Homework 1 (CS209, Spring 2017)

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We will use symbols T and F to indicate truth values True and False respectively.

1. Let A, B be distinct sentence symbols in $\mathcal{S}$. Determine if each of the following wffs is a tautology. If your answer is negative, find a truth assignment v that does not satisfy the wff and show the truth values of A, B under the assignment v.

   (a) $(((A \to B) \to B) \to B)$

   (b) $(((A \to B) \to B) \to A)$

   Solution: We will use truth table to check if the above wffs are tautologies.

A	B	$(((A \to B) \to B) \to B)$	$(((A \to B) \to B) \to A)$
T	T	T	T
T	F	**F**	T
F	T	T	**F**
F	F	T	T

   As we can clearly, see from the above table, there are truth assignments where the wffs evaluate to false. Specifically, the wff:$(((A \to B) \to B) \to B)$ is false when $A = T$ and $B = F$, the wff: $(((A \to B) \to B) \to A)$ is false when $A = F$ and $B = T$.

   So none of the wffs are tautology.

2. Let A, B, C be distinct sentence symbols in $\mathcal{S}$. Show that neither of the following two wffs tautologically implies the other:

   $$(A \leftrightarrow (B \leftrightarrow C))$$
   $$((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$$

   Note that you need to exhibit two truth assignments (i.e., not eight).

   Solution: Lets denote $\alpha = (A \leftrightarrow (B \leftrightarrow C))$ and $\beta = ((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C))))$.
   Consider the following truth assignment, $A = T$, $B = F$ and $C = F$, with this assignment the wff α evaluates to T, however β evaluates to F. This proves that $\alpha \not\models \beta$.

   Similarly, consider another truth assignment, $A = F$, $B = F$ and $C = F$. However, with this assignment the wff α evaluates to F and β evaluates to T. This proves that $\beta \not\models \alpha$.

   Thus $\alpha \not\models \beta$ and $\beta \not\models \alpha$.

3. Prove or disprove:

   THEOREM: For each natural number $n \geqslant 2$, there is a set Σ_n with n wff's such that (1) Σ_n is not satisfiable, and (2) each $(n-1)$-element subset of Σ_n is satisfiable.

   Solution: I will prove this by construction.

   Consider $A_1, A_2, A_3...$ be the countably infinite distinct sentence symbols in $\mathcal{S}$.

   Base case (n = 2): We can have $\Sigma_2 = \{A_1, \neg A_1\}$. All the possible subsets of size 1 (i.e. n-1) $\{A_1\}$ and $\{\neg A_1\}$ are satisfiable with truth assignments $A = T$ and $A = F$ respectively. However, $\{A_1, \neg A_1\}$ is not satisfiable.

   Now for any $n > 2$, we can create $\Sigma_n = \{A_1, A_2, ..., A_{n-1}, \alpha\}$, where $\alpha = ((\neg A_1) \vee (\neg A_2) \vee (\neg A_3) \vee ... \vee (\neg A_{n-1}))$.

   Consider all the subsets of size (n-1) of Σ_n, there are n subsets of size n-1, lets call them sub_1, sub_2, sub_3,sub_4,...,sub_{n-1}, sub_n. Where sub_1 is a set of all elements of Σ_n except A_1 i.e., $\Sigma_n - A_1 = \{A_2, ..., A_{n-1}, \alpha\}$, similarly $sub_2 = \Sigma_n - A_2$ and so on, finally $sub_n = \Sigma_n - \alpha$. Each of the subsets are satisfiable with mappings v_1, v_2,...,v_n respectively.

   where for $i \leq (n-1)$,

   $$v_i(B) = \begin{cases} F & if\ B = A_i \\ T & otherwise \end{cases} \tag{1}$$

   and $v_n = T$ for all sentence symbols in $\mathcal{S}$.

   Now, for Σ_n to be satisfiable we should have *all* sentence symbols $A_1, A_2, ..., A_{n-1}$ (condition 1) should be T and also α should be T, however, for α to be T, we should have atleast one sentence symbol from $A_1, A_2, ..., A_{n-1}$ to be F, contradicting condition 1, thus Σ_n is not satisfiable but all subsets of size (n-1) are satisfiable. Hence the proof.

4. Let Σ be a (possibly infinite) set of wffs and α, β two wffs.

   Prove: $\Sigma; \alpha \models \beta$ if and only if $\Sigma \models (\alpha \rightarrow \beta)$

   Note that the notation "$\Sigma; \alpha$" means $\Sigma \cup \{\alpha\}$.

   Solution: Lets assume $\Sigma; \alpha \models \beta$ (call this (1)), this means for all mappings $\bar{v}$ such that, if $\bar{v}(\Sigma) = T$ and $\bar{v}(\{\alpha\}) = T$ then $\bar{v}(\{\beta\}) = T$. Consider all the mappings $\bar{v_1}$ such that $\bar{v_1}(\Sigma) = T$, and $\bar{v_1}(\{\alpha\}) = F$, any mapping satisfying Σ should be either $\bar{v}$ or $\bar{v_1}$, from (1), $\bar{v}(\{\beta\}) = T$, and $\bar{v_1}(\{\alpha\}) = F$ or $\bar{v_1}(\{\neg\alpha\}) = T$, Hence $\Sigma \models (\beta \vee \neg\alpha)$ i.e., $\Sigma \models (\alpha \rightarrow \beta)$

   Consider, $\Sigma \models (\alpha \rightarrow \beta)$, this means all mappings $\bar{v}$, which evaluate $\bar{v}(\Sigma) = T$, if $\bar{v}(\{\alpha\}) = T$, then $\bar{v}(\{\beta\}) = T$. So, if $\bar{v}(\Sigma) = T$ and $\bar{v}(\{\alpha\}) = T$, then $\bar{v}(\{\beta\}) = T$, thus, $\bar{v}(\Sigma \cup \{alpha\}) = T$, then $\bar{v}(\{\beta\}) = T$, Hence $\Sigma; \alpha \models \beta$.

5. Write a complete proof for Case 3 (i.e., $\alpha = (\alpha_1 \vee \alpha_2)$) of the induction step in proving Lemma 3 (page marked "15" in the April 6's lecture notes).

   Solution: $\alpha = (\alpha_1 \vee \alpha_2)$

   $\alpha \in \Delta$

   $(\alpha_1 \vee \alpha_2) \in \Delta$

$\{\neg\alpha_1, \neg\alpha_2, \alpha_1 \vee \alpha_2\} \nsubseteq \Delta$, because Δ is fin sat.

$\neg\alpha_1 \notin \Delta$ or $\neg\alpha_2 \notin \Delta$

$\alpha_1 \in \Delta$ or $\alpha_2 \in \Delta$

$\bar{u}(\alpha_1) \; = \; T$ or $\bar{u}(\alpha_2) \; = \; T$

$\bar{u}(\alpha_1 \vee \alpha_2) \; = \; T$

6. (Duality) Let α be a wff whose only connectives are $\wedge, \vee$, and $\neg$. Let α^* be the resulting wff after interchanging $\wedge$ and $\vee$ and replacing each sentence symbol (e.g., A) by its negation (i.e., $(\neg A)$).

   THEOREM: *α^* is tautologically equivalent to $(\neg\alpha)$, i.e., $\alpha^* \models (\neg\alpha)$ and $(\neg\alpha) \models \alpha^*$.*

   Give a proof using mathematical induction.

   Solution:

   Base case: Consider $\alpha \; = \; A$, So, $\alpha^* \; = \; \neg A$. Now, $\neg\alpha \; = \; \neg A$. As we can see α^* is same as $\neg\alpha$. So, $\alpha^* \models (\neg\alpha)$ and $(\neg\alpha) \models \alpha^*$

   Induction: Consider, a wff α_1 Such that

   $\alpha_1^* \models (\neg\alpha_1)$ and $(\neg\alpha_1) \models \alpha_1^*$ (consider this 1)

   Now, Consider $\alpha = \neg\alpha_1$, So, $\alpha^* = (\neg\alpha_1)*$. So, $\alpha^* = \neg\alpha_1*$.

   Now, $\neg\alpha = \neg(\neg\alpha_1)$. From 1, we know $\neg\alpha_1$ and α_1^* are tautologically equivalent. So, $\neg\alpha = \neg\alpha_1^*$ Thus α^* and $\neg\alpha$ are equivalent.

   Consider another wff α_2, Such that

   $\alpha_2^* \models (\neg\alpha_2)$ and $(\neg\alpha_2) \models \alpha_2^*$ (consider this 2)

   Now, Consider $\alpha = \alpha_1 \vee \alpha_2$, So, $\alpha^* = \alpha_1^* \wedge \alpha_2^*$. From (1) and (2) we have, $\alpha^* = (\neg\alpha_1) \wedge (\neg\alpha_2) = \neg(\alpha_1 \vee \alpha_2) = \neg\alpha$. Thus α^* and $\neg\alpha$ are equivalent.

   Now, Consider $\alpha = \alpha_1 \wedge \alpha_2$, So, $\alpha^* = \alpha_1^* \vee \alpha_2^*$. From (1) and (2) we have, $\alpha^* = (\neg\alpha_1) \vee (\neg\alpha_2) = \neg(\alpha_1 \wedge \alpha_2) = \neg\alpha$. Thus α^* and $\neg\alpha$ are equivalent.