\omega_i \sim \omega_j u$.*

Proof. We have to prove the claim just when $u = a$, where $a : i \to j$ is an arrow. Let wa be some cycle along the face of Q. Then $wa \sim \omega_i$ and $aw \sim \omega_j$. It follows that $a\omega_i \sim awa \sim \omega_j a$. $\square$

Lemma 4.5. *Let u, v be two weak paths in $\widetilde{Q}$. Then $u \sim v$ if and only if they are equivalent in Q (that is, $\pi(u) \sim \pi(v)$) and have the same startpoints.*

Lemma 4.6. *Any weak cycle in $\widetilde{Q}$ is equivalent to ω^k for some $k \in \mathbb{Z}$.*

Proof. Assume that u is a weak cycle without self-intersections. Then it can be written as a product of cycles (taken in right direction) along the faces contained in u. But this product is a power of ω. In the general case, we can find a subcycle without self-intersections, represent it as a power of ω, and move it to the end of the cycle using the fact that ω commutes with paths. Then we repeat our procedure. $\square$

For any weak path u in Q we define its content $|u| \in \mathbb{Z}^{Q_1}$ by counting every arrow with a plus sign and its inverse with a minus sign. As for strict paths, we define $\mathrm{wt}(u) := \mathrm{wt}(|u|)$. For any weak path u in $\widetilde{Q}$, we define its weight by $\mathrm{wt}(u) := \mathrm{wt}(\pi(u)) \in \mathbb{Z}^{Q_1}$. Recall that $\mathrm{wt}(\omega) = \overline{\omega}$.

Lemma 4.7. *Assume that u is a weak path in $\widetilde{Q}$ such that $\mathrm{wt}(u) = 0$. Then u is a weak cycle and it is equivalent to the trivial path.*

Proof. Any arrow in $\widetilde{Q}$ defines a vector in the plane, and two arrows a, b define the same vector if $\pi(a) = \pi(b)$. It follows that the vector between $s(u)$ and $t(u)$ is determined by $\pi(u)$, or just $|\pi(u)|$. We can write $|\pi(u)| = \sum_{F \in Q_2} \lambda_F d_2(F) \in \mathbb{Z}^{Q_1}$, where $\sum \lambda_F = 0$. Any $d_2(F)$ determines the zero vector in the plane. It follows that $|\pi(u)|$ determines the zero vector in the plane and therefore u is a cycle. According to Lemma 4.6, $u \sim \omega^k$ for some $k \in \mathbb{Z}$. Then $\mathrm{wt}(u) = \mathrm{wt}(\omega^k) = k\overline{\omega} = 0$ and by the proof of Lemma 3.3, we have $k = 0$. $\square$

Proposition 4.8. *Weak paths in Q (or in $\widetilde{Q}$) having the same startpoints are equivalent if and only if their weights are equal.*

Proof. It follows immediately from the previous lemma. $\square$

Remark 4.9. The above result is very similar to the first assumption of Section 2. The only difference is that paths in Q having the same weight and the same startpoint are weakly equivalent but not necessarily strictly equivalent.

Remark 4.10. For any two nodes $i, j \in \widetilde{Q}_0$ and any two weak paths u, v between them we have $v \sim u\omega^k$ for some $k \in \mathbb{Z}$. It follows that there exists a shortest strict path v_{ij} between i and j, i.e. a path such that any other strict path between i and j is weakly equivalent to $v_{ij}\omega^k$, $k \geqslant 0$. It follows that the set of weak equivalence classes of strict paths starting at some fixed point $i_0 \in \widetilde{Q}_0$ can be identified with $\widetilde{Q} \times \mathbb{N}$, and the set of weak equivalence classes of weak paths starting at i_0 can be identified with $\widetilde{Q} \times \mathbb{Z}$.

Lemma 4.11. *The following conditions are equivalent:*

(1) *Given two paths u, v with the same endpoints and an arrow a with $s(a) = t(u)$, if au is strictly equivalent to av then u is strictly equivalent to v.*
(2) *Given two paths u, v with the same endpoints and an arrow a with $t(a) = s(u)$, if ua is strictly equivalent to va then u is strictly equivalent to v.*
(3) *The map $\mathcal{P}Q/(\partial W) \to \mathcal{P}_{\mathrm{w}}Q/(\partial W)$ is injective.*
(4) *Two paths in Q with the same startpoints are strictly equivalent if and only if their weights are equal.*

Proof. (1) $\Rightarrow$ (2). Assume that two paths u, v have the same endpoints i, j and $ua \sim va$ for some arrow $a : k \to i$. We can find a path w such that $aw \sim \omega_i$. Then

$$\omega_j u \sim u\omega_i \sim uaw \sim vaw \sim v\omega_i \sim \omega_j v.$$

Using condition (1) we obtain $u \sim v$.

(2) $\Rightarrow$ (3). Let us give a new description of the groupoid $\mathcal{P}_{\mathrm{w}}Q/(\partial W)$. It is obtained from the category $\mathcal{P}Q$ by adding morphisms $t_i : i \to i$, $i \in Q_0$, such that $at_i = t_j a$ for any arrow $a : i \to j$ and $t_i w = 1$ for any cycle w along a face starting at $i \in Q_0$. It follows that two strict paths u, v with the same endpoints are weakly equivalent if and only if v is obtained from u by first inserting words of the form $t_i w$ in u (say r times), then moving the t_i's inside the obtained word, and finally deleting r words of the form $t_i w$. Equivalently, we can insert r cycles along faces into u and v so that we obtain equal words. This implies that $u\omega^r$ and $v\omega^r$ are strictly equivalent. Condition (2) now implies that u and v are strictly equivalent.

(3) $\Rightarrow$ (4). This follows from Proposition 4.8.

(4) $\Rightarrow$ (1). Clear. $\square$

We will call the following condition the first consistency condition and we will assume it throughout the paper.

Condition 4.12. The equivalent conditions of Lemma 4.11 are satisfied.

Remark 4.13. It follows from [8, Lemma 5.3.1] that the condition (4) of Lemma 4.11 is satisfied if there exists an R-charge on the brane tiling. An R-charge is a collection $(R_a)_{a\in Q_1} \in (0,1)^{Q_1}$ such that for every face $F \in Q_2$ we have

$$\sum_{a\in F} R_a = 2$$

and for every node $i \in Q_0$ we have

$$\sum_{a\ni i} (1 - R_a) = 2.$$

An easily verified criterion for the existence of R-charges is given in [12]. We thank Alastair King for this remark.

Under the above condition and the nondegeneracy of the brane tiling, all the assumptions of Section 2 are satisfied. Then Lemma 2.6 implies

Proposition 4.14. *We have*

$$Z^i(A) = \sum_{\substack{\Omega\subset\Delta_i\\ \text{fin. ideal}}} \prod_{u\in\Omega} x_{t(u)} \in \mathbb{Q}[[x_j \mid j \in Q_0]].$$

This formula allows the computation of $Z^i(A)$ with the help of a computer. Let us discuss some examples that were considered earlier in the literature.

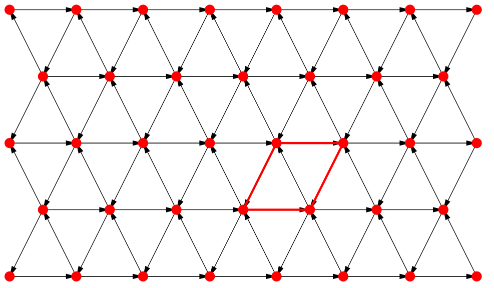


Fig. 1. The periodic quiver and a fundamental domain for $\mathbb{C}^3$.

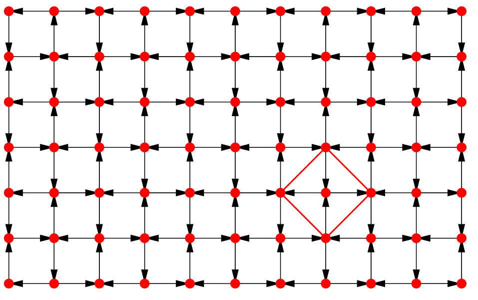


Fig. 2. The periodic quiver and a fundamental domain for the conifold.

Example 4.15. The simplest example is $\mathbb{C}^3$. (See Fig. 1.) It corresponds to the quiver with one node 0, three loops x, y, z, and potential $W = xyz - xzy$. The quiver potential algebra is $A = \mathbb{C}Q/(\partial W) \simeq \mathbb{C}[x, y, z]$. Every path in Q is equivalent to $x^k y^l z^m$ for some $(k, l, m) \in \mathbb{N}^3$ and therefore the poset of paths Δ_0 can be identified with $\mathbb{N}^3$. Ideals in Δ_0 correspond to the 3-dimensional Young diagrams. The generating function $Z^0(A)$ is given by the MacMahon function

$$M(1, x_0) = \prod_{n \geqslant 1} \left(\frac{1}{1 - x_0^n} \right)^n.$$

Example 4.16. Szendrői [18] considered the case of the conifold. (See Fig. 2.) It corresponds to the quiver with nodes $0, 1 \in \mathbb{Z}_2$ and arrows $x_i : i \to i + 1$, $y_i : i + 1 \to i$, $i = 0, 1$. The potential is given by

$$W = x_0 x_1 y_1 y_0 - x_1 x_0 y_0 y_1.$$

The poset Δ_0 corresponds to the pyramid arrangement from [18]. Ideals in Δ_0 correspond to the pyramid partitions from [18]. A nice closed formula for $Z^i(A)$ was conjectured by Szendrői [18, Theorem 2.7.2] and proved by Young [19].

Example 4.17. The orbifold $\mathbb{C}^3/\mathbb{Z}_n$ with a group action $\frac{1}{n}(1, 0, -1)$. (See Fig. 3.) The corresponding McKay quiver is given by

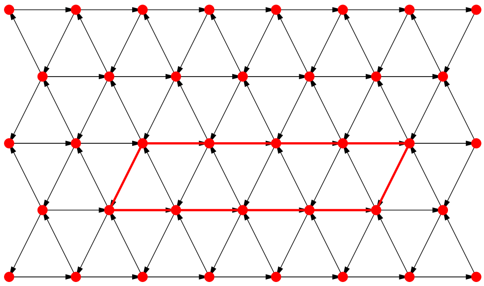


Fig. 3. The periodic quiver and a fundamental domain for $\mathbb{C}^3/\mathbb{Z}_4$.

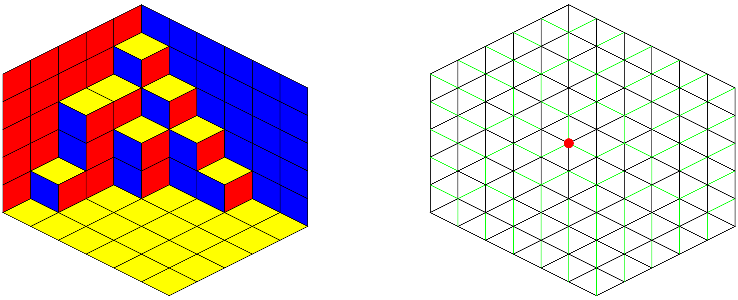
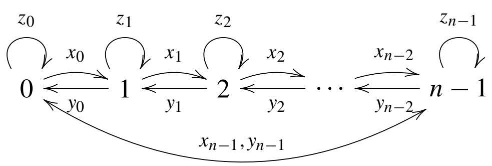


Fig. 4. The right picture is obtained from the left by drawing the shortest diagonals of the small parallelograms in green. (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)



The potential is given by

$$W = \sum_{i=1}^{n} (x_i y_i z_{i+1} - z_i y_i x_i).$$

The poset of paths Δ_0 can be identified with the lattice $\mathbb{N}^3$. New arrow x means increasing of x-coordinate, y – increasing of y-coordinate, z – increasing of z-coordinate. The ideals in Δ_0 correspond to 3-dimensional Young diagrams. The partition function $Z^0(A)$ is now an element of $\mathbb{Z}[[x_0, \ldots, x_{n-1}]]$. A simple formula for $Z^0(A)$ was also found by Young [19].

5. Path poset and perfect matchings

Let us first recall the nice correspondence between 3-dimensional Young diagrams and perfect matchings of the plane tiling by equilateral triangles (we use the dual point of view and call the set of edges a perfect matching if every face contains exactly one edge from this set). The following picture should explain this correspondence. (See Fig. 4.)

As we have seen in the last section, the 3-dimensional Young diagrams correspond to the ideals of the poset Δ_0 in the case of a quiver Q with one node 0, three arrows x, y, z and potential $W = xyz - zyx$. The quiver Q is embedded in the torus and the corresponding periodic quiver $\widetilde{Q}$ is embedded in the plane and defines there precisely the plane tiling by triangles discussed above.

We have thus all ingredients to generalize the above correspondence to arbitrary brane tilings. On the one hand we have ideals in the poset of equivalence classes of paths, and on the other hand we have perfect matchings of the plane tiling defined by the periodic quiver. The goal of this section is to construct such a correspondence.

So, let (Q, W) be a quiver potential arising from a brane tiling satisfying the conditions of the previous section. We fix some $i_0 \in \widetilde{Q}_0$. Let $\Delta = \Delta_{i_0}$ be the poset of equivalence classes of paths in $\widetilde{Q}$ starting at i_0 and let Δ' be the poset of equivalence classes of weak paths starting at i_0.

Remark 5.1. We can identify Δ with the set of equivalence classes of paths in Q starting at $\pi(i_0)$.

For any $i \in \widetilde{Q}_0$ and any subset $\Omega \subset \Delta'$, we define $\Omega_i \subset \Omega$ to be the set of all weak paths ending at i. For any point $i \in \widetilde{Q}_0$, we denote by $v_i \in \Delta_i$ the shortest path between i_0 and i (see Remark 4.10). Then any weak path in Δ'_i is equivalent to $\omega^k v_i$ for some $k \in \mathbb{Z}$. Therefore we can identify Δ' with $\widetilde{Q}_0 \times \mathbb{Z}$ and identify Δ with $\widetilde{Q}_0 \times \mathbb{N}$. For any ideal $\Omega \subset \Delta$, we define an ideal $\Omega' := \Omega \cup (\widetilde{Q}_0 \times \mathbb{Z}_{<0})$ in Δ'.

Proposition 5.2. *For any finite ideal $\Omega \subset \Delta$ the set*

$$I(\Omega) := \{a : i \to j \mid \exists u \in \Omega'_i,\ au \notin \Omega'_j\} \subset \widetilde{Q}_1$$

is a perfect matching of the plane tiling defined by $\widetilde{Q}$ (i.e. of the dual bipartite graph).

Proof. Let F be some face in $\widetilde{Q}$. We will show that some of its arrows are contained in $I(\Omega)$. If not, then for any arrow $a : i \to j$ and any path $u \in \Omega'_i$, we have $au \in \Omega'_j$. Going along the face F, we get $\omega u \in \Omega'_i$ and therefore $\omega^k u \in \Omega'_i$ for any $k \geqslant 0$. It follows that Ω_i is infinite.

Assume that there are two arrows $a_1, a_2 \in F$ contained in $I(\Omega)$. Let $w_2 a_2 w_1 a_1$ be the cycle along the face F. Let $u_i \in \Omega'_{s(a_i)}$ be such that $a_i u_i \notin \Omega'_{t(a_i)}$, $i = 1, 2$. Then $w_1 a_1 u_1 \notin \Omega'_{s(a_2)}$. This implies that $\omega u_2 \leqslant w_1 a_1 u_1$ and $w_2 a_2 u_2 \leqslant u_1$. Analogously $\omega u_1 \leqslant w_2 a_2 u_2$. This means that $\omega u_1 \leqslant u_1$, a contradiction. $\square$

We call the perfect matching $I_0 = I(\emptyset)$ the canonical perfect matching. We have

$$I_0 = \{a : i \to j \mid a v_i > v_j\}.$$

A perfect matching I will be called congruent to I_0 if the symmetric difference $(I \backslash I_0) \cup (I_0 \backslash I)$ is finite. It is clear that any $I(\Omega)$ is congruent to I_0.

The following condition will be called the second consistency condition and will be assumed throughout the paper.

Condition 5.3. For any two vertices $i, j \in \widetilde{Q}_0$, there exists an arrow $a : j \to k$ such that av_{ij} is the shortest path between i and k (here v_{ij} is the shortest paths between i and j, see Remark 4.10). Also the dual condition is satisfied.

Remark 5.4. It was proved by Davison [4] that Condition 5.3 follows from our first consistency condition. Earlier it was proved by Broomhead [3] that Condition 5.3 follows from the existence of an R-charge on the brane tiling (see Remark 4.13).

In the next theorem we will actually need the above condition just for $i = i_0$. We will see in the next section that the same condition is needed in order to ensure that the quiver potential algebra $\mathbb{C}Q/(\partial W)$ is a 3-Calabi–Yau algebra.

Theorem 5.5. *For any perfect matching I congruent to I_0, there exists a unique finite ideal Ω of Δ such that $I(\Omega) = I$. This ideal is defined as $\Omega' \cap \Delta$, where Ω' is the minimal ideal in Δ', containing $\widetilde{Q}_0 \times \mathbb{Z}_{<0}$ and such that for any arrow $a \notin I$ and any $u \in \Omega'_{s(a)}$, we have $au \in \Omega'_{t(a)}$.*

Proof. Let Ω' be the ideal in Δ' described above and let $\Omega = \Omega' \cap \Delta$. Let us show that Ω is finite. The union $I \cup I_0$ considered on the bipartite graph $\widetilde{G}$ consists of coinciding arrows and a finite number of cycles. On the periodic quiver $\widetilde{Q}$ these cycles go through a finite number of faces and intersect them alternatingly in arrows from I and I_0. We will call such a cycle in $\widetilde{Q}$ a trap. Every trap is determined by a sequence S of arrows in I and I_0 that it intersects.

We claim that every trap contains i_0. Assume that i_0 is outside of a trap. Let $au \in \Delta$ be one of the shortest paths among all paths to the points inside the trap. Then $s(a)$ is outside of the trap and $a \in S$. We have $v_{t(a)} = av_{s(a)}$, so $a \notin I_0$. This implies that all arrows from $S \cap I_0$ point outside the trap and all arrows from $S \cap I$ point inside the trap. It follows that there exists some point i in the trap such that for any arrow $a : i \to j$, we have $av_i > v_j$, which contradicts our assumption. Indeed, if this is not the case then using the fact that the number of points in the trap is finite, we can find a cycle $a_r \dots a_1$ such that $a_i v_{s(a_i)} \sim v_{t(a_i)}$ for all i and therefore $a_r \dots a_1 v_{s(a_1)} \sim v_{s(a_1)}$, a contradiction. So, the point i_0 is in the trap.

We claim that all arrows from $S \cap I_0$ point inside the trap. Let $au \in \Delta$ be one of the shortest paths among all paths to the points outside the trap. Then $av_{s(a)} = v_{t(a)}$, so $a \notin I_0$. It follows that $a \in I$ and all arrows from $S \cap I$ point outside the trap.

Starting with some path $u = (i, -1) \in \Omega'$, we apply to it all compatible arrows from $\widetilde{Q}_1 \backslash I$. It follows that either we stay in the same plane, or we apply some arrow from $I_0 \backslash I$ increasing the plane number by one and get automatically to some trap. Applying again the arrows from $\widetilde{Q}_1 \backslash I$, we cannot get outside of the trap (all arrows pointing outside the trap are contained in I). The only possibility to get outside of the trap is to use the property that Ω' is an ideal. So, we can move outside of the trap along (the inverse of) some arrow from I_0, but then the plane number will decrease by one.

We get an alternative description of Ω. The point $(i, k) \in \widetilde{Q}_0 \times \mathbb{N}$ is contained in Ω if and only if i is contained in more than k traps. From this description, it follows that Ω is finite.

Let us show that $I(\Omega) = I$. We have to check two properties:

(1) If $a : i \to j$ is in I then there exists $u \in \Omega'_i$ such that $au \notin \Omega'_j$.
(2) If $a : i \to j$ is not in I then for all $u \in \Omega'_i$, we have $au \in \Omega'_j$.

The second property follows from our construction. Consider some $a : i \to j$ in I. Let aw be a cycle along the face in $\widetilde{Q}$. Then no arrow from w is contained in I. It follows that for any $u \in \Omega'_j$, we have $wu \in \Omega'_i$. Let $u = (j, k) \in \Omega'_j$ be the path with the maximal $k \geqslant -1$. Then $wu \in \Omega'_i$, but $awu = (j, k+1) \notin \Omega'_j$. This proves the first property.

Let us show that Ω is uniquely determined. Assume that there exists some other finite ideal $\widetilde{\Omega} \supset \Omega$ with $I(\widetilde{\Omega}) = I$. Let

$$A = \{i \in \widetilde{Q}_0 \mid \text{there exists } u \in \widetilde{\Omega}_i \backslash \Omega_i\}.$$

We claim that for any arrow $a : i \to j$ not in I, we have $i \in A$ if and only if $j \in A$. If $i \in A$ then there exists some $u \in \widetilde{\Omega}_i \backslash \Omega_i$. It follows that $au \in \widetilde{\Omega}_j \backslash \Omega_j$ and $j \in A$. Conversely, assume that $j \in A$, so there exists some $k \geqslant 0$ with $(j, k) \in \widetilde{\Omega}_j \backslash \Omega_j$. If $a \notin I_0$ then $a(i, k) = (j, k)$ and $(i, k) \in \widetilde{\Omega}_i \backslash \Omega_i$, so $i \in A$. If $a \in I_0$ then (j, k) is in some trap, so $(j, 0) \in \Omega_j$ and therefore $k > 0$. It follows that $(i, k-1) = a^{-1}(j, k) \in \widetilde{\Omega}_i \backslash \Omega_i$ and $i \in A$.

The set $\widetilde{Q}_1 \backslash I$ connects all the nodes of $\widetilde{Q}$, so $A = \widetilde{Q}_0$. This contradicts the assumption that $\widetilde{\Omega}$ is finite. $\square$

For any perfect matching I of $\widetilde{Q}$, consider its characteristic function $\chi_I : \widetilde{Q}_1 \to \mathbb{Z}$. If I is congruent to I_0 then there exists a unique function $h_I : \widetilde{Q}_0 \to \mathbb{Z}$, called the height function, such that for every arrow $a : i \to j$, we have

$$h_I(i) - h_I(j) = \chi_I(a) - \chi_{I_0}(a),$$

and $h_I(i) = 0$ for i far enough from i_0. It is clear that $h_I(i)$ equals the number of traps (introduced in the theorem) containing i. In the course of the proof of the theorem, we have found an alternative description of the finite ideal Ω satisfying $I(\Omega) = I$. Namely,

$$\Omega = \bigl\{(i, k) \bigm| 0 \leqslant k \leqslant h_I(i) - 1\bigr\}.$$

Corollary 5.6. *We have*

$$Z^{i_0}(A) = \sum_{I\text{-}perf.mat.} \prod_{i \in \widetilde{Q}_0} x_{\pi(i)}^{h_I(i)}.$$

6. Calabi–Yau property

Let us study Condition 5.3 in more detail. We use the dual formulation.

Lemma 6.1. *The following conditions are equivalent:*

(1) *If u is the shortest path in $\widetilde{Q}$ (between its endpoints), then there exists some arrow a with $t(a) = s(u)$ such that ua is also the shortest path.*
(2) *If w is a weak path in $\widetilde{Q}$ (or in Q) such that wa is equivalent to a strict path for any arrow a with $t(a) = s(w)$ then w is equivalent to a strict path.*

Proof. Assume that the first condition is satisfied and let w be as in the second condition. We can write $w = \omega^k u$, where u is the shortest path. To show that w is equivalent to a strict path, we have to prove that $k \geqslant 0$. Assume that $k < 0$. By the first condition there exists some arrow a with $t(a) = s(u)$ such that ua is the shortest path. But this implies that $wa = \omega^k ua$ is not the strict path, as $k < 0$. This contradicts our assumption on w.

Assume that the second condition is true and let u be as in the first condition. We consider the weak path $w = \omega^{-1}u$. If ua is not the shortest path for some arrow a, then $\omega^{-1}ua = wa$ is equivalent to a strict path. It follows from the second condition that $w = \omega^{-1}u$ is equivalent to a strict path, and therefore u is not the shortest one, a contradiction. □

For any Λ-graded A-module M and for any $\lambda \in \Lambda$, we denote by $M[\lambda]$ a new Λ-graded A-module with $M[\lambda]_\mu = M_{\mu+\lambda}$. For any point $i \in Q_0$ we denote by S_i the one-dimensional A-module concentrated at i. We endow S_i with a Λ-grading so that all its elements have degree zero.

Proposition 6.2. *Assume that all consistency conditions are satisfied. Then for any $i \in Q_0$ there exists an exact sequence of Λ-graded A-modules*

$$0 \to P_i[-\overline{\omega}] \xrightarrow{\cdot b} \bigoplus_{b:k\to i} P_k[b-\overline{\omega}] \xrightarrow{\cdot \frac{\partial W}{\partial b} a^{-1}} \bigoplus_{a:i\to j} P_j[-a] \xrightarrow{\cdot a} P_i \to S_i \to 0.$$

Proof. Exactness of

$$\bigoplus_{b:k\to i} P_k[b-\overline{\omega}] \xrightarrow{\cdot \frac{\partial W}{\partial b} a^{-1}} \bigoplus_{a:i\to j} P_j[-a] \xrightarrow{\cdot a} P_i \to S_i \to 0$$

is known (see e.g. [2]). We just have to prove the exactness in the term $\bigoplus P_k[b-\overline{\omega}]$. A basis of P_k is given by the set Δ_k of equivalence classes of paths starting at k. So a general element of $\bigoplus_{b:k\to i} P_k[b-\overline{\omega}]$ can be uniquely written in the form

$$f = \sum_{t(b)=i} \sum_{u\in\Delta_{s(b)}} x_{b,u} u \otimes b^*,$$

where $x_{b,u} \in \mathbb{C}$, and we use the symbol b^* to denote the appropriate direct summand. Let us analyze the condition that for $a : i \to j$, the image of f in $P_j[-a]$ is zero. Let the terms of W that contain a be abv and acw. Then we have

$$\sum_{u\in\Delta_{s(b)}} x_{b,u} uv = \sum_{u'\in\Delta_{s(c)}} x_{c,u'} uw,$$

or, equivalently

$$\sum_{u\in\Delta_{s(b)}} x_{b,u} ub^{-1} = \sum_{u'\in\Delta_{s(c)}} x_{c,u'} u'c^{-1}.$$

It follows that if $x_{b,u} \neq 0$ then $ub^{-1} = u'c^{-1}$ for some path $u' \in \Delta_{s(c)}$ and $x_{c,u'} = x_{b,u}$. We have shown this for two arrows b, c with $t(b) = t(c) = i$ that belong to adjacent faces. If f is mapped to zero, then this is true for two arbitrary arrows b, c with $t(b) = t(c) = i$. If $x_{b,u} \neq 0$, then the weak path $\gamma = ub^{-1}$ satisfies the condition that for any arrow c with $t(c) = s(\gamma) = t(b) = i$, the weak path γc is a strict path. From Lemma 6.1 it follows that γ is a strict path. Then $x_{b,u}\gamma \in P_i[-\overline{\omega}]$ is mapped to $x_{b,u}u \otimes b^*$ in $P_k[b - \omega]$. This shows the exactness of the sequence in the second term. □

Theorem 6.3. *Assume that all consistency conditions are satisfied. Then the quiver potential algebra $\mathbb{C}Q/(\partial W)$ is a 3-Calabi–Yau algebra.*

Proof. We have to show the exactness of the sequence [7, Proposition 5.1.9, Corollary 5.3.3]

$$0 \to \bigoplus_i e_i A \otimes Ae_i \xrightarrow{j} \bigoplus_{b:k\to i} Ae_k \otimes e_i A \to \bigoplus_{a:i\to j} Ae_j \otimes e_i A \to \bigoplus_i Ae_i \otimes e_i A \to A \to 0.$$

The exactness should be actually shown just in the first and the second terms.

Let us show that j is injective. The map j is given by [7]

$$x \otimes y \mapsto \sum_{b:k\to i} yb \otimes x - y \otimes bx.$$

A general element in $\bigoplus_i e_i A \otimes Ae_i$ can be written in the form

$$\sum_{i\in Q_0} \sum_{u\in\Delta_i} x_u \otimes u,$$

where $x_u \in e_i A$ for $u \in \Delta_i$. If this element maps to zero then for any arrow $b : k \to i$, we have

$$\sum_{u\in\Delta_i} ub \otimes x_u = \sum_{u\in\Delta_k} u \otimes bx_u,$$

and therefore $x_u = bx_{ub}$ for any $u \in \Delta_i$. This easily implies that all x_u are zero.

To prove the exactness in the second term, we just have to show that the character of the complex of Λ-graded modules is zero. The character is defined as follows. For any Λ-graded vector space V that is finite-dimensional in any degree, we define $\operatorname{ch} V = \sum_{\lambda\in\Lambda} \dim V_\lambda t^\lambda$. For any finite complex $V_\cdot$ of Λ-graded vector spaces, we define $\operatorname{ch}(V_\cdot) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{ch}(V_k)$. For a fixed $i \in Q_0$ the corresponding summand of every component of the complex is obtained by tensoring the component of the previous proposition with $e_i A$. As the complex in the proposition is exact, the character of the i-th component is zero. □

Remark 6.4. For any finite-dimensional modules M, N over $\mathbb{C}Q/(\partial W)$, we have

$$\operatorname{Ext}^k(M, N) \simeq \operatorname{Ext}^{3-k}(N, M)^\vee, \quad k = 0, 1, 2, 3.$$

If M and N are Λ-graded modules, then we have an isomorphism of Λ-graded vector spaces

$$\operatorname{Ext}^k(M, N) \simeq \operatorname{Ext}^{3-k}\big(N, M[-\overline{\omega}]\big)^\vee, \quad k = 0, 1, 2, 3.$$

7. Donaldson–Thomas invariants

Let (Q, W) be a quiver potential arising from a brain tiling satisfying all consistency conditions and let $A = \mathbb{C}Q/(\partial W)$. Let $i_0 \in Q_0$ and $\alpha \in \mathbb{N}^{Q_0}$. We put $X = \mathrm{Hilb}^{\alpha}_{i_0}(A)$. It is known that X has an (equivariant) symmetric obstruction theory [18, Theorem 1.3.1]. According to Behrend and Fantechi [1, Theorem 3.4], the DT type invariant of X can be computed by the formula

$$\#^{vir} X = \sum_{(M,m)\in X^T} (-1)^{\dim T_{(M,m)} X},$$

where $T_{(M,m)}X$ is the tangent space at the T-fixed point (M, m). For any finite ideal $\Omega \subset \Delta_{i_0}$, let (M, m) be the corresponding T-fixed i_0-cyclic A-module and let d_Ω be the dimension of the tangent space at this point. Then we have

$$Z^{i_0}_{\mathrm{DT}}(A) = \sum_{\Omega\subset\Delta_{i_0}} (-1)^{d_\Omega} x^{\overline{\Omega}},$$

where $\overline{\Omega} = (\#\Omega_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$ and Ω_i is the set of paths in Ω with endpoint $i \in Q_0$.

Theorem 7.1. *For any finite ideal Ω, we have*

$$d_\Omega \equiv \overline{\Omega}_{i_0} + \langle\overline{\Omega}, \overline{\Omega}\rangle \pmod 2,$$

where $\langle -, -\rangle$ is the Ringel form of the quiver Q, given, for $\alpha, \beta \in \mathbb{Z}^{Q_0}$, by

$$\langle\alpha, \beta\rangle = \sum_{i\in Q_0} \alpha_i\beta_i - \sum_{a:i\to j} \alpha_i\beta_j.$$

Proof. Let (M, m) be an i_0-cyclic A-module corresponding to Ω. Consider an exact sequence

$$0 \to I \to P_{i_0} \to M \to 0.$$

Note that all the modules in this sequence are Λ-graded. One can show, in the same way as for Quot-schemes (see e.g. [10, Proposition 2.2.7]) that the tangent space $T_{(M,m)}X$ is isomorphic to $\mathrm{Hom}(I, M)$. Consider the exact sequence

$$0 \to \mathrm{Hom}(M, M) \to \mathrm{Hom}(P_{i_0}, M) \to \mathrm{Hom}(I, M) \to \mathrm{Ext}^1(M, M) \to 0.$$

The second map is actually zero, because any morphism $f : P_{i_0} \to M$ maps I to zero by degree reasons. It follows that

$$d_\Omega = \dim \mathrm{Hom}(I, M) = \dim \mathrm{Ext}^1(M, M).$$

In order to find $\dim \mathrm{Ext}^1(M, M)$, we will study the Euler characteristic $\chi(M, M)$. Usually it is defined as an alternating sum of dimensions of Ext-groups, but this gives a rather unsatisfactory output in the case of Calabi–Yau algebras. Therefore, for any finite-dimensional Λ-graded A-modules L, N, we define

$$\chi(L,N)=\sum_{k\geqslant 0}(-1)^k\operatorname{Ext}^k(L,N),$$

considered as an element of the Grothendieck group K_Λ of Λ-graded vector spaces. It follows from the results of the previous section that for any such modules L, N we have

$$\operatorname{Ext}^k(L,N)=\operatorname{Ext}^{3-k}\bigl(N,L[-\bar{\omega}]\bigr)^\vee=\operatorname{Ext}^{3-k}(N,L)^\vee[\bar{\omega}]$$

in K_Λ. For any $\lambda\in\Lambda$, we define $t^\lambda\in K_\Lambda$ to be the element corresponding to the one-dimensional vector space concentrated in degree λ. We define a homomorphism $D:K_\Lambda\to K_\Lambda$ by the formula $t^\lambda\mapsto t^{-\lambda-\bar{\omega}}$. Then

$$\operatorname{Ext}^k(L,N)=D\operatorname{Ext}^{3-k}(N,L).$$

It follows from Proposition 6.2 that for any $i,j\in Q_0$ we have

$$\chi(S_i,S_j)=\delta_{i,j}-\sum_{a:i\to j}t^{-a}+\sum_{b:j\to i}t^{b-\bar{\omega}}-\delta_{i,j}t^{-\omega}.$$

This implies for the module M:

$$\begin{aligned}\chi(M,M)=&\sum_{\substack{u,v\in\Omega\\ t(u)=t(v)}}t^{\mathrm{wt}(v)-\mathrm{wt}(u)}-\sum_{\substack{u,v\in\Omega\\ a:t(u)\to t(v)}}t^{\mathrm{wt}(v)-\mathrm{wt}(u)-a}\\&+\sum_{\substack{u,v\in\Omega\\ b:t(u)\to t(v)}}t^{\mathrm{wt}(v)-\mathrm{wt}(u)+b-\bar{\omega}}-\sum_{\substack{u,v\in\Omega\\ t(u)=t(v)}}t^{\mathrm{wt}(v)-\mathrm{wt}(u)-\bar{\omega}}.\end{aligned}$$

Therefore, for

$$A:=\sum_{\substack{u,v\in\Omega\\ t(u)=t(v)}}t^{\mathrm{wt}(v)-\mathrm{wt}(u)}-\sum_{\substack{u,v\in\Omega\\ a:t(u)\to t(v)}}t^{\mathrm{wt}(v)-\mathrm{wt}(u)-a}$$

we have $\chi(M,M)=A-DA$. Let us define

$$B:=\operatorname{Hom}(M,M)-\operatorname{Ext}^1(M,M).$$

Then we have $\chi(M,M)=B-DB$. Therefore

$$A-B=D(A-B).$$

If t^λ has coefficient n in $A-B$ then also $t^{-\lambda-\bar{\omega}}$ has coefficient n in $A-B$. Note that $\lambda\neq-\lambda-\bar{\omega}$ for any $\lambda\in\Lambda$. Indeed, if $\bar{\omega}=-2\lambda$ then it follows from the exact sequence (see Section 3)

$$0\to\mathbb{Z}\xrightarrow{1\mapsto\bar{\omega}}\Lambda\to\operatorname{coker}d_2\to 0$$

that coker d_2 has torsion, contradicting Lemma 3.3. This implies that the dimension of $A - B$ is even. Now we compute

$$\dim A = \sum_{i \in Q_0} \overline{\Omega}_i^2 - \sum_{a:i \to j} \overline{\Omega}_i \overline{\Omega}_j = \langle \overline{\Omega}, \overline{\Omega} \rangle,$$

$$\dim \operatorname{Hom}(M, M) = \dim \operatorname{Hom}(P_{i_0}, M) = \overline{\Omega}_{i_0},$$

and

$$\dim \operatorname{Ext}^1(M, M) \equiv \dim \operatorname{Hom}(M, M) - \dim A \equiv \overline{\Omega}_{i_0} + \langle \overline{\Omega}, \overline{\Omega} \rangle \pmod 2. \qquad \square$$

Remark 7.2. Our result immediately implies [19, Theorem A.3] on the signs in the cases $\mathbb{C}^3/\mathbb{Z}_n$ and $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and [18, Theorem 2.7.1] on the signs in the case of conifold.

Corollary 7.3. *Let $Z^{i_0}(A) = \sum_{\alpha \in \mathbb{N}^I} Z^{i_0}(A)_\alpha x^\alpha$. Then*

$$Z_{\mathrm{DT}}^{i_0}(A) = \sum_{\alpha \in \mathbb{N}^I} (-1)^{\alpha_{i_0} + \langle \alpha, \alpha \rangle} Z^{i_0}(A)_\alpha x^\alpha.$$

8. Rationality conjecture

In the last section we are going to discuss the structure of the generating function $Z^i(A) \in \mathbb{Q}[\![x_j \mid j \in Q_0]\!]$, where $A = \mathbb{C}Q/(\partial W)$ and $i \in Q_0$.

Let $R = \mathbb{Q}[\![x_1, \ldots, x_r]\!]$ and R^+ be its maximal ideal. We endow R with the structure of a λ-ring (see [6] or [15, Appendix]) by defining the Adams operations $\psi_n : R \to R$, $n \geqslant 1$

$$\psi_n\big(f(x_1, \ldots, x_r)\big) := f\big(x_1^n, \ldots, x_r^n\big).$$

We define a plethystic exponent $\operatorname{Exp} : R^+ \to 1 + R^+$ by the formula

$$\operatorname{Exp}(f) = \exp\left(\sum_{n \geqslant 1} \frac{1}{n} \psi_n(f)\right).$$

Its inverse, plethystic logarithm $\operatorname{Log} : 1 + R^+ \to R^+$, is given by

$$\operatorname{Log}(f) = \sum_{n \geqslant 1} \frac{\mu(n)}{n} \psi_n\big(\log(f)\big).$$

For example, the MacMahon function

$$M(x, z) = \prod_{n \geqslant 1} \left(\frac{1}{1 - xz^n}\right)^n = \prod_{n \geqslant 1} \operatorname{Exp}\big(xz^n\big)^n = \operatorname{Exp}\left(\sum_{n \geqslant 1} nxz^n\right) = \operatorname{Exp}\left(\frac{xz}{(1-z)^2}\right).$$

In [19] it was proved that the generating function $Z^i(A)$ in the cases of the orbifolds $\mathbb{C}^3/\mathbb{Z}_n$ (with a group action $\frac{1}{n}(1, 0, -1)$) and $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ (with a group action $\frac{1}{2}(1, 0, 1) \times \frac{1}{2}(0, 1, 1)$)

and in the case of the conifold can be represented as a product of MacMahon functions. This allows us to formulate

Conjecture 8.1. *Let G be a consistent brane tiling, (Q, W) be the corresponding quiver potential, $A = \mathbb{C}Q/(\partial W)$ be the quiver potential algebra and $i \in Q_0$ be some vertex. Let $r = \#Q_0$. Then the power series*

$$\operatorname{Log}\big(Z^i(A)\big)\big|_{x_1=\cdots=x_r=x} \in \mathbb{Q}[[x]]$$

is a rational function.

Remark 8.2. It is not true in general that the function $\operatorname{Log}(Z^i(A))$ is a rational function. This can be seen already in the case of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ with the group action $\frac{1}{3}(1,1,1)$. We thank Tom Bridgeland for this remark.

We have tested this conjecture using a computer in the case of a suspended pinch point (see e.g. [5]), which is given by the quiver

$$\begin{tikzcd}
 & 1 \arrow[loop above] \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
3 \arrow[rr, leftrightarrow] & & 2
\end{tikzcd}$$

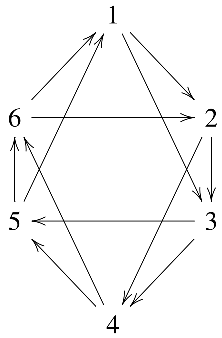
with potential

$$W = x_{21}x_{12}x_{23}x_{32} - x_{32}x_{23}x_{31}x_{13} + x_{13}x_{31}x_{11} - x_{12}x_{21}x_{11},$$

where x_{ij} is an arrow between the points i and j. Then

$$\begin{aligned}
&\operatorname{Log}\big(Z^1(A)\big)\big|_{x_1=x_2=x_3=x} \\
&\quad = \frac{x^{11} + 2x^{10} + 3x^9 + 2x^8 + 5x^7 + 6x^6 + 5x^5 + 2x^4 + 3x^3 + 2x^2 + x}{(1-x^6)^2}, \\
&\operatorname{Log}\big(Z^2(A)\big)\big|_{x_1=x_2=x_3=x} \\
&\quad = \frac{x^{11} + x^{10} + 3x^9 + 3x^8 + 5x^7 + 6x^6 + 5x^5 + 3x^4 + 3x^3 + x^2 + x}{(1-x^6)^2}.
\end{aligned}$$

We have also tested Model I of $dP3$ (see e.g. [5]), which is given by the quiver



with potential

$$W = x_{12}x_{23}x_{34}x_{45}x_{56}x_{61} + x_{13}x_{35}x_{51} + x_{24}x_{46}x_{62} \\ - x_{23}x_{35}x_{56}x_{62} - x_{13}x_{34}x_{46}x_{61} - x_{12}x_{24}x_{45}x_{51}.$$

In this case we have

$$\operatorname{Log}\big(Z^1(A)\big)\big|_{x_1=\cdots=x_6=x} \\ = \frac{x^{11} + x^{10} + 2x^9 + 2x^8 + 5x^7 + 6x^6 + 5x^5 + 2x^4 + 2x^3 + x^2 + x}{(1-x^6)^2}.$$

Of course it would be nice to be able to write down the rational function predicted by the conjecture just from the brane tiling data. At the moment we do not have such a construction.

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