Homework 1

CS209, Spring 2017

In this homework we consider the sentential logic language defined in the lecture, i.e., with the set $\mathcal{S}$ of sentence symbols and the set $\mathcal{S}^{\text{wff}}$ of well-formed formulas (wffs).

1. Let A, B be distinct sentence symbols in $\mathcal{S}$. Determine if each of the following wffs is a tautology. If your answer is negative, find a truth assignment v that does not satisfy the wff and show the truth values of A, B under the assignment v.

   (a) $(((A \to B) \to B) \to B)$

   (b) $(((A \to B) \to B) \to A)$

2. Let A, B, C be distinct sentence symbols in $\mathcal{S}$. Show that neither of the following two wffs tautologically implies the other:

$$(A \leftrightarrow (B \leftrightarrow C))$$
$$((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

   Note that you need to exhibit two truth assignments (i.e., not eight).

3. Prove or disprove:

   THEOREM: For each natural number $n \geqslant 2$, there is a set Σ_n with n wff's such that (1) Σ_n is not satisfiable, and (2) each $(n-1)$-element subset of Σ_n is satisfiable.

4. Let Σ be a (possibly infinite) set of wffs and α, β two wffs.

   Prove: $\Sigma; \alpha \models \beta$ if and only if $\Sigma \models (\alpha \to \beta)$

   Note that the notation "$\Sigma; \alpha$" means $\Sigma \cup \{\alpha\}$.

5. Write a complete proof for Case 3 (i.e., $\alpha = (\alpha_1 \vee \alpha_2)$) of the induction step in proving Lemma 3 (page marked "15" in the April 6's lecture notes).

6. (Duality) Let α be a wff whose only connectives are $\wedge, \vee$, and $\neg$. Let α^* be the resulting wff after interchanging $\wedge$ and $\vee$ and replacing each sentence symbol (e.g., A) by its negation (i.e., $(\neg A)$).

   THEOREM: α^* *is tautologically equivalent to* $(\neg\alpha)$, *i.e.,* $\alpha^* \models (\neg\alpha)$ *and* $(\neg\alpha) \models \alpha^*$.

   Give a proof using mathematical induction.