CM1103 Coursework 2015: Report

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***Abstract*—This report presents results of my investigation into fairness of two different scoring systems in sailing competitions based on results of conducted simulations.**

I. Introduction

I will start by defining both systems of scoring by referring to resources. Then I will present my (mathematical) findings, and conclusions which derive directly from it. My findings will be supported by date gathered by means of simulation and will be presented as a graph.

II. Main Part

To begin investigation of fairness of both systems for all combinations of skills and consistency, I need to explicitly define what fairness is. Assume that the system is said to be **fair** (fairness of system) if it favors sailor's *mean skill* μ over *standard deviation* σ.

One of the most successful features of the Olympic regatta was the system of scoring. This method, the invention of an Austrian scientist and yachtsman, is that the maximum points are awarded to the winner of a race.[1]

As shown in the findings section the word **maximum** is a key here. It puts emphasis on the fact that the difference in points which are awarded to two consecutive sailors decreases as the sailor's boat placing is worse. Such process is repeated for N races and the final score of competitor is sum of $N-1$ best marks.

Therefore, what is rewarded is a record of a consistently good performance over the whole series. It is of interest that visiting yachtsmen were so impressed with the fairness of this system that some of them have decided to use it for important events in their home waters.[1]

I will prove the truth of this statement and show how it compares with the first system which will be hereafter also referred as *Low Point System* (LPS).[2][3]

III. Findings

1) **(Range)** Given A - the number of boats participating in the competition and N - sailor's position in the race. The calculated scores in particular system will fall into given range as shown in Table I.

TABLE I
RANGE OF POINTS

Sailor's	Range
System I	$[1, N] \vee [-N, -1]$
System II	$[101, 101 + \log A]$

2) **(Difference)** Let k and l be any achieved position in a single race by a competitor. Then the difference in points between position k and l in *system I* and *system II* can be expressed as $|k - l|$ and $|1000 \log \frac{k}{l}|$ respectively. This implies that:
   - The difference in scoring between two consecutive position is constant in the first system and it is equal to $|k - (k-1)| = 1$. In contrary, it is not in the second scheme and is equal to $|1000 \log \frac{k}{k-1}|$.
   - Achieving top position in the second system is awarded with much greater amounts of points (in comparison to lower positions) that it would be in the first system. This leads to conclusion that scoring top potions not necessarily consistently is better than scoring middle positions in the second system.
   - However, since in the *first scheme* the difference is constant, the consistency is not as important as it would be in the second system.
3) **(Consistency)** For sailors with relatively high standard deviation σ) second system is relatively more beneficial in comparison with the first scheme. Assume that sailing competition consists of k races. Then let $n_1, \ldots n_k$ be arbitrary positions achieved by a sailor X in each race. Then the final scores expressed by formula of first and second system are equal to:

$$X_I = \sum_{\substack{i=1 \\ n_i \neq \max\{n_1, \ldots n_k\}}}^{k} n_i$$

$$X_{II} = \sum_{\substack{i=1 \\ n_i \neq \max\{n_1, \ldots n_k\}}}^{k} (101 + \log A - \log n_i)$$

Now assume that instead of achieving arbitrary position $n_1, \ldots n_k$ the sailor Y is consistent in about such a way that he always ends race at position:

$$m = \frac{X_I}{k-1}$$

Then his score expressed in *system I* and *system II* respectively would be equal to:

$$Y_I = \sum_{i=1}^{k-1} m = (k-1)m = (k-1)\frac{X_I}{k-1} = X_I$$

$$Y_{II} = \sum_{i=1}^{k-1} (101 + \log A - \log m)$$

Now it can be proven that for all $n_1, \ldots n_k$:

$$X_{II} \geq Y_{II}$$

$$\sum_{\substack{i=1 \\ n_i \neq \max\{n_1,\dots n_k\}}}^{k} (-\log n_i) \geq \sum_{i=1}^{k-1}(-\log m)$$

Let $h = \prod_{i=1}^{k} n_i$ for $n_i \neq \max\{n_1, \dots, n_k\}$, then:

$$-\log h \geq -(k-1)\log m \implies \log h \leq (k-1)\log m$$
$$\implies \log h \leq \log m^{k-1} \implies h \leq m^{k-1}$$

Finally:

$$\sqrt[k-1]{h} \leq m$$

That's true for all suitable value of k and h since geometric mean is always equal or less than arithmetic mean.
Since $X_I = Y_I$ and $X_{II} \geq Y_{II}$ it can be concluded that second system definitely values more top positions over lower ones, which cannot be said about the *system I*. Moreover in many cases it might be worth for competitor awarded under *system II* to sacrifice his position in one race to achieve much higher position in next race. However this is not true for the first system. This shows that in first scoring scheme each position carries even weight of points.

IV. Simulation

A series of simulations were conducted with the fleet of 100 boats and all combinations of skill and consistency. However I will present only results of two relevant simulations:

1) **(Simulation I)** Since *mean skill* $\mu \in [0, 100]$ it is reasonable to set range of *standard deviation* as $\sigma \in [0, \approx \frac{max\{\mu\}}{3}]$. This is because *normal distribution* is used for simulating players *performance* $\mathcal{N} \sim (\mu, \sigma^2)$.

TABLE II
Simulation I: Parameters

Sailor's	Range
Mean Skill μ	$\mu \in [0, 100]$
Standard Deviation σ	$\sigma \in [0, 20]$

Figure 1 presents results of **Simulation I**.

2) **(Simulation II)** Assuming that only sailors with *mean skill* value $\mu \geq 70$ and *standard deviation* $\sigma \leq 10$ can go through Olympics qualifications we get:

TABLE III
Simulation II: Parameters

Sailor's	Range
Mean Skill μ	$\mu \in [70, 100]$
Standard Deviation σ	$\sigma \in [0, 10]$

Figure 2 presents results of **Simulation II**.

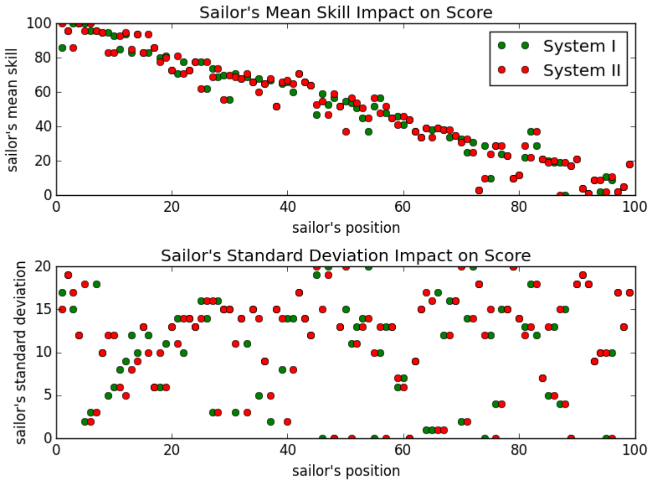

Fig. 1. Simulation results generated by `matplotlib` with assumptions as stated in Table II

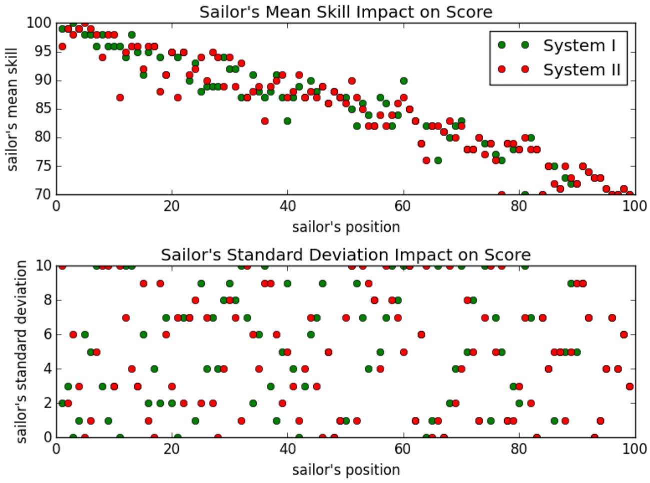

Fig. 2. Simulation results generated by `matplotlib` with assumptions as stated in Table III

V. Conclusion

After running several tests and compiling relevant figures it can be observed that:

1) **(Fairness)** As the standard deviation σ gets smaller the difference between the final ranking of sailing competition generated by *both* systems diminishes and is identical when $\sigma = 0$. Therefore, the fairness of the first scoring scheme is comparable with the fairness of the second system.
2) **(Consistency)** To conclude, both *Low Point System* (system I) and *High Point System* (system II) can be perceived as ones which favors mean skill over exceptional good performances. Since achieving top scores in the system is valued by far more than achieving the lower position, *system II* appreciate competitor's consistent over the whole competition.

References

[1] Organising Committee for the XIV Olympiad London 1948. *The Official Report of the Organising Committee for the XIV Olympiad London 1948 (pdf)*, pp. 513-517, 1951.
[2] Organising Authority. *London 2012 Olympic Sailing Competition Notice of Race" (PDF)*, pp. 6, retrieved 1 April 2014.
[3] *Appendix A* (refered in [2]. Appendix A refered to is avaible at: http://www.sailing.org/tools/documents/Appendices%20A%20-%20C-%5B444%5D.pdf