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A categorification of quantum sl(2)

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Abstract

We categorify Lusztig's $\dot{\mathbf{U}}$ — a version of the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{sl}_2)$. Using a graphical calculus a 2-category $\dot{\mathcal{U}}$ is constructed whose split Grothendieck ring is isomorphic to the algebra $\dot{\mathbf{U}}$. The indecomposable morphisms of this 2-category lift Lusztig's canonical basis, and the Homs between 1-morphisms are graded lifts of a semilinear form defined on $\dot{\mathbf{U}}$. Graded lifts of various homomorphisms and antihomomorphisms of $\dot{\mathbf{U}}$ arise naturally in the context of our graphical calculus. For each positive integer N a representation of $\dot{\mathcal{U}}$ is constructed using iterated flag varieties that categorifies the irreducible $(N+1)$-dimensional representation of $\dot{\mathbf{U}}$.

Keywords: Lusztig canonical basis; Categorification; Partial flag varieties; 2-representation; Diagrammatic algebra

Contents

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1. Introduction

It is quite natural to expect that a categorification of quantum groups should exist. The strongest evidence in support of this conjecture is Lusztig's important discovery of canonical bases which have surprising positivity and integrality properties [34]. The existence of these bases suggests that the representation theory of quantized enveloping algebras, and even the algebras themselves, can be realized as Grothendieck rings of some higher categorical structure where every object decomposes into a direct sum of objects lifting Lusztig's canonical basis. Here we use a diagrammatic calculus to give an entirely algebraic categorification in the context of $\mathfrak{sl}_2$, identifying the structures present in such a categorification, and providing concrete algebraic tools for studying this categorification.

The idea of obtaining an algebraic categorification of $\mathbf{U}_q(\mathfrak{sl}_2)$ using its canonical basis is by no means new. It originated in the work of Crane and Frenkel [14] where the term 'categorification' first appeared. There the authors had in mind a categorification of the Hopf algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ at a root of unity. They argue that such a categorification could lead to combinatorial 4-dimensional topological quantum field theories. In this paper we take a step towards achieving this goal by following an insightful conjecture of Igor Frenkel [16] to categorify the algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ at generic q using its canonical basis.

Prior to our work, most progress towards an algebraic categorification of $\mathbf{U}_q(\mathfrak{sl}_2)$ has been achieved by categorifying its representations. A program for categorifying the representation category of $\mathbf{U}_q(\mathfrak{sl}_2)$ has been formulated by Bernstein, Frenkel, and Khovanov (BFK) [7], and various steps in this program have already been implemented. The symmetric powers $V_1^{\otimes n}$ of the fundamental representation V_1 of $\mathbf{U}(\mathfrak{sl}_2)$ were categorified by BFK [7] and extended to the graded case of $\mathbf{U}_q(\mathfrak{sl}_2)$ by Stroppel [42]. Other tensor product representations of $\mathbf{U}_q(\mathfrak{sl}_2)$ were then categorified by Frenkel, Khovanov and Stroppel [17]. These algebraic categorifications were strongly motivated by the geometric categorifications of representations constructed using perverse sheaves [4,21]. Progress continues to be made in the area of geometric categorification. Recently, Zheng has constructed geometric categorifications of tensor products of simple $\mathbf{U}_q(\mathfrak{sl}_2)$-modules using perverse sheaves [44].

In this paper we take a different approach and categorify the algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ directly, rather than its representations. More precisely, we construct a 2-category $\dot{\mathcal{U}}$ whose split Grothendieck ring $K_0(\dot{\mathcal{U}})$ is isomorphic to the integral form of Lusztig's $\dot{\mathbf{U}}$. The algebra $\dot{\mathbf{U}}$ is a version of $\mathbf{U}_q(\mathfrak{sl}_2)$ best suited for studying representations that admit a decomposition into weight spaces. It was first introduced by Beilinson, Lusztig, and MacPherson [4], and was later generalized by Lusztig [34]. Lusztig's canonical basis $\dot{\mathbb{B}}$ for this algebra is such that all the structure constants are in $\mathbb{N}[q,q^{-1}]$. We show that all elements of Lusztig's $\dot{\mathbb{B}}$ can be realized as generators $[b] \in K_0(\dot{\mathcal{U}})$ corresponding to indecomposable 1-morphisms of $\dot{\mathcal{U}}$.

It may be surprising that a categorification of the algebra $\dot{\mathbf{U}}$ is given by a 2-category, rather than a category. The reason for this is that Lusztig's version of $\mathbf{U}_q(\mathfrak{sl}_2)$ is naturally a category. The algebra $\dot{\mathbf{U}}$ is obtained from the integral version of $\mathbf{U}_q(\mathfrak{sl}_2)$ by adjoining a collection of orthogonal idempotents 1_n, for $n \in \mathbb{Z}$, indexed by the weight lattice of $\mathbf{U}_q(\mathfrak{sl}_2)$. This decomposes the algebra $\dot{\mathbf{U}}$ into a direct sum $\oplus_{n,m\in\mathbb{Z}} 1_m\dot{\mathbf{U}}1_n$. The collection of $n \in \mathbb{Z}$ form the objects of the category $\dot{\mathbf{U}}$, and the hom sets from n to m are given by $1_m\dot{\mathbf{U}}1_n$; composition is given by multiplication, the identity morphisms are the idempotents 1_n. Thus, it is natural to expect that a categorification of $\dot{\mathbf{U}}$ would have the structure of a 2-category.

Beilinson, Lusztig and MacPherson [4] and Grojnowski and Lusztig [21] gave a geometric interpretation of Schur quotients of quantum $\mathfrak{sl}_n$ using perverse sheaves on partial flag varieties. This can be viewed as a geometric categorification of these quotients where the higher categorical structure can be realized in terms of the Ext-algebras between these sheaves. The 2-category $\dot{\mathcal{U}}$ should be related to a limiting or stable case of these quotients by a version of the parabolic Soergel functor. Such a relationship would supply a diagrammatic generators and relations description of the Ext-algebras between sheaves arising in the Beilinson-Lusztig-MacPherson construction.

Our approach to categorification most closely mirrors the work of Chuang and Rouquier [12]. They study categorifications of locally finite $\mathfrak{sl}_2$-representations using an approach that can be viewed as a direct categorification of the algebra $\mathfrak{sl}_2$. Many features of their approach have analogs in our work, such as biadjointness and Hecke algebra actions on 2-morphisms. This will be explained in greater detail below.

1.1. What to expect from categorification?

What makes a good candidate for a categorification $\dot{\mathcal{U}}$ of $\dot{\mathbf{U}}$? An outline of what to expect appears in the survey article [29]. At the very least there should be a 1-morphism b in $\dot{\mathcal{U}}$ for each element $[b]$ in Lusztig's canonical basis $\dot{\mathbb{B}}$. There should also be 1-morphisms $b\{s\}$ in $\dot{\mathcal{U}}$ for each $s \in \mathbb{Z}$ lifting the $\mathbb{Z}[q, q^{-1}]$-module structure of $\dot{\mathbf{U}}$. That is, $[b\{s\}] = q^s[b]$, so that multiplication by q lifts to the invertible functor $\{1\}$ of shifting by 1. We identify b with $b\{0\}$ and say that $b = b\{0\}$ has no shift. For the Grothendieck ring to be isomorphic to the algebra $\dot{\mathbf{U}}$ composition of 1-morphisms in $\dot{\mathcal{U}}$ must correspond to multiplication in $\dot{\mathbf{U}}$, that is, $[xy] = [x][y]$. Furthermore, in order to lift Lusztig's canonical basis every morphism in $\dot{\mathcal{U}}$ must have decomposition into a direct sum of indecomposables; the isomorphism classes of these indecomposables with no shift $\{s\}$ should bijectively correspond to the elements of $\dot{\mathbb{B}}$.

We would also expect a categorification of $\dot{\mathbf{U}}$ to have rich new features on the level of 2-morphisms, and that various maps on $\dot{\mathbf{U}}$ (that do not use minus signs) should have lifts to 2-functors on $\dot{\mathcal{U}}$. The grading shift operation on 1-morphisms suggests that the 2-morphisms of $\dot{\mathcal{U}}$ should have a grading as well. One might suspect that the homs $\dot{\mathcal{U}}(x, y)$ should form graded abelian groups, but this requirement is too strong. In particular, if $\dot{\mathcal{U}}(x, y)$ is a graded abelian group for all 1-morphisms $x, y \in \dot{\mathcal{U}}$, then the objects x and $x\{s\}$ would become isomorphic by the shifted identity map. However, this would imply that in the Grothendieck ring $[x] = q^s[x]$. Instead, we expect that the homs $\dot{\mathcal{U}}(x, y)$ in 2-category $\dot{\mathcal{U}}$ should consist of 2-morphisms that preserve the degree of the source and target. The presence of the grading suggests that $\dot{\mathcal{U}}$ should also have an 'enriched' hom functor that associates a graded abelian group $\dot{\mathcal{U}}^*(x, y) := \oplus_{s\in\mathbb{Z}}\dot{\mathcal{U}}(x\{s\}, y)$ to x and y.

The enriched hom $\dot{\mathcal{U}}^*(,)$ is a rigidly defined structure. This is because any choice of $\dot{\mathcal{U}}^*(,)$ will descend to a pairing $\langle [x], [y]\rangle$ on $\dot{\mathbf{U}}$ given by taking the graded rank rk_q of $\dot{\mathcal{U}}^*(x, y)$. That is,

$$\bigl\langle [x], [y]\bigr\rangle := \mathrm{rk}_q\, \dot{\mathcal{U}}^*(x, y) = \sum_{s\in\mathbb{Z}} q^s\, \mathrm{rk}\dot{\mathcal{U}}\bigl(x\{s\}, y\bigr),$$

where $\mathrm{rk}\dot{\mathcal{U}}(x\{s\}, y)$ is the usual rank of the abelian group $\dot{\mathcal{U}}(x\{s\}, y)$ of degree zero 2-morphisms. Notice the behaviour of this pairing with respect to the shift functor in each variable. If a map $f : x \to y$ has degree α, then the degree of the corresponding map $f : x\{r\} \to y$ will be $\alpha - r$. Similarly, the map $f : x \to y\{r'\}$ will have degree $\alpha + r'$. Hence,

$$\bigl\langle q^r[x], [y]\bigr\rangle = \mathrm{rk}_q\, \dot{\mathcal{U}}^*\bigl(x\{r\}, y\bigr) = q^{-r}\, \mathrm{rk}_q\, \dot{\mathcal{U}}^*(x, y) = q^{-r}\bigl\langle [x], [y]\bigr\rangle, \tag{1.1}$$

$$\bigl\langle [x], q^{r'}[y]\bigr\rangle = \mathrm{rk}_q\, \dot{\mathcal{U}}^*\bigl(x, y\bigl\{r'\bigr\}\bigr) = q^{r'}\, \mathrm{rk}_q\, \dot{\mathcal{U}}^*(x, y) = q^{r'}\bigl\langle [x], [y]\bigr\rangle \tag{1.2}$$

so that the pairing induced on $\dot{\mathbf{U}}$ must be semilinear, i.e. $\mathbb{Z}[q, q^{-1}]$-antilinear in the first slot, and $\mathbb{Z}[q, q^{-1}]$-linear in the second. Hence, the enriched hom on the 2-category $\dot{\mathcal{U}}$ must categorify a semilinear form on $\dot{\mathbf{U}}$.

1.2. What this paper does

In this paper we construct a 2-category that has all of these desirable properties mentioned above. A quick summary of our results is given below:

- We construct a 2-category $\dot{\mathcal{U}}$ such that the split Grothendieck ring $K_0(\dot{\mathcal{U}})$ is isomorphic to $\dot{\mathbf{U}}$ and show that the indecomposable 1-morphisms b of $\dot{\mathcal{U}}$ with no shift correspond to elements in Lusztig canonical basis $\dot{\mathbb{B}}$.

- We provide a graphical calculus that makes calculations inside $\dot{\mathcal{U}}$ easy to visualize. By studying the symmetries of our graphical calculus we find 2-functors that categorify many algebra homomorphisms defined on $\dot{\mathbf{U}}$.
- We define a semilinear form $\langle , \rangle$ on $\dot{\mathbf{U}}$ and construct an enriched hom on the 2-category $\dot{\mathcal{U}}$ that categorifies the semilinear form $\langle , \rangle$. All the 1-morphisms in $\dot{\mathcal{U}}$ have both left and right adjoints. This imposes additional requirements on the semilinear form illustrating the rigidity of our construction.
- Our categorification $\dot{\mathcal{U}}$ naturally admits an action of the nilHecke algebra on the endomorphism ring $\dot{\mathcal{U}}(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$, $a \in \mathbb{N}$, of the 1-morphisms $\mathcal{E}^a\mathbf{1}_n$ lifting the elements $E^a 1_n$ in $\dot{\mathbf{U}}$, providing an example of the richer structure expected from a categorification of $\dot{\mathbf{U}}$.
- Finally, for each positive integer N we define a representation Γ_N of $\dot{\mathcal{U}}$ on a 2-category $\mathbf{Flag}^*_N$. This 2-category is constructed using the cohomology rings of iterated flag varieties. We show that the representation Γ_N categorifies the irreducible $(N+1)$-dimensional representation of $\dot{\mathbf{U}}$.

We now elaborate on these points.

1.2.1. Categorifying $\dot{\mathbf{U}}$

To categorify $\dot{\mathbf{U}}$ we introduce a 2-category $\mathcal{U}^*$. This 2-category is primarily for the purpose of defining what we mean by the 'enriched hom' of a 2-category. The 2-category $\mathcal{U}^*$ has one object n, for $n \in \mathbb{Z}$ corresponding to the idempotents in $\dot{\mathbf{U}}$. A morphism from n to m is a formal direct sum of elements of the form $\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n$ for $\alpha_i, \beta_i \in \{0,1,2,\cdots\}$. For each such morphism x there is also a morphism $x\{s\}$ so that the shift map $\{s\}$ lifts the $\mathbb{Z}[q,q^{-1}]$ action with q^s acting on x by $x\{s\}$. For a pair of morphisms x and y of $\mathcal{U}^*$ the hom sets $\mathcal{U}^*(x,y)$ are graded abelian groups, ensuring that every 2-morphism has a grading associated to it. We also require that the shift map $\{s\}$ on 1-morphisms extends to 2-morphisms as well (so that the shift map is an invertible 2-functor on $\mathcal{U}^*$).

As we already mentioned above, the 2-category $\mathcal{U}^*$ will not have the correct structure on the Grothendieck group because shifting the identity map will give an isomorphism between x and $x\{s\}$. That is, the degree s map $x \to x\{s\}$ and the degree $-s$ map $x\{s\} \to x$ obtained from the identity map $x \to x$ establish an isomorphism $x \cong x\{s\}$. However, the 2-subcategory $\mathcal{U}$ obtained from $\mathcal{U}^*$ by restricting to degree-preserving 2-morphisms will not have the property that $x \cong x\{s\}$ since the shifted identity map is not degree preserving. The advantage of introducing the 2-category $\mathcal{U}^*$ is that it allows notions like the degree of a 2-morphism to have meaning in $\mathcal{U}$ and it also provides a way to associate a graded abelian group to any pair of 1-morphisms.

In the 2-category $\mathcal{U}$ the homs $\mathcal{U}(x,y)$ are no longer graded abelian groups. Rather, the collection of degree zero 2-morphisms is simply an abelian group. Because the 2-category $\mathcal{U}$ is a 2-subcategory of $\mathcal{U}^*$, there is a natural association of a graded abelian group to each pair of 1-morphisms — an enriched hom. This graded abelian group is given by the image of $\mathcal{U}(x,y)$ under the inclusion $\mathcal{U} \to \mathcal{U}^*$. That is, the enriched hom is given by the graded abelian group

$$\mathcal{U}^*(x,y) := \bigoplus_{s\in\mathbb{Z}} \mathcal{U}\bigl(x\{s\}, y\bigr).$$

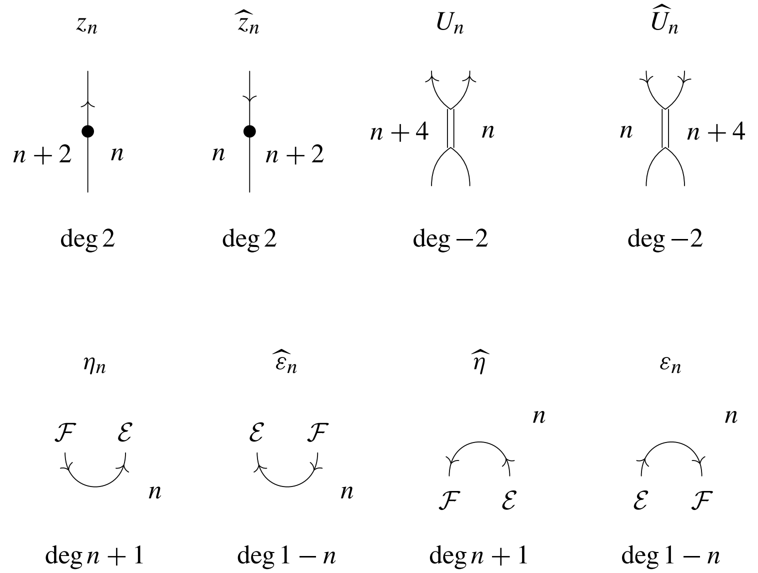
To understand the split Grothendieck ring of $\mathcal{U}$ we need to be able to decompose an arbitrary morphism x into a direct sum of indecomposables. This can be achieved by constructing a collection of primitive orthogonal idempotents $\{e_i\}_{i=1}^k$ in $A := \mathcal{U}(x,x)$ that decompose the identity

2-morphism $1_x = e_1 + e_2 + \cdots + e_k$. If the idempotents split, then we have $x = x_1 \oplus x_2 \oplus \cdots \oplus x_k$ where each $x_i = \mathrm{Im}(e_i)$ and $\mathrm{Hom}(e_i, e_j) = e_i A e_j$. While we can construct primitive orthogonal idempotents in $\mathcal{U}(x,x)$ that decompose the identity 1_x, we are not able to deduce that x decomposes into a sum of indecomposables. This is because we are working inside of an abstract 2-category and there is nothing to guarantee that the idempotents e_i split providing a direct sum decomposition of x.

To resolve the final issue of decomposing a 1-morphism into a direct sum of indecomposables we pass to the Karoubian envelope $Kar(\mathcal{U})$. The Karoubian envelope is an enlargement of the 2-category $\mathcal{U}$ inside of which all idempotents split. Our categorification of $\dot{\mathbf{U}}$ is given by $\dot{\mathcal{U}} = Kar(\mathcal{U})$. The Karoubian envelope is given by a universal property which provides a fully faithful embedding $\mathcal{U} \to \dot{\mathcal{U}}$. Using this embedding we can use the results derived for $\mathcal{U}^*$ and $\mathcal{U}$ inside of a framework where idempotents split. In Corollary 9.12 we show that the isomorphism classes of indecomposables with no shift are in a bijection with Lusztig's canonical basis, and in Theorem 9.13 we show that the split Grothendieck group of $\dot{\mathcal{U}}$ is isomorphic to $\dot{\mathbf{U}}$.

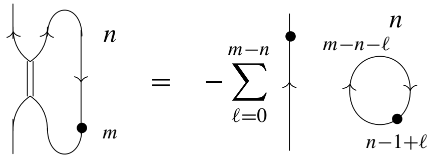
1.2.2. Graphical calculus

The 2-category $\mathcal{U}^*$ is defined by generators and relations. The 2-morphisms of $\mathcal{U}^*$ are specified using a graphical calculus of string diagrams common in the study of 2-categories. This calculus is explained in Section 4. To get a preview of what is to come, the generators of $\mathcal{U}^*$ are given below:



The second set of diagrams depicts the units and counits for the biadjoint structure on the morphisms $\mathcal{E}$ and $\mathcal{F}$.

The relations on the 2-morphisms of $\mathcal{U}^*$ are also conveniently expressed in this calculus. For example, one relation in $\mathcal{U}^*$ is the equation



that is used to reduce complex diagrams into simpler ones. Furthermore, topological intuition can be applied to these diagrams because the relations imposed on $\mathcal{U}^*$ ensure that any boundary preserving planar isotopy of a diagram results in the same 2-morphism.

Another advantage of formulating the definition of $\mathcal{U}^*$ in terms of these diagrams is that certain symmetries of the 2-category $\mathcal{U}^*$ become obvious. By a symmetry we mean certain invertible 2-functors on the category $\mathcal{U}^*$. We show in Theorem 9.16 that the invertible 2-functors constructed using the symmetries of the graphical calculus are graded lifts of well-known algebra maps on $\dot{\mathbf{U}}$. These algebra maps are recalled in Section 2.1 and Section 2.2.

1.2.3. Semilinear form

In Section 2.3 we define a semilinear form $\langle,\rangle$ on $\dot{\mathbf{U}}$ by twisting a certain bilinear form on $\dot{\mathbf{U}}$. We derive explicit formulas for the value of this semilinear form on elements in Lusztig's basis. We show in Theorem 9.18 that the enriched $\mathrm{hom}\dot{\mathcal{U}}^*(x,y)$ between any two 1-morphisms $x, y \in \dot{\mathcal{U}}$ is a categorification of this semilinear form in the sense that

$$\mathrm{rk}_q \dot{\mathcal{U}}^*(x,y) = \big\langle [x],[y] \big\rangle. \tag{1.3}$$

1.2.4. NilHecke action

There is an action of the nilHecke ring on the endomorphisms of $\mathcal{E}^a\mathbf{1}_n$. The biadjoint structure gives an action of the opposite of the nilHecke ring on the endomorphisms of $\mathcal{F}^a\mathbf{1}_n$. The *nilHecke algebra* denoted $\mathcal{NH}_a$ is the algebra with unit generated by u_i for $1 \leqslant i < a$, and pairwise commuting elements χ_i, for $1 \leqslant i \leqslant a$. The generators satisfy the relations

$$\begin{aligned}
&u_i^2 = 0 \quad (1 \leqslant i < a), && u_i\chi_j = \chi_j u_i \quad \text{if } |i-j|>1,\\
&u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \quad (1 \leqslant i < a-1), && u_i\chi_i = 1 + \chi_{i+1}u_i \quad (1 \leqslant i < a),\\
&u_i u_j = u_j u_i \quad \text{if } |i-j|>1, && \chi_i u_i = 1 + u_i\chi_{i+1} \quad (1 \leqslant i < a).
\end{aligned}$$

The relations in the left column are similar to the relations for the symmetric group except that $u_i^2 = 0$, rather than $u_i^2 = 1$. Many of the properties of the nilHecke algebra are collected in Section 3.

The algebra generated by the u_i subject to the relations in the left hand column is called the nilCoxeter algebra. This algebra has already appeared in the context of categorification. Khovanov uses this algebra to categorify the Weyl algebra in [27]. Its appearance again in the context of categorifying $\mathbf{U}_q(\mathfrak{sl}_2)$ is perhaps not a coincidence. In our context the nilHecke algebra action is built into the definition of the 2-category $\mathcal{U}^*$. One reason for this is that the nilHecke relations help to show that any diagram constructed from the generating 2-morphisms of $\mathcal{U}^*$

can be reduced to very simple diagrams. The nilpotency of the generators u_i in the nilCoxeter algebra control the size of the collection of 2-morphisms in $\dot{\mathcal{U}}$, effectively limiting the number of idempotent 2-morphisms. It is crucial to control idempotent 2-morphisms in order to have indecomposable 1-morphisms correspond bijectively to Lusztig canonical basis elements.

Another connection worth mentioning is the relationship to Chuang and Rouquier's $\mathfrak{sl}_2$ categorifications [12] (see also [39]). Within their framework for categorifying locally finite $\mathfrak{sl}_2$ representations they require an action of the degenerate affine Hecke algebra on the 2-morphisms of such a categorification. This action results in a beautiful classification theorem for $\mathfrak{sl}_2$-categorifications. They also provide many interesting examples which have an action of the degenerate affine Hecke algebra. Our goal is to extend their construction to the graded case, categorify $\mathbf{U}_q(\mathfrak{sl}_2)$ and Lusztig's canonical bases, all in the graphical framework emphasizing a new interplay between topology and algebra.

1.2.5. Iterated flag varieties

One drawback in defining the 2-morphisms of $\mathcal{U}^*$ by generators and relations is the difficulty in proving that the relations do not force the 2-morphisms to be trivial, or the other extreme, the relations might be so weak that the number of 2-morphisms becomes huge and unmanageable. To show that our relations appropriately control the size of the 2-morphisms in $\mathcal{U}^*$ we construct representations of $\mathcal{U}^*$ built from iterated flag varieties (Proposition 8.1). By restricting to the degree preserving 2-morphisms these representations provide representations of $\mathcal{U}$ as well.

What do we mean by a representation of a 2-category? When the algebra $\dot{\mathbf{U}}$ is regarded as a category, a representation is a functor from the category $\dot{\mathbf{U}}$ into some other category, like the category of vector spaces, or rings. Thus, a representation of the 2-category $\mathcal{U}^*$ is a 2-functor from $\mathcal{U}^*$ to some other 2-category, such as the 2-category **Bim** whose objects are rings, morphisms are bimodules, and 2-morphisms are bimodule maps.

For each positive integer N we construct representations $\Gamma_N : \mathcal{U}^* \to \mathbf{Flag}_N^*$ (Theorem 7.12). This induces a representation $\dot{\Gamma}_N : \dot{\mathcal{U}} \to \mathbf{Flag}_N^*$ (see Theorem 9.15). The 2-category $\mathbf{Flag}_N^*$ is a sub-2-category of **Bim** whose objects are the cohomology rings of certain Grassmannian varieties. The morphisms in $\mathbf{Flag}_N^*$ are bimodules constructed using the cohomology rings of iterated flag varieties, and the 2-morphisms are bimodule maps.

It is known that iterated flag varieties categorify irreducible representations of $\mathbf{U}_q(\mathfrak{sl}_2)$ [4,12, 17]. Here we extend this work to show that the 2-category $\mathbf{Flag}_N^*$, categorifying the $(N+1)$-dimensional irreducible representation of $\mathbf{U}_q(\mathfrak{sl}_2)$, carries the additional structure coming from the representation Γ_N. This includes the NilHecke action described above, biadjointness, and the rest of the relations of $\dot{\mathcal{U}}$. All of this structure is given explicitly.

Proving that the Γ_N preserve the relations of the 2-category $\mathcal{U}^*$ is a difficult task. To prove this we develop a different graphical calculus for performing calculations with the cohomology rings of iterated flag varieties. Using the representations Γ_N we are able to show that the 2-morphisms in $\mathcal{U}^*$ have the appropriate size. This is then used to show that the split Grothendieck ring of $\dot{\mathcal{U}}$ is isomorphic to the algebra $\dot{\mathbf{U}}$.

2. $\mathbf{U}_q(\mathfrak{sl}_2)$

2.1. Conventional $\mathbf{U}_q(\mathfrak{sl}_2)$

Definition 2.1. The quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$ is the associative algebra (with unit) over $\mathbb{Q}(q)$ with generators E, F, K, K^{-1} and relations

$$KK^{-1} = 1 = K^{-1}K, \tag{2.1}$$

$$KE = q^2 EK, \tag{2.2}$$

$$KF = q^{-2} FK, \tag{2.3}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{2.4}$$

For simplicity the algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ is written $\mathbf{U}$.

Define the quantum integer $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$ with $[0] = 1$ by convention. The quantum factorial is then $[a]! = [a][a-1]\dots[1]$, and the quantum binomial coefficient $\left[{a \atop b} \right] = \frac{[a]!}{[b]![a-b]!}$, for $0 \leqslant b \leqslant a$. For $a \geqslant 0$ define the divided powers $E^{(a)} = \frac{E^a}{[a]!}$ and $F^{(a)} = \frac{F^a}{[a]!}$. Denote the $\mathbb{Z}[q, q^{-1}]$ form of $\mathbf{U}$ by ${}_{\mathcal{A}}\mathbf{U}$. This $\mathbb{Z}[q, q^{-1}]$-algebra is the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathbf{U}$ spanned by products of elements in the set

$$\left\{ E^{(a)}, F^{(a)}, K^{\pm 1} \;\middle|\; a \in \mathbb{Z}_+ \right\}.$$

Important algebra automorphisms of $\mathbf{U}$*:*

Let $\bar{\ }$ be the $\mathbb{Q}$-linear involution of $\mathbb{Q}(q)$ which maps q to q^{-1}.

- The $\mathbb{Q}(q)$-antilinear algebra isomorphism $\psi : \mathbf{U} \to \mathbf{U}$ is given by

$$\psi(E) = E, \quad \psi(F) = F, \quad \psi(K) = K^{-1},$$
$$\psi(fx) = \bar{f}\psi(x) \quad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \mathbf{U}.$$

- There is a $\mathbb{Q}(q)$-linear algebra automorphism $\omega : \mathbf{U} \to \mathbf{U}$

$$\begin{aligned} &\omega(E) = F, && \omega(F) = E, \quad \omega(K) = K^{-1}, \\ &\omega(fx) = f\omega(x), && \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \mathbf{U}, \\ &\omega(xy) = \omega(x)\omega(y), && \text{for } x, y \in \mathbf{U}, \end{aligned}$$

  that is its own inverse.
- There is a $\mathbb{Q}(q)$-linear algebra antiautomorphism $\sigma : \mathbf{U} \to \mathbf{U}^{\mathrm{op}}$

$$\begin{aligned} &\sigma(E) = E, && \sigma(F) = F, \quad \sigma(K) = K^{-1}, \\ &\sigma(fx) = f\sigma(x), && \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \mathbf{U}, \\ &\sigma(xy) = \sigma(y)\sigma(x), && \text{for } x, y \in \mathbf{U}. \end{aligned}$$

- The algebra $\mathbf{U}$ has a $\mathbb{Q}(q)$-antilinear antiautomorphism $\tau : \mathbf{U} \to \mathbf{U}^{\mathrm{op}}$ described by

$$\tau(E) = qFK^{-1}, \qquad \tau(F) = qEK, \quad \tau(K) = K^{-1},$$
$$\tau(fx) = \bar{f}\tau(x), \qquad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \mathbf{U},$$
$$\tau(xy) = \tau(y)\tau(x), \qquad \text{for } x, y \in \mathbf{U}.$$

The inverse of τ is given by $\tau^{-1}(E) = q^{-1}FK$, $\tau^{-1}(F) = q^{-1}EK^{-1}$, $\tau^{-1}(K) = K^{-1}$.

- The $\mathbb{Q}(q)$-linear algebra antiautomorphism $\rho : \mathbf{U} \to \mathbf{U}^{\mathrm{op}}$ given by

$$\rho(E) = qKF, \qquad \rho(F) = qK^{-1}E, \quad \rho(K) = K,$$
$$\rho(fx) = f\rho(x), \qquad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \mathbf{U},$$
$$\rho(xy) = \rho(y)\rho(x), \qquad \text{for } x, y \in \mathbf{U},$$

is its own inverse, $\rho^2 = 1$. It is related to the antialgebra automorphism τ by

$$\rho = \psi\tau, \quad \rho = \tau^{-1}\psi, \quad \tau = \psi\rho. \tag{2.5}$$

2.2. *Lusztig's quantum* $\mathfrak{sl}_2$

Recall that $\mathbf{U}^0$ denotes the toral part of $\mathbf{U}$ which, in the case of $\dot{\mathbf{U}}(\mathfrak{sl}_2)$, is spanned by generators K and K^{-1}. The $\mathbb{Q}(q)$-algebra $\dot{\mathbf{U}}$ is a modified form of $\mathbf{U}$ in which $\mathbf{U}^0$ is replaced by a collection of orthogonal idempotents 1_n for $n \in \mathbb{Z}$

$$1_n 1_m = \delta_{n,m} 1_n, \tag{2.6}$$

indexed by the weight lattice of $\mathfrak{sl}_2$, such that

$$K1_n = 1_n K = q^n 1_n, \quad E1_n = 1_{n+2}E, \quad F1_n = 1_{n-2}F. \tag{2.7}$$

Similarly, the $\mathbb{Z}[q, q^{-1}]$-subalgebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ of $\dot{\mathbf{U}}$ is a modified form of ${}_{\mathcal{A}}\mathbf{U}$ where again the toral part ${}_{\mathcal{A}}\mathbf{U}^0$ is replaced by a collection of orthogonal idempotents (2.6), such that

$$K1_n = 1_n K = q^n 1_n, \quad E^{(a)}1_n = 1_{n+2a}E^{(a)}, \quad F^{(a)}1_n = 1_{n-2a}F^{(a)}. \tag{2.8}$$

There are direct sum decompositions of algebras

$$\dot{\mathbf{U}} = \bigoplus_{n,m\in\mathbb{Z}} 1_m \dot{\mathbf{U}} 1_n, \qquad {}_{\mathcal{A}}\dot{\mathbf{U}} = \bigoplus_{n,m\in\mathbb{Z}} 1_m ({}_{\mathcal{A}}\dot{\mathbf{U}}) 1_n$$

with $1_m({}_{\mathcal{A}}\dot{\mathbf{U}})1_n$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra spanned by $1_m E^{(a)} F^{(b)} 1_n$ and $1_m F^{(b)} E^{(a)} 1_n$ for $a, b \in \mathbb{Z}_+$.

The algebra $\dot{\mathbf{U}}$ does not have a unit since the infinite sum $\sum_{n\in\mathbb{Z}} 1_n$ is not an element in $\dot{\mathbf{U}}$; however, the system of idempotents $\{1_n \mid n \in \mathbb{Z}\}$ in some sense serves as a substitute for a unit. Algebras with systems of idempotents have a natural interpretation as pre-additive categories. In this interpretation, $\dot{\mathbf{U}}$ is a category with one object n for each $n \in \mathbb{Z}$ with homs from n to m given by the abelian group $1_n \dot{\mathbf{U}} 1_m$. The idempotents 1_n are the identity morphisms for this category and composition is given by the algebra structure of $\dot{\mathbf{U}}$. A similar interpretation of algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ as a pre-additive category also holds.

Some of the relations in ${}_{\mathcal{A}}\dot{\mathbf{U}}$ are collected below for later convenience:

$$E^{(a)}E^{(b)}1_n = \begin{bmatrix} a+b \\ a \end{bmatrix} E^{(a+b)}1_n, \tag{2.9}$$

$$F^{(a)}F^{(b)}1_n = \begin{bmatrix} a+b \\ a \end{bmatrix} F^{(a+b)}1_n, \tag{2.10}$$

$$F^{(a)}E^{(b)}1_n = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a-b-n \\ j \end{bmatrix} E^{(b-j)}F^{(a-j)}1_n, \tag{2.11}$$

$$E^{(a)}F^{(b)}1_n = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a-b+n \\ j \end{bmatrix} F^{(b-j)}E^{(a-j)}1_n. \tag{2.12}$$

Lusztig's canonical basis $\dot{\mathbb{B}}$ of $\dot{\mathbf{U}}$ consists of the elements

(i) $E^{(a)}1_{-n}F^{(b)}$ for $a, b, n \in \mathbb{N}$, $n \geqslant a+b$,
(ii) $F^{(b)}1_nE^{(a)}$ for $a, b, n \in \mathbb{N}$, $n \geqslant a+b$,

where $E^{(a)}1_{-a-b}F^{(b)} = F^{(b)}1_{a+b}E^{(a)}$. The importance of this basis is that the structure constants are in $\mathbb{N}[q, q^{-1}]$. In particular, for $x, y \in \dot{\mathbb{B}}$

$$xy = \sum_{x \in \dot{\mathbb{B}}} m_{x,y}^z z$$

with $z \in \dot{\mathbb{B}}$ and $m_{x,y}^z \in \mathbb{N}[q, q^{-1}]$. We rewrite Lusztig's basis $\dot{\mathbb{B}}$ of $\dot{\mathbf{U}}$ in the form:

(i) $E^{(a)}F^{(b)}1_n$ for $a,b \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, $n \leqslant b-a$,
(ii) $F^{(b)}E^{(a)}1_n$ for $a,b \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, $n \geqslant b-a$,

where $E^{(a)}F^{(b)}1_{b-a} = F^{(b)}E^{(a)}1_{b-a}$. Let ${}_m\dot{\mathbb{B}}_n$ denote the set of elements in $\dot{\mathbb{B}}$ belonging to $1_m\dot{\mathbf{U}}1_n$. Then the set $\dot{\mathbb{B}}$ is a union

$$\dot{\mathbb{B}} = \coprod_{n,m \in \mathbb{Z}} {}_m\dot{\mathbb{B}}_n.$$

The algebra maps ψ, ω, σ, τ and ρ all naturally extend to the integral form of $\mathbf{U}_q(\mathfrak{sl}_2)$ if we set

$$\psi(1_n) = 1_n, \quad \omega(1_n) = 1_{-n}, \quad \sigma(1_n) = 1_{-n}, \quad \rho(1_n) = 1_n, \quad \tau(1_n) = 1_n.$$

Taking direct sums of the induced maps on each summand $1_m\dot{\mathbf{U}}1_n$ allows these maps to be extended to $\dot{\mathbf{U}}$. For example, the antiautomorphism τ induces for each n and m in $\mathbb{Z}$ an isomorphism $1_m\dot{\mathbf{U}}1_n \to 1_n\dot{\mathbf{U}}1_m$. Restricting to the $\mathbb{Z}[q, q^{-1}]$-subalgebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ and taking direct sums, we obtain an algebra antiautomorphism $\tau : {}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\dot{\mathbf{U}}$ such that $\tau(1_n) = 1_n$, $\tau(1_{n+2}E1_n) = q^{-1-n}1_nF1_{n+2}$, and $\tau(1_nF1_{n+2}) = q^{1+n}1_{n+2}E1_n$ for all $n \in \mathbb{Z}$. These $\mathbb{Z}[q, q^{-1}]$-(anti)linear (anti)algebra homomorphisms are recorded below on some elements of ${}_{\mathcal{A}}\dot{\mathbf{U}}$:

$$\omega\colon \quad \begin{array}{l} q^s 1_m E^{(a)} F^{(b)} 1_n \mapsto q^s 1_{-m} F^{(a)} E^{(b)} 1_{-n} \\ q^s 1_m F^{(b)} E^{(a)} 1_n \mapsto q^s 1_{-m} E^{(b)} F^{(a)} 1_{-n} \end{array}, \tag{2.13}$$

$$\sigma\colon \quad \begin{array}{l} q^s 1_n E^{(a)} F^{(b)} 1_n \mapsto q^s 1_{-n} F^{(b)} E^{(a)} 1_{-m} \\ q^s 1_m F^{(b)} E^{(a)} 1_n \mapsto q^s 1_{-n} E^{(a)} F^{(b)} 1_{-m} \end{array}, \tag{2.14}$$

$$\psi\colon \quad \begin{array}{l} q^s 1_m E^{(a)} F^{(b)} 1_n \mapsto q^{-s} 1_m E^{(a)} F^{(b)} 1_n \\ q^s 1_m F^{(b)} E^{(a)} 1_n \mapsto q^{-s} 1_m F^{(b)} E^{(a)} 1_n \end{array}, \tag{2.15}$$

$$\tau\colon \quad \begin{array}{l} q^s 1_m E^{(a)} F^{(b)} 1_n \mapsto q^{-s-(a-b)(a-b+n)} 1_n E^{(b)} F^{(a)} 1_m \\ q^s 1_m F^{(b)} E^{(a)} 1_n \mapsto q^{-s-(a-b)(a-b+n)} 1_n E^{(b)} F^{(a)} 1_m \end{array}. \tag{2.16}$$

Notice that for $x \in \dot{\mathbb{B}}$ we have $\psi(x) = x$ so that ψ fixes elements in Lusztig's canonical basis.

2.3. The semilinear form on $\dot{\mathbf{U}}$

Proposition 2.2. *(See Lusztig [34, 26.1.1].) There exists a pairing $(,)\colon \dot{\mathbf{U}} \times \dot{\mathbf{U}} \to \mathbb{Z}[q, q^{-1}]$ with the properties:*

(a) $(,)$ *is bilinear, i.e., $(fx, y) = f(x, y)$, $(x, fy) = f(x, y)$, for $f \in \mathbb{Z}[q, q^{-1}]$ and $x, y \in \dot{\mathbf{U}}$,*
(b) $(1_n x 1_m, 1_{n'} y 1_{m'}) = 0$ *for all $x, y \in \dot{\mathbf{U}}$ unless $n = n'$ and $m = m'$,*
(c) $(ux, y) = (x, \rho(u)y)$ *for $u \in \mathbf{U}$ and $x, y \in \dot{\mathbf{U}}$,*
(d) $\langle E^{(a)} 1_n, E^{(a)} 1_n) = (F^{(a)} 1_n, F^{(a)} 1_n) = \prod_{s=1}^{a} \frac{1}{(1-q^{-2s})}$ *(see* [34], *Lemma* 1.4.4*),*
(e) *we have $(x, y) = (y, x)$ for all $x, y \in \dot{\mathbf{U}}$.*

Definition 2.3. Define a semilinear form $\langle,\rangle\colon \dot{\mathbf{U}} \times \dot{\mathbf{U}} \to \mathbb{Z}[q, q^{-1}]$ by

$$\langle x, y\rangle := \overline{\bigl(x, \psi(y)\bigr)} \quad \text{for all } x, y \in \dot{\mathbf{U}}.$$

From the proposition above this semilinear form has the properties given below:

Proposition 2.4. *The map $\langle,\rangle\colon \dot{\mathbf{U}} \times \dot{\mathbf{U}} \to \mathbb{Z}[q, q^{-1}]$ has the following properties*

(i) $\langle,\rangle$ *is semilinear, i.e., $\langle fx, y\rangle = \bar{f}\langle x, y\rangle$, $\langle x, fy\rangle = f\langle x, y\rangle$, for $f \in \mathbb{Z}[q, q^{-1}]$ and $x, y \in \dot{\mathbf{U}}$;*
(ii) Hom *property: $\langle 1_n x 1_m, 1_{n'} y 1_{m'}\rangle = 0$ for all $x, y \in \dot{\mathbf{U}}$ unless $n = n'$ and $m = m'$;*
(iii) *Adjoint property: $\langle ux, y\rangle = \langle x, \tau(u)y\rangle$ for $u \in \mathbf{U}$ and $x, y \in \dot{\mathbf{U}}$;*
(iv) *Grassmannian property: $\langle E^{(a)} 1_n, E^{(a)} 1_n\rangle = \langle F^{(a)} 1_n, F^{(a)} 1_n\rangle = \prod_{j=1}^{a} \frac{1}{(1-q^{2j})}$;*
(v) *We have $\langle x, y\rangle = \langle \psi(y), \psi(x)\rangle$ for all $x, y \in \dot{\mathbf{U}}$.*

Property (iv) is referred to as the Grassmannian property because the graded rank (denoted rk_q) of the cohomology ring of the Grassmannian $Gr(k, \infty)$ of k-planes in $\mathbb{C}^\infty$ is given by

$$\mathrm{rk}_q\bigl(H^*\bigl(Gr(k, \infty)\bigr)\bigr) = \prod_{j=1}^{k} \frac{1}{(1 - q^{2j})}.$$

We use the notation $g(a) := \prod_{j=1}^{a} \frac{1}{(1-q^{2j})}$ in what follows, so that

$$\langle E^{(a)}1_n, E^{(a)}1_n\rangle = \langle F^{(a)}1_n, F^{(a)}1_n\rangle = g(a).$$

Proof. The only nontrivial property to check is (iii). This follows from (2.5) since

$$\begin{aligned}\langle ux, y\rangle &:= \overline{(ux, \psi(y))} = \overline{(x, \rho(u)\psi(y))} = \overline{(x, \psi(\tau(u))\psi(y))} \\ &= \overline{(x, \psi(\tau(u)y))} =: \langle x, \tau(u)y\rangle. \qquad \square\end{aligned} \tag{2.17}$$

The bilinear and semilinear forms restrict to pairings

$$(,) : {}_{\mathcal{A}}\dot{\mathbf{U}} \times {}_{\mathcal{A}}\dot{\mathbf{U}} \to \mathbb{Z}[q, q^{-1}], \qquad \langle,\rangle : {}_{\mathcal{A}}\dot{\mathbf{U}} \times {}_{\mathcal{A}}\dot{\mathbf{U}} \to \mathbb{Z}[q, q^{-1}]. \tag{2.18}$$

Proposition 2.5. *We have*

$$\langle \omega(x), \omega(y)\rangle = \langle x, y\rangle, \tag{2.19}$$

$$\langle \sigma(x), \sigma(y)\rangle = \langle x, y\rangle \tag{2.20}$$

for all x, y in $\dot{\mathbf{U}}$ and ${}_{\mathcal{A}}\dot{\mathbf{U}}$.

Proof. This follows immediately from the corresponding property on the bilinear form $\langle,\rangle$ (see Lusztig [34], Propositions 26.1.4 and 26.1.6). $\square$

Consider the two subsets

$$\begin{aligned}\dot{\mathbb{B}}^{+} &:= \{E^{(a)}F^{(b)}1_n \mid a, b, \in \mathbb{Z}_+,\ n \in \mathbb{Z},\ n \leqslant b-a\}, \\ \dot{\mathbb{B}}^{-} &:= \{F^{(d)}E^{(c)}1_n \mid d, c, \in \mathbb{Z}_+,\ n \in \mathbb{Z},\ n \geqslant d-c\}\end{aligned} \tag{2.21}$$

of elements in $\dot{\mathbb{B}}$. Note that $\dot{\mathbb{B}}^{+} \cap \dot{\mathbb{B}}^{-}$ consists of those elements for which $n = b-a = d-c$.

Lemma 2.6. *For fixed $n, m \in \mathbb{Z}$, either ${}_m\dot{\mathbb{B}}_n \subset \dot{\mathbb{B}}^{+}$, or ${}_m\dot{\mathbb{B}}_n \subset \dot{\mathbb{B}}^{-}$. That is, ${}_m\dot{\mathbb{B}}_n$ only contains elements of same form, either $1_mE^{(a)}F^{(b)}1_n$, or $1_mF^{(d)}E^{(c)}1_n$, except when $n = b-a = d-c$.*

Proof. This is obvious. If $1_mE^{(a)}F^{(b)}1_n \in {}_m\dot{\mathbb{B}}_n$ and $1_mF^{(d)}E^{(c)}1_n \in {}_m\dot{\mathbb{B}}_n$, then $m = n + 2(a - b) = n + 2(c-d)$, so that $a-b = c-d$. Hence, $n \leqslant b-a = d-c \leqslant n$ which implies $n = b-a = d-c$. $\square$

The lemma implies that to determine the semilinear form on all elements in $\dot{\mathbb{B}}$ we need only compute the inner product on basis elements of the same form, i.e.

$$\langle E^{(a)}F^{(b)}1_n, E^{(c)}F^{(d)}1_n\rangle \quad \text{with } a-b = c-d \text{ and } n \leqslant b-a, \tag{2.22}$$

$$\langle F^{(b)}E^{(a)}1_n, F^{(d)}E^{(c)}1_n\rangle \quad \text{with } a-b = c-d \text{ and } n \geqslant b-a, \tag{2.23}$$

since when $n = b-a$ the basis elements $E^{(a)}F^{(b)}1_n$ and $F^{(b)}E^{(a)}1_n$ coincide.

We now determine an explicit formula for the semilinear form of Definition 2.3 for elements in $\dot{\mathbb{B}}$.

Proposition 2.7. *The value of the semilinear form $\langle x, y\rangle$ with $x, y \in \dot{\mathbb{B}}$ is given by the following formulas:*

$$\begin{aligned}
&\bigl\langle E^{(a)}F^{(b)}1_n, E^{(c)}F^{(d)}1_n\bigr\rangle \\
&\quad = \sum_{j=0}^{\min(a,c)} \begin{bmatrix} b+d-n \\ j \end{bmatrix}\begin{bmatrix} b+c-j \\ b \end{bmatrix}\begin{bmatrix} a+d-j \\ d \end{bmatrix} q^{(a+c-j)(b+d-j-n)} g(b+c-j), \\
&\bigl\langle F^{(b)}E^{(a)}1_n, F^{(d)}E^{(c)}1_n\bigr\rangle \\
&\quad = \sum_{j=0}^{\min(b,d)} \begin{bmatrix} a+c+n \\ j \end{bmatrix}\begin{bmatrix} a+d-j \\ a \end{bmatrix}\begin{bmatrix} b+c-j \\ c \end{bmatrix} q^{(b+d-j)(a+c-j+n)} g(a+d-j)
\end{aligned} \tag{2.24}$$

for $a-b=c-d$ to ensure that the values are nonzero.

Proof. Using the properties in Proposition 2.4, the relations in $\dot{\mathbf{U}}$, and the fact that $a-b=c-d$ the inner product is

$$\begin{aligned}
&\bigl\langle E^{(a)}F^{(b)}1_n, E^{(c)}F^{(d)}1_n\bigr\rangle \\
&\quad = \bigl\langle F^{(b)}1_n, \tau\bigl(E^{(a)}\bigr)E^{(c)}F^{(d)}1_n\bigr\rangle \\
&\quad = q^{-a(n+2(c-d)+a)}\bigl\langle F^{(b)}1_n, F^{(a)}E^{(c)}F^{(d)}1_n\bigr\rangle \\
&\quad = q^{-a(n+2(c-d)+a)} \sum_{j=0}^{\min(a,c)} \begin{bmatrix} a-c-(n-2d) \\ j \end{bmatrix}\bigl\langle F^{(b)}1_n, E^{(c-j)}F^{(a-j)}F^{(d)}1_n\bigr\rangle \\
&\quad = q^{a(2b-a-n)} \sum_{j=0}^{\min(a,c)} \begin{bmatrix} b+d-n \\ j \end{bmatrix}\bigl\langle \tau^{-1}\bigl(E^{(c-j)}\bigr)F^{(b)}1_n, F^{(a-j)}F^{(d)}1_n\bigr\rangle \\
&\quad = q^{(a+c-j)(b+d-j-n)} \sum_{j=0}^{\min(a,c)} \begin{bmatrix} b+d-n \\ j \end{bmatrix}\bigl\langle F^{(c-j)}F^{(b)}1_n, F^{(a-j)}F^{(d)}1_n\bigr\rangle
\end{aligned}$$

and using (2.9) and (2.10) we are done. A similar calculation establishes the other inner product formula. □

The following formulas for the semilinear form are often more convenient.

Proposition 2.8. *For all values of $a, b, c, d \in \mathbb{Z}_+$ with $a-b=c-d$ and $n \in \mathbb{Z}$ we have*

$$\begin{aligned}
&\left\langle E^{(a)}F^{(b)}1_n, E^{(c)}F^{(d)}1_n\right\rangle \\
&\quad = \sum_{j=\max(0,a-c)}^{\min(a,b)} q^{2j^2+(a-d+n)(a-c-2j)} g(a-j)g(b-j)g(j)g(c-a+j), \\
&\left\langle F^{(b)}E^{(a)}1_n, F^{(d)}E^{(c)}1_n\right\rangle \\
&\quad = \sum_{j=\max(0,a-c)}^{\min(a,b)} q^{2j^2+(b-c-n)(b-d-2j)} g(a-j)g(b-j)g(j)g(c-a+j).
\end{aligned}$$

Proof. This tedious proof appears in Section 10. □

3. Review of nilHecke ring and Schubert polynomials

In this section we introduce the nilHecke ring and recall some facts about Schubert polynomials.

3.1. The nilHecke ring $\mathcal{NH}_a$

Definition 3.1. The *nilCoxeter ring*, denoted $\mathcal{NC}_a$, is the unital ring generated by u_i for $1 \leqslant i < a$ subject to the relations:

$$u_i^2 = 0 \quad (1 \leqslant i < a), \tag{3.1}$$

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \quad (1 \leqslant i < a-1), \tag{3.2}$$

$$u_i u_j = u_j u_i \quad \text{if } |i-j| > 1. \tag{3.3}$$

In the symmetric group S_a we denote the elementary transposition that interchanges i and $i+1$ by s_i. For a permutation $w \in S_a$ an expression $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ of minimal possible length is called a *reduced word presentation* of w, and the length $\ell(w)$ of w is m. We denote by w_0 the element in S_a of maximal length. If an elementary permutation s_i is depicted as a crossing of the ith and $(i+1)$st strands

$s_i \quad \rightsquigarrow$  ,

with multiplication achieved by stacking such diagrams on top or each other, then a reduced word for w is represented by a diagram in which each strand never crosses itself and intersects other strands at most once. The element w_0 corresponds to a diagram where each strand crosses every other strand exactly once.

Given a permutation $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ in S_a we define the element $u_w \in \mathcal{NC}_a$ by $u_w = u_{i_1}u_{i_2}\cdots u_{i_m}$. Here it is important to choose a reduced word presentation of w because the element in $\mathcal{NC}_a$ would be zero otherwise by (3.1). Furthermore, it follows from the nilCoxeter relations that u_w does not depend on the particular choice of reduced word presentation.

The nilCoxeter ring $\mathcal{NC}_a$ is a graded ring with $\deg u_i = -2$. The set $\{u_w\}_{w \in S_a}$ forms a basis of $\mathcal{NC}_a$ as a free abelian group so that the graded rank of $\mathcal{NC}_a$ is given by

$$\begin{aligned}\mathrm{rk}_q\, \mathcal{NC}_a &= \left(1+q^{-2}\right)\left(1+q^{-2}+q^{-4}\right)\cdots\left(1+q^{-2}+\cdots+q^{-2(a-1)}\right)\\ &= \prod_{s=1}^{a}\frac{1-q^{-2s}}{1-q^{-2}} = \prod_{s=1}^{a} q^{1-s}[s] = q^{-a(a-1)/2}[a]! \end{aligned} \tag{3.4}$$

where $[a]!$ denotes the quantum factorial defined in Section 2.2.

Definition 3.2. The *nilHecke ring*, denoted $\mathcal{NH}_a$, is the ring with unit generated by u_i for $1 \leqslant i < a$, and pairwise commuting elements χ_i, for $1 \leqslant i \leqslant a$. The generators satisfy (3.1)–(3.3) and

$$u_i\chi_j = \chi_j u_i \quad \text{if } |i-j|>1, \tag{3.5}$$

$$u_i\chi_i = 1+\chi_{i+1}u_i \quad (1 \leqslant i < a), \tag{3.6}$$

$$\chi_i u_i = 1+u_i\chi_{i+1} \quad (1 \leqslant i < a). \tag{3.7}$$

We will often regard $\mathcal{NH}_a$ as a graded ring with $\deg u_i = -2$ and $\deg \chi_i = 2$.

When we work over a field we refer to the nilHecke ring as the nilHecke algebra. The nilHecke algebra was introduced by Kostant and Kumar [30]; an introduction to this algebra appears in the thesis of Postnikov ([38], Chapter 2).

The nilCoxeter ring $\mathcal{NC}_a$ and the polynomial ring $\mathbb{Z}[\chi_1,\dots,\chi_a]$ are both subrings of $\mathcal{NH}_a$. As a graded abelian group $\mathcal{NH}_a = \mathcal{NC}_a \otimes_{\mathbb{Z}} \mathbb{Z}[\chi_1,\dots,\chi_a]$ so that using (3.4) we have

$$\mathrm{rk}_q\, \mathcal{NH}_a = \mathrm{rk}_q\, \mathcal{NC}_a \cdot \mathrm{rk}_q\, \mathbb{Z}[\chi_1,\dots,\chi_a] = \left(q^{-a(a-1)/2}[a]!\right)\left(\frac{1}{1-q^2}\right)^a. \tag{3.8}$$

It is not hard to see that a basis of $\mathcal{NH}_a$ is given by the set of elements

$$\left\{ f_w(\chi)u_w \;\middle|\; w \in S_a \text{ and } f_w(\chi) \text{ monomials in } \mathbb{Z}[\chi_1,\dots,\chi_a]\right\}. \tag{3.9}$$

3.2. Divided difference operators and Schubert polynomials

Denote by $\mathcal{P}_a$ the graded polynomial ring $\mathbb{Z}[x_1,\dots,x_a]$ with $\deg x_i = 2$. Define the divided difference operators[1] $\partial_i : \mathcal{P}_a \to \mathcal{P}_a$ by

$$\partial_i := \frac{1-s_i}{x_i - x_{i+1}}. \tag{3.10}$$

From this definition it is clear that both the image and kernel of the operator ∂_i consist of polynomials which are symmetric in x_i and x_{i+1}. From the definition we have

[1] A generalization of divided differences has been studied in the context of the cohomology of generalized flag varieties independently by Bernstein, Gelfand and Gelfand [6] and Demazure [15].

$$\partial_i^2 = 0 \quad (1 \leqslant i < a), \tag{3.11}$$

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \quad (1 \leqslant i < a-1), \tag{3.12}$$

$$\partial_i \partial_j = \partial_j \partial_i \quad \text{if } |i-j| > 1, \tag{3.13}$$

so that the collection of divided difference operators on $\mathcal{P}_a$ provide a representation of $\mathcal{NC}_a$ with $u_i \mapsto \partial_i$. This extends to an action of the nilHecke ring $\mathcal{NH}_a$ on $\mathcal{P}_a$ with χ_i acting by multiplication by x_i and u_j acting by ∂_j. We denote the action of u_w by ∂_w.

Below we collect some well-known facts about the divided difference operators. For details, see for example Manivel [36, Chapter 2] and the references therein. It follows immediately from (3.11)–(3.13) that

$$\partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v), \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Definition 3.3. For $w \in S_a$ define the *Schubert polynomials* of Lascoux and Schützenberger [31] as

$$\mathfrak{S}_w(x) = \partial_{w^{-1}w_0} x^\delta \tag{3.15}$$

where w_0 is the permutation of maximal length and $x^\delta = x_1^{a-1} x_2^{a-2} \cdots x_{a-1}$.

The Schubert polynomial $\mathfrak{S}_w(x)$ is a homogeneous polynomial of degree $2\ell(w)$ in $\mathcal{P}_a$. The free graded abelian group generated by $\{\mathfrak{S}_w \mid w \in S_a\}$ has graded rank

$$\left(1+q^2\right)\left(1+q^2+q^4\right)\cdots\left(1+q^2+\cdots+q^{2(a-1)}\right) = \prod_{s=1}^{a} \frac{1-q^{2s}}{1-q^2} = q^{a(a-1)/2}[a]!\,. \tag{3.16}$$

This expression is most naturally expressed in terms of the nonsymmetric quantum integers

$$(j)_{q^2} := \frac{1-q^{2j}}{1-q^2} = \left(1+q^2+\cdots+q^{2(j-1)}\right),$$

which are related to the symmetric quantum integers by $(j)_{q^2} = q^{j-1}[j]$. The graded rank above is just the nonsymmetric quantum factorial $(a)_{q^2}^! = q^{a(a-1)/2}[a]!$.

The action of divided difference operators on the Schubert polynomials is given by

$$\partial_u \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u), \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

In particular, $\partial_w \mathfrak{S}_w = \mathfrak{S}_{ww^{-1}} = \mathfrak{S}_1 = 1$.

Denote by Λ_a the space of symmetric polynomials in the variables $x_1, \ldots, x_a$. The graded rank of Λ_a is

$$\operatorname{rk}_q \Lambda_a = \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2a-2})} = \frac{1}{(a)_{q^2}^!} \frac{1}{(1-q^2)^a}.$$

Proposition 3.4. *(See [36, Propositions 2.5.3 and 2.5.5].) The abelian subgroup $\mathcal{H}_a$ of $\mathcal{P}_a$ generated by monomials $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_a^{\alpha_a}$, with $\alpha_i \leqslant a-i$ for all $1 \leqslant i \leqslant a$, has rank $a!$. The Schubert polynomials $\mathfrak{S}_w$, as w runs through S_a, form an integral base of $\mathcal{H}_a$. The multiplication in $\mathcal{P}_a$ induces a canonical isomorphism of the tensor product $\mathcal{H}_a \otimes \Lambda_a$ with $\mathcal{P}_a$ as graded Λ_a-modules.*

The proposition together with (3.16) imply that the basis of Schubert polynomials expresses $\mathcal{P}_a$ as a free Λ_a-module of graded rank $(a)^!_{q^2}$. In this basis, every endomorphism of $\mathcal{P}_a$ can be expressed by an $a! \times a!$ sized matrix whose coefficients are in Λ_a. The ring of such matrices inherits a natural grading from the basis of Schubert polynomials. Hence, $\mathrm{Hom}_{\Lambda_a}(\mathcal{P}_a, \mathcal{P}_a)$ is isomorphic as a graded ring to the ring $\mathrm{Mat}((a)^!_{q^2}; \Lambda_a)$ of $(a)^!_{q^2} \times (a)^!_{q^2}$-matrices with coefficients in Λ_a. In particular, $\mathrm{rk}_q \mathrm{Mat}((a)^!_{q^2}; \Lambda_a) = (a)^!_{q^2}(a)^!_{q^{-2}} \mathrm{rk}_q \Lambda_a$. Furthermore, we get a homomorphism $\varphi : \mathcal{NH}_a \to \mathrm{Mat}((a)^!_{q^2}; \Lambda_a)$ given by letting $\mathcal{NH}_a$ act on $\mathcal{P}_a \cong \oplus_{(a)^!_{q^2}} \Lambda_a$ using the action defined above.

Proposition 3.5. *The homomorphism $\varphi : \mathcal{NH}_a \to \mathrm{Mat}((a)^!_{q^2}; \Lambda_a)$ given by letting $\mathcal{NH}_a$ act on $\mathcal{P}_a \cong \oplus_{(a)^!_{q^2}} \Lambda_a$ induces an isomorphism of graded rings.*

Proof. Order the basis of Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_a\}$ by length. To see that φ is injective suppose that for some collection of $f_w(\chi) \in \mathbb{Z}[\chi_1, \chi_2, \ldots, \chi_a]$ we have

$$\varphi\left(\sum_{w \in S_a} f_w(\chi) u_w\right) = \sum_{w \in S_a} f_w(x)\partial_w = 0 \tag{3.18}$$

with $f_w(x) \in \mathcal{P}_a$. In particular, for all polynomials $p \in \mathcal{P}_a$ we must have

$$\sum_{w \in S_a} f_w \partial_w(p) = 0. \tag{3.19}$$

Choose $v_0 \in S_a$ of minimal length in the above sum. Then we must have for $p = \mathfrak{S}_{v_0}$,

$$\sum_{w \in S_a} f_w \partial_w(\mathfrak{S}_{v_0}) = 0, \tag{3.20}$$

but by (3.17)

$$\partial_w \mathfrak{S}_{v_0} = \begin{cases} \mathfrak{S}_{v_0 w^{-1}} & \text{if } \ell(v_0 w^{-1}) = \ell(v_0) - \ell(w), \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Hence, the only contribution to (3.20) is from $w \in S_a$ with $\ell(w) = \ell(v_0)$. Then $\ell(v_0 w^{-1}) = \ell(v_0) - \ell(w) = 0$, but the only length zero element is the identity. Thus, $v_0 = w$ and (8.11) implies $f_{v_0} = 0$. Applying this argument inductively we have that all $f_w = 0$ proving injectivity.

To see that φ is surjective we show that the elementary matrices $E_{i,j}$ with a 1 in the ith row and jth column and zero everywhere else are in the image of φ. If $\mathfrak{S}_v$ is the ith basis element, then the elementary matrices $E_{i,a!}$ are given by

$$\varphi(\mathfrak{S}_v\partial_{w_0}) \quad \mapsto \quad \begin{array}{cc} & \begin{array}{cccc} \mathfrak{S}_1 & & & \mathfrak{S}_{w_0} \end{array} \\ \begin{array}{c} \mathfrak{S}_1 \\ \vdots \\ \\ \mathfrak{S}_v \\ \vdots \\ \mathfrak{S}_{w_0} \end{array} & \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \\ & & \vdots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}$$

since

$$\partial_{w_0}\mathfrak{S}_w = \begin{cases} 1 & \text{if } w = w_0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.22}$$

Similarly, if $w \in S_a$ is such that $\ell(w) = \ell(w_0) - 1$ so that $w = w_0 s_i$ for some generator $s_i \in S_a$, then

$$\partial_w \mathfrak{S}_u = \begin{cases} \mathfrak{S}_{s_i} & \text{if } u = w_0, \\ 1 & \text{if } u = w, \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

Hence,

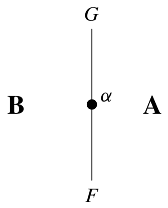
$$\mathfrak{S}_v\partial_w \quad \mapsto \quad \begin{array}{cc} & \begin{array}{cccccc} \mathfrak{S}_1 & & & & \mathfrak{S}_w & \mathfrak{S}_{w_0} \end{array} \\ \begin{array}{c} \mathfrak{S}_1 \\ \vdots \\ \\ \mathfrak{S}_v \\ \\ \vdots \\ \\ \mathfrak{S}_{s_i} \\ \vdots \\ \\ \mathfrak{S}_{w_0} \end{array} & \begin{pmatrix} 0 & & \cdots & & 0 & 0 \\ & \ddots & & & \vdots & \vdots \\ & & & & 0 & 0 \\ 0 & & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & & & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ \vdots & & & & 0 & 0 \\ 0 & & \cdots & & 0 & 1 \\ & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ 0 & & \cdots & & 0 & 0 \end{pmatrix} \end{array}$$

and the combination $\mathfrak{S}_v\partial_w - \mathfrak{S}_{s_i}\partial_{w_0}$ forms the elementary matrix with a single nonzero entry in the row corresponding to v and the column corresponding to w. Inductively applying this process shows that all elementary matrices in $\mathrm{Mat}((a)^!_{q^2}; \Lambda_a)$ are in the image of the map φ. Hence, φ is an isomorphism. $\square$

4. Graphical calculus

4.1. String diagrams for 2-categories

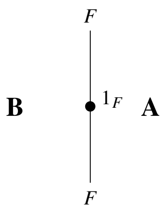
We use the graphical calculus of string diagrams to perform calculations inside of 2-categories. String diagrams represent natural transformations between functors, or more generally 2-morphisms inside some 2-category $\mathcal{K}$. A 2-morphism $\alpha : F \Rightarrow G$ between 1-morphisms $F, G : A \to B$ is depicted by the diagram:



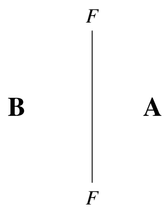
which is read from bottom to top and from right to left.

From the diagram all of the source and target information for the 2-morphism α can be read off immediately. Reading from bottom to top the bullet labelled by α divides lines labelled by F and G. This tells us that the 2-morphism α is a map from the 1-morphism F to the 1-morphism G. Going from right to left we pass from a region labelled A to a region labelled B. This tells us that the 1-morphisms F and G are maps from A to B. So in this sense, regions in the plane can be thought of as representing objects in the 2-category $\mathcal{K}$ and lines can be thought of as representing 1-morphisms in $\mathcal{K}$.

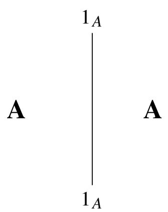
We write the identity 2-morphism



as

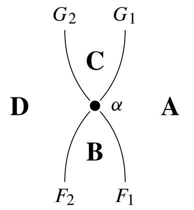


for simplicity. The identity 2-morphism of an identity 1-morphism $1_A : A \to A$ can be drawn as

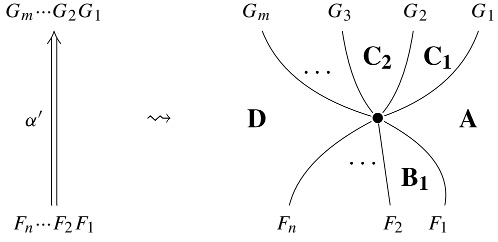


but to simplify the notation we represent this 2-morphism by the region labelled with an A omitting the line. 2-morphisms between composites of 1-morphisms, such as a 2-morphism α

between the composite $A \xrightarrow{F_1} B \xrightarrow{F_2} D$ and the composite $A \xrightarrow{G_1} C \xrightarrow{G_2} D$, can be represented as



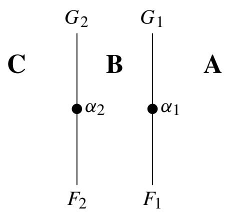
and between more general composites as



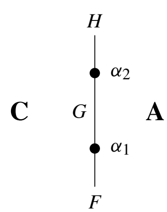
(4.1)

where the composites $F_n \cdots F_2 F_1$ and $G_m \cdots G_2 G_1$ are any composable string of 1-morphisms mapping A to D.

Horizontal composition of 2-morphisms is depicted by placing the diagrams side by side. For example,



and this is extended to 2-morphisms between arbitrary composites in the obvious way. Vertical composition of 2-morphisms $\alpha_1 : F \Rightarrow G$ and $\alpha_2 : G \Rightarrow H$ is achieved by stacking these diagrams on top of each other

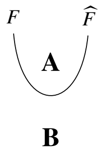


whenever the sources and targets are compatible.

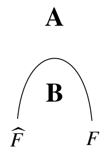
4.2. Graphical calculus for biadjoints

When the 1-morphism $F : A \to B$ is equipped with a specified left adjoint $\widehat{F} : B \to A$, written $\widehat{F} \dashv F$, the chosen unit $\eta : 1_B \Rightarrow F\widehat{F}$ and chosen counit $\varepsilon : \widehat{F}F \Rightarrow 1_A$ of this adjunction are depicted as follows:

$$\begin{array}{c} F\widehat{F} \\ \eta \Uparrow \\ 1_B \end{array} \quad \rightsquigarrow$$

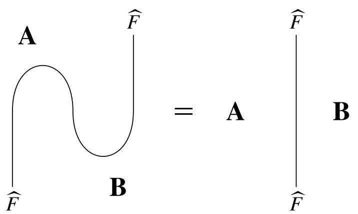


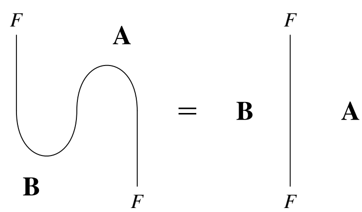
$$\begin{array}{c} 1_A \\ \varepsilon \Uparrow \\ \widehat{F}F \end{array} \quad \rightsquigarrow$$



(4.2)

omitting the vertex labelling the 2-morphisms and the lines corresponding to the identity 1-morphisms. The axioms of an adjunction require that the equalities between composites of 2-morphisms



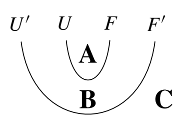


(4.3)

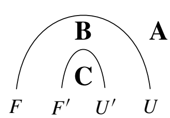
are satisfied. We call such equalities *zig-zag identities*.

Proposition 4.1. *If $\eta, \varepsilon : F \dashv U : A \to B$ and $\eta', \varepsilon' : F' \dashv U' : B \to C$ are adjunctions in the 2-category $\mathcal{K}$, then $FF' \dashv U'U$ with unit and counit given by the composites:*

$$\bar{\eta} := \left(1 \xRightarrow{\eta'} U'F' \xRightarrow{U'\eta F'} U'UFF'\right) \rightsquigarrow$$

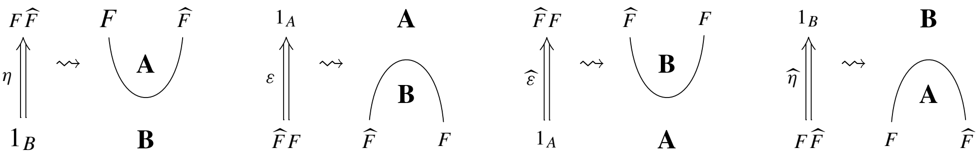


$$\bar{\varepsilon} := \left(FF'U'U \xRightarrow{F\varepsilon' U} FU \xRightarrow{\varepsilon} 1\right) \rightsquigarrow$$

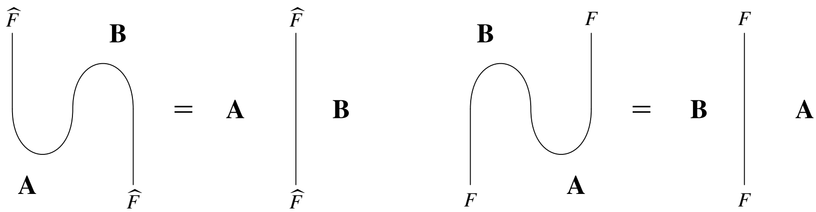


Proof. The zig-zag identities are straightforward to check using the string diagram calculus. □

If the 1-morphism F is biadjoint[2] to the 1-morphism $\widehat{F}$, so that $\widehat{F} \dashv F \dashv \widehat{F}$, we fix once and for all a choice of such a biadjoint structure. This is given by 2-morphisms $\eta: 1_B \Rightarrow F\widehat{F}$, $\varepsilon: \widehat{F}F \Rightarrow 1_A$, $\widehat{\varepsilon}: 1_A \Rightarrow \widehat{F}F$, $\widehat{\eta}: F\widehat{F} \Rightarrow 1_B$, depicted as



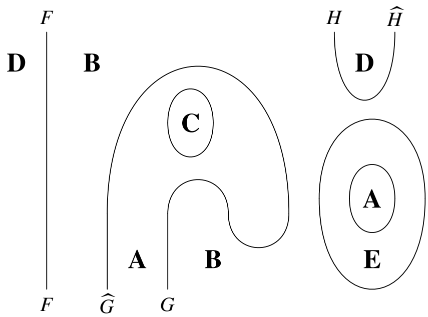
together with axioms asserting (4.3) and that the equalities



(4.4)

hold.

In general between any two objects A and B there may be many 1-morphisms between them with biadjoints. However, the biadjoint $\widehat{F}$ of a given 1-morphism F is unique up to isomorphism. An example of a typical diagram representing a 2-morphisms consisting of composites of units and counits for various biadjoints is given below:



Remark 4.2. The diagrammatic notation introduced in this section is the usual string diagram calculus common in 2-category theory [24,40,41]. This calculus is Poincaré dual to the typical globular diagrams used for representing 2-cells in a 2-category. For more on biadjoint 1-morphisms and their graphical calculus see [3,28,32,33,37].

[2] The author calls biadjoints ambidextrous adjoints in [32,33].

4.3. Mateship under adjunction

Here we recall the Australian 2-category theoretic notion of mateship under adjunction introduced by Kelly and Street [26]. This is a certain correspondence between 2-morphisms in the presence of adjoints.

Definition 4.3. Given adjoints $\eta, \varepsilon : F \dashv U : A \to B$ and $\eta', \varepsilon' : F' \dashv U' : A' \to B'$ in the 2-category $\mathcal{K}$, if $a : A \to A'$ and $b : B \to B'$, then there is a bijection M between 2-morphisms $\xi \in \mathcal{K}(bU, U'a)$ and 2-morphisms $\zeta \in \mathcal{K}(F'b, aF)$, where ζ is given in terms of ξ by the composite:

$$M : \mathcal{K}\big(bU, U'a\big) \longrightarrow \mathcal{K}\big(F'b, aF\big) \tag{4.5}$$

$$\xi \ \mapsto \ \Big(F'b \xRightarrow{F'b\eta} F'bUF \xRightarrow{F'\xi F} F'U'aF \xRightarrow{\varepsilon' aF} aF \Big) = \zeta, \tag{4.6}$$

and ξ is given in terms of ζ by the composite:

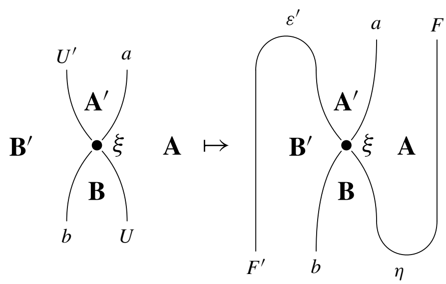
$$M^{-1} : \mathcal{K}\big(F'b, aF\big) \longrightarrow \mathcal{K}\big(bU, U'a\big) \tag{4.7}$$

$$\zeta \ \mapsto \ \Big(bU \xRightarrow{\eta' bU} U'F'bU \xRightarrow{U'\zeta U} U'aFU \xRightarrow{U'a\varepsilon} U'a \Big) = \xi. \tag{4.8}$$

Under these circumstances we say that ξ and ζ are *mates under adjunction*.

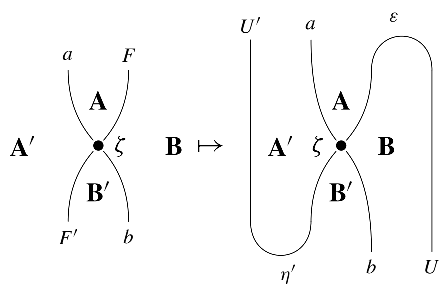
This bijection becomes much more enlightening when expressed diagrammatically.

$$M : \mathcal{K}\big(bU, U'a\big) \longrightarrow \mathcal{K}\big(F'b, aF\big),$$



(4.9)

$$M^{-1} : \mathcal{K}\big(F'b, aF\big) \longrightarrow \mathcal{K}\big(bU, U'a\big),$$

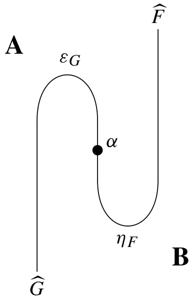


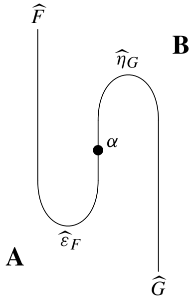
(4.10)

This bijection satisfies essentially all naturality axioms one could impose. The precise statement of naturality of this bijection can be expressed as an isomorphism of certain double categories (see [26], Proposition 2.2). This makes precise the idea that the association of mateship under adjunction respects composites and identities both of adjunctions and of morphisms in $\mathcal{K}$.

4.4. *Duals for 2-morphisms*

Given a pair of 1-morphisms $F, G : A \to B$ with chosen biadjoints $(\widehat{F}, \eta_F, \widehat{\eta}_F, \varepsilon_F, \widehat{\varepsilon}_F)$ and $(\widehat{G}, \eta_G, \widehat{\eta}_G, \varepsilon_G, \widehat{\varepsilon}_G)$, then any 2-morphism $\alpha : F \Rightarrow G$ has two obvious duals ${}^*\alpha, \alpha^* : \widehat{G} \Rightarrow \widehat{F}$, or mates, one constructed using the left adjoint structure, the other using the right adjoint structure. Diagrammatically the two mates are given by

${}^*\alpha :=$ 

$\alpha^* :=$ 

(4.11)

where we have inserted labels on the unit and counit 2-morphisms to avoid confusion. We will call α^* the right dual of α because it us obtained from α as its mate using the right adjoints of F and G. Similarly, ${}^*\alpha$ is called the left dual of α because it is obtained from α as its mate using the left adjoints of F and G.

In general there is no reason why ${}^*\alpha$ should be equal to α^*. To see this, suppose that ${}^*\alpha = \alpha^*$ for some choices of units and counits. Then for any invertible 2-morphism $\gamma : 1_B \Rightarrow 1_B$ we can twist the unit η_F by

$$\eta'_F := \left(1_B \xRightarrow{\gamma} 1_B \xRightarrow{\eta_F} F\widehat{F}\right).$$

If we also twist the counit by the inverse of γ so that

$$\varepsilon'_F := \left(\widehat{F}F = \widehat{F}1_B F \xRightarrow{\widehat{F}\gamma^{-1}F} \widehat{F}F \xRightarrow{\eta_F} 1_A\right),$$

then the pair $(\eta'_F, \varepsilon'_F)$ still satisfy the zig-zag identities (4.3). However, in the equation ${}^*\alpha = \alpha^*$ the left hand side has been twisted by a factor of γ^{-1}, so that the equation only remains valid when γ is the identity. Hence, it is clear that for $\alpha : F \Rightarrow G$, equality between ${}^*\alpha$ and α^* depends on the choice of biadjoint structure[3] on F and G.

Following Cockett, Koslowski and Seely [13], we call a 2-morphism $\alpha : F \Rightarrow G$ a *cyclic 2-morphism* if the equation ${}^*\alpha = \alpha^*$ holds for the chosen biadjoint structure on F and G. More precisely:

[3] Thanks to Bruce Bartlett for pointing out this example.

Definition 4.4. Given biadjoints $(F,\widehat{F},\eta_F,\widehat{\eta}_F,\varepsilon_F,\widehat{\varepsilon}_F)$ and $(G,\widehat{G},\eta_G,\widehat{\eta}_G,\varepsilon_G,\widehat{\varepsilon}_G)$ and a 2-morphism $\alpha : F \Rightarrow G$ define

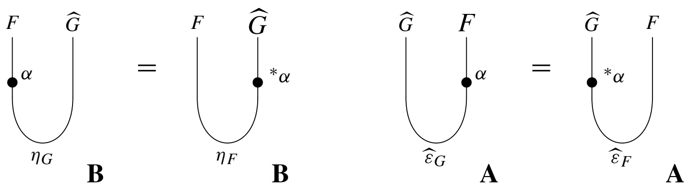
$$\begin{aligned} \alpha^* &:= \widehat{F}\widehat{\eta}_G.\widehat{\varepsilon}_F\widehat{G}, \\ {}^*\alpha &:= \varepsilon_G\widehat{F}.\widehat{G}\eta_F. \end{aligned} \tag{4.12}$$

Then a 2-morphism α is called a *cyclic 2-morphism* if the equation ${}^*\alpha = \alpha^*$ is satisfied, or either of the equivalent conditions ${}^{**}\alpha = \alpha$ or $\alpha^{**} = \alpha$ are satisfied.

The notions of a pivotal category [18], and pivotal 2-category [35] are related to the notion of cyclic 2-morphisms. In particular, fix an object A of a 2-category $\mathcal{K}$, and regard the 1-morphism category $\mathcal{K}(A,A)$ as a monoidal category whose objects are the endomorphisms of A, and whose morphisms are the 2-morphisms between such 1-morphisms. If all 1-morphisms $A \to A$ have biadjoints and all the 2-morphisms are composites of cyclic 2-morphisms, then the monoidal category $\mathcal{K}(A,A)$ is a pivotal monoidal category. In the definition of a pivotal 2-category with duals [35] the condition ${}^*\alpha = \alpha^*$ also appears.

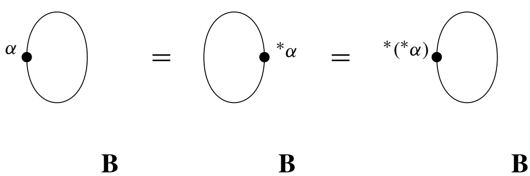
Cyclic 2-morphisms compose both horizontally and vertically to form cyclic 2-morphisms. This has the diagrammatic interpretation that the two twists shown in (4.11) are equal for any 2-morphism built from composites of cyclic 2-morphisms. This is referred to as the circuit flipping condition in [13].

The cyclic condition for 2-morphisms greatly simplifies their graphical calculus. If $\alpha : F \Rightarrow G$ and F and G have chosen biadjoints, then we always have equalities:



(4.13)

as well as upside down versions of the above. While the diagrams



(4.14)

always represent the same 2-morphism, when α is a cyclic 2-morphism the necessity of labelling can often be avoided since ${}^*({}^*\alpha) = \alpha$ so we always get back to where we started.

The following can be found throughout the literature in various guises, usually using the language of duals rather than adjunctions. As stated it follows from the axioms of an adjunction, the cyclic condition, and the interchange law relating horizontal and vertical composites of 2-morphisms.

Theorem 4.5. *(See Cockett, Koslowski and Seely [13].) Given a string diagram representing a cyclic* 2*-morphism between* 1*-morphisms with chosen biadjoints, then any isotopy of the diagram represents the same* 2*-morphism.*

5. The 2-category $\mathcal{U}$

In this section we define the 2-category $\mathcal{U}$ which is related to a categorification of the algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$. As an intermediate step we begin with a 2-category $\mathcal{U}^*$.

5.1. Some categorical preliminaries

Recall the definition of an enriched category from [25]. Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A *$\mathcal{V}$-category* $\mathcal{A}$ is defined in the same way as an ordinary category, except that the hom sets $\mathrm{Hom}(x, y)$ are replaced by objects $\mathrm{Hom}_{\mathcal{A}}(x, y) \in \mathcal{V}$, and composition and units maps are replaced by morphisms

$$\mathrm{Hom}_{\mathcal{A}}(x, y) \otimes \mathrm{Hom}_{\mathcal{A}}(y, z) \to \mathrm{Hom}_{\mathcal{A}}(x, z)$$

and

$$1_x : I \to \mathrm{Hom}_{\mathcal{A}}(x, x)$$

in $\mathcal{V}$. A *$\mathcal{V}$-functor* $F : \mathcal{A} \to \mathcal{B}$ between two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ is given by a function $\mathrm{ob}\,\mathcal{A} \to \mathrm{ob}\,\mathcal{B}$ together with a morphism $\mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{B}}(Fx, Fy)$ in $\mathcal{V}$ satisfying the usual axioms of a functor.

Examples abound in the literature: if $\mathcal{V}$ is the category of abelian groups, then a $\mathcal{V}$-category is known as a preadditive category. If $\Bbbk$ is a field and $\mathcal{V}$ is taken to be the category of $\Bbbk$-vector spaces, then a $\mathcal{V}$-category is exactly a $\Bbbk$-linear category and a $\mathcal{V}$-functor is a $\Bbbk$-linear functor. If $\mathcal{V}$ is the category of differential graded R-modules with the usual monoidal structure, then a $\mathcal{V}$-category is a differential graded category. The category **Cat** of categories and functors admits a monoidal structure given by the categorical product of categories. A category enriched in **Cat** with this monoidal structure is a strict 2-category and a $\mathcal{V}$-functor is a strict 2-functor.

A *graded preadditive category* $\mathcal{A}$ is a category enriched in the symmetric monoidal category of graded abelian groups with the monoidal structure given by the graded tensor product. This means that the hom set $\mathrm{Hom}(x, y)$ between any two objects $x, y, \in \mathcal{A}$ is a graded abelian group,

$$\mathrm{Hom}_{\mathcal{A}}(x, y) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_s(x, y)$$

where $\mathrm{Hom}_s(x, y)$ is the abelian group of homogeneous components of degree s; its elements are the morphisms of degree s. The composition map must also be degree homogeneous

$$\mathrm{Hom}_s(x, y) \otimes \mathrm{Hom}_{s'}(y, z) \to \mathrm{Hom}_{s+s'}(x, z).$$

An additive category is a preadditive category with a zero object and direct sums. An additive functor between two additive categories is an **Ab**-functor that preserves biproducts. That is, on hom sets an additive functor is a group homomorphism that preserves direct sums and the zero object. The category **Add-Cat** of additive categories, and additive functors forms a monoidal category with the usual tensor product of additive categories.

Similarly, a graded additive category is a graded preadditive category with a zero object and direct sums. Here zero objects and direct sums are defined as in the ungraded case, with the additional condition that the injections and projections are homogeneous. Again, we can form the monoidal category **Gr-Add-Cat** of graded additive categories, and graded additive functors. A graded additive functor is an additive functor such that

$$F\big(\mathrm{Hom}_s(x, y)\big) \subset \mathrm{Hom}_s(Fx, Fy). \tag{5.1}$$

A graded additive category is said to *admit translation* (see [22]) if for any object x and integer m there is an object $x\{m\}$ with an isomorphism $x \to x\{m\}$ of degree m. Given a graded additive category $\mathcal{A}$ we can always define an equivalent category $\mathcal{A}'$, were $\mathcal{A}'$ is a graded additive category with translation obtained by enlarging $\mathcal{A}$ in the obvious way. A graded additive functor between graded additive categories with translation will preserve translations since there is the degree $-s+s=0$ isomorphism

$$F(x\{s\}) \xrightarrow{F(\sim)} F(x) \xrightarrow{\sim} F(x)\{s\}. \tag{5.2}$$

Let **GAT** denote the monoidal category of graded additive categories with translation, together with graded additive functors.

Definition 5.1.

- An *additive 2-category*[4] is a 2-category enriched in the monoidal category **Ab-Cat** of additive categories.
- A *graded additive 2-category* is a 2-category enriched in the monoidal category **Gr-Ab-Cat** of graded additive categories.
- A graded additive 2-category is said to *admit translation* if it is a category enriched over the monoidal category **GAT** of graded additive categories that admit translation.

Hence, a graded additive 2-category admitting a translation is a strict 2-category $\mathcal{A}$ such that the hom categories ${}_a\mathcal{A}_b := \mathrm{Hom}_{\mathcal{A}}(a, b)$, between any two objects $a, b \in \mathcal{A}$, are graded additive categories that admit a translation.

If $x, y : a \to b$ are 1-morphisms in $\mathcal{A}$, then the graded abelian group $\mathrm{Hom}_{{}_a\mathcal{A}_b}(x, y)$ is written as $\mathcal{A}(x, y)$ with it understood that $\mathcal{A}(x, y)$ is zero if x and y do not have the same source and target. The decomposition into homogeneous elements is expressed as

$$\mathcal{A}(x, y) := \bigoplus_{s\in\mathbb{Z}} \mathcal{A}_s(x, y)$$

[4] Compare with 2-categories enriched over $\mathcal{V}$, or a $\mathcal{V}$-2-category of [20].

with $\mathcal{A}_s(x, y)$ the homogeneous elements of degree s. Composition and identities are given by graded additive functors

$$\mathcal{A}(a,b) \otimes \mathcal{A}(b,c) \to \mathcal{A}(a,c), \quad I \to \mathcal{A}(a,a),$$
$$\mathcal{A}_s(x,y) \otimes \mathcal{A}_{s'}(y,z) \mapsto \mathcal{A}_{s+s'}(x,z), \quad I \mapsto \mathcal{A}_0(x,x)$$

so that identity maps are always degree zero.

The above notions also make sense with minor modification in the case when the 2-categories are weak, that is, when composition is associative up to coherent isomorphism and identities act as identities up to coherent isomorphism. We will not explicitly distinguish between strict/weak in what follows.

A well-known example of an additive 2-category is the (weak) 2-category **Bim** whose objects are graded rings. If R and S are two such rings, then $\mathrm{Hom}_{\mathbf{Bim}}(R,S)$ is the additive category of graded (R,S)-bimodules and degree-zero bimodule homomorphisms. The composition functor

$$\mathrm{Hom}(S,T) \times \mathrm{Hom}(R,S) \to \mathrm{Hom}(R,T) \tag{5.3}$$

is given by the tensor product:

$$(M,N) \mapsto N \otimes_S M. \tag{5.4}$$

The 2-category $\mathbf{Bim}^*$ is the graded additive 2-category whose objects are graded rings. The graded additive categories $\mathrm{Hom}(R,S)$ are the categories of graded bimodules and all bimodule maps (which are finite sums of homogeneous maps). Composition is given by the tensor product. The 2-category $\mathbf{Bim}^*$ is enriched over graded additive 2-categories that admit a translation, since we can shift the degree of graded bimodules and bimodule maps.

An additive 2-functor $F : \mathcal{A} \to \mathcal{B}$ is a function $F : \mathrm{ob}\,\mathcal{A} \to \mathrm{ob}\,\mathcal{B}$, together with an additive functor $\mathcal{A}(a,b) \to \mathcal{B}(a,b)$ for all objects a and b, that preserves composition and identities up to isomorphisms.[5] Similarly, a graded additive 2-functor $F : \mathcal{A} \to \mathcal{B}$ is a function $F : \mathrm{ob}\,\mathcal{A} \to \mathrm{ob}\,\mathcal{B}$ together with a graded additive functor $\mathcal{A}(a,b) \to \mathcal{B}(a,b)$ preserving composition and identities up to isomorphisms.

5.2. The 2-category $\mathcal{U}^*$

The generators of $\mathcal{U}^*$

$\mathcal{U}^*$ is a graded additive 2-category with translation. The 2-category $\mathcal{U}^*$ has one object n for each $n \in \mathbb{Z}$. The 1-morphisms of $\mathcal{U}^*$ are formal direct sums of composites of the morphisms

$$\begin{aligned} \mathbf{1}_n &: n \to n, \\ \mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n &: n \to n+2, \\ \mathbf{1}_n\mathcal{F}\mathbf{1}_{n+2} &: n+2 \to n \end{aligned} \tag{5.5}$$

[5] An additive 2-functor is just a (weak) $\mathcal{V}$-functor for $\mathcal{V} = \mathbf{Ab\text{-}Cat}$.

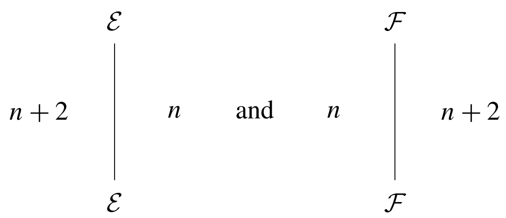
for each $n \in \mathbb{Z}$ together with their shifts $\{s\}$ for $s \in \mathbb{Z}$. The morphisms $\mathbf{1}_n$ are the identity 1-morphisms. The morphism $\mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n$ maps n to $n+2$ so we often simplify notation by writing only $\mathcal{E}\mathbf{1}_n$, or generically as $\mathcal{E}$, with it understood that $\mathcal{E}$ increases the subscript by two, passing from right to left. Similarly, the map $\mathbf{1}_n\mathcal{F}\mathbf{1}_{n+2}$ maps $n+2$ to n so we often write this morphism as $\mathcal{F}\mathbf{1}_{n+2}$, or $\mathcal{F}$. This simplification is extended to composites as well, so that $\mathcal{E}\mathcal{F}\mathbf{1}_n$ represents the composite $\mathbf{1}_n\mathcal{E}\mathbf{1}_{n-2} \circ \mathbf{1}_{n-2}\mathcal{F}\mathbf{1}_n$. When no confusion is likely to arise we simplify our notation even further and write simply $\mathcal{E}\mathcal{F}$.

More precisely, given objects n, m in $\mathcal{U}^*$, the graded additive category $\mathcal{U}^*(n,m)$ consists of

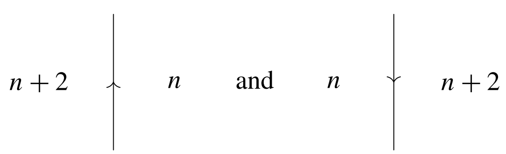
- objects of $\mathcal{U}^*(n,m)$: the objects are formal direct sums of composites

$$\mathbf{1}_m \mathcal{E}^{\alpha_1} \mathcal{F}^{\beta_1} \mathcal{E}^{\alpha_2} \dots \mathcal{F}^{\beta_{k-1}} \mathcal{E}^{\alpha_k} \mathcal{F}^{\beta_k} \mathbf{1}_n\{s\}$$

  where $m = n + 2(\sum \alpha_i - \sum \beta_i)$, and $s \in \mathbb{Z}$.
  Using the string diagram calculus introduced in Section 4 we depict the objects $n \in \mathcal{U}^*$ as regions labelled by n. The 1-morphisms $\mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n$ and $\mathbf{1}_n\mathcal{F}\mathbf{1}_{n+2}$ are depicted as

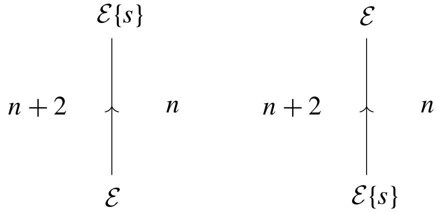


  We can omit the labels of $\mathcal{E}$ and $\mathcal{F}$ by introducing the convention that $\mathcal{E}$ is depicted by an upward pointing arrow and $\mathcal{F}$ is depicted by a downward pointing arrow.



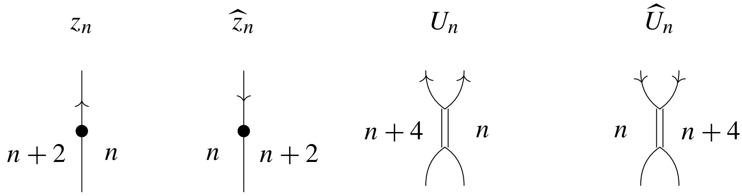
- morphisms of $\mathcal{U}^*(n,m)$: for 1-morphisms $x, y \in \mathcal{U}^*$ hom sets $\mathcal{U}^*(x,y)$ of $\mathcal{U}^*(n,m)$ are graded abelian groups given by $\mathbb{Z}$-linear combinations 2-morphisms built from composites of:
  (i) degree zero identity 2-morphisms 1_x for each 1-morphism x in $\mathcal{U}^*$,
  (ii) for each 1-morphism x, an isomorphism $x \simeq x\{s\}$ given by 2-morphisms $x \Rightarrow x\{s\}$ and $x\{s\} \Rightarrow x$ of degree s and $-s$, respectively. These are represented by the identity 2-

morphism together with a shift on the source or target. For example, the isomorphism $\mathcal{E}\mathbf{1}_n \simeq \mathcal{E}\mathbf{1}_n\{s\}$ is given by



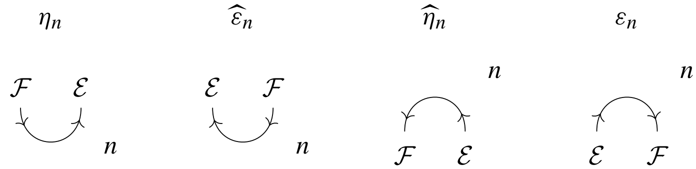
$\deg s$ $\qquad$ $\deg -s$

(iii) for each $n \in \mathbb{Z}$ the 2-morphisms[6]



$\deg 2$ $\qquad$ $\deg 2$ $\qquad$ $\deg -2$ $\qquad$ $\deg -2$

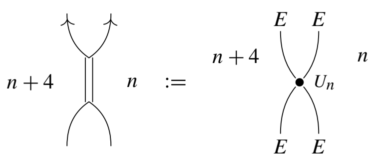
and



$\deg n+1$ $\qquad$ $\deg 1-n$ $\qquad$ $\deg n+1$ $\qquad$ $\deg 1-n$

- the graded additive composition functor $\mathcal{U}^*(n,n') \times \mathcal{U}^*(n',n'') \to \mathcal{U}^*(n,n'')$ is given on 1-morphisms of $\mathcal{U}^*$ by

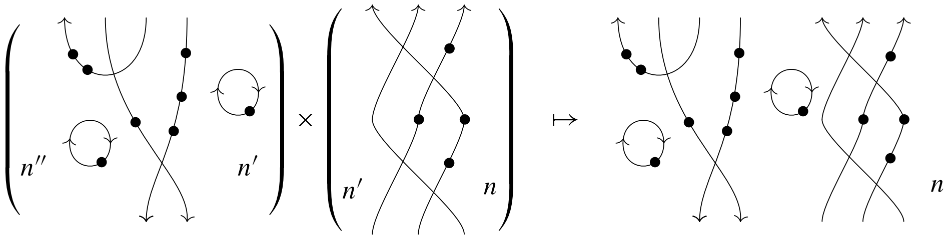
[6] Strictly speaking, the diagram used to represent U_n is a shorthand notation for the standard string diagram notation.



Similarly, we omit all labels in the string diagrams for z_n and $\widehat{z}_n$ since the axioms below ensure that this will lead to no ambiguity.

$$\mathcal{E}^{\alpha'_1}\mathcal{F}^{\beta'_1}\dots\mathcal{E}^{\alpha'_k}\mathcal{F}^{\beta'_k}\mathbf{1}_{n'}\{s'\} \times \mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\dots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_{n}\{s\}$$
$$\mapsto \mathcal{E}^{\alpha'_1}\mathcal{F}^{\beta'_1}\dots\mathcal{E}^{\alpha'_k}\mathcal{F}^{\beta'_k}\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\dots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_{n}\{s+s'\}$$

for $n' = n + 2(\sum \alpha_i - \sum \beta_i)$, and on 2-morphisms of $\mathcal{U}^*$ by juxtaposition of diagrams

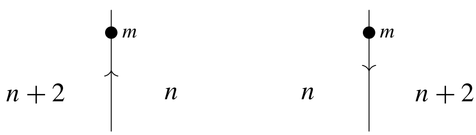


subject to relations given below.

We write $\mathcal{U}^*(x, y) = \sum_{s\in\mathbb{Z}} \mathcal{U}_s(x, y)$ where $\mathcal{U}_s(x, y)$ denote the homogeneous elements of degree s.

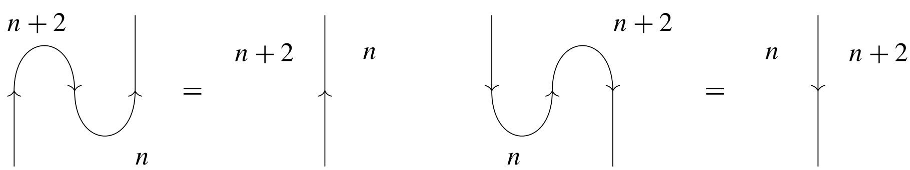
The relations on $\mathcal{U}^*$

For convenience, we denote the iterated vertical composites of z_n and $\widehat{z}_n$ as

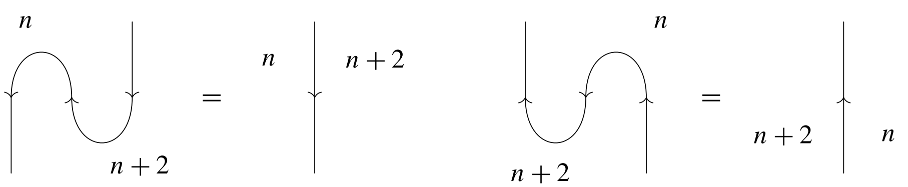


where the label of m indicated the number of iterated composites. Labels for regions in a diagram can be deduced from a single labelled region using the rules that crossing an upward pointing arrow from right to left increases the label by 2 and crossing a downward pointing arrow decreases the label by 2.

Biadjointness: We have biadjoint morphisms $\mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n \dashv \mathbf{1}_n\mathcal{F}\mathbf{1}_{n+2} \dashv \mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n$ with units and counits given by the pairs $(\eta_n, \varepsilon_{n+2})$ and $(\widehat{\varepsilon}_n, \widehat{\eta}_{n-2})$ for all $n \in \mathbb{Z}$. This is equivalent to requiring the following equalities



(5.6)



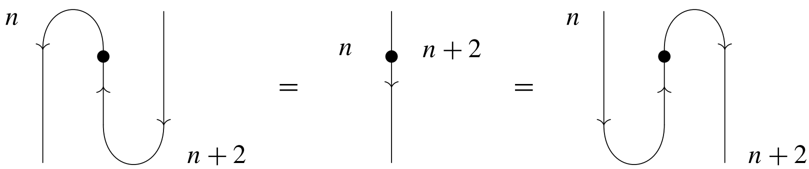
(5.7)

for all $n \in \mathbb{Z}$.

All morphisms in $\mathcal{U}^*$ are formal sums of composites and shifts of the morphisms $\mathcal{E}$ and $\mathcal{F}$; since the composite of a biadjoint morphisms is biadjoint by Proposition 4.1, we have that every morphism in $\mathcal{U}^*$ has a biadjoint. This biadjoint structure on a 1-morphism $\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\dots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}\mathbf{1}_n\{s\}$ is explicitly given by

$$\mathbf{1}_n\mathcal{E}^{\beta_m}\mathcal{F}^{\alpha_m}\dots\mathcal{E}^{\beta_1}\mathcal{F}^{\alpha_1}\mathbf{1}_m\{-s\} \dashv \mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\dots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}\mathbf{1}_n\{s\} \dashv \mathbf{1}_n\mathcal{E}^{\beta_m}\mathcal{F}^{\alpha_m}\dots\mathcal{E}^{\beta_1}\mathcal{F}^{\alpha_1}\mathbf{1}_m\{-s\}.$$

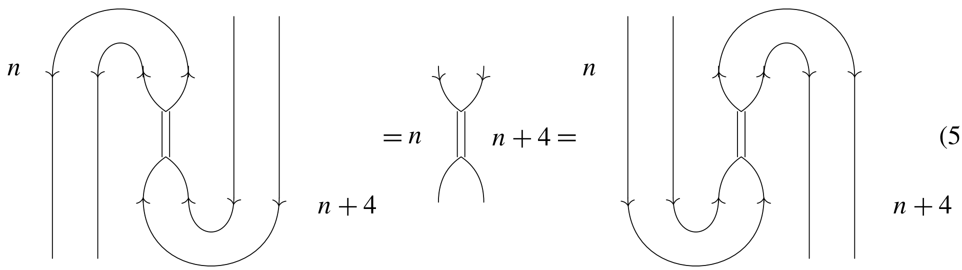
Duality for z_n and $\widehat{z}_n$: The two duals of the 2-morphism z_n under the above biadjoint structure are equal. More precisely, the equalities



(5.8)

hold for all $n \in \mathbb{Z}$. Note these equations together with (5.6)–(5.7) imply the same equations with the opposite orientation. They also imply that a dot on one strand of a cap or cup can be slid to the other side, and that sliding a dot around a closed diagram presents no ambiguity (see Section 4.4).

Duality for U_n and $\widehat{U}_n$: The two duals of the 2-morphism U_n under the above biadjoint structure are equal. More precisely, the equalities

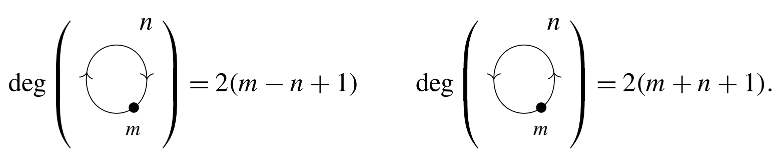


(5.9)

hold for all $n \in \mathbb{Z}$. Note these equations together with (5.6)–(5.7) imply the same equations with the opposite orientation. They also imply that the 2-morphism U_n is the 2-sided dual to the 2-morphism $\widehat{U}_n$.

The three axioms above imply that all the morphisms in $\mathcal{U}^*$ are cyclic 2-morphisms with respect to the biadjoint structure each 1-morphism inherits from the definitions above. Hence, these axioms ensure that topological deformations of a diagram that preserve the boundary result in a diagram representing the same 2-morphism.

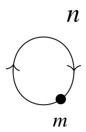
Positive degree of closed bubbles: From the degrees defined above we have



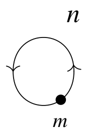
(5.10)

Diagrams of the above form are referred to as bubbles or dotted bubbles for the obvious reason.

In $\mathcal{U}^*$ we impose the relation that all bubbles of negative degree are zero. That is,



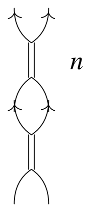
$$= 0 \quad \text{if } m < n - 1$$



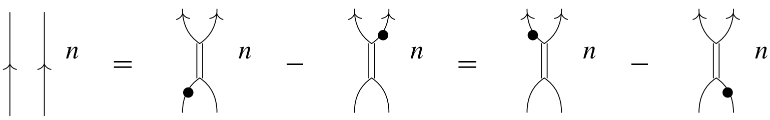
$$= 0 \quad \text{if } m < -n - 1$$

for all $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$. It is a non-obvious fact (Proposition 8.2) that the above condition ensures that any closed diagram of negative degree evaluates to zero.

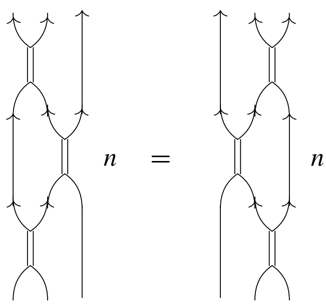
NilHecke action: The following equations hold:



$$= 0 \tag{5.11}$$



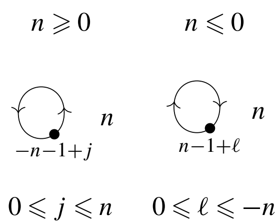
(5.12)



(5.13)

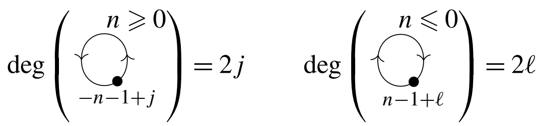
for all values of $n \in \mathbb{Z}$. These axioms ensure that the nilHecke algebra $\mathcal{NH}_a$ acts on $\mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ for all $n \in \mathbb{Z}$, with $\chi_i \mapsto z_{n+i}$ and $u_j \mapsto U_{n+j}$. Using the duality introduced above we also get an action of the opposite algebra $\mathcal{NH}_a^{\mathrm{op}}$ on $\mathcal{U}^*(\mathcal{F}^a\mathbf{1}_n, \mathcal{F}^a\mathbf{1}_n)$ for all $n \in \mathbb{Z}$, with $\chi_i \mapsto \widehat{z}_m$ and $u_j \mapsto \widehat{U}_j$. For this reason we will sometimes refer to the 2-morphisms U_j and $\widehat{U}_j$ in $\mathcal{U}^*$ as nilCoxeter generators.

The final two axioms are required in order to ensure that the 1-morphisms $\mathcal{E}$ and $\mathcal{F}$ lift the relation on E and F in $\mathbf{U}_q(\mathfrak{sl}_2)$. These axioms are defined recursively for each n. To state them in a convenient compact form we introduce the formal symbols:

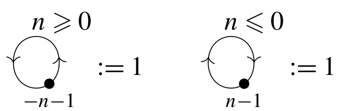


which a priori make no sense whatsoever. The integer labelling each bubble is a negative number. In the first case, $-n-1+j<0$ and in the second case $n-1+\ell<0$ by the assumption that $0 \leqslant j \leqslant n$ and $0 \leqslant \ell \leqslant -n$. This would correspond to vertically composing the 2-morphisms $\widehat{z}_m$ or z_m a negative number of times with themselves — impossible! For this reason we call the above symbols *fake bubbles*. The convenience of using fake bubbles is that they can be used to write down equations which seamlessly transition from the cases where $0 \leqslant j \leqslant n$ and $0 \leqslant \ell \leqslant -n$ to the case where $j>n$ and $\ell>-n$ where the dotted bubbles are well defined.

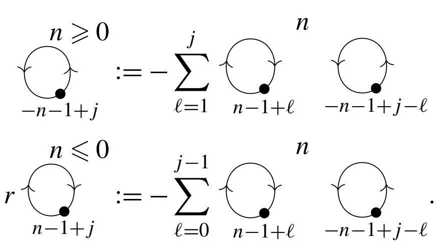
Each of the fake bubbles if taken literally have positive degree



so this convention does not contradict the 'positivity of bubbles' axiom above. However, the positivity of the degree of these bubbles with negative dots allows us to define them as sums of well-defined diagrams of positive degree — oppositely oriented bubbles with nonnegative dots. The symbols are defined by the condition that

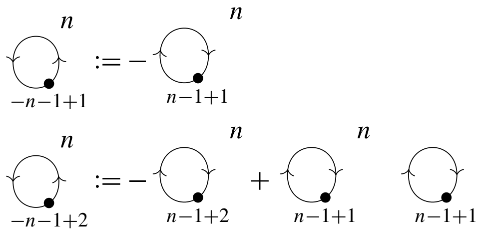


and for $1 \leqslant j \leqslant n$ recursively by



(5.14)

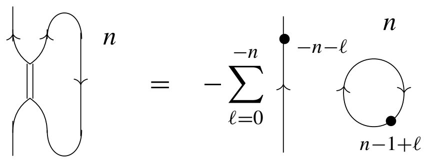
For example, suppose that $n \geqslant 0$. We then have



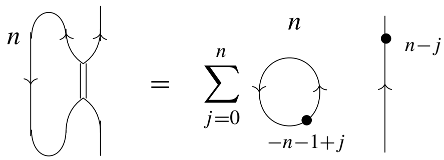
(5.15)

and so on. Note that none of the diagrams on the right hand side require negative labels; they are composites of the generating 2-morphisms in $\mathcal{U}^*$.

Reduction to bubbles: The equalities



(5.16)

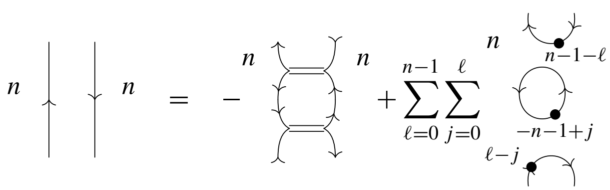


(5.17)

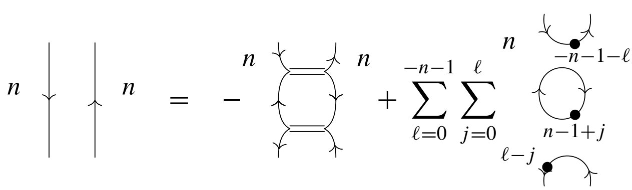
hold for all $n \in \mathbb{Z}$. All sums in this paper are increasing sums, or else they are taken to be zero. This means that in (5.16) when $-n < 0$ the term on the right hand side is zero. In (5.17) when $n < 0$ the right hand side is also zero. When these equations are nonzero we are making use of fake bubbles.

We can now state the final relation in $\mathcal{U}^*$. We emphasize again: **we do not allow dots with negative labels!** Bubble diagrams with negative labels and positive degree and have a formal meaning as explained above.

Identity decomposition: The equations



(5.18)



(5.19)

hold for all $n \in \mathbb{Z}$. Furthermore, since all summations are assumed to be increasing, when $n < 1$ the second term on the right of (5.18) vanishes, and when $n > 1$ the second term on the right of (5.19) vanishes. This means that, when the terms on the right hand sides of (5.18) and (5.19) are nonzero, the bubbles appearing in these terms are fake bubbles to be interpreted as explained above.

The reduction to bubbles, and identity decomposition axioms ensure that any closed diagram can be reduced to a sum of diagrams containing non-nested dotted bubbles of the same orientation (see Section 8).

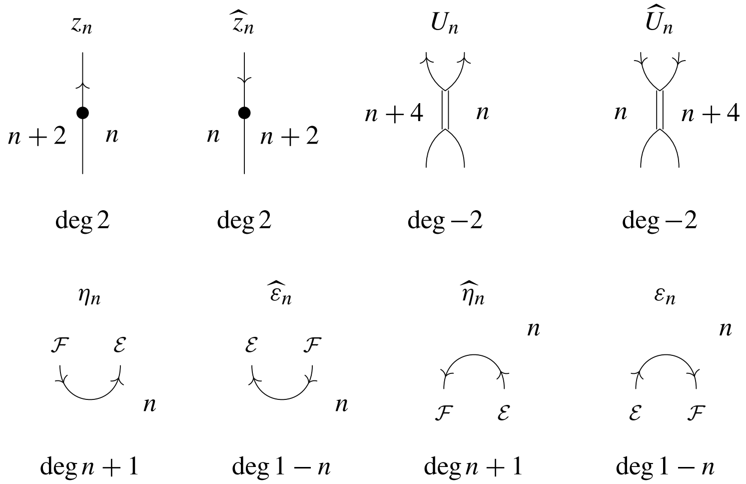
5.3. Summary

We use 2-categorical string diagrams to present the definition. The 2-category $\mathcal{U}^*$ consists of

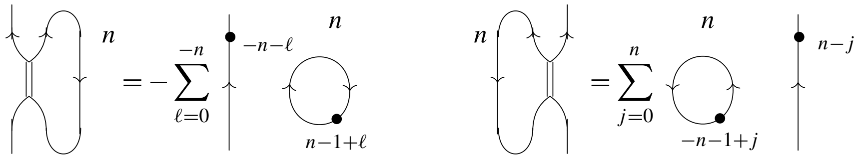
- objects: n for $n \in \mathbb{Z}$,
- 1-morphisms: formal direct sums of composites of

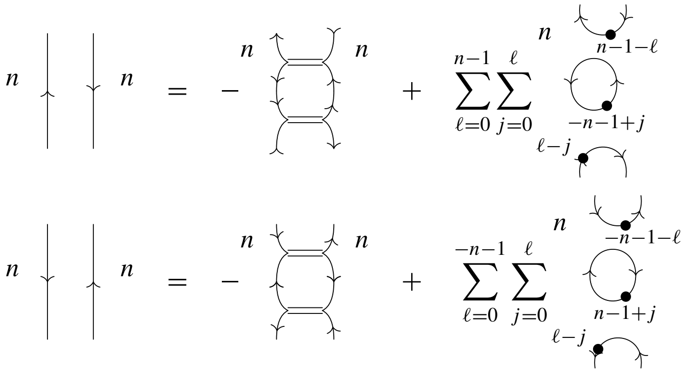
$$\mathbf{1}_m \mathcal{E}^{\alpha_1} \mathcal{F}^{\beta_1} \mathcal{E}^{\alpha_2} \cdots \mathcal{F}^{\beta_{k-1}} \mathcal{E}^{\alpha_k} \mathcal{F}^{\beta_k} \mathbf{1}_n \{s\}$$

  where $m = n + 2(\sum \alpha_i - \sum \beta_i)$, and $s \in \mathbb{Z}$.
- graded 2-morphisms



  together with identity 2-morphisms and isomorphisms $x \simeq x\{s\}$ for each 1-morphism x, such that
- $\mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n$ and $\mathbf{1}_n\mathcal{F}\mathbf{1}_{n+2}$ are biadjoints with units and counits given by the pairs $(\eta_n, \varepsilon_{n+2})$ and $(\widehat{\varepsilon}_n, \widehat{\eta}_{n-2})$.
- All 2-morphisms are cyclic with respect to the above biadjoint structure.
- All dotted closed bubbles of negative degree are zero.
- The nilHecke algebra $\mathcal{NH}_a$ acts on $\mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ and $\mathcal{U}^*(\mathcal{F}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ for all $n \in \mathbb{Z}$.
- The 1-morphisms $\mathcal{E}$ and $\mathcal{F}$ lift the relations of E and F in $\mathbf{U}_q(\mathfrak{sl}_2)$. This is ensured by requiring the equalities



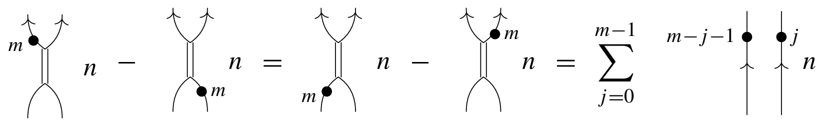


for all $n \in \mathbb{Z}$.

5.4. Helpful relations in $\mathcal{U}^*$

In this section we introduce some additional relations that follow from those in the previous section. These will be useful in manipulating the graphical calculus for $\mathcal{U}^*$.

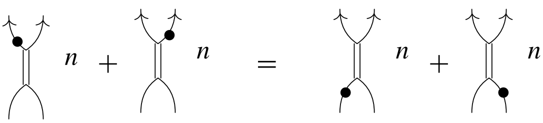
Proposition 5.2 *(Induction formula).*



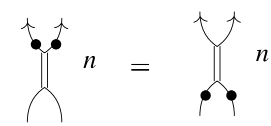
(5.20)

Proof. This follows by induction from (5.12). □

Proposition 5.3 *(Consequences of NilHecke relations).*



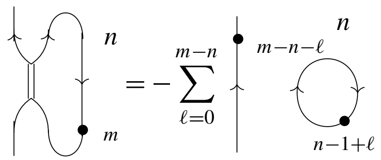
(5.21)



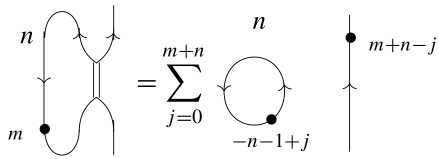
(5.22)

Proof. The first equation above is immediate from (5.12). The second equation also follows from (5.12) by composing with z_n on one of the strands and using (5.12) to simplify. □

Proposition 5.4 *(More reduction to bubbles). The equalities*



(5.23)



(5.24)

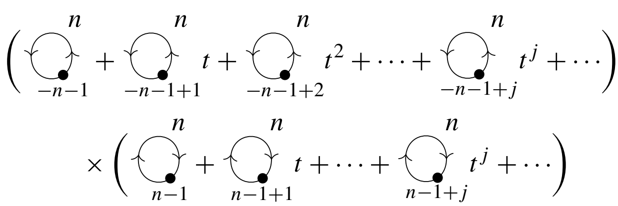
hold for all $n \in \mathbb{Z}$.

Recall that all sums in this paper are increasing sums, or else they are taken to be zero. This means that in (5.23) when $m - n < 0$ the term on the right hand side is zero. In (5.24) when $m + n < 0$ the right hand side is also zero. Depending on the value of n these equations make use of the fake bubbles. This demonstrates the convenience of fake bubbles; the above expression makes sense for all values of n.

Proof. This follows from the reduction to bubbles axiom and the induction formula above. □

The following proposition gives further motivation for the definition of these seemingly strange fake bubbles. It also serves as an example of a seamless transition from fake bubbles to actual bubbles as j grows larger than n.

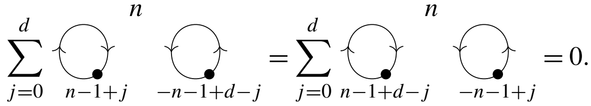
Proposition 5.5 *(Infinite Grassmannian relations). The following product:*



is equal to $\mathrm{Id}_{\mathbf{1}_n}$ *for all n, where t is a formal variable. This is in analogy with the generators of the cohomology ring $H^*(Gr(n,\infty))$ of the infinite Grassmannian (see Section* 6*):*

$$\big(1 + x_1 t + x_2 t^2 + \cdots + x_j t^j + \cdots\big)\big(1 + y_1 t + y_2 t^2 + \cdots + y_j t^j + \cdots\big) = 1.$$

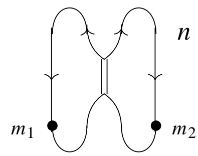
In particular, the above equation implies that for $d > 0$ we have



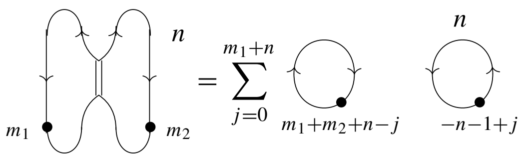
(5.25)

If $1 \leqslant d \leqslant n$ this equation is the defining equation for the fake bubbles. The content of the proposition is that this equation holds for all values of $d > 0$.

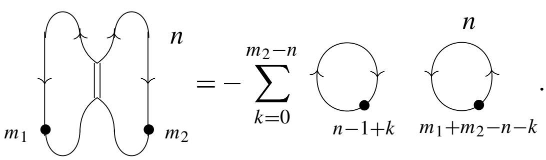
Proof. Choose m_1 and m_2 in $\mathbb{Z}_+$ so that $m_1 + m_2 + 1 > n$. Consider the two possible ways of decomposing the diagram



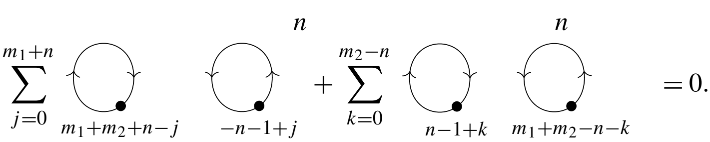
using (5.23) and (5.24). Using (5.24) we get



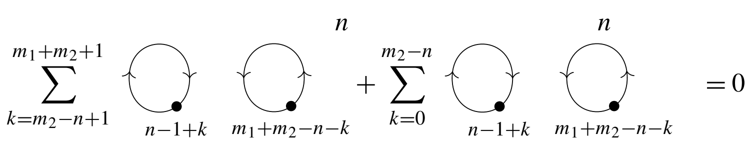
and using (5.23) we get



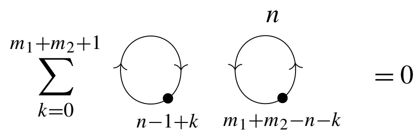
Consistency of the calculus requires that these two reductions are equal



Now in the first term make the change of variables $j \mapsto m_1 + m_2 - k + 1$ so that when $j = 0$ we have $k = m_1 + m_2 + 1$ and when $j = m_1 + n$ we have $k = m_2 - n + 1$. Reordering the terms we have

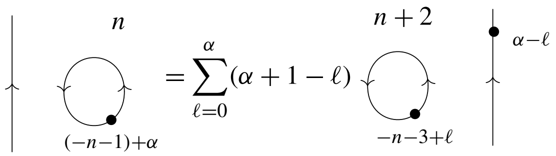


or after combining terms

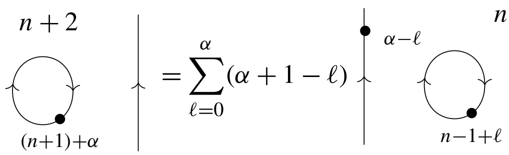


which is precisely the content of the proposition. □

Proposition 5.6 *(Bubble slides). The equalities*



(5.26)

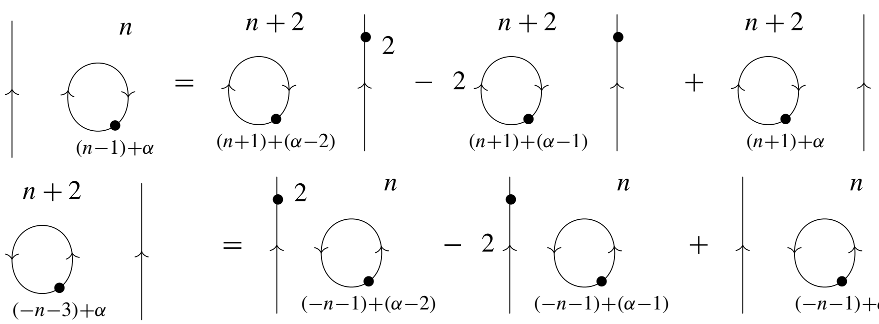


(5.27)

hold for all $n \in \mathbb{Z}$.

Proof. These equations follow from the reduction to bubbles (5.23), (5.24), and the identity decomposition (5.18) and (5.19). □

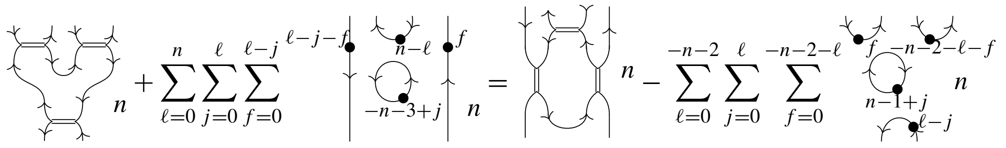
Proposition 5.7 *(Further bubble slides).*



Proof. These follow from the bubble slide formulas in Proposition 5.6. Just apply those formulas to the left hand sides of the equations above, shift appropriate indices, and cancel terms. □

One can check that sliding a dotted bubble from one side of a vertical line to the other and then back again results in the same dotted bubble. Bubble slides for downward pointing arrows readily follow from those above using biadjointness.

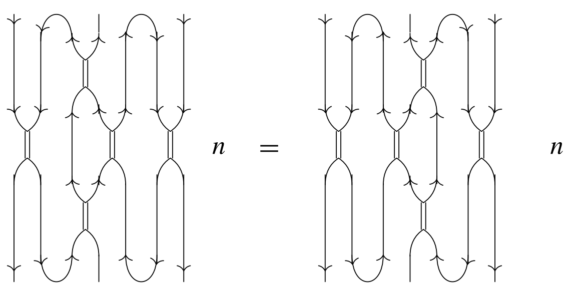
Proposition 5.8. *The equation*



holds for all $n \in \mathbb{Z}$.

The second term on the left hand side is zero when $n < 0$ by the rule that summations are increasing. Thus, this term utilizes the fake bubbles discussed above. Similarly, the second term on the right hand side is zero when $-2 < n$ and this term utilizes the fake bubbles as well.

Proof. From (5.13) we have



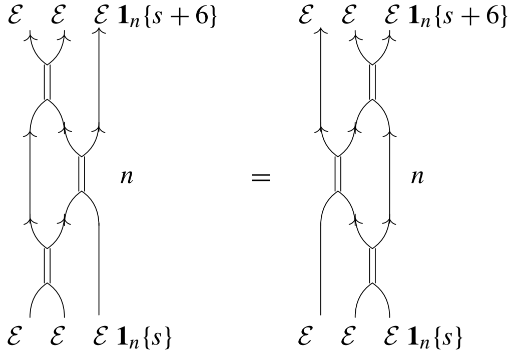
Using the reduction rules above this equality establishes the proposition. □

5.5. The 2-category $\mathcal{U}$

The 2-category $\mathcal{U}$ is the sub 2-category of $\mathcal{U}^*$ with the same objects and 1-morphisms of $\mathcal{U}^*$, but the 2-morphisms are degree preserving maps. That is, $\mathcal{U}(x, y) = (\mathcal{U}^*)_0(x, y)$.

The 2-category $\mathcal{U}$ is not enriched in graded additive categories because the sets $\mathcal{U}(x, y)$ are only abelian groups (not graded abelian groups). However, the 2-category $\mathcal{U}$ is enriched in additive categories and possesses a shift functor $\{\cdot\}$, except now the 1-morphism x is not isomorphic to $x\{m\}$ in $\mathcal{U}$ since this isomorphism is not degree zero.

The diagrammatic calculus used above naturally extends to $\mathcal{U}$. In this case, every diagram is interpreted as a degree zero diagram by shifting the source or target by an appropriate amount. All diagrammatic identities derived in this section remain true for any choice of grade shift on the source and target which make the 2-morphisms involved have degree zero. For example, the equality of degree (-6) 2-morphisms in (5.13) represents any of the degree zero equalities

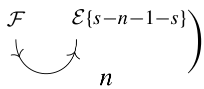


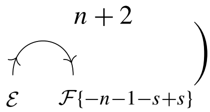
(5.28)

for $s \in \mathbb{Z}$.

The 2-category $\mathcal{U}$ is closely related to a categorification of ${}_{\mathcal{A}}\dot{\mathbf{U}}$ so we collect here the structure that $\mathcal{U}$ inherits from $\mathcal{U}^*$.

Almost biadjointness: The 1-morphism $\mathcal{E}\mathbf{1}_n$ no longer has a simultaneous left and right adjoint $\mathcal{F}\mathbf{1}_{n+2}$ because the units and counits which realize these biadjoints in $\mathcal{U}^*$ are not degree preserving. However, if we shift $\mathcal{F}\mathbf{1}_{n+2}$ by $\{-n-1\}$, then the unit and counit for the adjunction $\mathcal{E}\mathbf{1}_n \dashv \mathcal{F}\mathbf{1}_{n+2}\{-n-1\}$ become degree preserving. More generally, we have $\mathcal{E}\mathbf{1}_n\{s\} \dashv \mathcal{F}\mathbf{1}_{n+2}\{-n-1-s\}$ in $\mathcal{U}$ since the units and counits have degree:

deg() $= (1+n) + (-n-1) = 0$ (5.29)

deg() $= \big(1-(n+2)\big) - (-n-1) = 0$ (5.30)

and still satisfy the zig-zag identities. Similarly, $\mathcal{E}\mathbf{1}_n\{s\}$ has a left adjoint $\mathcal{F}\mathbf{1}_{n+2}\{n+1-s\}$ in $\mathcal{U}$. One can check that with these shifts the units and counits of the adjunction $\mathcal{F}\mathbf{1}_{n+2}\{n+1-s\} \dashv \mathcal{E}\mathbf{1}_n\{s\}$ become degree zero and are compatible with the zig-zag identities (4.3) and (4.4).

Notice that the left adjoint $\mathcal{F}\mathbf{1}_{n+2}\{n+1-s\}$ and right adjoint $\mathcal{F}\mathbf{1}_{n+2}\{-n-1-s\}$ of $\mathcal{E}\mathbf{1}_n\{s\}$ only differ by a shift. We call morphisms with this property *almost biadjoint*. This situation is familiar to those studying derived categories of coherent sheaves on Calabi–Yau manifolds. Functors with these properties are called 'almost Frobenius functors' in [28] where several other examples of this phenomenon are also given.

It is then clear that both $\mathcal{E}\mathbf{1}_n\{s\}$ and $\mathcal{F}\mathbf{1}_n\{s\}$ are almost biadjoint in $\mathcal{U}$ for all $s, n \in \mathbb{Z}$ with

$$\begin{aligned}
\mathbf{1}_n\mathcal{F}\mathbf{1}_{n+2}\{n+1-s\} &\dashv \mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n\{s\} \dashv \mathbf{1}_n\mathcal{F}\mathbf{1}_{n+2}\{-n-1-s\},\\
\mathbf{1}_n\mathcal{E}\mathbf{1}_{n-2}\{-n+1-s\} &\dashv \mathbf{1}_{n-2}\mathcal{F}\mathbf{1}_n\{s\} \dashv \mathbf{1}_n\mathcal{E}\mathbf{1}_{n-2}\{n-1-s\}.
\end{aligned}$$

Every morphism in $\mathcal{U}$ is the composite of $\mathcal{E}\mathbf{1}_n\{s\}$ and $\mathcal{F}\mathbf{1}_n\{s\}$ together with identities; by composing adjunctions as in Proposition 4.1 the right adjoints of composites can be computed. For example,

$$\mathbf{1}_{n+2}\mathcal{E}^{\alpha}\mathbf{1}_n\{s\} \dashv \mathbf{1}_n\mathcal{F}^{\alpha}\mathbf{1}_{n+2}\big\{-\alpha(n+\alpha) - s\big\}, \tag{5.31}$$

$$\mathbf{1}_{n-2}\mathcal{F}^{\beta}\mathbf{1}_n\{s\} \dashv \mathbf{1}_n\mathcal{E}^{\beta}\mathbf{1}_{n-2}\bigl\{\beta(n-\beta)-s\bigr\} \tag{5.32}$$

which leads to

$$\mathbf{1}_m\mathcal{E}^{\alpha}\mathcal{F}^{\beta}\mathbf{1}_n\{s\} \dashv \mathbf{1}_n\mathcal{E}^{\beta}\mathcal{F}^{\alpha}\mathbf{1}_m\bigl\{-(\alpha-\beta)(\alpha-\beta+n)-s\bigr\}. \tag{5.33}$$

Hence, the right adjoint of a general 1-morphism is given by

$$\begin{aligned}&\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\} \dashv \mathbf{1}_n\mathcal{E}^{\beta_k}\mathcal{F}^{\alpha_k}\cdots\mathcal{E}^{\beta_1}\mathcal{F}^{\alpha_1}\mathbf{1}_m\\ &\quad\times\left\{-\prod_{i=1}^{k}(\alpha_i-\beta_i)(\alpha_i-\beta_i+n)+2\prod_{i<j}^{k}(\alpha_i-\beta_i)(\alpha_j-\beta_j)-s\right\}.\end{aligned} \tag{5.34}$$

The left adjoint of a general 1-morphism can also be explicitly computed

$$\begin{aligned}&\mathbf{1}_n\mathcal{E}^{\beta_k}\mathcal{F}^{\alpha_k}\cdots\mathcal{E}^{\beta_1}\mathcal{F}^{\alpha_1}\mathbf{1}_m\left\{\prod_{i=1}^{k}(\alpha_i-\beta_i)(\alpha_i-\beta_i+n)\right.\\ &\quad\left.-2\prod_{i<j}^{k}(\alpha_i-\beta_i)(\alpha_j-\beta_j)-s\right\} \dashv \mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\}.\end{aligned} \tag{5.35}$$

Thus, it is clear that all morphisms in $\mathcal{U}$ have almost biadjoint.

Positivity of bubbles: As a consequence of the positivity of bubbles axiom, we have that there can only be one degree zero bubble mapping $\mathbf{1}_n \to \mathbf{1}_n$ and by definition this bubble is the identity. The degree $2m$ bubbles for $m > 0$ belong to the abelian groups $\mathcal{U}(\mathbf{1}_n\{s\}, \mathbf{1}_n\{s-2m\})$ for $s \in \mathbb{Z}$.

Pairing: The inclusion $\mathcal{U}$ into $\mathcal{U}^*$ associates to each pair of morphisms x and y in $\mathcal{U}$ the graded abelian group $\bigoplus_{s\in\mathbb{Z}}\mathcal{U}(x\{s\}, y) = \mathcal{U}^*(x, y)$.

NilHecke Algebra action: The nilHecke algebra $\mathcal{NH}_a$ acts on $\bigoplus_{s\in\mathbb{Z}}\mathcal{U}(\mathcal{E}^a\mathbf{1}_n\{s\}, \mathcal{E}^a\mathbf{1}_n)$ and $\bigoplus_{s\in\mathbb{Z}}\mathcal{U}(\mathcal{F}^a\mathbf{1}_n\{s\}, \mathcal{F}^a\mathbf{1}_n)$.

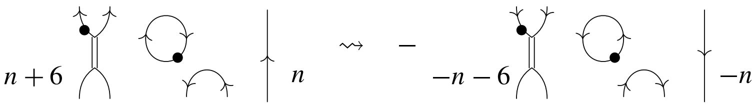
5.6. Symmetries of $\mathcal{U}$

We denote by $\mathcal{U}^{\mathrm{op}}$ the 2-category with the same objects as $\mathcal{U}$ but the 1-morphisms reversed. The direction of the 2-morphisms remain fixed. The 2-category $\mathcal{U}^{\mathrm{co}}$ has the same objects and 1-morphism as $\mathcal{U}$, but the directions of the 2-morphisms have been reversed. That is, $\mathcal{U}^{\mathrm{co}}(x, y) = \mathcal{U}(y, x)$ for 1-morphisms x and y. Finally, $\mathcal{U}^{\mathrm{co\,op}}$ denotes the 2-category with the same objects as $\mathcal{U}$, but the directions of the 1-morphisms and 2-morphisms have been reversed.

Using the symmetries of the diagrammatic relations imposed on $\mathcal{U}$ we construct 2-functors on the various versions of $\mathcal{U}$. In Section 9.3 we will show that these 2-functors are lifts of various $\mathbb{Z}[q, q^{-1}]$-(anti)linear (anti)automorphisms of the algebra $\dot{\mathbf{U}}$. The various forms of contravariant behaviour for 2-functors on $\mathcal{U}$ translate into properties of the corresponding homomorphism in $\dot{\mathbf{U}}$ as the following table summarizes:

2-functors	Algebra maps
$\mathcal{U} \to \mathcal{U}$	$\mathbb{Z}[q, q^{-1}]$-linear homomorphisms
$\mathcal{U} \to \mathcal{U}^{\mathrm{op}}$	$\mathbb{Z}[q, q^{-1}]$-linear antihomomorphisms
$\mathcal{U} \to \mathcal{U}^{\mathrm{co}}$	$\mathbb{Z}[q, q^{-1}]$-antilinear homomorphisms
$\mathcal{U} \to \mathcal{U}^{\mathrm{co\,op}}$	$\mathbb{Z}[q, q^{-1}]$-antilinear antihomomorphisms

Rescale, invert the orientation, and send $n \mapsto -n$: Consider the operation on the diagrammatic calculus that rescales the nilCoxeter generator $U_n \mapsto -U_n$, inverts the orientation of a diagram and sends $n \mapsto -n$:



This transformation preserves the degree of a diagram so by extending to sums of diagrams we get a 2-functor $\widetilde{\omega} : \mathcal{U} \to \mathcal{U}$ given by

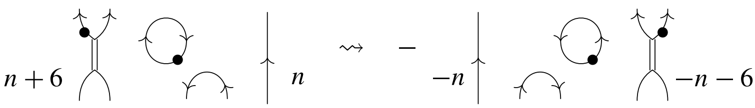
$$\widetilde{\omega} : \mathcal{U} \to \mathcal{U},$$

$$n \mapsto -n,$$

$$\mathbf{1}_m \mathcal{E}^{\alpha_1} \mathcal{F}^{\beta_1} \mathcal{E}^{\alpha_2} \cdots \mathcal{E}^{\alpha_k} \mathcal{F}^{\beta_k} \mathbf{1}_n\{s\} \mapsto \mathbf{1}_{-m} \mathcal{F}^{\alpha_1} \mathcal{E}^{\beta_1} \mathcal{F}^{\alpha_2} \cdots \mathcal{F}^{\alpha_k} \mathcal{E}^{\beta_k} \mathbf{1}_{-n}\{s\} \tag{5.36}$$

and on a 2-morphism α given by a formal sum of diagrams, $\widetilde{\omega}(\alpha)$ is the sum of diagrams obtained from α by applying the above transformation to each summand of α. It is straight forward to check that all composites are preserved and that all the relations imposed on $\mathcal{U}$ are invariant under this transformation, so that $\widetilde{\omega}$ is a strict 2-functor. In fact, it is a 2-isomorphism since its square is the identity.

Rescale, reflect across the y-axis, and send $n \mapsto -n$: The operation on the diagrammatic calculus that rescales the nilCoxeter generator $U_n \mapsto -U_n$, reflects a diagram across the y-axis, and sends n to $-n$ leaves invariant the relations on the 2-morphisms of $\mathcal{U}$. Observe that this operation



is contravariant for composition of 1-morphisms, but is covariant for composition of 2-morphisms. Furthermore, this transformation preserves the degree of a given diagram. Hence, this symmetry gives a 2-isomorphism

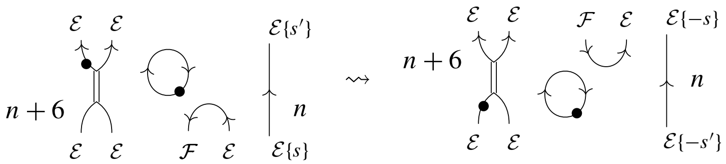
$$\widetilde{\sigma} : \mathcal{U} \to \mathcal{U}^{\mathrm{op}},$$

$$n \mapsto -n,$$

$$\mathbf{1}_m \mathcal{E}^{\alpha_1} \mathcal{F}^{\beta_1} \mathcal{E}^{\alpha_2} \cdots \mathcal{E}^{\alpha_k} \mathcal{F}^{\beta_k} \mathbf{1}_n\{s\} \mapsto \mathbf{1}_{-n} \mathcal{F}^{\beta_k} \mathcal{E}^{\alpha_k} \mathcal{F}^{\beta_{k-1}} \cdots \mathcal{F}^{\beta_1} \mathcal{E}^{\alpha_1} \mathbf{1}_{-m}\{s\}$$

and on 2-morphisms $\widetilde{\sigma}$ maps formal sums of diagrams to the formal sum of the diagrams obtained by applying the above transformation to each summand. Since, the relations on $\mathcal{U}$ are symmetric under this transformation, it is easy to see that $\widetilde{\sigma}$ is a 2-functor.

Reflect across the x-axis and invert orientation: Here we are careful to keep track of what happens to the shifts of sources and targets



The degree shifts on the right hand side are required for this transformation to preserve the degree of a diagram. This transformation preserves the order of composition of 1-morphisms, but is contravariant with respect to composition of 2-morphisms. Hence, by extending this transformation to sums of diagrams we get a 2-isomorphism given by

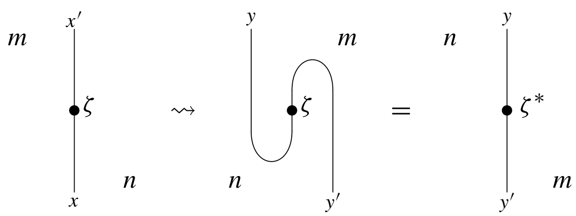
$$\widetilde{\psi}:\mathcal{U}\to\mathcal{U}^{\mathrm{co}},$$

$$n\mapsto n,$$

$$\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\}\mapsto\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{-s\} \tag{5.37}$$

and on 2-morphisms $\widetilde{\psi}$ reflects the diagrams representing summands across the x-axis and inverts the orientation. Again, the relations on $\mathcal{U}$ possess this symmetry so it is not difficult to check that $\widetilde{\psi}$ is a 2-functor. Furthermore, it is clear that $\widetilde{\psi}$ is invertible by the map which reflects across the x-axis and inverts the orientation in $\mathcal{U}^{\mathrm{co}}$.

Rotation by 180°*:* This transformation is a bit more subtle because it uses the almost biadjoint structure of $\mathcal{U}$, in particular, the calculus of mates (see Section 4.3). For each $\mathbf{1}_m x\mathbf{1}_n\in\mathcal{U}$ denote its right adjoint by $\mathbf{1}_n y\mathbf{1}_m$. The symmetry of rotation by 180° is realized by the 2-functor that sends a 1-morphism $\mathbf{1}_m x\mathbf{1}_n$ to its right adjoint $\mathbf{1}_n y\mathbf{1}_m$ and each 2-morphism $\zeta:\mathbf{1}_m x\mathbf{1}_n\Rightarrow\mathbf{1}_m x'\mathbf{1}_n$ to its mate under the adjunctions $\mathbf{1}_m x\mathbf{1}_n\dashv\mathbf{1}_n y\mathbf{1}_m$ and $\mathbf{1}_m x'\mathbf{1}_n\dashv\mathbf{1}_n y'\mathbf{1}_m$. That is, ζ is mapped to its right dual ζ^*. Pictorially,



Notice that this transformation is contravariant with respect to composition of 1-morphisms and 2-morphisms. We get a 2-functor

$$\widetilde{\tau}:\mathcal{U}\to\mathcal{U}^{\mathrm{co\,op}},$$

$$n\mapsto n,$$

$$\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\}\mapsto \text{the right adjoint in (5.34)},$$
$$\zeta\mapsto\zeta^* \tag{5.38}$$

where the degree shifts for the right adjoint (5.34) ensure that $\widetilde{\tau}$ is degree preserving. Inspection of the relations for $\mathcal{U}^*$ will reveal that they are invariant under this transformation so that $\widetilde{\tau}$ really is a 2-functor.

We can define an inverse for $\widetilde{\tau}$ given by taking left adjoints. We record this 2-morphism here.

$$\widetilde{\tau}^{-1}:\mathcal{U}\to\mathcal{U}^{\mathrm{co\,op}},$$
$$n\mapsto n,$$
$$\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\}\mapsto \text{the left adjoint in (5.35)},$$
$$\zeta\mapsto{}^*\zeta. \tag{5.39}$$

Lemma 5.9. *There are degree zero isomorphisms of graded abelian groups*

$$\mathcal{U}^*(fx,y)\to\mathcal{U}^*\big(x,\widetilde{\tau}(f)y\big), \tag{5.40}$$
$$\mathcal{U}^*(x,gy)\to\mathcal{U}^*\big(\widetilde{\tau}^{-1}(g)x,y\big) \tag{5.41}$$

for all 1*-morphisms* f,g,x,y *in* $\mathcal{U}$.

Proof. The isomorphism $\mathcal{U}^*(fx,y)\to\mathcal{U}^*(x,\widetilde{\tau}(f)y)$ is just the isomorphism M^{-1} defined in (4.10) given by taking mates under the right adjunctions. In particular, take $F\dashv U:=1\dashv 1$, $F'\dashv U':=f\dashv\widetilde{\tau}(f)$, $b=x$, and $a=y$ in Definition 4.3. Then $M^{-1}:\mathcal{U}^*(fx,y)\to\mathcal{U}^*(x,\widetilde{\tau}(f)y)$ gives a bijection as sets which extends to a homomorphisms since M^{-1} respects composites of 2-morphisms. Similarly, the isomorphism $\mathcal{U}^*(x,gy)\to\mathcal{U}^*(\widetilde{\tau}^{-1}(g)x,y)$ is just the map M from (4.9) with $F\dashv U:=1\dashv 1$, $F'\dashv U':=\widetilde{\tau}^{-1}(g)\dashv g$, $a:=y$, and $b:=x$. □

5.7. *Lifting the relations of E and F*

Here we show that the 1-morphisms $\mathcal{E}$ and $\mathcal{F}$ lift the relations from $\mathbf{U}_q(\mathfrak{sl}_2)$. For any morphism x in $\mathcal{U}$ and positive integer a, write $\oplus_{[a]}x$ for the direct sum of morphisms:

$$\oplus_{[a]}x:=x\{a-1\}\oplus x\{a-3\}\oplus\cdots\oplus x\{1-a\}. \tag{5.42}$$

Theorem 5.10. *There are decompositions of* 1*-morphisms:*

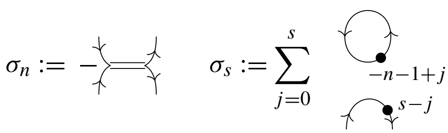
$$\mathcal{E}\mathcal{F}\mathbf{1}_n\cong\mathcal{F}\mathcal{E}\mathbf{1}_n\oplus_{[n]}\mathbf{1}_n\quad\textit{for } n\geqslant 0, \tag{5.43}$$
$$\mathcal{F}\mathcal{E}\mathbf{1}_n\cong\mathcal{E}\mathcal{F}\mathbf{1}_n\oplus_{[-n]}\mathbf{1}_n\quad\textit{for } n\leqslant 0, \tag{5.44}$$

given by systems of idempotents $e_j\in\mathcal{U}(\mathcal{E}\mathcal{F}\mathbf{1}_n,\mathcal{E}\mathcal{F}\mathbf{1}_n)$ *and* $\bar{e}_j\in\mathcal{U}(\mathcal{F}\mathcal{E}\mathbf{1}_n,\mathcal{F}\mathcal{E}\mathbf{1}_n)$.

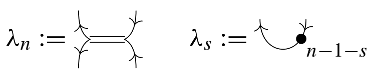
Proof. The decomposition $\mathcal{E}\mathcal{F}\mathbf{1}_n \cong \mathcal{F}\mathcal{E}\mathbf{1}_n \oplus_{[n]} \mathbf{1}_n$ for $n \geqslant 0$ is given by the system of 2-morphisms in $\mathcal{U}$

$$\begin{tikzcd}
 & & & & & \mathcal{E}\mathcal{F}\mathbf{1}_n & & & & \\
\mathcal{F}\mathcal{E}\mathbf{1}_n \arrow[urrrrr, "\lambda_n"] & \mathbf{1}_n\{n-1\} \arrow[urrrr, "\lambda_0"] & \oplus & \cdots & \oplus & \mathbf{1}_n\{n-1-2\ell\} \arrow[u, "\lambda_\ell"] & \oplus & \cdots & \oplus & \mathbf{1}_n\{1-n\} \arrow[ullll, "\lambda_{n-1}"'] \\
 & & & & & \mathcal{E}\mathcal{F}\mathbf{1}_n \arrow[ulllll, "\sigma_n"] \arrow[ullll, "\sigma_0"] \arrow[u, "\sigma_\ell"] \arrow[urrrr, "\sigma_{n-1}"'] & & & &
\end{tikzcd}$$

with



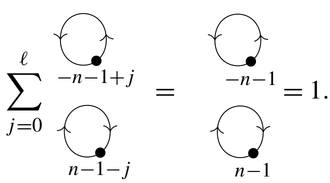
for $0 \leqslant s \leqslant n-1$, (5.45)



for $0 \leqslant s \leqslant n-1$. (5.46)

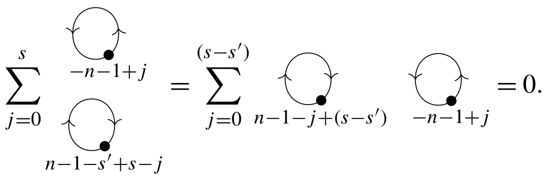
Notice that with the shifts, all of the above maps are degree zero. We claim that the maps $e_s := \lambda_s\sigma_s$ for $0 \leqslant s \leqslant -n$ form a collection of orthogonal idempotents decomposing the identity $\mathrm{Id}_{\mathcal{E}\mathcal{F}\mathbf{1}_n}$.

For $0 \leqslant s \leqslant n-1$ the composite $\sigma_s\lambda_s$ is

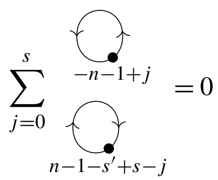


The first equality holds because the bottom bubble has negative degree when $j > 0$. Hence, all terms in the sum except $j = 0$ vanish. The composite $\sigma_n\lambda_n = \mathrm{Id}_{\mathcal{F}\mathcal{E}\mathbf{1}_n}$ by (5.19) using that $n \geqslant 0$. Hence, the 2-morphisms $\lambda_s\sigma_s$ are idempotent.

To show that these idempotents are orthogonal consider the composite $\sigma_s\lambda_{s'}$ with $0 \leqslant s' < s \leqslant n-1$:

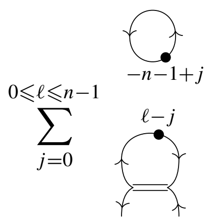


The first equality follows from the fact that the clockwise oriented bubble has negative degree for all $j > s - s'$ so these terms are zero. The second equality is just (5.25). For $0 \leqslant s < s' \leqslant n-1$ the composite is



which follows because $(s - s') < 0$ by assumption, so the clockwise bubble has negative degree for all j and is therefore equal to zero.

From (5.23) it follows that for $j \leqslant n-1$ the composite $\sigma_j\lambda_n$



is zero because the total degree of the bottom component is negative whenever $(\ell - j) < n$, but $0 \leqslant \ell \leqslant n-1$ and $j \leqslant \ell$. Thus, the idempotents are orthogonal. All that remains to be shown is that

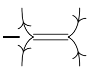
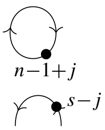
$$\sum_{s=0}^{n} \lambda_s \sigma_s = \mathrm{Id}_{\mathcal{E}\mathcal{F}\mathbf{1}_n},$$

but this is just the identity decomposition axiom (5.18).

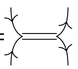
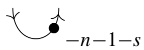
Similarly, the decomposition $\mathcal{F}\mathcal{E}\mathbf{1}_n \cong \mathcal{E}\mathcal{F}\mathbf{1}_n \oplus_{[-n]} \mathbf{1}_n$ for $n \leqslant 0$ is given by the system of maps

$$\begin{tikzcd}
 & & & \mathcal{F}\mathcal{E}\mathbf{1}_n & & \\
\mathcal{E}\mathcal{F}\mathbf{1}_n \arrow[urrr, "\lambda_{-n}"] & \mathbf{1}_n\{-n-1\} \oplus \arrow[urr, "\lambda_0"] & \cdots & \oplus\, \mathbf{1}_n\{-n-1-2\ell\} \arrow[u, "\lambda_\ell"] & \oplus \quad \cdots \quad \oplus & \oplus\mathbf{1}_n\{1+n\} \arrow[ull, "\lambda_{-n-1}"'] \\
 & & & \mathcal{F}\mathcal{E}\mathbf{1}_n \arrow[ulll, "\sigma_{-n}"] \arrow[ull, "\sigma_0"] \arrow[u, "\sigma_\ell"] \arrow[urr, "\sigma_{-n-1}"'] & &
\end{tikzcd}$$

with

$$\sigma_{-n} := -$$  $$\sigma_s := \sum_{j=0}^{s}$$ 

$$\text{for } 0 \leqslant s \leqslant -n-1, \qquad (5.47)$$

$$\lambda_{-n} :=$$  $$\lambda_s :=$$ 

$$\text{for } 0 \leqslant s \leqslant -n-1. \qquad (5.48)$$

By applying the 2-isomorphism $\widetilde{\omega}$ to the decomposition of $\mathrm{Id}_{\mathcal{E}\mathcal{F}\mathbf{1}_n}$ we see that the collection of maps $\lambda_s\sigma_s$ for $0 \leqslant s \leqslant -n$ form a collection of orthogonal idempotents decomposing the identity $\mathrm{Id}_{\mathcal{F}\mathcal{E}\mathbf{1}_n}$. $\square$

6. Graphical calculus for iterated flag varieties

Our goal in this section is to construct a graphical calculus for the cohomology of iterated flag varieties that will be used in the next section to construct representations of $\mathcal{U}^*$.

6.1. *Iterated flag varieties*

In this section we review some facts about the cohomology rings of flag varieties. Useful references for this material are Hiller [23, Chapter 3] and Fulton [19, Chapter 10].

6.1.1. *Grassmannians*

Fix a complex vector space W of dimension N. For $0 \leqslant k \leqslant N$ let G_k denote the variety of complex k-planes in W. In this notation we suppress the explicit dependence on N. If we wish to make this dependence explicit we use the notation $Gr(k, N)$. The cohomology ring of G_k has a natural structure of a $\mathbb{Z}$-graded algebra,

$$H^*(G_k, \mathbb{Q}) = \bigoplus_{0 \leqslant i \leqslant k(N-k)} H^i(G_k, \mathbb{Q}).$$

For simplicity we sometimes write $H_k := H^*(G_k, \mathbb{Q})$.

An explicit description of this cohomology ring can be given using Chern classes. The universal k-dimensional complex vector bundle $U_{k,N}$ on $Gr(k,N)$ has total space consisting of pairs (V,x) with $x \in V$ and $V \in Gr(k,N)$. Choose a hermitian metric on W. From the orthogonal complements of the fibres V of the bundle $U_{k,N}$ we can construct an $(N-k)$-dimensional complex vector bundle $U_{N-k,N}$ with the property that

$$U_{k,N} \oplus U_{N-k,N} \cong I^N$$

with I the trivial rank 1 bundle. The Chern classes $x_i := x_i(U_{k,N}) \in H^{2i}(Gr(k,N))$ for $1 \leqslant i \leqslant k$ and $y_j := y_j(U_{N-k,N}) \in H^{2j}(Gr(N-k,N))$ for $1 \leqslant j \leqslant N = k$ then satisfy

$$\left(1 + x_1 t + x_2 t^2 + \cdots + x_k t^k\right)\left(1 + y_1 t + y_2 t^2 + \cdots + y_{N-k} t^{N-k}\right) = 1 \tag{6.1}$$

by the Whitney sum axiom for Chern classes. Above t is a formal variable used to keep track of homogeneous elements. Borel [10] showed that the cohomology ring H_k is given by

$$H_k = \mathbb{Q}[x_1, \ldots, x_k, y_1, \ldots, y_{N-k}]/I_{k,N} \tag{6.2}$$

where $I_{k,N}$ is the ideal generated by the homogeneous terms in (6.1).

Our applications require many different interacting Grassmannians so when we want to emphasize the dependence on k and N we introduce a new parameter $n := 2k - N$ and let

$$H_k = \mathbb{Q}[x_{1,n}, \ldots, x_{k,n}, y_{1,n}, \ldots, y_{N-k,n}]/I_{k,N}, \tag{6.3}$$

where $x_{j,n} = x_j(U_{k,N})$ and $y_{j,n} = y_j(U_{N-k,N})$.

6.1.2. Partial flag varieties

For $0 \leqslant k < m \leqslant N$ let $G_{k,m}$ be the variety of partial flags

$$\left\{(L_k, L_m) \,\middle|\, 0 \subset L_k \subset L_m \subset W, \dim_{\mathbb{C}} L_k = k, \dim_{\mathbb{C}} L_m = m\right\}.$$

We also denote this same variety by $G_{m,k}$. Let $H_{k,m}$ be the cohomology algebra of $G_{k,m}$. Forgetful maps

$$G_k \xleftarrow{p_1} G_{k,m} \xrightarrow{p_2} G_m$$

induce maps of cohomology rings

$$H_k \xrightarrow{p_1^*} H_{k,m} \xleftarrow{p_2^*} H_m$$

which make the cohomology ring $H_{k,m}$ into a $H_k \otimes H_m$-module. Since the algebra H_m is commutative, we can turn a left H_m-module into a right H_m-module. Hence, we can make $H_{k,m}$ into a (H_k, H_m)-bimodule. In fact, $H_{k,m}$ is free as a graded H_k-module and as a graded H_m-module. This follows from the multiplicative property of spectral sequences of fibrations.

Let $k_1, \dots, k_m$ be a sequence of integers with $0 \leqslant k_i \leqslant N$ for all i. Form the (H_{k_1}, H_{k_m})-bimodule

$$H_{k_1,\dots,k_m} = H_{k_1,k_2} \otimes_{H_{k_2}} H_{k_2,k_3} \otimes_{H_{k_3}} \cdots \otimes_{H_{k_{m-1}}} H_{k_{m-1},k_m}.$$

Consider the partial flag variety $G_{k_1,\dots,k_m}$ which consists of sequences $(W_1, \dots, W_m)$ of linear subspaces of W such that the dimension of W_i is k_i and $W_i \subset W_{i+1}$ if $k_i \leqslant k_{i+1}$ and $W_i \supset W_{i+1}$ if $k_{i+1} \geqslant k_i$. The forgetful maps

$$G_{k_1} \xleftarrow{p_1} G_{k_1,\dots,k_m} \xrightarrow{p_2} G_{k_m}$$

induce maps of cohomology rings

$$H_{k_1} \xrightarrow{p_1^*} H(G_{k_1,\dots,k_m}, \mathbb{Q}) \xleftarrow{p_2^*} H_{k_m}$$

which make the cohomology ring $H(G_{k_1,\dots,k_m}, \mathbb{Q})$ into a graded (H_{k_1}, H_{k_m})-bimodule. As one might expect, there is an isomorphism

$$H^*(G_{k_1,\dots,k_m}, \mathbb{Q}) \cong H_{k_1,\dots,k_m} \tag{6.4}$$

of graded (H_{k_1}, H_{k_m})-bimodules.

6.1.3. One step iterated flag varieties

A special role is played in our theory by the one step iterated flag varieties

$$G_{k,k+1} = \{(W_k, W_{k+1}) \mid \dim_{\mathbb{C}} W_k = k,\ \dim_{\mathbb{C}} W_{k+1} = (k+1),\ 0 \subset W_k \subset W_{k+1} \subset W\}.$$

The cohomology ring $H_{k,k+1}$ again has a description using Chern classes:

$$H_{k,k+1} := \mathbb{Q}[x_1, x_2, \dots, x_k; \xi; y_1, y_2, \dots, y_{N-k-1}]/I_{k,k+1,N},$$

where $I_{k,k+1,N}$ is the ideal generated by the homogeneous elements in

$$\left(1 + x_1 + x_2 t^2 + \cdots + x_k t^k\right)(1 + \xi t)\left(1 + y_1 t + y_2 t^2 + \cdots + y_{N-k-1} t^{N-k-1}\right) = 1.$$

Here we have suppressed the dependence of the x_j and y_j on k, $k+1$, and N. A more appropriate notation might be $x_{j;k,k+1,N}$ but this seems a bit excessive.

The x_i are Chern classes of the tautological bundle U_k whose fibre over the partial flag $0 \subset W_k \subset W_{k+1} \subset W$ is W_k. Let U_{k+1} be the tautological bundle over $G_{k,k+1}$ whose fibre over the flag $0 \subset W_k \subset W_{k+1} \subset W$ is W_{k+1}. Then ξ is the first Chern class of the line bundle U_{k+1}/U_k. Finally, if W' is the trivial bundle with fibre W over $G_{k,k+1}$, then the y_j are Chern classes for the bundle W'/U_{k+1}.

The inclusions of rings

$$H_k \xrightarrow{p_1^*} H_{k,k+1} \xleftarrow{p_2^*} H_{k+1}$$

making $H_{k,k+1}$ an (H_k, H_{k+1})-bimodule are explicitly given as follows:

$$\begin{aligned} H_k &\hookrightarrow H_{k,k+1}, \\ x_{j,n} &\mapsto x_j \quad \text{for } 1 \leqslant j \leqslant k, \\ y_{1,n} &\mapsto \xi + y_1, \\ y_{\ell,n} &\mapsto \xi \cdot y_{\ell-1} + y_\ell \quad \text{for } 1 < \ell < N-k, \\ y_{N-k,n} &\mapsto \xi \cdot y_{N-k-1}, \end{aligned}$$

and

$$\begin{aligned} H_{k+1} &\hookrightarrow H_{k,k+1}, \\ x_{1,n+2} &\mapsto \xi + x_1, \\ x_{j,n+2} &\mapsto \xi \cdot x_{j-1} + x_j \quad \text{for } 1 < j < k+1, \\ x_{k+1,n+2} &\mapsto \xi \cdot x_k, \\ y_{\ell,n+2} &\mapsto y_\ell \quad \text{for } 1 \leqslant \ell \leqslant N-k-1. \end{aligned}$$

Using these inclusions we identify certain generators of H_k and H_{k+1} with their images in $H_{k,k+1}$ so that

$$H_{k,k+1} \cong \mathbb{Q}[x_{1,n}, x_{2,n}, \dots, x_{k,n}; \xi; y_{1,n+2}, y_{2,n+2}, \dots, y_{N-k-1,n+2}]/I_{k,k+1,N}$$

with $I_{k,k+1,N}$ the homogeneous elements in

$$\bigl(1 + x_{1,n} + x_{2,n}t^2 + \dots + x_{k,n}t^k\bigr)(1+\xi t)\bigl(1 + y_{1,n+2}t + \dots + y_{N-k-1,n+2}t^{N-k-1}\bigr) = 1. \quad (6.5)$$

6.1.4. Iterated flag varieties

The a-step iterated flag variety $G_{k,k+1,\dots,k+a}$ consists of $a+1$-tuples

$$\bigl\{(W_k, \dots, W_{k+a}) \bigm| \dim_{\mathbb{C}} W_{k+j} = (k+j), 0 \subset W_k \subset W_{k+1} \subset \dots \subset W_{k+a} \subset W\bigr\},$$

where $0 \leqslant k \leqslant k+a \leqslant N$. The cohomology ring $H_{k,k+1,\dots,k+a}$ admits a description using Chern classes of vector bundles. For $0 \leqslant j \leqslant a$ let U_{k+j} be the tautological bundle whose fibre over the element $0 \subset W_k \subset W_{k+1} \subset \dots \subset W_{k+a} \subset W$ in $G_{k,k+1,\dots,k+a+1}$ is W_{k+j}. Then

$$H_{k,k+1,\dots,k+a} \cong \mathbb{Q}[x_{1,n}, \dots, x_{k,n}; \xi_1, \xi_2, \dots, \xi_a; y_{1,n+2a}, \dots, y_{N-k-a,n+2a}]/I_{k,\dots,k+a} \quad (6.6)$$

where $I_{k,\dots,k+a}$ is the ideal generated by the homogeneous elements of

$$\begin{aligned}
&\big(1 + x_{1,n}t + \cdots + x_{k,n}t^k\big)(1 + \xi_1 t)\cdots(1 + \xi_a t) \\
&\quad \times \big(1 + y_{1,n+2a}t + \cdots + y_{N-k-a,n+2a}t^{N-k-a}\big) = 1. \qquad (6.7)
\end{aligned}$$

The $x_{i,n}$ are Chern classes of the bundle U_k, where we have taken the liberty of identifying these generator as the images of $x_{i,n} \in H_k$ under the natural inclusion. The generators ξ_j are the Chern classes of the line bundles U_{k+j}/U_{k+j-1}. Finally, the $y_{\ell,n}$ are the Chern classes of the bundle W'/U_{k+a} identified as the images of $y_{\ell,n+2a} \in H_{k+a}$ under the natural inclusion.

The generators ξ_i corresponding to Chern classes of the line bundles derived from iterated flag varieties are important in what follows. In particular, we show that the nilHecke ring $\mathcal{NH}_a$ acts on the collection generators ξ_i leading to bimodule maps $H_{k,\dots,k+a} \to H_{k,\dots,k+a}$.

6.1.5. Defining the 2-category $\mathbf{Flag}_N^*$

Recall the additive 2-category **Bim** whose objects are graded rings, morphisms are graded bimodules, and the 2-morphisms are degree-preserving bimodule maps. Idempotent bimodule homomorphisms split in **Bim**.

Let $\mathbf{Bim}^*$ denote the 2-category whose objects are graded rings, whose 1-morphisms are graded bimodules, and 2-morphisms are all bimodule maps. Just like the 2-category $\mathcal{U}^*$, $\mathbf{Bim}^*$ is enriched in graded additive categories with a translation; the shift functor given by the degree shift map on graded bimodules. Both $\mathbf{Bim}^*$ and **Bim** are weak 2-categories, or bicategories, since the composition of 1-morphisms is the tensor product of bimodules which is only associative up to coherent isomorphism.

We now define a sub 2-category $\mathbf{Flag}_N$ of **Bim** for each integer $N \in \mathbb{Z}_+$.

Definition 6.1. The additive 2-category $\mathbf{Flag}_N$ is the idempotent completion (see Section 9.1) inside of **Bim** of the 2-category consisting of

- objects: the graded rings H_k for each $0 \leqslant k \leqslant N$.
- morphisms: generated by the graded (H_k, H_k)-bimodule H_k, the graded (H_k, H_{k+1})-bimodule $H_{k,k+1}$ and the graded (H_{k+1}, H_k)-bimodule $H_{k+1,k}$ together with their shifts $H_k\{s\}$, $H_{k,k+1}\{s\}$, and $H_{k,k+1}\{s\}$ for $s \in \mathbb{Z}$. The bimodules $H_k = H_k\{0\}$ are the identity 1-morphisms. Thus, a generic morphism from H_{k_1} to H_{k_m} is a direct sum of graded (H_{k_1}, H_{k_m})-bimodules of the form
$$H_{k_1,k_2} \otimes_{H_{k_2}} \otimes_{H_{k_2,k_3}} \otimes_{H_{k_3}} \cdots \otimes_{H_{k_{m-1}}} H_{k_{m-1},k_m}\{s\}$$
where $k_{i+1} = k_i \pm 1$ for $1 < i \leqslant m$.
- 2-morphisms: degree-preserving bimodule maps.

There is a 2-subcategory $\mathbf{Flag}_N^*$ of $\mathbf{Bim}^*$ with the same objects and morphisms as $\mathbf{Flag}_N$, and with 2-morphisms

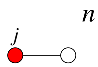
$$\mathbf{Flag}_N^*(x, y) := \bigoplus_{s \in \mathbb{Z}} \mathbf{Flag}_N\big(x\{s\}, y\big). \qquad (6.8)$$

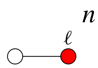
In Section 7 we show that $\mathbf{Flag}_N^*$ provides a representation of $\mathcal{U}^*$, and, by restriction to degree zero maps, that $\mathbf{Flag}_N$ provides a representation of $\mathcal{U}$.

6.2. Diagrammatics

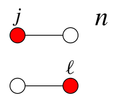
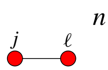
We now introduce a graphical calculus for computations in the cohomology rings of iterated flag varieties.

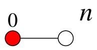
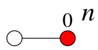
Calculus for H_k: For $n = 2k - N$ the calculus for the cohomology ring $H^*(Gr(k, N))$ is given by representing the generators $x_{j,n}$ and $y_{\ell,n}$ as coloured dumbbells together with a label floating in a region labelled n:

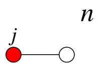
$x_{j,n} :=$  for $0 \leqslant j \leqslant k$, (6.9)

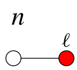
$y_{\ell,n} :=$  for $0 \leqslant \ell \leqslant N-k$. (6.10)

The coloured and labelled circles on the left and right of a dumbbell are called the weights of the dumbbell. An uncoloured circle corresponds to weight 0. The generators $x_{j,n}$ are given by dumbbells with left weight equal to j. The $y_{\ell,n}$ are depicted by dumbbells with the right weight ℓ. Products of Chern classes are depicted by putting multiple dumbbells in a region labelled n, or by combining them as follows:

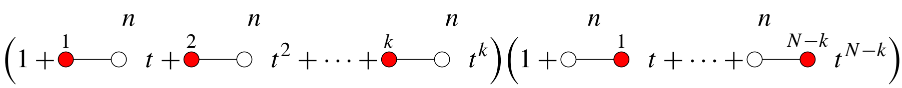
$x_{j,n}y_{\ell,n} =$  $=:$  . (6.11)

Note that  $= 1$ and  $= 1$. Define

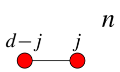
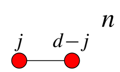
 $= 0$ for j not in the range $0 \leqslant j \leqslant k$, (6.12)

 $= 0$ for ℓ not in the range $0 \leqslant \ell \leqslant N-k$. (6.13)

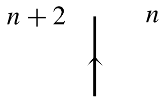
A dumbbell with weight equal to j has degree $2j$. Recall that the only relations on the generators of H_k were given by (6.1). In the graphical calculus this relation becomes

 $= 1.$ (6.14)

By examining the homogeneous elements we have the diagrammatic identity that for all $d > 0$

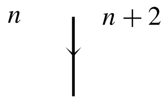
$\sum_{j=0}^{d}$  $= \sum_{j=0}^{d}$  $= 0.$ (6.15)

Calculus for $H_{k,k+1}$: The identity element in $H_{k,k+1}$ is represented by a vertical line

$H_{k,k+1} \ni 1 \quad :=$ 

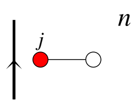
where the orientation indicates that we are regarding $H_{k,k+1}$ as an (H_k, H_{k+1})-bimodule. For fixed N, the n on the right hand side keeps track of the k value in the difference $2k - N$. Hence, having $n = 2k - N$ on the right hand side of the diagram indicates the left action of H_k on $H_{k,k+1}$. Similarly, the $n + 2 = 2(k + 1) - N$ on the left indicates the right action of H_{k+1} on $H_{k,k+1}$.

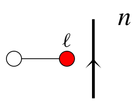
When we want to regard this bimodule as an (H_{k+1}, H_k)-bimodule we use the notation $H_{k+1,k}$ and depict it in the graphical calculus with the opposite orientation (a downward pointing arrow).

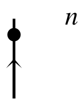
$H_{k+1,k} \ni 1 \quad :=$ 

The observant reader will have noticed that our convention is identical to that used in the previous section. Crossing an upward oriented arrow from right to left increases the value of n by two. Crossing a downward oriented arrow from right to left decreases the value by two. We often label only a single region of a diagram since these rules can be used to determine the labels on all other regions.

Eq. (6.5) shows that all of the generators from $H_{k,k+1}$, except for the one corresponding to the Chern class of the line bundle U_{k+1}/U_k, can be interpreted as either generators of H_k or H_{k+1} under the natural inclusions. This fact is represented in the graphical calculus as follows:

$H_{k,k+1} \ni x_{j,n} \quad :=$  (6.16)

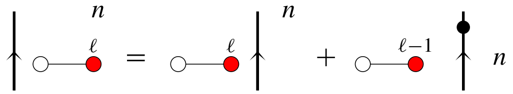
$H_{k,k+1} \ni y_{\ell,n+2} \quad :=$  (6.17)

$H_{k,k+1} \ni \xi \quad :=$  (6.18)

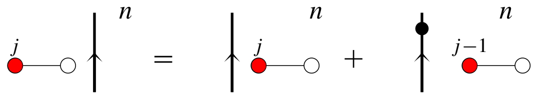
where each diagram inherits a grading from the Chern class it represents ($\deg x_{j,n} = 2j$, $\deg y_{\ell,n+2} = 2\ell$, and $\deg \xi = 2$). Equation (6.16) is meant to depicts the generator $x_{j,n} \in H_{k,k+1}$ as the element $x_{j,n} \in H_k$ acting on the identity of $H_{k,k+1}$. Likewise, the generator $y_{\ell,n+2} \in H_{k,k+1}$ is depicted as the element $y_{\ell,n+2} \in H_{k+1}$ acting on the identity of $H_{k,k+1}$. The generator ξ is represented by a dotted line so that ξ^α is represented by α dots on a line, but for simplicity we write this using a single dot and a label to indicate the power.

As explained in Section 6.1.3 the generators $y_{\ell,n} \in H_k$ do not map to our canonical generators of $H_{k,k+1}$ under the inclusion (unlike $x_{j,n} \in H_k$). Rather they are mapped to the sum $y_{\ell,n+2} +$

$\xi \cdot y_{\ell-1,n+2}$ where $y_{\ell-1,n+2}$ and $y_{\ell,n+2}$ are the images of generators from H_{k+1}. In terms of our graphical calculus, this fact becomes the identity

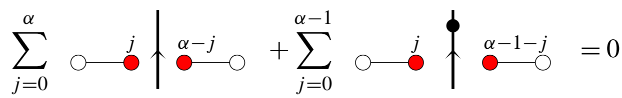

(6.19)

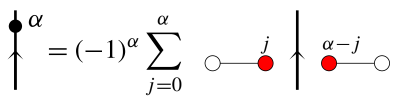
Similarly, since the generators $x_{j,n+2} \in H_{k+1}$ map to the sum $x_{j,n} + \xi \cdot x_{j-1,n}$ in $H_{k,k+1}$ we have

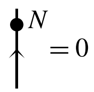

(6.20)

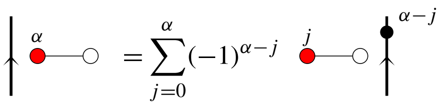
for all j in the appropriate range. Using that dumbbells with weight 0 are equal to 1, together with (6.12) and (6.13), we can afford to be less careful about restricting the range of j if we allow for the possibility that some diagrams may be zero. We use this observation throughout this section and the next.

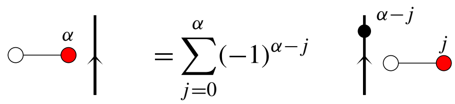
Some other graphical identities that follow from (6.5) are collected below:


(6.21)


(6.22)


(6.23)

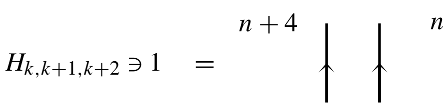

(6.24)


(6.25)

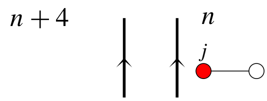
For example, (6.23) translated back into algebra just says that $\xi^N = 0$. The above relations by no means constitute a minimal set of relations; they are intended to be a list of convenient rules for diagrammatic calculations.

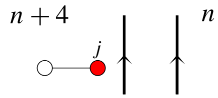
Remark 6.2. Note that the left hand side of (6.25) is zero if $\alpha > N - k - 1$ by (6.13). Similarly, the left hand side of (6.24) is zero if $\alpha > k$ by (6.12).

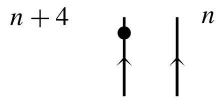
Calculus for $H_{k,k+1,k+2}$: The identity for the (H_k, H_{k+2})-bimodule $H_{k,k+1,k+2} = H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k+2}$ is

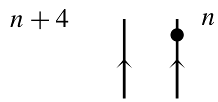


where the middle region carries a label of $n+2$. Again, most generators in $H_{k,k+1,k+2}$ can be identified as the images of generators in H_k and H_{k+2} under the action given by the natural inclusions. The two generators ξ_1 and ξ_2 corresponding to the first Chern classes of the line bundles U_{k+1}/U_k and U_{k+2}/U_{k+1} are represented by dots on one of the two lines:

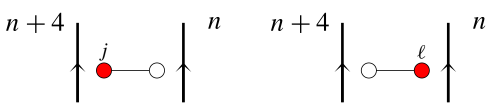
$H_{k,k+1,k+2} \ni x_{j,n} \quad :=$  (6.26)

$H_{k,k+1,k+2} \ni y_{j,n+4} \quad :=$  (6.27)

$H_{k,k+1,k+2} \ni \xi_1 \quad :=$  (6.28)

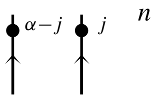
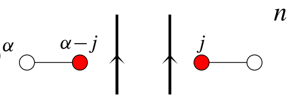
$H_{k,k+1,k+2} \ni \xi_2 \quad :=$  (6.29)

The tensor decomposition $H_{k,k+1,k+2} = H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k+2}$ has a natural depiction



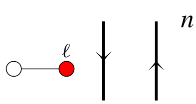
where the elements $x_{j,n+2}$ and $y_{\ell,n+2}$ in H_{k+1} can equivalently be regarded as acting on $H_{k,k+1}$ or $H_{k+1,k+2}$.

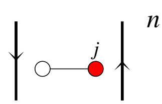
A helpful graphical identity derived from (6.7) is the following:

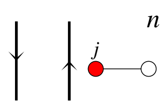
$\sum_{j=0}^{\alpha}$  $= \sum_{j=0}^{\alpha} (-1)^{\alpha}$  (6.30)

General calculus: The calculus for general tensor products iterated flag varieties is analogous to that described above. Tensoring by $- \otimes_{H_k} H_{k,k+1}$ adds an additional upward oriented line and tensoring with $- \otimes_{H_{k+1}} H_{k+1,k}$ adds an additional downward oriented line. For example $H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k}$ would have an upward pointing arrow on the right and a downward pointing arrow on the left. Dumbbells in the middle of these upward and downward pointing arrows represent the action of elements in H_{k+1}, and dumbbells on the outside represent the left or right action by elements in H_k.

Right action H_k — Action of H_{k+1} — Left action of H_k







Dots on the lines corresponding to tensor factors $H_{k,k+1}$ and $H_{k+1,k}$ represent the generator corresponding to the Chern class of the line bundle U_{k+1}/U_k.

In this way, any element of a tensor product of iterated flag varieties $H_{k_1,\dots,k_m}$ has a natural interpretation as a diagram in our calculus. Furthermore, the actions of the rings H_{k_i} have a natural interpretation as either generators or sums of generators in our calculus. In the next section we use this calculus to construct a representation of the 2-category $\mathcal{U}^*$.

6.3. Bimodule maps

For (H_{k_1}, H_{k_2})-bimodules H and H' the graphical calculus makes it easy to define bimodule maps $f : H \to H'$. A priori we would need to specify the bimodule map on all generators. However, using the relations and the identification of certain generators in H and H' with those in H_{k_1} and H_{k_2} we often need to specify far fewer. Loosely speaking, an element in H with a weighted dumbbell on the far right must be mapped to an element in H' with the same weighted dumbbell on the far right. Similarly, dumbbells on the far left must get mapped to elements with the same dumbbell on the far left.

For example, to define a (H_k, H_{k+1})-bimodule map $f : H_{k,k+1} \to H'$ we need only specify the image of 1 since f must preserve the action of H_k by $x_{j,n}$ and the action of H_{k+1} by $y_{\ell,n+2}$. By (6.5) the image of the generator ξ is determined by the relation that $\xi = -(x_{1,n} + y_{1,n+2})$. Hence, we do not need to specify the image of ξ or the image of any of its powers; they are determined by the action of H_k and H_{k+1}.

To define a (H_k, H_{k+2})-bimodule map $f : H_{k,k+1,k+2} \to H'$ we need only specify $f(\xi_1^\alpha)$ or $f(\xi_2^\alpha)$ for all nonzero powers α. All other generators are determined by identification of the generators of $H_{k,k+1,k+2}$ with those of H_k and H_{k+2}. A similar principle can be applied in general.

We use these facts many times in the next section to simplify our proofs.

7. Representing $\mathcal{U}^*$ on the flag 2-category

In this section we define for each positive integer N a weak 2-functor $\Gamma_N : \mathcal{U}^* \to \mathbf{Flag}_N^*$. The 2-functor Γ_N is degree preserving so that it restricts to a weak 2-functor $\Gamma_N : \mathcal{U} \to \mathbf{Flag}_N$.

7.1. Defining the 2-functor Γ_N

On objects the 2-functor Γ_N sends n to the ring H_k whenever n and k are compatible:

$$\begin{aligned} &\Gamma_N : \mathcal{U}^* \to \mathbf{Flag}_N^*, \\ &n \mapsto \begin{cases} H_k & \text{with } n = 2k - N \text{ and } 0 \leqslant k \leqslant N, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.1}$$

Morphisms of $\mathcal{U}^*$ get mapped by Γ to graded bimodules

$$\begin{aligned} &\Gamma_N : \mathcal{U}^* \to \mathbf{Flag}_N^*, \\ &\mathbf{1}_n\{s\} \mapsto \begin{cases} H_k\{s\} & \text{with } n = 2k - N \text{ and } 0 \leqslant k \leqslant N, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.2}$$

$$\mathcal{E}\mathbf{1}_n\{s\} \mapsto \begin{cases} H_{k,k+1}\{s+1-N+k\} & \text{with } n=2k-N \text{ and } 0 \leqslant k < N, \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

$$\mathcal{F}\mathbf{1}_n\{s\} \mapsto \begin{cases} H_{k+1,k}\{s+1-k\} & \text{with } n=2k-N \text{ and } 0 \leqslant k < N, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

Here $H_{k,k+1}\{s+1-k\}$ is the bimodule $H_{k,k+1}$ with the grading shifted by $s+1-k$ so that

$$\big(H_{k,k+1}\{s+1-k\}\big)_j = (H_{k,k+1})_{j+s+1-k}.$$

More generally, we have

$$\mathcal{E}^{\alpha}\mathbf{1}_n\{s\} \mapsto H_{k,k+1,k+2,\cdots,k+(\alpha-1),k+\alpha}\big\{s+\alpha(-N+k)+\alpha\big\},$$
$$\mathcal{F}^{\beta}\mathbf{1}_n\{s\} \mapsto H_{k,k-1,k-2,\cdots,k-(\beta-1),k-\beta}\{s-\beta k+2-\beta\}$$

so that

$$\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\dots\mathcal{F}^{\beta_{k-1}}\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\} \cong \mathcal{E}^{\alpha_1}\mathbf{1}_{n-\sum(\beta_j-\alpha_j)} \circ \dots \circ \mathcal{E}^{\alpha_k}\mathbf{1}_{n-2\beta_k} \circ \mathcal{F}^{\beta_k}\mathbf{1}_n\{s\} \tag{7.5}$$

is mapped to the graded bimodule

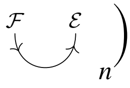
$$H_{k,k-1,\cdots,k-\beta_k,k-\beta_k+1,\cdots,k-\beta_k+\alpha_k,k-\beta_k+\alpha_k+1,\cdots,k-\sum_j(\beta_j+\alpha_j)}$$

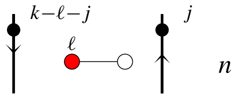
with grading shift $\{s+s'\}$, where s' is the sum of the grading shifts for each terms of the composition in (7.5). Formal direct sums of morphisms of the above form are mapped to direct sums of the corresponding bimodules.

It follows from (6.4) that Γ_N preserves composites of 1-morphisms up to isomorphism. Hence, the 2-functor Γ_N is a weak 2-functor or bifunctor. In what follows we will often simplify our notation and write Γ instead of Γ_N. We now proceed to define Γ on 2-morphisms.

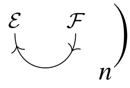
7.1.1. Biadjointness

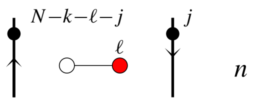
Definition 7.1. The 2-morphisms generating biadjointness in $\mathcal{U}^*$ are mapped by Γ to the following bimodule maps.

$\Gamma\Big($ 

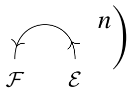
$\Big)$: $H_k \longrightarrow (H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k})\{1-N\}$, $\quad 1 \mapsto \sum_{\ell=0}^{k}\sum_{j=0}^{k-\ell}(-1)^{\ell}$ 

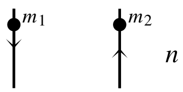
(7.6)

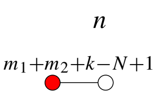
$\Gamma\Big($ 

$\Big)$: $H_k \longrightarrow (H_{k,k-1} \otimes_{H_{k-1}} H_{k-1,k})\{1-N\}$, $\quad 1 \mapsto \sum_{\ell=0}^{N-k}\sum_{j=0}^{N-k-\ell}(-1)^{\ell}$ 

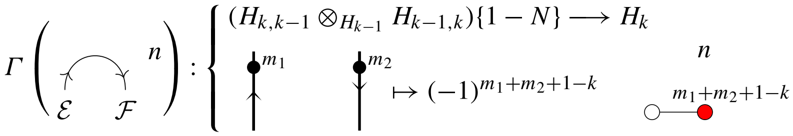
(7.7)

$\Gamma\Big($ 

$\Big)$: $(H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k})\{1-N\} \longrightarrow H_k$, 

$\mapsto (-1)^{m_1+m_2+k-N+1}$ 

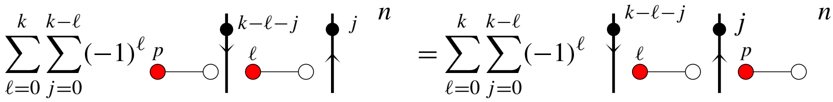
(7.8)



(7.9)

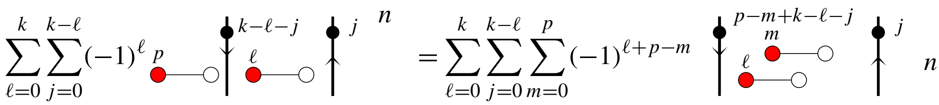
These definitions preserve the degree of the 2-morphisms of $\mathcal{U}^*$ defined in Section 5.2. In (7.6) the element 1 is in degree zero and is mapped to a sum of elements in degree $2k$ that have been shifted by $\{1-N\}$ for a total degree $2k+1-N = 1+n$. The degree in (7.7) is $2(N-k)+(1-N) = 1-n$. Similarly, in (7.8) a degree $2(m_1+m_2)$ element shifted by $1-N$ is mapped to a degree $2(m_1+m_2+k-N-1)$ element for a total degree of $1+n$. One can easily check that the map defined in (7.9) is of degree $1-n$.

To see that (7.6) and (7.7) are (H_k, H_k)-bimodule maps we must show that the left action of H_k on the image of $1 \in H_k$ is equal to the right action of H_k. For (7.6) we must show that



(7.10)

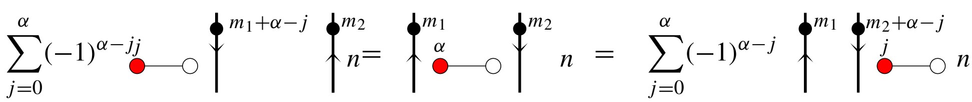
Using the rotated version of (6.24) we have



(7.11)

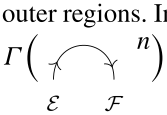
so that sliding the dumbbell labelled m to the right using (6.20), cancelling terms, and relabelling completes the proof. In a similar way Eq. (7.7) can be shown to define a bimodule map as well.

We have defined maps (7.8) and (7.9) on elements $\xi^{m_1} \otimes \xi^{m_2}$ with it implicit dumbbells on the far right and left are mapped to themselves. For these maps to be well defined (H_k, H_k)-bimodule maps the relations in $H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k}$ and $H_{k,k-1} \otimes_{H_{k-1}} H_{k-1,k}$ must be preserved. For example, the element

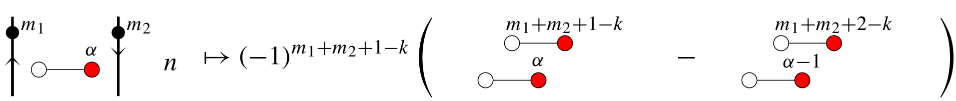


(7.12)

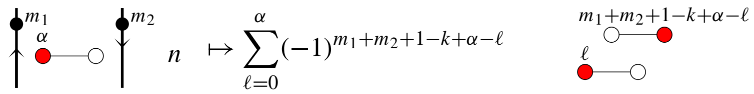
in $H_{k,k-1} \otimes_{H_{k-1}} H_{k-1,k}$ must be mapped to the same element in H_k. Since the maps (7.8) and (7.9) only depend on the sum m_1+m_2 it is easy to see that they give well-defined bimodule homomorphisms by using (6.19), (6.20), (6.24), and (6.25) to slide interior dumbbells to the outer regions. In particular, the images of dumbbells in the region $n-2$ under the bimodule map



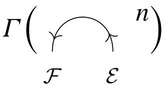
are given by

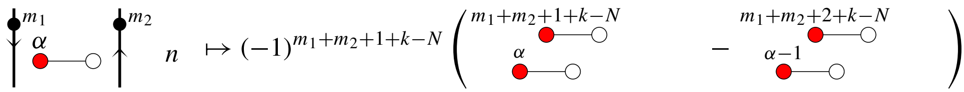


(7.13)

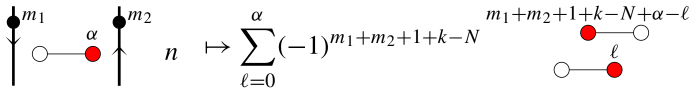


(7.14)

and the images of dumbbells in the region $n+2$ under the map  are given by

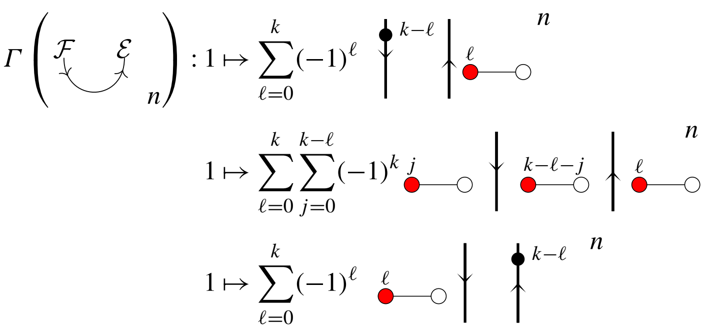


(7.15)



(7.16)

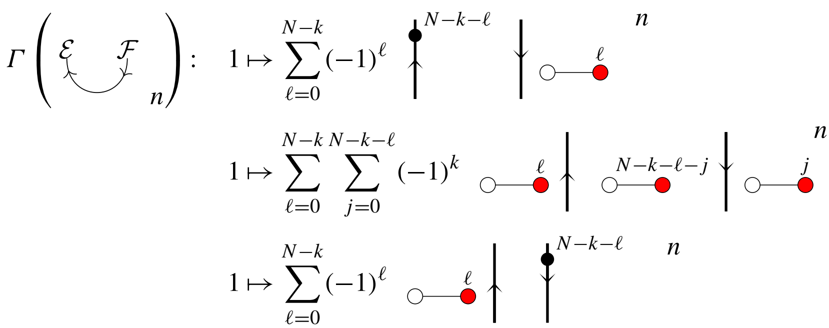
The following alternative definitions of the cups (7.6) and (7.7) will also be useful



(7.17)

(7.18)

(7.19)



(7.20)

(7.21)

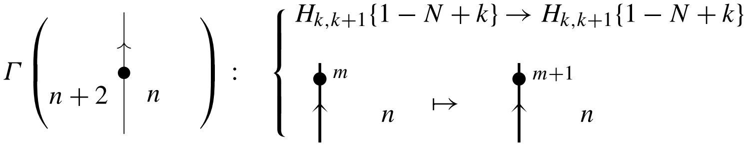
(7.22)

The proof that these are all equivalent is a straight forward application of the diagrammatic identities derived in Section 6.2.

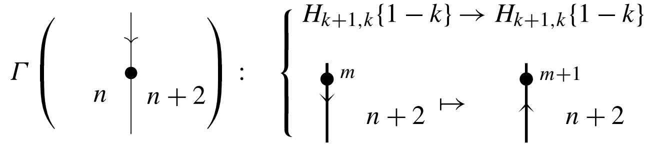
7.1.2. NilHecke generators

We show that the nilHecke algebra $\mathcal{NH}_a$ acts on $\mathrm{End}(H_{k,k+1,\dots,k+a})$ with χ_i acting by multiplication by ξ_i and u_i acting by the divided differences ∂_i operator on the variables ξ_i. Recall from Section 6.2 that the variables ξ_i correspond to Chern classes of line bundles naturally associated to iterated flag varieties.

Definition 7.2. The 2-morphisms z_n and $\widehat{z}_n$ in $\mathcal{U}^*$ are mapped by Γ_N to the graded bimodule maps:



(7.23)

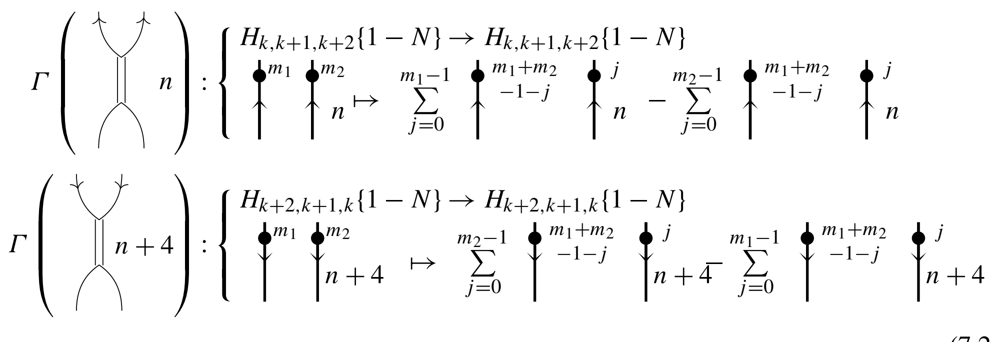


(7.24)

Note that these assignments are degree preserving since these bimodule maps are degree 2.

The nilCoxeter generator U_n is mapped to the bimodule map which acts as the divided difference operator in the variables ξ_j for $1 \leqslant j \leqslant a$.

Definition 7.3. The 2-morphisms U_n and $\widehat{U}_n$ are mapped by Γ_N to the graded bimodule maps:



(7.25)

The value of these maps on any other element is determined from the rules above together with the requirement that these maps preserve the actions of H_k and H_{k+2}. It is not hard to check that these assignments give well-defined bimodule homomorphisms. The maps $\Gamma(U_n)$ and $\Gamma(\widehat{U}_n)$ as defined above have degree -2 so that Γ preserves the degree of U_n and $\widehat{U}_n$.

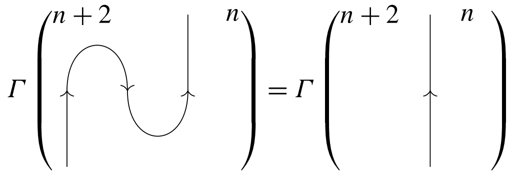
Remark 7.4. $\Gamma(U_n)(\xi_1^{m_1} \otimes \xi_2^{m_2})$ is zero when $m_1 = m_2$. This is clear from the definition. When the number of dots the upward oriented lines is equal the two sums in (7.25) cancel.

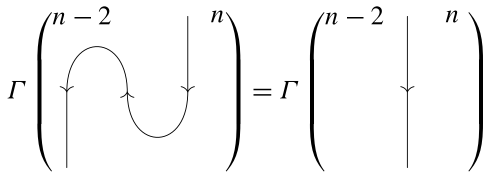
7.2. *Proving that Γ_N is a 2-functor*

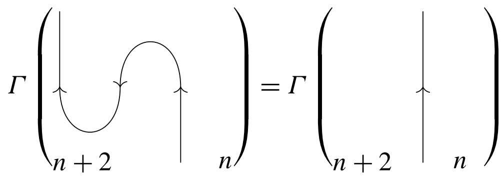
In this section we show that the relations of Section 5.2 for $\mathcal{U}^*$ are satisfied in $\mathbf{Flag}_N^*$, thus establishing that Γ_N is a 2-functor. From the definitions in the previous section it is clear that Γ_N preserves the degree associated to generators. For this reason, we often simplify our notation in this section by omitting the global grading shift of $\{1-N\}$.

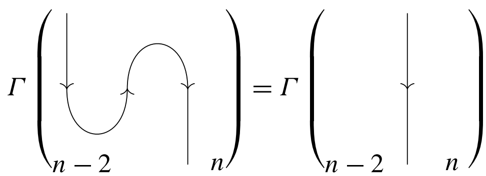
The proof that Γ_N is a 2-functor is long and labourious. The reader willing to take our word of this fact is advised to skip ahead to Section 8.

Lemma 7.5 *(Biadjointness). The following identities*



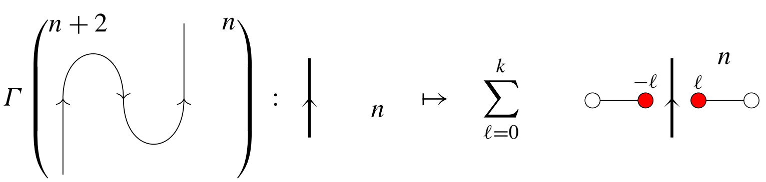






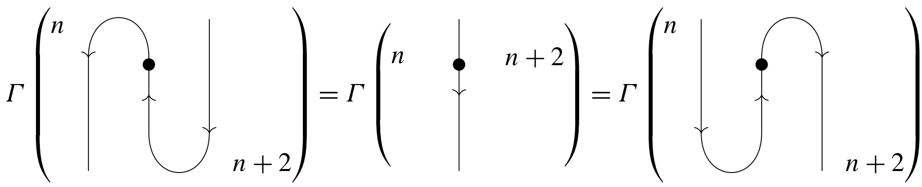
hold in $\mathbf{Flag}_N^*$ for all $n \in \mathbb{Z}$.

Proof. As explained in Section 6.3 it suffices to check these relations on the identity element of the bimodules $\Gamma(\mathbf{1}_{n-2}\mathcal{F}\mathbf{1}_n)$ and $\Gamma(\mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n)$, since all other elements are determined from this one by the bimodule property. Using the alternative definition of the cup from (7.17) we have



Hence the only nonzero value in the sum is when $\ell = 0$ proving the first identity. The rest are proven similarly. □

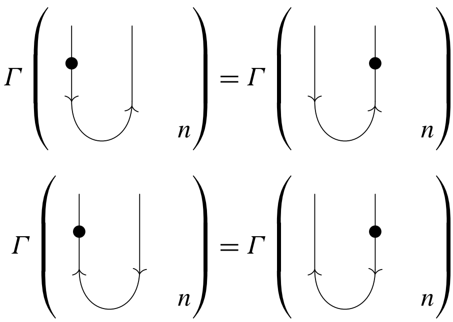
Lemma 7.6 *(Duality for z_n). The equations*



(7.26)

of bimodule maps hold in $\mathbf{Flag}_N^*$ *for all* $n \in \mathbb{Z}$.

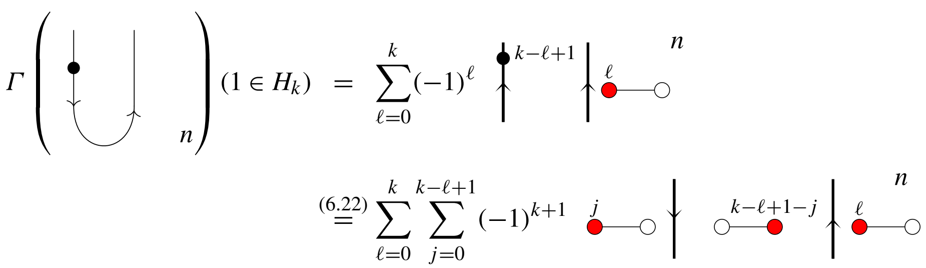
Proof. Since we have already established that Γ preserves the biadjoint structure of $\mathcal{U}^*$ in Lemma 7.5, the above (7.26) is equivalent to proving



(7.27)

(7.28)

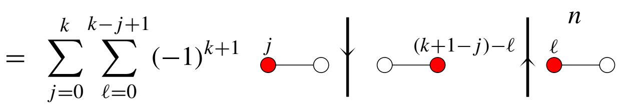
for all $n \in \mathbb{Z}$. We prove the first identity, the second can be proven similarly. We compare the images of both bimodule maps on the element $1 \in H_k$,



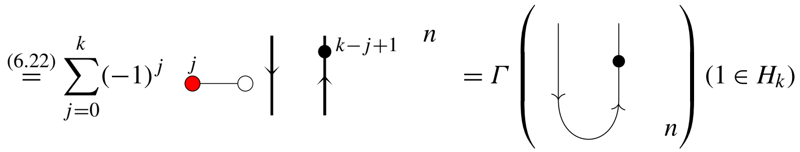
(7.29)

(7.30)

The dumbbell on the left is zero when the weight $j = k + 1$. Similarly, adding an $\ell = k + 1$ term to the sum is equivalent to adding zero since the dumbbell on the right hand side is zero when the weight $\ell = k + 1$. Switching the summation order we have



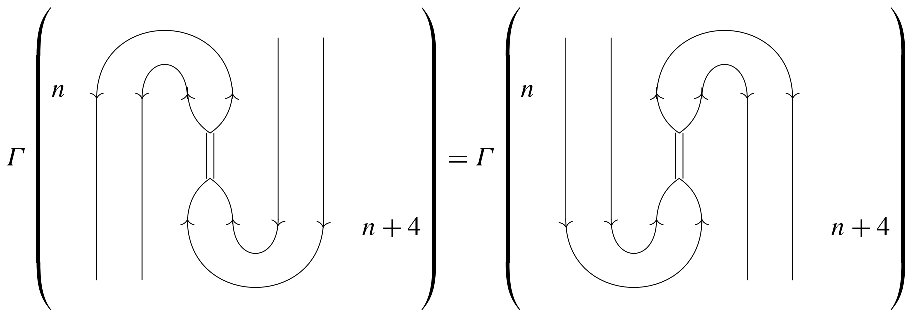
(7.31)



(7.32)

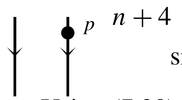
completing the proof. □

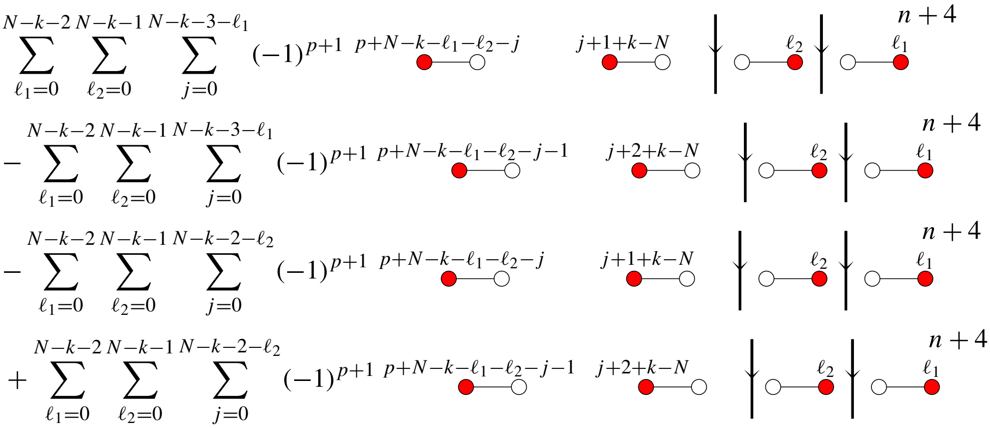
Lemma 7.7 *(Duality for U_n). The equation*



(7.33)

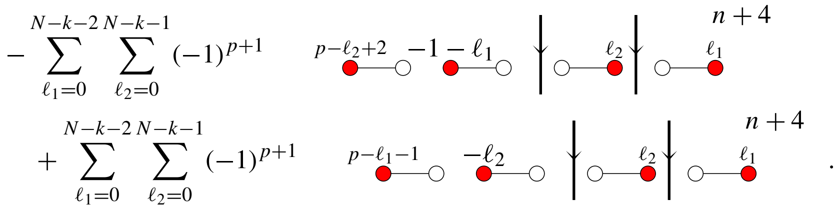
holds in $\mathbf{Flag}_N^*$ *for all* $n \in \mathbb{Z}$.

Proof. We show that the two maps agree on vectors of the form  since all other vectors are determined from the relations and the bimodule property. Using (7.20) twice for the maps $\widehat{\varepsilon}_{n+4}$ and $\widehat{\varepsilon}_{n+2}$ we compute the image of this vector under the first bimodule map



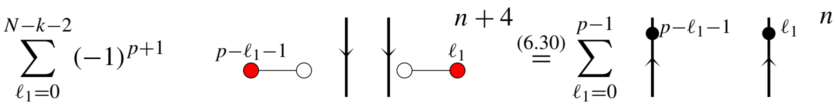
(7.34)

All terms in the first two factors cancel except for the $j=0$ term of the first factor and the $j = N-k-3-\ell_1$ term of the second factor. All terms of the third and fourth factors cancel except for the $j=0$ term of the third factor and the $j = N-k-2-\ell_2$ term of the fourth factor. Then the $j=0$ terms of the first and third factors cancel leaving only



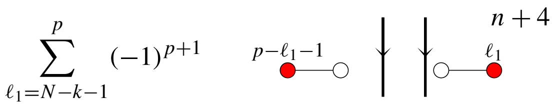
. (7.35)

The first factor is always zero because the dumbbell with weight $-1-\ell_1$ always has negative degree. In the second factor the only term for which the dumbbell with weight $-\ell_2$ is in positive degree is when $\ell_2=0$. This leaves only the terms



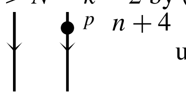
(7.36)

where in the last equality we may need to add terms

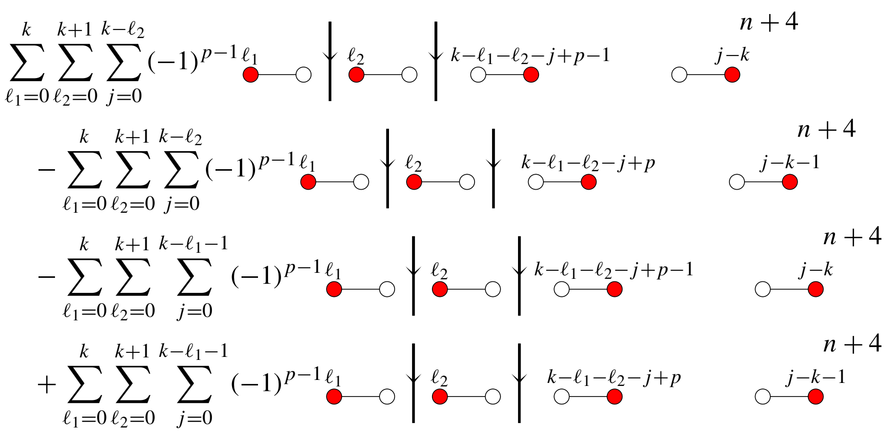


(7.37)

when $p>N-k-2$, but all of these terms are zero because the dumbbell on the right is zero whenever the right weight $\ell_1>N-k-2$ by (6.13).

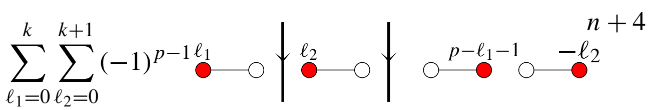
The image of the vector 

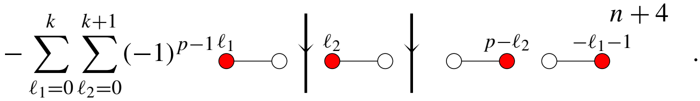
under the second bimodule map of (7.33) is



(7.38)

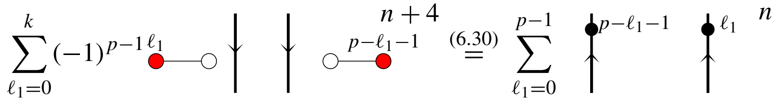
using (7.19) twice to compute the maps η_n and η_{n+2}. All terms in the first two factors cancel except for the $j=k-\ell_2$ term in the first and the $j=0$ in the second. All terms in the last two factors cancel except for the $j=k-\ell_1-1$ in the first and the $j=0$ term in the last. Again, the $j=0$ terms cancel leaving only





(7.39)

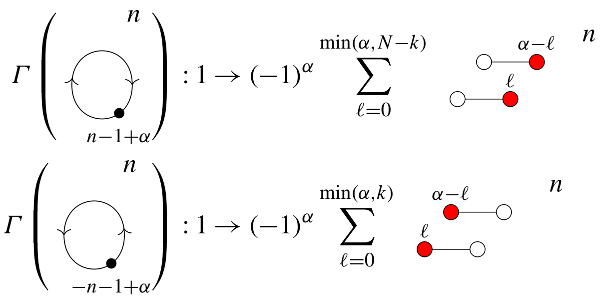
The terms in the second factor are always zero, while only the $\ell_2 = 0$ terms contribute to the first factor. Hence we have



(7.40)

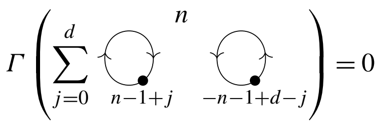
which agrees with (7.36) above. □

For the remaining identities it is helpful to compute



(7.41)

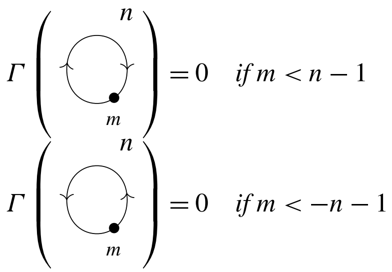
which follow from Definition 7.1. The same formulas define the image of fake bubbles under Γ. One can verify that the defining equation



(7.42)

is satisfied using the definitions above.

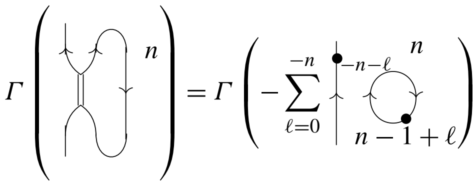
Lemma 7.8 *(Positive degree of closed bubbles). For all $m \geqslant 0$ we have*



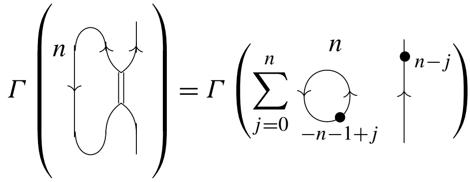
for all $n \in \mathbb{Z}$. Here 0 on the right hand side denotes the trivial bimodule map 0.

Proof. This is clear from the definitions above and the positive degree of all dumbbells. □

Lemma 7.9 *(Reduction to bubbles). The equations*



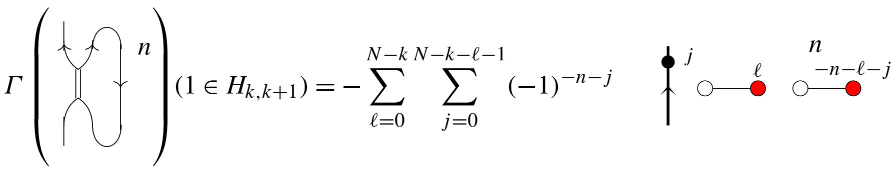
(7.43)



(7.44)

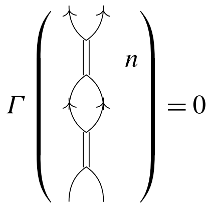
hold in $\mathbf{Flag}_N^*$ *for all* $n \in \mathbb{Z}$.

Proof. Consider the first identity on $1 \in H_{k,k+1}$. We compare the images of both bimodule maps on the element $1 \in H_{k,k+1}$,

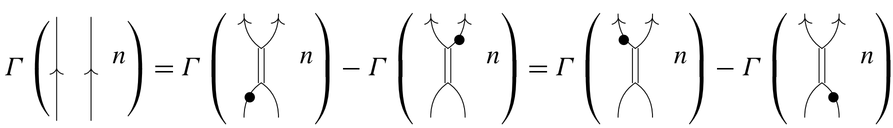


But the dumbbell with weight $-n-\ell-j$ is only nonzero when $j \leqslant -n-\ell = N-2k-\ell$. For $k>0$ this implies $-n-\ell \leqslant N-k-\ell-1$. After changing the j-summation to reflect this fact, so that $0 \leqslant j \leqslant -n-\ell$, the above is equal to the image of $1 \in H_{k,k+1}$ under the bimodule map on the right hand side of (7.43). Eq. (7.44) is proven similarly. □

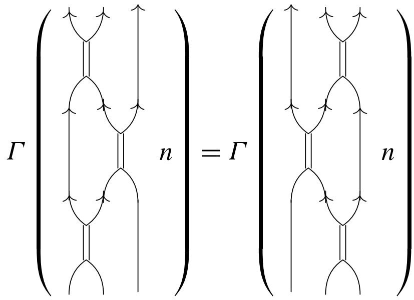
Lemma 7.10 *(NilHecke action). The equations*



(7.45)



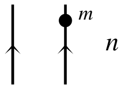
(7.46)



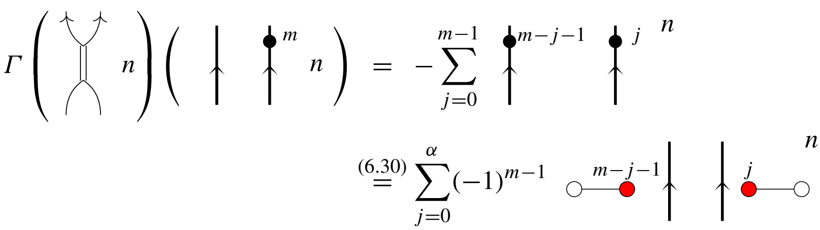
(7.47)

hold in $\mathbf{Flag}_N^*$ *for all* $n \in \mathbb{Z}$.

Proof. We compute the bimodule map (7.45) on the element

 $\in H_{k,k+1,k+2}$ (7.48)

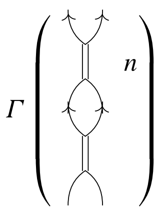
since this determines the module map on all other vectors. The first equation is proven as follows:



(7.49)

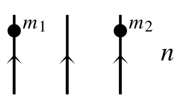
(7.50)

Hence, the composite

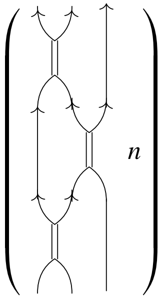


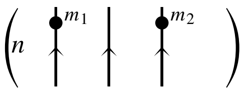
is zero by Remark 7.4. The second equation is a simple computation using the definition.

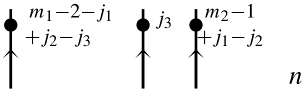
For the last relation we consider the element

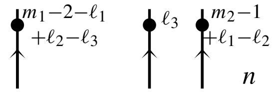
 $\in H_{k,k+1,k+2,k+3}$ (7.51)

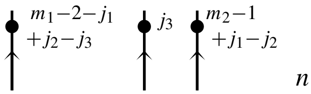
again, because all other vectors are determined from identities together with the bimodule property. By direct computation we have

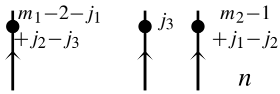
Γ 



$$= \sum_{j_1=1}^{m_1-2} \sum_{j_2=0}^{j_1-1} \sum_{j_3=0}^{m_1-2-j_1}$$


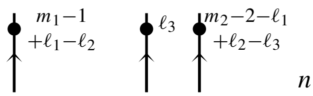
$$- \sum_{\ell_1=2}^{m_1-1} \sum_{\ell_2=1}^{\ell_1-1} \sum_{\ell_3=0}^{\ell_2-1}$$


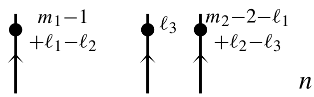
$$- \sum_{j_1=0}^{m_1-2} \sum_{j_2=0}^{m_2-1} \sum_{j_3=0}^{m_1-2-j_1}$$


$$+ \sum_{j_1=0}^{m_1-1} \sum_{j_2=1}^{m_2-1} \sum_{j_3=0}^{j_2-1}$$


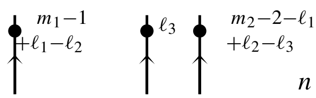
where we have written each summation to include only the nonzero terms. The first two terms cancel which is clear after the substitution $\ell_1 \mapsto m_1 - 1 - j_2$, $\ell_2 \mapsto m_1 - 1 - j_1$, and $\ell_3 \mapsto j_3$.

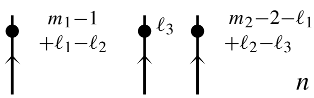
On the third term make the substitution $j_1 \mapsto m_1 - 1 - \ell_2$, $j_2 \mapsto m_1 + \ell_1 - 2\ell_2 + \ell_3$, and $j_3 \mapsto \ell_3$. On the fourth term make the substitution $j_1 \mapsto m_2 - 2 - 2\ell_1 + \ell_2 - \ell_3$, $j_2 \mapsto m_2 - 1 - \ell_1$, and $j_3 \mapsto \ell_3$. This leaves the terms

$$- \sum_{\ell_1=0}^{m_2-1} \sum_{\ell_2=1}^{m_1-1} \sum_{\ell_3=0}^{\ell_2-1}$$


$$+ \sum_{\ell_1=0}^{m_2-2} \sum_{\ell_2=0}^{m_1-1} \sum_{\ell_3=0}^{m_2-2-\ell_1} \qquad (7.52)$$


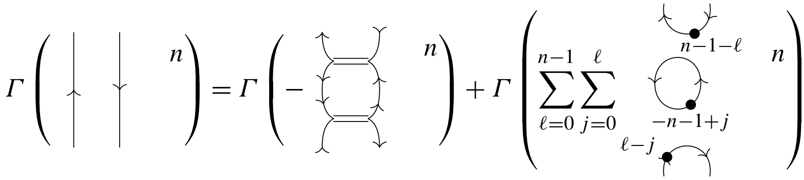
Add to this the expression

$$\sum_{\ell_1=2}^{m_2-1} \sum_{\ell_2=1}^{\ell_1-1} \sum_{\ell_3=0}^{\ell_2-1}$$


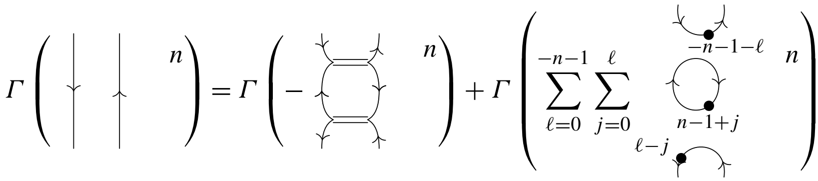
$$- \sum_{\ell_1=1}^{m_2-2} \sum_{\ell_2=0}^{\ell_1-1} \sum_{\ell_3=0}^{m_2-2-\ell_1} \qquad (7.53)$$


which can be shown to be equal to zero by shifting indices as above. This is precisely the nonzero terms of the bimodule map on the right hand side of (7.47) applied to the vector (7.51). □

Lemma 7.11 *(Identity decomposition). The equations*



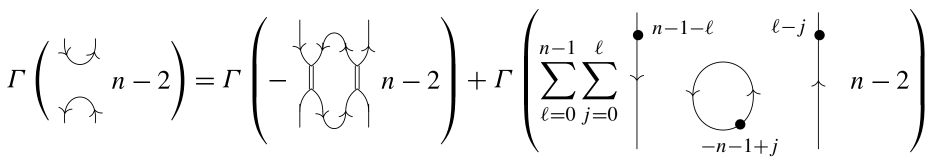
(7.54)



(7.55)

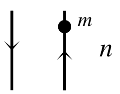
hold for all $n \in \mathbb{Z}$.

Proof. Using Lemma 7.5 proving the first identity is equivalent to proving



(7.56)

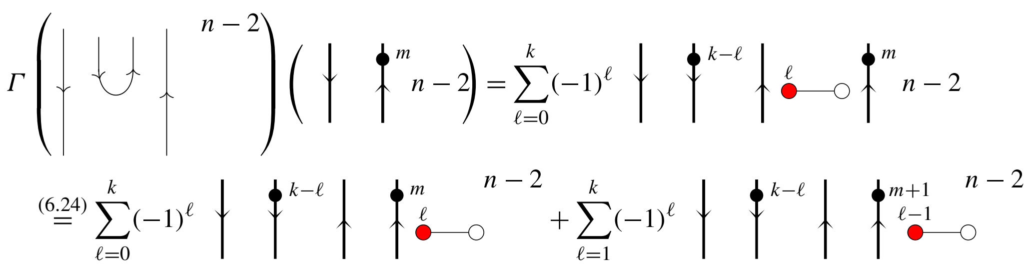
We compute the above maps on the elements



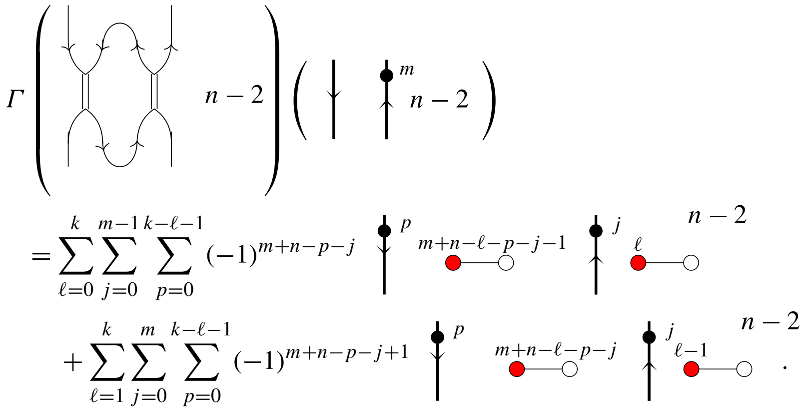
$$\in H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k} \tag{7.57}$$

for $m \geqslant 0$, since these elements, together with relations (6.22)–(6.19) and the bimodule property, determine the image on all other elements.

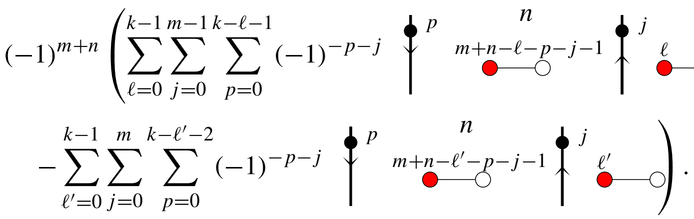
We begin by computing the image of the element (7.57) under the maps on the right hand side of (7.56).



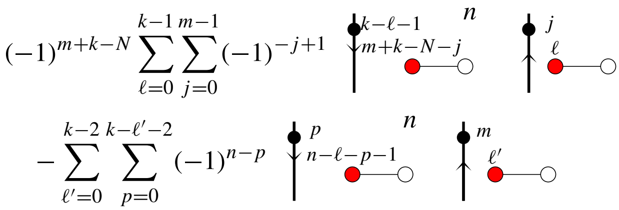
Hence,



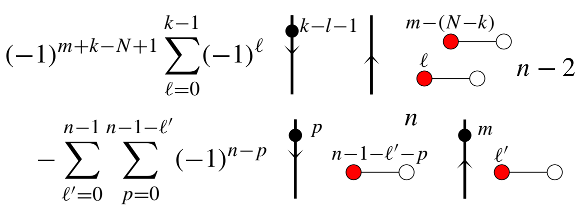
Now shift the second term $\ell' = \ell - 1$, and note the $\ell = k$ term of the first factor is zero by (6.12). This leaves



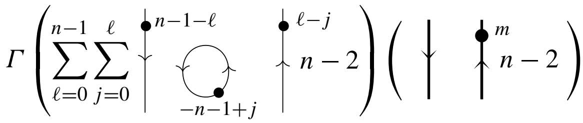
The only terms that do not cancel are the $p = k - \ell - 1$ term of the first factor and the $j = m$ term of the second factor

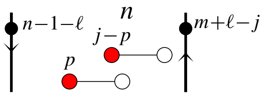


In the first term note that $m - 1 \geqslant m - (N - k)$ since $N - k \geqslant 1$, otherwise $k = N$. So we change the upper limit of the j summand to $m - (N - k)$ and apply (6.25) holding ℓ fixed. In the second summand we note that we must have $n - \ell' - p - 1 \geqslant 0$, and that $k - \ell - 2 \geqslant n - \ell - 1$ if $N - k \geqslant 1$. Hence, the ℓ' summation goes only as high as $n - 1$, and the p summation only as high as $n - 1 - \ell$ yielding

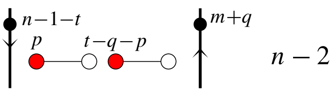


Now we compute the other maps involved in (7.56). The second map on the right hand side is

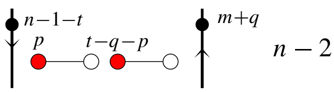


$$=\sum_{\ell=0}^{n-1}\sum_{j=0}^{\ell}\sum_{p=0}^{\min(j,k)}(-1)^j$$ 

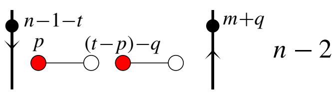
Now change the j summation by substituting $q=\ell-j$, and for later convenience set $t=\ell$

$$=\sum_{t=0}^{n-1}\sum_{q=0}^{t}\sum_{p=0}^{\min(t-q,k)}(-1)^{t-q}$$  (7.58)

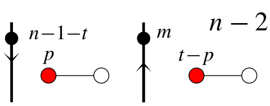
Note that $t-q\leqslant k$ since $t-q\leqslant n-1=2k-N-1$ and $k<N+1$. Hence, we can eliminate the min in the p summation.

$$=\sum_{t=0}^{n-1}(-1)^t\sum_{q=0}^{t}\sum_{p=0}^{t-q}(-1)^{-q}$$  (7.59)

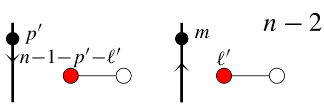
Switching the summation order on the p and q summation we have

$$=\sum_{t=0}^{n-1}(-1)^t\sum_{p=0}^{t}\sum_{q=0}^{t-p}(-1)^{-q}$$  (7.60)

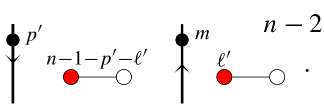
Holding t and p constant, (6.24) applied to the q summation yields

$$=\sum_{t=0}^{n-1}\sum_{p=0}^{t}(-1)^t$$  (7.61)

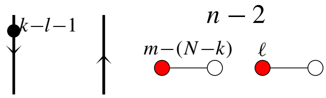
Finally, to make this expression match up with earlier expressions, make the substitutions $p'=n-1-t$, and $\ell'=t-p$

$$=\sum_{p'=0}^{n-1}\sum_{\ell'=0}^{n-1-p'}(-1)^{n-1-p'}$$  (7.62)

and switch the order of the summations so that we have

$$=\sum_{\ell'=0}^{n-1}\sum_{p'=0}^{n-1-\ell'}(-1)^{n-1-p'}$$  (7.63)

Hence, the image of the element (7.57) under the map on the right hand side of (7.56) is given by

$$(-1)^{m+k-N}\sum_{\ell=0}^{k-1}(-1)^{\ell}$$


which is precisely the image of (7.57) under the left hand side of (7.56). □

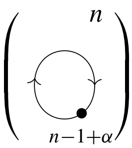
Theorem 7.12. *The assignments given in Subsection* 7.1 *define a graded additive* 2*-functor* $\Gamma_N : \mathcal{U}^* \to \mathbf{Flag}_N^*$. *By restricting to degree preserving* 2*-morphisms we also get an additive* 2*-functor* $\Gamma_N : \mathcal{U} \to \mathbf{Flag}_N$.

Proof. We have already seen that Γ_N preserves composites of 1-morphisms up to isomorphism. The lemmas above show that Γ_N preserve the defining relations of the 2-morphisms in $\mathcal{U}^*$. Therefore, Γ_N is a 2-functor. We have also shown that the assignments of Subsection 7.1 preserve the degree of the 2-morphisms in $\mathcal{U}^*$. Thus, it is clear that restricting to the degree preserving maps gives the restricted 2-functor $\Gamma_N : \mathcal{U} \to \mathbf{Flag}_N$. □

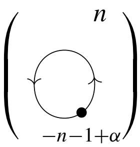
8. Size of 2-category $\mathcal{U}^*$

The rings H_k are defined over $\mathbb{Q}$. However, the following result holds for any field $\Bbbk$ with $\operatorname{char}\Bbbk \neq 2$.

Proposition 8.1. *The images of dotted bubbles under the representations* $\Gamma_N : \mathcal{U}^* \to \mathbf{Flag}_N^*$*:*

Γ


$$: 1 \to (-1)^{\alpha}\sum_{\ell=0}^{\min(\alpha,N-k)} y_{\ell,n}\,y_{\alpha-\ell,n}$$

Γ


$$: 1 \to (-1)^{\alpha}\sum_{\ell=0}^{\min(\alpha,k)} x_{\ell,n}\,x_{\alpha-\ell,n} \tag{8.1}$$

over all $\alpha \in \mathbb{Z}_+$ *generate the cohomology ring* H_k *for any choice of* N.

Proof. This is immediate when 2 is invertible. In that case, the counter-clockwise dotted bubble of degree α generates the Chern class $x_{\alpha,n}$ together with products of lower degree Chern classes. Likewise, the clockwise dotted bubble in degree α generates the Chern class $y_{\alpha,n}$ together with products of lower degree Chern classes. □

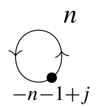
Recall that the cohomology rings H_k are spanned by the $x_{j,n}$ with $1 \leqslant j \leqslant k$ when $k \leqslant N-k$, and the $y_{\ell,n}$ with $1 \leqslant \ell \leqslant N-k$ when $N-k \geqslant k$. Without loss of generality assume $k \geqslant N-k$

and write all the $y_{\ell,n}$ in terms of $x_{j,n}$ using the Grassmannian relation. The remaining k relations on H_k are given by the first column of

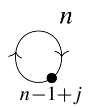
$$\begin{pmatrix} x_{1,n} & 1 & 0 & 0 & 0 \\ x_{2,n} & 0 & 1 & \ddots & 0 \\ x_{3,n} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & & 1 \\ x_{k,n} & 0 & & & 0 \end{pmatrix}^{N-k+1} \tag{8.2}$$

(see for example [23]). This implies that the relations on the generators $x_{j,n}$ are in degree $N-k+1,\ldots,N$. Since the representations Γ_N are defined for all positive integers N, it is always possible to choose N large enough so that there are no relations on the images of a given collection of non-nested dotted bubbles of the same orientation.

Proposition 8.2. *The assignment*



$$\mapsto v_{j,n} \quad \textit{for } n \geqslant 0 \tag{8.3}$$



$$\mapsto v_{j,n} \quad \textit{for } n \leqslant 0 \tag{8.4}$$

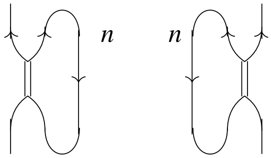
induces a graded ring isomorphism $\mathcal{U}^*(\mathbf{1}_n,\mathbf{1}_n) \to \mathbb{Z}[v_{1,n}, v_{2,n}, v_{3,n},\cdots]$ with $v_{i,n}$ in degree $2i$. Note that when $n=0$ we have two different isomorphisms related by the infinite Grassmannian relations in Proposition 5.5*. We sometimes suppress the dependence on the value of n and write v_i. This implies that every closed diagram can be reduced to a unique linear combination of diagram of non-nested dotted bubbles with the same orientation.*

Proof. We show that any closed diagram can be reduced to a unique sum of non-nested dotted bubbles of the same orientation; this together with the above observation that there are no relations among a collection of non-nested dotted bubbles of the same orientation completes the proof.

Using the bubble slide equations in Propositions 5.6 and 5.7 any dotted bubble can be pushed to the outside of a closed diagram. Using the Grassmannian relations of Proposition 5.5 all dotted bubbles can be made to have the same orientation. Either this process completes the proof, or we are left with a closed diagram, possibly consisting of several components, that contains no dotted bubbles.

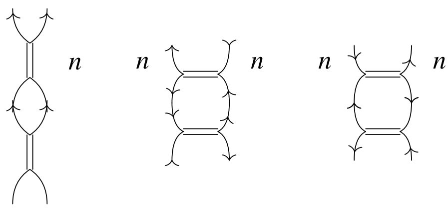
Consider the innermost diagram. We induct on the number of nilCoxeter generators U_n in this diagram to show that it can be reduced to sums of dotted bubbles. By contracting each double edge to a point and disregarding orientation and dots we are left with a connected planar 4-valent graph $\mathcal{G}$ with at least one vertex. First we show that if this graph contains a loop or a digon face then the number of nilCoxeter generators can be reduced. Then we show that if the graph contains no loops or digons, then using only relations which do not increase the number of nilCoxeter generators, it can be transformed into a diagram containing a digon face.

Loops in $\mathcal{G}$ arise from diagrams of the form



(8.5)

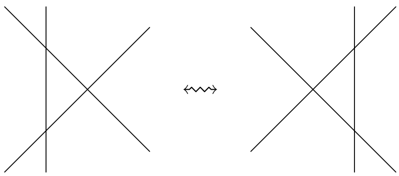
with some number of dots labelling each strand. Each diagram reduces using the reduction to bubbles axioms (5.23) and (5.24). Digon faces in $\mathcal{G}$ arise from diagrams of the form



(8.6)

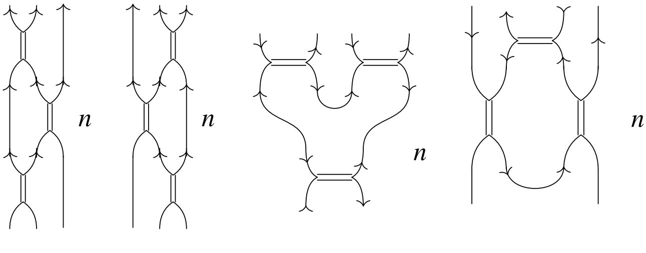
with some number of dots on each strand. Using the nilHecke relations these dots can be moved to the top of the diagram without increasing the number of nilCoxeter generators. The first diagram reduces using the nilCoxeter relation (5.11) and the second two can be reduced using the identity decomposition equations (5.18) and (5.19).

Now we appeal to a theorem from graph theory, see for example Carpentier [11, Lemma 2]. If a connected planar 4-valent graph has at least one vertex and does not contain a loop or digon face, then it is possible by a sequence of triangle moves



(8.7)

to transform the diagram into one containing a digon face. Hence, the proposition is complete if such moves can be performed on diagrams which produce graphs of the above form. The only possibilities arise from the diagrams

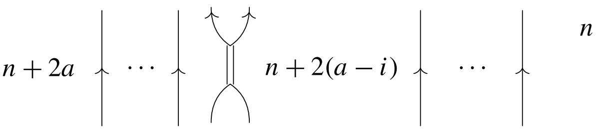


with some number of dots on each strand. Again, using the nilHecke relations these dots can be moved to the top of such diagrams producing sums of diagrams in which the number of nilCoxeter generators does not increase. The triangle move above then follows from Eq. (5.13) and Proposition 5.8. □

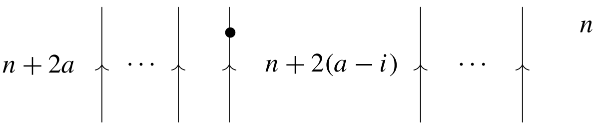
Theorem 8.3. *There is an isomorphism of graded rings*

$$\beta : \mathcal{NH}_a \otimes \mathbb{Z}[v_{1,n}, v_{2,n}, \ldots] \to \mathcal{U}^*\big(\mathcal{E}^a \mathbf{1}_n, \mathcal{E}^a \mathbf{1}_n\big)$$

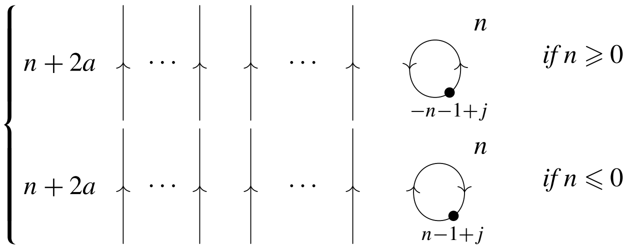
$u_i \otimes 1 \mapsto$



$\chi_i \otimes 1 \mapsto$



$1 \otimes v_{j,n} \mapsto$



with $v_{i,n}$ in degree $2i$.

Proof. The isomorphism constructed in Proposition 8.2 identifies the $v_{i,n}$ with dotted bubbles of a given orientation in the region labelled n. This, together with nilHecke action (5.11)–(5.13) built into the definition of the 2-category $\mathcal{U}^*$, shows that β is a homomorphism. Using graph theoretic arguments we now show that the image of the homomorphism β forms a spanning set for $\mathcal{U}^*(\mathcal{E}^a \mathbf{1}_n, \mathcal{E}^a \mathbf{1}_n)$ establishing surjectivity.

Given an element in $\mathcal{U}^*(\mathcal{E}^a \mathbf{1}_n, \mathcal{E}^a \mathbf{1}_n)$ represented by a linear combination of diagrams, we show that each diagram $\mathcal{D}$ in the sum can be reduced to a diagram in the image of β. Let $\mathcal{G}$ denote the 4-valent graph obtained from $\mathcal{D}$ by contracting each double edge to a point and disregarding the dots. Arguing as in Proposition 8.2, the diagram $\mathcal{D}$ can be written as a finite sum of diagrams $\mathcal{D}_i$ whose graphs $\mathcal{G}_i$ contain no loops or digons faces and all nested closed subdiagrams of $\mathcal{D}_i$ have been reduced to dotted closed bubbles with the same orientation and moved to the far right of the diagram to the region labelled by n. In this way, no closed diagram lies interior to the remaining 4-valent graph whose boundary consists of $2a$ points.

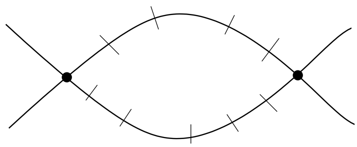
We argue that the diagrams $\mathcal{D}_i$ can be written as sums of diagrams $\mathcal{D}_{ij}$ whose graphs $\mathcal{G}_{ij}$ are such that any walk:

emanating from a boundary point crosses any other strand at most once and never crosses itself. By a walk we mean a path in the 4-valent graph extending from a point through each vertex to its opposite edge, until a closed loop is obtained, or a boundary point is reached.

Carpentier showed that any connected *closed* 4-valent graph with at least one vertex and no loops or digons can be transformed into a diagram containing a digon face using triangle moves [11]. The main argument used by Carpentier is the existence of a configuration of one of the two forms:

(8.8)

in the closed graph. It is argued that if this configuration is minimal (contains no configurations of the same form inside it), then the second case can be reduced to the first, and using triangle moves (8.7) this first configuration



can be transformed to a digon face by sliding all the internal edges outside the configuration. The requirement that the graph be closed is only used to prove the existence of on one of the configurations in (8.8).

If any strand in the graph $\mathcal{G}_i$ intersects itself, or a different strand twice, there must be one of the two possible configurations in (8.8). We then choose a minimal such configuration in $\mathcal{G}_i$ and by Carpentier's argument the diagram $\mathcal{D}_i$ reduces to a sum of diagrams $\mathcal{D}_{ij}$ whose graphs have fewer vertices. Applying this procedure inductively we can reduce each diagram $\mathcal{D}_i$ to a sum of diagrams whose graphs have no walks emanating from an open end that cross themselves or crosses any other strand more than once. Using Theorem 4.5 it is clear that edges in the resulting summands can be made strictly increasing using plane isotopies so that they correspond to elements in the image of β. The sum of diagrams $\mathcal{D}_{ij}$ is unique since for N large enough the 2-functor Γ_N is a faithful representation on homogeneous elements of a fixed degree.

To see that β is injective suppose that for some collections of polynomials $f_w(x) \in \mathbb{Z}[x_1, x_2, \ldots, x_a]$ and $g_w(v) \in \mathbb{Z}[v_1, v_2, \ldots]$ with $w \in S_a$ we have

$$\beta\biggl(\sum_{w\in S_a} f_w(x)u_w \otimes g_w(v)\biggr) = 0. \tag{8.9}$$

Choose N much larger than the degree of the 2-morphism (8.9) so that applying the 2-functor Γ_N implies

$$\Gamma_N\biggl(\beta\biggl(\sum_{w\in S_n} f_w(x)u_w \otimes g_w(v)\biggr)\biggr) = \sum_{w\in S_n} f_w(\xi)\partial_w \otimes G_w = 0 \tag{8.10}$$

for $f_w(xi)\partial_w : H_{k,\dots,k+a} \to H_{k,\dots,k+a}$ and some nonzero bimodule maps $G_w : H_k \to H_k$. Here we have used that Γ_N is a 2-functor mapping horizontal composites in $\mathcal{U}^*$ to tensor products of bimodule maps in $\mathbf{Flag}_N^*$. The G_w are nonzero since they are built from the images of non-nested dotted bubbles of the same orientation, and we have argued that each dotted bubble is mapped to a nonzero bimodule map. Since N was chosen large, the polynomial $f_w(\xi)$ in the variables ξ_i (corresponding to Chern classes of line bundles) is identical to the polynomial $f_w(x)$ with x_i replaced by ξ_i.

We show that the bimodule map $\sum_{w\in S_n} f_w\partial_w : H_{k,\dots,k+a} \to H_{k,\dots,k+a}$ must be the zero map. In particular, $f_w(\xi)$ are all zero; since N was chosen large, we deduce that the $f_w(x)$ are also all equal to zero.

Equation (8.10) implies that the image of any element $p_1 \otimes p_2 \in H_{k,\dots,k+a} \otimes H_k$ under the bimodule map $\sum_{w\in S_n} f_w\partial_w$ is

$$\sum_{w\in S_n} f_w\partial_w(p_1) \otimes G_w(p_2) = 0.$$

Take $v_0 \in S_a$ of minimal degree in the above sum and let $p_2 = 1$ and $p_1 = \mathfrak{S}_{v_0}(\xi)$. Here again, we use that N was chosen large enough so that the Schubert polynomial $\mathfrak{S}_{v_0}(\xi)$ can be formed. Then

$$\sum_{w\in S_n} f_w\partial_w(\mathfrak{S}_{v_0}) \otimes G_w(1) = 0, \tag{8.11}$$

but by (3.17)

$$\partial_w\mathfrak{S}_{v_0} = \begin{cases} \mathfrak{S}_{v_0 w^{-1}} & \text{if } \ell(v_0 w^{-1}) = \ell(v_0) - \ell(w), \\ 0 & \text{otherwise.} \end{cases} \tag{8.12}$$

Hence, the only contribution to (8.11) is from $w = v_0$. Since $G_{v_0}(1) \neq 0$ Eq. (8.11) implies $f_{v_0} = 0$.

Applying this argument inductively we have that all $f_w = 0$ proving injectivity. □

Remark 8.4. We can also obtain an isomorphism $\beta' : \mathbb{Z}[v_{1,n+2a}, v_{2,n+2a}, \dots] \otimes \mathcal{NH}_a \to \mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ by moving all the closed bubbles to the far left of a diagram. But the composite

$$\beta^{-1}\beta' : \mathbb{Z}[v_{1,n+2a}, v_{2,n+2a}, \dots] \otimes \mathcal{NH}_a \to \mathcal{NH}_a \otimes \mathbb{Z}[v_{1,n}, v_{2,n}, \dots]$$

is a nontrivial isomorphism due to the complicated bubble slide formulas.

Theorem 8.5. *There is an isomorphism of graded rings*

$$\beta' : \mathcal{NH}_a \otimes \mathbb{Z}[v_{1,n}, v_{2,n}, \dots] \to \mathcal{U}^*\big(\mathcal{F}^a\mathbf{1}_{n+2a}, \mathcal{F}^a\mathbf{1}_{n+2a}\big). \tag{8.13}$$

Proof. The isomorphism β' is given by the composite

$$\mathcal{NH}_a \otimes \mathbb{Z}[v_{1,n}, v_{2,n}, \dots] \xrightarrow{\beta} \mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n) \xrightarrow{\tilde{\psi}\tilde{\sigma}\tilde{\omega}} \mathcal{U}^*(\mathcal{F}^a\mathbf{1}_{n+2a}, \mathcal{F}^a\mathbf{1}_{n+2a}),$$

where the composite 2-functor $\widetilde{\psi}\widetilde{\sigma}\widetilde{\omega}$ has the interpretation as rotating the diagrams defining β by 180 degrees. $\square$

9. Categorification of Lusztig's $\dot{\mathrm{U}}$

We would like to be able to split idempotents in the 2-category $\mathcal{U}$. To do this in an abstract setting we require the notion of the Karoubi envelope of a category.

9.1. Karoubi envelope

An idempotent $e: b \to b$ in a category $\mathcal{C}$ is a morphism such that $e \circ e = e$. The idempotent is said to split if there exist morphisms

$$b \xrightarrow{g} b' \xrightarrow{h} b$$

such that $e = h \circ g$ and $g \circ h = 1_{b'}$.

The Karoubi envelope[7] $Kar(\mathcal{C})$ is a category whose objects are pairs (b, e) where $e: b \to b$ is an idempotent of $\mathcal{C}$ and whose morphisms are triples of the form

$$\left(e, f, e'\right): (b, e) \to \left(b', e'\right)$$

where $f: b \to b'$ in $\mathcal{C}$ making the diagram

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, "f"] & b' \arrow[d, "e'"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \tag{9.1}$$

commute [43]. Composition is induced from the composition in $\mathcal{C}$, but the identity morphism is $(e, e, e): (b, e) \to (b, e)$. When $\mathcal{C}$ is an additive category, the splitting of idempotents allows us to write $(b, e) \in Kar(\mathcal{C})$ as $\operatorname{im} e$ and we have $b \cong \operatorname{im} e \oplus \operatorname{im}(1 - e)$, see for example [2].

The identity map $1_b: b \to b$ is an idempotent and this allows us to define a fully faithful functor $\mathcal{C} \to Kar(\mathcal{C})$. In $Kar(\mathcal{C})$ all idempotents of $\mathcal{C}$ are split and this functor is universal with respect to functors which split idempotents in $\mathcal{C}$. This means that if $F: \mathcal{C} \to \mathcal{D}$ is any functor where all idempotents split in $\mathcal{D}$, then F extends uniquely (up to isomorphism) to a functor $\widetilde{F}: Kar(\mathcal{C}) \to \mathcal{D}$ (see for example [8, Proposition 6.5.9]). Furthermore, for any other functor $G: \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha: F \Rightarrow G$, α extends uniquely to a natural transformation $\widetilde{\alpha}: \widetilde{F} \Rightarrow \widetilde{G}$.

[7] Also known as the idempotent completion [1], or Cauchy-completion [8, Chapter 6.5]. The reader who finds the terminology Cauchy completion confusing see [9, 6.8.9].

9.2. The 2-category $\dot{\mathcal{U}}$

Let $\Bbbk$ be a field. For the remainder of this paper we work with $\Bbbk$ linear combinations of 2-morphisms in the 2-categories $\mathcal{U}$ and $\mathcal{U}^*$, rather than $\mathbb{Z}$-linear combinations. That is, for 1-morphisms x, y we have

$$\mathcal{U}^*(x, y) := \mathcal{U}^*(x, y) \otimes \Bbbk, \qquad \mathcal{U}(x, y) := \mathcal{U}(x, y) \otimes \Bbbk \tag{9.2}$$

so that $\mathcal{U}^*$ is a graded additive $\Bbbk$-linear 2-category and $\mathcal{U}$ is an additive $\Bbbk$-linear 2-category. In particular, $\mathcal{U}^*(x, y)$ is a graded $\Bbbk$-vector space and $\mathcal{U}(x, y)$ is a $\Bbbk$-vector space.

Definition 9.1. Define the 2-category $\dot{\mathcal{U}}$ to have the same objects as $\mathcal{U}$ and $\dot{\mathcal{U}}(n,m) = Kar(\mathcal{U}(n,m))$. The fully-faithful functors $\mathcal{U}(n,m) \to \dot{\mathcal{U}}(n,m)$ combine to form a 2-functor $\mathcal{U} \to \dot{\mathcal{U}}$ universal with respect to splitting idempotents in the hom categories $\dot{\mathcal{U}}(n,m)$. The composition functor $\dot{\mathcal{U}}(n,m) \times \dot{\mathcal{U}}(m,p) \to \dot{\mathcal{U}}(n,p)$ is induced by the universal property of the Karoubi envelope from the composition functor for $\mathcal{U}$.

The assignment of a graded abelian group to the homs between 1-morphisms in $\mathcal{U}$ induced by the inclusion $\mathcal{U} \to \mathcal{U}^*$ provides the 2-category $\dot{\mathcal{U}}$ with an enriched hom as well. For each pair of morphisms $x, y \in {}_m\dot{\mathcal{U}}_n$ there is a graded $\Bbbk$-vector space $\dot{\mathcal{U}}^*(x,y) := \bigoplus_{s\in\mathbb{Z}} \dot{\mathcal{U}}(x\{s\}, y)$. Proposition 9.17 shows that the graded $\Bbbk$-vector space $\dot{\mathcal{U}}^*(x,y)$ is a free graded module over the graded rings $\dot{\mathcal{U}}^*(\mathbf{1}_n, \mathbf{1}_n)$ and $\dot{\mathcal{U}}^*(\mathbf{1}_m, \mathbf{1}_m)$. The graded rank of the graded $\dot{\mathcal{U}}^*(\mathbf{1}_n, \mathbf{1}_n)$-module $\dot{\mathcal{U}}^*(x,y)$ is defined to be

$$\mathrm{rk}_q \dot{\mathcal{U}}^*(x, y) := \sum_{s\in\mathbb{Z}} q^s \,\mathrm{rk}\dot{\mathcal{U}}\bigl(x\{s\}, y\bigr) = \sum_{r\in\mathbb{Z}} q^{-r} \,\mathrm{rk}\dot{\mathcal{U}}\bigl(x, y\{r\}\bigr) \tag{9.3}$$

where $\mathrm{rk}\dot{\mathcal{U}}(x\{s\}, y)$ is the rank of the (ungraded) $\dot{\mathcal{U}}^*(\mathbf{1}_n, \mathbf{1}_n)$-module $\dot{\mathcal{U}}(x\{s\}, y)$. This definition is consistent because the degree zero maps $x \to y\{s\}$ are in one-to-one correspondence with degree zero maps $x\{-s\} \to y$.

Proposition 9.2. *The* 2*-functors*

$$\widetilde{\omega}:\mathcal{U} \to \mathcal{U}, \quad \widetilde{\sigma}:\mathcal{U} \to \mathcal{U}^{\mathrm{op}}, \quad \widetilde{\psi}:\mathcal{U} \to \mathcal{U}^{\mathrm{co}}, \quad \widetilde{\tau}:\mathcal{U} \to \mathcal{U}^{\mathrm{co\,op}}, \quad \widetilde{\tau}^{-1}:\mathcal{U} \to \mathcal{U}^{\mathrm{co\,op}} \tag{9.4}$$

extend to define 2*-functors on* $\dot{\mathcal{U}}$.

Proof. This follows immediately from the fact that 2-functors preserve composition of 1-morphisms, hence map idempotents to idempotents. □

Whenever the endomorphism ring $\dot{\mathcal{U}}(x,x)$ is a finite-dimensional $\Bbbk$-algebra, the Krull–Schmidt decomposition theorem implies that x can be written as a direct sum of indecomposables where the indecomposables and their multiplicities are unique up to isomorphism and reordering of the factors. Theorems 8.3 and 8.5 show that $\mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ and $\mathcal{U}^*(\mathcal{F}^a\mathbf{1}_n, \mathcal{F}^a\mathbf{1}_n)$ are finite-dimensional $\Bbbk$-algebras in each grade. This implies that both $\dot{\mathcal{U}}(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ and $\dot{\mathcal{U}}(\mathcal{F}^a\mathbf{1}_n, \mathcal{F}^a\mathbf{1}_n)$ are finite-dimensional $\Bbbk$-algebras. These results, together with dualities $\widetilde{\tau}$ and $\widetilde{\tau}^{-1}$ and the results of Section 5.7, imply the $\dot{\mathcal{U}}(x, y)$ is a finite-dimensional $\Bbbk$-algebra for any pair of 1-morphisms x, y of $\dot{\mathcal{U}}$.

Using the fully faithful embedding of $\mathcal{U}(n,m)$ into $\dot{\mathcal{U}}(n,m)$ we identify $x \in \mathcal{U}(n,m)$ with $(x,1) \in \dot{\mathcal{U}}(n,m)$ where 1 is the trivial idempotent. For any morphism x in $\dot{\mathcal{U}}$ and quantum integer $[a]$, write $\bigoplus_{[a]} x$ or $(x)^{\bigoplus [a]}$ for the direct sum of morphisms:

$$\bigoplus_{[a]} x = (x)^{\bigoplus [a]} := x\{a-1\} \oplus x\{a-3\} \oplus \cdots \oplus x\{1-a\}. \tag{9.5}$$

Furthermore, we extend this notation to quantum factorials $[a]!$ so that

$$\bigoplus_{[a]!} x = (x)^{\bigoplus [a]!} := \bigoplus_{j=0}^{a} \big(x\{a-1-2j\}\big)^{\binom{a}{j}} \tag{9.6}$$

where $\binom{a}{j}$ is the standard binomial coefficient denoting the multiplicity of the 1-morphism $x\{a-1-2j\}$ in the direct sum. More generally, write $x^{\oplus f}$, for a Laurent polynomial $f = \sum f_a q^a \in \mathbb{Z}[q, q^{-1}]$, for the direct sum over $a \in \mathbb{Z}$, of f_a copies of $x\{a\}$.

Let e_{w_0} be the idempotent in $\mathcal{U}(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ corresponding to the element $x^\delta \partial_{w_0} \in \mathcal{NH}_a$ under the isomorphism of Theorem 8.3. This element is idempotent because $x^\delta \partial_{w_0}(x^\delta)\partial_{w_0} = x^\delta \mathfrak{S}_1 \partial_{w_0} = x^\delta \partial_{w_0}$. We also denote the image of this element in $\mathcal{U}^*(\mathcal{F}^a\mathbf{1}_n, \mathcal{F}^a\mathbf{1}_n)$ under the isomorphism Theorem 8.5 as e'_{w_0}.

Definition 9.3. The categorifications of the divided powers are given by:

$$\mathcal{E}^{(a)}\mathbf{1}_n := \big(\mathcal{E}^a\mathbf{1}_n, e_{w_0}\big)\left\{\frac{a(1-a)}{2}\right\}, \tag{9.7}$$

$$\mathcal{F}^{(a)}\mathbf{1}_n := \big(\mathcal{F}^a\mathbf{1}_n, e'_{w_0}\big)\left\{\frac{a(1-a)}{2}\right\} \tag{9.8}$$

for $n \in \mathbb{Z}$ and e_{w_0}, e'_{w_0} the idempotents defined above.

Recall that in the basis of Schubert polynomials the polynomial algebra $\mathcal{P}_a$ is a free graded module over the symmetric polynomials Λ_a:

$$\mathcal{P}_a \cong \oplus_{(a)^!_{q^2}} \Lambda_a.$$

Since $\partial_{w_0}\mathfrak{S}_w = \delta_{w,w_0}$, the idempotent e_{w_0} acting on $\mathcal{P}_a$ projects onto a one-dimensional summand Λ_a corresponding to the basis element $\mathfrak{S}_{w_0}$. Thus, e_{w_0} is a minimal (or primitive) idempotent.

Proposition 9.4. *There are decompositions of* 1*-morphisms*

$$\mathcal{E}^a\mathbf{1}_n \cong \bigoplus_{[a]!} \mathcal{E}^{(a)}\mathbf{1}_n, \tag{9.9}$$

$$\mathcal{F}^a\mathbf{1}_n \cong \bigoplus_{[a]!}\mathcal{F}^{(a)}\mathbf{1}_n \tag{9.10}$$

in $\dot{\mathcal{U}}$ for all $n\in\mathbb{Z}$ and $a\in\mathbb{Z}_+$.

Proof. The graded abelian group $\dot{\mathcal{U}}^*(\mathcal{E}^a\mathbf{1}_n,\mathcal{E}^a\mathbf{1}_n)=\bigoplus_{s\in\mathbb{Z}}\dot{\mathcal{U}}(\mathcal{E}^a\mathbf{1}_n\{s\},\mathcal{E}^a\mathbf{1}_n)$ consists of all 2-morphisms in $\mathcal{U}^*$ from $\mathcal{E}^a\mathbf{1}_n$ to itself that commute (in the sense of (9.1)) with the identity morphism viewed as an idempotent. But this is just the graded abelian group $\mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n,\mathcal{E}^a\mathbf{1}_n)$:

$$\dot{\mathcal{U}}^*\big(\mathcal{E}^a\mathbf{1}_n,\mathcal{E}^a\mathbf{1}_n\big)=\mathcal{U}^*\big(\mathcal{E}^a\mathbf{1}_n,\mathcal{E}^a\mathbf{1}_n\big)\cong\mathcal{NH}_a\otimes\mathbb{Z}[v_1,v_2,\ldots]\cong\mathrm{Mat}\big((a)^!_{q^2};\Lambda_a\big)\otimes\mathbb{Z}[v_1,v_2,\ldots]$$

where the last isomorphism is established in Proposition 3.5.

All homogenous idempotents in $\dot{\mathcal{U}}^*(\mathcal{E}^a\mathbf{1}_n,\mathcal{E}^a\mathbf{1}_n)$ are of the form $e\otimes 1$, for $e\in\mathrm{Mat}((a)^!_{q^2};\Lambda_a)$. The matrix algebra $\mathrm{Mat}((a)^!_{q^2};\Lambda_a)$ has $a!$ minimal idempotents, $E_{1,1},E_{2,2},\ldots,E_{a!,a!}$, corresponding to elementary diagonal matrices. The minimal idempotent $e_{w_0}\in\mathrm{Hom}_{\Lambda_a}(\mathcal{P}_a,\mathcal{P}_a)\cong\mathrm{Mat}((a)^!_{q^2};\Lambda_a)$ projects onto a column vector corresponding to the largest degree basis element $\mathfrak{S}_{w_0}$ of $\mathcal{P}_a$, showing that $e_{w_0}=E_{a!,a!}$ in the basis of $\mathrm{Mat}((a)^!_{q^2};\Lambda_a)$ given by Schubert polynomials ordered by length (see Proposition 3.5).

Hence,

$$\operatorname{im}E_{a!,a!}\cong\operatorname{im}e_{w_0}:=\big(\mathcal{E}^a\mathbf{1}_n,e_{w_0}\big)=\mathcal{E}^{(a)}\mathbf{1}_n\left\{\frac{a(a-1)}{2}\right\}\in\dot{\mathcal{U}}. \tag{9.11}$$

The images of all other idempotents $E_{j,j}$ are isomorphic up to a grading shift to the image of the top degree idempotent $E_{a!,a!}$, so that

$$\mathcal{E}^a\mathbf{1}_n\cong\bigoplus_{j=0}^{a!}\operatorname{im}E_{j,j}\cong\bigoplus_{(a)^!_{q^{-2}}}\operatorname{im}E_{a!,a!}=\bigoplus_{(a)^!_{q^{-2}}}\operatorname{im}\mathcal{E}^{(a)}\mathbf{1}_n\left\{\frac{a(a-1)}{2}\right\} \tag{9.12}$$

in $\dot{\mathcal{U}}$. But $q^{a(a-1)/2}(a)^!_{q^{-2}}=[a]!$ establishing (9.9). A similar proof establishes the result for $\mathcal{F}^a\mathbf{1}_n$. □

Proposition 9.5. *There are decompositions of* 1*-morphisms*

$$\mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_n\cong\bigoplus_{\left[{a+b\atop a}\right]}\big(\mathcal{E}^{(a+b)}\mathbf{1}_n\big), \tag{9.13}$$

$$\mathcal{F}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\cong\bigoplus_{\left[{a+b\atop a}\right]}\big(\mathcal{F}^{(a+b)}\mathbf{1}_n\big) \tag{9.14}$$

in $\dot{\mathcal{U}}$ for all $n\in\mathbb{Z}$ and $a,b\in\mathbb{Z}_+$.

Proof. By the Krull–Schmidt theorem, the 1-morphism $\mathcal{E}^{a+b}\mathbf{1}_n$ of $\dot{\mathcal{U}}$ has a unique decomposition into indecomposables. Equation (9.13) follows from Proposition 9.4 by decomposing the composite $\mathcal{E}^a\mathcal{E}^b\mathbf{1}_n = \mathcal{E}^{a+b}\mathbf{1}_n$ in two different ways. Equation (9.14) is established similarly. □

Proposition 9.6. *There are decompositions of* 1*-morphisms*

$$\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \cong \bigoplus_{j=0}^{\min(a,b)} \left(\left(\mathcal{E}^{(a-j)}\mathcal{F}^{(b-j)}\mathbf{1}_n\right)^{\oplus \left[\begin{smallmatrix} b-a-n \\ j \end{smallmatrix}\right]}\right) \quad \text{if } n < -2a+2, \tag{9.15}$$

$$\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n \cong \bigoplus_{j=0}^{\min(a,b)} \left(\left(\mathcal{F}^{(b-j)}\mathcal{E}^{(a-j)}\mathbf{1}_n\right)^{\oplus \left[\begin{smallmatrix} a-b+n \\ j \end{smallmatrix}\right]}\right) \quad \text{if } n > 2b-2 \tag{9.16}$$

in $\dot{\mathcal{U}}$ *for* $n \in \mathbb{Z}$ *and* $a, b \in \mathbb{Z}_+$.

Proof. Recall from Theorem 5.10 that there are decompositions of 1-morphisms:

$$\mathcal{E}\mathcal{F}\mathbf{1}_n \cong \mathcal{F}\mathcal{E}\mathbf{1}_n \oplus_{[n]} \mathbf{1}_n \quad \text{for } n \geqslant 0, \tag{9.17}$$

$$\mathcal{F}\mathcal{E}\mathbf{1}_n \cong \mathcal{E}\mathcal{F}\mathbf{1}_n \oplus_{[-n]} \mathbf{1}_n \quad \text{for } n \leqslant 0. \tag{9.18}$$

The proposition follows by iteratively applying the above formula to $\mathcal{F}^b\mathcal{E}^a\mathbf{1}_n$ and $\mathcal{E}^a\mathcal{F}^b\mathbf{1}_n$ and using the unique decomposition property. □

Lemma 9.7. *The graded* $\Bbbk$*-vector spaces* $\dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{E}^{(a)}\mathbf{1}_n)$ *and* $\dot{\mathcal{U}}^*(\mathcal{F}^{(a)}\mathbf{1}_n, \mathcal{F}^{(a)}\mathbf{1}_n)$ *are free* $\dot{\mathcal{U}}^*(\mathbf{1}_n, \mathbf{1}_n)$*-modules whose graded ranks are given by*

$$\operatorname{rk}_q \dot{\mathcal{U}}^*\left(\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{E}^{(a)}\mathbf{1}_n\right) = \operatorname{rk}_q \dot{\mathcal{U}}\left(\mathcal{F}^{(a)}\mathbf{1}_n, \mathcal{F}^{(a)}\mathbf{1}_n\right) = g(a) \tag{9.19}$$

for all $n \in \mathbb{Z}$.

Proof. From Proposition 9.4, $\mathcal{E}^a\mathbf{1}_n = \oplus_{[a]!}\mathcal{E}^{(a)}\mathbf{1}_n$ so that

$$\dot{\mathcal{U}}\left(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n\right) = \dot{\mathcal{U}}\left(\bigoplus_{[a]!}\mathcal{E}^{(a)}\mathbf{1}_n, \oplus_{[a]!}\mathcal{E}^{(a)}\mathbf{1}_n\right). \tag{9.20}$$

Hence,

$$\operatorname{rk}_q \dot{\mathcal{U}}^*\left(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n\right) = [a]![a]! \operatorname{rk}_q \dot{\mathcal{U}}^*\left(\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{E}^{(a)}\mathbf{1}_n\right). \tag{9.21}$$

Proposition 8.2 establishes an isomorphism $\mathcal{U}^*(\mathbf{1}_n, \mathbf{1}_n) \cong \mathbb{Z}[v_{1,n}, v_{2,n}, \ldots]$ and Theorem 8.3 gives an isomorphism $\mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n) \cong \mathcal{NH}_a \otimes \mathbb{Z}[v_{1,n}, v_{2,n}, \ldots]$. Since $\dot{\mathcal{U}}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n) = \mathcal{U}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ and $\dot{\mathcal{U}}^*(\mathbf{1}_n, \mathbf{1}_n) = \mathcal{U}^*(\mathbf{1}_n, \mathbf{1}_n)$, we have that $\dot{\mathcal{U}}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ is a free $\dot{\mathcal{U}}^*(\mathbf{1}_n, \mathbf{1}_n)$-module of graded rank $\operatorname{rk}_q \dot{\mathcal{U}}^*(\mathcal{E}^a\mathbf{1}_n, \mathcal{E}^a\mathbf{1}_n)$ equal to the graded rank of the nilHecke ring $\mathcal{NH}_a$. This was calculated in (3.8) and is equal to $(q^{-a(a-1)/2}[a]!)(\frac{1}{1-q^2})^a$. Hence,

$$\operatorname{rk}_q \dot{\mathcal{U}}^*\big(\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{E}^{(a)}\mathbf{1}_n\big) = \frac{(q^{-a(a-1)/2}[a]!)}{[a]![a]!}\left(\frac{1}{1-q^2}\right)^a = \prod_{\ell=1}^{a}\frac{1}{1-q^{2\ell}} = g(a). \tag{9.22}$$

The graded rank of the free $\dot{\mathcal{U}}^*(\mathbf{1}_n,\mathbf{1}_n)$-module $\dot{\mathcal{U}}^*(\mathcal{F}^{(a)}\mathbf{1}_n, \mathcal{F}^{(a)}\mathbf{1}_n)$ is computed similarly. □

Proposition 9.8. *The graded $\Bbbk$-vector spaces $\dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n)$ for $n \leqslant b-a = d-c$ and $\dot{\mathcal{U}}^*(\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{F}^{(d)}\mathcal{E}^{(c)}\mathbf{1}_n)$ for $n \geqslant b-a = d-c$ are free $\dot{\mathcal{U}}^*(\mathbf{1}_n,\mathbf{1}_n)$-modules whose graded ranks are given by*

$$\operatorname{rk}_q \dot{\mathcal{U}}^*\big(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n\big) = \big\langle E^{(a)}F^{(b)}1_n, E^{(c)}F^{(d)}1_n\big\rangle, \tag{9.23}$$

$$\operatorname{rk}_q \dot{\mathcal{U}}^*\big(\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{F}^{(d)}\mathcal{E}^{(c)}\mathbf{1}_n\big) = \big\langle F^{(b)}E^{(a)}1_n, F^{(d)}E^{(c)}1_n\big\rangle \tag{9.24}$$

where $\langle,\rangle$ is the semilinear form defined in Section 2.3.

Proof. This proof is identical to the proof of Proposition 2.7 with $\langle,\rangle \mapsto \dot{\mathcal{U}}^*(,)$, $E \mapsto \mathcal{E}$, $F \mapsto \mathcal{F}$ and the antiautomorphisms τ and τ^{-1} replaced by the 2-functors defined in Proposition 9.2. The assumption that $n \leqslant b-a = d-c$ implies that $2d+a-c \leqslant d-c \leqslant n$ so that $n-2d \leqslant a-c$. We use this fact for the third equality in the following derivation of $\dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n)$:

$$\begin{aligned}
&= \dot{\mathcal{U}}^*\big(\mathcal{F}^{(b)}\mathbf{1}_n, \tilde{\tau}\big(\mathcal{E}^{(a)}\big)\mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n\big)\\
&= \dot{\mathcal{U}}^*\big(\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{F}^{(a)}\mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n\big\{-a\big(n+2(c-d)+a\big)\big\}\big)\\
&= \bigoplus_{j=0}^{\min(a,c)}\Big(\dot{\mathcal{U}}^*\big(\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(c-j)}\mathcal{F}^{(a-j)}\mathcal{F}^{(d)}\mathbf{1}_n\big\{-a\big(n+2(c-d)+a\big)\big\}\big)^{\oplus\left[\begin{smallmatrix} a-c-(n-2d)\\ j\end{smallmatrix}\right]}\Big)\\
&= \bigoplus_{j=0}^{\min(a,c)}\Big(\dot{\mathcal{U}}^*\big(\tilde{\tau}^{-1}\big(\mathcal{E}^{(c-j)}\big)\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{F}^{(a-j)}\mathcal{F}^{(d)}\mathbf{1}_n\big\{a(2b-a-n)\big\}\big)^{\oplus\left[\begin{smallmatrix} b+d-n\\ j\end{smallmatrix}\right]}\Big)
\end{aligned}$$

where $\tilde{\tau}^{-1}(\mathcal{E}^{(c-j)}\mathbf{1}_{n-2d+2(a-j)}) = \mathcal{F}^{(c-j)}\mathbf{1}_{n-2b}\{(c-j)(c-j-(n-2d+2(a-j)))\}$. Simplifying the overall degree shift, we have that $\dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n)$ is given by

$$\bigoplus_{j=0}^{\min(a,c)}\Big(\dot{\mathcal{U}}^*\big(\mathcal{F}^{(c-j)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{F}^{(a-j)}\mathcal{F}^{(d)}\mathbf{1}_n\big\{(a+c-j)(b+d-j-n)\big\}\big)^{\oplus\left[\begin{smallmatrix} b+d-n\\ j\end{smallmatrix}\right]}\Big)$$

which, after using Proposition 9.5 to combine the categorifications of the divided powers, shows that $\dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n)$ is a shifted direct sum of graded abelian groups of the form $\mathcal{U}^*(\mathcal{F}^{(p)}\mathbf{1}_n, \mathcal{F}^{(p)}\mathbf{1}_n)$ for various values of p. But it was shown in Lemma 9.7 that these graded abelian groups are free $\dot{\mathcal{U}}^*(\mathbf{1}_n,\mathbf{1}_n)$-modules. Hence, it follows that $\dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(c)}\mathcal{F}^{(d)}\mathbf{1}_n)$ is also a free graded $\dot{\mathcal{U}}^*(\mathbf{1}_n,\mathbf{1}_n)$-module.

Now using that $\operatorname{rk}_q \mathcal{U}^*(\mathcal{F}^{(p)}\mathbf{1}_n, \mathcal{F}^{(p)}\mathbf{1}_n) = g(p)$ together with the definition of the graded rank, the proposition follows. □

9.3. $\dot{\mathcal{U}}$ as a categorification of $\dot{\mathbf{U}}$

Proposition 9.9. *The* 1*-morphisms*

(i) $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\{s\}$ *for* $a,b \in \mathbb{N}$, $n, s \in \mathbb{Z}$, $n \leqslant b-a$,
(ii) $\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n\{s\}$ *for* $a,b \in \mathbb{N}$, $n, s \in \mathbb{Z}$, $n \geqslant b-a$,

are indecomposable. Furthermore, these indecomposables are not isomorphic unless $n = b-a$ *in which case* $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_{b-a}\{s\} \cong \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_{b-a}\{s\}$.

Proof. To see that the morphisms $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\{s\}$ and $\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n\{s\}$ are indecomposable we show that their endomorphism ring has no nontrivial idempotents. This happens when

$$\mathrm{rk}_q \dot{\mathcal{U}}^*\big(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\{s\}, \mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\{s\}\big) \in 1 + q\mathbb{N}[q],$$
$$\mathrm{rk}_q \dot{\mathcal{U}}^*\big(\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n\{s\}, \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n\{s\}\big) \in 1 + q\mathbb{N}[q].$$

By Proposition 9.8 these graded ranks are given by the semilinear form of Section 2.3. We can neglect the shift $\{s\}$ appearing in both terms since this will not contribute to the graded rank.

Using the version of this semilinear form given in Proposition 2.8 we have

$$\mathrm{rk}_q \dot{\mathcal{U}}^*\big(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\big) = \sum_{\ell=0}^{\min(a,b)} q^{2j(b-a+j-n)} g(a-j)g(b-j)g(j)g(j),$$

$$\mathrm{rk}_q \dot{\mathcal{U}}^*\big(\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n\big)$$
$$= \sum_{j=0}^{\min(a,b)} q^{2j(a-b+j+n)} q^{2j(b-a+j-n)} g(a-j)g(b-j)g(j)g(j).$$

The elements $g(s)$ are in $1 + q\mathbb{N}[q]$ for all $s \in \mathbb{Z}$, so that indecomposability is determined by the power of q in (9.25). Hence, $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n$ is indecomposable when $2j(b-a+j-n) \geqslant 0$ for all $0 \leqslant j \leqslant \min(a,b)$. This happens when $j = 0$ or $n \leqslant b-a+j \leqslant b-a$. The only contribution to the q^0 term comes from $j = n-(b-a) \leqslant 0$, that is, when $j = 0$. A similar calculation shows $\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n$ is indecomposable when $n \geqslant b-a$. Thus, the elements lifting Lusztig's canonical basis are indecomposable.

Now suppose that $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n$ and $\mathcal{E}^{(a+\delta)}\mathcal{F}^{(b+\delta)}\mathbf{1}_n$ are both indecomposable so that $n \leqslant b-a$. To see that these are not isomorphic we show that the graded rank of the Homs between them is strictly positive. From Proposition 2.8 we have $\mathrm{rk}_q \dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(a+\delta)}\mathcal{F}^{(b+\delta)}\mathbf{1}_n) = \mathrm{rk}_q \dot{\mathcal{U}}^*(\mathcal{E}^{(a+\delta)}\mathcal{F}^{(b+\delta)}\mathbf{1}_n, \mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n)$ is given by

$$\sum_{j=\max(0,-\delta)}^{\min(a,b)} q^{2j(j+b-a-n)+\delta^2+\delta(b-a-n+2j)} g(a-j)g(b-j)g(\delta+j)g(j).$$

When $(b-a-n) \geqslant 0$ and $j \geqslant 0$ the power of q in the exponent is greater than zero since

$$2j(j+b-a-n) + \delta^2 + \delta(b-a-n+2j) > 0$$

always holds for $\delta > 0$, and for $\delta < 0$ becomes the assertion

$$\delta^2 > \delta(b-a-n),$$

which is always true for $\delta < 0$. Similarly, it can be shown that none of the indecomposable $\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n$ are isomorphic to any $\mathcal{F}^{(b+\delta)}\mathcal{E}^{(a+\delta)}\mathbf{1}_n$. By Lemma 2.6 there are no maps from $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n$ $\mathcal{F}^{(d)}\mathcal{E}^{(c)}\mathbf{1}_n$ when these elements are both indecomposable, so all that remains to be shown is the isomorphism $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_{b-a}\{s\} \cong \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_{b-a}\{s\}$ when $n = b-a$. We give this isomorphism in Corollary 9.11. □

Recall from Theorem 5.10 that there are decompositions of 1-morphisms:

$$\mathcal{E}\mathcal{F}\mathbf{1}_n \cong \mathcal{F}\mathcal{E}\mathbf{1}_n \oplus_{[n]} \mathbf{1}_n \quad \text{for } n \geqslant 0, \tag{9.25}$$

$$\mathcal{F}\mathcal{E}\mathbf{1}_n \cong \mathcal{E}\mathcal{F}\mathbf{1}_n \oplus_{[-n]} \mathbf{1}_n \quad \text{for } n \leqslant 0. \tag{9.26}$$

Proposition 9.10.

(i) *Every* 1*-morphism x in ${}_m\dot{\mathcal{U}}_n$ decomposes as a direct sum of indecomposable* 1*-morphisms of the form*

$$\begin{aligned} &\mathbf{1}_m\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\{s\} \quad \text{for } a,b \in \mathbb{N},\ n,s \in \mathbb{Z},\ n \leqslant b-a, \\ &\mathbf{1}_m\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n\{s\} \quad \text{for } a,b \in \mathbb{N},\ n,s \in \mathbb{Z},\ n \geqslant b-a, \end{aligned} \tag{9.27}$$

where $m = n - 2(b-a)$.

(ii) *The direct sum decomposition of $x \in {}_m\dot{\mathcal{U}}_n$ is essentially unique, meaning that the indecomposables and their multiplicities are unique up to reordering the factors.*

(iii) *The morphisms in* (i) (9.27) *above are the only indecomposables in $\dot{\mathcal{U}}$ up to isomorphism.*

Proof. Once we have established (i), the Krull–Schmidt theorem then establishes (ii), and (iii) (see Chapter I of Benson [5]).

To prove (i) it suffices to show that any element $x = \mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\cdots\mathcal{F}^{\beta_{k-1}}\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\}$ in $\mathcal{U}$ decomposes as a sum of elements in (9.27). We use induction on $\sum_i(\alpha_i+\beta_i)$ — the total number of $\mathcal{E}$'s and $\mathcal{F}$'s appearing in x. The base case of $\sum_i(\alpha_i+\beta_i) \leqslant 1$ is covered by Proposition 9.4. Assume then that such a decomposition exists for all 1-morphisms with $\sum_i(\alpha_i+\beta_i) \leqslant \gamma$. We provide a decomposition for $\sum_i(\alpha_i+\beta_i) = \gamma+1$.

If $m = \sum_i(\beta_i-\alpha_i) \leqslant n$ move all $\mathcal{F}$'s appearing in x to the left hand side using Theorem 5.10.

$$\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\} \oplus (x') \cong \mathbf{1}_m\mathcal{F}^{\sum_i\beta_i}\mathcal{E}^{\sum_i\alpha_i}\mathbf{1}_n\{s\} \oplus (x'') \tag{9.28}$$

where x' and x'' are terms with $\sum_i(\alpha_i-\beta_i) \leqslant \gamma$. By hypothesis, x' and x'' decompose into a direct sum of elements in (9.27). The 2-morphism $\mathbf{1}_m\mathcal{F}^{\sum_i\beta_i}\mathcal{E}^{\sum_i\alpha_i}\mathbf{1}_n\{s\}$ decomposes into the direct sum of indecomposables

$$\mathbf{1}_m\mathcal{F}^{\sum_i\beta_i}\mathcal{E}^{\sum_i\alpha_i}\mathbf{1}_n\{s\} = \bigoplus_{[\sum_i\alpha_i]![\sum_i\beta_i]!} \mathbf{1}_m\mathcal{F}^{(\sum_i\beta_i)}\mathcal{E}^{(\sum_i\alpha_i)}\mathbf{1}_n\{s\}$$

by Proposition 9.4. Since $\sum_i(\beta_i-\alpha_i)\leqslant n$ this is a direct sum of indecomposables by Proposition 9.9. Now since the direct sum of x with the direct sum of indecomposables is equal to a sum of indecomposables by (9.28), the Krull–Schmidt theorem implies that x must be isomorphic to a sum of indecomposables as well.

If $m=\sum_i(\beta_i-\alpha_i)\geqslant n$ move all $\mathcal{F}$'s appearing in x to the right hand side and a similar argument shows that x decomposes into a direct sum of the indecomposables in Proposition 9.4. $\square$

Corollary 9.11. *The* 1*-morphisms $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_{b-a}\{s\}$ and $\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_{b-a}\{s\}$ are isomorphic in $\dot{\mathcal{U}}$.*

Proof. The proof of Proposition 9.10 shows that any element

$$\mathbf{1}_{a-b}\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\cdots\mathcal{F}^{\beta_{k-1}}\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_{b-a}\{s\}$$

with $\sum\alpha_i=a$ and $\sum\beta_i=b$ decomposes in two possible ways

$$\mathbf{1}_{a-b}\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_{b-a}\{s\}\oplus\mathbf{1}_{a-b}x\mathbf{1}_{b-a}\quad\text{and}\quad\mathbf{1}_{a-b}\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_{b-a}\{s\}\oplus\mathbf{1}_{a-b}x'\mathbf{1}_{b-a}\tag{9.29}$$

where $\mathbf{1}_{a-b}x\mathbf{1}_{b-a}$ and $\mathbf{1}_{a-b}x'\mathbf{1}_{b-a}$ are direct sums of the indecomposable morphisms given in (9.27). These two decompositions into indecomposables must be isomorphic by the Krull–Schmidt theorem. It was shown in Proposition 9.9 that none of the elements in (9.27) are isomorphic except for the possibility of $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_{b-a}\{s\}$ and $\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_{b-a}\{s\}$. Hence, for the two decompositions to be isomorphic we must have $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_{b-a}\{s\}\cong\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_{b-a}\{s\}$. $\square$

Corollary 9.12. *There is a bijective correspondence between $\dot{\mathbb{B}}$ the canonical basis of $\dot{\mathbf{U}}$ and a choice of representatives for the isomorphism classes of indecomposable* 1*-morphisms of $\dot{\mathcal{U}}$ with no shift given by*

$$\begin{aligned}E^{(a)}F^{(b)}1_n&\mapsto\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\quad\textit{for } a,b\in\mathbb{N},\ n\in\mathbb{Z},\ n<b-a,\\ F^{(b)}E^{(a)}1_n1_m&\mapsto\mathcal{F}^{(b)}\mathcal{E}^{(a)}1_n\quad\textit{for } a,b\in\mathbb{N},\ n\in\mathbb{Z},\ n>b-a,\\ E^{(a)}F^{(b)}1_{b-a}=F^{(b)}E^{(a)}1_{b-a}&\mapsto\mathcal{E}^{(a)}\mathcal{F}^{(b)}1_{b-a}\cong\mathcal{F}^{(b)}\mathcal{E}^{(a)}1_{b-a}.\end{aligned}\tag{9.30}$$

The collection of morphisms in the image of this bijection is written as $\dot{\mathcal{B}}$. Likewise, ${}_m\dot{\mathcal{B}}_n\subset\dot{\mathcal{B}}$ denotes those representative 1*-morphisms mapping n to m.*

Proof. This is immediate from the proposition and the corollary above. $\square$

Corollary 9.12 can be viewed as the statement that the 1-morphisms in $\dot{\mathcal{B}}$ are a categorification of Lusztig's canonical basis for $\dot{\mathbf{U}}$.

For each pair of objects n, m of $\dot{\mathcal{U}}$ the hom category ${}_m\dot{\mathcal{U}}_n$ is an additive category. We denote the split Grothendieck group of this additive category by $K_0({}_m\dot{\mathcal{U}}_n)$. This $\mathbb{Z}[q,q^{-1}]$-module is generated by symbols $[f]$ for each x in ${}_m\dot{\mathcal{U}}_n$ modulo the relations:

$$\begin{aligned}&[f]=[f_1]+[f_2]\quad\text{if } f=f_1\oplus f_2,\\ &\big[f\{s\}\big]=q^s[f].\end{aligned}$$

We have shown (Proposition 9.10) that the additive categories ${}_m\dot{\mathcal{U}}_n$ have the Krull–Schmidt property — all objects decompose into a unique sum of indecomposables. Hence, $K_0({}_m\dot{\mathcal{U}}_n)$ is generated as a $\mathbb{Z}[q,q^{-1}]$-module by the isomorphism classes of indecomposable 1-morphisms $b : n \to m$ with $b \in {}_m\dot{\mathcal{B}}_n$.

Define the Grothendieck ring $K_0(\dot{\mathcal{U}})$ of the 2-category $\dot{\mathcal{U}}$ to be the direct sum

$$K_0(\dot{\mathcal{U}}) := \oplus_{n,m} K_0({}_m\dot{\mathcal{U}}_n).$$

The multiplication in $K_0(\dot{\mathcal{U}})$ is induced by composition so that

$$[f] = [f_1][f_2] \quad \text{if } f = f_1 \circ f_2.$$

We have the following theorem:

Theorem 9.13. *The split Grothendieck ring $K_0(\dot{\mathcal{U}})$ is isomorphic as a $\mathbb{Z}[q,q^{-1}]$-module to ${}_{\mathcal{A}}\dot{\mathbf{U}}$. Multiplication by q corresponds to the grading shift $\{1\}$.*

Proof. Since $\dot{\mathcal{U}}$ has the Krull–Schmidt property (Proposition 9.10) its Grothendieck ring is freely generated as a $\mathbb{Z}[q,q^{-1}]$-module by the isomorphism classes of indecomposables with no shift. We have shown that these isomorphism classes of indecomposables correspond bijectively to elements in Lusztig's canonical basis (Corollary 9.12). Furthermore, the multiplicative structure of the split Grothendieck ring $K_0(\dot{\mathcal{U}})$ arises from composition in $\dot{\mathcal{U}}$ Hence, $K_0(\dot{\mathcal{U}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong \dot{\mathbf{U}}$ since the relations of $\dot{\mathbf{U}}$ lift to 2-isomorphisms in $\dot{\mathcal{U}}$ see (Section 5.7). Since $K_0(\dot{\mathcal{U}})$ is a free $\mathbb{Z}[q,q^{-1}]$-module the theorem follows. $\square$

Remark 9.14. Associated to the 2-category $\dot{\mathcal{U}}$ is a non-unital ring $\mathfrak{U}' := \oplus_{x,y \in \mathrm{Mor}\dot{\mathcal{U}}} \dot{\mathcal{U}}(x,y)$ obtained by taking direct sums of the abelian groups associated to the homs between any pair of 1-morphisms x and y. Rather than having a unit, this ring has an infinite collection of idempotents 1_x for each morphism x in $\dot{\mathcal{U}}$. The ring multiplication $\dot{\mathcal{U}}(x,y) \otimes \dot{\mathcal{U}}(z,w) \to \dot{\mathcal{U}}(x,w)$ is zero when $y \neq z$ and is given by composition when $y = z$. The ring $\mathfrak{U}'$ is Morita equivalent to the ring $\mathfrak{U} := \oplus_{b,b' \in \dot{\mathcal{B}}} \dot{\mathcal{U}}^*(b,b')$ obtained from the graded homs between indecomposable morphisms $b, b' \in {}_m\dot{\mathcal{B}}_n$ with no shift.

The additive category ${}_m\dot{\mathcal{U}}_n$ can be thought of as the category of finitely generated projective modules over the non-unital ring ${}_m\mathfrak{U}_n := \oplus_{b,b' \in {}_m\dot{\mathcal{B}}_n} \dot{\mathcal{U}}^*(b,b')$. For each $b \in {}_m\dot{\mathcal{B}}_n$ there is an indecomposable projective module P_b of ${}_m\mathfrak{U}_n$ given by

$$P_b = \bigoplus_{b' \in {}_m\dot{\mathcal{B}}_n} \dot{\mathcal{U}}^*(b',b). \tag{9.31}$$

These are the only indecomposable projectives since the primitive orthogonal idempotents for $\mathfrak{U}$ are in bijective correspondence with elements $b \in \dot{\mathcal{B}}$, with $e_b \longleftrightarrow 1_b : b \to b$.

The projective Grothendieck ring $K_0(\mathfrak{U}') \cong K_0(\mathfrak{U})$ is then identical to the split Grothendieck ring of the 2-category $\dot{\mathcal{U}}$ as defined above. In particular, $K_0(\mathfrak{U}) \cong \dot{\mathbf{U}}$ and the collection of indecomposable projective modules for $\mathfrak{U}$ are a categorification of Lusztig's canonical basis $\dot{\mathbb{B}}$ of $\dot{\mathbf{U}}$.

The next theorem shows that $\mathbf{Flag}_N$ is a categorification of the irreducible $(N+1)$-dimensional representation of $\mathbf{U}_q(\mathfrak{sl}_2)$. This result is not new. It was established in the ungraded case by Chuang and Rouquier [12], and in the graded case by Frenkel, Khovanov and Stroppel [17].

Theorem 9.15. *The representation $\Gamma_N : \mathcal{U} \to \mathbf{Flag}_N$ yields a representation $\dot{\Gamma}_N : \dot{\mathcal{U}} \to \mathbf{Flag}_N$. This representation categorifies the irreducible $(N+1)$-dimensional representation V_N of $\dot{\mathbf{U}}$.*

Proof. Idempotent bimodule maps split in the category of bimodules. Hence, idempotents split in the hom categories of $\mathbf{Flag}_N$. Thus, by the universal property of the Karoubi envelope we have

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[dr, "\Gamma_N"'] & \dot{\mathcal{U}} \arrow[d, dotted, "\dot{\Gamma}_N"] \\ & \mathbf{Flag}_N \end{tikzcd}$$

so that the representation Γ_N extends to a representation of $\dot{\mathcal{U}}$.

For each $0 \leqslant k \leqslant N$ recall the rings $H_k := H^*(Gr(k,N))$ forming the objects of $\mathbf{Flag}_N^*$. The rings H_k are graded local rings so that every finitely-generated projective module is free, and H_k has (up to isomorphism and grading shift) a unique graded indecomposable projective module. Let H_k-pmod denote the category of finitely generated graded projective H_k-modules. The split Grothendieck group of the category $\bigoplus_{j=0}^{N} H_j$-pmod is then a free $\mathbb{Z}[q,q^{-1}]$-module of rank $N+1$, freely generated by the indecomposable projective modules, where q^i acts by shifting the grading degree by i. Thus, we have

$$K_0\left(\bigoplus_{k=0}^{N} H_k\text{-pmod}\right) \cong {}_{\mathcal{A}}(V_N), \qquad K_0\left(\bigoplus_{k=0}^{N} H_k\text{-pmod}\right) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong V_N, \quad (9.32)$$

as $\mathbb{Z}[q,q^{-1}]$-modules, respectively $\mathbb{Q}(q)$-modules, where ${}_{\mathcal{A}}(V_N)$ is a representation of ${}_{\mathcal{A}}\dot{\mathbf{U}}$, an integral form of the representation V_N of $\dot{\mathbf{U}}$.

The bimodules $\Gamma_N(\mathbf{1}_n)$, $\Gamma_N(\mathcal{E}\mathbf{1}_n)$ and $\Gamma_N(\mathcal{F}\mathbf{1}_n)$ induce, by tensor product, functors on the graded module categories. More precisely, consider the restriction functors

$$\begin{aligned} \mathrm{Res}_k^{k,k+1} &: H_{k,k+1}\text{-pmod} \to H_k\text{-pmod}, \\ \mathrm{Res}_{k+1}^{k,k+1} &: H_{k,k+1}\text{-pmod} \to H_{k+1}\text{-pmod} \end{aligned}$$

given by the inclusions $H_k \to H_{k,k+1}$ and $H_{k+1} \to H_{k,k+1}$. For $0 \leqslant k \leqslant N$ and $n = 2k - N$ define functors

$$\mathbf{1}_n := H_k \otimes_{H_k} : H_k\text{-pmod} \to H_k\text{-pmod}, \quad (9.33)$$

$$\mathbf{E1}_n := \mathrm{Res}_{k+1}^{k,k+1} H_{k+1,k} \otimes_{H_k} \{1-N+k\} : H_k\text{-pmod} \to H_{k+1}\text{-pmod}, \quad (9.34)$$

$$\mathbf{F1}_{n+2} := \mathrm{Res}_{k}^{k,k+1} H_{k,k+1} \otimes_{H_{k+1}} \{-k\} : H_{k+1}\text{-pmod} \to H_k\text{-pmod}. \quad (9.35)$$

These functors have both left and right adjoints and commute with the shift functor, so they induce $\mathbb{Z}[q,q^{-1}]$-module maps on Grothendieck groups. Furthermore, the 2-functor $\dot{\Gamma}_N$ must preserve the relations of $\dot{\mathcal{U}}$, so by Theorem 5.10 these functors satisfy relations lifting those of $\dot{\mathbf{U}}$. □

Theorem 9.16. *The* 2*-functors*

$$\widetilde{\omega}:\dot{\mathcal{U}}\to\dot{\mathcal{U}},\quad \widetilde{\sigma}:\dot{\mathcal{U}}\to\dot{\mathcal{U}}^{\mathrm{op}},\quad \widetilde{\psi}:\dot{\mathcal{U}}\to\dot{\mathcal{U}}^{\mathrm{co}},\quad \widetilde{\tau}:\dot{\mathcal{U}}\to\dot{\mathcal{U}}^{\mathrm{co\,op}},\quad \widetilde{\tau}^{-1}:\dot{\mathcal{U}}\to\dot{\mathcal{U}}^{\mathrm{co\,op}} \tag{9.36}$$

are graded lifts of the corresponding algebra maps ω, σ, ψ, τ on $\dot{\mathbf{U}}$ defined in Eqs. (2.13)–(2.16).

Proof. Comparing the definitions of the 2-functors given in Section 5.6 with the definitions of the corresponding algebra homomorphisms (2.13)–(2.16) the proof is immediate since $[\mathbf{1}_m\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\{s\}]=q^s 1_m E^{(a)}F^{(b)}1_n$ and $[\mathbf{1}_m\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n\{s\}]=q^s 1_m F^{(b)}E^{(a)}1_n$. □

Proposition 9.17. *The graded abelian group $\dot{\mathcal{U}}^*(x,y)$ is a free graded $\mathbb{Z}[v_1,v_2,\cdots]$-module.*

Proof. The 1-morphisms x and y decompose into a sum of indecomposables $b\{s\}$ for $b\in\dot{\mathcal{B}}$ and some shift $\{s\}$. Hence, every 2-morphism $f:x\to y$ decomposes into a sum of 2-morphisms between shifts of elements in $\dot{\mathcal{B}}$. But we have already shown that $\dot{\mathcal{U}}^*(b,b')$ for $b,b'\in\dot{\mathcal{B}}$ are free graded $\mathbb{Z}[v_1,v_2,\cdots]$-modules. Thus, $\dot{\mathcal{U}}^*(x,y)$ is a free graded $\mathbb{Z}[v_1,v_2,\cdots]$ module as well. □

Theorem 9.18. *The graded abelian group $\dot{\mathcal{U}}^*(x,y)$ categorifies the semilinear form $\langle,\rangle$ of Proposition* 2.4. *That is* $\mathrm{rk}_q\,\dot{\mathcal{U}}(x,y)=\langle[x],[y]\rangle$.

Proof. We show that $\mathrm{rk}_q\,\dot{\mathcal{U}}(,)$ has the defining properties of the semilinear form given in Proposition 2.4. Since $\mathrm{rk}_q\,\dot{\mathcal{U}}^*(x\{s\},y)=q^{-s}\,\mathrm{rk}_q\,\dot{\mathcal{U}}^*(x,y)$ and $\mathrm{rk}_q\,\dot{\mathcal{U}}^*(x,y\{s'\})=q^{s'}\,\mathrm{rk}_q\,\dot{\mathcal{U}}^*(x,y)$ the natural structure of the graded Hom functor induces the semilinearity on Grothendieck rings. The Hom property (ii) follows from the definition of the vertical composition in $\dot{\mathcal{U}}$. A given pair of 2-morphisms are composable if and only if their sources and targets are compatible. The adjoint property (iii) follows from Lemma 5.9. Property (iv) of the semilinear form follows from Lemma 9.7. Property (v) follows from the isomorphism $\dot{\mathcal{U}}(x\{s\},y\{s'\})\cong\dot{\mathcal{U}}(y\{-s'\},x\{-s\})$ induced from the invertible 2-functor $\widetilde{\psi}$. □

Because the 2-functors $\widetilde{\omega}$ and $\widetilde{\sigma}$ on $\dot{\mathcal{U}}$ are isomorphisms they must induce isomorphisms of the abelian groups

$$\dot{\mathcal{U}}\big(\widetilde{\omega}(x),\widetilde{\omega}(y)\big)=\dot{\mathcal{U}}(x,y), \tag{9.37}$$

$$\dot{\mathcal{U}}\big(\widetilde{\sigma}(x),\widetilde{\sigma}(y)\big)=\dot{\mathcal{U}}(x,y) \tag{9.38}$$

for every pair of 1-morphisms $x,y\in\dot{\mathcal{U}}$. Taking the graded ranks of the enriched homs we obtain equalities

$$\big\langle\omega\big([x]\big),\omega\big([y]\big)\big\rangle=\big\langle[x],[y]\big\rangle, \tag{9.39}$$

$$\big\langle\sigma\big([x]\big),\sigma\big([y]\big)\big\rangle=\big\langle[x],[y]]\big\rangle \tag{9.40}$$

for all $[x], [y] \in \dot{\mathbf{U}}$. But this is precisely Proposition 2.5. Thus, we have categorified these equations as well.

10. Proof of Proposition 2.8

For $a, j > 0$ the identities

$$\begin{bmatrix} a+1 \\ j \end{bmatrix} = q^{-j}\begin{bmatrix} a \\ j \end{bmatrix} + q^{a-j+1}\begin{bmatrix} a \\ j-1 \end{bmatrix}, \tag{10.1}$$

$$\begin{bmatrix} a+1 \\ j \end{bmatrix} = q^{j}\begin{bmatrix} a \\ j \end{bmatrix} + q^{-a+j-1}\begin{bmatrix} a \\ j-1 \end{bmatrix} \tag{10.2}$$

can found in most treatments of quantum groups. Recall that $g(a) = \prod_{j=1}^{a} \frac{1}{(1-q^{2j})}$. For induction proofs the identities:

$$\begin{bmatrix} m \\ j+1 \end{bmatrix} \frac{1}{g(j+1)} = q^{m+1}\big(q^{2(j-m)} - 1\big)\begin{bmatrix} m \\ j \end{bmatrix} \frac{1}{g(j)}, \tag{10.3}$$

$$\begin{bmatrix} m \\ j+1 \end{bmatrix} \frac{1}{g(j+1)} = q^{m+1+j}\big(q^{-2a} - 1\big)\begin{bmatrix} m-1 \\ j \end{bmatrix} \frac{1}{g(j)} \tag{10.4}$$

are also useful.

Lemma 10.1. *The equation*

$$q^{a(3b+2(c-a)-2n)} \sum_{j \geqslant 0} \begin{bmatrix} a \\ j \end{bmatrix}\begin{bmatrix} b+c-j \\ b \end{bmatrix}\begin{bmatrix} 2b+c-a-n \\ j \end{bmatrix} q^{-j(b+c-n)} \frac{1}{g(j)}$$
$$= \sum_{s \geqslant 0} \begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} -a+b+c \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n)} \tag{10.5}$$

holds for all $a, b, c \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$.

Proof. We prove this by induction on a. The base case when $a = 0$ is trivial; both sides become $\begin{bmatrix} b+c \\ b \end{bmatrix}$. Now suppose that (10.5) holds for all $b, c \in \mathbb{Z}_+$, $n \in \mathbb{Z}$ for all values up to a. We show that (10.5) holds for $a+1$ as well:

$$q^{(a+1)(3b+2(c-a-1)-2n)} \sum_{j \geqslant 0} \begin{bmatrix} a+1 \\ j \end{bmatrix}\begin{bmatrix} b+c-j \\ b \end{bmatrix}\begin{bmatrix} 2b+c-a-1-n \\ j \end{bmatrix} q^{-j(b+c-n)} \frac{1}{g(j)}$$
$$= \sum_{s \geqslant 0} \begin{bmatrix} a+1 \\ s \end{bmatrix}\begin{bmatrix} -a-1+b+c \\ b-s \end{bmatrix} q^{-s(2(a+1)-3b-c+2n)}. \tag{10.6}$$

Using (10.2) the left hand side can be rewritten as

$$q^{-2-2a+3b+2c-2n}\left(q^{a(3b+2(c-a)-2(n+1))}\sum_{j\geqslant 0}\begin{bmatrix} a \\ j \end{bmatrix}\begin{bmatrix} b+c-j \\ b \end{bmatrix}\right.$$
$$\left.\times\begin{bmatrix} 2b+c-a-(n+1) \\ j \end{bmatrix}\frac{q^{-j(b+c-(n+1))}}{g(j)}\right)$$
$$+q^{-3-5a-2a^2+3b+3ab+2c+2ac-2n-2an}\sum_{j\geqslant 1}\begin{bmatrix} a \\ j-1 \end{bmatrix}$$
$$\times\begin{bmatrix} b+c-j \\ b \end{bmatrix}\begin{bmatrix} 2b+c-a-n-1 \\ j \end{bmatrix}\frac{q^{-j(b+c-(n+1))}}{g(j)}.$$

Use the induction hypothesis on the first term with $(a,b,c,n+1)$ and shift the second term by $j'=j-1$ to get

$$q^{-2-2a+3b+2c-2n}\sum_{s\geqslant 0}\begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} -a+b+c \\ b-s \end{bmatrix}q^{-s(2a-3b-c+2(n+1))}$$
$$+q^{-2-5a-2a^2+2b+3ab+c+2ac-n-2an}$$
$$\times\sum_{j'\geqslant 0}\begin{bmatrix} a \\ j' \end{bmatrix}\begin{bmatrix} b+c-j'-1 \\ b \end{bmatrix}\begin{bmatrix} 2b+c-a-n-1 \\ j'+1 \end{bmatrix}\frac{q^{-j'(b+c-(n+1))}}{g(j'+1)}.$$

Restricting our attention to the second term, (10.3) allows this term to be written as

$$-q^{-2-4a+4b+2c-2n}q^{a(3b+2((c-1)-a)-2n)}$$
$$\sum_{j'\geqslant 0}\begin{bmatrix} a \\ j' \end{bmatrix}\begin{bmatrix} b+(c-1)-j' \\ b \end{bmatrix}\begin{bmatrix} 2b+(c-1)-a-n \\ j' \end{bmatrix}\frac{q^{-j'(b+(c-1)-n)}}{g(j')}$$
$$+q^{-4a-2a^2+3ab+2ac-2an}\sum_{j'\geqslant 0}\begin{bmatrix} a \\ j' \end{bmatrix}\begin{bmatrix} b+c-j'-1 \\ b \end{bmatrix}\begin{bmatrix} 2b+c-a-n-1 \\ j' \end{bmatrix}$$
$$\times\frac{q^{-j'(b+c-n-3)}}{g(j')}.$$

On the first term use the induction hypothesis with $(a,b,c-1,n)$ and on the second term use (10.1) on the third quantum binomial to give

$$-q^{-2-4a+4b+2c-2n}\sum_{s\geqslant 0}\begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix}q^{-s(2a-3b-(c-1)+2n)}$$
$$\times q^{a(3b+2(c-1-a)-2(n+1))}\sum_{j\geqslant 0}\begin{bmatrix} a \\ j \end{bmatrix}\begin{bmatrix} b+(c-1)-j \\ b \end{bmatrix}\begin{bmatrix} 2b+(c-1)-a-(n+1) \\ j \end{bmatrix}$$
$$\times\frac{q^{-j(b+(c-1)-(n+1))}}{g(j)}$$

$$+\, q^{a(3b+2(c-1-a)-2(n+1))+2b+c-a-n-1} \sum_{j\geqslant 1} \begin{bmatrix} a \\ j \end{bmatrix} \begin{bmatrix} b+c-j-1 \\ b \end{bmatrix} \begin{bmatrix} 2b+c-a-n-2 \\ j-1 \end{bmatrix}$$

$$\times \frac{q^{-j(b+c-n-2)})}{g(j)}.$$

For the moment we restrict our attention to the second and third terms. Apply the induction hypothesis to the second term with $(a, b, c-1, n+1)$; shift the third term by letting $j' = j-1$ and apply (10.4) to the first quantum binomial to get

$$\sum_{s\geqslant 0} \begin{bmatrix} a \\ s \end{bmatrix} \begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix} q^{-s(2a-3b-(c-1)+2(n+1))} + q^{-2-2a+4b+2c-2n}\big(q^{-2a}-1\big)$$

$$\times\, q^{(a-1)(3b+2((c-a-1))-2(n+1))} \sum_{j\geqslant 0} \begin{bmatrix} a-1 \\ j \end{bmatrix} \begin{bmatrix} b+c-2-j \\ b \end{bmatrix} \begin{bmatrix} 2b+(c-a-1)-(n+1) \\ j-1 \end{bmatrix}$$

$$\times \frac{q^{-j(b+(c-2)-(n+1))})}{g(j)}.$$

Using the induction hypothesis on the last term for the values $(a-1, b, c-2, n+1)$, together with the identity

$$\begin{bmatrix} a-1 \\ s \end{bmatrix} = \frac{q^{s-2a}-q^{-s}}{q^{-2a}-1} \begin{bmatrix} a \\ s \end{bmatrix},$$

the last term can be written as

$$q^{-2-2a+4b+2c-2n} \sum_{s\geqslant 0} \begin{bmatrix} a \\ s \end{bmatrix} \begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+2)}\big(q^{s-2a}-q^{-s}\big).$$

Putting everything together, the left hand side of (10.5) becomes

$$q^{-2-2a+3b+2c-2n} \sum_{s\geqslant 0} \begin{bmatrix} a \\ s \end{bmatrix} \begin{bmatrix} -a+b+c \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+2)}$$

$$-\, q^{-2-4a+4b+2c-2n} \sum_{s\geqslant 0} \begin{bmatrix} a \\ s \end{bmatrix} \begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+1)}$$

$$+ \sum_{s\geqslant 0} \begin{bmatrix} a \\ s \end{bmatrix} \begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+3)}$$

$$+\, q^{-2-2a+4b+2c-2n} \sum_{s\geqslant 0} \begin{bmatrix} a \\ s \end{bmatrix} \begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+2)}\big(q^{s-2a}-q^{-s}\big).$$

The second term cancels with part of the fourth term leaving

$$q^{-2-2a+3b+2c-2n}\sum_{s\geqslant 0}\begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} -a+b+c \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+2)}$$
$$+\sum_{s\geqslant 0}\begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+3)}$$
$$-q^{-2-2a+4b+2c-2n}\sum_{s\geqslant 0}\begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} -a+b+c-1 \\ b-s \end{bmatrix} q^{-s(2a-3b-c+2n+3)}.$$

The first and third terms combine using (10.2) to give

$$q^{-a-2+3b+c-2n}\sum_{s\geqslant 0}\begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} -a+b+c-1 \\ b-s-1 \end{bmatrix} q^{-s(2a-3b-c+2n+3)}$$

which, after shifting by setting $s=s'-1$, combines with the second term to give the right hand side of (10.5) proving the lemma. □

Proof of Proposition 2.8. Setting $d=c-a+b$, Proposition 2.8 becomes the equation

$$q^{-(a+c)(a-2b-c+n)}\sum_{j=0}^{\min(a,c)}\begin{bmatrix} 2b+c-a-n \\ j \end{bmatrix}\begin{bmatrix} b+c-j \\ b \end{bmatrix}$$
$$\times\begin{bmatrix} b+c-j \\ a-j \end{bmatrix} q^{-j(2b+2c-j-n)} g(b+c-j)$$
$$=q^{(a-c)(2a-b-c+n)}\sum_{j=\max(0,a-c)}^{\min(a,b)} q^{2j(-2a+b+c+j-n)} g(a-j)g(b-j)g(j)g(c-a+j).$$

After making use of:

$$g(p)g(r)=\begin{bmatrix} p+r \\ p \end{bmatrix} q^{pr} g(p+r) \tag{10.7}$$

the equation is rewritten in the form

$$g(a)g(-a+b+c)q^{a(b-c-n)+c(2b+c-n)}\sum_{j\geqslant 0}\begin{bmatrix} a \\ j \end{bmatrix}\begin{bmatrix} b+c-j \\ b \end{bmatrix}$$
$$\times\begin{bmatrix} 2b+c-a-n \\ j \end{bmatrix} q^{-j(b+c-n)}\frac{1}{g(j)}$$
$$=g(a)g(-a+b+c)q^{(a-c)(2a-2b-c+n)}\sum_{j\geqslant 0}\begin{bmatrix} a \\ j \end{bmatrix}\begin{bmatrix} -a+b+c \\ b-j \end{bmatrix} q^{-j(2a-3b-c+2n)}$$

where we have made use of the fact that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]=0$ when k is larger than n. Simplifying where possible, the equation becomes (10.5) of Lemma 10.1. □

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References

[1] P. Balmer, M. Schlichting, Idempotent completion of triangulated categories, J. Algebra 236 (2) (2001) 819–834.
[2] D. Bar-Natan, S. Morrison, The Karoubi envelope and Lee's degeneration of Khovanov homology, Algebr. Geom. Topol. 6 (2006) 1459–1469 (electronic), math.GT/0606542.
[3] B. Bartlett, The geometry of unitary 2-representations of finite groups and their 2-characters, arXiv:0807.1329 [math.QA], 2008.
[4] A.A. Beĭlinson, G. Lusztig, R. MacPherson, A geometric setting for the quantum deformation of GL_n, Duke Math. J. 61 (2) (1990) 655–677.
[5] D.J. Benson, Representations and Cohomology. I: Basic Representation Theory of Finite Groups and Associative Algebras, second edition, Cambridge Stud. Adv. Math., vol. 30, Cambridge University Press, Cambridge, 1998.
[6] I.N. Bernšteĭn, I.M. Gel'fand, S.I. Gel'fand, Schubert cells and the cohomology of a flag space, Funktsional. Anal. i Prilozhen. 7 (1) (1973) 64–65.
[7] J. Bernstein, I. Frenkel, M. Khovanov, A categorification of the Temperley–Lieb algebra and Schur quotients of $u(\mathfrak{sl}_2)$ via projective and Zuckerman functors, Selecta Math. (N.S.) 5 (2) (1999) 199–241, math.QA/0002087.
[8] F. Borceux, Handbook of Categorical Algebra. 1, Encyclopedia Math. Appl., vol. 50, Cambridge University Press, Cambridge, 1994.
[9] F. Borceux, Handbook of Categorical Algebra. 2, Encyclopedia Math. Appl., vol. 51, Cambridge University Press, Cambridge, 1994.
[10] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. (2) 57 (1953) 115–207.
[11] R.P. Carpentier, From planar graphs to embedded graphs—a new approach to Kauffman and Vogel's polynomial, J. Knot Theory Ramifications 9 (8) (2000) 975–986, math.GT/0005128.
[12] J. Chuang, R. Rouquier, Derived equivalences for symmetric groups and sl_2-categorification, Ann. of Math. 167 (2008) 245–298, math.RT/0407205.
[13] J.R.B. Cockett, J. Koslowski, R.A.G. Seely, Introduction to linear bicategories, in: The Lambek Festschrift: Mathematical Structures in Computer Science, Montreal, QC, 1997, Math. Structures Comput. Sci. 10 (2) (2000) 165–203.
[14] L. Crane, I. Frenkel, Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases, in: Topology and Physics, J. Math. Phys. 35 (10) (1994) 5136–5154, hep-th/9405183.
[15] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, Invent. Math. 21 (1973) 287–301.
[16] I. Frenkel, Unpublished notes, circulation year, 1993.
[17] I. Frenkel, M. Khovanov, C. Stroppel, A categorification of finite-dimensional irreducible representations of quantum sl(2) and their tensor products, Selecta Math. (N.S.) 12 (3–4) (2006) 379–431, math.QA/0511467.
[18] P.J. Freyd, D.N. Yetter, Braided compact closed categories with applications to low-dimensional topology, Adv. Math. 77 (2) (1989) 156–182.
[19] W. Fulton, Young Tableaux: With Applications to Representation Theory and Geometry, London Math. Soc. Stud. Texts, vol. 35, Cambridge University Press, Cambridge, 1997.
[20] N. Ganter, M. Kapranov, Representation and character theory in 2-categories, Adv. Math. 217 (5) (2008) 2268–2300, math.KT/0602510.
[21] I. Grojnowski, G. Lusztig, On bases of irreducible representations of quantum GL_n, in: Kazhdan–Lusztig Theory and Related Topics, Chicago, IL, 1989, in: Contemp. Math., vol. 139, Amer. Math. Soc., Providence, RI, 1992, pp. 167–174.
[22] A. Heller, Homological algebra in abelian categories, Ann. of Math. (2) 68 (1958) 484–525.
[23] H. Hiller, Geometry of Coxeter Groups, Res. Notes Math., vol. 54, Pitman (Advanced Publishing Program), Boston, MA, 1982.

[24] A. Joyal, R. Street, The geometry of tensor calculus. I, Adv. Math. 88 (1) (1991) 55–112.

[25] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Math. Soc. Lecture Note Ser., vol. 64, Cambridge University Press, Cambridge, 1982.

[26] G.M. Kelly, R. Street, Review of the elements of 2-categories, in: Category Seminar, Proc. Sem., Sydney, 1972/1973, in: Lecture Notes in Math., vol. 420, Springer, Berlin, 1974, pp. 75–103.

[27] M. Khovanov, NilCoxeter algebras categorify the Weyl algebra, Comm. Algebra 29 (11) (2001) 5033–5052, math.RT/9906166.

[28] M. Khovanov, A functor-valued invariant of tangles, Algebr. Geom. Topol. 2 (2002) 665–741 (electronic), math.QA/0103190.

[29] M. Khovanov, V. Mazorchuk, C. Stroppel, A brief review of abelian categorifications, math.RT/0702746, 2007.

[30] B. Kostant, S. Kumar, The nilHecke ring and cohomology of G/P for a Kac–Moody group G, Adv. Math. 62 (3) (1986) 187–237.

[31] A. Lascoux, M.P. Schützenberger, Polynômes de Schubert, C. R. Acad. Sci. Paris Sér. I Math. 294 (13) (1982) 447–450.

[32] A.D. Lauda, Frobenius algebras and planar open string topological field theories, math.QA/0508349, 2005.

[33] A.D. Lauda, Frobenius algebras and ambidextrous adjunctions, Theory Appl. Categ. 16 (2006) 84–122, math.CT/0502550.

[34] G. Lusztig, Introduction to Quantum Groups, Progr. Math., vol. 110, Birkhäuser Boston Inc., Boston, MA, 1993.

[35] M. Mackaay, Spherical 2-categories and 4-manifold invariants, Adv. Math. 143 (2) (1999) 288–348.

[36] L. Manivel, Symmetric Functions, Schubert Polynomials and Degeneracy Loci, SMF/AMS Texts Monogr., vol. 6, American Mathematical Society, Providence, RI, 2001, translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3.

[37] M. Müger, From subfactors to categories and topology. I. Frobenius algebras in and Morita equivalence of tensor categories, J. Pure Appl. Algebra 180 (1–2) (2003) 81–157, math.CT/0111204.

[38] A. Postnikov, Enumeration in algebra and geometry, Ph.D. thesis, MIT, 1997 available at http://citeseer.ist.psu.edu/postnikov97enumeration.html, 1997.

[39] R. Rouquier, Derived equivalences and finite dimensional algebras, in: International Congress of Mathematicians, vol. II, Eur. Math. Soc., Zürich, 2006, pp. 191–221, math.RT/0603356.

[40] R. Street, Low-dimensional topology and higher-order categories, in: Proceedings of CT95, Halifax, July 9–15, 1995.

[41] R. Street, Categorical structures, in: Handbook of Algebra, vol. 1, North-Holland, Amsterdam, 1996, pp. 529–577.

[42] C. Stroppel, Categorification of the Temperley–Lieb category, tangles, and cobordisms via projective functors, Duke Math. J. 126 (3) (2005) 547–596.

[43] Wikipedia, Karoubi envelope — wikipedia, the free encyclopedia [accessed 09-july-2007].

[44] H. Zheng, A geometric categorification of tensor products of $u_q(\mathrm{sl}_2)$-modules, arXiv:0705.2630 [math.RT], 2007.