Chapter 11

G.E. in a Complete Market

Pareto Optimal：不损伤全部人的情况下，改进一部分人。
（没有资源闲置）

中央计划者 Central Planner.

$$\max_{\{c_{k0}, c_{k1}, \cdots, c_{kS}\}_{k=1}^{K}} \sum_{k=1}^{K} \mu_k \left[u_k(c_{k0}) + \sum_{s=1}^{S} \pi_s u_k(c_{ks}) \right]$$

k是人，s是state．
$\mu_k \geq 0$．

$$\text{s.t.} \quad \sum_{k=1}^{K} c_{k0} \leq \sum_{k=1}^{K} e_{k0} \quad \text{(period 0 消费} \leq \text{禀赋)}$$

$$\sum_{k=1}^{K} c_{ks} \leq \sum_{k=1}^{K} e_{ks}, \quad s = 1, \cdots, S$$

解拉格朗日方程：

$$\mathcal{L} = \sum_{k=1}^{K} \mu_k \left[u_k(c_{k0}) + \delta \sum_{s=1}^{S} \pi_s u_k(c_{ks}) \right] + \eta_0 \left[-\sum_{k=1}^{K} c_{k0} + \sum_{k=1}^{K} e_{k0} \right] + \sum_{s=1}^{S} \eta_s \left[-\sum_{k=1}^{K} c_{ks} + \sum_{k=1}^{K} e_{ks} \right].$$

在经济学里 lagrange 方程正负号：

① 如果是 max，约束是 $a \leq b$，$\Rightarrow \eta(b-a)$．
这样写，η 可以解读为影子价格（有经济含义）．

$$\frac{\partial \mathcal{L}}{\partial c_{k0}} = 0: \quad \mu_k u_k'(c_{k0}) = \eta_0. \Rightarrow \begin{cases} c_{k0} = u_k'^{-1}(\eta_0 / \mu_k) \\ c_{ks} = u_k'^{-1}(\eta_s / \delta \pi_s \mu_k). \end{cases} \Rightarrow \text{Pareto Optimal.}$$

$$\frac{\partial \mathcal{L}}{\partial c_{ks}} = 0: \quad \delta \mu_k \pi_s u_k'(c_{ks}) = \eta_s.$$

G.E. General Equilibrium. 上一讲里：

$$\begin{cases} u_k'(c_{k0}) = \lambda_k \\ \delta \pi_s u_k'(c_{ks}) = \lambda_k \psi_s \end{cases} \Rightarrow \begin{cases} c_{k0} = u_k'^{-1}(c_{k0}) \\ c_{ks} = u_k'^{-1}(\lambda_k \psi_s / \delta \pi_s) \end{cases} \quad \text{G.E.}$$

其中 $\lambda_K = \eta_0/\mu_K$，$\psi_s = \eta_s/\eta_0$. 则两个均衡相等. PO=GE.

λ_K: K个消费者的财富. ψ_s: Arrow Security 的价格.

⇒ 给一个价格 ⇒ 调整每个人初始财富：可以达到 Pareto Optimal.

第二福利定理：The 2nd Welfare Theorem

P.O. ⇒ G.E.

The 1st Welfare Theorem

G.E ⇒ P.O (市场均衡即是帕累托最优).

1st. +2nd. ⇒ P.O. ⇔ G.E.

① 从微观经济学角度来说：均衡就是好的.

② 分析帕累托最优,可以分析均衡 (后者的计算非常复杂)

Prop. 11.3：完备市场达到均衡时,消费者在不同状态中消费的波动只与各个状态中总禀赋的波动有关,而消费者自己的禀赋在各个状态波动无关

证明：$C_s = \sum_{K=1}^{K} C_{Ks} = \sum_{K=1}^{K} u_K'^{-1}(\eta_s/\delta\mu_K\pi_s)$. $u_K'^{-1}$: 减函数.

$\eta_s = g(c_s) = g(e_s)$. $C_{Ks} = u_K'^{-1}\left(\frac{g(e_s)}{\delta\mu_K\pi_s}\right)$. 且变量是 e_s.

μ_K: 受到人的财富影响.

$\mu_K \underbrace{(e_{K0} + \sum_{s=1}^{S}\psi_s e_{Ks})}_{\lambda_K}$.

Prop. 11.4. $\forall s, s'$. if $c_s > c_{s'}$.

Then $\forall K$, $C_{Ks} > C_{Ks'}$.

Prop. 11.5 Wilson Theorem.

绝对风险容忍度：$T(c) \triangleq \frac{1}{R_A(c)} = -\frac{u'(c)}{u''(c)}$.

$\frac{dC_{Ks}}{de_s} = T_K(C_{Ks}) / \sum_{K=1}^{K} T_K(C_{Ks})$. — 个人的风险容忍度/所有人的风险容忍度.

$$\max_{(\theta_1,\dots,\theta_J)} u(c_0)+\delta\sum_{s=1}^{S}\pi_s u(c_s).$$

$$\text{s.t.}\quad c_0=e_0-\sum_{j=1}^{J}p_j\theta_j$$

$$c_s=e_s+\sum_{j=1}^{J}x_j\theta_j.\quad (s=1,\dots,S)$$

代入目标函数: $\max\ u\left(e_0-\sum_{j=1}^{J}p_j\theta_j\right)+\delta\sum_{s=1}^{S}\pi_s u\left(e_s+\sum_{j=1}^{J}x_j\theta_j\right).$

对 θ_j 求导: $-p_j u'(c_0)+\delta\sum_{s=1}^{S}\pi_s u'(c_s)x_s^j.$

$$1=\delta\sum_{s=1}^{S}\pi_s\frac{u'(c_s)}{u'(c_0)}\cdot\frac{x_s^j}{p_j}$$ ⇒ 回报率 $+1=1+r_s^j$. 对任意 k 成立.

$$1=\delta_k\sum_{s=1}^{S}\pi_s\frac{u_k'(c_s)}{u_k'(c_0)}.$$

$$\Rightarrow 1=E\left[\delta\frac{u_k'(\tilde{c}_1)}{u_k'(\tilde{c}_0)}\cdot(1+\tilde{r}_j)\right]$$ 资产定价方程(带有回报率 $\tilde{r}_j$).

Representative Consumer 代表消费者.

Prop. 11.6 HARA ⇒ 认为只有一个消费者: $C=\sum_k c_k$.

$$\Rightarrow\quad 1=E\left[\delta\frac{u'(\tilde{c}_1)}{u'(\tilde{c}_0)}(1+\tilde{r}_j)\right].$$

定义 $\tilde{m}\triangleq\delta\frac{u'(\tilde{c}_1)}{u'(\tilde{c}_0)}$ 随机折现因子 Stochastic Discount Factor.

$1=E[\tilde{m}(1+\tilde{r}_j)]$. 对所有资产成立.

① 无风险: $1+r_f=1/E[\tilde{m}]$.

② $$0=E[\tilde{m}]+E[\tilde{m}\tilde{r}_j]-E[\tilde{m}]-E[\tilde{m}r_f].$$

$$=\sum_s\pi_s m_s r_{j,s}-\sum_s\pi_s m_s r_f.$$

$$=\sum_s\pi_s m_s(r_{j,s}-r_f)=E[\tilde{m}(\tilde{r}_j-r_f)].$$

$E[xy]=E(x)E(y)+Cov(x,y).$

$$\Rightarrow \; = E[\tilde{m}]\,E[\tilde{r}_j - r_f] + Cov(\tilde{m}, \tilde{r}_j - r_f)$$

$$= E[\tilde{m}]\,[E(\tilde{r}_j) - r_f] + Cov(\tilde{m}, \tilde{r}_j)$$

$$= \frac{E(\tilde{r}_j) - r_f}{1+r_f} + Cov(\tilde{m}, \tilde{r}_j)$$

$$\Rightarrow \; E(\tilde{r}_j) - r_f = -Cov(\tilde{m}, \tilde{r}_j)\cdot(1+r_f)$$

$$\Rightarrow \; E(\tilde{r}_j) - r_f = -(1+r_f)\cdot Cov\left(\delta\frac{u'(\tilde{c}_1)}{u'(c_0)}, \tilde{r}_j\right)$$

$$= \underbrace{-\frac{(1+r_f)\delta}{u'(c_0)}}_{小于0}\cdot Cov(u'(\tilde{c}_1), \tilde{r}_j)$$

边际效用

期望项取决于协方差项：$u'(\tilde{c}_1)$ 与 $\tilde{r}_j$

$\Rightarrow Cov(u'(\tilde{c}_1), \tilde{r}_j) > 0$ 雪中送炭的资产 价格↑，期望回报低

$Cov(u'(\tilde{c}_1), \tilde{r}_j) < 0$ 锦上添花的资产 价格↓，期望回报高

$u(c) = -ac^2 + bc \quad (a>0)$

$u'(c) = -2ac + b$

M: $x_s^M = e_s$. $r_M = \frac{\tilde{c}_1}{p_M} - 1$. $\quad \tilde{m} \triangleq \delta\frac{-2a\tilde{c}_1+b}{-2ac_0+b}$

$$E(\tilde{r}_j) - r_f = -(1+r_f)\,Cov\left(\delta\frac{-2a\tilde{c}_1+b}{-2ac_0+b}, \tilde{r}_j\right)$$

$$E(\tilde{r}_j) - r_f = \overbrace{\frac{2a\delta(1+r_f)\cdot p_M}{-2ac_0+b}}^{\Delta}\cdot Cov\left(\frac{\tilde{c}_1}{p_M} - 1, \tilde{r}_j\right)$$

$$= \Delta\cdot Cov\left(\frac{\tilde{c}_1}{p_M} - 1, \tilde{r}_j\right)$$

$$\frac{E(\tilde{r}_j)-r_f}{E(\tilde{r}_M)-r_f} = \frac{Cov(\tilde{r}_M, \tilde{r}_j)}{Var(\tilde{r}_M)}.$$

$$\Rightarrow E(\tilde{r}_j)-r_f = \beta_j\left[E(\tilde{r}_M)-r_f\right] \quad \text{CAPM}$$

$\Rightarrow$ CAPM 只是二次效用函数时，C-CAPM 的特例.

		state 1	state 2	
$\frac{1}{2}$	A	2	2	φ_a
$\frac{1}{2}$	B	3	1	φ_b.

Homework 11

(11.1) a

对消费者 A:

$$\max_{C_{A1}, C_{A2}} \frac{1}{2}\log C_{A1} + \frac{1}{2}\log C_{A2} \quad s.t.\ \varphi_1 C_{A1} + \varphi_2 C_{A2} = 2\varphi_1 + 2\varphi_2. \Rightarrow \begin{cases} \frac{1}{\lambda_1} = 2\varphi_a + 2\varphi_b. \\ C_{A1} = \varphi_a + \varphi_b / \varphi_a \\ C_{A2} = \varphi_a + \varphi_b / \varphi_b. \end{cases}$$

$$\max_{C_{B1}, C_{B2}} \frac{1}{2}\log C_{B1} + \frac{1}{2}\log C_{B2} \quad s.t.\ \varphi_1 C_{B1} + C_{B2}\varphi_2 = 3\varphi_a + \varphi_b \Rightarrow \begin{cases} \frac{1}{\lambda_2} = 3\varphi_a + \varphi_b \\ C_{B1} = 3\varphi_a + \varphi_b / 2\varphi_a \\ C_{B2} = 3\varphi_a + \varphi_b / 2\varphi_b. \end{cases}$$

$$\Rightarrow \begin{cases} C_{A1} + C_{A2} = 2+3 \\ C_{B1} + C_{B2} = 2+1 \end{cases} \Rightarrow \left\{ \frac{\varphi_b}{\varphi_a} = \frac{5}{3} \right. \quad \begin{cases} C_{A1} = \frac{8}{3} \\ C_{A2} = \frac{8}{5} \end{cases} \quad C_{B1} = \frac{7}{3} \quad C_{B2} = \frac{7}{5}.$$

市场出清 C_{A1} C_{B1}.

b). $\frac{1}{2}\ln\frac{8}{3} + \frac{1}{2}\ln\frac{8}{5} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln 16/15$ (A)

$\frac{1}{2}\cdot\ln\frac{7}{3} + \frac{1}{2}\ln\frac{7}{5} - \frac{1}{2}\ln 3 - \frac{1}{2}\ln 1 = \frac{1}{2}\cdot\ln 49/45$ (B).

c). A 会：快买保险就是交换.

效用↑. 波动↑ ⇒ 效用↓. Δu 是最大 >0.

(1.2). 重新来:

$$\max_{c_{A1}, c_{A2}} \ \frac{1}{3}\log c_{A1} + \frac{2}{3}\log c_{A2}$$

$$\text{s.t. } \varphi_1 c_{A1} + \varphi_2 c_{A2} = 2\varphi_1 + 2\varphi_2 .$$

$$\max_{c_{B1}, c_{B2}} \ \frac{2}{3}\log c_{B1} + \frac{1}{3}\log c_{B2}$$

$$\text{s.t. } \varphi_1 c_{B1} + \varphi_2 c_{B2} = 2\varphi_1 + 2\varphi_2 .$$

$$\Rightarrow \begin{cases} c_{A1} = \frac{4}{3} & c_{B1} = \frac{8}{3} \\ c_{A2} = \frac{8}{3} & c_{B2} = \frac{4}{3} \end{cases}$$

(b) 有差异. 消费者在不同状态存在差异. 效用函数不同.