Theorem 22A

Let v_1 and v_2 be two assignments that agree on all variables occurring free in a wff φ. Then

$$\models_{\mathfrak{A}} \varphi[v_1] \quad \textit{iff} \quad \models_{\mathfrak{A}} \varphi[v_2]$$

Proof:

By an induction on n = # of connectives & quantifiers

~~On what?~~ Basis: $n=0$, φ is atomic

φ is atomic $\Rightarrow$ each variable occuring in φ occurs free

$\Rightarrow v_1(x) = v_2(x)$ for all x in φ $\Rightarrow \underbrace{\bar{v}_1(t) = \bar{v}_2(t) \text{ if } t \text{ is in } \varphi}_{(*)}$

Case 1 $\varphi : \approx t_1 t_2$

$\models_{\mathfrak{a}} \varphi[v_1] \Leftrightarrow \models_{\mathfrak{a}} \approx t_1 t_2 [v_1] \Leftrightarrow \bar{v}_1(t_1) = \bar{v}_1(t_2) \Leftrightarrow \bar{v}_2(t_1) = \bar{v}_2(t_2)$ (By $*$)

$\Leftrightarrow \models_{\mathfrak{a}} \approx t_1 t_2 [v_2]$

Case 2: $\varphi : P t_1 \cdots t_n$

$\models_{\mathfrak{a}} P t_1 \cdots t_n [v_1] \Leftrightarrow (\bar{v}_1(t_1), \ldots \bar{v}_1(t_n)) \in P^{\mathfrak{a}} \Leftrightarrow \ldots$

Induction Step:

I.H. Assume that $\models_{\mathfrak{a}} \varphi_i[v_1]$ iff $\models_{\mathfrak{a}} \varphi_i[v_2]$ $(i=1,2)$

when φ_i has $<n$ connectives and quantifiers

Case 1 $\varphi : \neg\varphi_1$

$\models_{\mathfrak{a}} \varphi[v_1] \Leftrightarrow \models_{\mathfrak{a}} \neg\varphi_1[v_1] \Leftrightarrow \not\models_{\mathfrak{a}} \varphi_1[v_1] \Leftrightarrow \not\models_{\mathfrak{a}} \varphi_1[v_2]$

$\Leftrightarrow \models_{\mathfrak{a}} \neg\varphi_1[v_2] \Leftrightarrow \models_{\mathfrak{a}} \varphi[v_2]$

Case 2: $\varphi : \varphi_1 \wedge \varphi_2$

Omited

Case 3 : φ : $\forall x \varphi_1$

$\models_{\mathfrak{A}} \varphi[v_1] \iff \models_{\mathfrak{A}} \forall x \varphi_1[v_1] \iff$ for each $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \varphi_1[v_1(x/d)]$

since $v_1 v_2$ agree on all free var in $\forall x \varphi_1$

$v_1(x/d)$, $v_2(x/d)$ agree on all free vars in φ_1

$\iff \models_{\mathfrak{A}} \varphi_1 [v_2(x/d)] \iff \models_{\mathfrak{A}} \forall x \varphi_1[v_2] \iff \models_{\mathfrak{A}} \varphi[v_2]$

for all $d \in |\mathfrak{A}|$.

Theorem (Corollary 22B)

For each sentence σ, either

(a) $\mathfrak{A}$ satisfies σ with every assignment, or

(b) $\mathfrak{A}$ does not satisfy σ with every assignment.

Proof: Follow from the previous theorem ∎

- For sentences σ,
  we use $\models_{\mathfrak{A}} \sigma$ to mean
  $\models_{\mathfrak{A}} \sigma[v]$ for some (all) assignments v

Models

A structure $\mathfrak{A}$ is a *model* of a sentence σ
  if $\mathfrak{A}$ satisfies σ with every assignment, i.e. $\models_{\mathfrak{A}} \sigma$

$\mathfrak{A}$ is a *model* of a set Σ of sentences
  if $\mathfrak{A}$ is a model of every sentence in Σ

A set Σ of sentences is *satisfiable*
  if it has a model

$\mathcal{L}_1 = \langle 0, 1, <, +, \times \rangle$, $\langle \mathbb{R}; 0, 1, <, +, \times \rangle$

$\forall x \forall y \forall z$ $\quad x \times (y+z) \approx x \times y + x \times z$

$\langle \mathfrak{a}, 0, 1, <, +, \times \rangle$

$\mathcal{L}_2 = \langle 0, < \ + \ \times \rangle$ $\quad \mathbb{R}$, $\models_{\mathfrak{a}} \varphi(x)[v]$ iff $v(x) = 1$

$\varphi(x) = \forall y \ y \times x = y$

$\mathcal{L}_3 = \langle P \rangle$

$\forall x \forall y \ x \approx y$

$\forall x \forall y \ Pxy \to Pyx$

$|\mathfrak{a}_1| = \{a\}$ $\quad |\mathfrak{a}_2| = \{a\}$

$P^{\mathfrak{a}_1} = \{(a,a)\}$ $\quad P^{\mathfrak{a}_2} = \{\}$

Logical Implication

Let Γ be a set of wffs and φ a wff of a language $\mathcal{L}$

Γ *logically implies* φ, denoted as $\Gamma \models \varphi$,
if for every structure $\mathfrak{A}$ (of $\mathcal{L}$), every assignment v
such that $\mathfrak{A}$ satisfies every wff in Γ with v,
$\mathfrak{A}$ also satisfies φ with v

φ and ψ are *logically equivalent*, $\varphi \models\!\!=\!\!| \; \psi$,
if $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$

A wff φ is *valid* if $\emptyset \models \varphi$ $\iff$ for all $\mathfrak{A}$ all v $\models_{\mathfrak{A}} \varphi[v]$

Notation: $\{\varphi\}$ is denoted as φ, $\emptyset \models \varphi$ as $\models \varphi$

$x \approx x$

$\forall x (Px \lor \neg Px)$