Chapter 5.

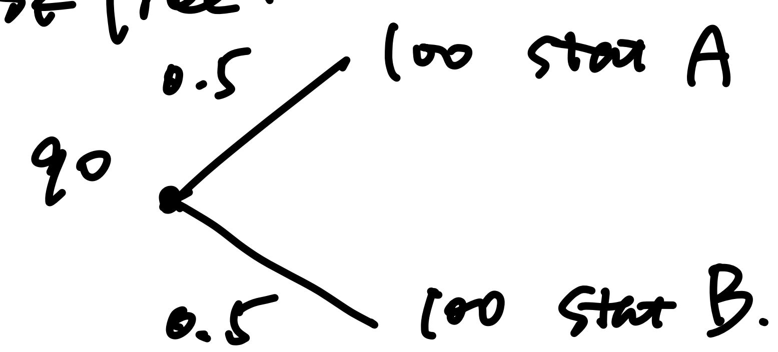
Markowitz. 1952. Portfolio Selection — CAPM 才对贴现率 r 有一个比较好的估计.

Covariance.

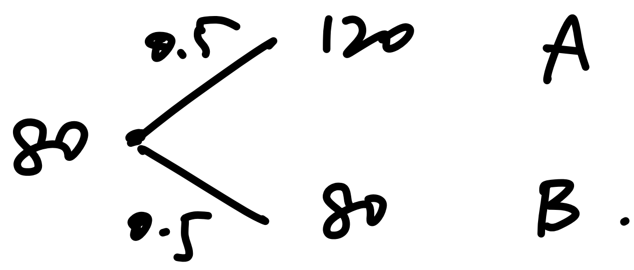
Mean 和 Variance 来刻画了 Portfolio. Mean-Variance Analysis

Markowitz 说: 最后的组合必须一起看, 不能单独看每一个资产.

risk-free:



risky:



ex-ante rate of return. 事前回报率 = expected rate of return 期望

$E(\tilde r) = 0.5\times(\frac{120}{80}-1)+0.5\times(\frac{80}{80}-1)=25\%$.

$E(r_f) = 100/90 - 1 = 11.1\%$.

ex-post rate of return:

$r_a = 50\%$ $\qquad r_{fa}=r_{fb}=\frac{100}{90}-1=11.1\%$.

$r_b = 0\%$

对市场的利率 r 是在 period 0 采取的这么一个事前回报率.

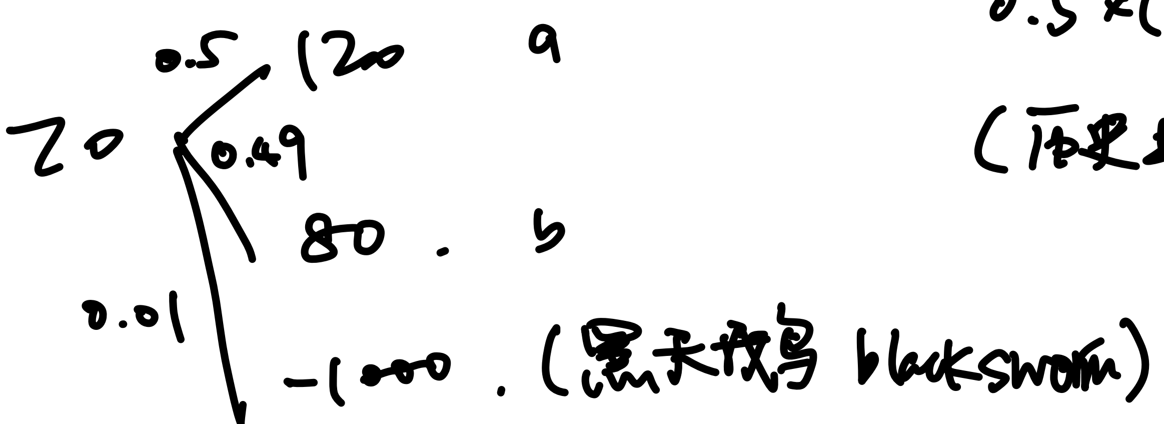
risk premium: 风险溢价. (只包含在预期回报率中).

$E(\tilde r) - r_f = 25\% - 11\% = 14\%$

historical ex-post 历史数据来展现预期

Variance$(r_f)=0$? 是对期望

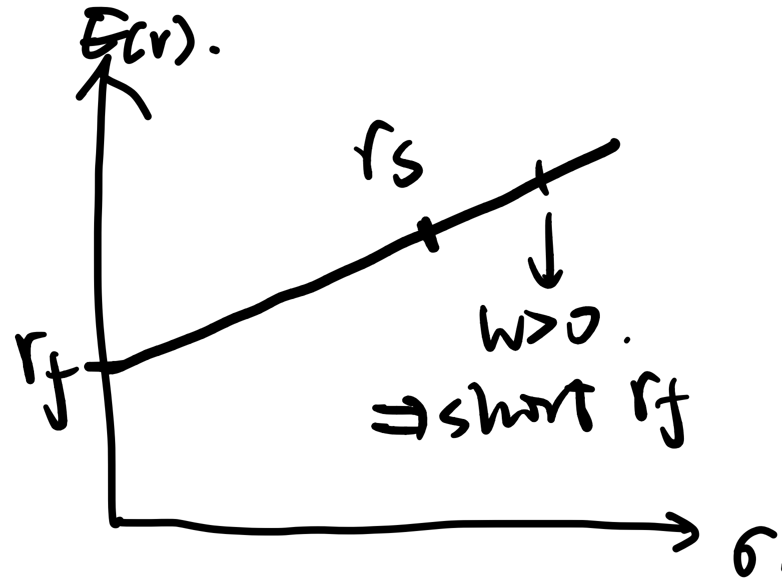
幸存者偏差:



$0.5\times(\frac{120}{70}-1)+0.49(\frac{120}{80}-1)=43\%$

$0.5\times(\frac{120}{70}-1)+0.49(\frac{120}{80}-1)+0.01\times(\frac{-1000}{80}-1)$

(历史上数据没有黑天鹅出现). $= 27\%$.



Portfolio：资产组合.

$(w_1, w_2, \cdots, w_n)$. n-number of assets.

w_i: share of mls on asset i.

$\sum_{i=1}^{n} w_i = 1$. $w_i > 0$ long 做多

$w_i < 0$ short 做空.

为何用σ，不用σ^2：

因为量纲一样.

$$\overline{r_p} = E[(1-w)r_f + w\cdot\tilde{r}_s]$$
$$= (1-w)r_f + w\cdot\overline{r_s}$$

$r_f + \tilde{r}_s$.

$(1-w, w)$.

$$\sigma_p^2 = E[(1-w)r_f + w\tilde{r}_s - (1-w)r_f - w\overline{r_s}]^2 = w^2\sigma_s^2.$$

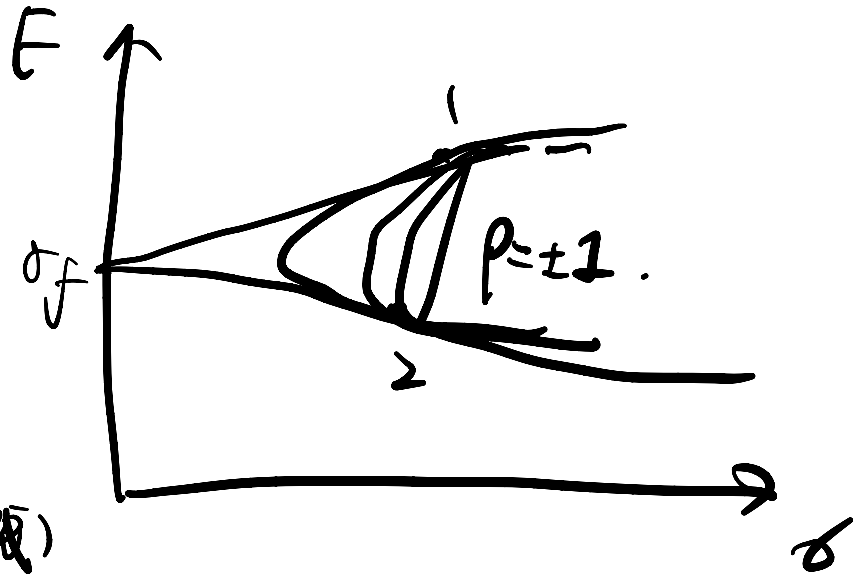
$$\Rightarrow \overline{r_p} = r_f + \frac{\overline{r_s} - r_f}{\sigma_s}\cdot\sigma_p$$ 直线方程

类似的：两个风险资产：

$\tilde{r}_1 + \tilde{r}_2$.

$(w, 1-w)$.

$$\overline{r_p} = w\overline{r_1} + (1-w)\overline{r_2}$$
$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2(1-w)w\sigma_1\sigma_2$$ (双曲线)



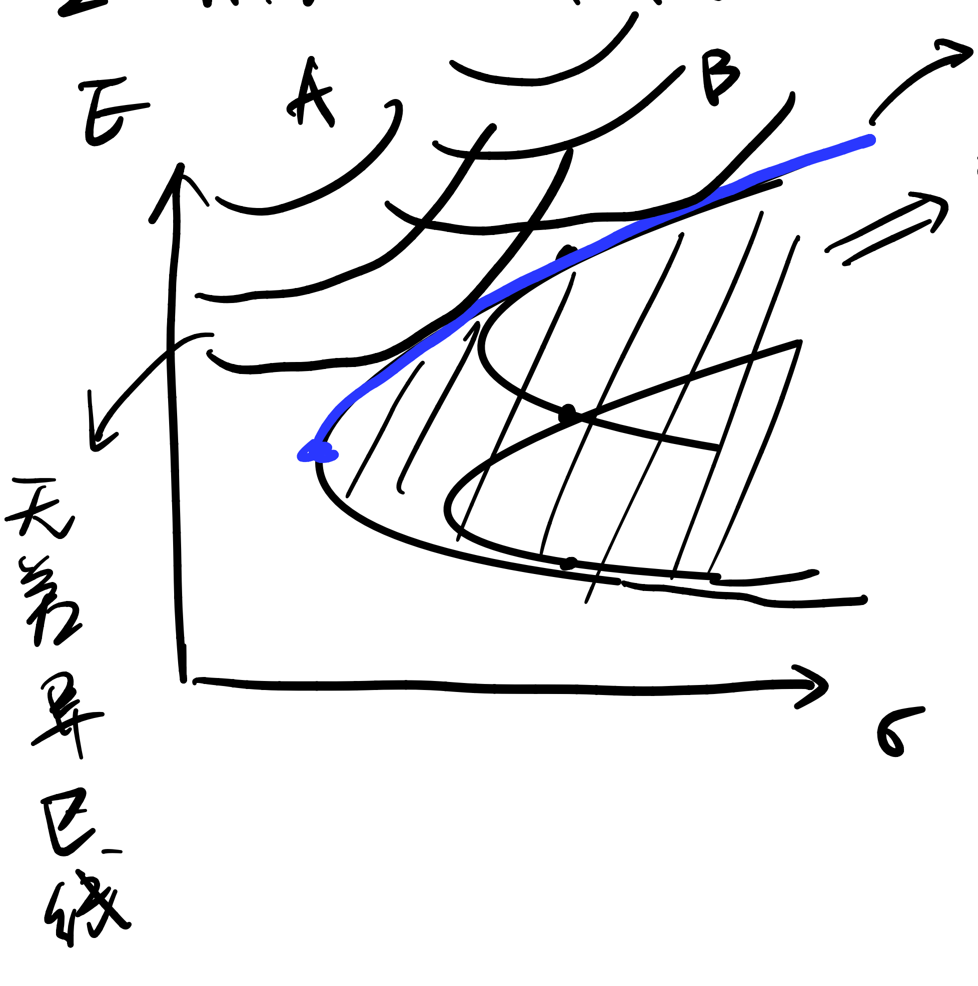
扁平程度取决于两类资产的相关程度.

2→n. 数学上算即可.

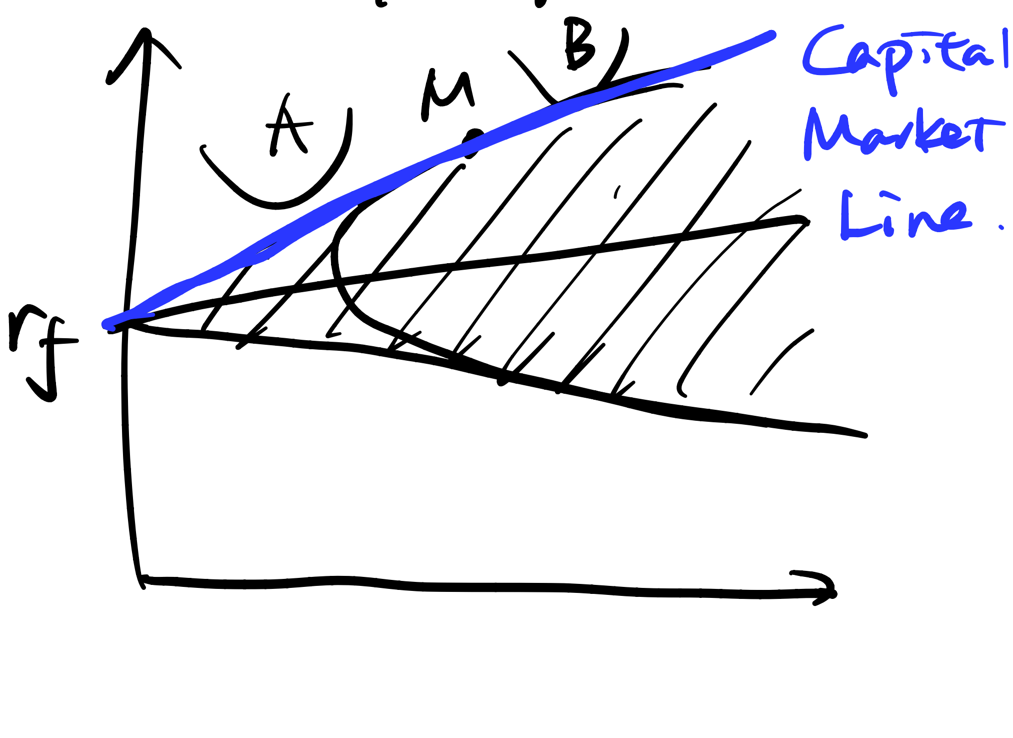
efficient frontier 有效前沿.

(也是双曲线).

⇒ 可选择区域.



加入r_f后、变成线



M: Market Portfolio：无论是风险厌恶，抑或风险喜好，均应该用 M 来选择 Portfolio.

$\boxed{\text{Mutual Fund Separation Theorem}}$:

Step 1：市场组合 M.

Step 2：市场组合和无风险资产进行分配.

Homework:

	A	B
E	0.1	0.15
σ	0.15	0.2
	w	1-w

$E_A = 0.1 \quad \sigma_A = 0.15$

$E_B = 0.15 \quad \sigma_B = 0.2$

$$\sigma_p^2 = E[(w r_A + (1-w) r_B) - (w E_A + (1-w) E_B)]^2$$

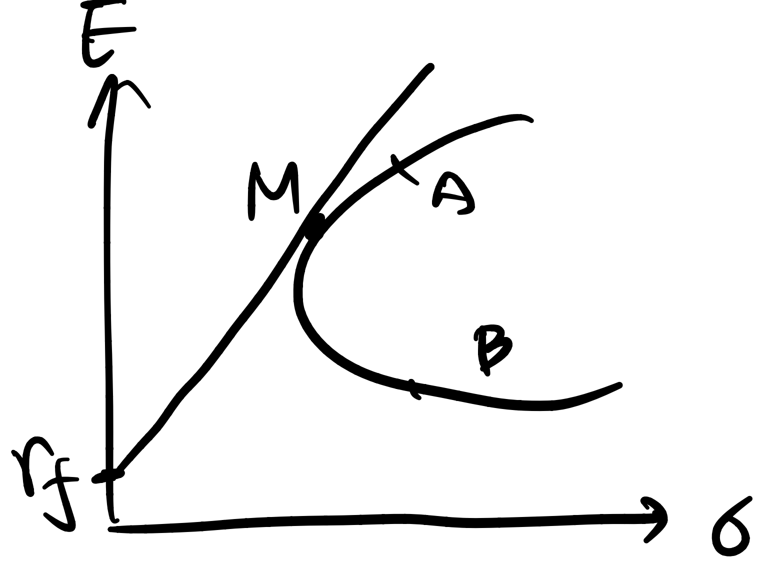
$$= w^2 \sigma_A^2 + (1-w)^2 \sigma_B^2 + 2w(1-w)\,Cov.$$

$$\frac{\partial \sigma_p^2}{\partial w} \Rightarrow \quad w = \frac{\sigma_B^2 - Cov}{\sigma_A^2 + \sigma_B^2 - 2Cov} \qquad 1-w = \frac{\sigma_A^2 - Cov}{\sigma_A^2 + \sigma_B^2 - 2Cov}$$

ρ	σ_{AB}	w^*	σ_p
-1	-0.03	0.57	0
0	0	0.64	0.12
0.5	0.015	0.77	0.14

ρ↑ w^*↑. 原因是：相关系数越大，两个资产会倾向于增大波动率更小的资产的权重.

(b). 先求解 AB 双曲线：



AB的双曲线：

$$E_p = wE_A + (1-w)E_B$$

$$\sigma_p^2 = w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_{12}.$$

但是这里是参数方十

变换思路：让M点 shape ratio 最大 (shape ratio：

$$S_p^2 = \frac{(E_p - r_f)^2}{\sigma_p^2}$$

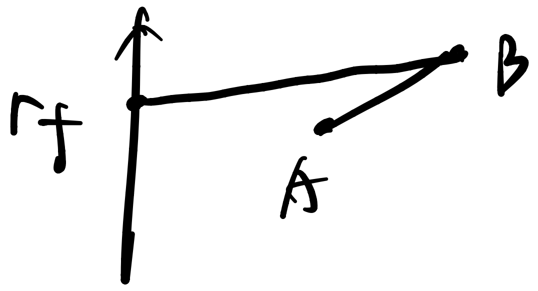
$$\frac{\partial S_p^2}{\partial w} = \frac{1}{\sigma_p^4}\left[\cancel{2}(E_p - \cancel{r_f})\frac{\partial E_p}{\partial w}\cdot\sigma_p^{\cancel{2}} - (E_p - r_f)^{\cancel{2}}\cdot\cancel{2}\frac{\partial \sigma_p}{\partial w}\cancel{\sigma_p}\right] = 0$$

$$\Rightarrow\ 2\frac{\partial E_p}{\partial w}\cdot\sigma_p^2 = (E_p - r_f)\cdot\frac{\partial \sigma_p^2}{\partial w}.$$

$$2(E_A - E_B)\left(w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_{12}\right) = (wE_A + (1-w)E_B - r_f)\cdot\left(2w\sigma_A^2 - 2(1-w)\sigma_B^2 + (2-2w)\cdot\sigma_{12}\right)$$

$$\left(2(E_A - E_B)\sigma_A^2 + \sigma_B^2 - 2\sigma_{12}\right)\cdot w^2$$
$$+ \left(-2(E_A - E_B)\cdot\sigma_B^2 - 2(E_A - E_B)\sigma_{12} - E_A + E_B - 2\sigma_A^2 - 2\sigma_B^2 + 2\sigma_{12}\right)w$$
$$+ 2(E_A - E_B)\sigma_B^2 - E_B + r_f + 2\sigma_B^2 - 2\sigma_{12} = 0.$$

		MVP		ORP	
ρ_{12}	cov	A头寸	B头寸	A头寸	B头寸
-1	-0.03	0.57	0.43	0.57	0.43
0	0	0.64	0.36	0.47	0.53
0.5	0.015	0.77	0.23	0.25	0.75



最低风险 最大收益 给B更多

$(r_1, r_2) \Rightarrow$ CML 方程推导

$w \quad 1-w$.

$$E_p = w(E_1 - E_2) + E_1$$

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)Cov \Rightarrow \frac{d\sigma_p^2}{dw} = (2\sigma_1^2 + 2\sigma_2^2 - 4Cov)w + 2Cov - 2\sigma_2^2$$

$$SR_p = \frac{E_p - r_f}{\sigma_p} \Rightarrow SR_p^2 = \frac{(E_p - r_f)^2}{\sigma_p^2}$$

$$\Rightarrow \frac{dSR_p}{dw} = 0 \Rightarrow \frac{1}{\sigma_p^4}\left(2(E_p - r_f)\cdot\frac{dE_p}{dw}\cdot\sigma_p^2 - (E_p - r_f)^2\cdot\frac{d\sigma_p^2}{dw}\right) = 0$$

$$\Rightarrow 2\sigma_p^2\cdot\frac{dE_p}{dw} = (E_p - r_f)\cdot\frac{d\sigma_p^2}{dw}$$

$$\Rightarrow \cancel{2}\left(w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)Cov\right)(E_1 - E_2) = \left(w(E_1 - E_2) + E_2 - r_f\right)\left((\cancel{2}\sigma_1^2 + \cancel{2}\sigma_2^2 - 4Cov)w + \cancel{2}Cov - \cancel{2}\sigma_2^2\right)$$

$\Rightarrow$ w^2 的系数:

$$(\sigma_1^2 + \sigma_2^2 + 2Cov)(E_1 - E_2) - (E_1 - E_2)(\sigma_1^2 + \sigma_2^2 - 2Cov) = 0$$

w 的系数:

$$(-2\sigma_2^2 + 2Cov)(E_1 - E_2) - (E_1 - E_2)(Cov - \sigma_2^2) - (E_2 - r_f)(\sigma_1^2 + \sigma_2^2 - 2Cov)$$

$$(E_1 - E_2)(Cov - \sigma_2^2) - (E_2 - r_f)(\sigma_1^2 + \sigma_2^2 - 2Cov)$$

$$= E_1(Cov - \sigma_2^2) + E_2(\sigma_2^2 - Cov - \sigma_1^2 - \sigma_2^2 + 2Cov) + r_f(\sigma_1^2 + \sigma_2^2 - 2Cov)$$

$$= E_1(Cov - \sigma_2^2) + E_2(Cov - \sigma_1^2) + r_f(\sigma_1^2 + \sigma_2^2 - 2Cov)$$

$$= -(E_1 - r_f)\sigma_2^2 - (E_2 - r_f)\sigma_1^2 + (E_1 + E_2 - 2r_f)Cov$$

常数:

$$\sigma_2^2(E_1 - E_2) - (E_2 - r_f)(Cov - \sigma_2^2)$$

$$= \sigma_2^2(E_2 - r_f) - Cov(E_2 - r_f)$$

$$\Rightarrow w^* = \frac{\sigma_2^2(E_2 - r_f) - Cov(E_2 - r_f)}{(E_1 - r_f)\sigma_2^2 + (E_2 - r_f)\sigma_1^2 - (E_1 + E_2 - 2r_f)Cov} \quad *$$

$\Downarrow$ 这就是 (r_1, r_2) 组合时，CML切点M所需要的 weight w^*.