A Corollary

Theorem:

If $\Sigma \models \tau$, then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$.

Proof: by contradiction.

Suppose that $\Sigma_0 \not\models \tau$ for all finite $\Sigma_0 \subseteq \Sigma$

$\Rightarrow \Sigma_0; \neg\tau$ is satisfiable for all finite $\Sigma_0 \subseteq \Sigma$

(Note that $\Gamma \models \sigma$ iff $\Gamma; \neg\sigma$ is unsatisfiable)

$\Rightarrow \Sigma; \neg\tau$ is finitely satisfiable

$\Rightarrow \Sigma; \neg\tau$ is satisfiable

$\Rightarrow \Sigma \not\models \tau$ ∎

Summary

- Sentential logic: syntax and semantics
- Compactness theorem
  a tool for proving an infinite set is satisfiable
- Limited expressiveness

CS 209
First Order Logic

Outline

- First order languages
- Structures and semantics
- Models and logical implication
- Definability
- Homomorphism theorem

A First Order Language(s): Alphabet

A first order logic language consists of the following *symbols* :

Logic symbols

1. Parentheses : (,)
2. Sentential connectives: $\wedge$, $\neg$
3. Variables: x, y, z, x_1, x_2, ...
4. Equality symbol (optional): $\approx$

Parameters

1. Quantifier symbol: $\forall$
2. Constant symbols: c_1, c_2, ...
3. For each positive integer n,
   n-ary *function* symbols: f, g, ...
4. For each positive integer n,
   n-ary *predicate* symbols: P, Q, ...

Example FO Logic Languages

- Concerning numbers
  - Number theory : $\langle 0, \approx, <, succ, +, \times \rangle$
  - Rational numbers and linear arithmetic : $\langle (q)_{q\in\mathbb{Q}}, \approx, <, + \rangle$
  - Real closed field : $\langle 0, 1, \approx, <, +, \times \rangle$
- Database query languages
  - $\langle \mathbf{dom}, Employees, Products, ..., \rangle$
- Suggestions?

Terms

The set of *terms* consists of the following elements:

- Constant symbols,
- Variables, and
- Expressions of form $ft_1 \cdots t_n$,
  where f is an n-ary function symbol and
  t_1, ..., t_n are terms.

We may also write a term $ft_1 \cdots t_n$ as $f(t_1, \ldots, t_n)$ for readability

Well-Formed Formulas

A *well-formed formula* (*wff*) is one of the following:

- (atomic) $\approx t_1 t_2$
  where t_1, t_2 are terms,
- (atomic) $Pt_1 \cdots t_n$
  where P is an n-ary predicate symbols
  and $t_1, \ldots, t_n$ are terms,
- $(\varphi \wedge \psi)$, $(\neg\varphi)$, or $(\forall x \varphi)$
  where φ, ψ are wffs and x is a variable

$$((\forall x_2 (\forall x_1 \approx x_1 x_2)) \wedge P x_1)$$

Free Variables

A variable x *occurs free* in a wff φ if one of the following holds:

- φ is atomic and x occurs in φ,
- φ is $(\psi_1 \wedge \psi_2)$ and x occurs free in ψ_1 or ψ_2,
- φ is $(\neg\psi)$ and x occurs free in ψ, or
- φ is $(\forall y \psi)$, x occurs free in ψ, and x, y are distinct variables

A wff φ is a *sentence* if no variables occur free in it

$$(\forall x \ \varphi_{prime}(x) \vee (\exists y_1 (\exists y_2 \approx x \times y_1 y_2) \wedge y_1 > 1 \wedge \ldots)$$

$$(\forall x \ > x \ 1)$$

First Order Structure

A *structure* of a first order language $\mathcal{L}$ is a mapping $\mathfrak{A}$ over the parameters of $\mathcal{L}$ such that $\mathfrak{A}$ maps

1. the quantifier $\forall$ to a nonempty set denoted as $|\mathfrak{A}|$ (called the *universe*),
2. each constant symbol c to an element $c^{\mathfrak{A}}$ in the universe $|\mathfrak{A}|$,
3. each n-ary function symbol f to an n-ary function $f^{\mathfrak{A}} : |\mathfrak{A}|^n \to |\mathfrak{A}|$, and
4. each n-ary predicate symbol P to an n-ary relation $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$

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$\langle \forall, 0, succ, \approx, <, +, \times \rangle$

$\mathbb{N}$ $\quad |\mathbb{N}|,$

$= \{0, 1, 2, \ldots\}$

$0^{\mathbb{N}}$

$succ^{\mathbb{N}}(i) = i{+}{+}$

$+^{\mathbb{N}}$

$\times^{\mathbb{N}}$

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