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Note on summable trigonometric series : S. Verblunsky.

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Congruence properties of the coefficients in the expansions of certain modular functions : J. R. Wilton.

A PROOF OF FOURIER'S THEOREM

J. R. WILTON*.

1. The suggestion for the proof of Fourier's theorem which is given below came to me from a remark by Prof. Hardy that certain conditions which I have given for the truth of Poisson's summation formula† might be generalized by using the Stieltjes integral instead of the Lebesgue integral.

I prove the theorem in the form:

If $0 \leqslant a < b \leqslant 2\pi$, $0 \leqslant x < 2\pi$, *and if* $f(t)$ *is of bounded variation over* (a, b), *then*

(1) $$\frac{1}{\pi} \sum_{n=0}^{\infty}{}' \int_a^b f(t) \cos n(x-t)\, dt = F(x),$$

* Received 1 October, 1930; read 13 November, 1930.

† J. R. Wilton, *Journal London Math. Soc.*, 5 (1930), 276–279.

where the dash denotes that the term $n = 0$ is to be halved, and

$$(2)\quad \begin{cases} F(x) = 0, & \textit{if } x < a \textit{ or if } x > b, \\ F(x) = \tfrac{1}{2}\{f(x+0)+f(x-0)\}, & \textit{if } a < x < b, \\ F(a) = \tfrac{1}{2}f(a+0), & \textit{unless both } a = 0 \textit{ and } b = 2\pi, \\ F(b) = \tfrac{1}{2}f(b-0), & \textit{if } b \neq 2\pi, \\ F(0) = \tfrac{1}{2}\{f(+0)+f(2\pi-0)\}, & \textit{if } a = 0 \textit{ and } b = 2\pi. \end{cases}$$

The apparatus which I require is, first, the elementary theorem:

$$(3)\qquad \sum_{n=1}^{\infty} \frac{\sin nt}{n\pi} = \begin{cases} \dfrac{1}{2}-\dfrac{t}{2\pi}, & \textit{if } 0 < t < 2\pi, \\ -\dfrac{1}{2}-\dfrac{t}{2\pi}, & \textit{if } -2\pi < t < 0, \end{cases}$$

and the series (3) *is boundedly convergent.*

Second, the following properties of the Lebesgue-Stieltjes integral (LS-integral),

$$\int_a^b \phi df = \int_a^b \phi(t)\, df(t),$$

in which $f(t)$ is of bounded variation over (a, b), and it will be sufficient to assume that $\phi(t)$ is continuous, except for at most two simple discontinuities in (a, b).

(i) *If* $a < c < b$, *then*

$$(4)\qquad \int_a^b \phi df = \int_a^c \phi df + \int_c^b \phi df,$$

*and**

$$(5.1)\qquad \int_a^b \phi df = \phi(a)\{f(a+0)-f(a)\} + \int_{a+0}^b \phi df$$

$$(5.2)\qquad = \phi(b)\{f(b)-f(b-0)\} + \int_a^{b-0} \phi df;$$

(ii) *If $\phi(t)$ is an indefinite integral of a function $\phi'(t)$ when $a \leqslant t \leqslant b$, then*

$$(6)\qquad \int_a^b \phi df = \phi(b)f(b) - \phi(a)f(a) - \int_a^b f\phi'\, dt,$$

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* By $\int_{a+0}^b \phi\, df$ I mean, of course, $\lim\limits_{\epsilon\to 0} \int_{a+\epsilon}^b \phi\, df$.

where it is to be understood that if, for instance, the lower limit of the integral on the left of (6) is $a+0$, then the interval within which $\phi(t)$ is an integral of $\phi'(t)$ is $a < t \leqslant b$, and we must write $\phi(a+0)f(a+0)$ for $\phi(a)f(a)$ on the right;

(iii) *If, as* $n \to \infty$, $\lim \phi_n(t) = \phi(t)$, *and if* $|\phi_n(t)| \leqslant K$ *for every* n *and for all values of* t *in* (a, b), *then*

$$\lim_{n\to\infty} \int_a^b \phi_n(t)\, df(t) = \int_a^b \phi(t)\, df(t), \tag{7}$$

provided that all the integrals exist as LS-integrals.*

Since (3) is boundedly convergent, we have, by (7),

$$\sum_1^\infty \int_a^b \frac{\sin n(t-c)}{n}\, df(t) = \int_a^b \sum_1^\infty \frac{\sin n(t-c)}{n}\, df(t). \tag{8}$$

2. *Proof of Fourier's theorem.*

LEMMA. *If* $\phi(t)$ *is an indefinite integral of* $\phi'(t)$ *in* $a \leqslant t < c$ *and in* $c < t \leqslant b$, *and if* $\phi(t)$ *has a simple discontinuity at* $t = c$, *then*

$$\int_a^b \phi df = \phi(b)f(b) - \phi(a)f(a) - f(c-0)\{\phi(c)-\phi(c-0)\} \\ - f(c+0)\{\phi(c+0)-\phi(c)\} - \int_a^b f\phi' dt. \tag{9}$$

This formula is correct when $c = a$ *and when* $c = b$, *if we put*

$$f(a-0) = f(b+0) = 0.$$

When $c = a$ we have, by (5 . 1) and (6),

$$\int_a^b \phi df = \phi(a)\{f(a+0)-f(a)\} + \int_{a+0}^b \phi df \\ = \phi(b)f(b) - \phi(a)f(a) - f(a+0)\{\phi(a+0)-\phi(a)\} - \int_a^b f\phi' dt.$$

This is the particular case of (9) when $c = a$; the case $c = b$ follows in the same way from (5 . 2); and the general case is then a consequence of (4).

Now let $$\phi(t) = -\sum_{n=1}^\infty \frac{\sin n(x-t)}{n\pi};$$

* E. W. Hobson, *Theory of functions of a real variable*, 1, 606.

then, by (8) and (6),

$$\int_a^b \phi df = -\frac{1}{\pi}\int_a^b \left(\sum_1^\infty \frac{\sin n(x-t)}{n}\right) df(t)$$
$$= -\sum_1^\infty \int_a^b \frac{\sin n(x-t)}{n\pi}\, df(t)$$
$$= -\frac{1}{\pi}\sum_1^\infty \left[\frac{\sin n(x-b)}{n} f(b) - \frac{\sin n(x-a)}{n} f(a) + \int_a^b f(t)\cos n(x-t)\, dt\right]$$
$$= \phi(b) f(b) - \phi(a) f(a) - \frac{1}{\pi}\sum_1^\infty \int_a^b f(t)\cos n(x-t)\, dt. \tag{10}$$

On the other hand, if a, b, x satisfy the inequalities stated in the theorem, then, by (3), $\phi(t)$ is an integral of $-1/(2\pi)$ except at the one point $t=x$ when $a \leqslant x \leqslant b$, or, in the special case when $a = x = 0$ and $b = 2\pi$, at the two points $t = 0$ and $t = 2\pi$; and, since $\phi(x) = 0$ $\phi(x+0) = \frac{1}{2}$, and $\phi(x-0) = -\frac{1}{2}$, it follows from (9) that

$$\int_a^b \phi df = \phi(b) f(b) - \phi(a) f(a) - F(x) + \frac{1}{2\pi}\int_a^b f dt, \tag{11}$$

where $F(x)$ is defined by (2). The theorem is a consequence of (10) and (11).

ON THE REPRESENTATIONS OF A NUMBER AS THE SUM OF TWO NUMBERS NOT DIVISIBLE BY k-th POWERS

T. ESTERMANN*.

Let a positive integer be called a k-number if it is not divisible by the k-th power of any integer greater than 1 (the 2-numbers are usually called "quadratfrei"), and let $Q_k(n)$ denote the number of representations of n as the sum of two k-numbers.

My object is to give a short and elementary proof of the following theorem† :—

Let k be any integer greater than or equal to 2, and let

$$c_k = \prod_p (1 - 2p^{-k}), \tag{1}$$

* Received 31 May, 1930; read 19 June, 1930.

† The theorem is due to Linfoot and Evelyn, who are publishing their proof in the *Journal für Mathematik*.