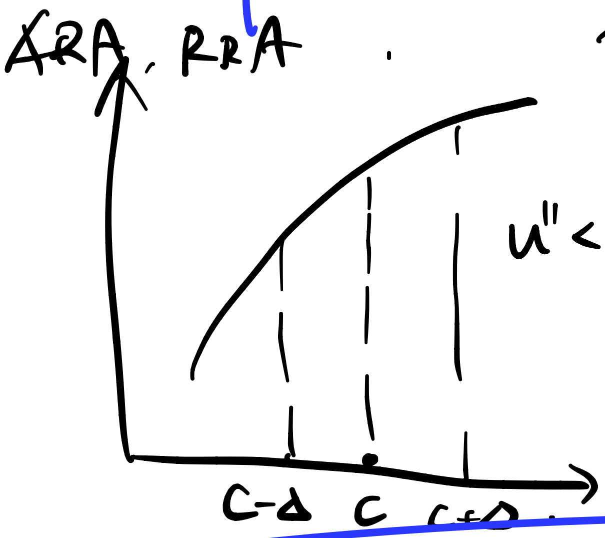
Chapter 9. ⇒ 风险厌恶. Portfolio Optimization.



$\tilde{r}$ 风险资产, r_f 无风险资产.

人什么时候会买风险资产?

$E(\tilde{r})-\Delta = r_f$ $\Delta>0$? $(\tilde{r}, r_f)$

a, w_0-a.

这里就预应注意一定要减去 w_0

$$\max_a E[u(\tilde{w})] = \max_a E\{u[w_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

期末总财富

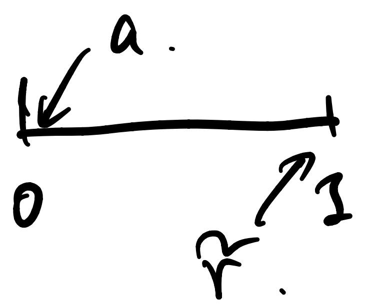
FOC: 一阶条件

$$= \max_a \sum_{n=1}^{N} p_N u[w_0(1+r_f)+a(r_n-r_f)].$$

$$\Rightarrow E\{u'[w_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\} = 0 \quad (\text{FOC}=0).$$

Prop. 9.1: (i) $a^*>0 \Leftrightarrow E(\tilde{r})>r_f$.

Proof: Def: $V(a) = E\{u[w_0(1+r_f)+a(\tilde{r}-r_f)]\}$.



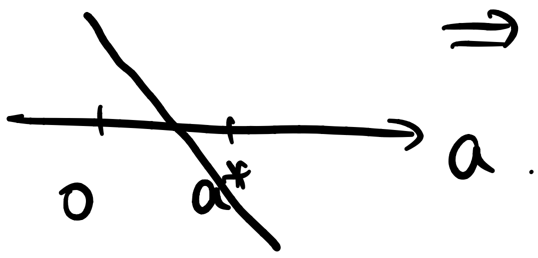
FOC: $V'(a^*)=0$

$V''(a)=E\{u''[w_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)^2\}<0$.

$\Rightarrow V'(a)$ 是减函数.

$a^*>0 \Leftrightarrow V'(0)>0 \Rightarrow V'(0)=E\{u'[w_0(1+r_f)](\tilde{r}-r_f)\}>0$.

$\Rightarrow V'(0)=u'[w_0(1+r_f)]\cdot E(\tilde{r}-r_f)>0$ 得证.



Arrow-Pratt Aproximation.

$a=0 \to a>0$ ① $Eu\uparrow \sim a$ 因为: $E(\tilde{r})>r_f$.

② $EU\downarrow \sim a^2$ 风险导致

所以 负面效应小于正面效应.

Prop. 9.2. $E(\tilde{r})>r_f$. $u''(\cdot)<0$.

(i) $a^{*\prime}(w_0)<0 \Leftrightarrow R_A'(\cdot)<0$ (DARA) (decreasing ARA)

$R_A'(\cdot)<0$ 意味着 财富增加 ⇒ 绝对风险规避系数越小.

财富越多. 越不怕风险.

$E[u''(\tilde{w})(\tilde{r}-r_f)]$ 正负号不好判断 (证明技巧).

$\sum_{n=1}^{N} P_n(\cdot\cdot)$ ⇒ 一项一项讨论正负号.

$e(w_0) \triangleq \frac{da^*}{a} / \frac{dw_0}{w_0}$ 弹性: 初始财富增加比例与投在风险资产上增加的比例.

Prop. $E(\tilde{r})>0$. $u''(\cdot)<0$.

(i) $e(w_0)>1 \Leftrightarrow R_R'(\cdot)<0$ (DRRA). → 财富↑. $\tilde{r}$↑.

$=1$ $\quad =0$ (CRRA)

<1 $\quad >0$ (IRRA). 财富↑. r↓.

财富越多. 相对风险规避系数 <0.

现实中的人. 大多数是 CRRA ($u(c)=\frac{c^{1-\gamma}-1}{1-\gamma}$).

Risk Neutral: $u(c)=\alpha c \Rightarrow E(\tilde{r})>r_f \Rightarrow a\to\infty$.

Saving Without Uncertainty

$W=(W-S)+R$. $\quad$ W: wealth. $\quad R\uparrow \to S$?

S: saving

R: return.

储蓄率上升. 应该多储蓄还是多消费?

$\max\ u(W-S)+\delta u(SR)$ $\quad \delta$: 贴现因子. 常数

FOC: 对S: $-u'(W-S)+\delta R u'(SR)=0$. $\quad \frac{dS}{dR}$?

再对 R 求导. S是R函数:

$$-\frac{ds}{dR}u''(w-s) = \delta u'(sR) + \delta R^2\left[u''(sR)\left(s+R\frac{ds}{dR}\right)\right].$$

$$\Rightarrow \frac{ds}{dR} = \frac{\delta u'(sR) + \delta sR\, u''(sR)}{-u''(w-s) - \delta R\, u''(sR)}.$$

$$\text{numerator} = \delta u'(sR)\Big[1 + \underbrace{\frac{sR\,u''(sR)}{u'(sR)}}_{RRA}\Big] = \delta u'(sR)\left[1 - R_R(sR)\right].$$

$\frac{ds}{dR} > 0 \Leftrightarrow R_R(sR) < 1$ sub. 相对风险规避系数<1，消费上升

$\frac{ds}{dR} < 0 \Leftrightarrow R_R(sR) > 1$ Income. 相对风险规避系数<1. 消费下降.

Substituting effect 替代效应. 储蓄↑越来越有利可图.

Income effect. 储蓄越多. 未来消费越多.

Intertemporal: 跨期:

$$\max\ u(w_1) + \delta u(w_2) \quad \text{s.t.}\ w_1 + w_2 = w.$$

FOC: $u'(w_1) = \delta u'(w_2)$.

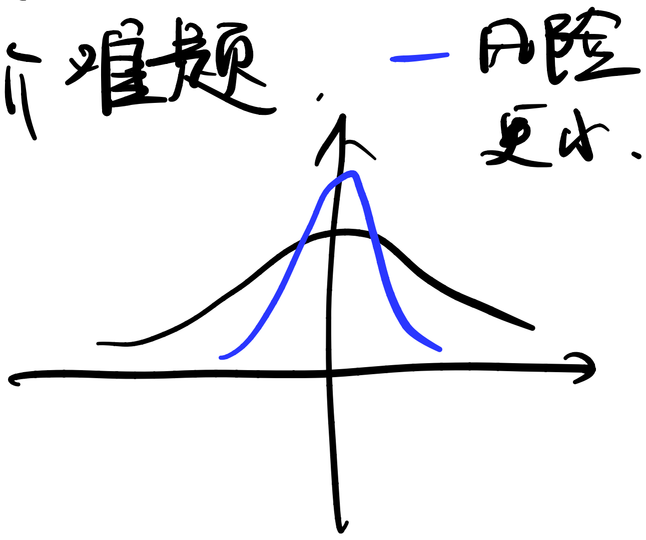
Maximization Under Uncertainty.

$$\max\ p_1 u(w_1) + p_2 u(w_2) \quad \text{s.t.}\ w_1 + w_2 = w.$$

FOC: $p_1 u'(w_1) = p_2 u'(w_2)$.

形式类似. Consumption Smoothing.

Equity Premium Puzzle 风险溢价难题.



Saving Under Uncertainty

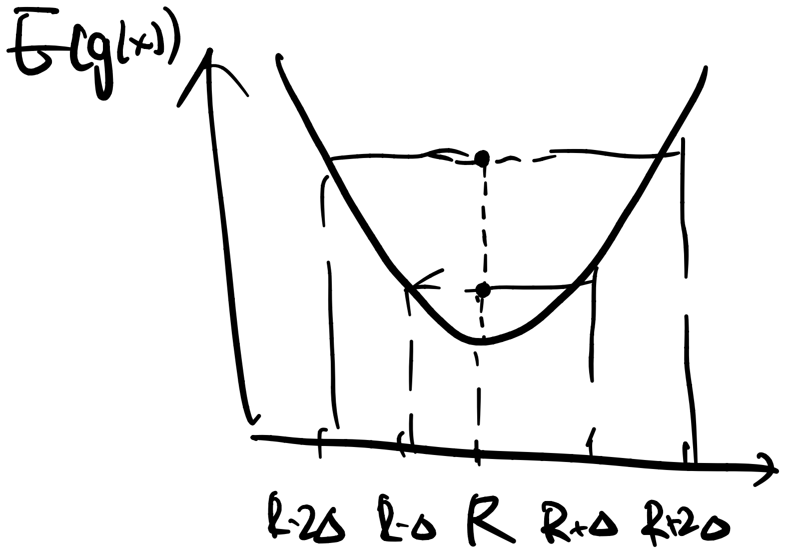
$$\max_s\ u(w-s) + \delta E[u(sR)].$$

FOC: $u'(w-s) = \delta E[R u'(sR)]$

左边: $\frac{d(u'(w-s))}{ds} > 0$. RHS↑ $\Rightarrow$ s↑.

$g(R) \triangleq R\,u'(sR)$

$g(R)$ Need to be a Convex Function. $g'' > 0$.



$$g''(R) = 2su''(sR) + s^2 R u'''(sR)$$

$$= su''(sR)[2 - P_R(sR)].$$

$$P_R(y) \triangleq -\frac{y u'''(y)}{u''(y)}$$ 相对审慎系数.

Kimball ⇒ Prudence. 1990

$P_R(sR) > 2 \Rightarrow g''(R) > 0 \Rightarrow \sigma^2(R)\uparrow \Rightarrow s\uparrow$.

当相对审慎系数 >2 ⇒ g(R) 是凸 ⇒ 未来风险变大 ⇒ 储蓄增加.

储蓄风险↑ ⇒ 未来消费削弱↑ ⇒ 审慎的人则增加储蓄

⇒ 未来投资无力可图 ⇒ 审慎的人则减少储蓄

综合来看最后结果.

CRRA:

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$$

$R_R = \gamma$. $P_R = \gamma + 1$.

$\gamma = 1$ 时 $u(c) = \ln c$.
$R_P = 1$
$P_R = 2$

$\gamma = 1$ 时: 刚好

Homework 9.

9.1). 定义:

绝对风险厌恶系数: ARA: $R_A(y) \triangleq -\frac{u''(y)}{u'(y)}$.

相对风险厌恶系数: RRA: $R_R(y) \triangleq -\frac{y u''(y)}{u'(y)}$

相对审慎系数: $P_R(y) = -\frac{y u'''(y)}{u''(y)}$.

绝对审慎系数: $P_A(y) = \frac{u'''(y)}{u''(y)}$.

① $u(c)=ac$. $u'=a$ $u''=0$ $u'''=0$.

$R_A(y)=0$ $R_R(y)=0$. $P_A(y)$ 不存在. $P_R(y)$ 不存在.

② $u(c)=c-Ac^2$. $u'=1-2Ac$. $u''=-2A$ $u'''=0$.

$R_A(y)=\frac{2A}{1-2Ay}$. $R_R(y)=\frac{2Ay}{1-2Ay}$. $P_A(y)=0$ $P_R(y)=0$.

③. $u(c)=-\frac{1}{c}$. $u'=\frac{1}{c^2}$. $u''=-\frac{2}{c^3}$. $u'''=+\frac{6}{c^4}$.

$R_A(y)=\frac{2}{y}$ $R_R(y)=2$. $P_A(y)=\frac{3}{y}$ $R_R(y)=3$

④. $u(c)=\ln c$, $u'=\frac{1}{c}$. $u''=-\frac{1}{c^2}$. $u'''=+\frac{2}{c^3}$.

$R_A(y)=1/y$ $R_R(y)=1$. $P_A(y)=2/y$ $P_R(y)=2$.

⑤. $u(c)=\frac{c^{1-\rho}-1}{1-\rho}$. $u'=c^{-\rho}$. $u''=-\rho c^{-\rho-1}$ $u'''=\rho(\rho+1)\cdot c^{-\rho-2}$

$R_A(y)=\rho/y$ $R_A=\rho$. $P_A(y)=(\rho+1)/y$ $P_R(y)=\rho+1$

9.2). 证明:

$$\max_a E\{u[w_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

$$a^{*\prime}(w_0)<0 \Longleftrightarrow R_A'(\cdot)>0 \text{ (IARA)}.$$

证明:

FOC: $E\{u'[w_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\}=0$

对 w 求导: $E\{u''[w_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)$
$[(1+r_f)+(\tilde{r}-r_f)\frac{da}{dw_0}]\}=0$.

$$\Rightarrow \frac{da}{dw_0} = -\frac{(1+r_f)\cdot E[u''(w_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)]}{E[u''(w_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)]}$$ $\Rightarrow$ 小于0.

$\because u''(\cdot)<0$.

$\therefore \frac{da}{dw_0}$ 的符号与 $E\{u''(w_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\}$ 一致.

$E\{u''[w_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\}$

$= E\{-u'[w_0(1+r_f)+a(\tilde{r}-r_f)]R_A(w_0(1+r_f)+a(\tilde{r}-r_f))\cdot(\tilde{r}-r_f)\}$.

$= \sum_{n=1}^{N} p_n\cdot[-u'(w)\cdot R_A(w)(\tilde{r}-r_f)]$.   IARA.

① 若: $\tilde{r}\geq r_f$ 时: $a^*>0$.

$\therefore W_n \geq w_0(1+r_f)$.

IARA 性质: $R_A(W_n)\geq R_A(w_0(1+r_f))$.

$\Rightarrow R_A(W_n)\cdot(\tilde{r}-r_f)\geq R_A(w_0(1+r_f))\cdot(\tilde{r}-r_f)$.

② 若: $\tilde{r}<r_f$ 时:

$\alpha^*>0 \iff W_n\leq w_0(1+r_f)$.

IARA: $R_A(W_n)\leq R_A(w_0(1+r_f))$.

$\Rightarrow R_A(W_n)\cdot(r_n-r_f)\geq R_A(w_0(1+r_f))(r_n-r_f)$

$\therefore$ 综上, 只要满足: IARA 也即:

$\Rightarrow R_A(w_0(1+r_f)+\alpha(r-r_f))(\tilde{r}-r_f)\geq R_A(w_0(1+r_f))(\tilde{r}-r_f)$.

$\because -u'(w_0(1+r_f)+a(\tilde{r}-r_f))<0$.

$\therefore \forall n$: $-u'(\tilde{w})\cdot R_A(\tilde{w})(\tilde{r}-r_f) < -u'(\tilde{w})R_A(w_0(1+r_f))\cdot(\tilde{r}-r_f)$.

$\therefore E[u''(\tilde{w})(\tilde{r}-r_f)] < \sum_{n=1}^{N} p_n - u'(\tilde{w})R_A(w_0(1+r_f))(\tilde{r}-r_f)$

$$= -R_A\underbrace{E[u'(\tilde{w})(\tilde{r}-r_f)]}_{=0\ \text{因为一阶条件}} = 0.$$

$\therefore \ a^{*\prime}(w) < 0$.

Period 0	Period 1
w_0	$(w_0 - s) \Rightarrow$ 50% 8%, 50% 4%
$-s$	

9.3).

我的期望效用为: $\ln c$.

$w_0 = c_1 + s = 10$.

$\max_{a,s} \ \ln c_1 + \delta E \ln c_2$. 风险资产 a, 无风险 $s-a$.

$$\Leftrightarrow \max_{a,s} \ \ln(w_0 - s) + \frac{\delta}{2}\log\left(a(1+0.04) + (s-a)(1+r_f)\right) + \frac{\delta}{2}\log\left(a(1+0.08) + (s-a)(1+r_f)\right) = u$$

$$\Leftrightarrow \max_{a,s} \ \ln(10-s) + \frac{0.9}{2}\log(1.04a + (s-a)\cdot 1.05) + \frac{0.9}{2}\log(1.08a + (s-a)1.05)$$

$$\max_{a,s} \ \ln(10-s) + \frac{0.9}{2}\log(1.05s - 0.01a) + \frac{0.9}{2}\log(1.05s + 0.03a)$$

$$\Rightarrow \frac{\partial u}{\partial a} = 0 \Rightarrow 0 = \frac{0.9}{2}\cdot\frac{-0.01}{1.05s - 0.01a} + \frac{0.9}{2}\times\frac{0.03}{1.05s + 0.03a}$$

$$\Rightarrow 105s - a = 35s + a \Rightarrow 35s = a$$

$$\Rightarrow \frac{\partial u}{\partial s} = 0 \Rightarrow \frac{-1}{10-s} + \frac{0.9}{2}\frac{1.05}{1.05s - 0.01a} + \frac{0.9}{2}\frac{1.05}{1.05s + 0.03a} = 0$$

$$\Rightarrow \frac{1}{10-s} + \frac{0.9}{2}\frac{105}{105s - a} + \frac{0.9}{2}\frac{105}{105s + 3a} = 0$$

$$\Rightarrow \frac{1}{10-s} + \frac{0.9}{2}\frac{105}{105s - 35s} + \frac{0.9}{2}\frac{105}{105s + 105s} = 0$$

$$\Rightarrow \frac{1}{10-s} + \frac{0.9}{2}\frac{105}{70}\frac{1}{s} + \frac{0.9}{2}\frac{105}{210}\frac{1}{s} = 0$$

$$\Rightarrow \frac{1}{10-s} = -0.9\frac{1}{s} \Rightarrow s = 9 - 0.9s \Rightarrow s = 4.74$$

$x = 165.9$. $\quad c_1 = 5.26 \quad \underbrace{c_2 = 6.636}_{E}$