Homework #2

CS209, Spring 2017

1. Consider a (first-order) logic language with the following parameters: $\forall$, Px (x is a person), Ty (y is a time instant), Fxy (you can fool person x at time instant y), Translate the following English sentences into this logic language. (If a sentence is ambiguous and you will need more than one translation.)
   (a) You can fool some of the people all of the time.
   (b) You can fool all of the people some of the time.
   (c) You can't fool all of the people all of the time.
2. Let Γ be a set of wffs, φ and ψ be wffs in some first-order language $\mathcal{L}$. Prove the following:
   (a) $\Gamma; \varphi \models \psi$ iff $\Gamma \models (\varphi \to \psi)$.
   (b) $\varphi \models\!\dashv \psi$ iff $\models (\varphi \leftrightarrow \psi)$.
3. Let α and β be two wffs. Prove the following: $\{\forall x(\alpha \to \beta), \forall x \alpha\} \models \forall x \beta$.
4. Show that a wff θ is valid iff $\forall x \theta$ is valid.
5. Let $\mathcal{L}$ be a logic language with one unary predicate symbol R, no contant symbols, no function symbols. Let V be a set of variables. We define the set of wffs $\Gamma = \{\neg Ry \mid y \in V\} \cup \{\exists x\, Rx\}$. Is Γ satisfiable? Prove your answer.
6. Let $\mathfrak{R}$ be the structure $(\mathbb{R}, +, \times)$ for real numbers for the language consists of $\forall, +$, and $\times$ (no constants). The addition and multiplication operations in $\mathfrak{R}$ are the usual operations.
   (a) Write a formula that defines the set $\{0\}$ (i.e., containing a single element 0).
   (b) Write a formula that defines the interval $[0, \infty)$ (a set of real numbers).
   (c) Write a formula that defines the set $\{2\}$.
7. Consider a new quantifier: $\exists! x \alpha$ (read "there exists a unique x such that α") is to be satisfied in a structure $\mathfrak{A}$ with an assignment v iff there is one and only one element $d \in |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \alpha[v(x/d)]$. Assume that the language has the equality symbol. Find a formula in the original language that is logically equivalent to $\exists! x \alpha$.
8. Prove Part (3) and Part (4) of the Homomorphism Theorem.
9. Let h be an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{B}$. Show that there is a structure $\mathfrak{C}$ isomorphic to $\mathfrak{B}$ such that $\mathfrak{A}$ is a substructure of $\mathfrak{C}$.
   Hint: Let g be a one-to-one function with domain $|\mathfrak{B}|$ such that $g(h(a)) = a$ for all $a \in |\mathfrak{A}|$. Define the structure $\mathfrak{C}$ such that g is an isomorphism onto $\mathfrak{C}$.
10. Consider the structure structure $(\mathbb{R}, +)$ of real numbers for the language consisting of $\forall, +$ (no multiplication nor constants). The addition operation is the usual operations. Prove that the set $\{1\}$ cannot be defined by any formula.
    Hint: Consider automorphisms defined by linear functions.