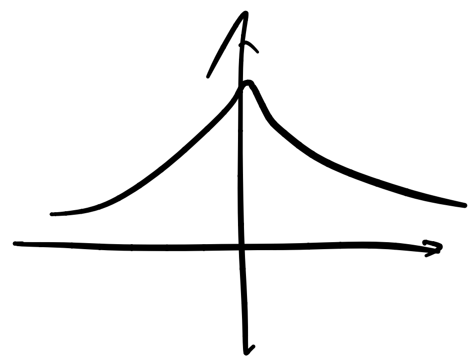
Chapter 18.

Black-Scholes-Merton 公式:

天全书分析、理论的钥匙.

- Normal, Log-normal
- Brownian Motion (布朗运动). (Wiener Process).
- Stochastic Differentiation (Ito's lemma). Integration.
- Black-scholes Partial Differential Equation.
- Martingale Method to B-S.

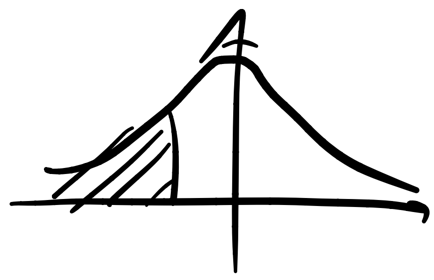
Central Limit Theorem.



$x \sim \phi(\mu, \sigma^2)$. $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$t \triangleq x-\mu/\sigma^2 \sim \phi(0,1)$. $f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

密度函数 $\longrightarrow$



$N(v) = Pr(t \le v)$.

$N(-v) = 1 - N(v)$.

Log-Normal:

$\log S_T - \log S_0 \sim \Phi(\mu, \sigma^2)$.

$= \log(S_T/S_0) = \log(1 + (S_T - S_0)/S_0) \approx \frac{S_T - S_0}{S_0}$ (股票对数差分满足正态分布).

$\Rightarrow S_T = S_0 \cdot e^x$. $x \sim \phi(\mu, \sigma^2)$. $\boxed{\begin{array}{l} t = \frac{x-\mu}{\sigma^2} \\ x = \sigma t + \mu \end{array}}$

$E(e^x) = \int_{-\infty}^{+\infty} e^x \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx$

$= \int_{-\infty}^{\infty} e^{\sigma t + \mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2}} d(\sigma t + \mu)$

$= e^{\mu} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{\sigma t - \frac{t^2}{2}} dt = e^{\mu} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-\sigma)^2 - \sigma^2}{2}} dt.$

$$= e^{\mu+\frac{1}{2}\sigma^2} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(t-\sigma)^2}{2}} \, d(t-\sigma)$$ 正态分布密度积分.

$$= e^{\mu+\frac{1}{2}\sigma^2}.$$

总结：如果：$\log X \sim \phi(\mu, \sigma^2)$.

$$E(e^X) = e^{\mu+\frac{1}{2}\sigma^2}.$$

1827年布朗运动： 1905 Einstein 水分子热运动.

1923年 Wiener Wiener Process. 维纳运动.

离散时间的连续化.

Random Walk: i.i.d. $\varepsilon_t \sim \phi(0,1)$.

$\{Z_t\}$ $\quad Z_1 - Z_0 = \varepsilon_0$

$\vdots$

$Z_{t+1} - Z_t = \varepsilon_t$.

$$E(Z_t - Z_0) = E\left(\sum_{j=1}^{t} \varepsilon_{t-j}\right) = 0$$

$$Var(Z_t - Z_0) = E\left(\sum_{j=1}^{t} \varepsilon_{t-j}\right)^2 = t.$$

$$\sigma(Z_t - Z_0) = \sqrt{t}.$$

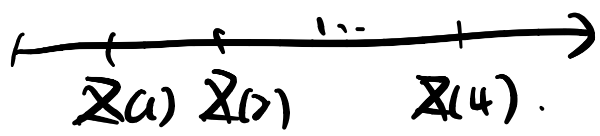
$Z_{t+\Delta} - Z_t \sim \phi(0, \Delta)$. 取$\Delta$为正数即为随机连续时间游走

Definition 18.1 $\{X(t), t \geq 0\}$. Brownian Motion (or Wiener Process).

(i) Process with independent movement

(ii) $\forall\, s, t > 0$ $\quad X(s+t) - X(s) \sim \phi(0, \sigma^2 t)$.

(iii) Continuous. 独立增量过程.



$X(4) - X(3)$ 与 $X(2) - X(1)$ 不相关

Stochastic Differentiation.

$dZ_t \triangleq \lim_{\Delta \to 0+} (Z_{t+\Delta} - Z_t)$. 看 dZ_t 的期望和方差.

（连续时间 dZ_t 对应离散 ε_t）.

$E(dz_t)=0$.

$Var(dz_t) = E[(dz_t - E(dz_t))^2] = E[dz_t^2] \sim \sqrt{dt}$. 是和dt量级一样.

$$\frac{dz_t}{dt} = \lim_{\Delta\to 0} \frac{\sqrt{\Delta}}{\Delta}$$

$dx_t = \mu dt + \sigma dz_t$. 带漂移的布朗运动.

$\frac{dS_t}{S_t}$ (变化率) $= \mu dt + \sigma dz_t$. (股价变化率)

衍生品价格和股价的关系:

Ito's Lemma: 解决了: 一个 x 做布朗运动. f(x) 做什么样的运动?

	dz_t	dt
dz_t	dt	0
dt	0	0

假设 x 做布朗运动:

$y_t = f(x_t)$.

① Ito's lemma: y_t Taylor Expansion:

$$dy_t = \frac{\partial f}{\partial x} \cdot dx_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \cdot dx_t^2 + \cdots$$

$$= \frac{\partial f}{\partial x}(\mu dt + \sigma dz_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \cdot (\mu^2 dt^2 + 2\mu\sigma\, dt\, dz_t + \sigma^2 (dz_t)^2)$$

$$= \frac{\partial f}{\partial x}(\mu dt + \sigma dz_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\sigma^2 \cdot dt).$$

$$dy_t = \underbrace{\left(\frac{\partial f}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \cdot \sigma^2\right)}_{\text{漂移项}} dt + \underbrace{\frac{\partial f}{\partial x}}_{\text{随机项}} \cdot \sigma dz_t.$$

也是一个布朗运动. 所以连续时间可以更容易.

随机积分：Stochastic Integration.

$$\int_{t=0}^{T} dZ_t = \lim_{\Delta\to 0}[Z_\Delta - Z_0 + \dots + Z_T - Z_{T-\Delta}] = Z_T - Z_0 \sim \phi(0, T).$$

$$\int_{t=0}^{T} dX_t = \int_{t=0}^{T} \mu\, dt + \int_{t=0}^{T} \sigma\, dZ_t.$$

$$X_T - X_0 = \mu T + \sigma \int_{t=0}^{T} dZ_t.$$

$$X_T = X_0 + \mu T + \sigma \int_{t=0}^{T} dZ_t \sim \phi(X_0 + \mu T, \sigma^2 T)$$

$dS_t = \mu S_t dt + \sigma S_t dZ_t$ 股票，几何布朗.

$dB_t = r B_t dt$ 债券(无风险) $\Rightarrow$ 单利计算.

$C_t = f(t, S_t)$. 衍生品价格.

构造：Risk Free Portfolio. (Stock, Derivative)

(D , -1).

$$V(t, S_t) = -f(t, S_t) + D \cdot S_t \Rightarrow -f(t, S_t) + \frac{\partial f}{\partial S_t} \cdot S_t.$$

$$\frac{\partial V}{\partial S_t} = -\frac{\partial f}{\partial S_t} + D = 0 \text{ (无风险)} \Rightarrow D = \frac{\partial f}{\partial S_t}.$$

$$dV(t, S_t) = -df(t, S_t) + \frac{\partial f}{\partial S} \cdot dS_t.$$

$$= -\left[\frac{\partial f}{\partial t} \cdot dt + \cancel{\frac{\partial f}{\partial S_t} dS_t} + \frac{1}{2}\frac{\partial f}{\partial S^2}(dS_t)^2\right] + \cancel{\frac{\partial f}{\partial S} dS_t}$$

$$= -\frac{\partial f}{\partial t} dt - \frac{1}{2}\frac{\partial f^2}{\partial S^2}(\mu S_t dt + \sigma S_t dZ_t)^2.$$

$$= -\frac{\partial f}{\partial t} dt - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2 (dZ_t)^2$$

$$= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\cdot \sigma^2 \cdot S_t^2\right)\cdot dt$$ 没有 dz_t，无风险.

$$dV(t,S_t) = rV(t,S_T)\,dt$$ 无风险

$$\Rightarrow -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\cdot \sigma^2 S_t^2\right) = -\left(rf + r\frac{\partial f}{\partial S_t}\cdot S_t\right)$$

$$\Rightarrow \frac{\partial f}{\partial t} + rS_t\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S^2} = rf$$ B-S PDE 偏微分方程.

European Call: $f(T, S_T) = \max\{S_T - K, 0\}$

给定边界条件来求解 Black-Scholes 公式

Martingale Method

$$d(\log B_t) = \frac{1}{B_t}\,dB_t = r\,dt$$

$$dB_t = rB_t\,dt$$

$$\Rightarrow \int_{t=0}^{T} d(\log B_t) = \int_{t=0}^{T} r\,dt$$

$$\Rightarrow \log B_T - \log B_0 = rT$$

$$\Rightarrow B_T = B_0 e^{rT}$$ (复利公式). 无风险债券定价.

$$d(\log S_t) = \frac{1}{S_t}\cdot dS_t - \frac{1}{2}\cdot\frac{1}{S_t^2}\cdot dS_t^2$$

$$= \mu\,dt + \sigma\,dz_t - \frac{1}{2}\frac{1}{S_t^2}\cdot \sigma^2 S_t^2\,dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\,dz_t$$ $\Rightarrow$ 股票的 log 也是布朗运动.

$$\Rightarrow \int_{t=0}^{T} d(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\int_{t=0}^{T} dt + \int_{t=0}^{T}\sigma\,dz_t$$

$$\log S_T - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\int_{t=0}^{T} dz_t$$

$\Rightarrow S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)T + \sigma\int_{t=0}^{T} dZ_t)$ 对数正态分布.

$E[S_T] = S_0 e^{(\mu-\frac{1}{2}\sigma^2)T + \frac{1}{2}\sigma^2 T}$.

$= S_0 e^{\mu T}$ $\Rightarrow$ 不满足鞅性. 不是 $S_0 e^{rT}$.

在真实世界里不是鞅. 很正常.

$\Rightarrow$ 存在等价鞅测度 存在且唯一: 贴现为鞅:

$\hat{E}(S_T) = S_0 e^{rT}$.

$S_T = S_0 \exp((r - \frac{1}{2}\sigma^2)T + \sigma\int_0^T d\hat{Z}_t)$.

$\Rightarrow$ ① μ 变为 r

② dZ_t 变成 $d\hat{Z}_t$. 是在等价鞅测度的方程.

$C_0 = e^{-rT} \cdot \hat{E}[\max\{S_T - K, 0\}]$.

EMM : $\log S_T \sim \Phi[(r - \frac{1}{2}\sigma^2)T + \log S_0, \sigma^2 T]$.

Define: $\log S_T = a + bu$.

$a = \ln S_0 + (r - \frac{1}{2}\sigma^2)T$.

$b = \sigma\sqrt{T}$. $\qquad S_T = e^{a+bv} = K$ (行权价阈值).

$\Rightarrow v = \frac{\log K - a}{b} = \log K - \log S_0 - (r - \frac{1}{2}\sigma^2)T / \sigma\sqrt{T}$.

$\hat{E}[\max\{S_T - K, 0\}]$

$= \int_v^{+\infty} (e^{a+bu} - K)\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$.

$= \int_v^{+\infty} \frac{1}{\sqrt{2\pi}} \exp[a + bu - \frac{u^2}{2}] du - K\int_v^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$.

$$= e^{a+\frac{1}{2}b^2}\int_{v-b}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(u-b)^2}{2}\right]d(u-b) - K(1-N(v)).$$

$$= S_0 e^{rT} N(b-v) - K N(-v)$$

$$C_0 = e^{-rT}\hat{E}[\max\{S_T - K, 0\}].$$

$$= S_0 N(b-v) - e^{-rT} K N(-v).$$

$$C_0 = S_0 N(d_1) - e^{-rT} K N(d_2)$$

$$d_1 = b - v = \frac{\ln S_0/K + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Black-Scholes Formula.

经济含义. $N(d_2)$: 期权行权概率.

$K\cdot N(d_2)$: 支付期望.

$S_0 N(d_1)$: 贴现值, 在 $S_T > K$ 时 等于 S_T. 其他时候等于 0.

Homework 18:

$$\begin{cases} dS_t = S_t(\mu dt + \sigma dz_t) \\ dB_t = rB_t dt \end{cases}$$

无风险债券: $B_T = B_0 e^{rT}$.

股价: $S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\int_0^T d\tilde{z}_t}$.

等价鞅测度下: $C_0 = e^{-rT}\tilde{E}\{\max\{S_T - K, 0\}\}$.

在0时刻下:

但这是对 T 时刻到期的买入欧式期权.

我们运用:

$d(\ln S_t) = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\, dz_t$.

$\Rightarrow$. $S_T = S_t\, e^{(\mu-\frac{1}{2}\sigma^2)(T-t)+\sigma\int_t^T dz_t}$.

$\ln S_T = \ln S_t + (\mu-\frac{1}{2}\sigma^2)(T-t) + \sigma\int_t^T dz_t \sim N(\ln S_t + (\mu-\frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$

$\Rightarrow E(e^{S_T}) = e^{\ln S_t + (\mu-\frac{1}{2}\sigma^2)(T-t) + \frac{1}{2}\sigma^2(T-t)}$.

$= S_t e^{\mu(T-t)}$. 不符合鞅性 $\Rightarrow$ $\mu = r$.

$\Rightarrow$. 令: $S_T = a + bu$.

$a = \ln S_t + (r - \frac{1}{2}\sigma^2)(T-t)$

$b = \sqrt{T-t}\cdot\sigma$.

当: $u = u_0$ 时. $S_T = K$.

$C_T = \tilde{E}[\max\{S_T - K, 0\}]$.

$\Rightarrow$. $C_T = e^{r(T-t)} S_t N(d_1) - KN(d_2)$

$C_t = S_t N(d_1) - e^{-r(T-t)} KN(d_2)$.