LETTER

Different adiabatic quantum optimization algorithms for the NP-complete exact cover problem

Altshuler et al. (1) claim that adiabatic quantum optimization (AQO) fails for random instances of the NP-complete exact cover 3 (EC3) problem. They showed that a specific AQO algorithm failed for random instances of EC3 because of Anderson localization (AL).

For any problem, there are three components (initial Hamiltonian, problem Hamiltonian, and evolution path) specifying an AQO algorithm. A different component (particularly a different problem Hamiltonian) gives a different AQO algorithm for the same problem. It is incorrect to conclude from the failure of one specific AQO algorithm that all algorithms fail for the same problem. In this letter, we describe an AQO algorithm for EC3 in which the argument by Atshuler et al. (1) is inapplicable and does not prove that this algorithm also fails.

An instance of EC3 is given by a three-conjunctive normal form Boolean formula $\Psi(x_1,\ldots,x_n) = C_1 \wedge \ldots \wedge C_m$ with n binary variables and m clauses, where each clause $C_i = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ consists of three variables. Ψ is satisfiable if there exists an assignment such that exactly one variable in each clause takes value 1. The nonnegative cost function,

$$\mathcal{E}_\Psi(x_1,\ldots,x_n) = \sum_{i=1}^{m} (x_{i_1} + x_{i_2} + x_{i_3} - 1)^2$$

gives zero cost only for a satisfying assignment of Ψ. The corresponding problem Hamiltonian in the article by Atshuler et al. (1) is $\mathcal{H}_A = \sum_{i\in \mathrm{V}(G_{\mathrm{EC}})} B_i \sigma_i^z + \sum_{ij\in \mathrm{E}(G_{\mathrm{EC}})} I_{ij}\sigma_i^z\sigma_j^z$, where B_i is the number of clauses containing variable x_i, I_{ij} is the number of clauses containing both x_i and x_j, $\mathrm{V}(G_{\mathrm{EC}}) = \{1,\ldots,n\}$, and $\mathrm{E}(G_{\mathrm{EC}}) = \{ij: x_i \text{ and } x_j \text{ appear in a clause}\}$.

For each clause to be satisfied, exactly one variable must take value 1, whereas the other two are 0. Thus, $x_i x_j = 0$ ($ij \in \mathrm{E}(G_{\mathrm{EC}})$) for the satisfying assignment. Hence, for arbitrary $D_{ij} > 0$, the cost function,

$$\mathcal{E}'_\Psi(x_1,\ldots,x_n) = \mathcal{E}_\Psi(x_1,\ldots,x_n) + \sum_{ij\in \mathrm{E}(G_{\mathrm{EC}})} D_{ij} x_i x_j$$

can also serve as the energy function for EC3. The corresponding Hamiltonian is $\mathcal{H}_C = \mathcal{H}_A + \sum_{i\in \mathrm{V}(G_{\mathrm{EC}})} \sum_{j\in \mathrm{nbr}(i)} D_{ij}\sigma_i^z + \sum_{ij\in \mathrm{E}(G_{\mathrm{EC}})} D_{ij}\sigma_i^z\sigma_j^z$, as first derived by Choi (2).

The argument in the article by Atshuler et al. (1) necessarily depends on the energy function of $\mathcal{H}_A$. The authors claim that the correctness of their argument relies not on the specific form of the problem Hamiltonian for EC3 but only on the properties of the problem instance B_i and I_{ij}. Our problem Hamiltonian, $\mathcal{H}_C$, which depends on the additional D_{ij}, belies their claim, however. The possibility of the problem Hamiltonian, $\mathcal{H}_C$, or the cost function, $\mathcal{E}'_\Psi$, with a flexible parameter D_{ij} was overlooked in the article by Atshuler et al. (1), resulting in their incorrect claim. To show that the AQO algorithm with $\mathcal{H}_C$ fails, one would need to show that the algorithms corresponding to all possible values of D_{ij} fail. Moreover, considering alternative NP-complete reductions would give additional different problem Hamiltonians, and thus different AQO algorithms for EC3, with different time complexities. An example of how different problem Hamiltonians can avoid AL, and thus change the time complexity drastically, is shown in Fig. 1.

To conclude, the time complexity of AQO algorithms for random instances of EC3 or the general NP-complete problem remains open.

Vicky Choi[1]
Department of Computer Science, Virginia Polytechnic Institute and State University, Falls Church, VA 22043

1. Altshuler B, Krovi H, Roland J (2010) Anderson localization makes adiabatic quantum optimization fail. *Proc Natl Acad Sci USA* 107:12446–12450.
2. Choi V (2010) Different adiabatic quantum optimization algorithms for the NP-complete Exact Cover and 3SAT problems. ArXiv:quant-ph/1004.2226, 1–20.
3. Amin MHS, Choi V (2009) First order phase transition in adiabatic quantum computation. *Phys Rev A* 80:062326–062330.

Author contributions: V.C. designed research; performed research; and wrote the paper.
The author declares no conflict of interest.

[1]E-mail: vchoi@cs.vt.edu.

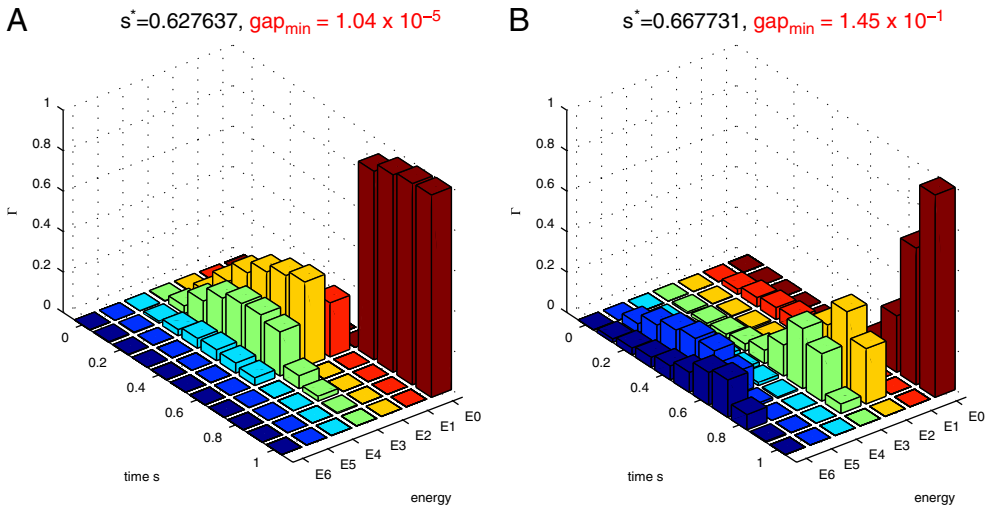

Fig. 1. Different problem Hamiltonians can drastically change the minimum spectral gap, gap_{min} (and hence the time complexity). The figures show the decomposed states (of the ground state) evolution of different problem Hamiltonian algorithms for the problem described by Amin and Choi (3). Let $\mathcal{H}_C = -\sum_{i\in V(G)} 2w_i + \sum_{i\in V(G)}\sum_{j\in \text{nbr}(i)} D_{ij}\sigma_i^z + \sum_{ij\in E(G)} D_{ij}\sigma_i^z\sigma_j^z$, where G is depicted in figure 2 of the article by Amin and Choi (3), and $w_i = 1$ for $1 \le i \le 6$ and $w_i = 1.8$ for $7 \le i \le 15$. (*A*) $D_{ij} \equiv 2$. (*B*) $D_{ij} \equiv 20$. There is a level anticrossing at s^* resulting in $\text{gap}_{\text{min}} \sim 10^{-5}$ in *A*, whereas $\text{gap}_{\text{min}} \sim 10^{-1}$ in *B* (no level anticrossing). The *x* axis represents time. The *y* axis represents the lowest seven energy levels. Each color corresponds to an energy level. The *z* axis is the percent Γ of corresponding states participating in the instantaneous ground state. As time *s* increases, one can see how Γ of each energy level evolves. For example, Γ of the lowest energy level (brown bar) changes from ~0 before $s = 0.6$ to >0.9 at $s = 0.7$ when it undergoes a level anticrossing at s^* in *A*, whereas it changes gradually in *B*.