Well-Formed Formulas

A *well-formed formula* (*wff*) is one of the following:

- (atomic) $\approx t_1 t_2$
  where t_1, t_2 are terms,

- (atomic) $P t_1 \cdots t_n$
  where P is an n-ary predicate symbols
  and $t_1, \ldots, t_n$ are terms,

- $(\varphi \wedge \psi)$, $(\neg\varphi)$, or $(\forall x \varphi)$
  where φ, ψ are wffs and x is a variable

$\langle \approx, E \rangle$

$(\exists x \exists y \exists z \, (E(x,y) \wedge E(y,z)))$

Free Variables

A variable x *occurs free* in a wff φ if one of the following holds:

- φ is atomic and x occurs in φ,
- φ is $(\psi_1 \wedge \psi_2)$ and x occurs free in ψ_1 or ψ_2,
- φ is $(\neg\psi)$ and x occurs free in ψ, or
- φ is $(\forall y \psi)$, x occurs free in ψ, and x, y are distinct variables

A wff φ is a *sentence* if no variables occur free in it

First Order Structure

A *structure* of a first order language $\mathcal{L}$ is a mapping $\mathfrak{A}$ over the parameters of $\mathcal{L}$ such that $\mathfrak{A}$ maps

1. the quantifier $\forall$ to a nonempty set denoted as $|\mathfrak{A}|$ (called the *universe*),
2. each constant symbol c to an element $c^{\mathfrak{A}}$ in the universe $|\mathfrak{A}|$,
3. each n-ary function symbol f to an n-ary function $f^{\mathfrak{A}} : |\mathfrak{A}|^n \to |\mathfrak{A}|$, and
4. each n-ary predicate symbol P to an n-ary relation $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$

V_3: $x \to 2$, $y \to 2$, $z \to 2$, $x_1 \to 3$

$\models_{\mathfrak{a}_1} \varphi_2[v_3]$?

$\langle \approx, E \rangle$ $|\mathfrak{a}_1| = \{1, 2, 3\}$, $E^{\mathfrak{a}_1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$

v_1: $x \mapsto 1$, $y \to 2$, $z \to 3$

$\models_{\mathfrak{a}_1} \varphi_2[v_1]$

v_2: $x \mapsto 1$, $y \to 1$, $z \to 1$

$\boxed{\models_{\mathfrak{a}_1} \varphi_2[v_2] \,?}$

$|\mathfrak{a}_2| = \{1, 2, 3, 4\}$ $E^{\mathfrak{a}_2} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 1 \end{bmatrix}$

$\varphi_2 = \exists x \exists y \exists z \, (E(x,y) \wedge E(y,z))$

$\models_{\mathfrak{a}_1} \varphi_2 \wedge E(x,y)$ $[v_{1,2,3}]$

Assignments

Let $\mathfrak{A}$ be a structure for a first order language $\mathcal{L}$

An *assignment* is a mapping v from variables to the universe $|\mathfrak{A}|$, $v : V \to |\mathfrak{A}|$

An assignment v is *extended* to a mapping $\bar{v}$ from terms to $|\mathfrak{A}|$ as follows:

- For each variable x, $\bar{v}(x) = v(x)$
- For each constant symbol c, $\bar{v}(c) = c^{\mathfrak{A}}$
- If $t_1, \ldots, t_n$ are terms and f an n-ary function symbol, $\bar{v}(ft_1 \cdots t_n) = f^{\mathfrak{A}}(\bar{v}(t_1), \ldots, \bar{v}(t_n))$

Satisfaction

A structure $\mathfrak{A}$ *satisfies* a wff φ with an assignment v, denoted as $\models_{\mathfrak{A}} \varphi[v]$, if

- $\models_{\mathfrak{A}} \approx t_1 t_2[v]$ if $\bar{v}(t_1) = \bar{v}(t_2)$
- $\models_{\mathfrak{A}} P t_1 \cdots t_n[v]$ if $(\bar{v}(t_1), \ldots, \bar{v}(t_n)) \in P^{\mathfrak{A}}$
- $\models_{\mathfrak{A}} (\neg\varphi)[v]$ if $\not\models_{\mathfrak{A}} \varphi[v]$
- $\models_{\mathfrak{A}} (\varphi_1 \wedge \varphi_2)[v]$ if $\models_{\mathfrak{A}} \varphi_1[v]$ and $\models_{\mathfrak{A}} \varphi_2[v]$
- $\models_{\mathfrak{A}} (\forall x \varphi)[v]$
  if $\models_{\mathfrak{A}} \varphi[v(x/d)]$ for every element d in the universe $|\mathfrak{A}|$, where $v(x/d)$ is the following assignment modified from v:

$$v(x/d)(y) = \begin{cases} d & \text{if } y = x \\ v(y) & \text{otherwise} \end{cases}$$

Abbreviations

$(\alpha \vee \beta)$ abbreviates $(\neg((\neg\alpha) \wedge (\neg\beta)))$

$(\alpha \rightarrow \beta)$ abbreviates $(\neg(\alpha \wedge (\neg\beta)))$

$(\alpha \leftrightarrow \beta)$ abbreviates $((\neg(\alpha \wedge (\neg\beta))) \wedge (\neg(\beta \wedge (\neg\alpha))))$

$(\exists x\, \alpha)$ abbreviates $(\neg(\forall x\, (\neg\alpha)))$

Extending *satisfaction* naturally, e.g.:

- $\models_{\mathfrak{A}} (\varphi_1 \vee \varphi_2)[v]$ if $\models_{\mathfrak{A}} \varphi_1[v]$ or $\models_{\mathfrak{A}} \varphi_2[v]$,
- $\models_{\mathfrak{A}} (\exists x\, \varphi)[v]$ if for some element $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \varphi[v(x/d)]$

Proof of $\models_{\mathfrak{A}} \exists x\, \alpha[v]$

$\models_{\mathfrak{A}} \exists x\, \alpha[v]$

iff for some element $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \alpha[v(x/d)]$

$\models_{\mathfrak{A}} \exists x\, \alpha[v]$

iff $\models_{\mathfrak{A}} \neg\forall x \neg\alpha[v]$

iff $\not\models_{\mathfrak{A}} \forall x \neg\alpha[v]$

iff it is not true that for all d in $|\mathfrak{A}|$, $\models_{\mathfrak{A}} \neg\alpha[v(x/d)]$

iff it is not true that for all d in $|\mathfrak{A}|$, $\not\models_{\mathfrak{A}} \alpha[v(x/d)]$

iff for some element $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \alpha[v(x/d)]$

Theorem 22A

Let v_1 and v_2 be two assignments that agree on all variables occurring free in a wff φ. Then

$$\models_{\mathfrak{A}} \varphi[v_1] \quad \textit{iff} \quad \models_{\mathfrak{A}} \varphi[v_2]$$

Proof:

By an induction.

On what?

Theorem (Corollary 22B)

For each sentence σ, either

(a) $\mathfrak{A}$ satisfies σ with every assignment, or

(b) $\mathfrak{A}$ does not satisfy σ with every assignment.

Proof: Follow from the previous theorem ∎

- For sentences σ,
  we use $\models_{\mathfrak{A}} \sigma$ to mean
  $\models_{\mathfrak{A}} \sigma[v]$ for some (all) assignments v