Introduction to simulations and Monte Carlo methods	02.12.2024
Project nr 2	
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1 Brownian motion and geometric Brownian motion

DISCLAIMER: This is by no means a full introduction to Brownian motion. It is a *minimalist* introduction for the purposes of this project.

1.1 Brownian motion

Roughly speaking, a stochastic process $\mathbf{B} = (B(t))_{t \le T}$ is a **Brownian motion** if $B(t_0) = 0$ at $t_0 = 0$, and for any $0 \le t_1 < \ldots < t_n \le T$, the vector $(B(t_1), \ldots, B(t_n))$ is a zero-mean multivariate normal random variable $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ with covariance matrix

$$\boldsymbol{\Sigma}(i,j) = \mathrm{Cov}\left(B(t_i), B(t_j)\right) = \min(t_i, t_j), \quad i,j = 1, \ldots, n.$$

In this project, we consider $T = 1$ and equally spaced time points $(t_1, t_2, \ldots, t_n) = \left(\frac{1}{n}, \frac{2}{n}, \ldots, 1\right)$.

1.2 Stratified sampling of a multivariate normal $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ random variable

Suppose we want to sample a random variable $\mathbf{B} = (B_1, \ldots, B_n)^T \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ using m strata. Let $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ be a multivariate standard normal random variable. The strata will be defined by ascending rings $A^1, \ldots, A^m$, which are determined by balls A'_i centered at $(0, \ldots, 0)$ with suitable radii such that $\mathbb{P}(\mathbf{Z} \in A^i) = 1/m$. Thus, let

- A'_1 be an n-dimensional ball such that $\mathbb{P}(\mathbf{Z} \in A'_1) = 1/m$;
- A'_2 be a ball such that $\mathbb{P}(\mathbf{Z} \in A'_2 \setminus A'_1) = 1/m$;
- etc.

Set $A^1 = A'_1$, $A^2 = A'_2 \setminus A'_1$, …, $A^m = A'_m \setminus A'_{m-1}$.

Let $\mathbf{A}$ be such that $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^T$ (Cholesky decomposition).

Define the i-th stratum by $S^i = \{\mathbf{A}\mathbf{z} : \mathbf{z} \in A^i\}$.

Assume that $\mathbf{Z}^i \stackrel{D}{=} (\mathbf{Z} \,|\, \mathbf{Z} \in A^i)$. Then $\mathbf{B}^i = \mathbf{A}\mathbf{Z}^i$ is from stratum S^i.

It remains to show how to sample $\mathbf{Z}^i \stackrel{D}{=} (\mathbf{Z} \,|\, \mathbf{Z} \in A^i)$. For $n = 2$ and $m = 1$, the method was presented in the lecture (which *de facto* is the Box-Muller method). For general $n \geq 2$, let $\xi_1, \ldots, \xi_n$ be i.i.d. standard normal $\mathcal{N}(0,1)$ random variables. Denote $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T$. Let $D > 0$. Then the vector

$$\left(D \frac{\xi_1}{||\boldsymbol{\xi}||}, \ldots, D \frac{\xi_n}{||\boldsymbol{\xi}||} \right)^T$$

has a uniform distribution on a sphere with radius D. We have the following proposition:

Proposition 1 *Let $\mathbf{Z} = (Z_1, \ldots, Z_n)$ be a standard multivariate normal random variable. Then the square of the length of $\mathbf{Z}$ is $D^2 = Z_1^2 + \ldots + Z_n^2$ and has a χ^2_n distribution (χ^2 with n degrees of freedom).*

Recall that the density and c.d.f. of χ^2_n are as follows:

$$f_{\chi^2_n}(r) = \frac{1}{2^{n/2}\Gamma(n/2)} r^{n/2-1} e^{-r/2}, \quad F_{\chi^2_n}(r) = \frac{1}{\Gamma(n/2)} \gamma_{n/2}(r/2),$$

where Γ is the gamma function, and γ is the *incomplete gamma function*.[1] For $n = 2$, the random variable D has the so-called Rayleigh distribution. Admittedly, there is no explicit formula for the inverse function of $F_{\chi^2_n}(r)$ for general n, but numerically this inverse is available in several libraries.[2]

Summing up, sampling $\mathbf{B}^i \stackrel{D}{=} (\mathbf{B} \,|\, \mathbf{B} \in A^i)$ is as follows:

1. Perform Cholesky decomposition: $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^T$.
2. Sample $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T$, where $\xi_i \sim \mathcal{N}(0,1)$ i.i.d. Set
$$\mathbf{X} = (X_1, \ldots, X_n)^T = \left(\frac{\xi_1}{||\boldsymbol{\xi}||}, \ldots, \frac{\xi_n}{||\boldsymbol{\xi}||} \right)^T.$$
3. Sample $U \sim \mathcal{U}(0,1)$. Set
$$D^2 = F^{-1}_{\chi^2_n} \left(\frac{i-1}{m} + \frac{1}{m} U \right).$$
4. Set $\mathbf{Z} = (Z_1, \ldots, Z_n) = (DX_1, \ldots, DX_n)$.
5. Return $\mathbf{B}^i = \mathbf{A}\mathbf{Z}$.

1.2.1 Stratified sampling of a Brownian motion

We can simply use the procedure described in Section 1.2. Recall that $\mathbf{B} = (B(1/n), B(2/n), \ldots, B(1))$ is a multivariate normal random variable $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ with the covariance matrix

$$\boldsymbol{\Sigma}(i,j) = \frac{1}{n} \min(i,j).$$

[1] https://en.wikipedia.org/wiki/Incomplete_gamma_function

[2] E.g., `scipy.stats.chi2.ppf` in `Python` or `chi2inv` in `Matlab`

We can perform the Cholesky decomposition $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$, where

$$\mathbf{A}(i,j) = \begin{cases} \dfrac{1}{\sqrt{n}} & \text{if } j \le i \\ 0 & \text{otherwise.} \end{cases}$$

In Figure 1, 5000 points within 4 strata were simulated using the above method.

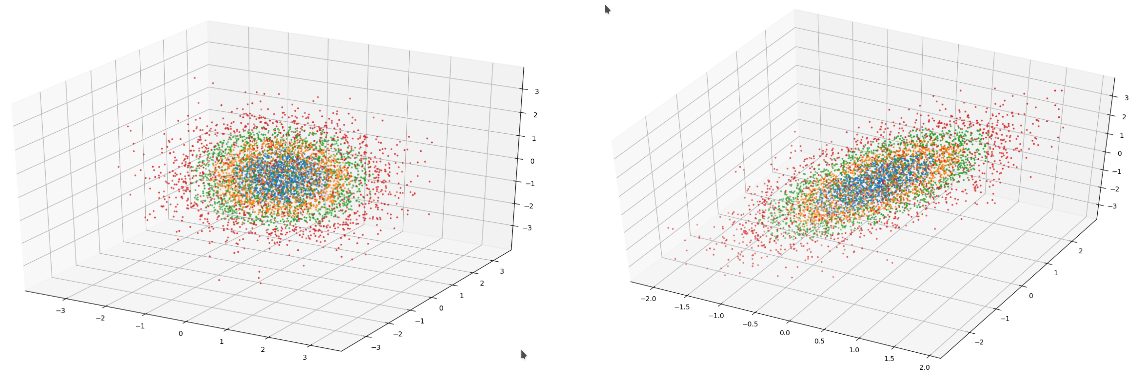

Figure 1: 5000 points from a 3-dimensional standard normal distribution obtained using stratified (4 strata) sampling (left). Points from a 3-dimensional normal distribution with covariance matrix $\mathbf{\Sigma}(i,j) = \min(i,j)/3$ (right).

1.3 Geometric Brownian motion

The evolution of stocks (assets) is often modeled as geometric Brownian motion—GBM(μ, σ)—which is defined by

$$S(t) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B(t)\right), \qquad 0 \le t \le T, \tag{1.1}$$

where $B(t)$ $(0 \le t \le T)$ is Brownian motion. In computing option prices, often the interest rate r and volatility σ are known; we then make computations for GBM(r, σ). Denote $\mu^* = r - \sigma^2/2$. Then we have

$$S(t) = S(0) \exp\left(\mu^* t + \sigma B(t)\right), \qquad 0 \le t \le T. \tag{1.2}$$

2 European and Asian call options

We are interested in estimating the following (called an *option*, with discounted payoff at time 1) with price given by the formula

$$I = e^{-r} E(A_n - K)_+, \tag{2.3}$$

where

$$A_n = \frac{1}{n} \sum_{i=1}^{n} S(i/n)$$

and $S(t)$ is given in (1.2).

In the case $n = 1$, this is called a **European call option**; otherwise, it is called an **Asian call option**.

2.1 Black-Scholes formula

In the case $n = 1$ (i.e., European call option), the exact value of $E(A_1 - K)_+ = E(S(1)-K)_+$ is provided by the Black-Scholes formula (where Φ is the c.d.f. of $\mathcal{N}(0,1)$):

$$E(S(1) - K)_+ = S(0)\Phi(d_1) - Ke^{-r}\Phi(d_2), \tag{2.4}$$

where

$$d_1 = \frac{1}{\sigma}\left[\log\left(\frac{S(0)}{K}\right) + r + \frac{\sigma^2}{2}\right],$$

and

$$d_2 = d_1 - \sigma.$$

3 Task

Fix the parameters: $r = 0.05$, $\sigma = 0.25$ (thus $\mu^* = r - \sigma^2/2 = -0.0125$), $S(0) = 100$, and $K = 100$.

Estimate the I given in (2.3) using

- a) Crude Monte Carlo estimator.
- b) Stratified estimator. Consider separately $n = 1$ and $n \geq 2$.
- c) For $n = 1$: Antithetic estimator. You may take (Z_{2i-1}, Z_{2i}) with $Z_{2i} = -Z_{2i-1}$, where Z_{2i-1}, $i = 1, \ldots, R/2$, are i.i.d. standard normal $\mathcal{N}(0,1)$.
- d) For $n = 1$: Control variate estimator. As a control variate, you may take $X = B(1)$.

Compare the results. For the case $n = 1$, compare estimations with the exact value using the Black-Scholes formula (2.4). For stratified estimators, consider proportional and optimal allocation schemes. Provide a report in a `.pdf` file and the working implementation you used.