

Homework 1 (CS209, Spring 2017)

Aravind Machiry

We will use symbols T and F to indicate truth values True and False respectively.

1. Let A, B be distinct sentence symbols in \mathcal{S} . Determine if each of the following wffs is a tautology. If your answer is negative, find a truth assignment v that does not satisfy the wff and show the truth values of A, B under the assignment v .

(a) $((A \rightarrow B) \rightarrow B) \rightarrow B$

(b) $((A \rightarrow B) \rightarrow B) \rightarrow A$

Solution: We will use truth table to check if the above wffs are tautologies.

| A | B | $((A \rightarrow B) \rightarrow B) \rightarrow B$ | $((A \rightarrow B) \rightarrow B) \rightarrow A$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | T | F |
| F | F | T | T |

As we can clearly, see from the above table, there are truth assignments where the wffs evaluate to false. Specifically, the wff: $((A \rightarrow B) \rightarrow B) \rightarrow B$ is false when $A = T$ and $B = F$, the wff: $((A \rightarrow B) \rightarrow B) \rightarrow A$ is false when $A = F$ and $B = T$.

So none of the wffs are tautology.

2. Let A, B, C be distinct sentence symbols in \mathcal{S} . Show that neither of the following two wffs tautologically implies the other:

$$(A \leftrightarrow (B \leftrightarrow C))$$
$$((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

Note that you need to exhibit two truth assignments (i.e., not eight).

Solution: Lets denote $\alpha = (A \leftrightarrow (B \leftrightarrow C))$ and $\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$.

Consider the following truth assignment, $A = T, B = F$ and $C = F$, with this assignment the wff α evaluates to T , however β evaluates to F . This proves that $\alpha \not\models \beta$.

Similarly, consider another truth assignment, $A = F, B = F$ and $C = F$. However, with this assignment the wff α evaluates to F and β evaluates to T . This proves that $\beta \not\models \alpha$.

Thus $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

3. Prove or disprove:

THEOREM: For each natural number $n \geq 2$, there is a set Σ_n with n wff's such that (1) Σ_n is not satisfiable, and (2) each $(n-1)$ -element subset of Σ_n is satisfiable.

Solution: I will prove this by construction.

Consider A_1, A_2, A_3, \dots be the countably infinite distinct sentence symbols in \mathcal{S} .

Base case ($n = 2$): We can have $\Sigma_2 = \{A_1, \neg A_1\}$. All the possible subsets of size 1 (i.e. $n-1$) $\{A_1\}$ and $\{\neg A_1\}$ are satisfiable with truth assignments $A = T$ and $A = F$ respectively. However, $\{A_1, \neg A_1\}$ is not satisfiable.

Now for any $n > 2$, we can create $\Sigma_n = \{A_1, A_2, \dots, A_{n-1}, \alpha\}$, where $\alpha = ((\neg A_1) \vee (\neg A_2) \vee (\neg A_3) \vee \dots \vee (\neg A_{n-1}))$.

Consider all the subsets of size $(n-1)$ of Σ_n , there are n subsets of size $n-1$, let's call them $sub_1, sub_2, sub_3, sub_4, \dots, sub_{n-1}, sub_n$. Where sub_1 is a set of all elements of Σ_n except A_1 i.e., $\Sigma_n - A_1 = \{A_2, \dots, A_{n-1}, \alpha\}$, similarly $sub_2 = \Sigma_n - A_2$ and so on, finally $sub_n = \Sigma_n - \alpha$. Each of the subsets are satisfiable with mappings v_1, v_2, \dots, v_n respectively.

where for $i \leq (n-1)$,

$$v_i(B) = \begin{cases} F & \text{if } B = A_i \\ T & \text{otherwise} \end{cases} \quad (1)$$

and $v_n = T$ for all sentence symbols in \mathcal{S} .

Now, for Σ_n to be satisfiable we should have *all* sentence symbols A_1, A_2, \dots, A_{n-1} (condition 1) should be T and also α should be T , however, for α to be T , we should have at least one sentence symbol from A_1, A_2, \dots, A_{n-1} to be F , contradicting condition 1, thus Σ_n is not satisfiable but all subsets of size $(n-1)$ are satisfiable. Hence the proof.

4. Let Σ be a (possibly infinite) set of wffs and α, β two wffs.

Prove: $\Sigma; \alpha \models \beta$ if and only if $\Sigma \models (\alpha \rightarrow \beta)$

Note that the notation " $\Sigma; \alpha$ " means $\Sigma \cup \{\alpha\}$.

Solution: Let's assume $\Sigma; \alpha \models \beta$ (call this (1)), this means for all mappings \bar{v} such that, if $\bar{v}(\Sigma) = T$ and $\bar{v}(\{\alpha\}) = T$ then $\bar{v}(\{\beta\}) = T$. Consider all the mappings \bar{v}_1 such that $\bar{v}_1(\Sigma) = T$, and $\bar{v}_1(\{\alpha\}) = F$, any mapping satisfying Σ should be either \bar{v} or \bar{v}_1 , from (1), $\bar{v}(\{\beta\}) = T$, and $\bar{v}_1(\{\alpha\}) = F$ or $\bar{v}_1(\{\neg\alpha\}) = T$, Hence $\Sigma \models (\beta \vee \neg\alpha)$ i.e., $\Sigma \models (\alpha \rightarrow \beta)$

Consider, $\Sigma \models (\alpha \rightarrow \beta)$, this means all mappings \bar{v} , which evaluate $\bar{v}(\Sigma) = T$, if $\bar{v}(\{\alpha\}) = T$, then $\bar{v}(\{\beta\}) = T$. So, if $\bar{v}(\Sigma) = T$ and $\bar{v}(\{\alpha\}) = T$, then $\bar{v}(\{\beta\}) = T$, thus, $\bar{v}(\Sigma \cup \{\alpha\}) = T$, then $\bar{v}(\{\beta\}) = T$, Hence $\Sigma; \alpha \models \beta$.

5. Write a complete proof for Case 3 (i.e., $\alpha = (\alpha_1 \vee \alpha_2)$) of the induction step in proving Lemma 3 (page marked "15" in the April 6's lecture notes).

Solution:

$$1) \alpha = (\alpha_1 \vee \alpha_2)$$

$$2) \leftrightarrow \alpha \in \Delta$$

- 3) $\leftrightarrow (\alpha_1 \vee \alpha_2) \in \Delta$
 4) $\rightarrow \{\neg\alpha_1, \neg\alpha_2, \alpha_1 \vee \alpha_2\} \not\subseteq \Delta$, because Δ is fin sat.
 5) $\leftrightarrow \neg\alpha_1 \notin \Delta$ or $\neg\alpha_2 \notin \Delta$
 6) $\leftrightarrow \alpha_1 \in \Delta$ or $\alpha_2 \in \Delta$.
 7) $\leftrightarrow \bar{u}(\alpha_1) = T$ or $\bar{u}(\alpha_2) = T$
 8) $\leftrightarrow \bar{u}(\alpha_1 \vee \alpha_2) = T$

From equation 6 and 3 are equivalent, as shown below:

$$\alpha_1 \in \Delta \text{ or } \alpha_2 \in \Delta.$$

$$3) \leftrightarrow (\alpha_1 \vee \alpha_2) \in \Delta$$

Thus establishing the equivalence through out.

6. (Duality) Let α be a wff whose only connectives are \wedge, \vee , and \neg . Let α^* be the resulting wff after interchanging \wedge and \vee and replacing each sentence symbol (e.g., A) by its negation (i.e., $(\neg A)$).

THEOREM: α^* is tautologically equivalent to $(\neg\alpha)$, i.e., $\alpha^* \models (\neg\alpha)$ and $(\neg\alpha) \models \alpha^*$.

Give a proof using mathematical induction.

Solution:

Base case: Consider $\alpha = A$, So, $\alpha^* = \neg A$. Now, $\neg\alpha = \neg A$. As we can see α^* is same as $\neg\alpha$. So, $\alpha^* \models (\neg\alpha)$ and $(\neg\alpha) \models \alpha^*$

Induction: Consider, a wff α_1 Such that

$$\alpha_1^* \models (\neg\alpha_1) \text{ and } (\neg\alpha_1) \models \alpha_1^* \text{ (consider this 1)}$$

Now, Consider $\alpha = \neg\alpha_1$, So, $\alpha^* = (\neg\alpha_1)^*$. So, $\alpha^* = \neg\alpha_1^*$.

Now, $\neg\alpha = \neg(\neg\alpha_1)$. From 1, we know $\neg\alpha_1$ and α_1^* are tautologically equivalent. So, $\neg\alpha = \neg\alpha_1^*$
 Thus α^* and $\neg\alpha$ are equivalent.

Consider another wff α_2 , Such that

$$\alpha_2^* \models (\neg\alpha_2) \text{ and } (\neg\alpha_2) \models \alpha_2^* \text{ (consider this 2)}$$

Now, Consider $\alpha = \alpha_1 \vee \alpha_2$, So, $\alpha^* = \alpha_1^* \wedge \alpha_2^*$. From (1) and (2) we have, $\alpha^* = (\neg\alpha_1) \wedge (\neg\alpha_2) = \neg(\alpha_1 \vee \alpha_2) = \neg\alpha$. Thus α^* and $\neg\alpha$ are equivalent.

Now, Consider $\alpha = \alpha_1 \wedge \alpha_2$, So, $\alpha^* = \alpha_1^* \vee \alpha_2^*$. From (1) and (2) we have, $\alpha^* = (\neg\alpha_1) \vee (\neg\alpha_2) = \neg(\alpha_1 \wedge \alpha_2) = \neg\alpha$. Thus α^* and $\neg\alpha$ are equivalent.