

The polyhedral product functor: A method of decomposition for moment-angle complexes, arrangements and related spaces

A. Bahri^a, M. Bendersky^b, F.R. Cohen^{c,*}, S. Gitler^d

^a Department of Mathematics, Rider University, Lawrenceville, NJ 08648, USA

^b Department of Mathematics CUNY, East 695 Park Avenue, New York, NY 10065, USA

^c Department of Mathematics, University of Rochester, Rochester, NY 14625, USA

^d Department of Mathematics, Cinvestav, San Pedro Zacatenco, Mexico, D.F. CP 07360 Apartado Postal 14-740, Mexico

Received 17 January 2009; accepted 25 March 2010

Communicated by Mark Hovey

Abstract

This article gives a natural decomposition of the suspension of generalized moment-angle complexes or *partial product spaces* which arise as *polyhedral product functors* described below. The geometrical decomposition presented here provides structure for the stable homotopy type of these spaces including spaces appearing in work of Goresky–MacPherson concerning complements of certain subspace arrangements, as well as Davis–Januszkiewicz and Buchstaber–Panov concerning moment-angle complexes. Since the stable decompositions here are geometric, they provide corresponding homological decompositions for generalized moment-angle complexes for any homology theory.

© 2010 Elsevier Inc. All rights reserved.

MSC: primary 13F55, 14F45, 32S22, 52C35, 55U10; secondary 16E05, 57R19

Keywords: Arrangements; Cohomology ring; Moment-angle complex; Simplicial complex; Simplicial sets; Stable splittings; Stanley–Reisner ring; Suspensions; Toric varieties

* Corresponding author.

E-mail addresses: bahri@rider.edu (A. Bahri), mbenders@xena.hunter.cuny.edu (M. Bendersky), cohf@math.rochester.edu (F.R. Cohen), sgitler@math.cinvestav.mx (S. Gitler).

Contents

1. Introduction	1635
2. Statement of results	1636
3. Toric manifolds	1652
4. Preparation	1655
5. Proof of Theorem 2.10	1658
6. Proof of Theorem 2.12	1660
7. Proof of Theorem 2.19	1662
8. Proof of Theorem 2.35	1662
9. Proof of Theorem 2.46	1663
10. Connection to Kurosh's theorem	1664
Acknowledgments	1666
References	1667

1. Introduction

Spaces which now are called (*generalized*) *moment-angle complexes*, have been studied by topologists since the 1960's thesis of G. Porter [41]. In the 1970's E.B. Vinberg [47] and in the late 1980's S. Lopez de Medrano developed some of their features [33]. In seminal work during the early 1990's, M. Davis and J. Januszkiewicz introduced quasi-toric manifolds, a topological generalization of projective toric varieties which were being studied intensively by algebraic geometers [14]. They observed that every quasi-toric manifold is the quotient of a moment-angle complex by the free action of a real torus. The moment-angle complex is denoted $Z(K; (D^2, S^1))$ where K is a finite simplicial complex, as defined below. The spaces $Z(K; (D^2, S^1))$ are at the confluence of work of many people. A short introduction to some of these connections is given next.

Let R denote the ring given by \mathbb{Z} , \mathbb{Q} , or a finite field. Given K , there is an associated ring known as the Stanley–Reisner ring of K , defined below, and denoted $R[K]$ here. The ring $R[K]$ is a quotient of a finitely generated polynomial ring $P(K) = R[v_1, \dots, v_m]$ with generators v_i for each vertex of K . M. Hochster, in purely algebraic work, calculated the Tor-modules $\text{Tor}_{P(K)}(R[K], R)$ in terms of the full subcomplexes of K [25]. In this work Hochster also produced an algebraic decomposition of these Tor-modules.

Subsequently, and independently, Goresky–MacPherson [22] studied the cohomology of complements of subspace arrangements $U(\mathcal{A})$ and related decompositions of their cohomology. These spaces included complements of certain coordinate subspace arrangements [22]. A more direct proof was subsequently given by Ziegler–Zivaljević [50].

Later, as well as independently, Davis and Januszkiewicz introduced manifolds now called quasi-toric varieties, a topological generalization of projective toric varieties [14]. They proved that a certain choice of Borel construction for the space $Z(K; (D^2, S^1))$ which they define precisely had cohomology ring given by $R[K]$ for $R = \mathbb{Z}$, the Stanley–Reisner ring of K . These spaces are now known as the Davis–Januszkiewicz spaces.

Buchstaber and Panov [9] synthesized these different developments by proving that the spaces $Z(K; (D^2, S^1))$ are strong deformation retracts of complements of certain coordinate subspace arrangements $U(\mathcal{A})$ appearing earlier in work of Goresky and MacPherson. They also proved

that the cohomology algebra of $Z(K; (D^2, S^1))$ is isomorphic to $\text{Tor}_{P(K)}(\mathbb{Z}[K], \mathbb{Z})$ which had been considered earlier by Hochster.

Among others who have worked extensively on moment-angle complexes are Notbohm and Ray [38], Grbic and Theriault [23], Strickland [45], Baskakov [5,6], Buchstaber and Panov [9], Panov [39], Baskakov, Buchstaber and Panov [7], Buchstaber, Panov and Ray [11], Franz [19, 20], Panov, Ray and Vogt [40] and Kamiyama and Tsukuda [30]. Work in [13,15,16,26,28,29, 32,43,46] is closely related. Applications to robotics are the focus of [24,12]. The direction of this paper is guided by the development in elegant work of Denham and Suciu [17]. In addition, generalized moment-angle complexes had been defined earlier in work of Anick [2].

Among the results given here is a natural decomposition for the suspension of the generalized moment-angle complex, the value of the suspension of the “polyhedral product functor.” Since the decomposition is geometric, an analogous homological decomposition for a generalized moment-angle complex applies for any homology theory. This last decomposition specializes to the homological decompositions in the work of several authors cited above. Furthermore, this decomposition gives an additive decomposition for the Stanley–Reisner ring of a finite simplicial complex extended to other natural settings and which arises from generalizations of certain homotopy theoretic results of Porter [41] and Ganea [21]. Descriptions for some of the decompositions below appear in [4]. Extensions of these structures to a simplicial setting as well as to the structure of ordered, commuting n -tuples in a Lie group G are given in [1].

Generalized moment-angle complexes are given by functors determined by simplicial complexes with values given by subspaces of products of topological spaces. Thus, these may be regarded as “polyhedral product functors” indexed by abstract simplicial complexes, terminology for which the authors thank William Browder (also the inventor of the name “orbifold”).

The authors thank the referee as well as Peter Landweber for useful suggestions which improved the presentation of this article.

2. Statement of results

In this article, a polyhedron is defined to be the geometric realization of a simplicial complex. Generalized moment-angle complexes can be regarded loosely as a functor from simplicial complexes with values given by subspaces of products of topological spaces. Thus these may be regarded as “polyhedral product functors.” Generalized moment-angle complexes are defined next.

- (1) Let $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)_{i=1}^m\}$ denote a set of triples of connected CW-complexes with base-point x_i in A_i .
- (2) Let K denote an abstract simplicial complex with m vertices labeled by the set $[m] = \{1, 2, \dots, m\}$. Thus, a $(k-1)$ -simplex σ of K is given by an ordered sequence $\sigma = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$ such that if $\tau \subset \sigma$, then τ is a simplex of K . In particular the empty set \emptyset is a subset of σ and so it is in K . The set $[m]$ is minimal in the sense that every $i \in [m]$ belongs to at least one simplex of K . The length k of σ is denoted $|\sigma|$.
- (3) Let $\Delta[m-1]$ denote the abstract simplicial complex given by the power set of $[m] = \{1, 2, \dots, m\}$. Let $\Delta[m-1]_q$ denote the subset of the power set of $[m]$ given by all subsets of cardinality at most $q+1$. Thus $\Delta[m-1]_q$ is the q -skeleton of $\Delta[m-1]$.

Define two functors from the category K of simplicial complexes with morphisms simplicial maps to the category CW_* of connected, based CW-complexes and based continuous maps as follows.

Definition 2.1. As above, let $(\underline{X}, \underline{A})$ denote the collection $\{(X_i, A_i, x_i)\}_{i=1}^m$.

The *generalized moment-angle complex or polyhedral product functor* determined by $(\underline{X}, \underline{A})$ and K denoted

$$Z(K; (\underline{X}, \underline{A}))$$

is defined using the functor

$$D: K \rightarrow CW_*$$

as follows: For every σ in K , let

$$D(\sigma) = \prod_{i=1}^m Y_i, \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \in [m] - \sigma \end{cases}$$

with $D(\emptyset) = A_1 \times \cdots \times A_m$.

The generalized moment-angle complex is

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \operatorname{colim} D(\sigma)$$

where the colimit is defined by the inclusions, $d_{\sigma, \tau}$ with $\sigma \subset \tau$ and $D(\sigma)$ is topologized as a subspace of the product $X_1 \times \cdots \times X_m$. The *generalized pointed moment-angle complex* is the underlying space $Z(K; (\underline{X}, \underline{A}))$ with base-point $\ast = (x_1, \dots, x_m) \in Z(K; (\underline{X}, \underline{A}))$.

Note that the definition of $Z(K; (\underline{X}, \underline{A}))$ did not require spaces to be either based or CW-complexes. In the special case where $X_i = X$ and $A_i = A$ for all $1 \leq i \leq m$, it is convenient to denote the generalized moment-angle complex by $Z(K; (X, A))$ to coincide with the notation in [17].

The smash product

$$X_1 \wedge X_2 \wedge \cdots \wedge X_m$$

is given by the quotient space $(X_1 \times \cdots \times X_m)/S(X_1 \times \cdots \times X_m)$ where $S(X_1 \times \cdots \times X_m)$ is the subspace of the product with at least one coordinate given by the base-point $x_j \in X_j$. Spaces analogous to generalized moment-angle complexes are given next where products of spaces are replaced by smash products, a setting in which base-points are required.

Definition 2.2. Given a generalized pointed moment-angle complex $Z(K; (\underline{X}, \underline{A}))$ obtained from $(\underline{X}, \underline{A}, \ast)$, the *generalized smash moment-angle complex*

$$\widehat{Z}(K; (\underline{X}, \underline{A}))$$

is defined to be the image of $Z(K; (\underline{X}, \underline{A}))$ in the smash product $X_1 \wedge X_2 \wedge \cdots \wedge X_m$.

The image of $D(\sigma)$ in $\widehat{Z}(K; (\underline{X}, \underline{A}))$ is denoted by $\widehat{D}(\sigma)$ and is

$$Y_1 \wedge Y_2 \wedge \cdots \wedge Y_m$$

where

$$Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \in [m] - \sigma. \end{cases}$$

As in the case of $Z(K; (\underline{X}, \underline{A}))$, note that $\widehat{Z}(K; (\underline{X}, \underline{A}))$ is the colimit obtained from the spaces $\widehat{D}(\sigma)$ with $\widehat{D}(\sigma) \cap \widehat{D}(\tau) = \widehat{D}(\sigma \cap \tau)$.

Remark 2.3. The constructions $Z(K; (\underline{X}, \underline{A}))$ and $\widehat{Z}(K; (\underline{X}, \underline{A}))$ are bifunctors, natural for morphisms of $(\underline{X}, \underline{A})$ as well as $(\underline{X}, \underline{A}, *)$, and for injections in K .

The next example provides elementary cases of moment-angle complexes together with associated toric manifolds.

Example 2.4. Let K denote the simplicial complex given by $\{\{1\}, \{2\}, \emptyset\}$ with

$$(X_i, A_i) = (D^{n_i}, S^{n_i-1}) \quad \text{for } i = 1, 2.$$

(1) The first example is

$$(X_i, A_i) = (D^{2n}, S^{2n-1}) \quad \text{for } i = 1, 2$$

where D^{2n} is the $2n$ -disk and S^{2n-1} is its boundary sphere. Then

$$Z(K; (D^{2n}, S^{2n-1})) = D^{2n} \times S^{2n-1} \cup S^{2n-1} \times D^{2n}$$

with the boundary of $D^{2n} \times D^{2n}$ given by $\partial(D^{2n} \times D^{2n}) = D^{2n} \times S^{2n-1} \cup S^{2n-1} \times D^{2n}$ so $Z(K; (D^{2n}, S^{2n-1})) = S^{4n-1}$.

(2) The second example is $(X_1, A_1) = (D^2, S^1)$ and $(X_2, A_2) = (D^3, S^2)$. Then

$$Z(K; (\underline{X}, \underline{A})) = D^2 \times S^2 \cup S^1 \times D^3 = S^4.$$

(3) The third example exhibits the connection to toric manifolds where K' is a simplicial complex with m vertices. Regard (D^{2n}, S^{2n-1}) as a pair of subspaces of \mathbb{C}^n in the standard way with the standard S^1 -action restricting to an action on the pair (D^{2n}, S^{2n-1}) . Thus T^m acts on $Z(K'; (D^{2n}, S^{2n-1}))$. Specialize to $K' = K$. Notice that the S^1 diagonal subgroup $\Delta(T^2)$ acts freely on $Z(K; (D^{2n}, S^{2n-1})) = S^{4n-1}$ where the quotient space $S^{4n-1}/\Delta(T^m)$ is \mathbb{CP}^{2n-1} , complex projective space of dimension $(2n-1)$. If $n = 1$, then $(D^{2n}, S^{2n-1}) = (D^2, S^1)$ and

$$Z(K; (D^2, S^1)) = S^3$$

is a case related to toric topology. That is, \mathbb{CP}^1 is the toric manifold quotient by $\Delta(T^2)$ of its associated moment-angle complex.

(4) More generally, let $K = \Delta[m-1]_q$ be the q -skeleton of the $(m-1)$ -simplex $\Delta[m-1]$ consisting of all the simplices of $\Delta[m-1]$ of dimension $\leq q$. Since

$$\partial \Delta[m-1] = \Delta[m-1]_{m-2},$$

there is a homeomorphism

$$Z(\partial\Delta[m-1]; (D^{2n}, S^{2n-1})) \rightarrow \partial(D^{2n})^m = S^{2mn-1}.$$

Definition 2.5. Consider an ordered sequence $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$ together with pointed spaces Y_1, \dots, Y_m . Then

- (1) the length of I is $|I| = k$,
- (2) the notation $I \subseteq [m]$ means I is any increasing subsequence of $(1, \dots, m)$,
- (3) $Y^{[m]} = Y_1 \times \dots \times Y_m$,
- (4) $Y^I = Y_{i_1} \times Y_{i_2} \times \dots \times Y_{i_k}$,
- (5) $\widehat{Y}^I = Y_{i_1} \wedge \dots \wedge Y_{i_k}$.

Two more conventions are listed next.

- (1) The symbol $X * Y$ denotes the join of two topological spaces X and Y . If X and Y are pointed spaces of the homotopy type of a CW-complex, then $X * Y$ has the homotopy type of the suspension $\Sigma(X \wedge Y)$.
- (2) Given the family of pairs $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ and $I \subseteq [m]$, define

$$(\underline{X}_I, \underline{A}_I) = \{(X_{i_j}, A_{i_j})\}_{j=1}^{|I|}$$

which is the subfamily of $(\underline{X}, \underline{A})$ determined by I .

Standard constructions for simplicial complexes and associated posets are recalled next.

Definition 2.6. Let K denote a simplicial complex with m vertices.

- (1) Recall that the empty simplex \emptyset is required to be in K .
 - (2) Given a sequence $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$, define $K_I \subseteq K$ to be the *full subcomplex* of K consisting of all simplices of K which have all of their vertices in I , that is $K_I = \{\sigma \cap I \mid \sigma \in K\}$.
 - (3) Let $|K|$ denote the geometric realization of the simplicial complex K .
 - (4) Associated to a simplicial complex K , there is a partially ordered set (poset) \overline{K} given as follows. A point σ in \overline{K} corresponds to a simplex $\sigma \in K$ with order given by *reverse* inclusion of simplices. Thus $\sigma_1 \leq \sigma_2$ in \overline{K} if and only if $\sigma_2 \subseteq \sigma_1$ in K . The empty simplex \emptyset is the unique maximal element of \overline{K} .
- Let P be a poset with $p \in P$. There are further posets given by

$$P_{\leq p} = \{q \in P \mid q \leq p\}$$

as well as

$$P_{< p} = \{q \in P \mid q < p\}.$$

Thus

$$\overline{K}_{< \sigma} = \{\tau \in \overline{K} \mid \tau < \sigma\} = \{\tau \in K \mid \tau \supset \sigma, \tau \neq \sigma\}.$$

On the other hand, given a poset P , there is an associated simplicial complex $\Delta(P)$ called the order complex of P which is defined as follows.

Definition 2.7. Given a poset P , the *order complex* $\Delta(P)$ is the simplicial complex with vertices given by the set of points of P and k -simplices given by the ordered $(k+1)$ -tuples $(p_1, p_2, \dots, p_{k+1})$ in P with $p_1 < p_2 < \dots < p_{k+1}$. It follows that $\Delta(\bar{K}) = \text{cone}(K')$ where K' denotes the barycentric subdivision of K .

The following decomposition is well-known [36,27] for which \bigvee denotes the wedge, and $\Sigma(X)$ denotes the reduced suspension of X .

Theorem 2.8. Let (Y_i, y_i) be pointed, connected CW-complexes. There is a pointed, natural homotopy equivalence

$$H: \Sigma(Y_1 \times \dots \times Y_m) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{Y}^I\right)$$

where I runs over all the non-empty subsequences of $(1, 2, \dots, m)$. Furthermore, the map H commutes with colimits.

Remark 2.9. The natural homotopy equivalence $H: \Sigma(Y_1 \times \dots \times Y_m) \rightarrow \Sigma(\bigvee_{I \subseteq [m]} \widehat{Y}^I)$ in Theorem 2.8 is defined and used in the proof of Theorem 2.10 below.

Recall from Definition 2.5 that $(\underline{X}_I, \underline{A}_I)$ denotes the subcollection of $(\underline{X}, \underline{A})$ determined by I . An application of Theorem 2.8 yields the following splitting theorem.

Theorem 2.10. Let K be an abstract simplicial complex with m vertices. Given $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ where (X_i, A_i, x_i) are pointed triples of CW-complexes for which X_i , and A_i are connected for all i , the homotopy equivalence of Theorem 2.8 induces a natural pointed homotopy equivalence

$$H: \Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X}_I, \underline{A}_I))\right).$$

Remark 2.11. The spaces $Z(K; (\underline{X}, \underline{A}))$ generally do not decompose as a wedge before suspending. One example is given where K is the simplicial complex determined by a square, with 4 vertices and 4 edges, and where $Z(K; (D^2, S^1))$ is $S^3 \times S^3$.

The next result is a determination of the homotopy type of the $\widehat{Z}(K; (\underline{X}, \underline{A}))$ in case the inclusions $A_i \hookrightarrow X_i$ are null-homotopic for every $i \in [m]$.

Theorem 2.12. Let K be an abstract simplicial complex with m vertices and \bar{K} its associated poset. Let $(\underline{X}, \underline{A})$ have the property that the inclusion $A_i \subset X_i$ is null-homotopic for all i . Then there is a homotopy equivalence

$$\widehat{Z}(K; (\underline{X}, \underline{A})) \rightarrow \bigvee_{\sigma \in K} |\Delta(\bar{K}_{<\sigma})| * \widehat{D}(\sigma).$$

This theorem is a generalization of the wedge lemma of Welker, Ziegler and Živaljević in [49], applied to generalized moment-angle complexes. Substituting Theorem 2.12 in Theorem 2.10 yields the next result.

Theorem 2.13. *Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that the inclusion $A_i \subset X_i$ is null-homotopic for all i . Then there is a homotopy equivalence*

$$\Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_I \left(\bigvee_{\sigma \in K_I} |\Delta((\bar{K}_I)_{<\sigma})| * \widehat{D}(\sigma)\right)\right).$$

Two special cases of Theorem 2.13 are presented next where either (i) the A_i are contractible for all i or (ii) the X_i are contractible for all i . In the case for which all A_i are contractible, the decomposition given in Theorem 2.10 has the property that $\widehat{D}(\sigma)$ is contractible for all $\sigma \neq [m]$. The one remaining summand in Theorem 2.10 occurs for $\sigma = [m]$ and thus $\widehat{D}(\sigma) = \widehat{X}^{[m]}$ as given in Definition 2.5. The next result follows at once.

Theorem 2.14. *Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence*

$$\widehat{Z}(K; (\underline{X}, \underline{A})) = \begin{cases} * & \text{if } K \text{ is not the simplex } \Delta[m-1], \text{ and} \\ \widehat{X}^{[m]} & \text{if } K \text{ is the simplex } \Delta[m-1]. \end{cases}$$

Notice that the spaces $\widehat{Z}(K_I; (\underline{X}_I, \underline{A}_I))$ of Theorem 2.10 are all contractible unless K_I is a simplex of K by Theorem 2.14. The simplices of K have been identified with certain increasing sequences $I \subseteq [m]$. Thus, the next result follows immediately from Theorems 2.10 and 2.14.

Theorem 2.15. *Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence*

$$\Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_{I \in K} \widehat{X}^I\right).$$

In the situation where $(X_i, A_i, x_i) = (X, A, x_0)$ for all i , the next result follows at once.

Corollary 2.16. *If (1) $X_i = X$ and $A_i = A$ for all i are connected CW-complexes with A contractible, and (2) K_1 and K_2 are simplicial complexes both having m vertices as well as the same number of simplices in every dimension, then there is a homotopy equivalence*

$$\Sigma(Z(K_1; (X, A))) \rightarrow \Sigma(Z(K_2; (X, A))).$$

Remark 2.17. The condition in Corollary 2.16 that two simplicial complexes K_1 and K_2 both have m vertices as well as the same number of simplices in every dimension is the definition of having the same f -vectors.

Let $\bar{P}(Y)$ denote the reduced Poincaré series for a finite, connected CW-complex.

Corollary 2.18. *If (X, A, x) is a triple of pointed finite CW-complexes for which X is connected, A is contractible and $|K|$ is connected, then*

$$\overline{P}(Z(K; (X, A))) = \sum_{k=0}^n f_k(\overline{P}(X))^{k+1}$$

where n is the dimension of K , and f_k is the number of k -simplices of K .

When specialized to $(\underline{X}, \underline{A})$ for which the spaces X_i are contractible, there is a precise identification of $\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X}_I, \underline{A}_I))$ obtained by appealing to the “Wedge Lemma” in work of Welker, Ziegler and Živaljević in [49] on homotopy colimits of spaces. The next theorem addresses the case for which all of the X_i are contractible. Recall the join of two spaces $X * Y$ as well as the notation $\widehat{A}^{[m]} = A_1 \wedge \cdots \wedge A_m$ from Definition 2.5.

Theorem 2.19. *Let K be an abstract simplicial complex with m vertices. Let*

$$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$$

denote m choices of triples of pointed CW-complexes for which X_i , and A_i are connected for all i . If all of the X_i are contractible, then there are homotopy equivalences

$$\widehat{Z}(K; (\underline{X}, \underline{A})) \rightarrow |K| * \widehat{A}^{[m]} \rightarrow \Sigma(|K| \wedge \widehat{A}^{[m]}).$$

Since all of the X_i are contractible, all of the $\widehat{D}(\sigma)$ are contractible with the possible exception of $\widehat{D}(\emptyset) = A_1 \wedge \cdots \wedge A_m$. Since the element \emptyset is the maximal element under reverse inclusion in \overline{K} , the order complex $\Delta(\overline{K}_{<\emptyset})$ is equal to the barycentric subdivision of K , denoted by K' . Furthermore, there are homeomorphisms on the level of geometric realizations $|\Delta(\overline{K})| \rightarrow |K'| \rightarrow |K|$.

Remark 2.20. For the case $(X, A) = (D^2, S^1)$ and K a simplicial complex with m vertices, David Stone [44] has shown that the homotopy equivalence

$$\widehat{Z}(K; (D^2, S^1)) \rightarrow \Sigma^{m+1}|K|$$

given by Theorem 2.19 extends to a homeomorphism. He does this by “linearizing” a smash product of discs using joins. This result may be interpreted as giving, for any simplicial complex K on m vertices, a model of $\Sigma^{m+1}|K|$ inside an m -fold smash product of two-discs which preserves the combinatorial structure of K in a natural way.

The next result is now a consequence of Theorems 2.10 and 2.19. The result becomes clear on noticing that whenever $I \subseteq [m]$ corresponds to a simplex in K , then $|K_I|$ is contractible.

Theorem 2.21. *Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that the X_i are contractible for all i . Then there is a homotopy equivalence*

$$\Sigma Z(K; (\underline{X}, \underline{A})) \rightarrow \Sigma \left(\bigvee_{I \notin K} |K_I| * \widehat{A}^I \right).$$

Remark 2.22. An important case of Theorem 2.21 is given by setting X_i equal to the cone on A_i .

Corollary 2.23. Let (X_i, A_i, x_i) denote the triple $(D^2, S^1, *)$ for all i . Then there are homotopy equivalences

$$\Sigma(Z(K; (D^2, S^1))) \rightarrow \Sigma\left(\bigvee_{I \notin K} |K_I| * S^{|I|}\right) \rightarrow \bigvee_{I \notin K} \Sigma^{2+|I|} |K_I|.$$

An analogous consequence is given next.

Corollary 2.24. Let (X_i, A_i, x_i) denote the triple $(D^{n+1}, S^n, *)$ for all i , $n \geq 1$. Then there are homotopy equivalences

$$\Sigma(Z(K; (D^{n+1}, S^n))) \rightarrow \Sigma\left(\bigvee_{I \notin K} |K_I| * S^{n+|I|}\right) \rightarrow \bigvee_{I \notin K} \Sigma^{2+n+|I|} |K_I|.$$

Remark 2.25.

- (1) A simplicial complex K determines the complement of a complex coordinate subspace arrangement as outlined in [9]. This space is $Z(K; (\mathbb{C}, \mathbb{C}^*))$ which has $Z(K; (D^2, S^1))$ as a strong deformation retract [9]. Similarly, $Z(K; (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} - \{0\}))$ is the complement of a certain arrangement of real subspaces having $Z(K; (D^{n+1}, S^n))$ as a strong deformation retract.
- (2) The associated additive decomposition for singular homology given in Corollary 2.23 is due Goresky and MacPherson [22] via different methods. The work of Hochster [25] in combination with work of Buchstaber and Panov [9] as described in the Introduction implied analogous homological decompositions.
- (3) In addition, Goresky and MacPherson [22] gave a more general homological decomposition for complements of complex arrangements which do not follow from the decompositions above.
- (4) The cohomology ring structure for $Z(K; (D^2, S^1))$ does not follow from the above Corollary 2.23. The cohomology ring structure was studied rationally by Buchstaber and Panov in [9] and integrally by Franz in [18] and [19] as well as Baskakov, Buchstaber and Panov in [7].

Shifted simplicial complexes are defined next.

Definition 2.26. A simplicial complex K on m vertices is *shifted* if there exists a labeling of the vertices by 1 through m such that for any simplex of K , replacing any vertex of that simplex with a vertex of smaller label and not in that simplex results in a collection which is also a simplex. For a shifted complex K , the geometric realization of K and every K_I is homotopy equivalent to a wedge of spheres.

The next corollary follows.

Corollary 2.27. If K is a shifted complex, then $\Sigma Z(K; (D^{n+1}, S^n))$ is homotopy equivalent to a wedge of spheres.

Remark 2.28. In the special case for which K is a shifted complex, a stronger result than that of Corollary 2.27 was proven by Grbic and Theriault [23]. They prove that $Z(K; (D^2, S^1))$ is homotopy equivalent to a wedge of spheres without suspending.

Let $(\underline{CX}, \underline{X}) = \{(CX_i, X_i, x_i)\}_{i=1}^m$, where CX_i denotes the reduced cone on X_i . The theorems of Porter [41] and Grbic and Theriault [23] support the following.

Conjecture 2.29. *If K is a shifted complex, then*

$$Z(K; (\underline{CX}, \underline{X})) \quad \text{and} \quad \bigvee_{I \neq K} |K_I| * \widehat{X}^I$$

are of the same homotopy type.

Denham and Suciu prove an elegant Lemma 2.9 in [17] which relates fibrations and moment-angle complexes. That lemma is stated next.

Lemma 2.30. *Let $p: (E, E') \rightarrow (B, B')$ be a map of pairs, such that both $p: E \rightarrow B$ and $p|_{E'}: E' \rightarrow B'$ are fibrations, with fibers F and F' , respectively. Suppose that either $F = F'$ or $B = B'$. Then the product fibration, $p^n: E^n \rightarrow B^n$, restricts to a fibration $Z(K; (F, F')) \rightarrow Z(K; (E, E')) \rightarrow Z(K; (B, B'))$. Moreover, if $(F, F') \rightarrow (E, E') \rightarrow (B, B')$ is a relative bundle (with structure group G), then the above bundle has structure group G^n .*

Notice that Lemma 2.30 extends easily to generalized moment-angle complexes as follows (with details of proof omitted).

Lemma 2.31. *Let $p_i: (E_i, E'_i) \rightarrow (B_i, B'_i)$ be a map of pairs, such that $p: E_i \rightarrow B_i$ and $p|_{E'_i}: E'_i \rightarrow B'_i$ are fibrations with fibers F_i and F'_i respectively. Let K be a simplicial complex with m vertices.*

- (1) *If $B_i = B'_i$ for all i then, $Z(K; (\underline{B}, \underline{B}')) = B_1 \times \cdots \times B_m$.*
- (2) *If $F_i = F'_i$ for all i , then $Z(K; (\underline{F}, \underline{F}')) = F_1 \times \cdots \times F_m$.*
- (3) *In either of these two cases, the following is a fibration:*

$$Z(K; (\underline{F}, \underline{F}')) \rightarrow Z(K; (\underline{E}, \underline{E}')) \rightarrow Z(K; (\underline{B}, \underline{B}')).$$

Natural consequences of Lemma 2.31 arose earlier in the work of G. Porter [41], T. Ganea [21] and A. Kurosh [31] as discussed next. Porter considered the cases $\{(X_i, *_i, *_i)\}_{i=1}^m$. In this case, consider the natural filtration of the product $X_1 \times \cdots \times X_m$ as defined by G.W. Whitehead with j th-filtration given by the space

$$W_j \left(\prod_{1 \leq i \leq m} X_i \right) = \{(y_1, \dots, y_m) \mid y_i = *_i, \text{ the base-point of } X_i \text{ for at least } m - j \text{ values of } i\}.$$

The subspace $W_j(\prod_{1 \leq i \leq m} X_i)$ of the product is precisely a choice of a generalized moment-angle complex as follows.

Let $\Delta[m-1]_q$ denote the q -skeleton of $\Delta[m-1]$, that is $\Delta[m-1]_q = \{I \subset [m] \mid |I| \leq q+1\}$. Then

$$W_j\left(\prod X_i\right) = Z(\Delta[m-1]_{j-1}; (\underline{X}, *)).$$

In this case, Porter's result gives an identification of the homotopy theoretic fiber of the inclusion

$$W_j\left(\prod_{1 \leq i \leq m} X_i\right) \subset \prod_{1 \leq i \leq m} X_i,$$

a result which follows directly from the classical path-loop fibration over X and Lemma 2.31, features elucidated next.

Two standard constructions are used here. Recall that the path-space PX of a pointed space $(X, *)$ is the space of pointed continuous maps $f: [0, 1] \rightarrow X$ with $f(0) = *$, the base-point of X . The evaluation map

$$e_1: PX \rightarrow X$$

is defined by $e_1(f) = f(1)$ and is a fibration with fiber ΩX the space of continuous maps $f: [0, 1] \rightarrow X$ with $f(0) = * = f(1)$. Observe that there is a fibration of pairs

$$(\Omega X, \Omega X) \rightarrow (PX, \Omega X) \xrightarrow{e_1} (X, *).$$

Next consider the (unreduced) cone CY over a space Y defined as

$$C(Y) = [0, 1] \times Y / \approx,$$

the quotient obtained from the equivalence relation $(1, a) \approx (1, b)$ for all $a, b \in Y$ with equivalence classes denoted $[t, y]$. Observe that there is a map

$$\kappa: X \rightarrow X \times C(PX)$$

with

$$\kappa(x) = (x, [0, f_x])$$

where $f_x: [0, 1] \rightarrow X$ with $f_x(t) = x$. The map κ is evidently a homotopy equivalence.

Consider the pair

$$(X \times C(PX), PX)$$

for which PX is the subspace of $X \times C(PX)$ given by pairs $(f(1), [0, f]) \in X \times C(PX)$. Observe that there is a fibration of pairs

$$(C(PX), \Omega X) \rightarrow (X \times C(PX), PX) \xrightarrow{\pi_X \times e_1} (X, X)$$

for which $\pi_X : X \times C(PX) \rightarrow X$ is the natural projection. Also, observe that there is an equivalence of pairs

$$(X, *) \rightarrow (X \times C(PX), PX).$$

Thus by Lemma 2.31, there is a fibration

$$Z(K; (\underline{PX}, \underline{\Omega X})) \rightarrow Z(K; (\underline{X} \times \underline{PX}, \underline{PX})) \xrightarrow{\pi_X \times e_1} Z(K; (\underline{X}, \underline{X}))$$

as well as the associated fibration (up to homotopy)

$$Z(K; (\underline{PX}, \underline{\Omega X})) \rightarrow Z(K; (\underline{X}, *)) \xrightarrow{\pi_X \times e_1} Z(K; (\underline{X}, \underline{X}))$$

a remark recorded as the next corollary. Recall that $X^{[m]}$ denotes the product $\prod_{i=1}^m X_i$.

Corollary 2.32. *If all of the X_i are path-connected, the homotopy theoretic fiber of the inclusion $Z(K; (\underline{X}, *)) \subset Z(K; (\underline{X}, \underline{X}))$ is $Z(K; (\underline{PX}, \underline{\Omega X}))$. Thus, the homotopy theoretic fiber of the inclusion $W_j(X^{[m]}) \subset X^{[m]}$ is*

$$Z(\Delta[m-1]_{j-1}; (\underline{PX}, \underline{\Omega X})).$$

For a more general version of this result, see [40], Proposition 5.1.

A second theorem of Porter [41] gives the structure of the fiber $Z(\Delta[m-1]_{j-1}; (\underline{PX}, \underline{\Omega X}))$. After one suspension, this result also follows from the next theorem given by a decomposition of the suspension of $Z(\Delta[m-1]_{q-1}; (\underline{CY}, \underline{Y}))$ where CY denotes the cone over a connected CW-complex Y . Let $(\underline{CY}, \underline{Y}) = \{(CY_i, Y_i, x_i)\}_{i=1}^m$ be a family of connected, pointed CW-pairs and recall that \widehat{Y}^I is $Y_{i_1} \wedge \cdots \wedge Y_{i_k}$ if $I = (i_1, \dots, i_k)$. The next result follows from Theorem 2.19.

Theorem 2.33. *Let $(\underline{CY}, \underline{Y}) = \{(CY_i, Y_i, x_i)\}_{i=1}^m$ be a family of connected, pointed CW-pairs. Then there is a homotopy equivalence*

$$\Sigma(Z(\Delta[m-1]_q; (\underline{CY}, \underline{Y}))) \rightarrow \Sigma\left(\bigvee_{|I|>q+1} \bigvee_{t_I} (\Sigma^{|I|+1} \widehat{Y}^I)\right)$$

where t_I is the binomial coefficient $\binom{|I|+1}{q+1}$.

When $q = m-2$, $Z(\Delta[m-1]_{m-2}; (\underline{X}, *))$ is the fat wedge $W_{m-1}(IX_i)$ and the homotopy fiber is $Z(\Delta[m-1]_{m-2}; (\underline{PX}, \underline{\Omega X}))$. In this case, it follows that there is a homotopy equivalence

$$\Sigma(Z(\Delta[m-1]_{m-2}; (\underline{PX}, \underline{\Omega X}))) \rightarrow \Sigma(\Omega X_1 * \cdots * \Omega X_m).$$

T. Ganea [21] had identified the homotopy theoretic fiber of the natural inclusion

$$X \vee Y \rightarrow X \times Y$$

as the join $\Omega(X) * \Omega(Y)$ in case X and Y are path-connected, pointed spaces of the homotopy type of a CW-complex. This example is a special case of the homotopy theoretic fiber for the

inclusion of a generalized moment-angle complex in a product corresponding to the simplicial complex K given by two disjoint points.

For the case that each A_i is contractible, the next result gives a determination of the cohomology ring of $Z(K; (\underline{X}, \underline{A}))$ under certain conditions.

Definition 2.34. Let R be a ring.

- (1) A finite set of spaces X_1, \dots, X_m is said to satisfy the *strong form* of the Künneth theorem over R provided that the natural map

$$\bigotimes_{1 \leq j \leq k} H^*(X_{i_j}; R) \rightarrow H^*(X_{i_1} \times \cdots \times X_{i_k}; R)$$

is an isomorphism for every sequence of integers $1 \leq i_1 < i_2 < \cdots < i_k \leq m$.

- (2) A finite set of path-connected spaces X_1, \dots, X_m is said to satisfy the *strong smash form* of the Künneth theorem over R provided that the natural map

$$\bigotimes_{1 \leq j \leq k} \bar{H}^*(X_{i_j}; R) \rightarrow \bar{H}^*(X_{i_1} \wedge \cdots \wedge X_{i_k}; R)$$

is an isomorphism for every sequence of integers $1 \leq i_1 < i_2 < \cdots < i_k \leq m$.

Observe that if a finite set of path-connected spaces X_1, \dots, X_m satisfies the strong form of the Künneth theorem over R , then they satisfy the strong smash form of the Künneth theorem over R . A similar result holds for any cohomology theory. Examples in the case $R = \mathbb{Z}$ are given by spaces X_i of finite type which have torsion-free cohomology over \mathbb{Z} .

Notice that there is a natural inclusion $j: Z(K; (\underline{X}, \underline{A})) \subset \prod_{i=1}^m X_i$. Theorem 2.15 implies that

$$j^*: H^*\left(\prod X_i; R\right) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})); R)$$

is onto and that the kernel of j^* is defined to be the *generalized Stanley–Reisner ideal* $I(K)$ which is generated by all elements $x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_l}$ for which $x_{j_i} \in \bar{H}^*(X_{j_i}; R)$ and the sequence $J = (j_1, \dots, j_l)$ is not a simplex of K . This construction provides a useful extension of the Stanley–Reisner ring [14] as stated next.

Theorem 2.35. Let K be an abstract simplicial complex with m vertices and let

$$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$$

be m pointed, connected CW-pairs. If all of the A_i are contractible and coefficients are taken in a ring R for which the spaces X_1, \dots, X_m satisfy the strong form of the Künneth theorem over R , then there is an isomorphism of algebras

$$\bigotimes_{i=1}^m H^*(X_i; R)/I(K) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})); R).$$

Furthermore, there are isomorphisms of underlying abelian groups given by

$$E^*(Z(K; (\underline{X}, \underline{A}))) \rightarrow \bigoplus_{I \in K} E^*(\widehat{X}^I)$$

for any reduced cohomology theory E^* .

Remark 2.36. An analogous result for the cohomology ring structure holds for any cohomology theory $E^*(-)$ for which X_1, \dots, X_m satisfy the analog of the strong form of the Künneth theorem with respect to $E^*(-)$ [3].

Denham and Suciu [17] consider an action of a topological group G on the pair (X, A) together with the natural induced action of G^m on $Z(K; (X, A))$ where K has m vertices. The associated Borel construction (homotopy orbit space) is

$$EG^m \times_{G^m} Z(K; (X, A)).$$

The applications here will be restricted to topological groups G which are CW-complexes. Regard G as a subspace of EG given by the orbit of a point corresponding to the identity in G . Denham and Suciu proved the following using their Lemma 2.10 [17].

Theorem 2.37. *The space $Z(K; (BG, *))$ is homotopy equivalent to the homotopy orbit space*

$$EG^m \times_{G^m} Z(K; (EG, G)).$$

Furthermore, the natural projection

$$EG^m \times_{G^m} Z(K; (EG, G)) \rightarrow BG^m$$

is a fibration with fiber

$$Z(K; (EG, G)).$$

Recall the T^m -action on $Z(K; (D^2, S^1))$ induced by the natural S^1 -action on the pair (D^2, S^1) .

Definition 2.38. Define the Davis–Januszkiewicz space

$$\mathcal{DJ}(K) = ET^m \times_{T^m} Z(K; (D^2, S^1)) \approx Z(K; (BS^1, *)).$$

The space $\mathcal{DJ}(K)$ is an example of the construction addressed above by Denham and Suciu [17]. The cohomology ring of $\mathcal{DJ}(K)$ was computed first in work of Davis and Januszkiewicz [14] as well as work of Buchstaber and Panov [9]. An elegant proof was given in work of Denham and Suciu [17].

To state their results, let $I = (i_1, \dots, i_k)$ denote a simplex in $\Delta[m-1]$. Then, let $x_I = x_{i_1} \dots x_{i_k}$ for which the x_i , $1 \leq i \leq m$, are the natural choice of algebra generators for the integral cohomology ring of BT^m .

Theorem 2.39. *The integral cohomology ring of the Davis–Januszkiewicz space*

$$\mathcal{DJ}(K) = Z(K; (BS^1, *))$$

is isomorphic as an algebra to

$$\mathbb{Z}[x_1, \dots, x_m]/I(K)$$

where $I(K)$ denotes the ideal generated by the elements $x_I = x_{i_1} \dots x_{i_k}$ for which $I = (i_1, \dots, i_k)$ is not a simplex in K . In particular, $\mathbb{Z}[x_1, \dots, x_m]/I(K)$ is the Stanley–Reisner ring of K .

Remark 2.40. Theorem 2.39 is a special case of Theorem 2.35 where $(X_i, x_i) = (BS^1, *)$.

The decompositions for the suspensions of generalized moment-angle complexes given in Theorem 2.10 specialize to decompositions for $Z(K; (BG, *))$ as follows.

Corollary 2.41. *There is a homotopy equivalence*

$$\Sigma(Z(K; (BG, *))) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (BG, *))\right).$$

Thus in case $G = T = S^1$, there is an additive decomposition of the Stanley–Reisner ring given by the cohomology algebra of $\mathcal{DJ}(K) = ET^m \times_{T^m} Z(K; (D^2, S^1))$ together with an isomorphism of underlying abelian groups

$$H^i(\mathcal{DJ}(K); \mathbb{Z}) \rightarrow H^i\left(\bigvee_{I \in K} \widehat{(BS^1)^I}; \mathbb{Z}\right).$$

Furthermore, there is an isomorphism

$$E_*(\mathcal{DJ}(K); \mathbb{Z}) \rightarrow E_*\left(\bigvee_{I \in K} \widehat{(BS^1)^I}\right)$$

for any homology theory E_ .*

Remark 2.42. An analogous algebraic decomposition was given by Björner and Sarkaria in [8] for the special case of (D^2, S^1) and singular cohomology with integer coefficients. The decomposition in Theorem 2.35 is also closed with respect to the Steenrod operations in mod- p cohomology arising from the geometric splitting of the suspension. It is not apparent that the decomposition of Björner and Sarkaria after reduction mod- p preserves the action of the mod- p Steenrod algebra.

Some features of generalized moment-angle complexes together with their stable decompositions in Theorem 2.10 extend directly to simplicial spaces X_* and their geometric realizations denoted $|X_*|$. To do so, some notation is given next by analogy with Definition 2.1.

Definition 2.43. Let $(\underline{X}_*, \underline{A}_*)$ denote the collection of pairs of simplicial spaces

$$\{(X_*(i), A_*(i))\}_{i=1}^m$$

and let $(\underline{X}_n, \underline{A}_n)$ denote the collection of pairs $\{(X_n(i), A_n(i))\}_{i=1}^m$ where $X_n(i)$ denotes the n th space in $X_*(i)$. Define

$$Z(K; (\underline{X}_*, \underline{A}_*))_n = Z(K; (\underline{X}_n, \underline{A}_n)).$$

The next lemma gives the property that the collection of spaces $Z(K; (\underline{X}_*, \underline{A}_*))_n$ for all $n \geq 0$ is naturally a simplicial space which is denoted by $Z(K; (\underline{X}_*, \underline{A}_*))$.

Lemma 2.44. *The natural inclusions*

$$i_n : Z(K; (\underline{X}_*, \underline{A}_*))_n \rightarrow Z(K; (\underline{X}_n, \underline{A}_n)) \rightarrow X_n(1) \times \cdots \times X_n(m)$$

are closed with respect to the face and degeneracy operations in the product

$$X_*(1) \times \cdots \times X_*(m).$$

That is, the following properties are satisfied:

- (1) $d_i : Z(K; (\underline{X}_*, \underline{A}_*))_n \rightarrow Z(K; (\underline{X}_*, \underline{A}_*))_{n-1}$ and
- (2) $s_j : Z(K; (\underline{X}_*, \underline{A}_*))_n \rightarrow Z(K; (\underline{X}_*, \underline{A}_*))_{n+1}$.

Hence

- (1) $s_j(x) \in A_{n+1}(i)$ if and only if $x \in A_n(i)$, and
- (2) *the natural inclusion gives a map of simplicial space*

$$i_* : Z(K; (\underline{X}_*, \underline{A}_*)) \rightarrow X_*(1) \times \cdots \times X_*(m).$$

The geometric realization of a simplicial space X_* is recalled next for convenience of the reader.

Definition 2.45. The geometric realization $|X_*|$ is the quotient

$$\bigcup_{0 \leq i} \Delta[i] \times X_i / (\sim)$$

where $\Delta[i]$ denotes the i -simplex and \sim is the equivalence relation generated by

- (1) $(u, d_i x) \sim (\partial_i u, x)$ for $u \in \Delta[r-1]$ and $x \in X_r$, and
- (2) $(\sigma_j(v), y) \sim (v, s_j(x))$ for $v \in \Delta[r+1]$ and $y \in X_r$.

Let $[u, x]$ in $|X_*|$ denote the equivalence class of the pair (u, x) in $\Delta[r] \times X_r$.

Given the collection of pairs of simplicial spaces

$$(\underline{X}_*, \underline{A}_*) = \{(X_*(i), A_*(i))\}_{i=1}^m,$$

let $(|\underline{X}_*|, |\underline{A}_*|)$ denote the collection of spaces $\{(|X_*(i)|, |A_*(i)|)\}_{i=1}^m$ where $|X_*(i)|$ and $|A_*(i)|$ denote the respective geometric realizations.

It is a classical fact [37] that there are homeomorphisms

$$\pi : |X_*(1) \times \cdots \times X_*(m)| \rightarrow |X_*(1)| \times \cdots \times |X_*(m)|$$

for simplicial spaces $X_*(i)$ for $1 \leq i \leq m$ in case either of the following are satisfied.

- (1) Each $X_*(i)$ is countable, or
- (2) the spaces $|X_*(i)|$ are locally finite.

A variation in the underlying point-set topology is given in [34] where it suffices to assume that all spaces are compactly generated and weak Hausdorff.

The homeomorphism π is induced by the product of the natural projection maps

$$\pi_j : X_*(1) \times \cdots \times X_*(m) \rightarrow X_*(j)$$

which gives a map

$$|\pi_j| : |X_*(1) \times \cdots \times X_*(m)| \rightarrow |X_*(j)|.$$

The inverse of $\pi = |\pi_1| \times \cdots \times |\pi_m|$

$$\eta : \prod_{1 \leq i \leq m} |X_*(i)| \rightarrow \left| \prod_{1 \leq i \leq m} X_*(i) \right|$$

is induced by sending the class of $\prod_{1 \leq i \leq m} |u_i, x_i|$ to $|w, \prod_{1 \leq i \leq m} s_{J_i}(x_i)|$ where w and the possibly iterated degeneracies $s_{J_i}(x_i)$ are specified in [37].

Since the natural inclusion $i_n : Z(K; (X_n, A_n)) \rightarrow X_n(1) \times \cdots \times X_n(m)$ is a monomorphism of simplicial spaces by Lemma 2.44, the realization

$$|Z(K; (\underline{X}_*, \underline{A}_*))|$$

is a subspace of $|X_*(1) \times \cdots \times X_*(m)|$. Furthermore, let

$$i : Z(K; (|\underline{X}_*|, |\underline{A}_*|)) \rightarrow |X_*(1)| \times \cdots \times |X_*(m)|$$

denote the natural inclusion. The following statement is proven by repeating a proof in [37].

Theorem 2.46. *There is a commutative diagram*

$$\begin{array}{ccc} Z(K; (|\underline{X}|_*, |\underline{A}|_*)) & \xrightarrow{h} & |Z(K; (\underline{X}_*, \underline{A}_*))| \\ \downarrow i & & \downarrow |i_*| \\ |X_*(1)| \times \cdots \times |X_*(m)| & \xrightarrow{\eta} & |X_*(1) \times \cdots \times X_*(m)| \end{array}$$

where h is the restriction of η ,

$$h = \eta|_{|Z(K; \underline{X}_*)|}.$$

Furthermore, if either

- (1) the simplicial spaces $X_*(i)$ are countable,
- (2) the spaces $|X_*(i)|$ are locally finite, or
- (3) the spaces $X_n(i)$ are assumed to be compactly generated and weak Hausdorff,

then the map

$$h : Z(K; (|\underline{X}|_*, |\underline{A}|_*)) \rightarrow |Z(K; (\underline{X}_*, \underline{A}_*))|$$

is a homeomorphism.

3. Toric manifolds

This section is a brief introduction to properties of toric manifolds with an emphasis on relationships with moment-angle complexes $Z(K; (D^2, S^1))$. Following Davis and Januszkiewicz [14] and Buchstaber and Panov [9] define a toric manifold starting with T^n the real n -torus

$$T^n = S^1 \times \cdots \times S^1 \quad (n \text{ factors}).$$

Notice that T^n acts on \mathbb{C}^n , n -space by:

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, t_2 z_2, \dots, t_n z_n)$$

where $t_i z_i$ is the natural multiplication of two complex numbers. The quotient space \mathbb{C}^n / T^n is homeomorphic to

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}, x_i \geq 0\}.$$

Let M^{2n} be a $2n$ -manifold which has an action of T^n . Say that T^n acts on M^{2n} with *locally standard action* if (i) every point $x \in M^{2n}$ has a coordinate neighborhood U_x invariant under the action of T^n , (ii) there is a homeomorphism $\varphi_x : U_x \rightarrow V_x \subset \mathbb{R}^{2n} = \mathbb{C}^n$ with V_x open, and (iii) for every $t \in T^n$ and $y \in U_x$, the following equality

$$\varphi_x(t \cdot y) = \alpha_x(t) \cdot \varphi_x(y)$$

holds, where α_x is an automorphism of T^n with the standard action on the right-hand side. It follows that M^{2n}/T^n is the union of spaces \mathbb{R}_+^n . Recall that a convex polytope P^n of dimension n is said to be *simple* if every vertex of P is the intersection of exactly n facets [10]. Say that M^{2n} is a *toric manifold over P^n* , if the orbit space is homeomorphic to a simple n -polytope P^n in such a way that the isotropy subgroup is an $(n - k)$ -torus on the inverse image of the interior of each k -face.

Recall that P^n can be described as the intersection of m half-spaces. Notice that P^n has faces of all codimensions from 1 to n . Assume that there are m codimension-1 faces, F_1, \dots, F_m . Each non-empty intersection $F_{i_0} \cap \dots \cap F_{i_k} \neq \emptyset$ for $1 \leq i_0 < i_1 < \dots < i_k$ is a codimension k -face.

Let $\sigma = (i_0, \dots, i_k)$ denote an increasing sequence in $[m]$. Consider the intersection defined by $F_\sigma = F_{i_0} \cap \dots \cap F_{i_k}$ which will be assumed to be non-empty. Then the set of all such σ for which F_σ is not empty defines a simplicial complex K (which contains the empty set by definition). If K contains an element σ which is not the empty sequence, then K is the dual complex of the boundary complex of the simple convex polytope P^n . Moreover, $|K| = S^{n-1}$. Note that this particular K is denoted K_P in [14] and [10].

Recall that T^n acts on M^{2n} with isotropy specified below. Let $\pi: M^{2n} \rightarrow P^n$ denote the quotient map. Observe that T^n acts on $\pi^{-1}(\text{Int}(F_\sigma))$ with constant isotropy subgroup, denoted $\lambda(F_\sigma)$, a $(k + 1)$ -subtorus of T^n , where $k + 1 = |\sigma| = \text{length of } \sigma$ and $\text{Int}(F_\sigma)$ denotes the interior of F_σ .

In particular T^n acts freely on $\pi^{-1}(\text{Int}(P^n))$. However, T^n acting on M^{2n} has fixed points which are the inverse images of the vertices of P^n . The 1-dimensional tori $\lambda(F_i)$ are represented uniquely up signs in entries of the vector by

$$\lambda(F_i) = \{(e^{2\pi i \lambda_{i1}}, \dots, e^{2\pi i \lambda_{in}})\}$$

for $i = 1, \dots, m$, where $\lambda_i = (\lambda_{i1}, \dots, \lambda_{in})$ is a primitive vector of \mathbb{Z}^n , that is λ_i can be extended to a basis for \mathbb{Z}^n . Since P^n is a simple polytope, every face is given by a unique intersection of codimension-1 faces. Thus, the vectors $(\lambda_{i1}, \dots, \lambda_{ik})$ are linearly independent for $k \leq n$ if F_i is a face of P^n .

Consider the family of codimension-2 submanifolds $M(i) = \pi^{-1}(F_i)$. Define

$$M(\sigma) = M(i_0) \cap \dots \cap M(i_k)$$

where $\sigma = (i_0, \dots, i_k)$. Since this intersection is transverse, $M(\sigma) = \pi^{-1}(F_\sigma)$ is a codimension- $2(k + 1)$ submanifold of M^{2n} .

The $m \times n$ matrix $\wedge = (\lambda_{ij})$ defines an epimorphism

$$\lambda: T^m \rightarrow T^n$$

with the additional property that the group

$$T_\sigma^m = \{(t_1, \dots, t_m) \mid t_i = 1 \text{ if } i \notin \sigma\}$$

is a natural subgroup of T^m and $\lambda|_{T_\sigma^m}: T_\sigma^m \rightarrow \lambda(F_\sigma)$ is an isomorphism. The kernel of λ , denoted $\text{Ker}(\lambda)$, acts freely on $Z(K; (D^2, S^1))$, where K is the dual of ∂P^n and there is a map

$$Z(K; (D^2, S^1))/\text{Ker}(\lambda) \rightarrow M^{2n}$$

which is a homeomorphism. Furthermore, the epimorphism $\lambda: T^m \rightarrow T^n$ is split so that T^m is isomorphic to $\text{Ker}(\lambda) \times T^n$ as a topological group. A direct consequence is recorded next for use below.

Lemma 3.1. *The induced map*

$$\rho: ET^m \times_{T^m} Z(K; (D^2, S^1)) \rightarrow ET^n \times_{T^n} M^{2n}$$

is a homeomorphism.

Proof. Observe that $\lambda: T^m \rightarrow T^n$ induces a principal fibration

$$\text{Ker}(\lambda) \rightarrow Z(K; (D^2, S^1)) \rightarrow M^{2n}.$$

Next, observe that T^m splits as $T^n \times \text{Ker}(\lambda)$ as a topological group. Notice that (i) T^m acts on $Z(K; (D^2, S^1))$, and (ii) the map

$$Z(K; (D^2, S^1))/\text{Ker}(\lambda) \rightarrow M^{2n}$$

is a homeomorphism. Thus T^m acts on $Z(K; (D^2, S^1))$ with

$$ET^m \times_{T^m} Z(K; (D^2, S^1))$$

homeomorphic to

$$ET^n \times_{T^n} Z(K; (D^2, S^1))/\text{Ker}(\lambda).$$

The lemma follows. \square

This setting is clearer with the following equivalent construction of M^{2n} and $Z(K; (D^2, S^1))$ due to Davis and Januszkiewicz in [14]. Here, M^{2n} is the quotient space

$$T^n \times P^n / \sim$$

where \sim is the equivalence relation:

$$(t, x) \sim (t', x') \quad \text{if } x = x' \quad \text{and} \quad t't^{-1} \in \lambda(F_\sigma),$$

where F_σ is the unique face such that $x \in \text{Int}(F_\sigma)$. Now $Z(K; (D^2, S^1)) \approx T^m \times P^n / \sim$, where \sim is the equivalence relation

$$(t, x) \sim (t', x')$$

if $x = x'$ and $t't^{-1} \in T_\sigma^m$ where x lies in the interior of the unique face, F_σ .

Since the map

$$\rho: ET^m \times_{T^m} Z(K; (D^2, S^1)) \rightarrow ET^n \times_{T^n} M^{2n}$$

is a homeomorphism by Lemma 3.1, and is equivariant with respect to the natural T^m -action, it follows that there is a morphism of fibrations (up to homotopy) as given next.

$$\begin{array}{ccccc}
 \text{Ker}(\lambda) & \longrightarrow & E(\text{Ker}(\lambda)) & \longrightarrow & B(\text{Ker}(\lambda)) \\
 \downarrow & & \downarrow & & \downarrow \\
 Z(K; (D^2, S^1)) & \xrightarrow{j_1} & ET^m \times_{T^m} Z(K; (D^2, S^1)) & \xrightarrow{\pi_1} & BT^m \\
 \downarrow & & \downarrow \rho & & \downarrow B\lambda \\
 M^{2n} & \xrightarrow{j_2} & ET^n \times_{T^n} M^{2n} & \xrightarrow{\pi_2} & BT^n.
 \end{array}$$

One consequence of this diagram of fibrations and earlier remarks is the following theorem concerning the cohomology of toric manifolds which has been proven in [10] and [13].

Theorem 3.2. *The cohomology ring $H^*(M^{2n}; \mathbb{Z})$ is isomorphic to*

$$\mathbb{Z}[x_1, \dots, x_m] / (I(K) + I(\lambda))$$

where $I(\lambda)$ is the ideal generated by $\pi_2^*(H^2(BT^n))$.

4. Preparation

The proof given below for the stable decompositions of moment-angle complexes relies on work of Welker, Ziegler and Živaljević on homotopy colimits of spaces [49]. Relevant features of that work are recalled next for purposes of exposition.

- (1) Let CW_* denote the category of CW-complexes and pointed, continuous maps.
- (2) Let P denote a finite poset (partially ordered set).

Given a poset P , define a diagram D of CW-complexes over P to be a functor

$$D: P \rightarrow CW_*$$

so that for every $p \leq p'$ in P there is a map

$$d_{pp'}: D(p') \rightarrow D(p)$$

for which d_{pp} is the identity. Thus, it follows that

$$d_{pp'} \circ d_{p'p''} = d_{pp''}$$

when $p \leq p' \leq p''$.

The colimit of D is the space

$$\text{colim}(D) = \coprod_{p \in P} D(p) / \sim$$

for which \sim denotes the equivalence relation generated by requiring that for each $x \in D(p')$, $x \sim d_{pp'}(x)$ for every $p < p'$.

Recall the poset associated to a simplicial complex K with order given by reverse inclusion of simplices and denoted by the symbol \bar{K} in Definition 2.6. Given a simplicial complex K , there are two diagrams D and \hat{D} over the poset \bar{K} given by $D(\sigma)$ and $\hat{D}(\sigma)$ with all the diagram maps induced by obvious inclusions. It follows that

$$Z(K; (\underline{X}, \underline{A})) = \operatorname{colim}(D(\sigma))$$

and

$$\hat{Z}(K; (\underline{X}, \underline{A})) = \operatorname{colim}(\hat{D}(\sigma)).$$

Homotopy colimits are defined next using the order complex $\Delta(P)$ as described in the introduction. Given a poset P and $p \in P$, define a new poset given by

$$P_{\leq p} = \{q \in P \mid q \leq p\}.$$

The order complex $\Delta(P_{\leq p})$ is a subcomplex of $\Delta(P)$. Also, whenever $p \leq p'$, the natural inclusions induce an injection

$$i_{p'p} : \Delta(P_{\leq p}) \hookrightarrow \Delta(P_{\leq p'}).$$

For all $p_1 < p_2$, there are two natural maps α and β determined by $i_{p_2p_1}$ and $d_{p_1p_2}$ given by

$$\alpha = 1 \times d_{p_1p_2} : \Delta(P_{\leq p_1}) \times D(p_2) \rightarrow \Delta(P_{\leq p_1}) \times D(p_1)$$

and

$$\beta = i_{p_2p_1} \times 1 : \Delta(P_{\leq p_1}) \times D(p_2) \rightarrow \Delta(P_{\leq p_2}) \times D(p_2).$$

The homotopy colimit of a diagram $D(P)$ is defined as

$$\operatorname{hocolim}(D(P)) = \left\{ \coprod_{p \in P} \Delta(P_{\leq p}) \times D(p) \right\} / \sim$$

where \sim is the equivalence relation defined by identifying every pair of points $\alpha(x, u)$ and $\beta(x, u)$.

The next theorem, a restatement of the ‘Projection Lemma’ stated in [49] as Lemma 3.1 or in [42] permits replacement of colimits by homotopy colimits in the case of CW-diagrams (in particular). One feature of this process is that homotopy colimits frequently have more useful homotopy properties than colimits in the proofs below.

Theorem 4.1. *Let $D(P)$ be a diagram over P having the property that the map*

$$\operatorname{colim}_{q > p} D(q) \hookrightarrow D(p)$$

is a closed cofibration. Then the natural projection map

$$\pi(D) : \operatorname{hocolim}(D(P)) \rightarrow \operatorname{colim}(D(P))$$

induced by the projection

$$\Delta(P_{\leq p}) \times D(p) \rightarrow D(p)$$

is a homotopy equivalence.

The last ingredient required here is the next result, proven in [50] as Lemma 1.8, and stated in [49] as Lemma 4.9 (Wedge Lemma).

Theorem 4.2. *Let P be a poset with maximal element \mathfrak{m} . Let $D(P)$ be a diagram over P so that there exist points $c_p \in D(p)$ for all $p < \mathfrak{m}$ such that $d_{pp'}$ is the constant map with $d_{pp'}(x) = c_p$ for all $p' > p$, then there is a homotopy equivalence*

$$\operatorname{hocolim}(D, P) \rightarrow \bigvee_{p \in P} (\Delta(P_{< p}) * D(p)).$$

An additional result is developed next. Let $\Sigma : CW_* \rightarrow CW_*$ be the suspension functor. By the definition of colim , if $D : P \rightarrow CW_*$ is a diagram, there is a natural inclusion

$$h(p) : D(p) \rightarrow \operatorname{colim}(D(P))$$

which induces a map

$$\Sigma(h(p)) : \Sigma(D(p)) \rightarrow \Sigma(\operatorname{colim}(D(P)))$$

and thus a map

$$h : \operatorname{colim}((\Sigma \circ D)(P)) \rightarrow \Sigma(\operatorname{colim}(D(P))).$$

Theorem 4.3. *Let $D : P \rightarrow CW_*$ be a diagram of finite CW-complexes. Then the map*

$$h : \operatorname{colim}((\Sigma \circ D)(P)) \rightarrow \Sigma(\operatorname{colim}(D(P)))$$

is a homotopy equivalence.

Proof. Since homology commutes with finite colimits by a check of the Mayer–Vietoris spectral sequence, there is a commutative diagram where the vertical maps are isomorphisms:

$$\begin{array}{ccc}
 H_*(\operatorname{colim}((\Sigma \circ D)(P))) & \xrightarrow{h_*} & H_*(\Sigma(\operatorname{colim}(D(P)))) \\
 \downarrow c & & \downarrow 1 \\
 \operatorname{colim}(H_*(\Sigma \circ D(p))) & \longrightarrow & H_*(\Sigma(\operatorname{colim}(D(P)))) \\
 \downarrow s_* & & \downarrow s_* \\
 \operatorname{colim}(H_{*-1}(D(p))) & \xrightarrow{\cong} & H_{*-1}(\operatorname{colim}(D)).
 \end{array}$$

Here s_* is the inverse of the suspension isomorphism in homology, and c is the commutation isomorphism. Thus h_* is an isomorphism in homology. Since $\operatorname{colim}((\Sigma \circ D)(P))$ and $\Sigma(\operatorname{colim}(D(P)))$ are both 1-connected, J.H.C. Whitehead's theorem gives that h is a homotopy equivalence. \square

The following ‘Homotopy Lemma’ given in [41,48] and [49] is useful in what follows.

Theorem 4.4. *Let D and E be two diagrams over P with values in CW_* . If f is a map of diagrams over P such that for every p , $f_p : D(p) \rightarrow E(p)$ is a homotopy equivalence, then f induces a homotopy equivalence*

$$\bar{f} : \operatorname{hocolim}(D(P)) \rightarrow \operatorname{hocolim}(E(P)).$$

The previous theorem coupled with Theorem 4.1, implies the following corollary.

Corollary 4.5. *Let D and E be two diagrams over P with values in CW_* for which the maps $\operatorname{colim}_{q>p} D(q) \hookrightarrow D(p)$, and $\operatorname{colim}_{q>p} E(q) \hookrightarrow E(p)$ are closed cofibrations. If f is a map of diagrams over P such that for every p , $f_p : D(p) \rightarrow E(p)$ is a homotopy equivalence, then f induces a homotopy equivalence*

$$f : \operatorname{colim}(D(P)) \rightarrow \operatorname{colim}(E(P)).$$

5. Proof of Theorem 2.10

Recall the functor $D : K \rightarrow CW_*$ of Definition 2.1 as follows: For every σ in K , let

$$D(\sigma) = \prod_{i=1}^m Y_i, \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \in [m] - \sigma. \end{cases}$$

Then

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \operatorname{colim} D(\sigma).$$

The analogous functor with values in smash products of spaces is given in Definition 2.2 by $\widehat{D}: K \rightarrow CW_*$ defined by

$$\widehat{D}(\sigma) = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_m \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \in [m] - \sigma. \end{cases}$$

As in Definition 2.5, let I denote an ordered sequence $I = (i_1, \dots, i_k)$ which satisfies $1 \leq i_1 < \cdots < i_k \leq m$. Specialize K to K_I , the full subcomplex of K given by

$$K_I = \{\sigma \cap I \mid \sigma \in K\}.$$

Associated to K_I is the diagram $\widehat{D}_I: K_I \rightarrow CW_*$ satisfying

$$\widehat{D}_I(\sigma \cap I) = Y_{i_1} \wedge Y_{i_2} \wedge \cdots \wedge Y_{i_k} \quad \text{where } Y_{i_j} = \begin{cases} X_{i_j} & \text{if } i_j \in \sigma \cap I, \\ A_{i_j} & \text{if } i_j \in I - \sigma \cap I. \end{cases}$$

Theorem 2.8 states that there is a natural pointed homotopy equivalence

$$H: \Sigma D(\sigma) \rightarrow \Sigma \left(\bigvee_{I \subseteq [m]} \widehat{D}_I(\sigma \cap I) \right).$$

Definition 5.1. The diagram

$$E: K \rightarrow CW_*$$

is defined by the values on objects

$$E(\sigma) = \bigvee_{I \subseteq [m]} \widehat{D}_I(\sigma \cap I)$$

together with naturality.

Observe that the map H of Theorem 2.8 gives a map of diagrams $H: \Sigma D \rightarrow \Sigma E$ which induces a homotopy equivalence

$$H: \Sigma D(\sigma) \rightarrow \Sigma E(\sigma)$$

for every σ . It follows from Corollary 4.5 that there is a homotopy equivalence

$$H: \operatorname{colim}(\Sigma D) \rightarrow \operatorname{colim}(\Sigma E).$$

Theorem 4.3 allows the commutation of the suspension with the colimits giving a homotopy equivalence

$$H: \Sigma(\operatorname{colim}(D)) \rightarrow \Sigma(\operatorname{colim}(E)).$$

Now,

$$\operatorname{colim}(D(\sigma)) = Z(K; (\underline{X}, \underline{A})) \quad \text{and} \quad \operatorname{colim}(E(\sigma)) = \bigvee_{I \subseteq [m]} \widehat{Z}(K; (\underline{X}_I, \underline{A}_I)),$$

thus there is a homotopy equivalence

$$H: \Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{Z}(K; (\underline{X}_I, \underline{A}_I))\right)$$

which is natural for morphisms of pairs $(\underline{X}, \underline{A})$ and embeddings in K , completing the proof of Theorem 2.10.

6. Proof of Theorem 2.12

The hypotheses of Theorem 2.12 are that the inclusion maps $A_i \subset X_i$ are null-homotopic for all i .

Definition 6.1. Consider the diagram

$$\widehat{D}: \overline{K} \rightarrow CW_*,$$

where \overline{K} is the poset associated to K described in Definition 2.6, defined by

$$\widehat{D}(\sigma) = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_m \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \in [m] - \sigma \end{cases}$$

with $\hat{d}_{\sigma, \tau}: \widehat{D}(\tau) \rightarrow \widehat{D}(\sigma)$ given by the natural inclusions when $\sigma < \tau$, that is $\tau \subset \sigma$.

Consider the diagram

$$A_k \xrightarrow{i_k} X_k \xrightarrow{\operatorname{id}} X_k.$$

By hypothesis, i_k is null-homotopic, so there is a null-homotopy $F_k: A_k \times I \rightarrow X_k$ with $F_k(a, 0) = i_k(a)$ and $F_k(a, 1) = x_k$ for all $a \in A_k$.

By the homotopy extension theorem, there exists $G_k: X_k \times I \rightarrow X_k$ with $G_k(x, 0) = x$, $G_k(x, 1) = g_k(x)$ such that $g_k(a_k) = x_k$, the base-point in X_k , for all $a_k \in A_k$. Thus there is a commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\operatorname{id}} & A_k \\ i_k \downarrow & & \downarrow c_k \\ X_k & \xrightarrow{g_k} & X_k \end{array}$$

where c_k is the constant map to x_k the base-point of X_k .

Definition 6.2. Define a new diagram

$$\widehat{E}: \overline{K} \rightarrow CW_*$$

by $\widehat{E}(\sigma) = \widehat{D}(\sigma)$ with diagram maps $\widehat{e}_{\sigma,\tau}: \widehat{E}(\tau) \rightarrow \widehat{E}(\sigma)$ given by constant maps to base-points.

Next, define maps $\alpha(\sigma)$ as follows.

Definition 6.3. For every $\sigma \in \overline{K}$, define

$$\alpha(\sigma): \widehat{D}(\sigma) \rightarrow \widehat{E}(\sigma)$$

by $\alpha(\sigma) = \alpha_1(\sigma) \wedge \cdots \wedge \alpha_m(\sigma)$ where

$$\alpha_k(\sigma) = \begin{cases} g_k: X_k \rightarrow X_k & \text{if } k \in \sigma, \\ \text{id}: A_k \rightarrow A_k & \text{if } k \in [m] - \sigma. \end{cases}$$

Since the g_k are homotopy equivalences, so is $\alpha(\sigma)$ for all $\sigma \in \overline{K}$. Furthermore, if $\sigma < \tau$ (that is, $\tau \subset \sigma$), the following diagram commutes:

$$\begin{array}{ccc} \widehat{D}(\tau) & \xrightarrow{\alpha(\tau)} & \widehat{E}(\tau) \\ \hat{d}_{\sigma,\tau} \downarrow & & \downarrow \widehat{e}_{\sigma,\tau} \\ \widehat{D}(\sigma) & \xrightarrow{\alpha(\sigma)} & \widehat{E}(\sigma). \end{array}$$

Hence the maps $\alpha(\sigma)$ give a map of diagrams

$$\alpha: \widehat{D}(\overline{K}) \rightarrow \widehat{E}(\overline{K})$$

which by Corollary 4.5 gives a homotopy equivalence

$$\alpha: \text{colim}(\widehat{D}(\overline{K})) \rightarrow \text{colim}(\widehat{E}(\overline{K})).$$

Finally, Theorem 4.2 gives a homotopy equivalence

$$\text{colim}(\widehat{E}(\overline{K})) \rightarrow \bigvee_{p \in P} |\Delta(P_{<p})| * D(p).$$

Theorem 2.12 follows since

$$\text{colim}(\widehat{D}(\overline{K})) = \widehat{Z}(K; (\underline{X}, \underline{A})).$$

7. Proof of Theorem 2.19

Let (X_i, A_i, x_i) denote a triple of finite CW-complexes with base-point x_i in A_i and with all X_i contractible for $1 \leq i \leq m$. In particular, the inclusions $A_i \subset X_i$ are all null-homotopic and hence Theorem 2.12 applies to this situation. This gives a homotopy equivalence

$$\widehat{Z}(K; (\underline{X}, \underline{A})) \rightarrow \bigvee_{\sigma \in K} |\Delta(\overline{K}_{<\sigma})| * \widehat{D}(\sigma)$$

where the diagram $\widehat{D}: \overline{K} \rightarrow CW_*$ is described in Definition 6.1. Notice now that if $\widehat{D}(\sigma)$ contains an X_i as one of its factors then it is contractible. Hence, the only non-contractible $\widehat{D}(\sigma)$ possible is $\widehat{D}(\phi) = \widehat{A}^{[m]}$. In this case,

$$\Delta(\overline{K}_{<\phi}) = K'$$

where K' is the barycentric subdivision of K . Finally, since $|K'|$ and $|K|$ are homeomorphic we obtain homotopy equivalences

$$\widehat{Z}(K; (\underline{X}, \underline{A})) \rightarrow |K| * \widehat{A}^{[m]} \rightarrow \Sigma(|K| \wedge \widehat{A}^{[m]})$$

completing the proof of Theorem 2.19.

8. Proof of Theorem 2.35

Consider the natural inclusion

$$Z(K; (\underline{X}, \underline{A})) \rightarrow X_1 \times \cdots \times X_m = X^{[m]}$$

together with $I = (i_1, \dots, i_t)$ for $1 \leq i_1 < \cdots < i_t \leq m$. Recall the notational convention of Definition 2.5 that $\widehat{X}^I = X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_t}$. Since all A_i are contractible by assumption, apply Theorems 2.10 and 2.14 to obtain a homotopy equivalence

$$\Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_{I \in K} \widehat{X}^I\right).$$

By Theorem 2.8, there is a homotopy equivalence

$$\Sigma(X_1 \times \cdots \times X_m) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{X}^I\right).$$

Naturality implies that the map

$$\Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma(X_1 \times \cdots \times X_m)$$

is split with cofiber

$$\Sigma\left(\bigvee_{I \notin K} \widehat{X}^I\right).$$

Hence there is a split cofibration

$$\Sigma\left(\bigvee_{I \in K} \widehat{X}^I\right) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{X}^I\right) \rightarrow \Sigma\left(\bigvee_{I \notin K} \widehat{X}^I\right).$$

Next assume that coefficients are taken in a ring R for which the spaces X_1, \dots, X_m satisfy the strong form of the Künneth theorem over R .

Remark 8.1. One case for which the conditions above are satisfied is given by $R = \mathbb{Z}$ and spaces X_i of finite type with torsion-free integral cohomology.

Thus the reduced cohomology $\overline{H}^*(X^{[m]}; R)$ is isomorphic to the direct sum

$$\bigoplus_{I \subseteq [m]} \overline{H}^*(X_{i_1}; R) \otimes \cdots \otimes \overline{H}^*(X_{i_t}; R).$$

The natural inclusion map $\Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma(X_1 \times \cdots \times X_m)$ induces a map

$$\overline{H}^*(X^{[m]}; R) \rightarrow \overline{H}^*(Z(K; (\underline{X}, \underline{A})); R)$$

which corresponds to the projection map

$$\bigoplus_{I \subseteq [m]} \overline{H}^*(X_{i_1}; R) \otimes \cdots \otimes \overline{H}^*(X_{i_t}; R) \rightarrow \bigoplus_{I \in K} \overline{H}^*(X_{i_1}; R) \otimes \cdots \otimes \overline{H}^*(X_{i_t}; R)$$

with kernel exactly

$$\bigoplus_{I \notin K} \overline{H}^*(X_{i_1}; R) \otimes \cdots \otimes \overline{H}^*(X_{i_t}; R)$$

which is the Stanley–Reisner ideal $I(K)$ by inspection. The last part of Theorem 2.35 follows by applying the cohomology theory E^* to the result of Theorem 2.15.

9. Proof of Theorem 2.46

The first preparatory proof in this section is that of Lemma 2.44 which follows from the fact that $Z(K; (\underline{X}, \underline{A}))$ is natural for morphisms with respect to morphisms of pairs, $(\underline{X}, \underline{A}) \rightarrow (\underline{X}', \underline{A}')$.

Proof. The first part of the lemma is to check that

- (1) $d_i : Z(K; (\underline{X}_*, \underline{A}_*))_n \rightarrow Z(K; (\underline{X}, \underline{A})_*)_n$ and
- (2) $s_j : Z(K; (\underline{X}_*, \underline{A}_*))_n \rightarrow Z(K; (\underline{X}, \underline{A})_*)_{n+1}$.

So it suffices to check that if (x_1, x_2, \dots, x_m) is in the subset

$$Z(K; (\underline{X}_*, \underline{A}_*))_n \subset X_n(1) \times \cdots \times X_n(m),$$

then

- (1) $d_i(x_1, x_2, \dots, x_m) = (d_i(x_1), d_i(x_2), \dots, d_i(x_m))$ is in $Z(K; (\underline{X}_*, \underline{A}_*))_{n-1}$, and
- (2) $s_j(x_1, x_2, \dots, x_m) = (s_j(x_1), s_j(x_2), \dots, s_j(x_m))$ is in $Z(K; (\underline{X}_*, \underline{A}_*))_{n+1}$.

Notice that a point (x_1, x_2, \dots, x_m) is in $Z(K; (\underline{X}_*, \underline{A}_*)) \subseteq X_n(1) \times \dots \times X_n(m)$ provided $x_t \in A_n(t)$ for $\sigma \in K$ and $t \notin \sigma$. But then $d_i(x_t) \in A_{n-1}(t)$ for $t \notin \sigma$ as $A_*(t)$ is a simplicial subcomplex of $X_*(t)$ by assumption. Similarly, notice that $s_j(x_t)$ is in $A_{n+1}(t)$ if and only if $x_t = d_j s_j(x_t)$ is in $A_n(t)$. The lemma follows. \square

The next proof is that of Theorem 2.46.

Proof. Let $|u_i, x_i|$ denote the equivalence class of a point in the geometric realization $|X_*(i)|$. The homeomorphism

$$\eta: |X_*(1)| \times \dots \times |X_*(m)| \rightarrow |X_*(1) \times \dots \times X_*(m)|$$

is induced by $\eta(|u_1, x_1|, \dots, |u_m, x_m|) = |w, s_{I_1}(x_1), \dots, s_{I_m}(x_m)|$ for choices of w and s_{I_i} given in [37,34] as well as in the proof of Theorem 11.5 in [35]. Notice that $s_{I_j}(x_j)$ is in $A_*(j)$ if and only if x_j is in $A_*(j)$ by Lemma 2.44. Thus there is a commutative diagram

$$\begin{array}{ccc} Z(K; (|\underline{X}|_*, |\underline{A}|_*)) & \xrightarrow{h} & |Z(K; (\underline{X}_*, \underline{A}_*))| \\ \downarrow i & & \downarrow |i_*| \\ |X_*(1)| \times \dots \times |X_*(m)| & \xrightarrow{\eta} & |X_*(1) \times \dots \times X_*(m)| \end{array}$$

as stated in Theorem 2.46.

Since (1) η is a homeomorphism, (2) $|i_*|$ is a monomorphism, and (3) $s_{I_j}(x_j)$ is in $A_*(j)$ if and only if x_j is in $A_*(j)$, it follows that h is a surjection. Since h is a continuous, bijective, open map, it is a homeomorphism and the theorem follows. \square

10. Connection to Kurosh's theorem

The purpose of this section is to point out that certain fibrations above involving the polyhedral product functor give natural, useful covering spaces. For example, Ganea's theorem, a special case of the Denham–Suciu fibration, implies a statement equivalent to a classical theorem within group theory due to A. Kurosh [31]. This section is purely expository development of this connection.

Kurosh's theorem addresses the structure of a free product of discrete groups

$$\coprod_{1 \leq \alpha \leq n} G_\alpha.$$

One statement of the theorem is as follows. If G is a subgroup of the free product of discrete groups A and B , $A \coprod B$, then G is a free product of the following groups:

- (1) a subgroup of G conjugate in $A \amalg B$ to a subgroup of A ,
- (2) a subgroup of G conjugate in $A \amalg B$ to a subgroup of B , and/or
- (3) a free group.

The first observation is the following corollary of Kurosh's theorem.

Proposition 10.1. *Let G be a subgroup of the free product $A \amalg B$. Then G surjects to a subgroup of $A \times B$ with kernel a free group.*

The fibration of Denham–Suciu stated here as Lemma 2.30 provides a simple reformulation of this corollary by inspection as given next. The homotopy fiber of the inclusion

$$X \vee Y \rightarrow X \times Y$$

is a generalized moment-angle complex which has the homotopy type of the join

$$\Omega(X) * \Omega(Y).$$

Furthermore, the induced map

$$W : \Omega(X) * \Omega(Y) \rightarrow X \vee Y$$

is the Whitehead product

$$[E_X, E_Y]$$

where $E_X : \Sigma\Omega(X) \rightarrow X \vee Y$ denotes the composite

$$\Sigma\Omega(X) \xrightarrow{e_X} X \xrightarrow{i_X} X \vee Y$$

with the natural evaluation map e_X and with the natural inclusion i_X .

Recall that for two pointed topological spaces U and V , each of the homotopy type of a CW-complex, their join $U * V$ has the homotopy type of the suspension $\Sigma(U \wedge V)$. This is applied next. Consider the case of

$$X = K(\pi, 1) \quad \text{and} \quad Y = K(\Gamma, 1)$$

for discrete groups π and Γ to obtain

$$\Omega(X) = K(\pi, 0) \quad \text{and} \quad \Omega(Y) = K(\Gamma, 0).$$

Thus $\Omega(X) * \Omega(Y)$ has the homotopy type of

$$\Sigma((\pi \times \Gamma)/(\pi \vee \Gamma))$$

which is a bouquet of circles, one for each pair $\{(\alpha, \beta) \in \pi \times \Gamma \mid \alpha \neq 1, \beta \neq 1\}$.

The next proposition follows easily.

Proposition 10.2. *The kernel of the natural projection map*

$$q: \pi \coprod \Gamma \rightarrow \pi \times \Gamma$$

*is the fundamental group of $\Sigma((\pi \times \Gamma)/(\pi \vee \Gamma))$, a free group. Furthermore, the Whitehead product map $W: \Omega(X) * \Omega(Y) \rightarrow X \vee Y$ induces the map from $\pi_1(\Sigma((\pi \times \Gamma)/(\pi \vee \Gamma))) \rightarrow \pi \coprod \Gamma$ which is an isomorphism onto the kernel of the map $q: \pi \coprod \Gamma \rightarrow \pi \times \Gamma$.*

The proof of Proposition 10.1 is given next. Let G be a subgroup of $A \coprod B$. Project to A and to B , then let the images of G be G_A and G_B respectively. Define the map

$$j_A: G \rightarrow A \coprod B$$

as the composite

$$G \rightarrow G_A \xrightarrow{i_A} A \xrightarrow{\gamma_A} A \coprod B$$

where i_A and γ_A are natural inclusions. Define

$$j_B: G \rightarrow A \coprod B$$

similarly. Thus the kernel of the natural map

$$j_A \times j_B: G \rightarrow G_A \times G_B$$

is a subgroup of the kernel of

$$q: A \coprod B \rightarrow A \times B,$$

a free group.

A standard consequence is recorded next.

Corollary 10.3. *Let*

$$f: G_1 \coprod \cdots \coprod G_n \rightarrow H$$

be a group homomorphism which when restricted to each G_i is a monomorphism. Then the kernel of f is free.

Acknowledgments

The authors would like to thank the Institute for Advanced Study for providing a wonderful environment to develop this mathematics. The first author was partially supported by the award of a research leave from Rider University. The third author was partially supported by NSF grant number 0340575 and DARPA grant number 2006-06918-01. The fourth author was partially supported by the Department of Mathematics at Princeton University.

References

- [1] A. Adem, A. Bahri, M. Bendersky, F.R. Cohen, S. Gitler, On stable decompositions for simplicial spaces, arXiv:0904.4699.
- [2] D. Anick, Connections between Yoneda and Pontrjagin algebras, in: *Algebraic Topology*, Aarhus, 1982, in: *Lecture Notes in Math.*, vol. 1051, Springer-Verlag, 1984, pp. 331–350.
- [3] L. Astey, in preparation.
- [4] A. Bahri, M. Bendersky, F.R. Cohen, S. Gitler, Decompositions of the polyhedral product functor with applications to moment-angle complexes and related spaces, *Proc. Natl. Acad. Sci. USA* 106 (2009) 12241–12244.
- [5] I. Baskakov, Cohomology of K -powers of spaces and the combinatorics of simplicial divisions, *Russian Math. Surveys* 57 (5) (2002) 989–990.
- [6] I. Baskakov, Triple Massey products in the cohomology of moment-angle complexes, *Russian Math. Surveys* 58 (5) (2003) 1039–1041.
- [7] I. Baskakov, V. Buchstaber, T. Panov, Algebras of cellular cochains, and torus actions, *Russian Math. Surveys* 59 (3) (2004) 562–563.
- [8] A. Björner, K.S. Sarkaria, The zeta function of a simplicial complex, *Israel J. Math.* 103 (1998) 29–40.
- [9] V. Buchstaber, T. Panov, Actions of tori, combinatorial topology and homological algebra, *Russian Math. Surveys* 55 (5) (2000) 825–921.
- [10] V. Buchstaber, T. Panov, *Torus Actions and Their Applications in Topology and Combinatorics*, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002.
- [11] V. Buchstaber, T. Panov, N. Ray, Spaces of polytopes and cobordism of quasitoric manifolds, *Mosc. Math. J.* 7 (2) (2007) 219–242.
- [12] F.R. Cohen, D.E. Koditschek, Hybrid control over the coordination complex, in preparation.
- [13] V.I. Danilov, The geometry of toric varieties, *Uspekhi Mat. Nauk* 33 (1978) 85–134 (in Russian). English translation in *Russian Math. Surveys* 33 (1978) 97–154.
- [14] M. Davis, T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, *Duke Math. J.* 62 (2) (1991) 417–451.
- [15] C. De Concini, C. Procesi, Wonderful models of subspace arrangements, *Selecta Math. (N.S.)* 1 (3) (1995) 459–494.
- [16] P. Deligne, M. Goresky, R. MacPherson, L’algèbre de cohomologie du complément, dans un espace affine, d’une famille finie de sous-espaces affines, *Michigan Math. J.* 48 (2000) 121–136.
- [17] G. Denham, A. Suciu, Moment-angle complexes, monomial ideals and Massey products, *Pure Appl. Math. Q.* 3 (1) (2007) 25–60.
- [18] M. Franz, The integral cohomology of smooth toric varieties, arXiv:math/0308253v1.
- [19] M. Franz, The integral cohomology of toric manifolds, in: *Proceedings of the Keldysh Conference*, Moscow, 2004, *Proc. Steklov Inst. Math.* 252 (2006) 53–62.
- [20] M. Franz, Koszul duality for tori, Thesis, Universität Konstanz, 2001.
- [21] T. Ganea, A generalization of the homology and homotopy suspension, *Comment. Math. Helv.* 39 (1965) 295–322.
- [22] R. Goresky, R. MacPherson, *Stratified Morse Theory*, *Ergebnisse Der Mathematik Und Ihrer Grenzgebiete* 3rd series, vol. 14, Springer-Verlag, Berlin, 1988.
- [23] J. Grbic, S. Theriault, Homotopy type of the complement of a coordinate subspace arrangement of codimension two, *Russian Math. Surveys* 59 (3) (2004) 1207–1209.
- [24] G.C. Haynes, F.R. Cohen, D. Koditschek, Gait transitions for quasi-static hexapedal locomotion on level ground, in: *14th International Symposium on Robotics Research (ISRR 2009)*, http://repository.upenn.edu/ese_papers/503, submitted for publication.
- [25] M. Hochster, Cohen–Macaulay rings, combinatorics, and simplicial complexes, in: *Ring Theory, II*, *Proc. Second Conf.*, Univ. Oklahoma, Norman, Okla., 1975, in: *Lect. Notes Pure Appl. Math.*, vol. 26, Dekker, New York, 1977, pp. 171–223.
- [26] Y. Hu, On the homology of complements of arrangements of sub-spaces and spheres, *Proc. Amer. Math. Soc.* 122 (1994) 285–290.
- [27] I. James, Reduced product spaces, *Ann. of Math.* 62 (1955) 170–197.
- [28] K. Jewell, Complements of sphere and sub-space arrangements, *Topology Appl.* 56 (1994) 199–214.
- [29] K. Jewell, P. Orlik, B.Z. Shapiro, On the complements of affine sub-space arrangements, *Topology Appl.* 56 (1994) 215–233.
- [30] Y. Kamiyama, S. Tsukuda, The configuration space of the n -arms machine in the Euclidean space, *Topology Appl.* 154 (2007) 1447–1464.

- [31] A. Kurosh, Lectures on General Algebra, published in Russian in 1960 and an English translation published in Chelsea Publishing Company, New York, 1963.
- [32] M. de Longueville, C. Schultz, The cohomology rings of complements of subspace arrangements, *Math. Ann.* 319 (4) (2001) 625–646.
- [33] S. Lopez de Medrano, Topology of the intersection of quadrics in \mathbb{R}^n , in: *Algebraic Topology* (Arcata, CA), in: *LNM*, vol. 1370, Springer-Verlag, 1989, pp. 280–292.
- [34] J.P. May, The Geometry of Iterated Loop Spaces, *LNM*, vol. 268, Springer-Verlag, Berlin, New York, 1972.
- [35] J.P. May, *Simplicial Objects in Algebraic Topology*, *Math. Studies*, vol. 11, Van Nostrand, 1967.
- [36] J. Milnor, On the construction $F[K]$, in: J.F. Adams (Ed.), *A Student's Guide to Algebraic Topology*, in: *Lecture Notes of the London Mathematical Society*, vol. 4, 1972, pp. 119–136.
- [37] J. Milnor, The geometrical realization of a semi-simplicial complex, *Ann. of Math.* 65 (1957) 357–362.
- [38] D. Notbohm, N. Ray, On Davis–Januszkiewicz homotopy types. I. Formality and rationalization, *Algeb. Geom. Topol.* 5 (2005) 31–51.
- [39] T. Panov, Cohomology of face rings, and torus actions, *London Math. Soc. Lecture Notes* 347 (2008) 165–201, *Surveys in Contemporary Mathematics*, Cambridge, arXiv:math/0506526.
- [40] T. Panov, N. Ray, R. Vogt, Colimits, Stanley–Reisner algebras, and loop spaces, in: *Categorical Decomposition Techniques in Algebraic Topology*, in: *Progr. Math.*, vol. 215, Birkhäuser, Basel, 2004, pp. 261–291.
- [41] G. Porter, The homotopy groups of wedges of suspensions, *Amer. J. Math.* 88 (1966) 655–663.
- [42] G. Segal, Classifying spaces and spectral sequences, *Publications Mathématiques de l'IHÉS* 34 (1968) 105–112.
- [43] R.P. Stanley, *Combinatorics and Commutative Algebra*, second ed., *Progr. Math.*, vol. 41, Birkhäuser, Boston, MA, 1996.
- [44] D. Stone, Private communication.
- [45] N. Strickland, Notes on toric spaces, preprint, 1999, available at Strickland's webpage.
- [46] E. Swartz, Topological representations of matroids, *J. Amer. Math. Soc.* 16 (2003) 427–442.
- [47] E.B. Vinberg, Discrete linear groups generated by reflections, *Izv. Akad. Nauk SSSR Ser. Mat.* 35 (1971) 1072–1112 (in Russian). English translation in *Math. USSR Izv.* 5 (1971) 1083–1119.
- [48] R. Vogt, Homotopy limits and colimits, *Math. Z.* 134 (1973) 11–52.
- [49] V. Welker, G. Ziegler, R. Živaljević, Homotopy colimits-comparison lemmas for combinatorial applications, *J. Reine Angew. Math.* 509 (1999) 117–149.
- [50] G. Ziegler, R. Živaljević, Homotopy types of sub-space arrangements via diagrams of spaces, *Math. Ann.* 295 (1993) 527–548.