7

Quantum Measurement

> A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.
>
> – *Max Planck, 1936*

The only way quantum theory allows us to interact with quantum systems is via quantum processes. Thus, the only way we can extract information about the state of a quantum system is to apply some non-deterministic process and observe which of the branches happened. For that reason, many people refer to the act of applying certain quantum processes as *measurement* and sometimes even as *observation*. A consequence of this unfortunate terminology is that one of the most touted 'strange' features of quantum theory is often misleadingly described as follows: 'The mere act of observing a quantum state changes it.'

'Misleading', since the concept of measurement described in the previous paragraph is far from the passive concept of observation familiar from our macroscopic world, but is rather a non-trivial process that will almost always drastically affect the quantum state. So, the mysterious aspect of quantum theory is not that 'observation' alters the quantum state but, rather, that it is impossible, even in principle, to 'observe' a quantum system in the classical sense.

Given this fundamental restriction on how we can extract information from a quantum system, what can we learn about that system by means of a quantum process? The answer is, in fact, very little! First, performing a particular measurement could, for the vast majority of quantum states, yield any outcome i with a non-zero probability. In that case, obtaining an outcome i doesn't tell us much about what the state of the system was. Second, we say 'was', because the measurement will moreover irreversibly change the state according to the outcome. So rather than revealing the state of a system, a quantum measurement typically erases that state from history!

Nonetheless, these quantum measurements are crucial to quantum theory since they constitute the only interface between us, in our classical world, and the quantum world. The great insight of the quantum computing community is that the non-deterministic changes induced by these quantum measurements are not a nuisance, but rather an extremely useful resource. Indeed, in the previous chapter we saw how any quantum map can be realised by means of a non-deterministic quantum process. In fact, the quantum process we used to demonstrate this was the simplest kind of quantum measurement.

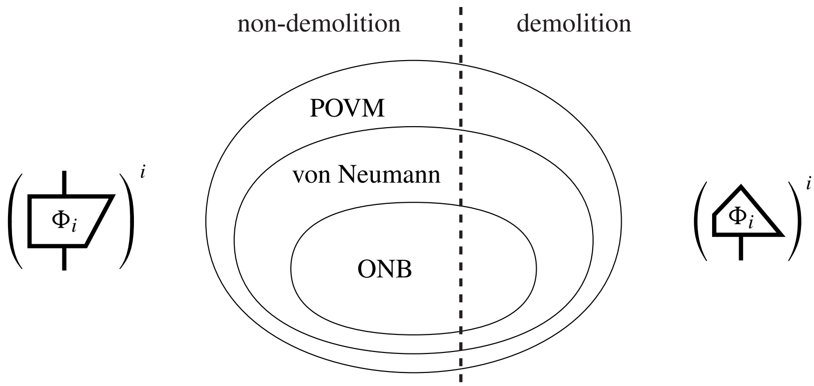
In the quantum teleportation protocol, it is Aleks' (non-deterministic) realisation of a cap via measurement that allows Bob to receive Aleks' state after performing a correction. We'll see in Section 7.2 that there are other very useful protocols that use a similar trick. For example, we can use measurements to glue short bits of entanglement into long bits of entanglement and give the crucial ingredient for *measurement-based quantum computation*, which we'll develop fully in Chapter 12.

When exploiting quantum measurement in these ways, the quantum algorithm and protocol designers have to be really clever in order to cancel out the resulting non-determinism (e.g. with unitary corrections in the case of quantum teleportation). This balancing act is pretty much the crux of the art of quantum algorithm and protocol design.

So what is a measurement exactly? From our point of view, there is nothing really to distinguish a measurement from any other non-deterministic quantum process, so in this chapter we will discuss three important families of processes that have been called measurements. These families are:

1. *ONB measurements*
2. *von Neumann measurements*
3. *POVM measurements*.

Each of these is more general than the previous one, and comes in two flavours, demolition and non-demolition measurements:



ONB measurements are the most specific, 'purest' kind of measurement, whereas POVM measurements, like impure quantum states, are much more general. In between these two extremal cases are von Neumann measurements, which over the past 80 years or so have been the most dominant notion of measurement. As the name suggests, they were the measurements that featured in von Neumann's original formulation of quantum theory.

One of the main reasons for this chapter is that we will be providing many concrete examples of quantum measurements, starting with the Stern–Gerlach device in Section 7.1.1. We give so many in fact that we won't bother putting them in special 'Example' sections.

7.1 ONB Measurements

The physicist working in the laboratory may be a bit confused by all this discussion about measurement. This is because, even though the result of performing a measurement on a state hardly resembles what one would expect from simple 'observation', in the laboratory many of these measurements boil down to observing something, for example, the position of a needle or a spot on a photo plate.

So there is a big mismatch between, on the one hand, how one acts on a system, i.e.:

- *one intends to observe it,*

and, on the other hand, what actually happens to that system, i.e.:

- *it undergoes a non-deterministic radical change.*

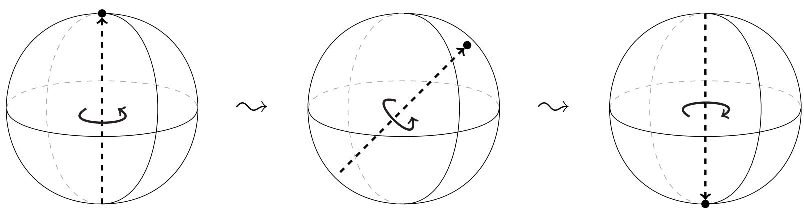
In order to see where this tension comes from, it helps to know what kinds of devices can actually perform quantum measurements.

7.1.1 A Dodo's Introduction to Measurement Devices

One of the earliest examples of a quantum measurement was performed by a *Stern–Gerlach apparatus*. We will explain here how this works in some detail.

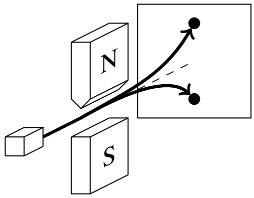
A standard exercise in a course on classical electromagnetism is to study what happens to a spinning magnet as it passes through a magnetic field. If the field is the same everywhere, i.e. *homogeneous*, all the forces cancel out, and the spinning magnet flies straight through. However, if it is stronger in some places than others, i.e. *inhomogeneous*, then it will get deflected according to the direction it is spinning. Classical physics predicts that it would be deflected in one direction (say up) if it is spinning clockwise (relative to the magnetic field), and in the other direction (say down) if it is spinning anticlockwise. The details of this calculation are not important. The important fact is that the state of our tiny magnet can encode (at least) one bit of information: say 0 for clockwise and 1 for anticlockwise. The way we can measure that bit is to send it through an inhomogeneous magnetic field and see which way it deflects.

We can now gradually tilt the rotation axis of the little spinning magnet, until the initial clockwise rotation has become an anticlockwise rotation:



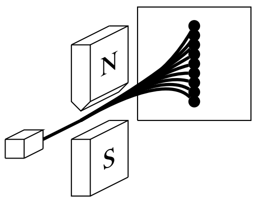
In this case, the deflection in an inhomogeneous magnetic field will also gradually change from up-deflection to down-deflection, and in particular, at some point passing through a state of no-deflection.

In the early twentieth century, it was discovered that electrons behave the same way as spinning magnets when they travel through a magnetic field. We can exploit this fact to design a device for measuring this ‘deflect-up’ versus ‘deflect-down’ property, called the Stern–Gerlach apparatus:

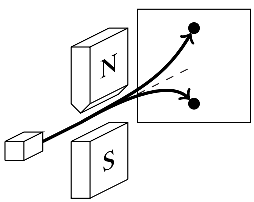


This device sends atoms through an inhomogeneous magnetic field (which is indicated by the fact that one of the magnets, here the N magnet, is pointed), then measures the direction they are deflected on a screen. Suppose, for simplicity, we choose hydrogen atoms, which have just one electron. If we send an atom whose electron is in the ‘deflect-up’ state, the atom is deflected upwards, and if it is ‘deflect-down’, the atom is deflected downwards. By analogy to the classical case, this property of electrons is called *spin*, though it doesn’t have anything to do with something actually spinning.

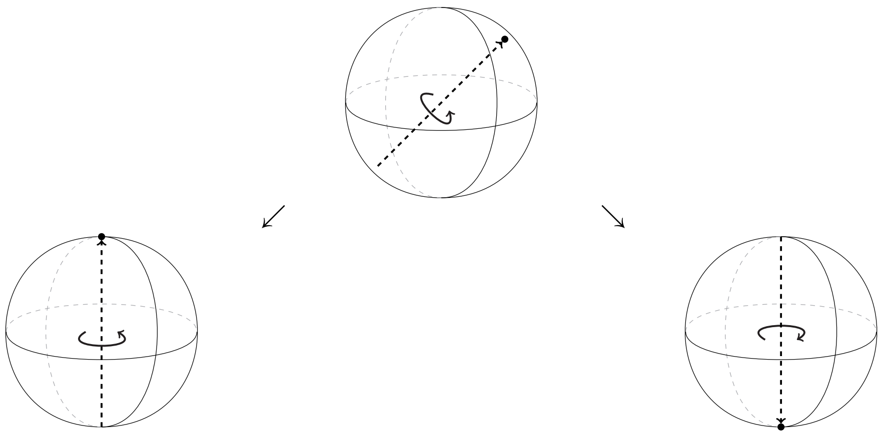
By analogy to the classical case, we expect that, as we gradually tilt the spin-axis, we see a gradual change from up-deflection to down-deflection. So if we send a stream of atoms whose electrons have random spin-axes through such a device, we expect to get a continuous line of different possible deflections:



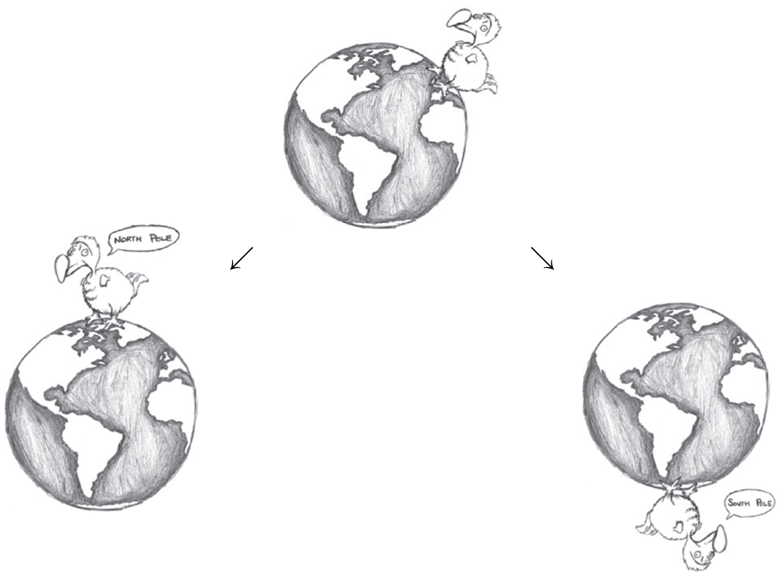
But the truly shocking thing that Stern and Gerlach discovered is that rather than getting a continuous line of outcomes, we always get one of precisely two outcomes, spin-up or spin-down:



Thus, the spin-axis of an electron, which we can plot as an arbitrary point on a sphere, gets projected to one of just two points:



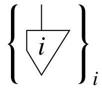
Does this look familiar? That's right, this is just like the dramatic travels of Dave the Dodo:



The fact that there are only two possible outcomes when measuring the spin of an electron with a Stern–Gerlach device is analogous to what Max Planck discovered (cf. Section 6.7) for energy, namely that it comes in certain packages called quanta. Here, there are only two such packages, namely *spin-up* and *spin-down*. This example is a member of the the simplest family of quantum processes referred to as measurements, which are called demolition ONB measurements.

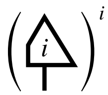
7.1.2 Demolition ONB Measurements

Proposition 6.92 from the previous chapter guarantees that for any ONB:



(7.1)

the corresponding doubled effects:



(7.2)

constitute a quantum process. We give such processes a special name.

Definition 7.1 A *demolition ONB measurement* is a quantum process of the form (7.2). The indices i are called the *measurement outcomes*.

This is called a demolition measurement because after the process happens, the system no longer exists. A concrete physical example of a such a process is the ‘observation’ of a photon, by means of an old-fashioned photographic plate; an active-pixel sensor (APS), which can be found in most digital cameras; or a photomultiplier, which is often used in laboratories to detect single photons (see Fig. 7.1). Each of these absorbs the photon in order to produce an easily detectable witness for the photon: a spot on the photo plate, an activated pixel in the APS, or a click of the detector.

The measurement outcome i corresponds to what we observe using our measurement device. For example, in the case of the APS, outcomes correspond to pixels that could be activated when they are struck by a photon. The number of pixels determines the number of

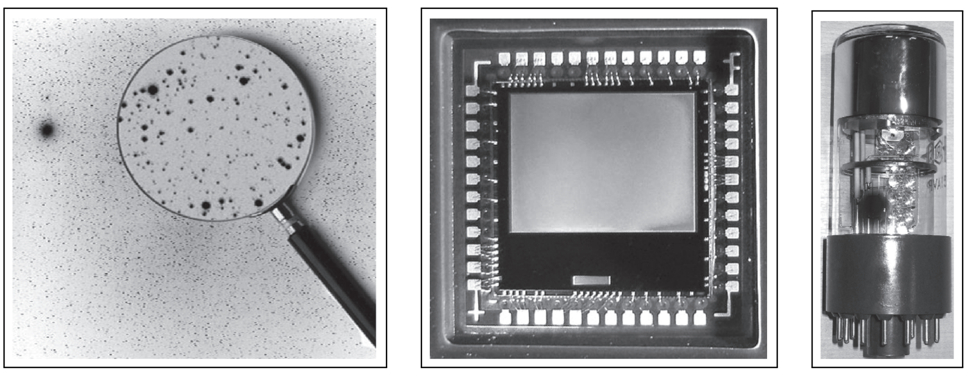
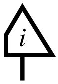


Figure 7.1 Three devices to measure photons (left to right): a photoplate; an active-pixel sensor (APS), as found in most digital cameras; and a photomultiplier.

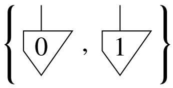
basis states, and hence the dimension of the quantum system $\widehat{A}$. The process of the photon being detected at the i-th pixel corresponds to the effect:



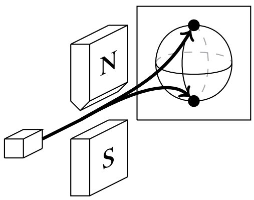
As we will see later, all other measurements can be derived from ONB measurements by coarse-graining them (Section 7.3.1) or by applying them to part of a larger system (Section 7.3.4).

In Proposition 5.38, we saw that unitary processes are precisely those that send ONBs to ONBs. It furthermore implied that any ONB can be obtained by means of a unitary applied to some fixed ONB. This suggests that the set of ONB measurements on a system is tightly intertwined with the unitaries that one can apply to that system.

Consider a measurement performed by a Stern–Gerlach apparatus from the previous section. We can model this as measuring the ONB:

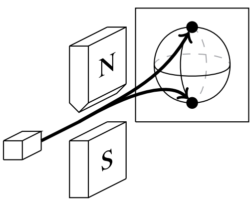


The outcomes can then be pictured as one of two antipodal points on the Bloch sphere, just like the corresponding ONB states (cf. Section 6.1.2):

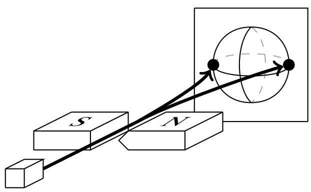


In this particular measurement, 0 means ‘deflects up’ and 1 means ‘deflects down’ along the Z-axis. We can now vary the ONB measurement simply by rotating the Stern–Gerlach apparatus relative to the particle source:

Z-measurement

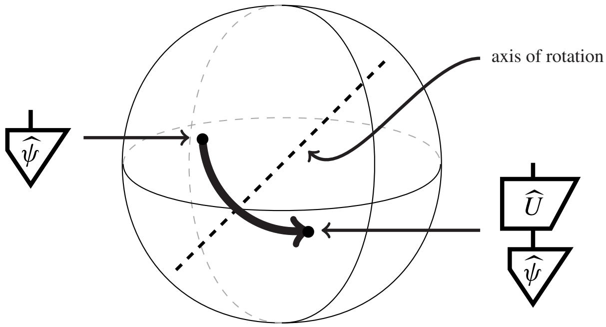


X-measurement

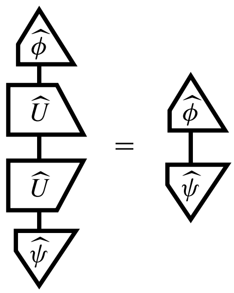


On the Bloch sphere this corresponds to a rotation, and in our mathematical model, to a unitary by the following fact.

Proposition 7.2 Unitaries $\widehat{U}$ from $\widehat{C^2}$ to $\widehat{C^2}$ correspond exactly to rotations of the Bloch sphere:

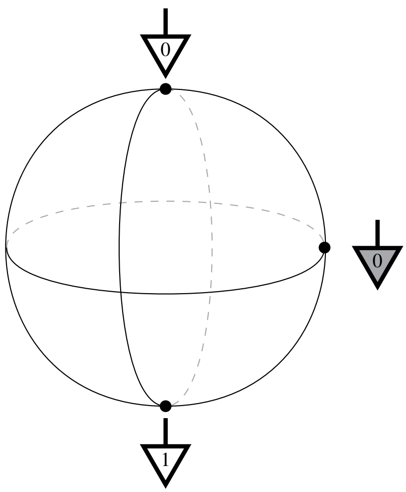


Proof (sketch) One can directly prove this fact by doing some horrible trigonometric mumbo-jumbo extending what we did in Section 6.1.2 or by some sophisticated representation theory for groups. However, intuitively this follows from two facts. The first is that unitaries send pure states to pure states. The second is that unitaries preserve the Born rule:

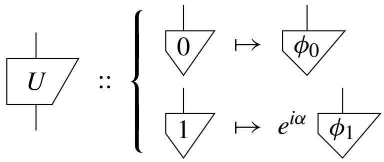


The Born rule measures the distances between points on the surface of the sphere, so unitaries must preserve those distances. The only things that preserve distances on a sphere are rotations and reflections, and of these two, the only things coming from linear maps are rotations.

Conversely, any rotation on a sphere is uniquely fixed by where any three distinct points on the surface go. We can choose for example:

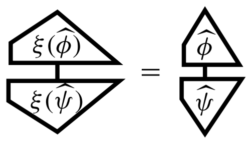


A rotation sends antipodes to antipodes, so the doubled Z-basis states will get sent to another doubled ONB $\{\widehat{\phi_j}\}_j$. The X-basis state will then get sent to an arbitrary location halfway between these two. Obviously we can build a unitary U such that $\widehat{U}$ does this (see Corollary 5.39). Less obviously, by choosing the phase $e^{i\alpha}$ in:



we can send the doubled X-basis state to any location halfway between the two antipodes without affecting the doubled Z-basis states. □

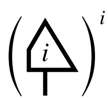
Remark* 7.3 A much more general statement of Proposition 7.2 is called *Wigner's theorem*. It states that any function ξ on pure quantum states (with no a priori requirement of being a linear map) that preserves the Born rule, that is:



corresponds to either a unitary or a so-called *anti-unitary*, which is the composite of a unitary and conjugation. Now, on the Bloch sphere, the Born rule translates into the distance between points, and the maps on points that preserve this distance exactly correspond to either a rotation or a rotation composed with a reflection. It is then easy to see that the first corresponds to unitaries, while the latter corresponds to anti-unitaries.

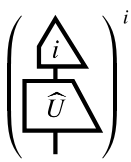
From this intuitive geometric representation on the Bloch sphere it is clear that all ONB measurements on qubits can be obtained via unitaries by rotating two antipodal points to any other two antipodal points. In fact, this holds in general.

Proposition 7.4 Given an ONB measurement on a system:



(7.3)

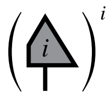
all other ONB measurements on that system are obtained as follows:



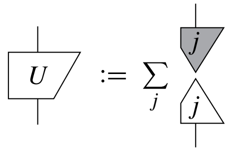
(7.4)

where $\widehat{U}$ is a unitary quantum map.

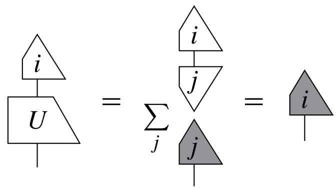
Proof Given any such ONB measurement:



the linear map:



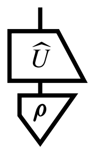
is unitary by Corollary 5.39. We moreover have:



so (7.4) follows by doubling. Conversely, by Proposition 5.38, (1) $\Leftrightarrow$ (2), any quantum process of the form (7.4) is an ONB measurement. □

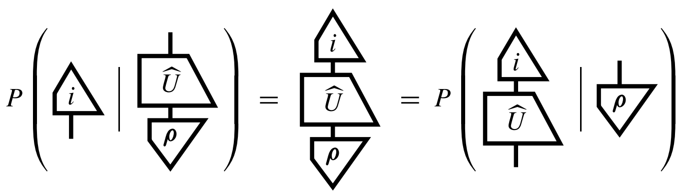
There is a clear relationship between rotations we could apply to the Stern–Gerlach device and rotations we could apply to the state of the incoming particles. Namely, we will observe the same probabilities regardless of whether we rotate our measurement by $\widehat{U}$ or we rotate our state by the adjoint of $\widehat{U}$.

Proposition 7.5 The probabilities for measurement (7.3) on the state:



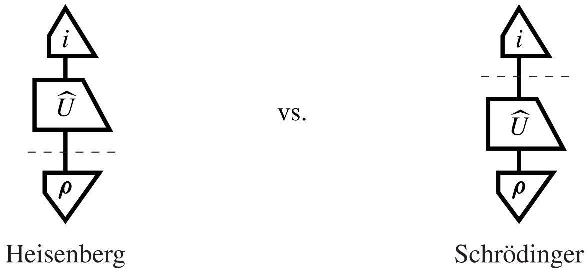
are the same as those for measurement (7.4) on the state:

Proof We have:



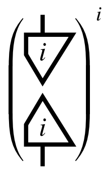
so the probabilities indeed coincide. □

In fact, these are two equivalent ways to think about how a quantum system changes over time. Thinking of this change as something happening to a measurement device is called the *Heisenberg picture*, whereas considering it as something happening to the quantum state is called the *Schrödinger picture*. Of course, the only difference is where you ‘draw the line’:



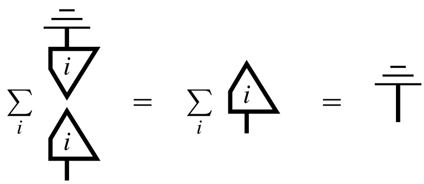
7.1.3 Non-demolition ONB Measurements

Rather than just considering doubled ONB-effects, we can instead consider effect–state pairs:



(7.5)

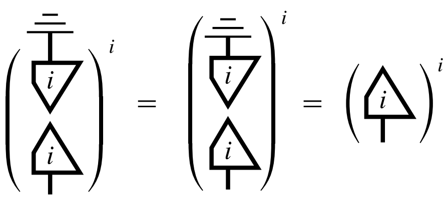
Again we obtain a quantum process:



After such a process is performed, we still have a quantum system left behind. Thus, this is an example of a *non-demolition process*.

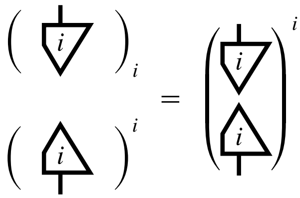
Definition 7.6 A *non-demolition ONB measurement* is a quantum process of the form (7.5). The resulting state is called the *outcome state*.

Given a non-demolition ONB measurement, we can recover its related demolition measurement by simply discarding the resulting system:

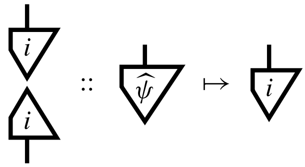


(7.6)

Conversely, any non-demolition ONB measurement can be regarded as a demolition ONB measurement followed by a controlled preparation:



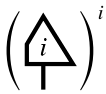
Consequently, each branch of the non-demolition ONB measurement causes the state of the system to change into the corresponding outcome state of the measurement (ignoring numbers):



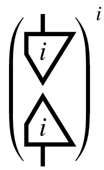
(7.7)

So while non-demolition measurements do not destroy the system itself, they do irreversibly destroy almost every state of the system. The only exceptions are the *eigenstates* of that measurement.

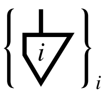
Definition 7.7 Given an ONB measurement:



or



the states in the set:



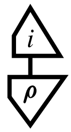
are called the *eigenstates* for this ONB measurement. A state that is not an eigenstate for a given ONB measurement is called a *superposition state*.

We defined eigenstates of generic processes back in Definition 5.42 as those states that were left unchanged (up to a number) by the process. The spirit of Definition 7.7 is much the same, in that eigenstates are precisely those states that get sent to themselves with certainty by (the non-demolition form of) an ONB measurement.

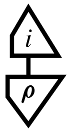
7.1.4 Superposition and Interference

Let's first have a look at which outcomes one can obtain for an ONB measurement when the system is in a certain state. The probability of obtaining an outcome i is, as always, given by the Born rule:

$$P(i \mid \rho) \;:=$$



This probability is 0 if and only if $\boldsymbol{\rho}$ and the i-th ONB state are orthogonal:



$$= 0$$

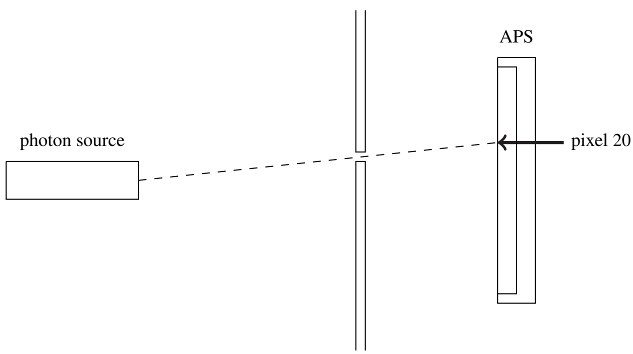
In the case of qubits, there is only one such state, depicted on the Bloch sphere as the antipodal point. So for the vast majority of states, all of the outcomes i are possible! Hence, each state may lead to most of the outcomes, and each outcome may occur for most of the states. So, measurement seems to look more like playing the lottery than actual 'observation'.

In the case of an APS (Fig. 7.1) this means that for most states a photon may actually be detected by each of the pixels. But then, before it is destroyed, one may ask:

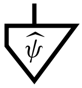
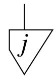
Where was that photon?

According to the probabilities that the Born rule gives, it's typically a bit everywhere at the same time, and then, when the measurement process is launched, a lottery decides where it ends up being detected. But hold on a second here, how do digital cameras (or our eyes, for that matter!) produce pictures that are not just pure randomness?

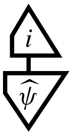
Thankfully, even though the photon is a bit everywhere, it tends to be 'mostly' where we expect it to be. For example, suppose we take this setup:



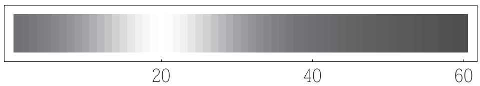
The state of the photon by the time it hits the APS can be written in this form:

 $= \text{double}\left(\sum_j r_j e^{i\alpha_j}$  $\right)$

Here, the basis state j corresponds to the j-th pixel on the APS, and we've written the complex numbers in polar form (see Section 5.3.1). For $\widehat{\psi}$, the real number r_j is large when $j = 20$, but falls off quickly as j moves away from 20. Of course, the phase $e^{i\alpha_j}$ also depends on j, but when we compute the probabilities for each pixel of the APS firing the phase disappears:

$P\left(j \,\middle|\, \widehat{\psi}\right) :=$  $= \overline{(r_j e^{i\alpha_j})}(r_j e^{i\alpha_j}) = (r_j)^2$

If we plot these probabilities, we will therefore see something like this:

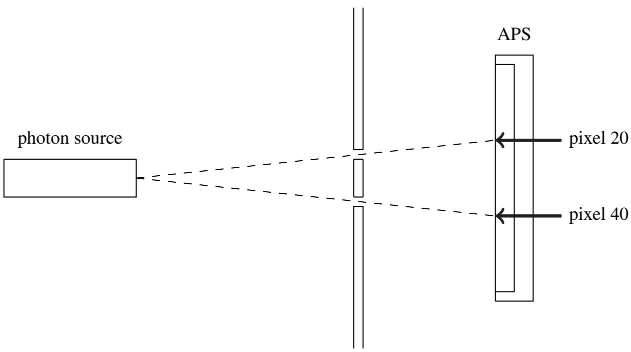


(7.8)

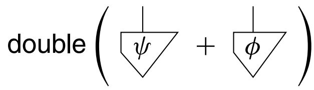
where 'white' means probability 1 at pixel j and 'black' means 0. Similarly, if we open a slit in front of pixel 40, we produce another state $\widehat{\phi}$. The probabilities for $\widehat{\psi}$ will be concentrated around 40:

(7.9)

So, why are (7.8) and (7.9) really bad 'observations' of the states $\widehat{\psi}$ and $\widehat{\phi}$? We saw above that $P(j \mid \widehat{\psi})$ only depends on r_j and similarly for $P(j \mid \widehat{\phi})$. Notably, there is <u>no trace</u> of the phases α_j. But so what? Maybe these phases, like the global phases we killed back in Section 6.1.2, don't effect anything anyway. We might think this is so, until we open both slits:

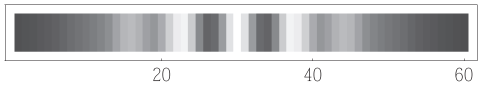


effectively forming the state:



(7.10)

When we look at the probabilities for the APS, we get a bit of a surprise:



(7.11)

Rather than two nice, even bumps, we get a bunch of alternating bands of light and dark. This is because the phases from $\widehat{\psi}$ and $\widehat{\phi}$, which were invisible in the first two measurements, start to *interfere* with each other in the third measurement. This produces a result that we would never have expected from (7.8) and (7.9) alone.

Remark 7.8 This kind of setup is called the *double slit* experiment, which is used to demonstrate how light behaves both like a particle and like a wave. Even though we get interference fringes like one would expect when studying waves, in each run of the experiment, the photon is only detected at a single pixel, like a particle (see Fig. 7.2).

Now, suppose instead we are able to detect which slit the photon goes through each time. In this case (by ignoring the outcome of the 'which slit' measurement), we get a mixture instead of a superposition:

(7.12)

Of course, we have known since Section 6.1.5.2 that doubling does not preserve sums, and indeed this extra measurement has killed all of the interesting interference from (7.11), yielding probabilities:

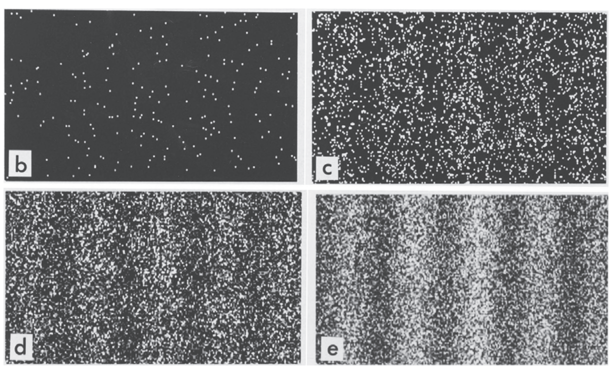


Figure 7.2 A more traditional double-slit experiment, involving a photographic plate. Note how interference fringes appear even when photons are sent one by one through a double slit. This exposes wave-like behaviour even for single photons.

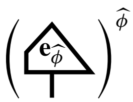
So, ONB measurements will typically throw away lots of data about the quantum state and can even kill interesting quantum behaviours. As such, ONB measurements are really bad at 'observing' quantum states. In spite of this they are the next best thing to observation.

7.1.5 The Next Best Thing to Observation

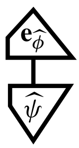
The bottom line concerning quantum measurement is that it is impossible, even in principle, to 'observe' a quantum system in the classical sense:

Theorem 7.9 'Observing' is not a quantum process.

Let's provide Theorem 7.9 with some formal content. Ideally, an 'observation' would be a quantum process that tells us the exact state of a system. More explicitly, an observation:



should be a process with the property that, for every pure state $\widehat{\phi}$, there is a corresponding outcome obtained with certainty if and only if the system is in the state $\widehat{\phi}$:

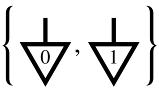


$$= \begin{cases} 1 & \text{if } \widehat{\psi} = \widehat{\phi} \\ 0 & \text{otherwise} \end{cases} \tag{7.13}$$

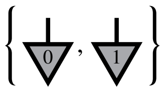
Note that we restrict to pure states. We wouldn't expect this to hold for impure states because some (possibly important) part of the state has already been discarded. However, even after making this restriction, we can still prove the following.

Lemma 7.10 No quantum process satisfies (7.13).

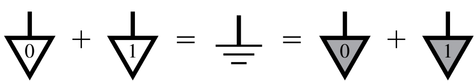
Proof Let:



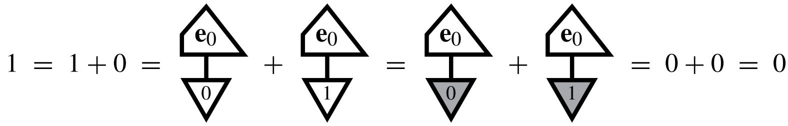
and



be pure states formed from two different (and disjoint) ONBs, e.g. the Z- and X-bases from Section 5.3.4. By reflecting (6.37) we have:



Now, suppose we have an effect $\mathbf{e}_0$ that identifies the white 0-state. Applying $\mathbf{e}_0$ to each of the states in the LHS and the RHS we obtain:

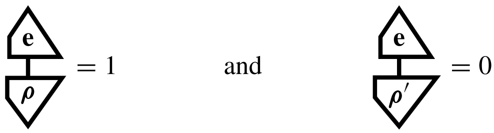


which is a contradiction. Thus no such effect exists. □

We might also wish to define the process of ‘observing’ as something that leaves the system unaffected after identifying its state. If such a process existed, then we could obtain (7.13) just by discarding its output. Thus, proving the non-existence of this ‘demolition’ observation process proves the non-existence of any observation process.

If there is no such thing as observing in quantum theory, what then is the the next best thing to observing? There is actually more than one possible answer to this question, depending on what we hope to accomplish by means of a quantum measurement (see e.g. Section 7.4.2 for an alternative point of view to the one outlined in this section). However, one choice for the next best thing to an ‘observation’ is a process that behaves like (7.13) for as many pairs of states as possible. This turns out to put a pretty steep restriction on a pair of states.

Exercise* 7.11 Show that two states are perfectly distinguishable by a quantum process if and only if they are orthogonal. That is, suppose there exists a quantum process containing an effect $\mathbf{e}$ as one of its branches, where:

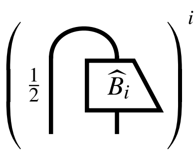


for causal states $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$. Show then that $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ must be orthogonal.

As a consequence, ONB measurements are genuinely the next best thing to observing in the sense that they allow for the maximum number of states to be perfectly distinguished, i.e. ‘observed’, simultaneously.

7.2 Measurement Dynamics and Quantum Protocols

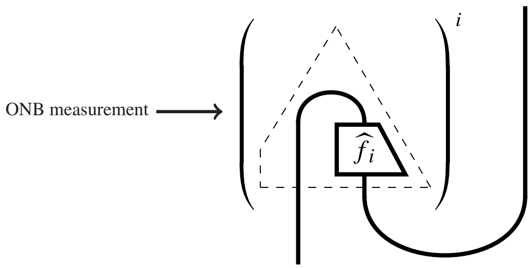
While ONB measurements may not be very good at observing the state of a system, which is what they were intended for, they are extremely good at something they were not intended for! We saw in Section 6.4.6 that, in the teleportation protocol, Aleks’ non-deterministic effect does (almost) all of the work:



But what does ‘work’ precisely mean here? Work here means effectively acting on systems and thereby invoking radical, often non-local, changes. The use of ‘radical’ is by no means an overstatement, given that for a physical system to dynamically evolve from one state to another takes time, while these measurement-induced changes happen instantaneously. We’ll see that small bits of entanglement can instantaneously become long bits of entanglement and that arbitrary linear maps can instantaneously be applied, all by means of quantum measurements.

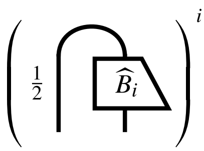
7.2.1 Measurement-Induced Dynamics I: Backaction

We already saw an example in Section 6.4.2 of ONB measurements changing the state of a system in useful ways, where we used (what we now know is) an ONB measurement to realise any quantum map non-deterministically:



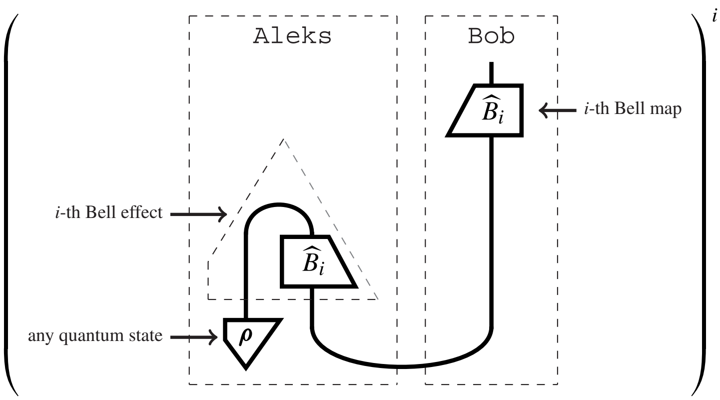
(7.14)

Similarly, in the case of qubits, we can realise quantum teleportation by means of a *demolition Bell measurement*:

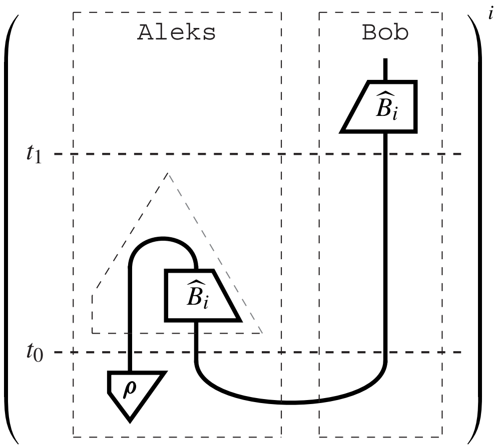


(7.15)

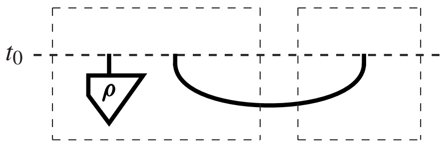
Since each of the B_i are unitary, we can appropriately correct the error:



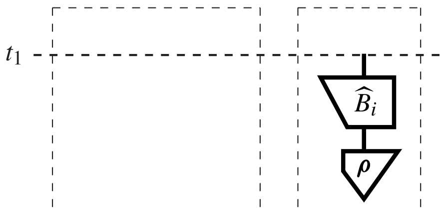
Let's break this into smaller steps and see how the state changes over time. When performing the measurement, the overall state at times t_0 and t_1 in the scenario:



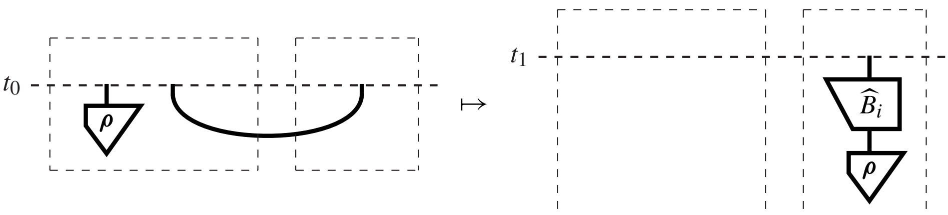
is altered radically. At time t_0 we have:



while at time t_1 we have:



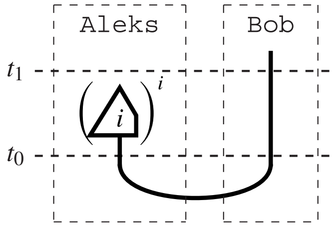
That is, we have an <u>instant transition</u>:



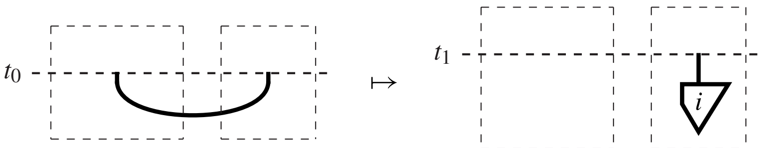
where the state ρ pops over from Aleks to Bob, and an error is created.

However, we already learned in Section 6.3.2 that due to some subtle balancing act between quantum theory and the theory of relativity, instant transitions like this cannot be exploited to achieve faster-than-light signalling. Nonetheless, they enable one to teleport and are very useful for other things too, as we will see shortly.

The above scenario is not the simplest one that exhibits the instant transition phenomenon. A simplification that only involves two systems consists of an ONB measurement performed on one of a pair of systems in a Bell state:



in which case we have the following instant transition:



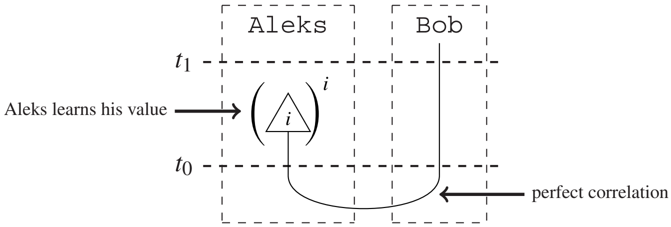
Here, Aleks' measurement outcome determines Bob's resulting state. So the simple performance of a measurement changes the state of the system so drastically that whatever effect Aleks observes ends up as a state at Bob's end.

Definition 7.12 The instant transition induced by a demolition measurement on part of a composite system is called its *backaction*.

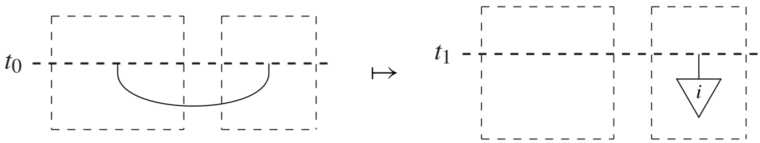
We call this 'backaction' because the measurement's influence seems to travel backwards in time, according to the 'logical reading' of diagrams that we discussed back in Section 4.4.3. Importantly, this measurement-induced backaction is not just 'a bit' of dynamics, but some really mind-blowing whole lot of dynamics, which has no counterpart anywhere else in physics!

In spite of this, we can compare this situation with something much less surprising that occurs in classical probability theory. Probability theory can be seen as a way of modelling our state of knowledge about a system. As soon as some part of the system (e.g. a random variable) becomes known, this could induce a global change of our knowledge about the system (e.g. when that variable is correlated with others). This change to our state of knowledge is referred to as *Bayesian updating*.

Consider a situation similar to one we first encountered in Example 4.91 where Aleks and Bob each have a sealed envelope containing the same message, which is taken randomly from a set of possible messages $\{1, \ldots, n\}$. Then, we can treat Aleks opening his envelope as a non-deterministic process:



Now, since the content of both envelopes is the same, the content of Bob's envelope is known instantaneously once Aleks performs his 'measurement' process:

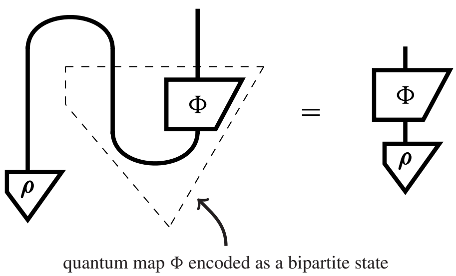


However, the crucial difference between quantum theory and classical probability theory is that in the case of the latter what is being altered is nothing more than our knowledge about the content of the envelope, not the content itself. In contrast, according to the standard conception, in quantum theory it is the (pure) state of the system itself that changes, not just our knowledge about it. Many have attempted to provide a similar explanation for the quantum case in terms of knowledge. However, quantum non-locality forbids any such model to be *local*, as we shall explain in Section 11.1. In other words, something must be happening instantaneously between Aleks and Bob, even though they are far apart.

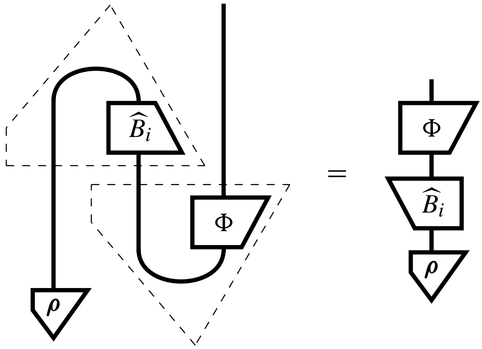
7.2.2 Example: Gate Teleportation

Teleportation exploits the backaction of an ONB measurement to pass a state from one place to another, leaving it unchanged, i.e. applying the identity process to it. Would it also be possible to apply some other quantum map Φ to that state? In other words, can we encode an entire 'computation' in a quantum state, then 'perform' the computation just by measuring? Astoundingly, the answer turns out to be yes, although it does require some cleverness.

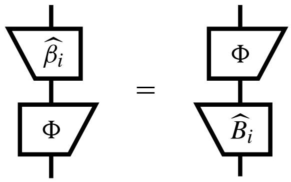
The key to solving this problem is process–state duality. The quantum map Φ can be encoded as a bipartite state. We can then modify the quantum teleportation protocol by using this bipartite state instead of the Bell state:



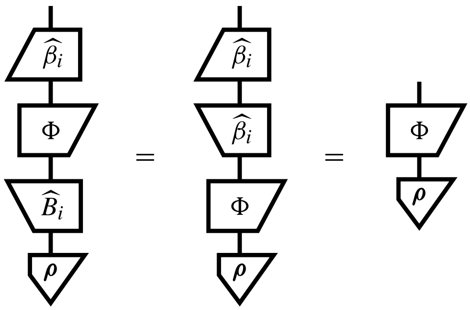
Accounting for the error, this becomes:



So, we have nearly applied the quantum map Φ that was encoded in the quantum state. But there's a bit of a problem: the error $\widehat{B}_i$ is stuck behind Φ, so it cannot be corrected directly. However, if there exists another unitary $\widehat{\beta}_i$ such that:



then we are back in business:

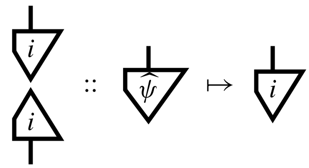


This might seem like quite a restriction on the kinds of maps we can realise this way, but it turns out that this trick suffices to produce a universal quantum computational model, called *measurement-based quantum computation*, where the 'dynamics' is entirely measurement induced. In Section 12.3 we will describe precisely how this model of computation works.

7.2.3 Measurement-Induced Dynamics II: Collapse

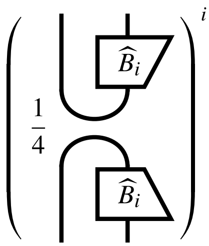
As compared with their demolition counterparts, non-demolition ONB measurements exhibit two kinds of measurement-induced dynamics:

- the backaction discussed in Section 7.2.1, and
- the *collapse* we saw in (7.7):

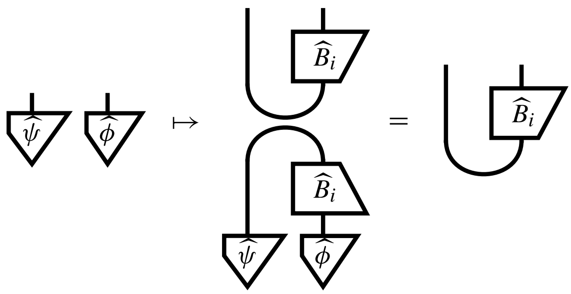


which could also be called ‘forward action’.

For example, the *non-demolition Bell measurement*:

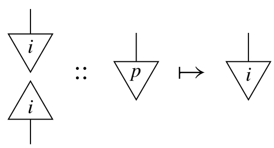


which has the same backaction as the corresponding demolition version, causes two unentangled systems to become entangled (where again we ignore numbers):



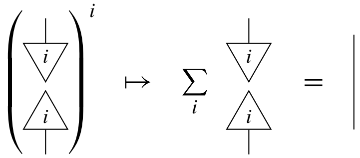
(7.16)

Just as there exists a counterpart to backaction in classical probability theory, the same is true for collapse. In classical probability theory, collapse:



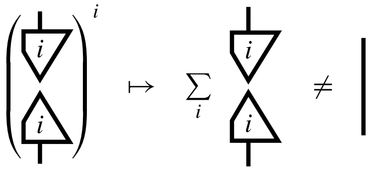
simply means that the content of the envelope goes from being unknown (i.e. in some probability distribution p) to known (i.e. in a point distribution at i). As with backaction, nothing changes except our knowledge when a collapse occurs, whereas collapse is much more destructive in the quantum case. For example, consider what happens if after opening a sealed envelope, we decide to seal the envelope up again and forget the value we saw.

Forgetting the outcome of a non-deterministic process just corresponds to mixing the branches together:



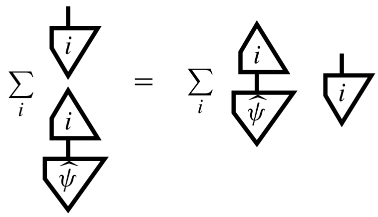
(7.17)

Since our knowledge hasn't changed, nothing changes. However, if we do the same for a quantum measurement, we get something quite different:

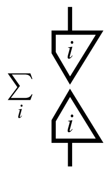


(7.18)

The resulting quantum process will send any quantum state to a mixture of basis states. In particular, a pure state $\widehat{\psi}$ indeed becomes a mixture of the basis states:



weighted by the corresponding probabilities. The process:



is called *decoherence*, and we will study it extensively in Section 8.3.2, once we give it a simple (sum-free) diagrammatic presentation.

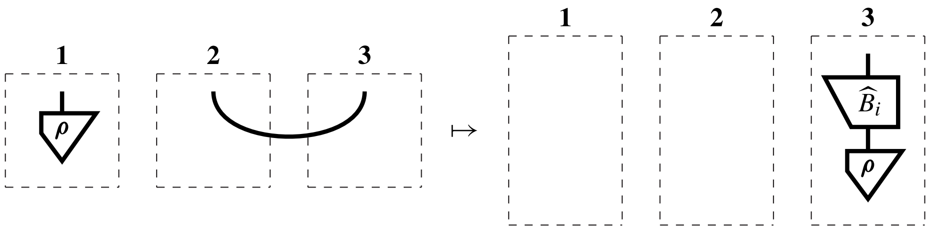
Exercise 7.13 Show that decoherence is indeed not the identity by writing both as sums of ONB states.

Despite its destructive nature for quantum states, collapse can also be very useful. One use is *state preparation*, where one obtains a state $\widehat{\psi}$ simply by performing an ONB measurement that has state $\widehat{\psi}$ as a possible outcome state. Performing the measurement a sufficient number of times on a state for which the outcome state $\widehat{\psi}$ is possible will yield $\widehat{\psi}$ at some point. This is exactly how a polarising filter produces polarised light, for example.

We will now see a protocol that exploits both the backaction and the collapse of a non-demolition Bell-basis measurement.

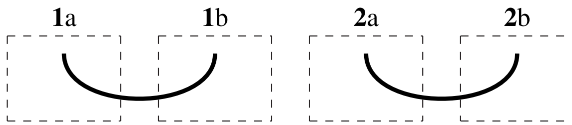
7.2.4 Example: Entanglement Swapping

In Section 7.2.1, we saw how teleportation depends on the backaction of a (demolition) Bell-basis measurement. That is, when the measurement is applied to systems **1** and **2**,

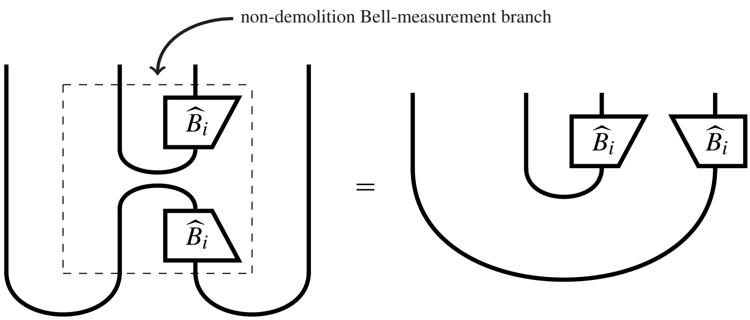


ρ shifts from system **1** to system **3**, despite the fact that **3** is never directly acted on (except later to make a correction). We also saw in (7.16) how the forward action of a non-demolition Bell measurement allows us to entangle two previously unentangled systems.

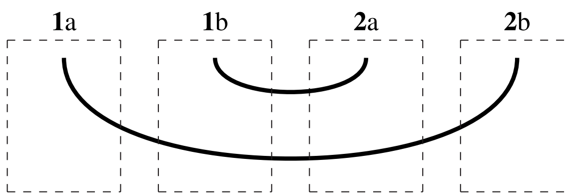
Let us now try something that combines these two ideas. Suppose we start with a pair of Bell states:



and we apply a non-demolition Bell-basis measurement to systems **1b** and **2a**. Then we obtain the following state:



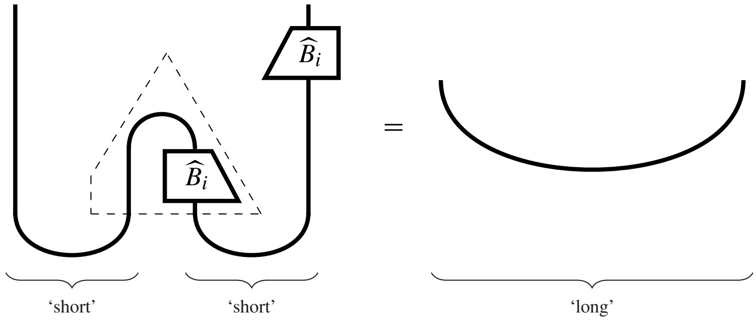
After applying the necessary corrections, we end up with:



While originally system **1a** was entangled with **1b** and **2a** with **2b**, the final state has **1a** entangled with **2b** and **1b** with **2a**. In other words, the entanglements have been 'swapped'. We call this procedure *entanglement swapping*. The amazing bit about this is that **1a** becomes entangled with **2b**, although these systems were never acted on together, or in other words, quantum theory allows for:

$\boxed{\textit{entangling without touching}}$

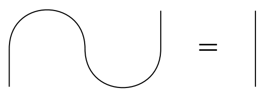
A practical use of this procedure for quantum technologies is to generate entanglement over a large distance, given that one possesses some entangled states over shorter distances:



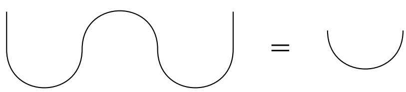
A device that performs entanglement swapping for this particular purpose is sometimes called a *quantum repeater* and is a crucial component to the feasibility of producing high-quality entangled states over long distances. It is called a repeater by analogy to classical signal repeaters, which make it possible to send messages long distances by occasionally capturing the signal and 'repeating' an amplified version of the signal down the wire.

Exercise 7.14 Given n short pieces of entanglement, design a protocol that produces a 'long' piece of entanglement, while minimising the amount of needed corrections.

Remark 7.15 While the teleportation protocol captures in essence the defining equation of string diagrams:



which involves two caps/cups in the LHS, entanglement swapping is the next one in line. It captures an equation involving three caps/cups in the LHS:

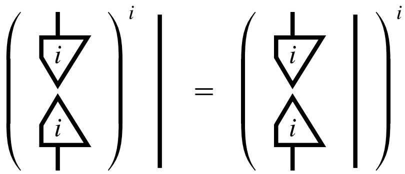


7.3 More General Species of Measurement

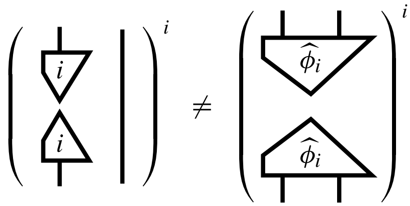
In this section, we will look at more general kinds of measurements that arise once one considers ONB measurements within the broader context of a process theory. For example, what happens if we measure just one sub-system of a quantum state or compose an ONB measurement with other processes?

7.3.1 Von Neumann Measurements

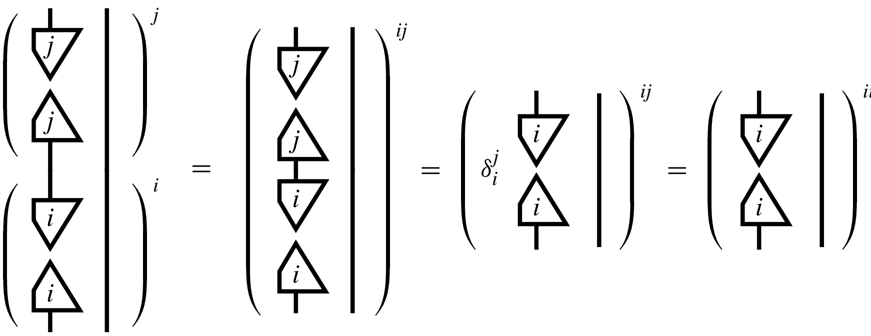
Consider the quantum process obtained by doing a non-demolition ONB measurement on one system, while doing nothing on another system:



This process cannot itself be an ONB measurement, simply because its branches are not $\circ$-separable:

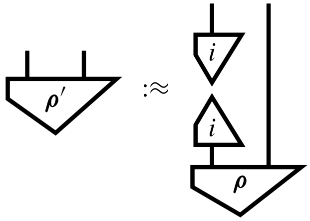


However, it does share an important characteristic with an ONB measurement, which we can see if we perform the process twice:



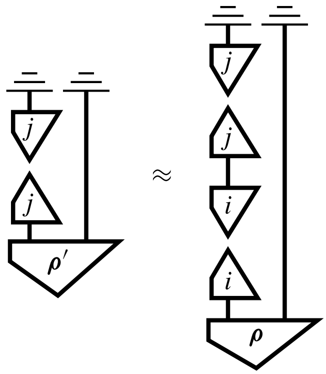
The reason why we could take $i = j$ is that for $i \neq j$ we get the impossible process, which, of course, will never happen. Hence, in the two consecutive processes we get the same outcome, and the overall process is the same as as what we would have gotten if we did the process only once.

It may be helpful to see what happens if these processes are applied to a state. Suppose we perform this process on a state ρ and get outcome i. Then the new state will be:



Then, if we immediately perform the process again, all of the branches $j \neq i$ occur with probability 0:

$$P\left(j \neq i \mid \rho'\right) \overset{(6.65)}{=}$$



$$= 0$$

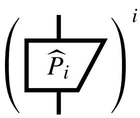
so we are guaranteed to get outcome i again. Moreover, it doesn't matter if we perform this process 1 time, 2 times, or 100 times. After one measurement, the damage is done, so the state no longer changes after successive measurements.

This idea that a process can cause a quantum state to 'collapse' in such a way that repeating the process will always yield the same thing was considered by von Neumann to be the characterising feature of quantum measurements, yielding the following.

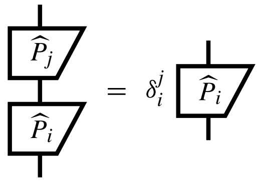
Von Neumann's collapse postulate: After a non-demolition measurement is performed and an outcome i is obtained, performing the same measurement again yields the same outcome with certainty, and does not further affect the state of the system.

This can be put in the language of quantum processes as follows.

Definition 7.16 A (non-demolition) *von Neumann measurement* is a quantum process:



such that:

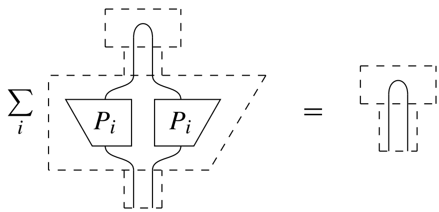


(7.19)

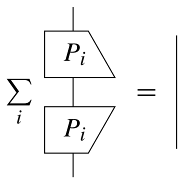
Von Neumann measurements are also sometimes called *projective measurements*, for the following reason.

Proposition 7.17 For any von Neumann measurement, the branches $\widehat{P}_i$ are projectors.

Proof We already know from (7.19) that the $\widehat{P}_i$ are idempotent, so it only remains to show they are self-adjoint. The collection of quantum maps $\widehat{P}_i$ together forms a quantum process, so it is causal. Unfolding the causality condition yields:

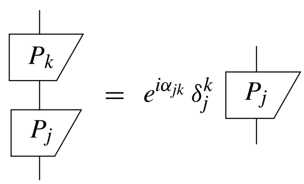


so we obtain:



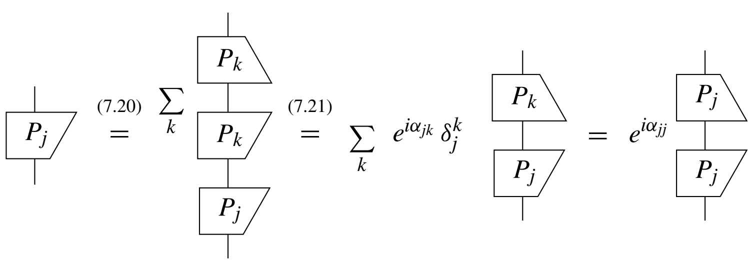
(7.20)

By Theorem 6.17 we can replace the doubled equations (7.19) by non-doubled equations, up to a global phase $e^{i\alpha_{jk}}$ for all values of j and k:

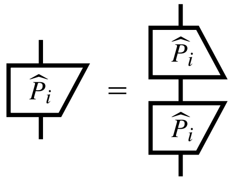


(7.21)

Hence:

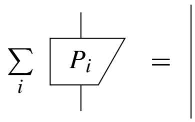


Doubling then eliminates the global phase:

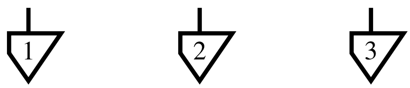


So, by Proposition 4.70 each $\widehat{P}_i$ is a projector. □

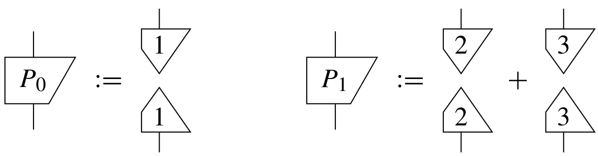
Projectors are called *orthogonal* precisely when they satisfy (7.19), so we could have just as well defined a von Neumann measurement as a quantum process consisting of mutually orthogonal projectors. Moreover, since we are dealing with projectors, causality as in (7.20) simplifies to the underlying projectors (cf. Proposition 6.21) forming a resolution of the identity:



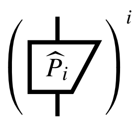
Alternatively, one can obtain a von Neumann measurement from an ONB measurement, by combining multiple measurement outcomes into one, or *coarse-graining*. For example, suppose we have a three-dimensional quantum system; then perhaps rather than performing a measurement to tell which of these states:



the system is in, we just devise a measurement to check whether the system is in state 1. We can do this by fixing projectors:



then measuring:



(7.22)

More generally, fix any *partition* of the set of outcomes:

$$I := \{1, \ldots, D\}$$

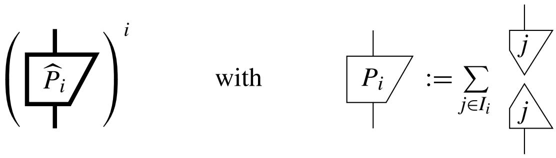
that is, a collection of subsets of I:

$$\{I_1, \ldots, I_n\}$$

which satisfy:

$$I_1 \cup \cdots \cup I_n = I \qquad \text{and} \qquad \forall i \neq j : I_i \cap I_j = \emptyset$$

Then we obtain a von Neumann measurement as follows:

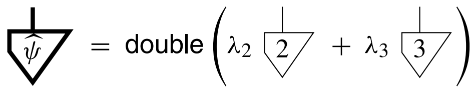


(7.23)

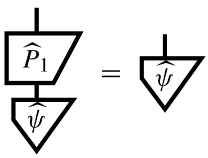
Clearly each of the $\widehat{P}_i$ are projectors, and mutual orthogonality is a consequence of the fact that each of the sets I_i is disjoint.

Exercise 7.18 Show that for any partition of an ONB, (7.23) is a quantum process and hence a von Neumann measurement. Conversely, use the spectral theorem from Section 5.3.3.1 to show that any von Neumann measurement can be expressed as (7.23) for some partition of an ONB.

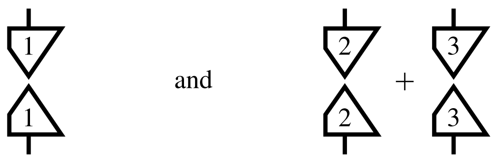
A coarse-grained measurement teaches us less than an ONB measurement, but it also does less damage. For example, any state of the form:



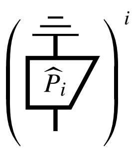
is kept intact by the measurement (7.22) because $\widehat{P}_0$ will never occur and:



This is because we were careful to coarse-grain at the level of undoubled processes P_i. If we instead performed a quantum process consisting of:



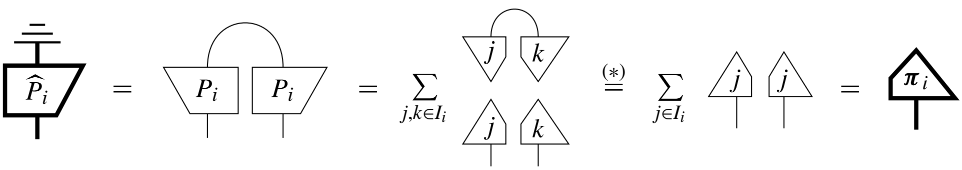
then $\widehat{\psi}$ would no longer even get sent to a pure state (unless λ_2 or λ_3 is 0), much less itself. On the other hand, if we consider *demolition von Neumann measurement*, that is quantum processes of the form:



such that (7.19) holds, the distinction disappears. Consequently, unlike demolition ONB-measurements, the branches of a demolition von Neumann measurement need not be pure.

Proposition 7.19 A quantum process is a demolition von Neumann measurement if and only if it is of the form:

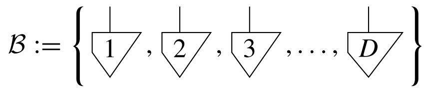
Proof Let the projections P_i be defined as in (7.23). Then we have:



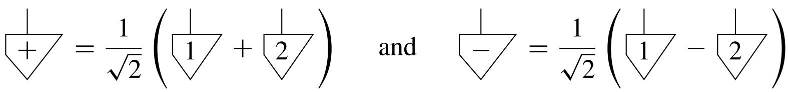
where in $(*)$ we rely on orthonormality of basis states. $\square$

The final thing to note is that for both demolition and non-demolition von Neumann measurements, this coarse-graining is not unique. That is, a single von Neumann measurement can arise from coarse-graining more than one ONB.

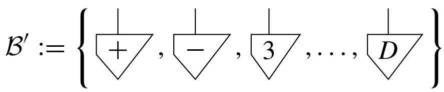
Exercise 7.20 For an ONB:



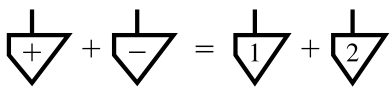
set:



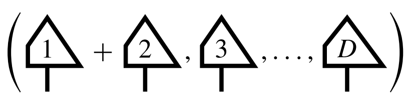
in order to obtain another ONB:



Now show that we have:



and hence that the measurement:



arises by coarse-graining ONB measurements for $\mathcal{B}$ or $\mathcal{B}'$. Can you characterise all of the ONBs that yield this measurement via coarse-graining?

7.3.2 Von Neumann's Quantum Formalism

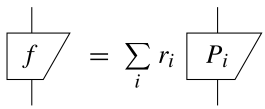
Von Neumann measurements make up the core of the quantum formalism as it is still found in most textbooks, which is quite different from the one that we have presented. For one thing, one typically distinguishes between *pure state quantum theory* and *mixed state quantum theory*. Pure states are taken to be primitive, while mixed states are considered to be an (optional) derived concept. Pure state quantum theory is given as three postulates.

Postulate 1: systems. A *quantum system* is represented by a Hilbert space. The *state* of a quantum system then corresponds to an equivalence class of normalised vectors that are equal up to a global phase (cf. Section 6.1.2). *Composite systems* are represented by the tensor product of the Hilbert spaces representing the sub-systems (cf. Section 5.4.2).

Postulate 2: evolution. *Deterministic, reversible quantum processes* are represented by unitaries acting on the Hilbert space (cf. Corollary 6.58).

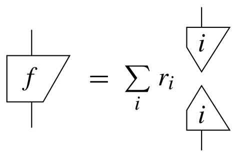
Postulate 3: measurements. Quantum *measurements* are represented by self-adjoint linear maps acting on the Hilbert space.

But wait a minute here. The third postulate looks totally different from anything we have called a measurement, much less a von Neumann measurement, which presumably lies at the heart of the von Neumann formalism. However, with a bit of help from the spectral theorem (Section 5.3.3.1), we can see that this isn't so different. Starting with a von Neumann measurement, we can wrap up a set of projectors into a single map:



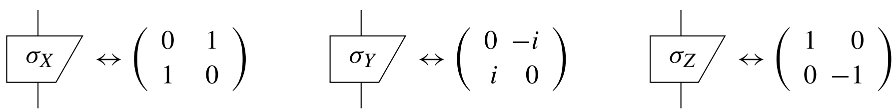
(7.24)

where all r_i are distinct real numbers. The resulting map is self-adjoint, and we can always recover the projectors P_i via the spectral theorem, which guarantees the existence of an ONB such that:

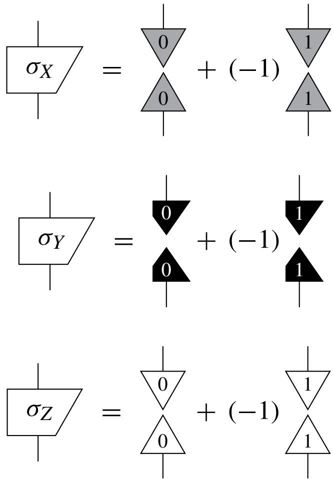


In general, some numbers r_i might be repeated. This induces a partition $\{I_1, \ldots, I_n\}$ where each distinct real number r_i corresponds to a particular set I_i. We can then use (7.23) to recover the decomposition (7.24).

Example 7.21 The Pauli maps:

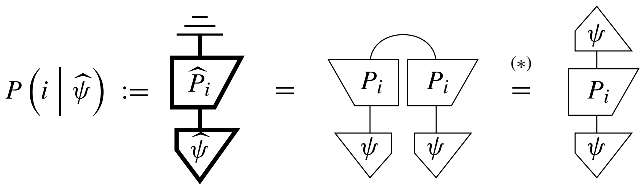


whose matrices we first encountered in Remark 5.99, are self-adjoint linear maps that represent measurements for the X-basis, Y-basis (see Exercise 6.7), and Z-basis, respectively:



So, the resulting self-adjoint linear map should be considered not as a process itself, but rather as one way to present the actual processes, i.e. the projectors.

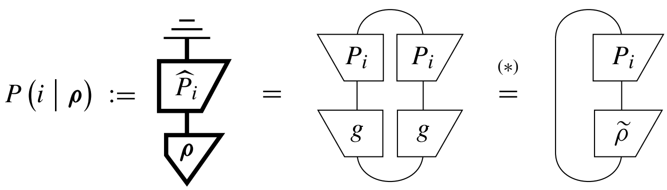
Postulate 3: measurements (continued). When a measurement takes place the state of the system changes (i.e. 'collapses') according to the action of one of the projectors, and the probability of each of the projectors for doing so is given by the Born rule:



(7.25)

where $(*)$ comes from the fact that P_i is a projector.

Remark 7.22 In the case of impure states, the Born rule becomes:

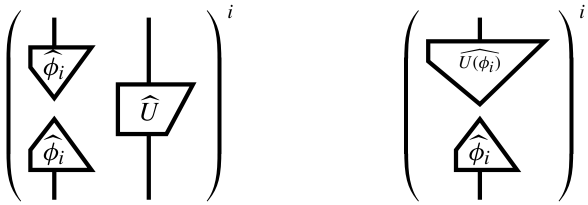


where $\widetilde{\rho} := g \circ g^{\dagger}$ is the density operator associated with ρ (see Remark 6.43). So in the more traditional notation, the Born rule probabilities for pure and mixed states, respectively, become:

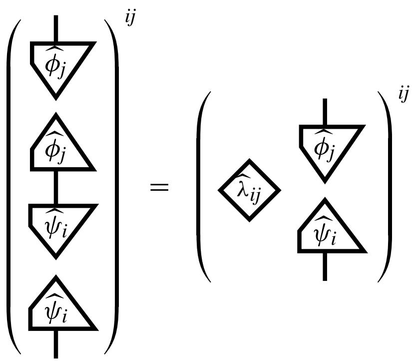
$$\langle \psi | P_i | \psi \rangle \qquad \text{and} \qquad \operatorname{tr}\left(P_i \widetilde{\rho}\right)$$

What is most notable about this formulation of pure quantum theory is that unitaries and von Neumann measurements are singled out as very special processes. One reason for singling out unitaries is Stinespring's dilation theorem in the form of Corollary 6.63, which states that any deterministic quantum process can be realised by means of a state, a unitary, and discarding. Moreover, in the following section we will see that all quantum processes can be realised if we additionally consider ONB measurements (see Section 7.3.4).

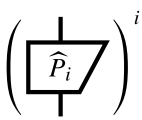
On the other hand, to obtain a quantum process theory, we necessarily have to consider more general processes than unitaries and von Neumann measurements because these operations are simply not closed under composition. For example, composing a state and a unitary in parallel yields a proper isometry. Similarly, when composing a non-demolition von Neumann measurement with a unitary, either in sequence or in parallel, one gets a quantum process that is no longer a von Neumann measurement, for example:



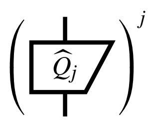
Also when composing two non-demolition von Neumann measurements sequentially the result is usually not a von Neumann measurement. For example, composing two distinct ONB measurements yields:



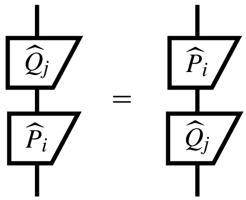
Exercise 7.23 Prove that when composing two non-demolition von Neumann measurements:



and



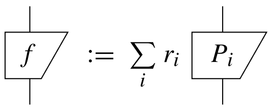
sequentially, one again gets a von Neumann measurement if and only if the measurements *commute*. That is, for all i and j we have:



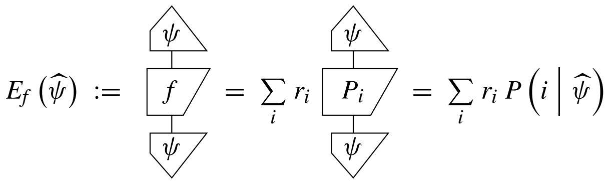
So we can conclude the following.

Theorem 7.24 Von Neumann's formulation of quantum theory is not closed under forming diagrams. In particular, it is not closed under parallel and sequential composition of processes.

Remark 7.25 One advantage of presenting von Neumann measurements as self-adjoint linear maps is that the r_is can be taken to be actual physical quantities associated with a projector, e.g. a position or a momentum. If we replace P_i in the RHS of (7.25) above with a self-adjoint linear map:



instead of a probability, we get a weighted-average of these numbers r_i:



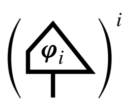
which is called the *expectation value*.

Remark* 7.26 Von Neumann considered projectors on a Hilbert space to be the quantum analogue to the more familiar notion of 'proposition' from classical logic. This insight led him to co-found the field of *quantum logic*, which we discuss in Section* 7.6.2.

7.3.3 POVM Measurements

The most general kind of demolition measurement is simply any quantum process into the trivial system, i.e. a collection of effects that is only constrained by being jointly causal.

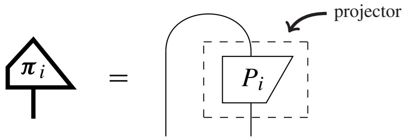
Definition 7.27 Any quantum process of the form:



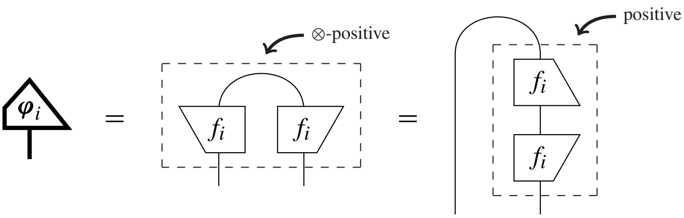
is called a *demolition POVM measurement*.

The abbreviation ‘POVM’ stands for ‘positive operator-valued measure’. We briefly explain this terminology, as it is a bit of a departure from the terminology used in this book.

- Why ‘positive operator’? While for projective measurements we have:



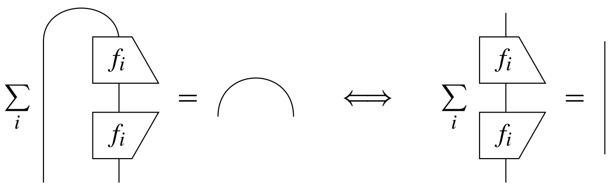
now we have:



(7.26)

So, this just refers to the fact that the elements of this quantum process are represented by positive operators, a.k.a. positive linear maps.

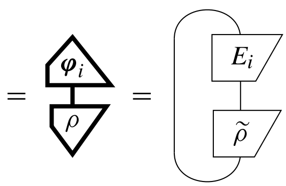
- Why ‘measure’? In probability theory, a finite ‘probability measure’ is an assignment of positive numbers $P(i)$ to each element $i \in \{1, \ldots, D\}$ of a finite set, such that these numbers add up to 1. A ‘positive operator-valued measure’ is a generalisation, in that it assigns to each i a positive map, such that these maps add up to the identity. In other words, this just means causality, since we have:



(7.27)

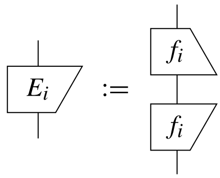
Remark 7.28 As with von Neumann measurements, we can write the Born rule in terms of a density matrix and the trace:

$P(i \mid \rho) =$



(7.28)

where:



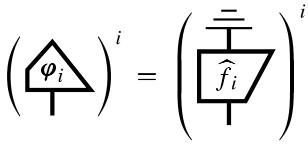
(7.29)

In non-graphical notation (7.28) becomes:

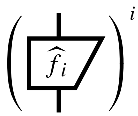
$$P(i \mid \rho) \;=\; \mathrm{Tr}(E_i\,\widetilde{\rho}\,)$$

which is the Born rule for POVMs found in the standard literature.

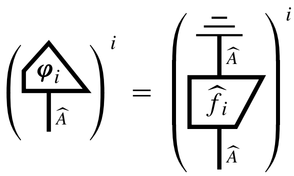
Though it's not a standard concept, we could also consider 'non-demolition' POVM measurements. That is, we look at a family of quantum processes such that we can obtain any demolition POVM measurement by discarding the output. Such a family can be obtained via purification (cf. Section 6.4.3):



Thus, it suffices to consider just quantum processes with pure branches to recover all demolition POVM measurements. So, 'non-demolition POVM measurement' is just a synonym for pure quantum process:



Remark 7.29 Though it won't be necessary for what follows, to really be called a <u>non-demolition</u> measurement, the maps $\widehat{f}_i$ should have the same output system-type as input. In other words, for a POVM measurement on $\widehat{A}$, we should be able to chose the $\widehat{f}_i$ to have output type $\widehat{A}$:

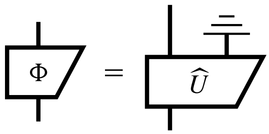


By Exercise 6.97, this is always possible.

Of course, many non-demolition POVM measurements produce the same demolition POVM measurement when discarding the output. This comes from the fact that there are many ways to decompose φ_i as in (7.26). So the ultimate fate of the quantum state depends on the $\widehat{f}_i$ themselves, not just $\boldsymbol{\varphi}_i$. From this, we can deduce that many different non-demolition POVM measurements could reproduce the same probabilities for outcomes, but may act differently on the quantum system itself.

7.3.4 Naimark and Ozawa Dilation

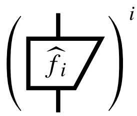
Now, recall that by Stinespring dilation (cf. Theorem 6.61) every causal quantum map Φ (i.e. every deterministic quantum process) arises from some isometry $\widehat{U}$ by discarding one of its outputs:



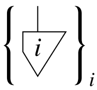
(7.30)

A similar dilation result holds for POVM measurements. We mentioned that any quantum process can be associated with a single isometry. We first show this for pure quantum processes, where it is called *Naimark dilation*.

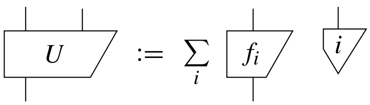
Lemma 7.30 For any pure quantum process with D branches:



and any ONB with D basis states:

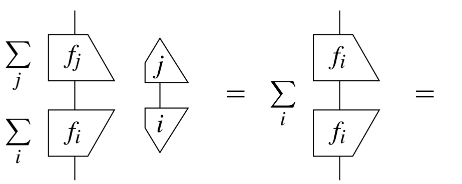


the following is an isometry:



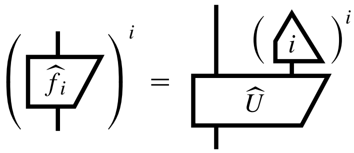
and hence so is the quantum map obtained by doubling.

Proof We have:



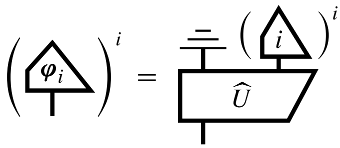
where the last equality holds by causality as in (7.27). □

Theorem 7.31 (Naimark dilation) Every non-demolition POVM measurement arises as an isometry $\widehat{U}$ with a ONB measurement at one of its outputs:



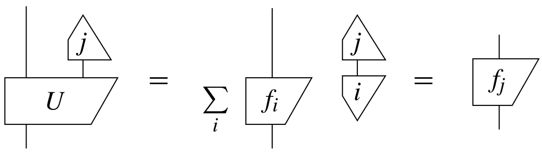
(7.31)

Consequently, every demolition POVM measurement arises as an isometry $\widehat{U}$ with one of its outputs discarded and an ONB measurement at the other output:



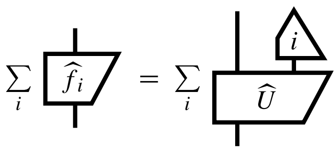
(7.32)

Proof For the isometry constructed in Lemma 7.30 we have:

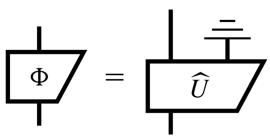


which, when doubled, yields (7.31). □

Remark 7.32 We can obtain (7.30) from (7.31) by summing over the branches of the ONB measurement:



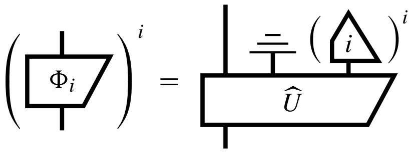
which corresponds to measuring then forgetting the outcome:



So, we saw that 'demolition POVMs' are just arbitrary quantum processes into the trivial system, and 'non-demolition POVMs' are just pure quantum processes from a system to itself. Thus, when we introduced POVMs, all we really did was give a fancier name to stuff we already knew about. But we did establish something important, namely, that we can reproduce all probabilities generated by arbitrary quantum processes in terms of nothing more than an isometry and an ONB measurement. That is the one important thing to remember from this section.

If we combine this with Stinespring dilation, we get a general form for <u>all</u> quantum processes:

Theorem 7.33 (Ozawa dilation) Every quantum process arises from some isometry $\widehat{U}$ with one of its outputs discarded and an ONB measurement at another output:



(7.33)

Exercise 7.34 Prove Theorem 7.33.

So one reason to consider general POVM measurements is that they arise naturally when considering an ONB measurement on part of a larger system, as in Theorem 7.31. The following are two more reasons.

- POVM measurements arise due to imperfections in the measurement procedure due to noise or limited access to the physical system. From this perspective, it is natural to think of (proper) POVM measurements as *mixed measurements*.
- POVM measurements can perform tasks that no von Neumann measurement can. We will see an example of this in Section 7.4.2, where special measurements called *informationally complete* POVM measurements are used to perform a task called *quantum state tomography*.

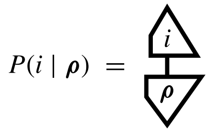
7.4 Tomography

A single measurement typically destroys a quantum state without giving us much information in return. However, if we have lots of identical copies of a quantum state, we can do better. By performing a series of carefully chosen measurements, one can actually infer the state of a system from the probability distributions over all of the measurement outcomes. This procedure is known as *tomography*. In this section, we will look at how tomography works, and what sorts of measurements are required to achieve it.

7.4.1 State Tomography

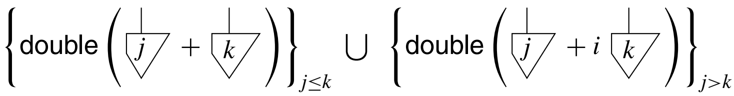
Obviously, 'observing', as we defined it in Section 7.1.5, is the ultimate form of tomography. So for process theories where an observation process is available the notion of tomography is more or less redundant. On the other hand, given that 'single-shot observing' is not available in quantum theory, tomography is a very relevant concept.

Since a single measurement isn't good enough, the next thing we might try are many applications of the same ONB measurement. However, that doesn't work either. Since a doubled ONB is not an ONB (see Section 6.1.5.3), the probabilities:

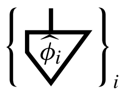


will never suffice to uniquely fix the state ρ.

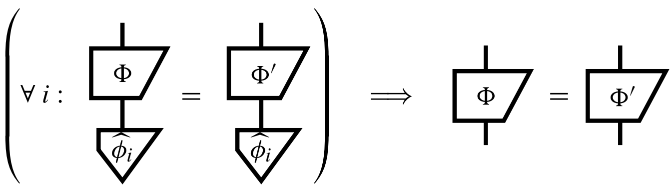
However, there do exist larger sets of states that serve as (non-orthonormal) bases for quantum systems. We even explicitly constructed one in Theorem 6.24:



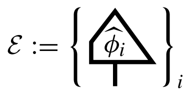
In other words, there exists a set of quantum states



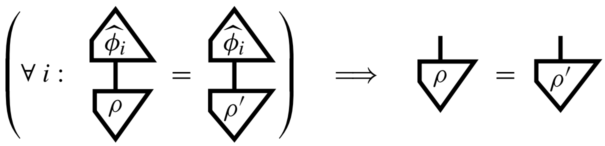
such that:



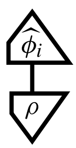
It follows from this that the associated set of quantum effects



suffices to distinguish states:



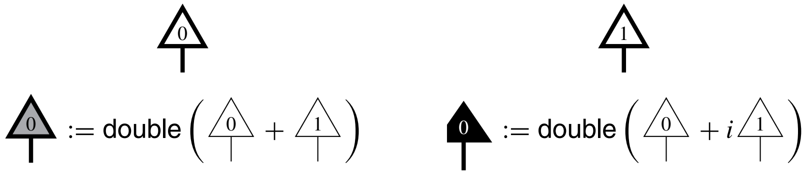
Fixing the probabilities:



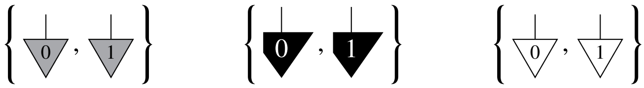
(7.34)

for every $\widehat{\phi}_i$ in $\mathcal{E}$ thus uniquely fixes the state ρ. Of course, we cannot realise effects deterministically, but for any quantum map we can find a non-deterministic process that realises it. So in particular, we can find a collection of measurements that together include all of the effects in $\mathcal{E}$.

Example 7.35 For qubits, the following four effects will uniquely fix any state:



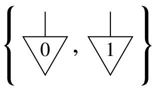
so we can perform state tomography on qubits via measurements in the X-, Y-, and Z-bases (defined in Exercise 6.7):



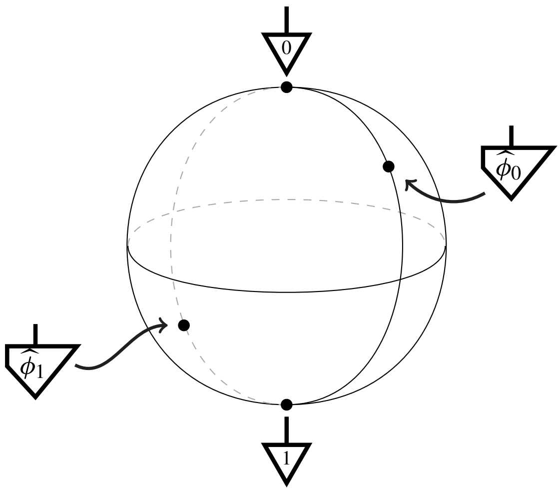
In fact, no smaller set of ONB measurements than the one given in Example 7.35 does the job. To prove this, we can again exploit the geometry of the Bloch sphere.

Proposition 7.36 Qubit state tomography by means of ONB measurements requires measurements in at least three different ONBs.

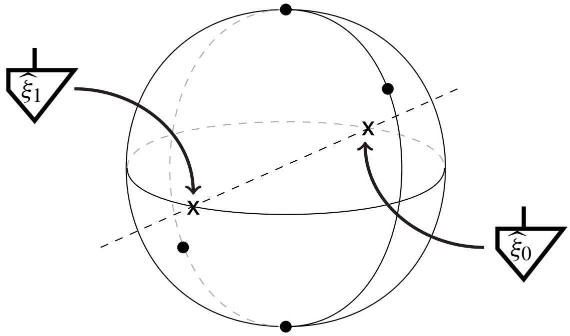
Proof We will give a geometric proof, recalling from Section 6.1.2 that the inner product (7.34) determines the distance between two states on the Bloch sphere. It suffices to show that for any two ONB measurements, there exist at least two states ρ and ρ' that cannot be distinguished. Assume without loss of generality that the first ONB is:



since otherwise we could just express everything to follow in terms of some other basis, rather than the Z-basis. Fix a second ONB $\{\phi_0, \phi_1\}$. Then $\widehat{\phi_0}$ is an arbitrary point on the Bloch sphere, and $\widehat{\phi_1}$ is its antipode:



To find two states that cannot be distinguished, it suffices to find two states that are equally far from all four of the basis states. To do this, draw a line perpendicular to the plane made by the four points, and mark the two places this line crosses the Bloch sphere:



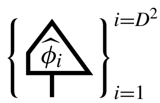
Then, measuring ξ_i in either ONB yields either outcome with probability $1/2$. Thus, with just two basis measurements, we cannot possibly distinguish ξ_0 from ξ_1. □

7.4.2 *Informationally Complete Measurements*

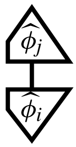
For a qubit one needs three distinct ONB measurements to realise state tomography. Somewhat surprisingly, one can do better when not restricting to ONB measurements and allowing for general POVM measurements. In fact, there is a kind of POVM measurement called *symmetric informationally complete* (SIC), which can do state tomography all by itself. It may be somewhat counterintuitive that a 'mixed' process would be better at anything than a pure one, but here you have a compelling example of that fact.

To understand how this can be the case, assume that we consider the three ONB measurements required for qubit tomography. Now, define a (single) new quantum process as follows. Roll a die, and if the outcome is 1 or 2, then perform the first ONB measurement; if the outcome is 3 or 4, then perform the second ONB measurement; and if the outcome is 5 or 6, then perform the third ONB measurement. Clearly, if we do this new quantum process many times, each of the three ONB measurements will have been performed a sufficient number of times for the sake of doing qubit tomography. A SIC-POVM measurement improves on this quantum process by doing the same job in a more direct, and geometrically elegant, way.

A SIC-POVM measurement consists of pure (but unnormalised) quantum effects. In order to be *informationally complete*, that is, sufficing for tomography, we need to require that this quantum process has (at least) D^2 branches, since this is the number of quantum states we need to form a basis for the doubled system (cf. Theorem 6.24). For such a set consisting of D^2 effects:

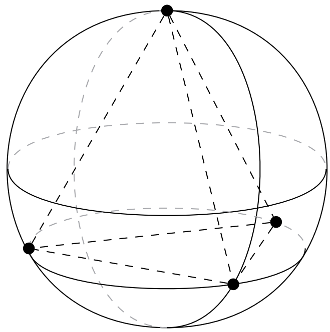


it is impossible for these to all be orthogonal. In the case of qubits, this is like trying to find four points on the Bloch sphere that are all antipodal! Thus, we do the ‘next best thing’ and require that all effects overlap by the same amount:

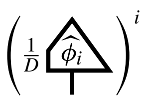


$$= \begin{cases} 1 & \text{if } i = j \\ \lambda & \text{if } i \neq j \end{cases} \qquad (7.35)$$

for some fixed number λ. Geometrically, this means that we choose them to be evenly spaced out across the state space. For the Bloch sphere, a set of four evenly spaced points is always a tetrahedron:



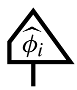
This condition is what the *symmetric* part of ‘symmetric informationally complete’ stands for. For this set of effects to also be causal, we will need to scale down each effect, thus obtaining the quantum process:



(7.36)

where causality becomes:

$$\sum_{i=1}^{i=D^2} \frac{1}{D}$$



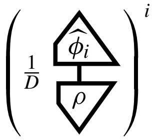
$$=$$

(7.37)

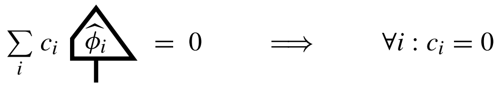
Definition 7.37 A *SIC-POVM measurement* is a causal quantum process of the form (7.36) that satisfies (7.35) for some fixed number λ.

Finally, and most importantly, one can prove that SIC-POVM measurements are informationally complete. That is, the probabilities produced by a SIC-POVM measurement uniquely characterise a quantum state.

Exercise* 7.38 Show that for any SIC-POVM measurement the probabilities:



totally characterise the state ρ. Hint: since a SIC-POVM measurement consists of D^2 effects, an equivalent characterisation of being a basis is the following condition, known as 'linear independence':



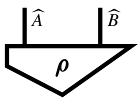
To show all of the c_i must be zero, start by showing they must all be the same.

If you find a solution to the next exercise, let us know!

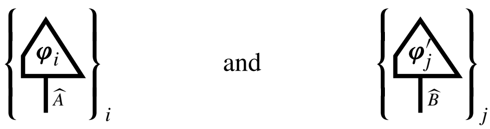
Exercise* 7.39 Is there a (nice) diagrammatic characterisation of SIC-POVM measurements?

7.4.3 Local Tomography = Process Tomography

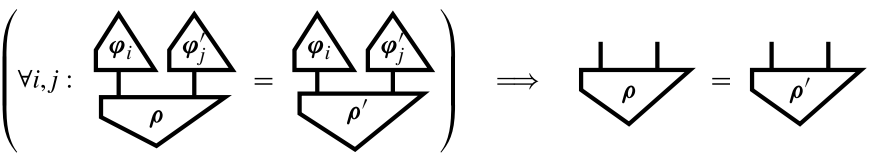
Quantum theory allows enough ONB measurements to identify quantum states. A related question is: can we identify quantum states on multiple systems using only *local measurements*? That is, can we identify a state:



just by measuring each system individually? Following the previous section, this amounts to asking whether there exist sets of effects:

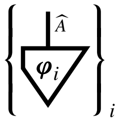


such that:

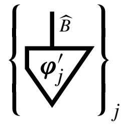


One usually refers to this property as *local tomography*.

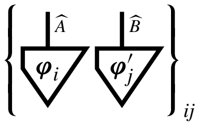
The answer for quantum theory is a clear ‘yes’. Such local measurements do exist, and it is very easy to see why. Since we can find bases:



and



we can define the desired local measurements in terms of the product basis:

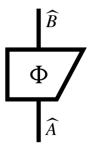


We showed in Section 5.2 that products of ONBs give ONBs, but in fact for **linear maps** this works for arbitrary bases as well. Hence we have the following theorem.

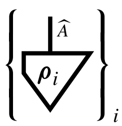
Theorem 7.40 Quantum theory obeys local tomography.

Once one becomes accustomed to the concept of product bases, local tomography seems like such a natural assumption that one wonders how it can possibly fail. However, we do not need to stray too far from quantum theory for this to no longer hold. For instance, a variation of quantum theory built on matrices of real instead of complex numbers does not have this property. This is shown in Section* 7.6.3.

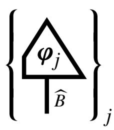
A related, but (seemingly) quite different tomographic notion is *process tomography*. Process tomography is a procedure by which we try to identify a black-box process:



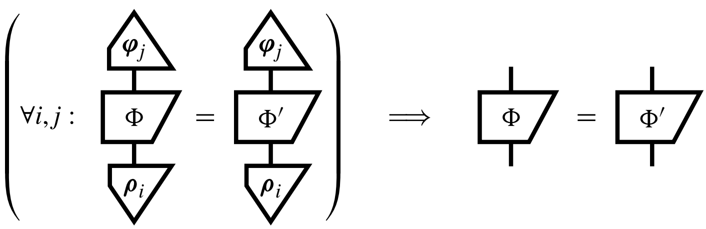
by feeding in states and performing measurements. It amounts to finding states and effects:



and



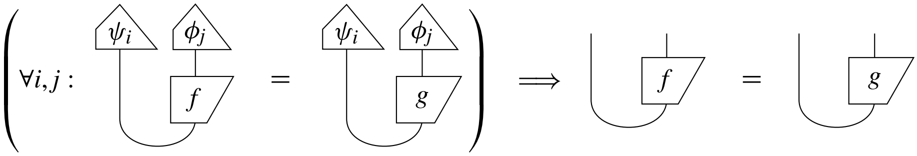
such that:



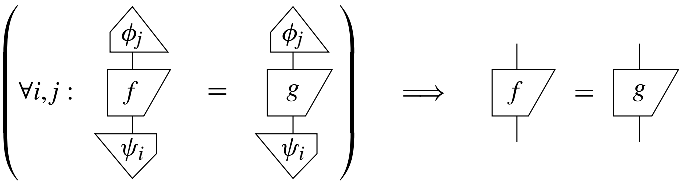
For process theories described by string diagrams, these two notions of tomography are actually equivalent.

Theorem 7.41 A process theory that admits string diagrams has local tomography if and only if it has process tomography.

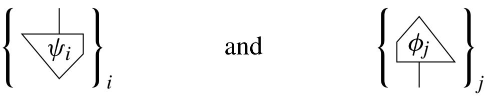
Proof This simply follows from process–state duality. Assume a process theory admits local tomography. Then for any processes f and g we have:



from which we get:



Thus the processes f and g are distinguishable via the following set of states and effects:

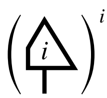


The converse is proved similarly. □

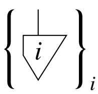
7.5 Summary: What to Remember

1. 'Observing' is not a quantum process.
2. Several species of quantum processes are called *quantum measurements*.

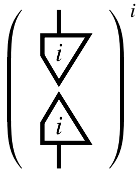
- A *demolition ONB measurement* is a quantum process of the form:



where:

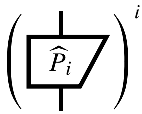


is any ONB, and a corresponding *non-demolition ONB measurement* is a quantum process of the form:

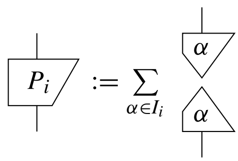


Any doubled ONB state is called an *eigenstate*, and any other state a *superposition* for that measurement.

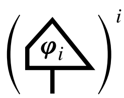
- A *von Neumann measurement* is a quantum process of the form:



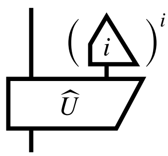
where the branches are mutually orthogonal projectors. Equivalently, von Neumann measurements coarse-grain ONB measurements:



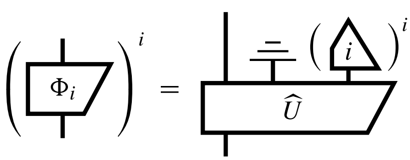
- A *POVM measurement* is any quantum process of the form:



The probability distribution produced by a POVM measurement can also be achieved by means of an isometry and an ONB measurement:

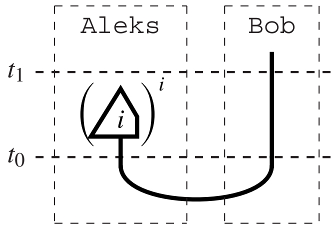


which one refers to as *Naimark dilation*. More generally, every quantum process arises from some isometry $\widehat{U}$ with one of its outputs discarded and an ONB measurement at another output:

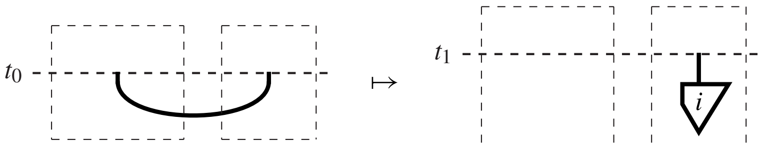


3. Quantum measurements induce two kinds of 'dynamics':

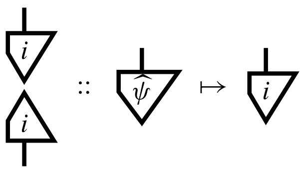
1. *Backaction*, which causes radical instantaneous changes in the state of systems other than those measured. For example, in the scenario:



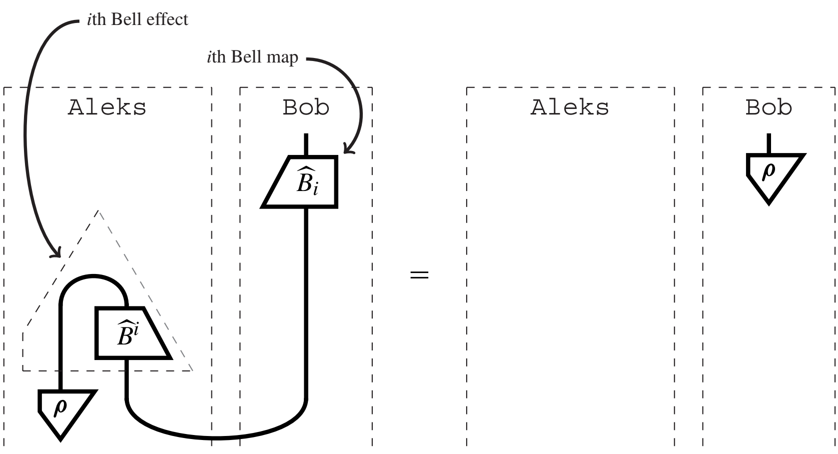
we have the following dynamics between times t_0 and t_1:



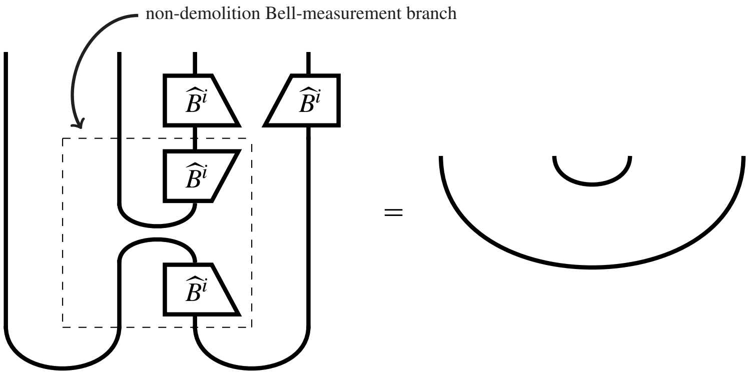
2. *Collapse*, which causes the state being measured to instaneously become an eigenstate of the measurement:



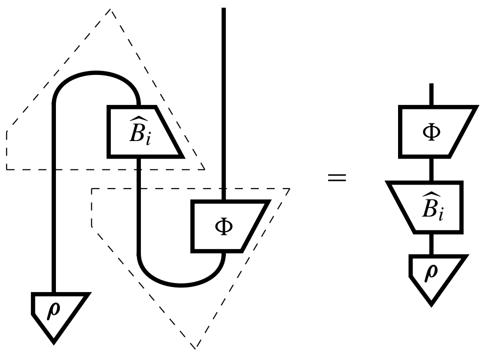
4. These dynamics are exploited in quantum protocols, including *quantum teleportation*:



entanglement swapping:

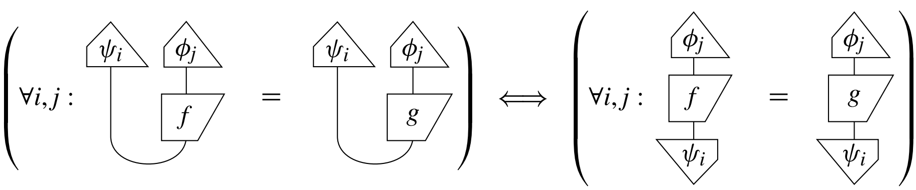


which allows one to create long bits of entanglement from short ones, and *gate teleportation*:



which allows one to apply an arbitrary quantum map Φ, encoded in a bipartite state, to $\boldsymbol{\rho}$. This last trick forms the basis of *measurement-based quantum computation*.

5. If we have many copies of a system, all in the same quantum state, then we can infer that state by means of *tomography*. Certain POVM measurements, called SIC-POVM measurements, allow one to do that by only relying on one particular measurement. For process theories that admit string diagrams, local tomography and process topography coincide:



7.6 Advanced Material*

We'll now have a look at some foundational aspects that have surrounded quantum measurement. We start with a more philosophical discussion of the *measurement problem*. As the name already indicates, there are indeed some sticky issues to discuss. That's also where logic entered the picture for the first time, under the name *quantum logic*, which takes as its starting point the idea that projectors (as used in von Neumann measurements) should be treated as the analogue to propositions in ordinary logic. We contrast this 'logic' with our 'logic of interaction' (cf. Section 1.2.3). We end with an example – taken from work in quantum foundations – showing how something as practical as tomography identifies a fundamental difference between a theory based on complex numbers and one based on real numbers.

*7.6.1 Do Quantum Measurements Even Exist?**

Quantum measurements are clearly indispensable, as they provide our only access to the quantum world. But what are they as compared with their classical counterparts? In particular, is the name 'measurement' truly justified? In Section 7.3.4 we saw that all quantum processes can be seen as some kind of measurement, which seems to render the name 'measurement' redundant. However, in this section we embark on a more philosophical path, contemplating what a quantum measurement is truly about.

Why is there no observation process in quantum theory? We have already seen a proof that such a thing doesn't exist in Section 7.1.5, but this assumes quite a lot of mathematical structure from quantum theory. What we would like is a conceptual justification for why one should not expect there to be an 'observation' process. An answer that traces back to the early days of quantum theory, mostly associated with Niels Bohr and Werner Heisenberg, is:

Any attempt to observe is bound to disturb.

Many would argue that in prequantum theories observing comes for free, and thus the absence of an observation process in quantum theory is a clear departure from our everyday intuition. But is it actually true that in prequantum theories observing comes for free? Consider just bare-bones Newtonian mechanics, before we add any fancy stuff like electromagnetism (and, in particular, light). Since there is no light, we don't have things like cameras (or eyes!) for making observations. If this stripped-down version of Newtonian mechanics admits an observation process, then we should be able to find an object in a dark room without disturbing it. Clearly, if that object is very light, say a balloon, then it is virtually impossible to locate it without moving it a bit. More specifically, in order for us to locate an object, it must exert a force on us, and according to the action–reaction principle, it will experience a force too, and hence move a bit.

Of course, if we now bring electromagnetism, i.e. light and eyes, back into the game, then we can effectively observe the balloon without disturbing it in any noticeable way.

In contrast, for quantum systems there is unfortunately no analogue to the role light and eyes play for mechanical objects. It is unfortunate indeed, but not that surprising anymore. One could say that we suffer from *quantum blindness* and that we can only probe that quantum world by means of some invasive interaction, just like what we have to do in search of an object in a dark room.

Many scientists do find it really hard to give up on the idea that a measurement genuinely represents an observation, and out of pure desperation even invoke the conscious human act of deciding to make an observation within quantum theory. This gave rise to things like the famous *Schrödinger's cat* and the perhaps less famous *Wigner's friend* paradoxes.

Given that there is no observation process in quantum theory, the difficulty of accessing the quantum state then raises the questions about what a quantum state actually represents. For example: does a quantum state represent actual properties of a system, or is it more like a probability distribution, i.e. something that merely reflects the state of our knowledge about the system? In many ways, the second interpretation would be more palatable, because like quantum states, probability distributions can never be observed perfectly, and can even 'collapse' when new information is gained, via *Bayesian updating* (Section 7.2). However, several no-go theorems substantially obstruct this interpretation of the quantum state. Most notably, the recent *Pusey–Barrett–Rudolph theorem* states that, under a few assumptions about quantum systems (which many consider reasonable), this 'state of knowledge' interpretation is totally incorrect.

A closely related problem to that of interpreting the quantum state is the following.

Definition 7.42 The *measurement problem* comprises two questions concerning a (typically von Neumann) measurement:

m1 What causes the measurement process and, in particular, the collapse of the quantum state, to take place?
m2 What 'decides' the outcome i of this process?

Attempts to address these questions are usually referred to as *interpretations*. Some of these deny that there is a change of the state, hence rejecting von Neumann's collapse postulate, in a desire to rescue the idea of there being an observation process. This usually comes at a very high cost; for example, in the case of *Everett's many-worlds* interpretation one has to accept the existence of a humongous number of completely independent parallel universes. We refer to the vast body of existing literature for these interpretations, and provide some pointers in Section 7.7. Here we will restrict ourselves to recasting the measurement problem when viewing quantum theory as a process theory, and how some natural strands for resolving it present themselves.

Early solutions to **m1** included the idea that being coupled to a *measurement device* causes the collapse or, more generally, that any *macroscopic system* causes the collapse. However, this requires a clear-cut definition of what a measurement device is and where the micro-world ends and the macro-world starts. Both of these ideas proved very difficult, if not impossible, to fully develop. Nothing in our experience suggests the existence of a 'wall' that separates micro and macro levels, and it is even more ridiculous to think that a particular human-made machine would play a leading role in fundamental physics which pre-existed humanity.

Alternatively, one could say that a measurement process takes place when the system is in an environment that causes this particular process to happen. Nothing more, nothing less. Taking this stance, **m1** amounts to providing an explicit description of such an environment that causes a measurement process to happen. This point of view fits particularly well with the description of quantum systems as a wire in a diagram, where a system is characterised by its behaviour in context. This is also closely related to the late-Wittgensteinian concept of *meaning in context*, which states that the meaning of something only comes about when one also considers the context in which this thing is considered. Translated to quantum theory, this means that we should conceive of quantum systems not as isolated entities, but rather as entities in interaction with a context, i.e. the rest of a diagram, which in particular includes the measurement.

A related question concerns the conceptual status of an ONB in a measurement process. If we want to learn something about a quantum system we can only do so in terms of things that we are able to perceive, e.g. locations in spacetime. But maybe spacetime is not the natural habitat of a quantum system, and maybe the measurement process is then all about 'forcing' the system into the spacetime theatre. One could refer to such a process as *classicization*. The idea that spacetime may not be the theatre in which all of reality takes place, but rather a form of human experience, traces back to philosopher Immanuel Kant and was further refined by mathematician Henri Poincaré, who went as far as attributing the role of geometry in physics in part to human intuition.

Turning our attention now to **m2**, we address what causes a particular measurement outcome i to occur. The first attempted solutions, under the impetus of Einstein, who famously stated that 'God does not play dice', suggested that there is more to a quantum system than just the quantum state. This 'more' was referred to as *hidden variables*. The main idea is that additional variables associated to the quantum system could determine the measurement outcomes. Again, a string of no-go theorems excludes the existence of many kinds of hidden variables. Most notably, the *Bell–Kochen–Specker theorem* rules out *non-contextual* hidden variables, whose values are in some sense 'real' and independent of measurement choices. Some hidden variable theories survive these no-go theorems, notably *Bohm's hidden variable model*. However, these hidden variables are very different from what Einstein had in mind. In particular, they are necessarily non-local, as we will demonstrate in Section 11.1.

On the other hand, if one associates additional variables to the context of a state rather than the state itself, which includes the interaction between the quantum system and its environment, then there is no obstruction to attributing the outcomes to the pair consisting of the quantum state and these additional variables. Somewhat surprisingly, this option has never really entered the mainstream. While all of this sounds very reasonable to us, again driven by a desire to rescue the process of observation, many scientists refuse to accept the idea that there may be environmental interference to what one learns about a system.

A major obstacle to swallowing this idea is that science has traditionally been built on the presupposition that any system subject to scientific investigation must be sufficiently isolated from its environment when being probed. Maybe the key lesson from quantum theory is that this stance cannot be retained and that one has to opt for something more along the lines of *relationalism*, in that the fabric of reality is about relationships between things, rather than their individual attributes. Diagrams of course provide the natural language for such a relational universe.

*7.6.2 Projectors and Quantum Logic**

In Remark 7.26 we mentioned that projectors can be thought of as propositions about quantum systems, resulting in the field of *quantum logic*. The starting point of quantum logic is not to follow Schrödinger's vision that the beating heart of quantum theory is the manner in which quantum systems compose, but rather to follow von Neumann's vision that understanding quantum measurements is the key.

In classical logic, we typically consider systems to be sets of states X. Then, a *proposition* is just a subset of states $P \subseteq X$, which should be thought of as the set of all states for which P holds. For example, if our system is a potato, P might be 'is boiled', which we represent formally as the set of all possible states of a potato where the potato is boiled.

As subsets, propositions come with a natural ordering, namely subset inclusion. So, for instance, if Q stands for 'is cooked', any boiled potato is also cooked, so we have:

$$P \subseteq Q$$

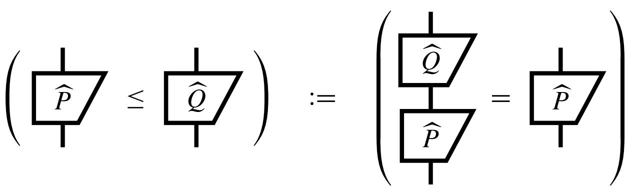
In other words, we can deduce Q from P. This ordering, which represents 'deduction', is the cornerstone of any logical system. We can also represent conjunction ('P and Q'), disjunction ('P or Q'), and negation ('not P') straightforwardly as operations on subsets, respectively:

$$P \cap Q \qquad\qquad P \cup Q \qquad\qquad P^{\perp} := X \backslash P \tag{7.38}$$

These operations give the set of all propositions the mathematical structure of a *boolean lattice*. Boolean lattices are sets with operations $\cup, \cap, (\ )^{\perp}$ satisfying various equations. Most notably, they are *distibutive*:

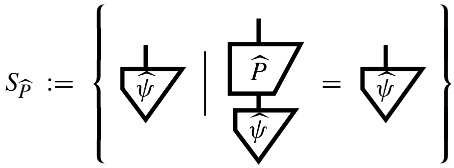
$$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$$

Projectors in some sense play the role of propositions in quantum theory, in that they are the 'verifiable propositions'. Namely, for any $\widehat{P}$, we can fix a von Neumann measurement that checks whether $\widehat{P}$ holds. As with propositions, we can give a 'deduction' ordering to projectors:



(7.39)

So, how is this like the ordering on propositions from before? We can associate to each projector $\widehat{P}$ a set of states for which $\widehat{P}$ holds (with certainty):



Then it's easy to check that (7.39) is equivalent to:

$$S_{\widehat{P}} \subseteq S_{\widehat{Q}}$$

So, for a given state ψ, 'satisfying' $\widehat{P}$ implies 'satisfying' $\widehat{Q}$. However, these $S_{\widehat{P}}$ aren't just arbitrary sets, but rather *subspaces*.

Exercise 7.43 Show that, for any projector $\widehat{P}$ on a quantum system $\widehat{H}$, there exists a subspace $H_P \subseteq H$ such that $S_{\widehat{P}}$ consists of all the states $\widehat{\psi}$ for $\psi \in H_P$. Conversely, show that every subspace of H corresponds to a (unique) projector in this way.

We can get some mileage out of this. For example:

$$S_{\widehat{P}} \cap S_{\widehat{Q}}$$

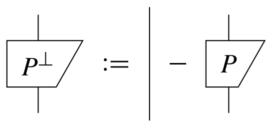
also comes from a subspace, so we can let '$\widehat{P}$ and $\widehat{Q}$' just be its associated projector, which we denote by:

$$\widehat{P} \wedge \widehat{Q}$$

However:

$$S_{\widehat{P}} \cup S_{\widehat{Q}}$$

is <u>not</u> a subspace, so we need to get a bit creative. Letting:

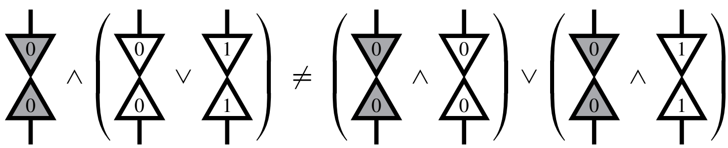


we get negation. Then by de Morgan's law, 'not (not P and not Q)' should be the same as 'P or Q', so let:

$$\widehat{P} \vee \widehat{Q} \;:=\; (\widehat{P}^{\perp} \wedge \widehat{Q}^{\perp})^{\perp}$$

Unfortunately, $\wedge$ and $\vee$ are *not distributive*.

Exercise 7.44 Show that:



Therefore, instead of distributivity, quantum logicians usually assume something a lot weaker, called *orthomodularity*:

$$\widehat{P} \leq \widehat{Q} \quad \Longrightarrow \quad \widehat{Q} = \widehat{P} \vee \left(\widehat{P}^{\perp} \wedge \widehat{Q}\right)$$

The goal of quantum logic is to reason about quantum systems to the greatest extent possible just using this very weak logical structure.

This perspective clears away a lot of the 'noise' associated with Hilbert spaces and can provide new insights, and in this way chimes well with the goals of this book. However, as we stressed in Section 1.2.4, quantum logic aims to characterise quantum theory by the failure of something, which explains the failure of this research program. Interestingly, the issue on which quantum logic failed mostly was the description of composite systems, which is precisely where we started off in this book, and that's perhaps the most important lesson to be taken from quantum logic.

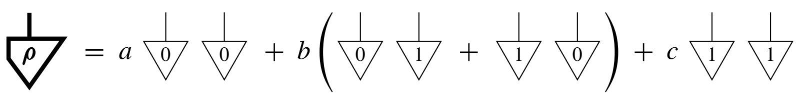
*7.6.3 Failure of Local Tomography**

More recently, much of the interesting quantum foundations research has embraced the fact that many characteristic features of quantum theory are tightly intertwined with the interaction of multiple systems. One very remarkable such result is the fact that one can affirm the key role complex numbers play by means of tomography on two systems.

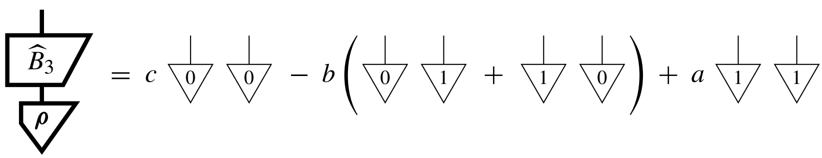
Suppose we define a new process theory called $\mathbb{R}$**-quantum maps**, consisting of only those quantum maps that involve real numbers. More precisely, rather than doubling **matrices**($\mathbb{C}$) one doubles **matrices**($\mathbb{R}$). One would expect this theory to be quite similar to that of **quantum maps**, but the fact that we leave out the imaginary part of complex numbers has some dramatic consequences.

Theorem 7.45 The theory of $\mathbb{R}$**-quantum maps** does not admit process tomography and, hence, does not admit local tomography.

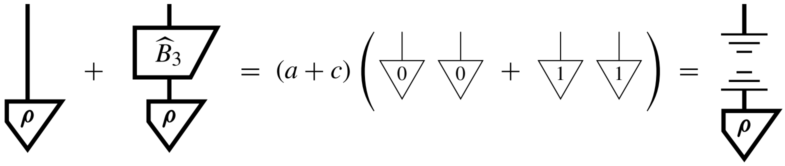
Proof As with **quantum maps**, any state ρ in the theory of $\mathbb{R}$**-quantum maps** is a $\otimes$-positive linear map; so in particular, it is self-conjugate. Thus any two-dimensional state in $\mathbb{R}$**-quantum maps** (a.k.a. 'rebit' state) can be written as:



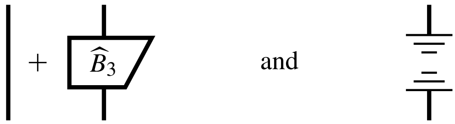
for $a, b, c \in \mathbb{R}$. The Bell maps have real matrices, so $\widehat{B}_3$ is a process in $\mathbb{R}$**-quantum maps**. Applying it to ρ yields:



When we add the resulting state to $\boldsymbol{\rho}$ itself, we get:



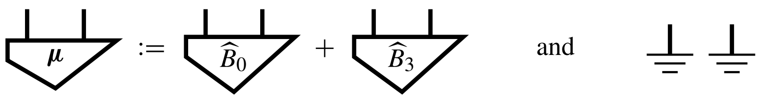
Thus, the processes:



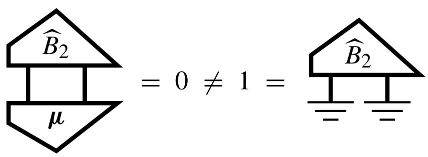
(7.40)

agree on all states in $\mathbb{R}$**-quantum maps**, yet they are clearly not equal. □

We cannot tell the difference between the processes in (7.40) just by applying these processes to states. So what? It could have been the case that we will just never see the difference in these two processes, in which case they would be, for all intents and purposes, the same. But this is not the case! We can see this immediately if we turn these processes into bipartite states. The equivalence of process tomography and local tomography comes from process–state duality. Applying process–state duality to the processes in (7.40) tells us that we cannot distinguish the bipartite states:



by means of local effects. However, we can still distinguish them by means of a global effect, namely any of the other Bell-basis effects:



So we can conclude that $\mathbb{R}$**-quantum maps** is truly a very different theory from **quantum maps**. In particular, tomography of processes and composite systems would be an entirely different ballgame.

Exercise 7.46 Give a diagram in which the two processes in (7.40) are distinguishable (i.e. yield different probabilities via the Born rule) in the theory of $\mathbb{R}$**-quantum maps**.

7.7 Historical Notes and References

Quantum measurements as described here, and von Neumann measurements in particular, are evidently due to von Neumann (1932), in particular including the idea of the collapse, and that this collapse is induced by projectors.

Naimark's theorem (which is also sometimes spelled Neumark's theorem) for POVMs first appeared in Neumark (1943). Ozawa (1984) stated a more general version of this theorem for what he calls 'quantum instruments', which include quantum processes as we have defined them. Entanglement swapping was first proposed in Zukowski et al. (1993), and teleportation-based universal quantum computing was proposed in Gottesman and Chuang (1999). Wigner's theorem, which was referred to in Remark* 7.3, is taken from Wigner (1931).

The local tomography axiom in the form that we presented it here is taken from Chiribella et al. (2010). A similar formulation appeared earlier in Barrett (2007), which in turn traced back to a reconstruction of quantum theory in Hardy (2001). SIC-POVMs trace back to Lemmens and Seidel (1973). Exercise 7.39 was suggested to us by Chris Fuchs, who strongly believes that SIC-POVMs have a fundamental role in quantum theory (Fuchs, 2002). This stance is one aspect of an interpretation of quantum theory called *quantum Bayesianism*, or *QBism* (Fuchs et al., 2014).

The measurement problem grew out of the Bohr–Einstein debate, which mainly followed onwards from the EPR papers (Einstein et al., 1935; Einstein, 1936) and Niels Bohr's reply (Bohr, 1935). Heisenberg's and Bohr's positions were already stated much earlier in their respective books Heisenberg (1930) and Bohr (1931). Schrödinger's cat paradox appeared first in Schrödinger (1935), and Wigner's friend in Wigner (1995a). Many textbooks have been dedicated to the measurement problem, a selection being Jammer (1974), Redhead (1987), and Bub (1999). For original texts on interpretations that are currently still popular we refer to Everett (1957) and Bohm (1952a,b), respectively, for the many-worlds interpretation and an example contextual hidden variable theory. Constraint theorems on the interpretation of the quantum state are Jauch and Piron (1963), Kochen and Specker (1967), and Pusey et al. (2012). In fact, Kochen and Specker (1967) is a fairly straightforward corollary to the theorem of Gleason (1957), as explained in Belinfante (1973).

Planck's quote at the beginning of this chapter dates back to 1936, from his address on the twenty-fifth anniversary of the Kaiser Wilhelm Society for the Advancement of Science (see Macrakis, 1993), which after being implicated in Nazi scientific operations, was dissolved and had its functions taken over by the Max Planck Society. Evidently, it was motivated by the aforementioned interpretational difficulties that caused many physicists not to accept the theory.

Poincaré's modified Kantian views mentioned in Section 7.6.1 are taken from Poincaré (1902). Wittgenstein's 'meaning in context' appeared first in Wittgenstein (1953). Relational views on space and time trace back to Leibniz (see e.g. Rickles, 2007).

Already in his book on quantum theory, von Neumann (1932) attributed a fundamental significance to projectors, in that they represented the propositions of the quantum world. This then became the basis of so-called quantum logic (Birkhoff and von Neumann, 1936). For von Neumann, quantum logic was a path towards a better formalism for quantum theory. This in part drove him to the introduction of *von Neumann algebras* and in particular to the study of *Type II factors* therein. All of this is detailed in Redei (1996). Interestingly, while quantum logicians adopted the orthomodular law, von Neumann insisted on the stronger *modular law*:

$$\widehat{P} \leq \widehat{Q} \quad \Longrightarrow \quad \widehat{P} \vee \big(\widehat{R} \wedge \widehat{Q}\big) = \big(\widehat{P} \vee \widehat{R}\big) \wedge \widehat{Q}$$

which holds only for finite-dimensional quantum theory. His reason was that this law holds for projective geometries and that the lattice of closed subspaces of any Hilbert space naturally embeds in a modular one, which then, via the fundamental theorem of projective geometry, yields a vector space representation (Piron, 1976; Stubbe and van Steirteghem, 2007).

For many quantum logicians, quantum logic wasn't really about logic, but rather about probability theory and algebra. An operational variant was initiated by Mackey (1963) and was further conceptually and philosophically underpinned in Piron (1976) and Moore (1999). Constantin Piron (1964) also proved what could be considered as the first reconstruction of quantum theory from operational principles.

As already hinted at in the preface to this book, one of the authors of this book ended up realising the importance of processes through quantum logic. In particular, rather than thinking of static propositions, passing to how propositions evolve allows one to derive the linearity of processes (Faure et al., 1995). The same argument provides the most compelling interpretation for orthomodularity too, corresponding to the fact that projectors are actual processes (Coecke et al., 2001; Coecke and Smets, 2004), via an argument that generalises *weakest precondition semantics* from computer science (Dijkstra, 1968; Hoare and He, 1987). This line of research is further developed in Baltag and Smets (2005). Our love for quantum logic is not entirely gone, as witnessed by a recent attempt to reconcile diagrams and quantum logic (Coecke et al., 2013b).