11

Quantum Foundations

> Mermin once summarized a popular attitude towards quantum theory as ‘Shut up and calculate.’ We suggest a different slogan: ‘Shut up and contemplate!’
>
> – *Lucien Hardy and Rob Spekkens, 2010*

This chapter is dedicated to the foundations of quantum theory, or as it’s more fashionably called these days, quantum foundations. Here, we will use all of the things we’ve learned so far to probe some very deep questions:

1. What features of nature are imposed on us by quantum theory?
2. Conversely, what features of a physical theory are imposed on us by (our current understanding of) nature?
3. Which of these features are ‘properly quantum’, in the sense that they have no counterpart in any classical physical theory?

We’ll address these questions by looking at one of those most celebrated (and historically controversial) properties of quantum theory: *quantum non-locality*. First, we will give a precise definition of non-locality and prove that it exists within the theory of **quantum processes** and in fact already within the comparatively tiny subtheory of causal **Clifford maps**. Then, we will present a new process theory called **spek**, which has locality built right in. A remarkable thing is that the two theories of **Clifford maps** and **spek** are identical in every respect except one: the phase group of a single system. And (another spoiler alert!) it is indeed this one difference that kills the proof of non-locality that works for quantum theory.

11.1 Quantum Non-locality

Quantum non-locality is probably still the least understood of all the new quantum features, in both philosophical and structural terms. Our upbringing in a seemingly ‘classical’ world, and especially our undeniably corrupting ‘classical’ scientific education, tends to make us expect two things from a physical theory:

1. *Realism*: physical systems have real pre-existing properties, and hence the outcome of 'measuring' such a property is fixed in some way prior to the measurement.
2. *Locality*: it is impossible for one system to affect another distant system instantaneously.

Very early on, something made Einstein, the father of relativity theory, extremely uncomfortable with quantum theory. He realised that something really weird was going on:

Quantum theory is <u>not</u> a local-realistic theory.

It was the first reaction of many (including Einstein) to think this simply meant quantum theory was 'incomplete'. Since of course any theory should be local and realistic, the failure of quantum theory to be so was simply a bug that needed to be fixed.

But, as we'll see shortly, we will have to learn to live in absence of localrealism. The failure of <u>any</u> local-realistic theory to reproduce the predictions of quantum theory is what we call *quantum non-locality*.

11.1.1 Refinements of Quantum Theory

Typically we consider physical systems to have certain properties even if they are not observed, and when we observe the system, it is these properties that we witness. For example, the colour of a pencil won't change when we don't look at it. Realism stands for the assumption that something like this is true in quantum theory, namely, that what we learn in quantum measurements is not just created out of the blue during the measurement process but has some cause in the past.

Of course, when we started this book with Dave's travels to the North and South Poles, we made it clear that measurements change the state of the system non-deterministically and that the measurement outcome does not faithfully reflect what the state of the system was. We know now that this is just what the standard quantum formalism tells us.

However, there is no reason a priori that we couldn't refine quantum theory in such a way that each measurement outcome can be traced back to something pre-existing. It could be the case that somebody had already put Dave in a rocket aimed at the South Pole, and they were just waiting for us to ask where he was. That is, there could be some *hidden variables* at work here, and it is our ignorance of them that leads to the apparent non-determinism. We can thus refine our theory, putting those extra variables in, and poof! – no non-determinism.

Such a refinement is sometimes called a *hidden variable model*, or more recently, an *ontological model*. To avoid any philosophical baggage that comes with each of these terms, we will stick to *refinement*.

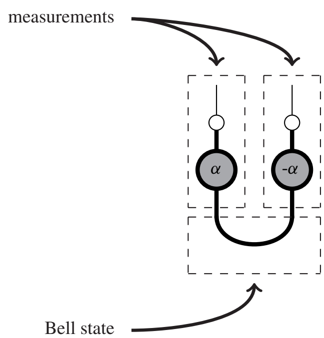
The crucial feature of a refinement is that even though it may be adding additional variables to account for measurement outcomes, it should retain the predictions of quantum theory. A famous example of such a refinement is *de Broglie–Bohm theory*, which postulates the existence of particles flying around that always have precise positions and momenta. The catch is, they are being pushed around by something else spread out in space, called the *pilot wave*, which explains the characteristically quantum behaviour we saw in Section 7.1.4 whenever we try to measure those particles. Hence the theory keeps realism, at the cost of dropping locality.

On the other hand, as we saw in Section 6.3, relativity theory is derived from the principle that nothing travels faster than the speed of light. Therefore, many would consider dropping locality too high of a price to pay for realism.

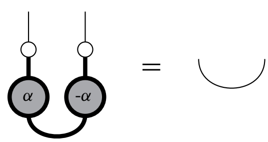
In that case, we should require that any refinement of quantum theory that provides it with predetermined measurement outcomes should be compatible with relativity theory. Quantum theory is of course already compatible with relativity theory, thanks to the causality postulate (cf. Section 6.3.2). To keep this compatibility intact, any newly introduced variables should not travel faster than the speed of light.

So in particular, any correlation that may occur for spatially separated systems must have some common cause in the past. In other words, they should respect *Reichenbach's common cause* principle. This principle states that every correlation is either a consequence of a direct causal link or due to a common cause. An example of the first is that being shot by a gun causes pain (or death). An example of the second is the strong correlation in the previous century between the spread of televisions in households and the death of hedgehogs. The common cause is the spread of wealth, which caused people not only to buy televisions, but also to buy cars, and these cars killed hedgehogs. Bummer.

In order to establish a contradiction between quantum theory and local realism, we will proceed as follows. We will consider carefully chosen *measurement scenarios*; that is, we fix a particular quantum state and measure each of its systems in a number of different ways. We compute the probabilities for each scenario, and hence the *correlations* between the outcomes in each measurement, and study the properties these correlations obey. For example, if we were to consider two systems in a Bell state and measure each system with the following measurements:



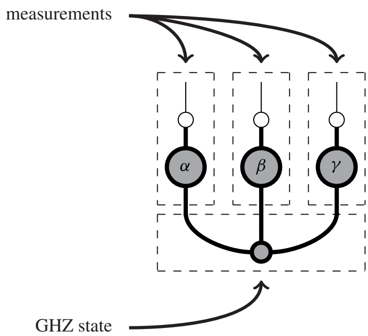
simply using phase spider fusion we learn that the outcomes for the measurements on each of the systems will always be the same; i.e. they are perfectly correlated:



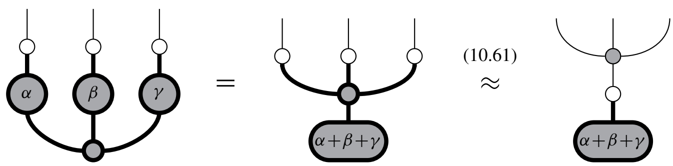
Despite the fact that this scenario involves quantum processes, we could of course also produce these same correlations by some totally classical (and hence local) process. In order to establish a contradiction with local realism, we will need to consider a scenario consisting of several different choices of measurement on the same quantum state.

11.1.2 GHZ-Mermin Scenarios

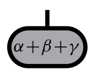
Consider again the following measurements on a GHZ-state:



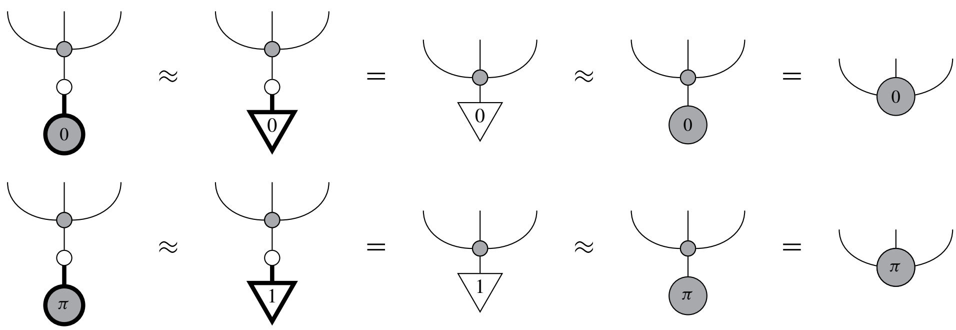
Each of the measurements seems to depend (locally) on a phase, but by using phase spider fusion and strong complementarity we obtain:



In the particular cases that $\alpha + \beta + \gamma$ is either 0 or π, then the phase state:



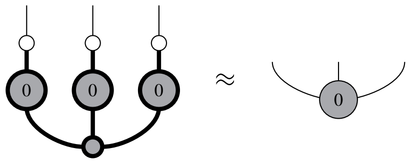
is in the classical subgroup for ○, and hence, by (10.69) and (10.26):



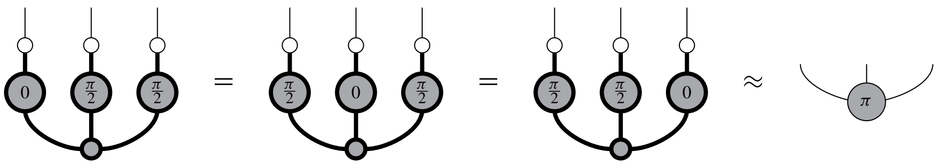
That is, we obtain the even-parity state and odd-parity state, respectively (cf. Section 9.3.5).

Now, let's consider some fixed choices for the phases α, β, and γ. If we let the phase be 0, we obtain the usual Z-measurement, whereas if it is $\frac{\pi}{2}$, the result is a Y-measurement:

In order to produce a contradiction with local realism, we will need to consider one choice of measurements that yields the even-parity state, namely:



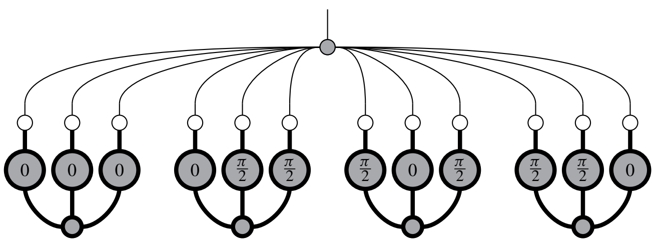
as well as three choices that yield the odd-parity state, namely:



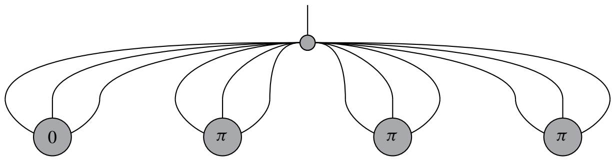
In other words, we consider the following measurement choices:

	system A	system B	system C
scenario 1	Z	Z	Z
scenario 2a	Z	Y	Y
scenario 2b	Y	Z	Y
scenario 2c	Y	Y	Z

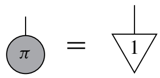
We'll use a particular property of all of these scenarios together that allows us to draw a contradiction with local realism. That property is the 'overall parity':



Substituting the parities for the individual scenarios:



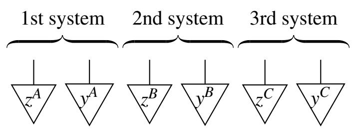
by phase spider fusion we obtain:



So the overall parity is odd.

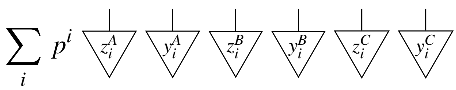
11.1.3 Drawing a Contradiction

Local realism assumes that all measurement outcomes have some common cause in the past. So, we construct a refined model, whose classical values already 'know in advance' what outcome they will provide for either measurement:



(11.1)

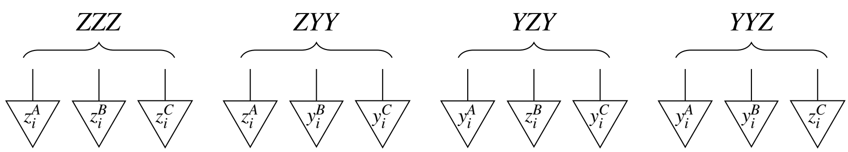
So, for instance, if we measure Z on the first system, we will get the outcome $z^A \in \{0, 1\}$; if we measure Y on the third system, we will get y^C; and so on. A generic state in this model is then a probability distribution over these classical values:



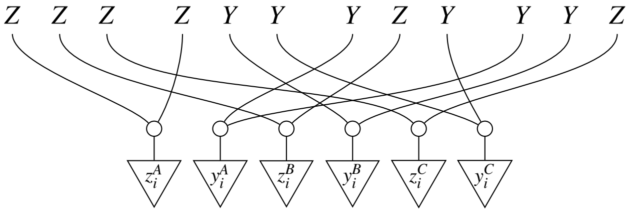
(11.2)

Now, we know that quantum theory predicts that the overall parity for the four measurement choices is always odd. Hence, to be consistent with quantum theory, it should be the case that every possible value in the above probability distribution yields an odd overall parity. But, as we shall now see, not one of them does!

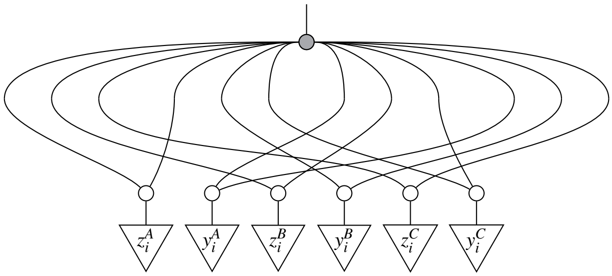
By definition, each value in (11.2) gives the following outcomes for each of the four measurement choices:



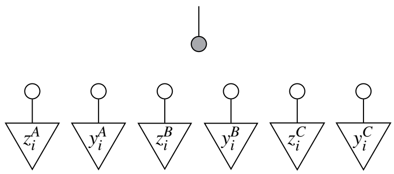
Combining duplicate states via copy spiders, we obtain:



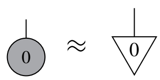
Then, the overall parity is given by:



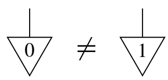
Looking closely at this ‘locality Spaghetti Monster’, we see that exactly two legs connect each ○-spider to the ●-spider. So, applying the complementarity equation (10.49), we obtain:



which is equal to:



That is, for every state, the overall parity is <u>even</u>. Since even is not equal to odd:



quantum theory is non-local.

11.2 Quantum-like Process Theories

One way to understand a thing is by understanding how it relates to similar things. For example, suppose we wished to list the most remarkable traits of a dodo. If we compared dodos with people, we would find lots of totally uninteresting distinctions, like ‘dodos don’t have fingers’. However, if we compare dodos with other birds, we find their unique characteristics start to stand out more. For instance, we will immediately note their inability to fly and their legendary tastiness (which gives us some clues toward their extinction!).

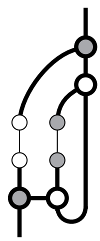
Something similar has been done for quantum theory, by considering it as part of a broader class of theories. This can be done in many different ways, depending on what features of the theory one wishes to study and contrast with others. For example, a lot of effort has been put into understanding quantum theory as an instance of a *generalised probabilistic theory*. In this general setting, systems always have convex sets of states (e.g., the Bloch ball in the case of qubits), but many of the other characteristics of quantum theory start to break down.

In another direction, one can look at process theories that admit similar diagrammatic (i.e compositional) behaviour to quantum theory, which is what we (of course!) will focus on here. We saw in Chapter 9 that many quantum features can be expressed in terms of spiders, which are purely diagrammatic creatures. Hence, just by reinterpreting these spider diagrams in other process theories, we can see how things such as measurement, complementarity, and even non-locality arguments look in 'quantum-like' process theories.

11.2.1 Complementarity in relations

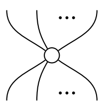
In Chapter 5, we learned that **relations** were totally boring when it comes to ONBs. Namely, each system has exactly one ONB (cf. Example 5.6), given just by the set of singletons. However, by Chapter 8, ONBs were superseded by spiders for our purposes, and indeed in the theory of **linear maps**, fixing an ONB is exactly the same as fixing a family of spiders. Of course, if that were true for **relations**, then spiders would be just as boring as ONBs, since there would only be spiders of one colour around.

However, it turns out that the theory of **relations** is a lot wilder than one might imagine at first, and there are lots of things that behave like spiders that do not arise from an ONB. For example, even for the system $\mathbb{B}$ there are already spiders of two colours. Among many other things, this means that the diagram:



makes perfect sense within the theory of **relations** (or, more precisely, the theory of 'cq-relations' built in the analogous way to **cq-maps**), and we can calculate with these spiders just as we have been doing all along.

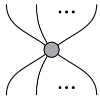
Spiders of the first colour are indeed the ones that arise from the unique ONB for $\mathbb{B}$:



$$:: \begin{cases} (0, \ldots, 0) \mapsto (0, \ldots, 0) \\ (1, \ldots, 1) \mapsto (1, \ldots, 1) \end{cases}$$

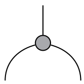
and it's straightforward to see that these creatures indeed fuse in the appropriate manner. But what about the spiders of the second colour?

Recall from Section 9.3.5 that we gave a characterisation of ●-spiders in terms of the parity of ○-basis states. While the X-basis doesn't carry over into **relations** (thanks to that pesky minus sign in the second basis state), ●-spiders do! That is, we let:



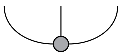
$$:: (b_1, \ldots, b_m) \mapsto (b'_1, \ldots, b'_n)$$

if and only if the number of 1s in $b_1, \ldots, b_m, b'_1, \ldots, b'_n$ is even. Particular cases of these parity spiders are the relational versions of the examples we saw in Section 9.3.5, e.g. relational XOR:



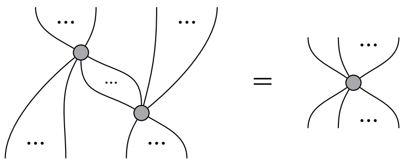
$$:: \ (i, j) \mapsto i \oplus j$$

and the relational three-system parity state:

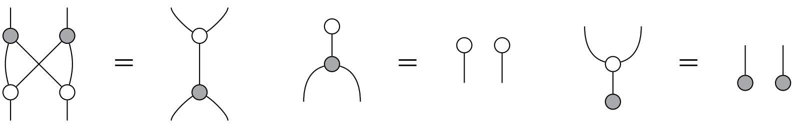


$$:: \ * \mapsto \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$$

Exercise 11.1 Show that the ● spiders defined above indeed behave like spiders, that is, that we have:



as well as invariance under leg-swapping and conjugation and that the pair ○/● is strongly complementary:



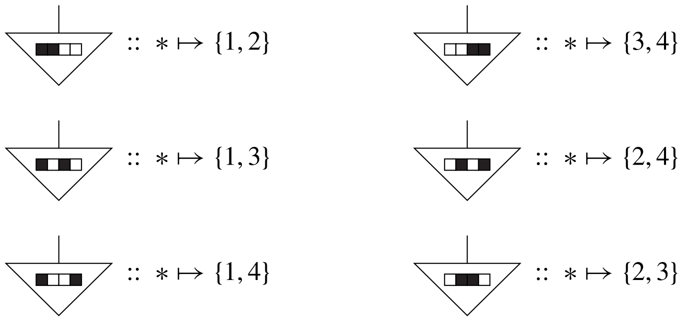
11.2.2 Spekkens' Toy Quantum Theory

While it has non-separable states, and even strongly complementary spiders, the theory of **relations** doesn't really look much like quantum theory. With this in mind, it may come as a surprise that there exists a subtheory of **relations** that does exhibit many quantum features. For instance, the states of the smallest non-trivial system do organise themselves into a sphere, which looks a whole lot like the six-state restriction of the Bloch sphere we encountered in Section 9.4.2 with the **Clifford maps**. We'll first define this theory concretely (i.e. 'the hard way'), and later show that it can be obtained equivalently as a small modification to ZX-calculus.

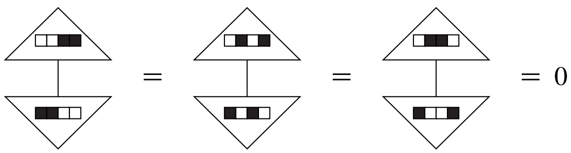
The basic system in this new theory, called **spek**, is the four-element set:

$$\mathrm{IV} := \{1,2,3,4\}$$

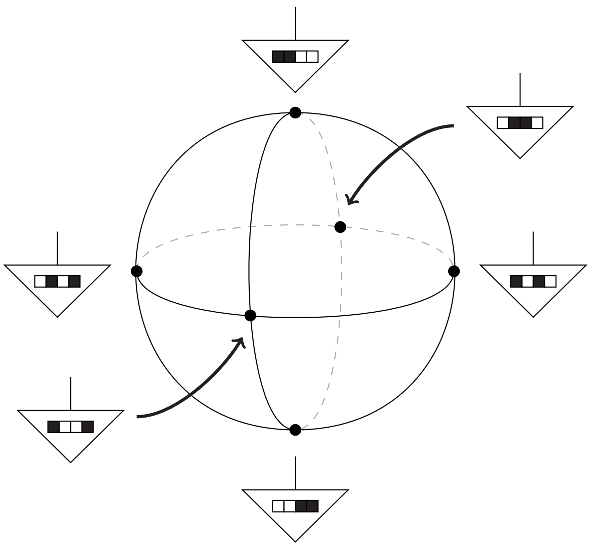
Since this will be a subtheory of **relations**, the states of IV are given by subsets. Rather than taking all $2^4 = 16$ subsets of IV, we'll take our (non-zero) states to be just the six two-element subsets:



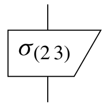
Notably, these organise themselves into three pairs of orthogonal states:



so we can, somewhat suggestively, place them on the 'spek sphere':

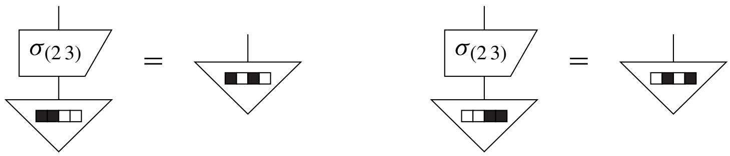


By analogue to the Bloch sphere, we will refer to the states on the Z-axis as the spek-Z states, those on the X-axis as spek-X states, and those on the Y-axis as spek-Y states. These are the states of our process theory, but what about processes? On a single system, we allow arbitrary permutations of IV, e.g.:



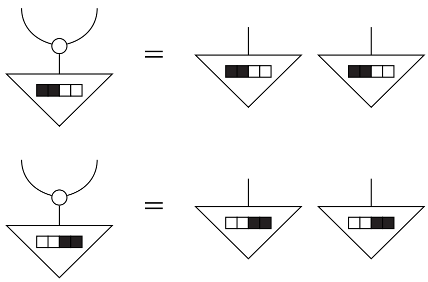
$$:: \; 1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 2,\ 4 \mapsto 4$$

Since permutations cannot change the number of filled-in boxes, they will always send two-element subsets to other two-element subsets. For example:



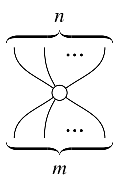
Furthermore, they will always send orthogonal states to orthogonal states; i.e. they preserve antipodes on the 'spek sphere'. Hence, these permutations, which are easily seen to be unitary, are the analogues to the one-qubit unitaries in **Clifford maps**.

That's it for processes on a single system. To get all the processes in **spek**, we just add a single family of spiders, namely the spiders that copy the **spek** spek-Z states:



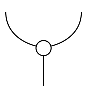
(11.3)

We call this family of spiders the $\{1,2\}$-*parity spiders*. Explicitly, a $\{1, 2\}$-parity spider is given by the relation:



$$:: \begin{cases} (a_1,\ldots,a_m) \mapsto (b_1,\ldots,b_n) \\ (a_1+2,\ldots,a_m+2) \mapsto (b_1+2,\ldots,b_n+2) \end{cases}$$

for all $a_i, b_i \in \{1,2\}$ such that the number of 2s in $a_1,\ldots,a_m,b_1,\ldots,b_n$ is even. In fact, this is the unique family of a spiders whose copy relation:

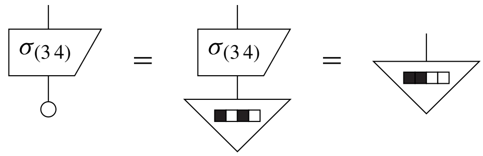


$$:: \begin{cases} 1 \mapsto \{(1,1),(2,2)\} \\ 2 \mapsto \{(1,2),(2,1)\} \\ 3 \mapsto \{(3,3),(4,4)\} \\ 4 \mapsto \{(3,4),(4,3)\} \end{cases}$$

satisfies equations (11.3). Furthermore:

 $:: \ * \mapsto \{1,3\}$

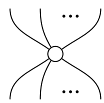
is one of the six states on the ‘spek-sphere’, so we can reach any other state by means of permutations, e.g.:



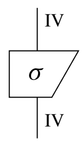
Since we can recover the ‘spek-sphere’ using just spiders and permutations, we can define the full process theory **spek** as follows.

Definition 11.2 The theory **spek** is the following subtheory of **relations**:

- The systems consist of n copies of IV.
- The processes are string diagrams made up of:
  - $\{1,2\}$-parity spiders:



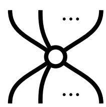
  - all permutations on a single system:



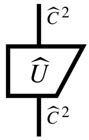
Now compare this with the following characterisation of **Clifford maps**.

Proposition 11.3 The theory **Clifford maps** can equivalently be defined as the following subtheory of **pure quantum maps**:

- The systems consist of n copies of $\widehat{\mathbb{C}^2}$.
- The processes are string diagrams made up of:
  - the Z quantum spiders:

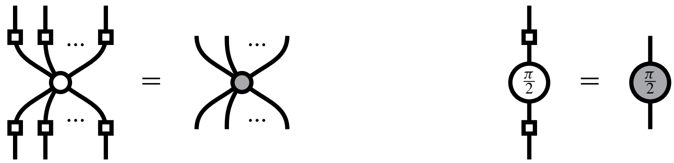


- all Clifford unitaries on a single system:



Proof Clearly any string diagram consisting of ○-spiders and Clifford unitaries is a Clifford diagram (cf. Definition 9.104), and hence gives a Clifford map. Conversely, the following are Clifford unitaries on a single system:

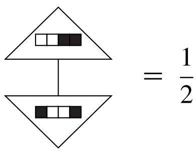
From these and ○-spiders, we can obtain:



which gives us all the pieces we need to construct an arbitrary Clifford diagram. □

Not too bad, eh?

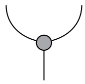
Remark 11.4 One apparent difference between **Clifford maps** and **spek** is that the numbers are different; namely, in **Clifford maps** these are the positive real numbers, while in **spek** these are the booleans. We say 'apparent', since this can be easily adjusted. A simple way to do this is to take 'equality up to a number' as equality in **Clifford maps**, so that only two non-equal numbers (namely, 0 and 1) remain. It's a bit more tedious, but also possible, to augment the numbers in **spek** with the positive real numbers and adjust the definition of 'wiring processes together' a bit, so that, for example:

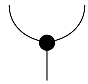


And the analogy between **Clifford maps** and **spek** goes even further. Just like we constructed the ○ spiders (a.k.a. spek-Z spiders), we can construct two other families of spiders (a.k.a. spek-X spiders and spek-Y spiders) that copy the other two orthogonal pairs of states.

Exercise 11.5 Using ○-spiders and permutations, define new spiders:

such that:





copy the spek-X and spek-Y states, respectively.

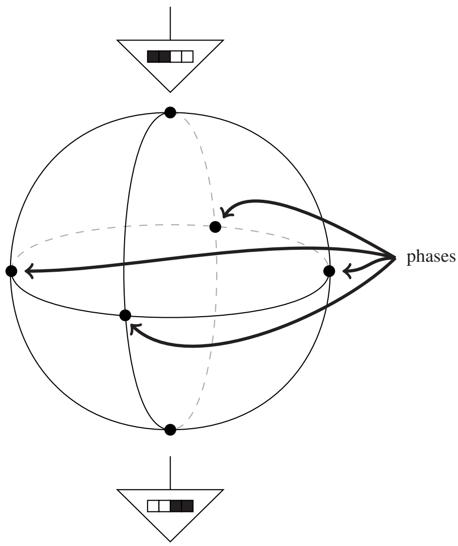
And of course these interact the way one would hope.

Exercise 11.6 Show that the spek ○-spiders and the spek ●-spiders are strongly complementary.

Let's see how far this analogy goes.

11.2.3 Phases in **spek**

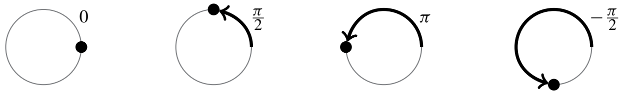
Since we laid out our six states on a sphere it is tempting to try to think of the points on the equator as phases:



and try to fit them together into a (commutative) phase group. In fact, as we'll soon see, this is possible. And furthermore, once we start decorating a pair of strongly complementary spiders with these phases, we get universality for **spek**, a Y-rule that relates two different ways to copy the spek-Y basis, and even a corresponding completeness theorem! How on Earth could there be any difference between **spek** and **Clifford maps**?

Those who know a bit of group theory (or who have read Section* 9.3.6) may have guessed the answer. There are precisely two commutative groups with four elements, and as it turns out, **Clifford maps** has one and **spek** has the other! In the case of **Clifford maps**, the phase group is $\mathbb{Z}_4$, the four-element cyclic group. That is, it has four elements $\{0, 1, 2, 3\}$, where the group-sum is addition modulo 4.

'Hang on!' you might say, 'I thought the phase group consisted of rotations around the Bloch sphere equator':

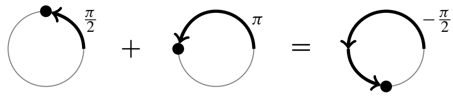


– and you'd be right! But just by giving these group elements different names:

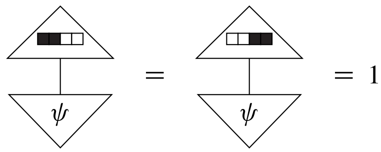
$$0 \leftrightarrow 0 \qquad 1 \leftrightarrow \frac{\pi}{2} \qquad 2 \leftrightarrow \pi \qquad 3 \leftrightarrow -\frac{\pi}{2}$$

we see that this is in fact the same group, e.g.:

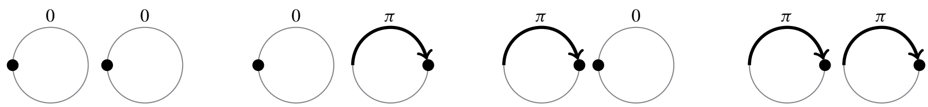
$1 + 2 = 3 \qquad \leftrightarrow$



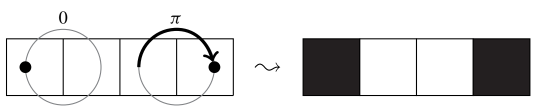
In order to be a phase for the spek-Z states, it must be the case that:



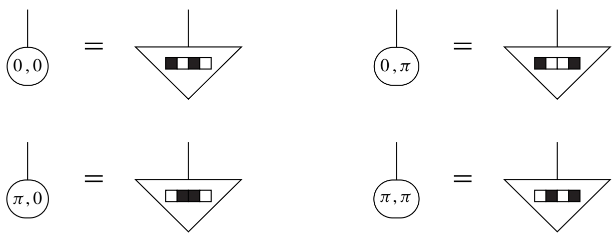
So, for a phase state, we must pick one element from $\{1, 2\}$ and one element from $\{3, 4\}$, which indeed gives the four remaining spek states. As with qubits, we can make this into a group by picturing little wheels. However, this time, we should picture two wheels rather than one, which can each be set to 0 or π:



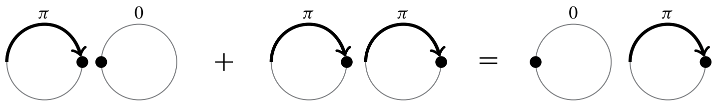
To see which state each of these corresponds to, just colour in the boxes where the black dots land:



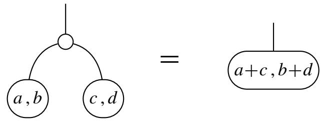
hence the four phase states are:



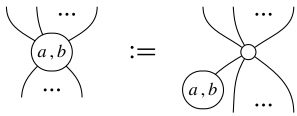
The group-sum is just adding the angles element-wise, e.g.:



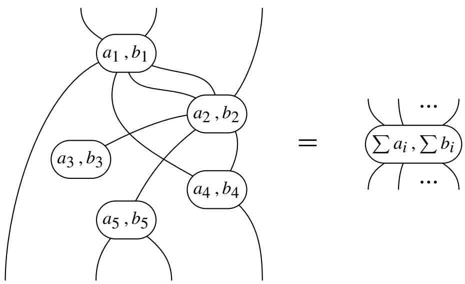
and indeed we can check that:



As in the quantum case, we can use these new phase states to decorate a new breed of spiders:

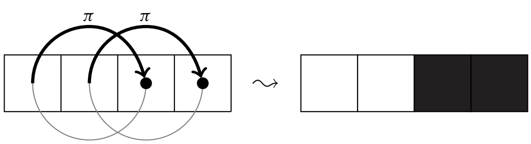


which satisfy 'spek spider fusion':

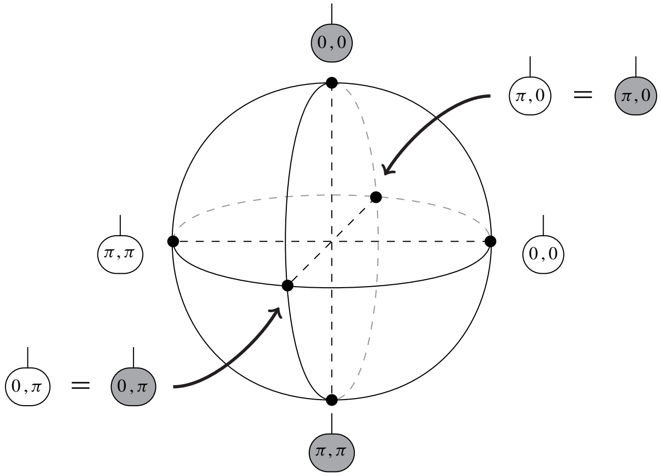


Whereas the phase group for **Clifford maps** is $\mathbb{Z}_4$, this new phase group is called $\mathbb{Z}_2 \times \mathbb{Z}_2$. Instead of a four-element group $\{0, 1, 2, 3\}$ where we do addition modulo 4, this is a pair of identical two-element groups $\{0, 1\}$ where we do addition modulo 2.

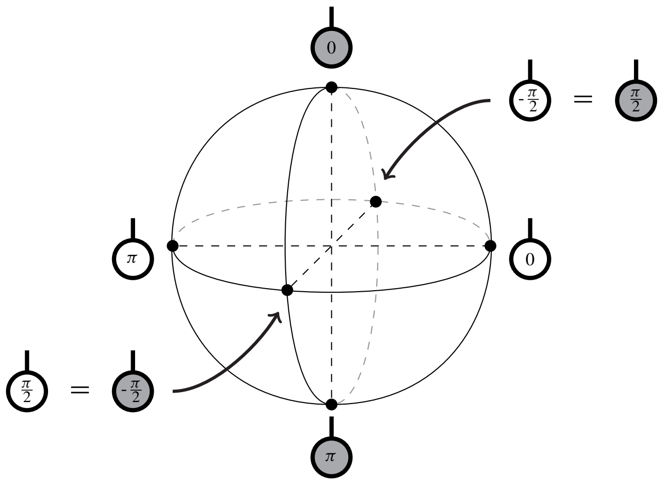
To form phase states for the spek-X states, we should choose one element from $\{1, 3\}$ and one element from $\{2, 4\}$. To get this from an element of the phase group, we just realign the little wheels accordingly:



which gives us enough phase states to fill up the spek sphere:



Let's just have a quick peek at the Bloch sphere again, shall we?

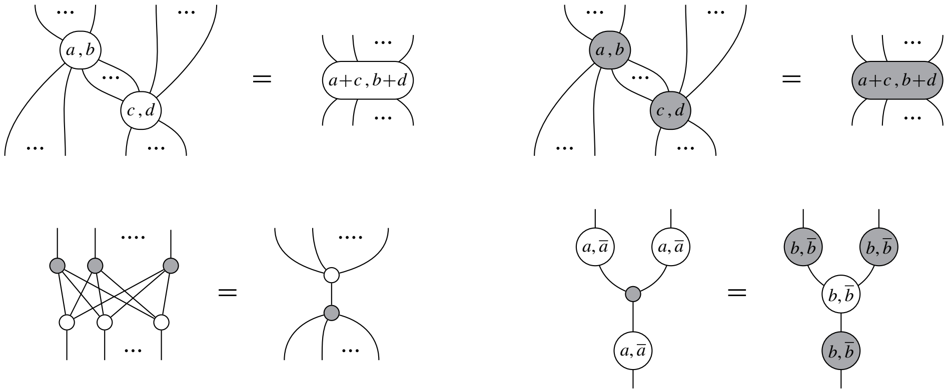


Wow! Like brothers from another mother! It's really starting to look like the only difference between these two theories is the phase group. In fact, this is indeed the case. But what precisely does it mean to 'change the phase group' of a whole process theory? We can make this precise with the help of our old friend, the ZX-calculus.

11.2.4 ZX-Calculus for spek

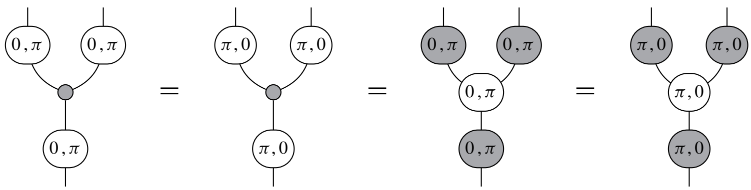
We now know that **spek** admits a phase group and a version of phase spider fusion. We furthermore know that ○ and ● are strongly complementary. Hence, we have nearly established a fully fledged ZX-calculus for **spek**. The only thing missing is the Y-rule, which we will add now.

Definition 11.7 The *spek ZX-calculus* consists of the following rules:

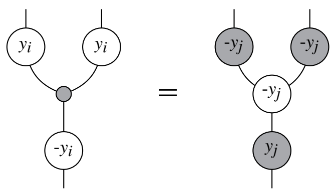


where $a, b, c, d \in \{0, \pi\}$ and $\overline{a}$ is shorthand for $\pi + a$.

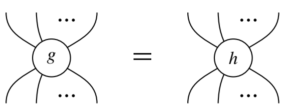
This definition is almost identical to the definition of the ZX-calculus for **Clifford maps**. Even the 'extended' Y-rule, which in the case of **spek** relates four variations of the Y-copy, rather than two:



is in fact perfectly analogous to the Y-rule for **Clifford maps**. That is, they both arise, for chosen elements y_1, y_2 of the phase group, as:



for all $i, j \in \{1, 2\}$. If we assume $y_1 \neq y_2$ and that spider decorations are unique, i.e.:

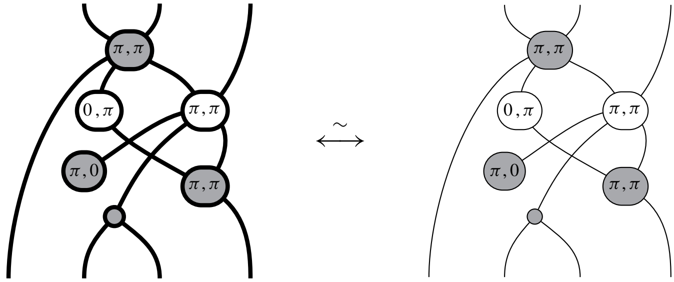


$$\Longrightarrow \qquad g = h$$

then y_1 and y_2 are already uniquely fixed by the calculus (up to possibly renaming some elements of the phase group). For **Clifford maps**, they must be $y_1 := \frac{\pi}{2}$ and $y_2 := -\frac{\pi}{2}$ and for **spek**, $y_1 := (0, \pi)$ and $y_2 := (\pi, 0)$.

Exercise* 11.8 In **Clifford maps**, the extra variations on the Y-rule are redundant. Are they indeed necessary in the case of **spek**?

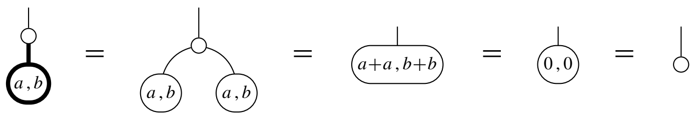
Example 11.9 Since doubling the processes in any process theory just eliminates global phases (cf. Remark 6.19) and the only global phase in **spek** is 1, doubling the processes in **spek** just gives **spek** again:



However, doubling enables us to encode measurements:

whose outcomes we can compute using the phase group. For example, if we ○-measure each of two spek-Z states, we get:

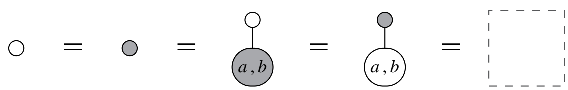
which stand for 'definitely in $\{0, 1\}$' and 'definitely in $\{2, 3\}$', respectively. The other states on the spek sphere are ○-phase states, so if we ○-measure them we get:



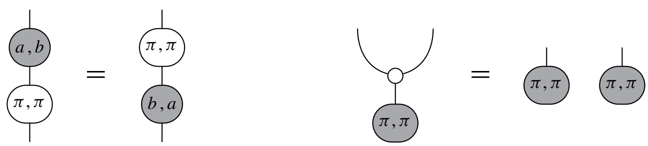
which stands for 'I have no idea'.

Just as with the ZX-calculus for **Clifford maps**, the first thing we want to do is derive some convenient rules.

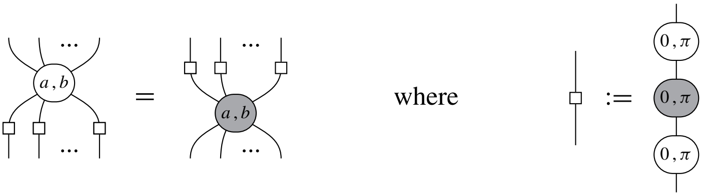
Exercise 11.10 Assuming the analogous 'little rules' to equation (9.107):



show that the spek ZX-calculus obeys its own versions of the 'dodo rules' derived in Section 9.4.3:



and the colour-change rule:



Like its Clifford cousin, the following is also true for the spek ZX-calculus.

Theorem 11.11 The spek ZX-calculus is complete for **spek**.

In fact, the proof (which we won't go into here) works in much the same way as the proof for **Clifford maps**. Namely, the processes in **spek** admit a natural notion of *graph form*, into which any process can be translated. Then, using the spek version of the local complementation rule (9.129), it is possible to translate two graph-form diagrams into one another if and only if they are equal as processes in **spek**.

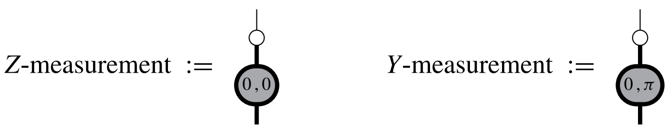
That means, just as in the case of **Clifford maps**, instead of defining **spek** concretely, we could just as well define it as a theory whose processes are diagrams of phase spiders, modulo the equations in Definition 11.7. Hence we have the following theorem.

Theorem 11.12 The only difference between **Clifford maps** and **spek** is the choice of phase group: $\mathbb{Z}_4$ versus $\mathbb{Z}_2 \times \mathbb{Z}_2$.

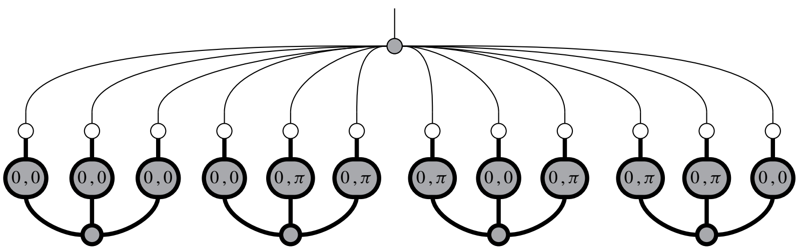
So, is this one little difference such a big deal?

11.2.5 Non-locality in spek*?*

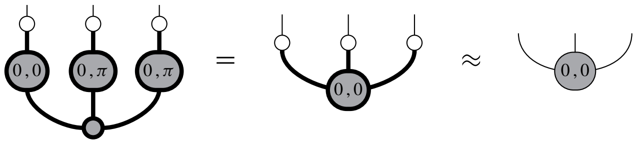
Is this difference in phase groups a big deal? Yes, it is! Let's try to reproduce the GHZ-Mermin scenario from Section 11.1.2 in **spek** by doubling and fixing some measurements (cf. Example 11.9):



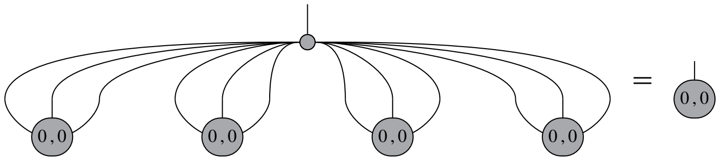
Previously, a local-realistic theory predicted that the overall parity:



should be even, i.e. a ● phase of 0, in **spek** a.k.a. $(0,0)$. Since in **Clifford maps** we obtained a ● phase of π, quantum theory is non-local. However, since for **spek** the phase group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, all pairs of $(0,\pi)$-phases all cancel out, e.g.:

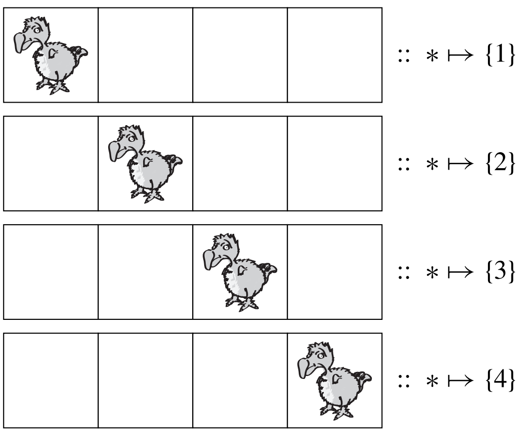


So we obtain:

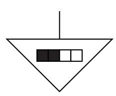


and the contradiction vanishes! In fact, a non-locality argument is doomed in **spek** no matter what choice of measurements we make. This is because **spek** is, by construction, a locally realistic theory!

To see this, all we need to do is think about what the states in **spek** actually mean. As we saw in Section 3.4.1, states in **relations** represent non-determinism. That is, the set IV can be thought of as a (classical) system, which has one of four possible states. For instance, it could be a collection of four boxes, where Dave is hiding in exactly one box:



Since this is representing some actual state of affairs, these are called *ontic* states. On the other hand, it could be that we don't know exactly where Dave is, but we do know he's in one of the first two boxes:



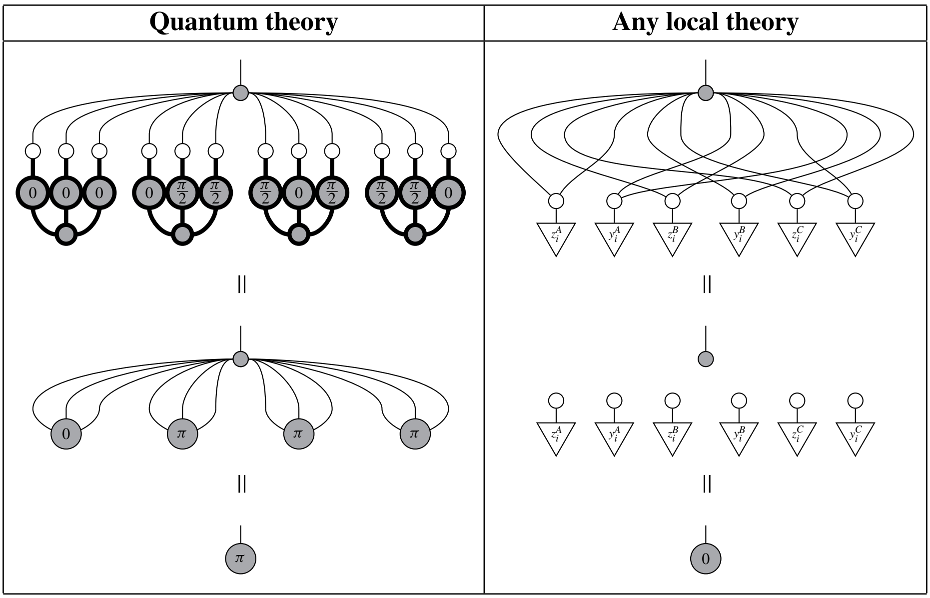
Since this doesn't represent the real state of the system, but merely our knowledge about that system, this is called an *epistemic* state.

The crucial thing about **spek** is we put certain limitations on which epistemic states are allowed, so we never have perfect knowledge about which box Dave is in. Out of this comes many seemingly quantum features, such as (strong) complementarity, unseparability of systems, and something that looks a whole lot like the Bloch sphere. But, ultimately, this theory was designed from the start to always admit a refinement into a local realistic theory, by simply admitting the underlying ontic states as hidden variables.

Exercise* 11.13 Transform the ZX-calculus into a graphical calculus for a quantum system other than qubits. What are the theories that one can obtain by changing the phase group?

11.3 Summary: What to Remember

1. Quantum theory is not a *local-realistic theory*; that is, any refinement of quantum theory in which all measurement outcomes have some common cause in the past must violate locality.
2. This fact can be established by drawing a contradiction diagrammatically:



3. In the theory of **relations** there are complementary spiders even on the two-element set $\mathbb{B}$.
4. Spekkens' toy theory, a theory that closely resembles qubit quantum theory, can be formulated as a process theory **spek**, which is a subtheory of **relations**. It can also be formulated entirely diagrammatically, as a modification of ZX-calculus simply by replacing the group $\mathbb{Z}_4$ by the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.
5. Quantum non-locality, and the GHZ-Mermin scenario in particular, is tightly intertwined with the fact that the phase group for qubits includes $\mathbb{Z}_4$. In contrast, the phase group in Spekkens' toy theory does <u>not</u> contain $\mathbb{Z}_4$.

6. More generally, studying **quantum theory** within a wider space of process theories teaches us which ingredients of quantum theory cause its remarkable features.

11.4 Historical Notes and References

The GHZ-Mermin argument was initially proposed in a somewhat more complicated form by Greenberger et al. (1990) and was put in its present form by Mermin (1990). A first overly complicated diagrammatic proof was produced by Coecke et al. (2012). The same authors did sober up not too long after and produced the proof presented here (Coecke et al., 2016). Generalisations of this argument can be found in Gogioso and Zeng (2015).

The first non-locality proof was due to John Bell (1964), and Einstein's related concern, which instigated the discussions leading to Bell's theorem, appeared in (Einstein et al., 1935; Einstein, 1936). The first experimental verification was due to Aspect et al. (1981), but only recently an experiment took place that is widely accepted to be 'loophole-free' (Hensen et al., 2015). David Bohm's hidden variable model was first published in (Bohm, 1952a,b).

The fact that one can represent complementarity in **relations** was observed by Coecke and Edwards (2011), where also **spek** was first presented as a subtheory of **relations**. Pavlovic (2009) classified all spiders in **relations**, and all pairs of strongly complementary spiders were classified by Evans et al. (2009) and, independently, also by Edwards. For a bestiary of sets and relations, see Gogioso (2015a).

Spekkens presented his toy theory in Spekkens (2007). An earlier very similar but less developed toy theory was presented by Hardy (1999). Rather than a full presentation of a theory, Spekkens gave a recipe to produce all states and processes, without evidence that a consistent theory would emerge in this manner. Key to that recipe was a so-called knowledge-balance principle, which restricted the amount of knowledge one could ever have about a system. That Spekkens' recipe produced a consistent theory was established in Coecke and Edwards (2012), by relating Spekkens' recipe to **spek**. That paper also contained a picture of process-theoretic graffiti under an Oxford bridge:

The two camps joined forces to figure out that it is the phase group that captures the true difference between quantum theory and the toy theory (Coecke et al., 2011b). This body of work then became the content of the DPhil thesis of Edwards (2009), which also contains some further elaborations. Edwards' current whereabouts are unknown to us.

Backens and Nabi Duman (2015) realised that one could adjust the ZX-calculus in order to obtain **spek** and also provided a corresponding completeness theorem, hence establishing an entirely diagrammatic presentation of Spekkens' toy theory.

The quote at the beginning of this chapter is taken from Hardy and Spekkens (2010). See also the discussion of Mermin's quote in Section 1.3.