Examples of Logical Implication

$\forall x\, Qx \models Qy$

Proof:

Let $\mathfrak{A}$ be a structure and v an assignment.

$\models_{\mathfrak{A}} \forall x Qx[v]$

$\Rightarrow$ for each $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} Qx[v(x/d)]$

$\Rightarrow$ for each $d \in |\mathfrak{A}|$, $d \in Q^{\mathfrak{A}}$

$\Rightarrow$ letting $d = v(y)$, $\models_{\mathfrak{A}} Qy[v]$

(Note that y is not a free variable in $\forall x Qx$)

$Qy \not\models \forall x Qx$

Proof?

$\lvert\mathfrak{A}\rvert = \mathbb{R}$	$\{1, 2\}$
$Q^{\mathfrak{A}} = [0, 5]$	$\{1\}$
$v(y) = 1$	$v(y) = 1$

$\models \exists x(Qx \to \forall x Qx)$?

Equivalently: is $\exists x(Qx \to \forall x Qx)$ valid ?

Yes, proof?

$\iff$ for every structure $\mathfrak{A}$, every assignment v, $\models_{\mathfrak{A}} \exists x(Qx \to \forall x Qx)[v]$

$\iff$ " " " " there exists $d \in |\mathfrak{A}|$ $\models_{\mathfrak{A}} Qx \to \forall x Qx[v(x/d)]$

$\iff$ " " " " there exists $d \in |\mathfrak{A}|$, either $\not\models_{\mathfrak{A}} Qx[v(x/d)]$, or $\models_{\mathfrak{A}} \cancel{\forall x Qx[v(x/d)]}$ $\models_{\mathfrak{A}} \forall x Q(x)[v]$

$\iff$ " , $\mathfrak{A}$ " " v " d, either $d \notin Q^{\mathfrak{A}}$, or for each $d' \in |\mathfrak{A}|$ $\underline{\models_{\mathfrak{A}} Qx[v(x/d')]}$

$d' \in Q^{\mathfrak{A}}$

$\iff$ true

Example: $\exists x \forall y Pxy \models \forall y \exists x Pxy$? Yes

$\dashv$?

Homomorphism

Let $\mathfrak{A}$, $\mathfrak{B}$ be two structures (of some logic language)

A mapping $h : |\mathfrak{A}| \to |\mathfrak{B}|$ is a *homomorphism* of $\mathfrak{A}$ *into* $\mathfrak{B}$ if

1. for each n-ary predicate symbol P and elements $a_1, \ldots, a_n$, $(a_1, \ldots, a_n) \in P^{\mathfrak{A}}$ iff $(h(a_1), \ldots, h(a_n)) \in P^{\mathfrak{B}}$,

2. for each n-ary function symbol f and each n-tuple $(a_1, \ldots, a_n)$, $h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$, and

3. for each constant symbol c, $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$

$\langle +, \times \rangle$ $\langle \mathbb{N}, +, \times \rangle$ $\mathfrak{B}$ $\{e, o\}$ $+^{\mathfrak{B}}$ $\begin{bmatrix} & e & e \\ e & o & o \\ o & e & o \\ o & o & e \end{bmatrix}$ $\times^{\mathfrak{B}}$ $\begin{bmatrix} e & e & e \\ e & o & e \\ o & e & e \\ o & o & o \end{bmatrix}$

Language

$h(n) = \begin{cases} e & \text{if } n \text{ is even} \\ o & \text{" "} \end{cases}$

$<$ $\langle \mathbb{N}\ < \rangle \leftarrow \mathbb{P}$ positive numbers $<$

$h_1(n) = n$ not onto

$h_2(n) = n - 1$ onto

Isomorphism

Let h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

- If h is 1-to-1, it is an *isomorphism* of $\mathfrak{A}$ *into* $\mathfrak{B}$
- If h is 1-to-1 and onto,
  it is an *isomorphism* of $\mathfrak{A}$ *onto* $\mathfrak{B}$
- $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*, denoted as $\mathfrak{A} \cong \mathfrak{B}$,
  if there is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$

Homomorphism Theorem

$$\begin{tikzcd} V \arrow[r, "v"] \arrow[d, "h\circ v"'] & |\mathfrak{A}| \arrow[dl, "h"] \\ |\mathfrak{B}| & \end{tikzcd}$$

Theorem:

Let h be a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ and v an assignment for $\mathfrak{A}$ (mapping from variables to $|\mathfrak{A}|$).

1. For each term t, we have $h(\bar{v}(t)) = \overline{h \circ v}(t)$
2. For each quantifier-free formula φ not containing the equality symbol, $\models_{\mathfrak{A}} \varphi[v]$ iff $\models_{\mathfrak{B}} \varphi[h \circ v]$
3. If h is one-to-one, then "not containing the equality symbol" in (2) can be dropped
4. If h is onto, then "quantifier-free" in (2) can be dropped

1) By induction on # fn symbols in terms

Basis : $n = 0$

Case 1 $t = c$: $h(\bar{v}(c)) = h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$

$\overline{h\circ v}(c) = c^{\mathfrak{B}}$

Case 2 : $t = x$ $h(\bar{v}(x)) = h(v(x)) = h\circ v(x)$

$= \overline{h\circ v}(x)$