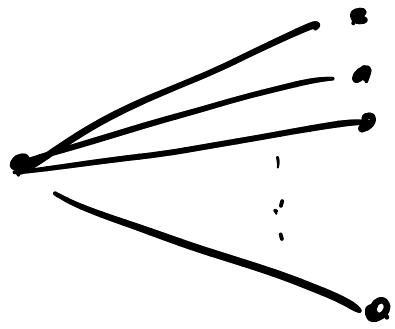
# Chapter 10.

Assumption: (i) Consumption Non-storable.

(ii). Endowment economy 禀赋经济(外在给予).

储蓄行为通过资产买卖来代替.

period 0 period 1 state $s \in S$ (即是数目、也是集合).



$0 < \pi_s < 1$, $\sum_{s=1}^{S} \pi_s = 1$.

Random Variable: $\tilde{r} = (r_1, r_2, \cdots, r_S)^T$.

$\tilde{a} = (a_1, a_2, \cdots, a_S)^T$.

一个资产 Asset j 它的支付: $\begin{bmatrix} x_1^j \\ x_2^j \\ \vdots \\ x_S^j \end{bmatrix} \in \mathbb{R}^S$. $1 \le j \le J$.

共存J种资产. ⇒ J个资产的 Payoff Matrix: $X = \begin{bmatrix} x_1^1 & \cdots & x_1^J \\ \vdots & & \vdots \\ x_S^1 & \cdots & x_S^J \end{bmatrix} \in \mathbb{R}^{S \times J}$

(Payoff是在未来的支付.也即在Period 1).

在Period 0 的价格:

$P = [p_1, \cdots, p_J]$. Asset Pricing问题: 给定 $X$ ⇒ 得到 $P$.

Portfolio: $(\theta_1, \cdots, \theta_J)^T$. Period 0 的价格:

$$\sum_{j=1}^{J} p_j \theta_j = P\theta.$$

$$\begin{array}{c} 1 \\ \vdots \\ S \end{array} \begin{bmatrix} \sum_{j=1}^{J} \theta_j x_1^j \\ \vdots \\ \sum_{j=1}^{J} \theta_j x_S^j \end{bmatrix}$$ 支付向量.

$= X\theta$.

Definition of Compete Market: (完备市场).

$\forall\, c = (c_1, c_2, \cdots, c_S)^T$ $\exists\, \theta = (\theta_1, \cdots, \theta_J)^T$.

消费

s.t. $c_s = \sum_{j=1}^{J} \theta_j x_s^j$, $s = 1, 2, \cdots, S$

资产 $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\forall c_1, c_2$. $\exists \theta_1, \theta_2$. 均存在.

$$\begin{cases} c_1 = \theta_1 + 3\theta_2 \\ c_2 = 2\theta_1 + 4\theta_2 \end{cases} \Rightarrow \begin{cases} \theta_1 = -2c_1 + \frac{3}{2}c_2 \\ \theta_2 = c_1 - \frac{1}{2}c_2 \end{cases}$$

Compete 的条件：$X$ 是满秩.

Compete：任何一个消费计划都可以被满足

$\Rightarrow$ 任何两期之间，均可以任意调配.

$X \in \mathbb{R}^{S \times J}$. $J \geq S$.

$$\begin{cases} c_1 = \theta_1 \\ c_2 = \theta_2 \\ \vdots \quad \vdots \\ c_S = \theta_S \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \triangleq I \quad (\theta_1, \cdots, \theta_S)^T$$

Arrow Securities.
阿罗证券.

Arrow-Debreu Market.
所有完备市场等价于阿罗市场.

State price：$\varphi_1, \cdots, \varphi_S$. 阿罗证券在 0 时刻的价格.

$\varphi \triangleq [\varphi_1, \cdots, \varphi_S]$ $X^j = (x_1^j, \cdots, x_S^j)$.

例子：$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. 阿罗证券：$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $X = 3I_1 + 2I_2 + I_3$.

$I_1 \; I_2 \; I_3$.

$\therefore P = 3\varphi_1 + 2\varphi_2 + \varphi_3$ 是价格.

$\therefore p^j = \sum_{s=1}^{S} x_s^j \varphi_s$. 无风险资产：$p = \sum_{s=1}^{S} \varphi_s$.

现在问题是求 $\varphi_s$ 是多少：

$$\max_{\theta_1, \cdots, \theta_J} u^k(c_0) + \sum_{s=1}^{S} \pi_s u^k(c_s) \cdot \delta^k$$

$c_0$：零时刻消费.
$c_s$：其实应该写成 $c_{1,s}$
$\delta$：贴现因子.

s.t. $c_0 = e_0 - \sum_{j=1}^{J} \underset{\sim\sim}{p_j\theta_j}$ 消费 period 0.

$c_s = e_s + \sum_{j=1}^{J} \underset{\sim\sim}{x_s^j\theta_j}$ 购买的支付. $s=1,\dots,S$ period 1.

将其改写成一个 Arrow Debreu Market.

$\Rightarrow$ max $u(c_0) + \delta\sum_{s=1}^{S}\pi_s u(c_s)$.

s.t. $c_0 = e_0 - \sum_{s=1}^{S}\varphi_s\theta_s$.

$c_s = e_s + \theta_s$. ($\because x$ 是单位的 AD Security).

构造拉格朗日函数.

$$\mathcal{L} = u(c_0) + \delta\sum_{s=1}^{S}\pi_s u(c_s) + \lambda_1\left[-c_0 + e_0 - \sum_{s=1}^{S}\varphi_s(c_s - e_s)\right].$$

FOC: $\frac{\partial L}{\partial c_0} = 0 \Rightarrow u'(c_0) = \lambda_1$

$\frac{\partial L}{\partial c_s} = 0 \Rightarrow \delta\,\pi_s{}'(c_s) = \lambda_1\varphi_s$.

$$\Rightarrow \frac{\pi_s u'(c_s)}{\pi_{s'} u'(c_{s'})} = \frac{\varphi_s}{\varphi_{s'}}.$$

不同状态下：边际效应 * 概率 的比值是等于价格之比.

Example:

$u_1 = \log c$. $u_2 = 2\sqrt{c}$ (都是CRRA) $\delta = 1$

$$\begin{array}{c} a \\ b \end{array}\begin{bmatrix} 1 & 0.5 \\ 1 & 2 \end{bmatrix}$$

1 unit of consumption good at period 1.
1 stock.

$0.5\varphi_a + 2\varphi_b$.

Step 1: Individual Optimization Problem.

Step 2: Market Clear.

$$\mathcal{L} = \log c_{1,0} + \left(\tfrac{1}{2}\log c_{1,a} + \tfrac{1}{2}\log c_{1,b}\right) + \lambda_1\left[-c_{1,0} - \varphi_a c_{1,a} - \varphi_b c_{1,b} + 1\right].$$

$\frac{\partial L}{\partial c_{1,0}} = 0 \Rightarrow \frac{1}{c_{1,0}} = \lambda_1$, $\frac{\partial L}{\partial c_{1,a}} = 0 \Rightarrow \frac{1}{2c_{1,a}} = \lambda_1\varphi_a$, $\frac{1}{2c_{1,b}} = \lambda_1\varphi_b$.

⇒ 带回约束条件, 得到:

$c_{1,0}=\frac{1}{2}$, $c_{1,a}=\frac{1}{4\varphi_a}$. $c_{1,b}=\frac{1}{4\varphi_b}$.

Example 2: $\max\ 2\sqrt{c_{2,0}}+\frac{1}{2}c_d\sqrt{c_{2,a}}+\frac{1}{2}2\sqrt{c_{2,b}}$.

s.t. $c_{2,0}+\varphi_a c_{2,a}+\varphi_b\cdot c_{2,b}=0.5\varphi_a+2\varphi_b$.

$c_{1,0}+c_{2,0}=1$, $c_{1,a}+c_{2,a}=0.5$ $c_{1,b}+c_{2,b}=2$.

$\begin{cases}\varphi_a\approx 0.81 \\ \varphi_b\approx 0.32\end{cases}$ $\varphi_a+\varphi_b=1.13$. stock price $=1.04$.

Home Work:

(0.1). $\mathrm{Rank}(X)\leq J$. $\mathrm{Rank}(X)\leq S$.

(0.2) ⓐ $\mathrm{Rank}\begin{bmatrix}1&0&0\\2&4&2\\2&2&1\end{bmatrix}=\mathrm{Rank}\begin{bmatrix}1&0&0\\0&4&2\\0&2&1\end{bmatrix}=\mathrm{Rank}\begin{bmatrix}1&0&0\\0&2&1\\0&0&0\end{bmatrix}$ 不是.

ⓑ.

$\mathrm{Ranks}\begin{bmatrix}1&2&0&0\\1&4&1&2\\1&2.5&1&2\end{bmatrix}=3$. 是:

ⓒ

$\begin{bmatrix}uS_a & dS_a\\u^2S_b & d^2S_b\end{bmatrix}$. $u\neq d$ 是完备 $u=d$ 不是完备

(0.3). $\begin{bmatrix}3&1&1\\2&0&2\\1&3&0\end{bmatrix}$

① $\begin{bmatrix}3&1&1\\2&0&2\\1&3&0\end{bmatrix}\Leftrightarrow\begin{bmatrix}1&3&0\\3&1&1\\2&0&2\end{bmatrix}\Leftrightarrow\begin{bmatrix}1&3&0\\0&-8&1\\0&-6&2\end{bmatrix}\Leftrightarrow\begin{bmatrix}1&3&0\\0&-8&1\\0&-3&1\end{bmatrix}$ 是完备的.

需搞清楚关系：

$$X = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 3 & 0 \end{bmatrix}. \qquad p = \begin{bmatrix} 1.3 \\ 1.1 \\ 0.6 \end{bmatrix}.$$

∴ 有 $D = X\cdot\theta \Rightarrow \theta = \begin{bmatrix} -0.4 \\ 1.8 \\ 2.4 \end{bmatrix}$.

$\Rightarrow P_D = p^T\cdot\theta = 2.9$.

c) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} \Theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \Theta = \begin{bmatrix} 0.6 & -0.3 & -0.2 \\ -0.2 & 0.1 & 0.4 \\ -0.6 & 0.8 & 0.2 \end{bmatrix}$.

$P_{AS_1} = \Theta_1 \times \theta = 0.2$

$P_{AS_2} = \Theta_2 \times \theta = 0.2$

$P_{AS_3} = \Theta_3 \times \theta = 0.3$.

d) $\varphi\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 0.2\times 3 + 0.2\times 4 + 0.3\times 5 = 2.9$.

一致.

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(0.4). $u_1(c) = \ln c$. $u_2(c) = 2c^{\frac{1}{2}}$.

$$\begin{array}{ccc} & \text{bond} & \text{stock} \\ a & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} 1 & 1.5 \\ 1 & 0.5 \end{bmatrix}$}} \\ b & & \end{array}$$

a) $\max\limits_{c_0, c_a, c_b} \log c_0 + \frac{1}{2}\log c_a + \frac{1}{2}\log c_b$.

s.t. $c_0 + \psi_a c_a + \psi_b c_b = 2$.

$L = \log c_0 + \frac{1}{2}\log c_a + \frac{1}{2}\log c_b + \lambda(2 - c_0 - \psi_a c_a - \psi_b c_b)$.

$$\left.\begin{aligned}
\frac{\partial L}{\partial C_0} = 0 &\Leftrightarrow \frac{1}{C_0} = \lambda \\
\frac{\partial L}{\partial C_a} = 0 &\Leftrightarrow C_a = \frac{1}{2\varphi_a \lambda} \\
\frac{\partial L}{\partial C_b} = 0 &\Leftrightarrow C_b = \frac{1}{2\varphi_b \lambda}
\end{aligned}\right\} \Rightarrow \begin{aligned} &\lambda = 1 \\ &C_0 = 1,\ C_a = \frac{1}{2\varphi_a},\ C_b = \frac{1}{2\varphi_b} \end{aligned}$$

消费者2：

$$\max_{C_0, C_a, C_b} \ 2\sqrt{C_0} + \sqrt{C_a} + \sqrt{C_b} \quad \text{s.t.} \quad C_0 + \varphi_a C_a + \varphi_b C_b = 1.5\varphi_a + 0.5\varphi_b$$

$$\Rightarrow \begin{cases} C_0 = \frac{1}{\lambda^2} \\ C_a = \frac{1}{4\varphi_a^2\lambda^2} \\ C_b = \frac{1}{4\varphi_b^2\lambda^2} \end{cases} \Rightarrow \begin{cases} C_0 = \frac{1.5\varphi_a + 1.5\varphi_b}{1 + 1/4\varphi_a + 1/4\varphi_b} \\ C_a = \frac{1.5\varphi_a + 1.5\varphi_b}{(1 + 1/4\varphi_a + 1/4\varphi_b)\,4\varphi_a^2} \\ C_b = \frac{1.5\varphi_a + 1.5\varphi_b}{(1 + 1/4\varphi_a + 1/4\varphi_b)\,4\varphi_b^2} \end{cases}$$

市场出清 $2 = 1 + \frac{1.5\varphi_a + 1.5\varphi_b}{1 + 1/4\varphi_a + 1/4\varphi_b}$

$$\begin{aligned} 1 &= C_{1o} + C_{2o} \\ 1.5 &= C_{1a} + C_{2a} \\ 0.5 &= C_{1b} + C_{2b} \end{aligned} \Rightarrow \begin{cases} \varphi_a = 0.608 \\ \varphi_b = 1.366 \end{cases} \therefore \begin{cases} \text{stock} = \frac{1.5\varphi_a + 0.5\varphi_b}{2} = 0.798 \\ \text{bond} = \frac{\varphi_a + \varphi_b}{2} = 0.987 \end{cases}$$

b). $\frac{1}{2}(\varphi_a + 2\varphi_b) = 1.67$

c). 不改变 消费量仅由禀赋和偏好决定