8

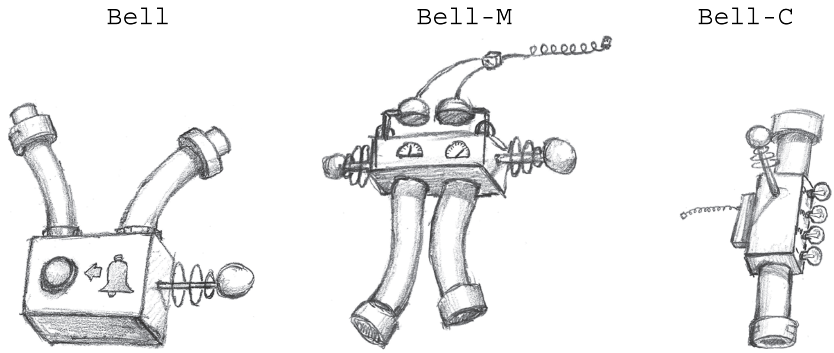
Picturing Classical-Quantum Processes

> ... she sprinkled her with the juice of Hecate's herb, and immediately at the touch of this dark poison, Arachne's hair fell out. With it went her nose and ears, her head shrank to the smallest size, and her whole body became tiny. Her slender fingers stuck to her sides as legs, the rest is belly, from which she still spins a thread, and, as a spider, weaves her ancient web.
>
> – *Ovid, The Metamorphoses, 8 AD*

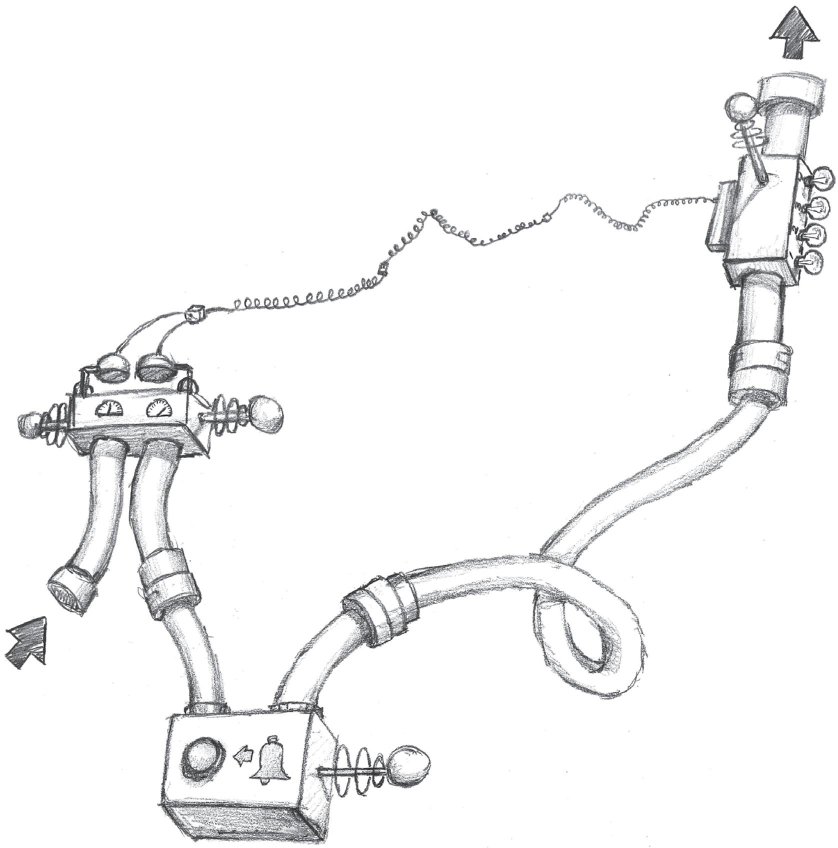
Most quantum protocols rely on the interplay between quantum systems and classical data. For instance, measurements extract classical data from a quantum system, whereas controlled operations use classical data to affect a quantum system. Moreover, given that truth is in the eye of the beholder, we want to understand quantum theory relative to our perception of reality, which is classical, and hence want to understand how the two relate. Somewhat surprisingly, it turns out to be much easier to represent the classical world relative to quantum processes, than the other way around.

One way to get a handle on this interaction is to express as much of it as possible in a purely diagrammatic form. Previously, we have drawn diagrams of quantum processes, then used some sort of 'external' means of describing the classical data flow, i.e. with indices and brackets, which can't really be plugged together like pieces of a diagram. Even worse, in most standard textbooks, classical data is not even part of the actual formalism, but is described in words.

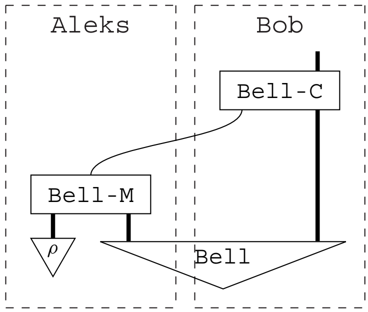
Rather than describing this interplay of quantum systems and classical data using lots of 'blah blah blah', or a cross-breed between diagrams and symbols, can we instead just give a diagram of all of the devices involved and how they are wired together? For example, suppose we have a device '`Bell`' that prepares Bell states, another device '`Bell-M`' that performs Bell measurements, and a third device '`Bell-C`' that does Bell corrections, depicted very realistically as follows:



Now, suppose we want to describe to a technician how to wire these devices together to do teleportation:

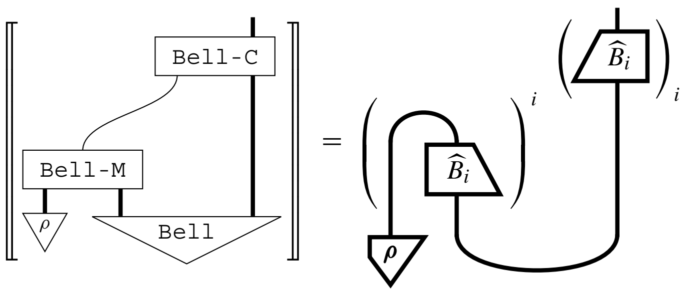


We could describe this using a *specification language*, that is, a diagrammatic language where the boxes correspond to devices and the wires correspond to literally ‘wiring up’ the devices:



where we now distinguish quantum wires and classical wires.

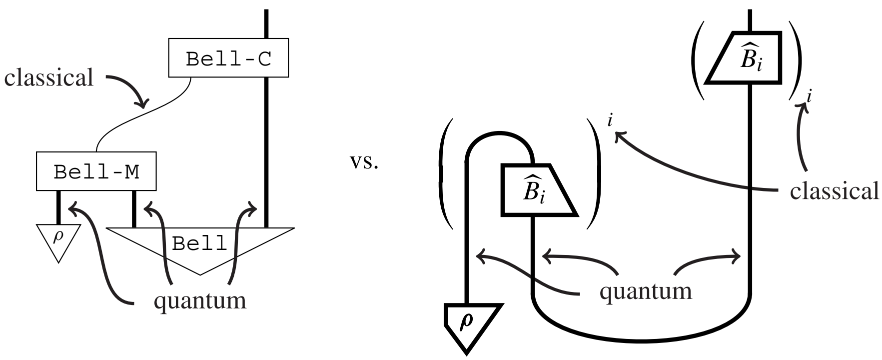
Every well-formed specification of a protocol, that is, a diagram in the specification language, should have a corresponding mathematical model, which the theorist can use to predict what a protocol will actually do. We can picture this mapping between specification and model via *interpretation brackets* $[\![\ ]\!]$. Then, for teleportation we have:



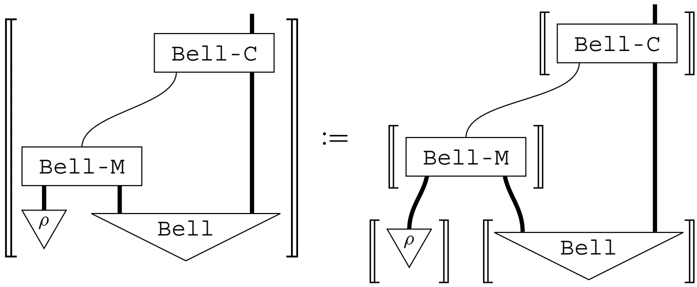
(8.1)

So, the LHS describes how the technician sets up the equipment, whereas the RHS is what the theorist predicts the equipment will do. One could say the real predictive power of quantum theory amounts to giving a definition of $[\![-]\!]$.

As we have noted before, there is a slight disconnect here, since quantum systems are wires, but classical systems are something else entirely:

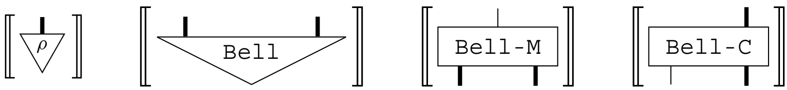


What we would really like is to treat classical and quantum systems both as wires, at which point the recipe for interpreting a specification diagram amounts to simply interpreting each of its components:



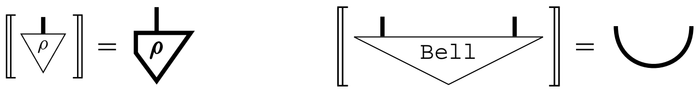
(8.2)

That is, modelling the overall process amounts to modelling these four smaller processes:



and then composing them together.

We already know how to do this for purely quantum boxes and wires:

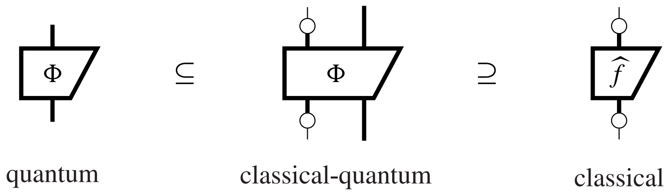


so to complete the picture, we need to figure out how to interpret boxes that have some classical wires as well:

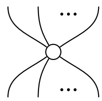
We have already benefitted from the fact that doubling makes space for some extra processes, namely impure quantum maps. Now we will see how it makes space for some extra, classical systems as well. By extending **quantum maps** with classical wires, we obtain a new process theory of **classical-quantum maps**, or **cq-maps** for short. After imposing causality, we obtain the full theory of **quantum processes**, which is really what this whole book is about. The blanks above can then be filled in as follows:

Just as in the specification language, the thick wires represent quantum systems, whereas the thin wires represent classical systems. A state of a quantum system is the same as before, but as we will see shortly, a state of a classical system is a probability distribution.

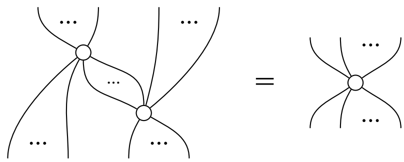
Thus, cq-maps simultaneously generalise quantum maps and the natural notion of mappings between classical probability distributions (i.e. *stochastic maps*):



As classical data can be copied and deleted, it is natural to allow classical wires to split and merge in various ways. These points where multiple classical wires meet are called *spiders*:



Spiders can be seen as a generalisation of caps and cups, whose behaviour is dictated by a single 'fusion' rule:



While a wire is something that connects two ends together, a spider is a generalisation that connects many ends together. Consequently, the spider-fusion rule, which subsumes the yanking equations of caps and cups, is the embodiment of the concept that 'only connectivity matters'.

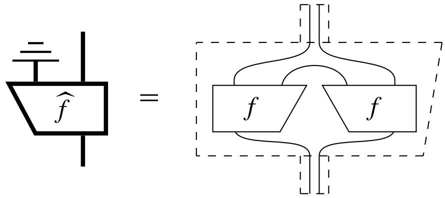
While we introduce spiders in order to reason about classical data, they will also be used to construct new quantum maps and classical-quantum hybrids. In fact, these spiders are so powerful that we will be able to give the entire story of quantum theory in terms of them. Now-familiar concepts such as *measurement*, *classical control*, and *mixedness*, as well as new notions such as *classical copying*, (non-pure state) *entanglement*, and *decoherence* will all be expressed in terms of these new kinds of processes. Indeed, from this chapter onwards, this book will be full of spiders!

8.1 Classical Systems as Wires

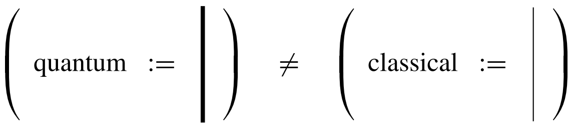
In this section we show how classical systems, just like quantum systems, can be represented in terms of linear maps. Thus, rather than needing to treat classical inputs and outputs of a quantum process 'externally' with indices, we can simply express them as wires that can be plugged together. This allows us to give a simple presentation for the theory of **quantum processes** and recover the elegance of the original (sum-free) version of the causality postulate.

8.1.1 Double versus Single Wires

The theory of **quantum maps** can be seen as restricting the theory of **linear maps** such that we only allow maps of a very particular form:



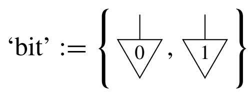
We now wish to construct a theory that includes maps where quantum and classical systems interact with each other. Quantum systems are of a fundamentally different type from classical systems, so there should be two distinct kinds of wires. Since quantum systems are already represented by thick doubled wires, to introduce a new kind of system, we simply put some single wires back in:



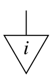
And as we will soon see, this turns out to be a perfect fit for representing classical systems. So:

$$\frac{\text{classical}}{\text{quantum}} = \frac{\text{thin/single wires}}{\text{thick/double wires}}$$

We encode classical data on a single wire by representing classical values as basis states of a fixed ONB. So for an ONB on a classical system of dimension D, this means that the corresponding classical data has D possible values. For example, a bit corresponds to an ONB on a two-dimensional system, i.e.:

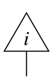


More precisely, we interpret the states of this ONB as:

- 

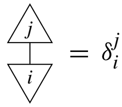
:= 'providing classical value *i*'

and for the corresponding effects we then have:

- 

:= 'testing for classical value *i*'

Orthonormality:

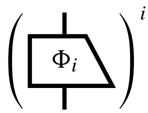


(8.3)

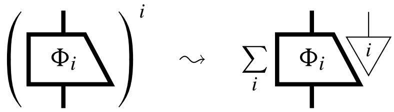
then makes perfect sense in light of this interpretation. We obtain probability 0 (i.e. 'impossible') when we are testing for the value $j \neq i$ on the value i, and we obtain probability 1 (i.e. 'certain') when we are testing for the value i on the value i.

Remark 8.1 We'll use self-conjugate ONBs for classical data, since conjugation has no classical counterpart; e.g. 'conjugating a bit' is meaningless.

When one applies a quantum process such as this one:



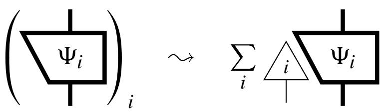
two things happen. Of course, one of the branches Φ_i gets applied to the quantum system, but also a classical value i pops out, telling us which branch happened. To capture this with a classical wire, we can represent a quantum process like this:



(8.4)

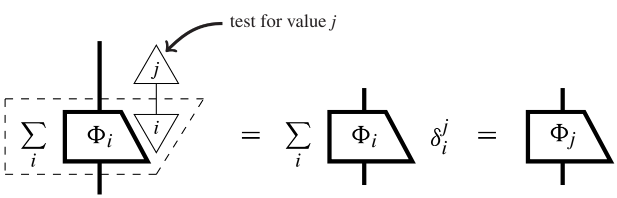
that is, we make the creation of classical output explicit and represent the whole process as one big linear map.

Similarly, if the outcomes of quantum processes are controlled by a classical input, we can rely on the i-effects to represent the fact that it tests for an input value:

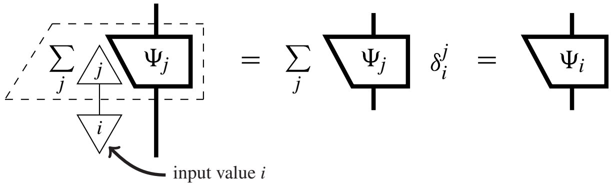


(8.5)

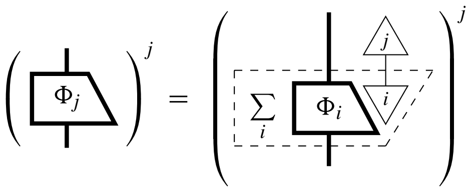
It is easy to see that these give us faithful representations of the respective quantum processes. We can still access the individual branches of this process by composing it with the appropriate classical effect, i.e. subjecting it to the appropriate test:



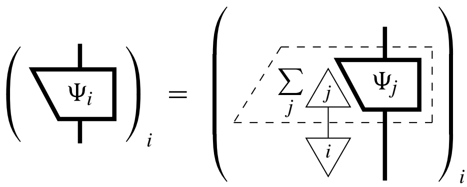
and for a controlled process, we can access each component by providing the corresponding input:



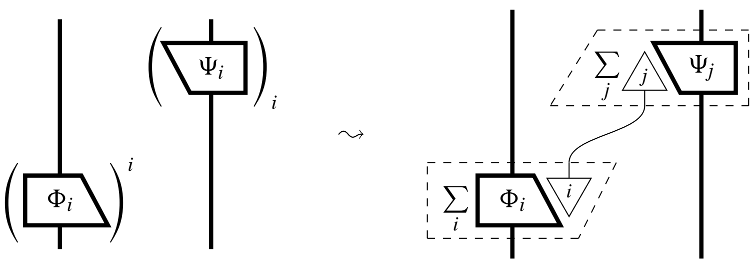
So we can indeed reconstruct (8.4) via:



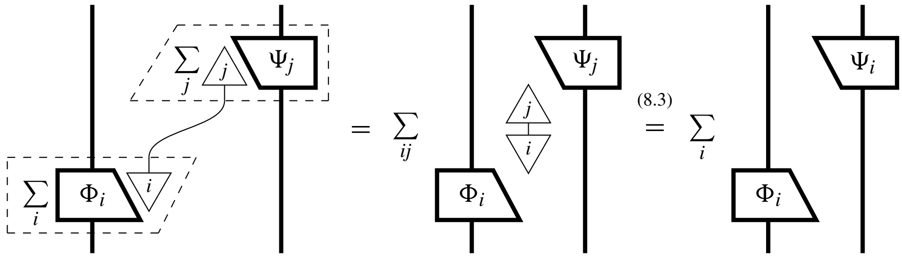
and (8.5) as:



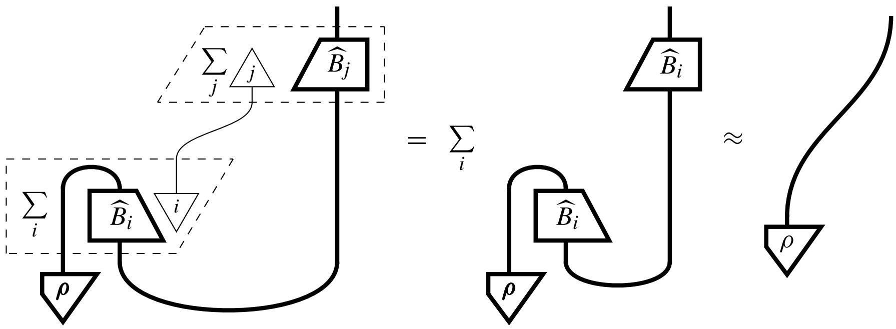
What's more, we can now connect the i-state to the i-effect by a wire to indicate the actual classical communication:



Note how, in the diagram on the right, the indices i and j only occur locally. In other words, the two linear maps in the dashed boxes are completely independent processes. However, by the presence of a classical wire connecting the two, i and j are forced to become the same, thanks to orthonormality:



Teleportation now becomes:

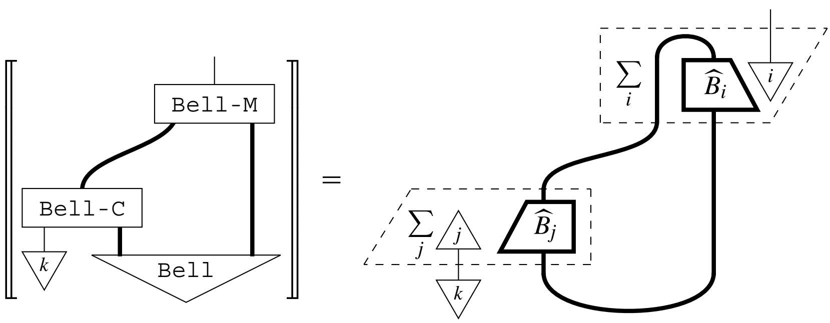


In particular, the entire picture is now a linear map of a certain kind, which we will define shortly. Clearly, such maps properly generalise quantum maps, since now there are also classical systems involved. So we have now been able to eliminate syntactic garbage that lived 'outside' **linear maps** and return to working just with diagrams of processes, which has some big advantages.

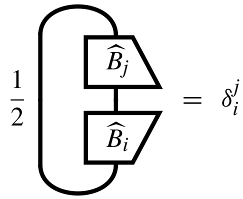
8.1.2 Example: Dense Coding

Now that classical systems have their own wires, we can consider protocols with classical inputs and classical outputs. While teleportation relied on classical communication (with the help of a Bell state) to send a quantum state, we now ask the converse question: can we use a quantum state to send a classical one? Also, do we gain anything by doing so?

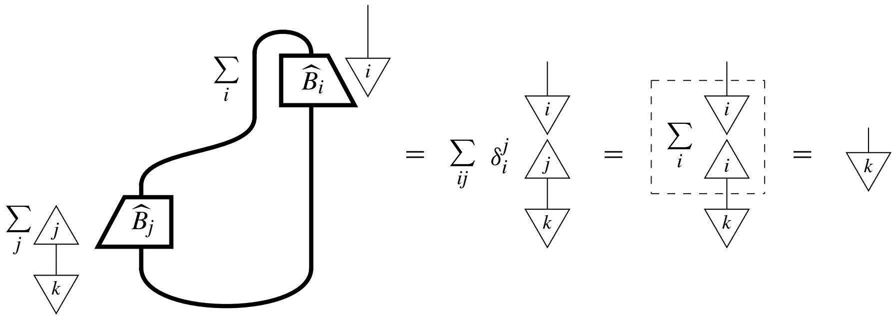
Of course, the magic of teleportation wouldn't have been possible without the Bell state, so we will assume that here too. This yields the following specification diagram and corresponding interpretation:



Then, by orthonormality of the Bell basis:



we obtain:



So what does this protocol achieve? When viewed as a communication protocol, communicating classical data by means of quantum systems seems a bit heavy handed. However, the quantum systems we are dealing with (i.e. the bold wires in the pictures) are two dimensional. On the other hand, there are four Bell states, so the measurement generates one of four values as classical output (i.e. the thin wires in the pictures). So, while Aleks only sends one qubit, he succeeds in communicating two classical bits. This result straightforwardly generalises to D-dimensional bold wires, which enable one to communicate D^2-valued classical data.

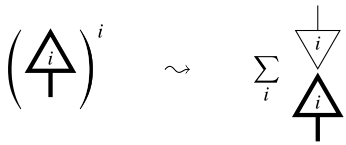
Remark 8.2 It is sometimes said that teleportation and dense coding are dual. This is because there is an exchange of roles played by the classical and the quantum channels (while the entangled state stays quantum of course). However, it is worth noting that the two aspects of the Bell matrices – the fact that (i) they form unitaries and (ii) their associated states form an ONB – play different roles in the two protocols. Whereas teleportation most obviously depends on (i), the key to dense coding is (ii). What is true is that if one possesses an entangled state in advance, then:

- a 'D-dimensional quantum channel' and
- a 'D^2-dimensional classical channel'

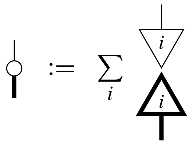
become equivalent, in that teleportation allows us to convert the classical channel into a quantum channel, while dense coding allows us to convert the quantum channel into a classical channel.

8.1.3 Measurement and Encoding

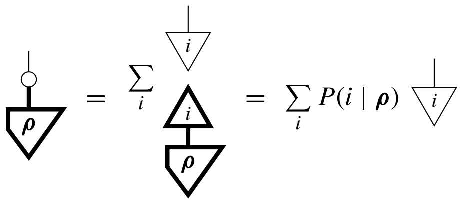
We now convert the quantum process for an ONB measurement into this new representation using classical wires:



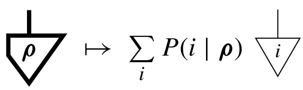
This linear map is so important that we give it its own notation:



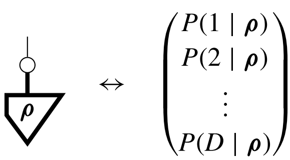
So what does this map do? Let's apply it to an arbitrary quantum state:



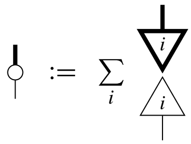
Unsurprisingly, these numbers are exactly the probabilities for the ONB measurement according to the Born rule. So the linear map representing an ONB-measurement sends a quantum state to a probability distribution:



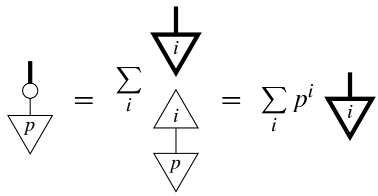
over all of the possible measurement outcomes. Seen as a matrix, this state is:



Now that an ONB measurement is represented by a linear map, which from now on we'll call *measure*, we can also consider its adjoint:

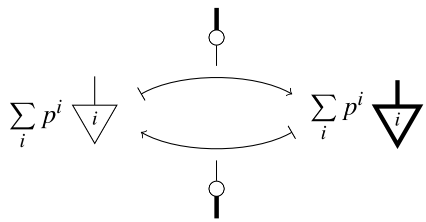


Applying this linear map to an arbitrary probability distribution, we obtain:

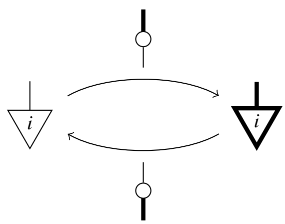


that is, we obtain the encoding of a probability distribution as a quantum state given in Proposition 6.74. For that reason, we'll call this linear map *encode*.

Measuring undoes this operation. Hence, these two processes interconvert the two representations of a classical probability distribution:



In particular, they interconvert classical values and their corresponding quantum states:

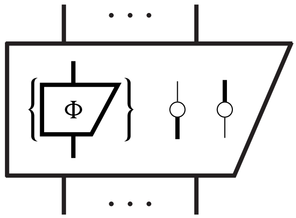


(8.6)

8.1.4 Classical-Quantum Maps

Though they may look special, it turns out that these measure/encode processes are in fact all we need to add to get a full theory of classical-quantum maps. Just as we defined **quantum maps** as the process theory where we took **pure quantum maps** and added one new ingredient (discard), we can define classical-quantum maps by adding two new ingredients to **quantum maps**.

Definition 8.3 A classical-quantum map (cq-map) is a linear map obtained by composing quantum maps, encoding, and measurement:



(8.7)

We call the associated process theory **cq-maps**.

Before we do anything else, we first show the following theorem.

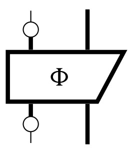
Theorem 8.4 The theory of **cq-maps** admits string diagrams.

Proof Since measure is the adjoint of encode, the theory of **cq-maps** inherits adjoints from **linear maps**. We saw back in Chapter 6 how to build cups and caps for the quantum systems, so we only need to show them for the classical systems. These arise as follows:

In the next section we will see a very easy way to prove such equations, so for now we leave these (still pretty easy) proofs as an exercise. □

Also, just like what we established in Proposition 6.46 for quantum maps, we can always put cq-maps into a 'normal form'.

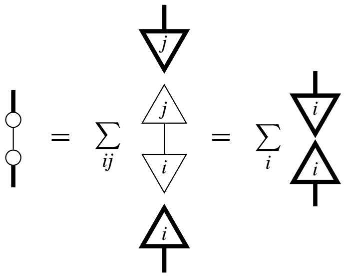
Proposition 8.5 All cq-maps are of the form:



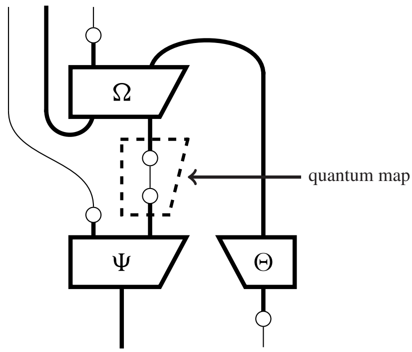
(8.8)

up to reordering some input/output wires.

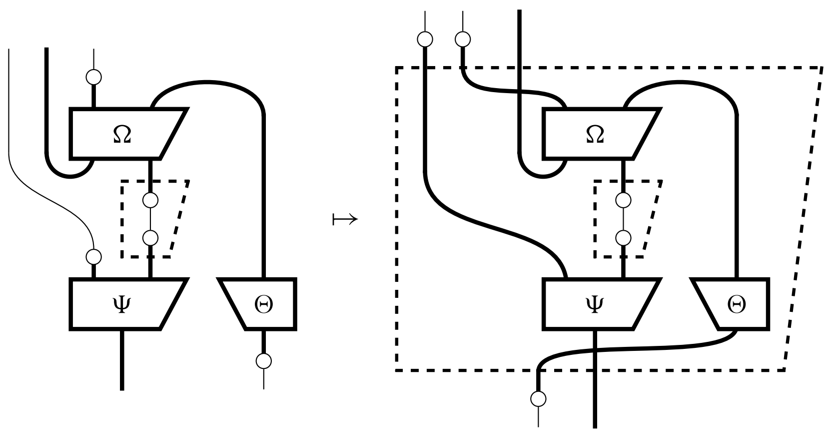
Proof The proof goes much like that of Proposition 6.46 giving a standard form for quantum maps. First, note that any time the classical wires from a measure and an encode connect, this results in a quantum map:



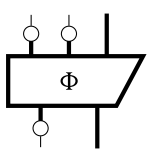
Hence any cq-map can be written as a quantum map with some measures on its outputs and some encodes on its inputs:



Then, by reordering some inputs/outputs, we can bring all of the measure/encode processes together:

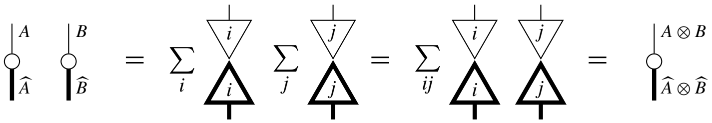


which gives a process of the form:



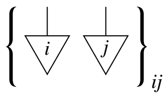
(8.9)

Just as with discarding, multiple measure processes can be combined into one process on a larger system:



(8.10)

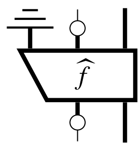
where the RHS is the measure process for the ONB:



Encode processes can be combined similarly. Thus, combining the measure/encode processes in (8.9) yields a process of the form (8.8). □

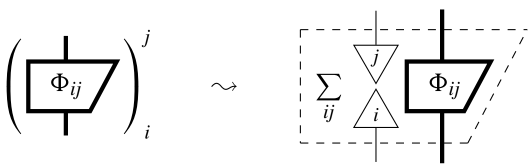
Then, just by purifying the quantum map Φ above, we have:

Corollary 8.6 Every cq-map is of the form:



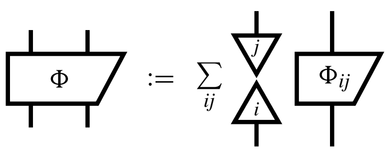
Remark 8.7 In this chapter, we'll assume that every classical system carries its own, 'preferred' ONB, which we'll always write using white triangles. Hence, the ONBs on the systems A and B in equation (8.10) will be different if $A \neq B$. In Chapter 9, where we study interaction between different ONBs for the same system, we will remove this assumption, but we will then need to be a bit careful with the classical wires. We'll explain this in Section 9.2.

So what does Definition 8.3 have to do with our efforts from the previous sections? Before, we showed how to express quantum processes as single linear maps:



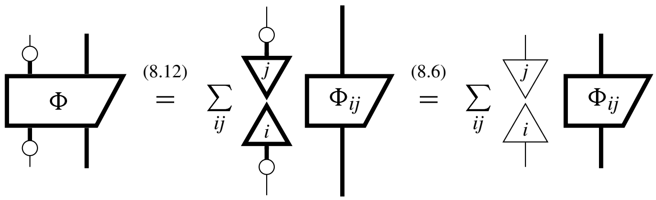
(8.11)

We can now show that such linear maps are indeed cq-maps. First roll all of the Φ_{ij} together with doubled ONB states/effects:

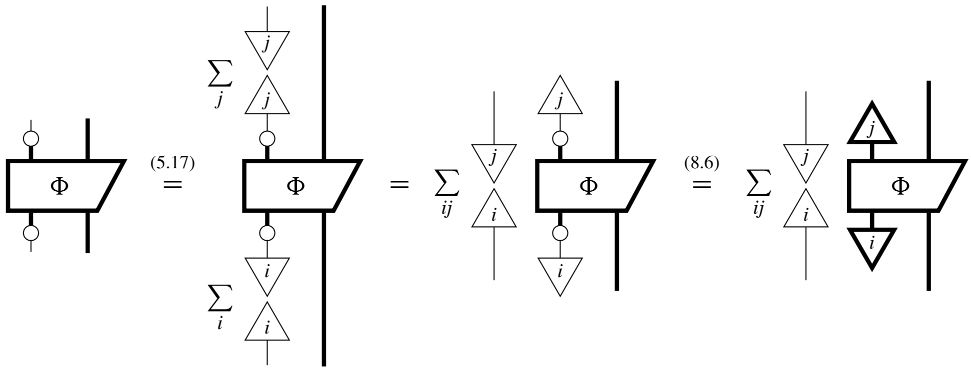


(8.12)

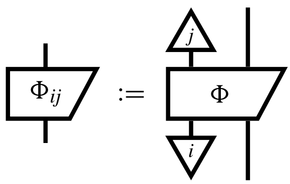
Since this is a sum of quantum maps, by Theorem 6.67, Φ is itself a quantum map. So, the linear map (8.11) is indeed a cq-map:



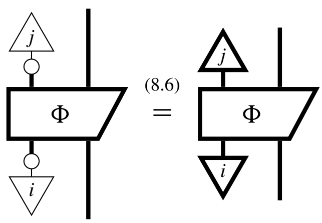
Conversely, any cq-map is of the form (8.11):



where:



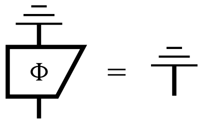
So for a generic cq-map, the quantum maps Φ_{ij} are precisely the branches of that map, which have been selected via classical inputs/outputs:



However, Definition 6.100 of quantum processes included one important extra ingredient, which so far cq-maps are lacking. Namely, there is nothing to guarantee that these branches, taken together, satisfy the causality postulate (6.77). To treat this in an elegant, sum-free fashion, we will require one more diagrammatic ingredient.

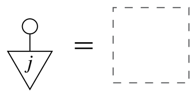
8.1.5 Deleting and Causality

Back before non-determinism came into the picture, we had a beautifully simple equation that captured causality:



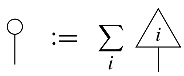
If we discard everything coming out of a process, we might as well just not do the process and discard everything coming in. To extend this to cq-maps, we just need to say what it means to 'discard' a classical system. For this, we will introduce a new process called *deleting*:

It is clear from the name what such a process should do to any classical value j:



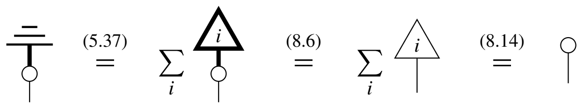
(8.13)

but then, of course, since all of these classical values together form an ONB this totally fixes deleting:



(8.14)

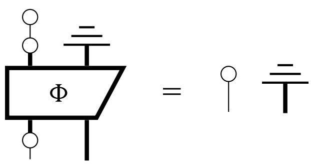
One might think that we need to put this into the theory of **cq-maps** by hand, but in fact, it's already there:



(8.15)

which of course makes perfect sense. If we encode some classical data as a quantum system, then discard the system, this is the same as just deleting the data. The following definition then pretty much writes itself.

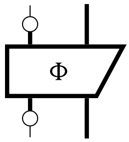
Definition 8.8 A cq-map is *causal* if:



(8.16)

And this is indeed equivalent to how we defined causality before.

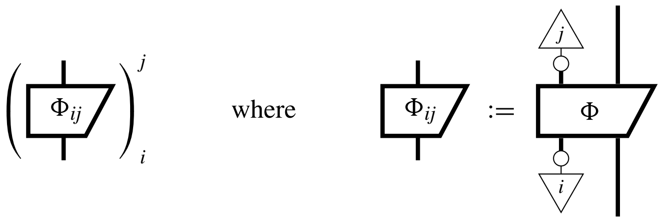
Proposition 8.9 For a cq-map:



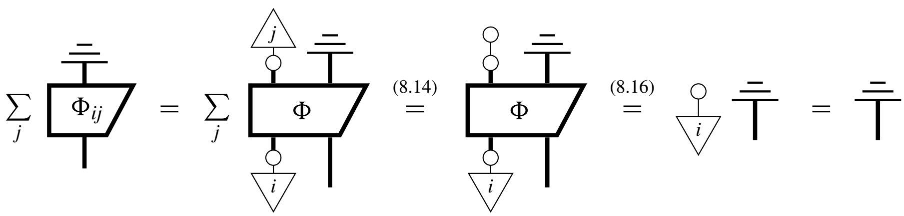
(8.17)

Definition 6.100 and Definition 8.8 for causality coincide.

Proof The cq-map (8.17) encodes a quantum process:

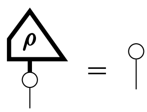


Assuming causality as in Definition 8.8, we obtain the following for all i:

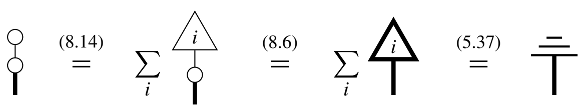


which is precisely the statement of causality from Definition 6.100. The proof of the converse proceeds similarly. □

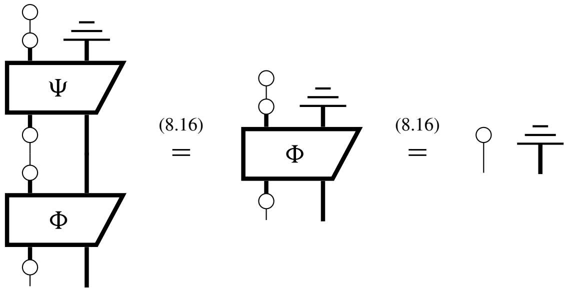
Causal quantum maps now arise as a special case of this definition. Just like with discarding for quantum systems, deleting is the unique causal effect for a classical system. That is, for any causal effect on a classical system, equation (8.16) reduces to:



Measure and encode are also causal. We already 'accidentally' showed encoding was causal in equation (8.15). The proof for measuring is very similar:



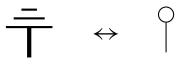
An important additional benefit of this new presentation of causality is that many proofs about causality can now be done in exactly the same way as the purely quantum case. For example, proving that the sequential composition of two causal cq-maps is again causal is simply:



More generally, it is therefore also easily seen that any circuit diagram of causal cq-maps is again a causal cq-map. So, at last we are ready to define the most important process theory in the book.

Definition 8.10 **Quantum processes** is the subtheory of **cq-maps** consisting of all causal cq-maps.

Since deleting is just the classical counterpart to discarding:



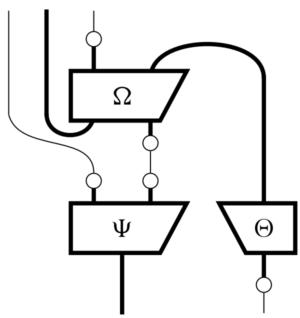
Definition 8.8 is just a minor update to our original slogan for causality:

If we discard/delete all of the quantum/classical outputs of a quantum process, it may as well have never happened.

Thus we have succeeded (as promised) in extending the interpretation of causality for quantum maps of Section 6.2.4, to quantum processes that may also involve classical inputs and outputs. So classical and quantum inputs, in the light of causality, are now on equal footing. Consequently, statements about causality for quantum maps transfer straightforwardly to quantum processes. For example, the results about non-signalling established in Section 6.3.2 now also apply to general quantum processes.

8.2 Classical Maps from Spiders

We now have a proper process theory of **quantum processes** capturing both classical and quantum systems as wires. This allows us to reason about these processes using diagrams:



Unfortunately, these diagrams contain some very specific linear maps, namely measure and encode, and in order to establish all equations between diagrams we will unavoidably have to use their explicit forms:

which involve ONB-states and sums. This is a bit of a pain.

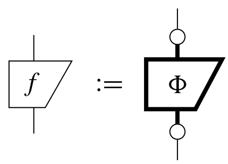
In this section and the next we will do better than this. We will establish measure and encode as purely diagrammatic entities that allow us to derive equations between diagrams of **quantum processes** without ever needing the explicit matrix forms of measure and encode again.

The key to doing so, it turns out, is to better understand what classical processes are. While we have built up a pretty good repertoire of quantum maps, we have still said fairly little about their classical counterparts. We have seen one classical map already (deleting), and we will see a few more of these before establishing that they all emerge as special cases in the next diagrammatic revolution in this book: the rule of the spiders!

8.2.1 Classical Maps

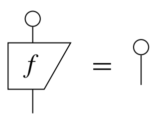
By just restricting Definition 8.3, we have the following definition.

Definition 8.11 A *classical map* is a cq-map with only classical inputs and classical outputs, that is, a linear map f of the form:

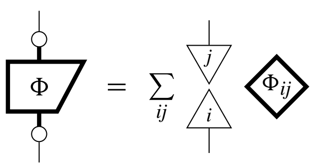


(8.18)

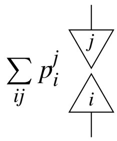
and a *classical process* is a classical map that is causal:



Writing out a classical map as a sum over its branches as in (8.11), we obtain:

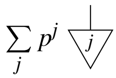


Of course, quantum maps with no inputs or outputs are just positive numbers, so a classical map is really nothing more than a matrix of positive numbers:



$$\leftrightarrow \quad \begin{pmatrix} p_1^1 & p_2^1 & \cdots & p_m^1 \\ p_1^2 & p_2^2 & \cdots & p_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ p_1^n & p_2^n & \cdots & p_m^n \end{pmatrix}$$

In particular, classical states are just vectors of positive real numbers:



$$\leftrightarrow \quad \begin{pmatrix} p^1 \\ p^2 \\ \vdots \\ p^n \end{pmatrix}$$

For classical processes, causality reduces to:

$$\forall i \; : \; \sum_j p_i^j \; = \; 1$$

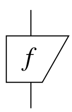
That is, classical processes have matrices with positive entries where each column sums to 1. For classical states causality becomes:

$$\sum_j p^j \; = \; 1$$

so causal classical states are exactly probability distributions!

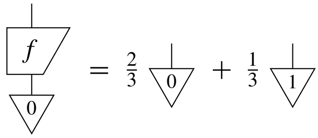
The matrices of classical processes are often referred to as *stochastic matrices*, and the linear maps themselves as *stochastic maps*. 'Stochastic' is essentially a synonym for 'random', and stochastic maps correspond to processes acting on probability distributions where there can be some element of randomness involved. Much like quantum processes are the most general maps that send causal quantum states to other causal quantum states, stochastic maps are the most general maps that send probability distributions to probability distributions.

Example 8.12 Imagine a process that takes in a classical bit, and with probability $1/3$ flips the bit (thus leaving the bit fixed with $2/3$ probability). We could describe this with the following stochastic map:

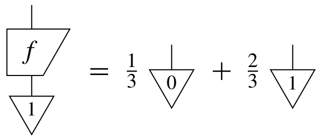


$$\leftrightarrow \quad \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{8.19}$$

If we input bit zero:



we get a bit 0 out with probability $\frac{2}{3}$ and a bit 1 with $\frac{1}{3}$. If we input bit 1, we get the opposite:

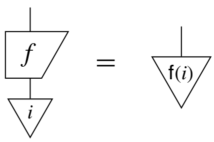


A deterministic classical process sends each ONB state to one (and only one) ONB state, and hence acts like a function on classical values. Thus, they take the following form.

Definition 8.13 A classical process f is called *deterministic* if there exists an underlying function:

$$\mathsf{f} : \{1, \ldots, m\} \to \{1, \ldots, n\}$$

such that:

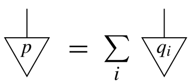


(8.20)

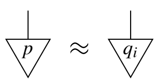
We will refer to deterministic classical processes as *function maps* for short. To see that any linear map described by (8.20) is automatically a classical process, it suffices to examine its matrix. Entries are all either 0 or 1, so of course they are all positive. Furthermore, there is precisely one 1 in each column, so causality follows immediately.

Restricting Definition 8.13 to states, we conclude that *deterministic classical states* are just ONB states, a.k.a. point distributions (cf. Section 5.1.4). Deterministic states are the classical analogue to pure quantum states, since they don't arise from (non-trivial) probabilistic mixing.

Exercise 8.14 Show that when:



p is a deterministic classical state if and only if for all i we have:



Exercise 8.15 Recall from Remark 8.7 that all of the notions in this section depend on a particular choice of ONB for each classical type. Indeed, if we express the matrix of a classical map or state in a different ONB, typically the entries will no longer be positive. Find an ONB in which the matrix of the stochastic map (8.19) has non-positive entries.

We'll now have a look at some very special classical processes.

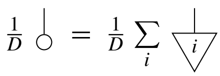
8.2.2 Copying and Deleting

In addition to deleting, which we have seen already, we can also copy classical data. What may come as a surprise is that some discarding-related features of quantum processes fail to have a classical deleting-related counterpart, most notably purification. In other words, the ability to purify is a characteristic feature of quantum processes. On the other hand, copying has no counterpart for quantum systems (cf. Section 4.4.2), so this process witnesses classicality. Hence, rather than considering no-cloning to be a shortcoming of quantum systems, we will consider *copiability* (or *cloneability*) to be the characterising feature for classical ones.

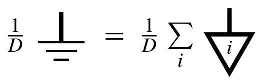
8.2.2.1 Deleting

As we saw above, deleting is the classical counterpart to discarding and is therefore the unique causal effect for a classical system.

Similarly, the adjoint of deleting:

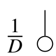


is the classical counterpart to the maximally mixed state:



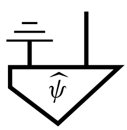
This classical state has a standard name.

Definition 8.16 The classical state:

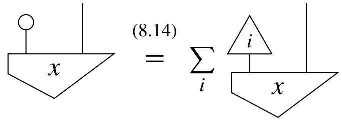


is called the *uniform probability distribution*.

Just as we had a notion of reduced quantum states (see Proposition 6.33):



given a state on two classical systems, if we delete one of the systems:

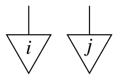


we obtain a familiar operation from probability theory, called *marginalisation*. The classical state (a.k.a. probability distribution) x with one of the systems deleted is called the *marginal* distribution.

While reduced states have natural classical counterparts, which have many practical applications, here's a remarkable shortcoming of the classical world:

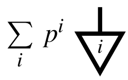
Purification of quantum states has no classical counterpart.

In other words, it is not the case that any classical state can be realised as the marginal of a deterministic classical state. The only deterministic classical states on two systems are of the form:

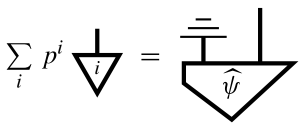


Then, clearly deleting one system will again result in a deterministic state. Hence, the only states that can be 'purified' to a deterministic state were deterministic in the first place!

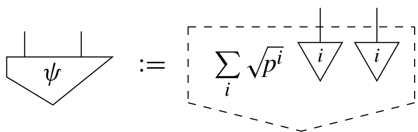
Comparing this situation with the quantum case, we can represent probability distributions as quantum states (cf. Proposition 6.74):



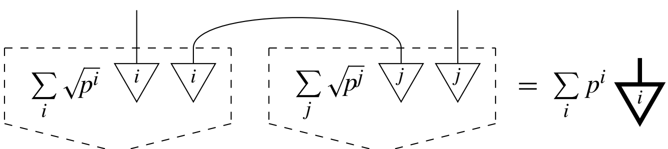
This is indeed a quantum state and hence can be purified:



Explicitly, for:



then we indeed have:

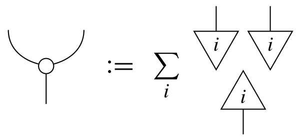


So the magic of purification comes from the fact that quantum states have two ways of combining processes via summation, mixing, and superposition:

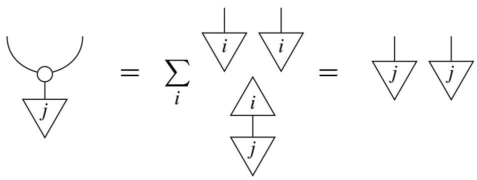
and it is the latter that gives us enough extra flexibility to purify any quantum state.

8.2.2.2 *Copying*

Definition 8.17 Copying is the following classical map:



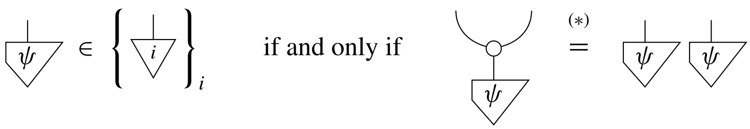
This map also behaves as advertised on basis states:



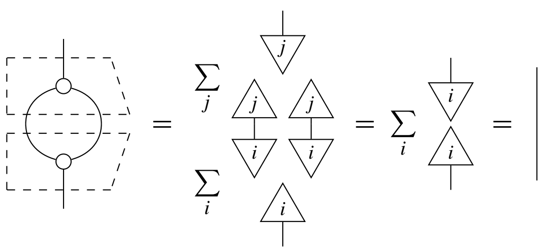
(8.21)

In fact, a state is copied by  if and only if it is a basis state.

Theorem 8.18 Copying uniquely fixes an ONB. More specifically, for any non-zero state ψ:

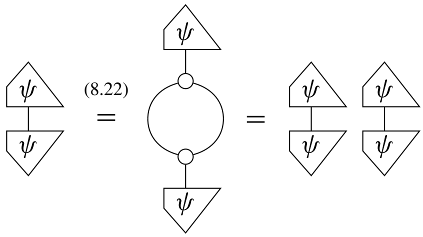


Proof First note that copying is an isometry:



(8.22)

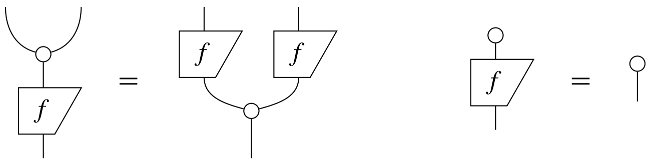
Assuming ($*$), this implies that:



The only numbers such that $p = p^2$ are 0 and 1, so ψ must be normalised. By Theorem 4.85 we know that normalised states that are jointly cloneable by an isometry must be orthogonal. Since by (8.21), the ONB states are also copied, it follows that ψ must either be an ONB state or be orthogonal to every ONB state. But the only state that is orthogonal to every state in an ONB is 0, so ψ must be equal to exactly one of the ONB states. $\square$

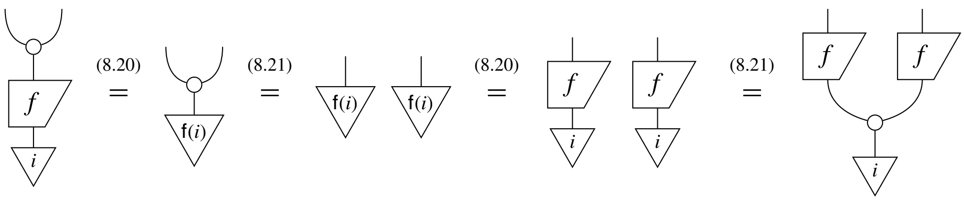
So, not only does fixing an ONB of classical states yield a copying map, the copying map uniquely fixes an ONB. Moreover, copiability gives a diagrammatic characterisation for deterministic classical processes.

Proposition 8.19 A linear map f is a function map (i.e. a deterministic classical process) if and only if it satisfies the following two equations:

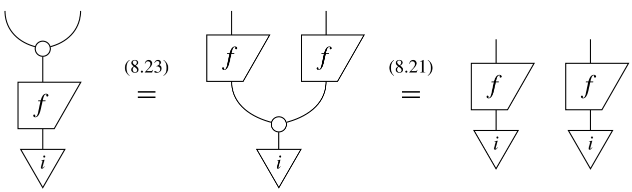


(8.23)

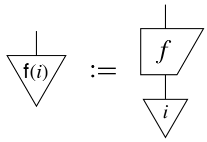
Proof First, assume f is a function map. By Definition 5.2 linear maps are equal if they agree on an ONB, so we can prove the two equations above by composing with ONB states:



The second equation is shown similarly. Conversely, we have:



So $f \circ i$ is copied by the copy map, and it is furthermore non-zero by the second equation in (8.23). Thus, by Theorem 8.18, the state $f \circ i$ must be an ONB state. Thus, we can define the underlying function f as follows:

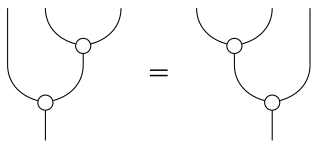


□

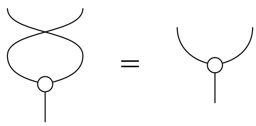
The input and output systems of f (and hence the copy operations on the LHS and RHS above) could be different. In particular, copying ONB states is just the special case where the input of f is trivial.

One would expect that if we copy classical data, then the two identical copies can be swapped freely. Also, if we copy one of the two resulting copies again, both ways of making three copies are equal. This is indeed the case.

Proposition 8.20 We have:

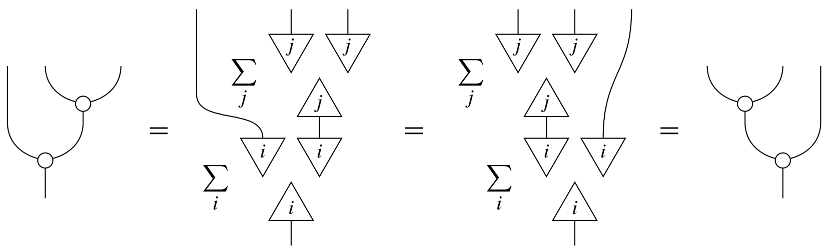


(8.24)

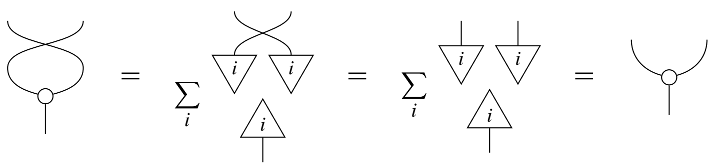


(8.25)

Proof Unfolding the definitions, we have:



and:

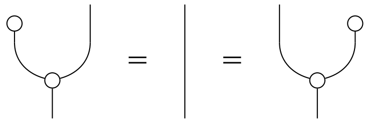


□

8.2.2.3 Copying and Deleting

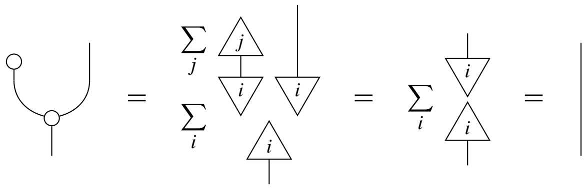
One would also expect that if we copy classical data, then delete one copy, this is the same as doing nothing. Again, this is true.

Proposition 8.21 We have:



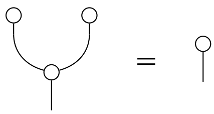
(8.26)

Proof Unfolding the definitions, we have:



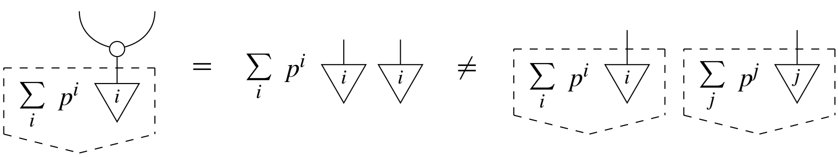
□

Note that this immediately implies that copying is causal, since deleting the outputs is indeed the same as deleting the input:

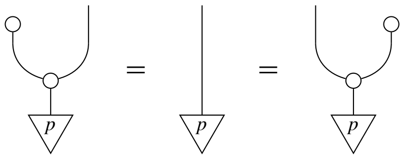


Remark* 8.22 Earlier in Remark 3.17 we said that copying is an example of a *comultiplication*. Equations (8.24) and (8.25) tell us that copying is *coassociative* and *cocommutative*, which are the coalgebraic counterparts to associativity and commutativity in algebra, respectively. Equation (8.26) tells us that deleting is the corresponding *counit*, which is the coalgebraic counterpart to the usual notion of a unit in algebra. The equations in Proposition 8.19 then tell us that function maps are *comonoid homomorphisms*.

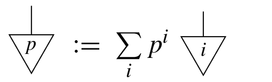
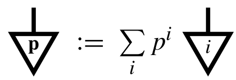
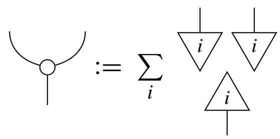
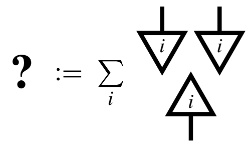
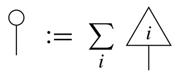
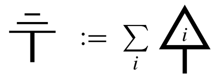
In (6.48) we saw that cloning fails for general probability distributions, and indeed:



However, probability distributions can be broadcast. Indeed, we can interpret *broadcasting* as the existence of a map we can apply to any classical state p such that Aleks and Bob can each recover p just by deleting the other system:

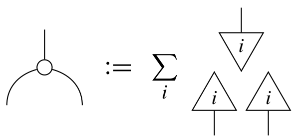


So, we have two equivalent ways to express probability distributions, as well as the associated copying and deleting operations:

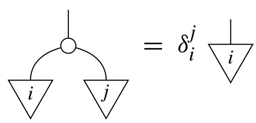
state		
copy/broadcast		
delete/discard		

8.2.2.4 *Matching*

Matching is the adjoint of copying:



This classical map takes in two ONB states. If those states are the same, it sends that state out; otherwise it goes to zero:

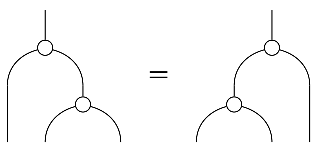


(8.27)

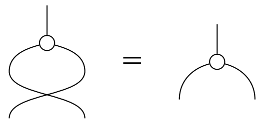
Remark 8.23 Matching, unlike its adjoint, is not causal. In particular, it does not send probability distributions to probability distributions. It has a clear meaning and will be a useful operation nonetheless.

We can now simply take the adjoint of equations (8.24), (8.25), and (8.26) to obtain corresponding equations for matching.

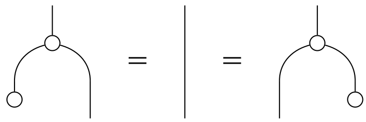
Proposition 8.24 We have:



(8.28)

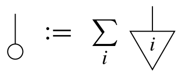


(8.29)



(8.30)

where the adjoint to deleting is:

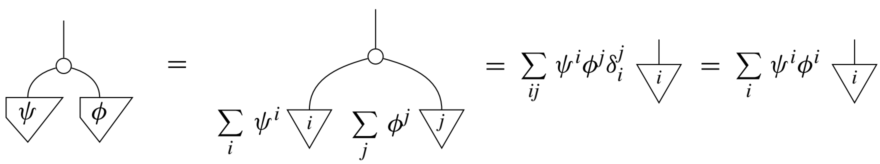


(8.31)

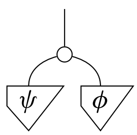
Remark* 8.25 The equations (8.28), (8.29), and (8.30) are the less exotic, *algebraic* versions of the coalgebraic equations explained in Remark 8.22, namely *associativity*, *commutativity*, and *unitality*.

On arbitrary states:

matching multiplies matrix entries pointwise:



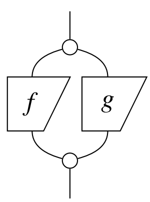
Written as an operation on matrices:



$$\leftrightarrow \begin{pmatrix}\psi^1\\ \vdots \\ \psi^D\end{pmatrix} \star \begin{pmatrix}\phi^1\\ \vdots \\ \phi^D\end{pmatrix} := \begin{pmatrix}\psi^1\phi^1\\ \vdots \\ \psi^D\phi^D\end{pmatrix} \tag{8.32}$$

This $\star$-operation is sometimes called the *Hadamard product* or the *Schur product*, and it extends to arbitrary matrices.

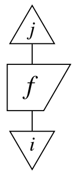
Exercise 8.26 Show that for any two linear maps f and g of the same type, the diagram:

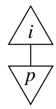


yields the Hadamard product of matrices:

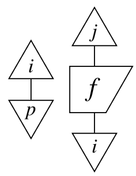
$$\begin{pmatrix} f_1^1 & \cdots & f_D^1 \\ \vdots & \ddots & \vdots \\ f_1^D & \cdots & f_D^D \end{pmatrix} \star \begin{pmatrix} g_1^1 & \cdots & g_D^1 \\ \vdots & \ddots & \vdots \\ g_1^D & \cdots & g_D^D \end{pmatrix} = \begin{pmatrix} f_1^1 g_1^1 & \cdots & f_D^1 g_D^1 \\ \vdots & \ddots & \vdots \\ f_1^D g_1^D & \cdots & f_D^D g_D^D \end{pmatrix}$$

Example 8.27 Another use for the copying map is turning causal classical maps into causal bipartite classical states. In probability theory, the former are called *conditional probability distributions*, whereas the latter are called *joint probability distributions*. Suppose we write down some probabilities in terms of a classical map and a particular input state:

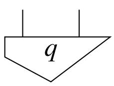
$P(j \mid i) :=$ 

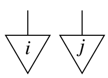
$P(i) :=$ 

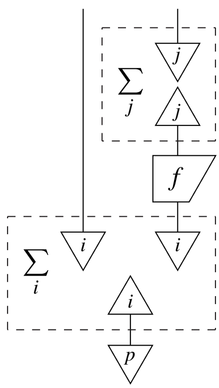
In the language of probability theory, the probabilities $P(i)$ are known as *priors*. Along with the conditional probabilities, they are used to compute the joint probabilities $P(ij)$, i.e. those of 'i and j both happening':

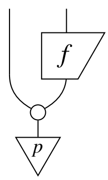
$P(ij) = P(i)P(j \mid i) =$ 

Since this gives a probability distribution, we can form a new classical state on two systems as follows:



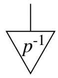
$:= \sum_{ij} P(ij)$ 

$=$ 

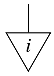
$=$ 

which gives us an expression of the joint distribution. The resulting state is causal because f, p, and copying are.

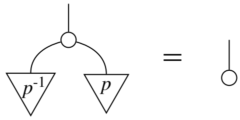
We can also go the other direction and turn a joint distribution into a conditional distribution. For this, we need the *inverse state*:



$$:= \sum_i \frac{1}{p^i}$$

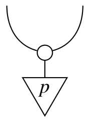


or equivalently, the unique classical state satisfying:

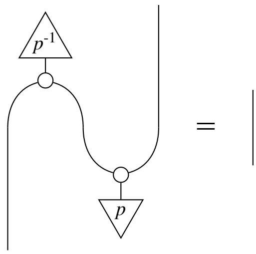


(8.33)

(If you are worried about dividing by zero, see Convention 8.29 below.) Then, thinking of:

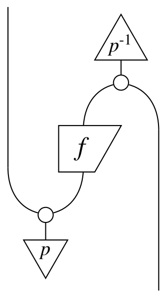


as a 'cup', we can build the associated 'cap' using p^{-1}:

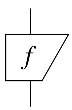


(8.34)

This equation follows directly from (8.33) and the 'spider-fusion' rule, which we will soon unveil. But in the meantime one can use the definitions of copying/merging to prove (8.34) concretely. Combining the cap and the cup gives us a way to turn $P(j \mid i)$ into $P(i \mid j)$, which is called *Bayesian inversion*:



Exercise* 8.28 For the classical map:



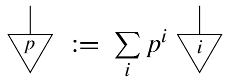
$$\leftrightarrow \begin{pmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{pmatrix}$$

compute the associated joint distributions for the following priors:

$$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}1/2\\1/2\end{pmatrix}$$

Then, compute f's Bayesian inverse for these states.

Convention 8.29 In forming the inverse state above, we have implicitly assumed that our probability distribution has *full support*, that is:



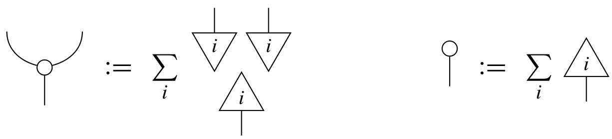
where $\quad \forall i: \; p^i \neq 0 \qquad$ (8.35)

When we are free to choose the dimension of our classical system, this is no loss of generality: we can always pass from any probability distribution to one with full support in a lower dimension by getting rid of classical values that occur with probability zero.

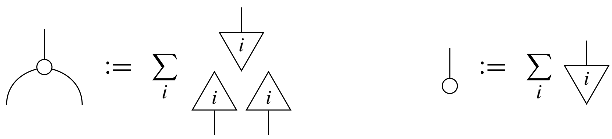
8.2.3 Spiders

We encountered the following classical maps, which all admit a natural interpretation as a classical data operation:

- *copying* and *deleting*:



- *matching* and the (unnormalised) *uniform state*:

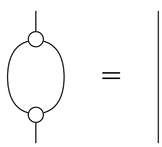


There is no need to stop here. We could also add, e.g.:

- states/effects representing *perfect correlation* of two classical systems:

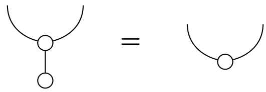
We can derive various equations (and their adjoints), which have natural interpretations similar to (8.24), (8.25), and (8.26), e.g.:

- Copying followed by matching equals doing nothing:



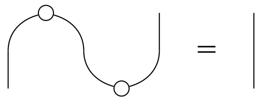
 https://doi.org/10.1017/9781316219317.009

- ‘Copying’ the uniform state yields the perfectly correlated state:



Some of these equations even look familiar:

- Classical data admits yanking:

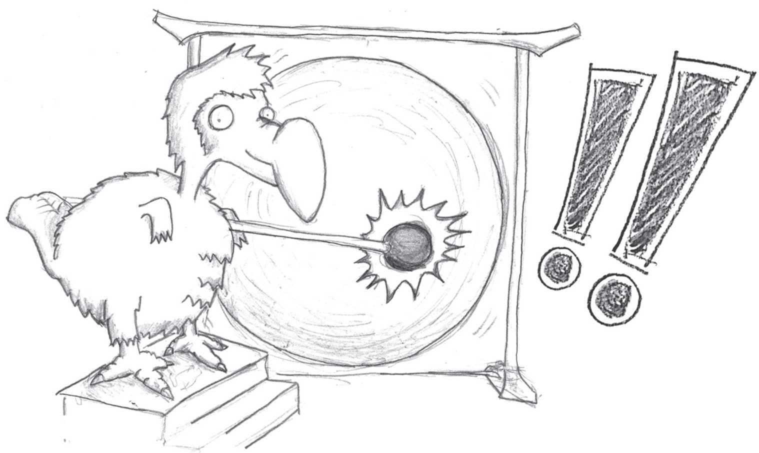


(8.36)

Exercise 8.30 Prove the above equations between classical maps.

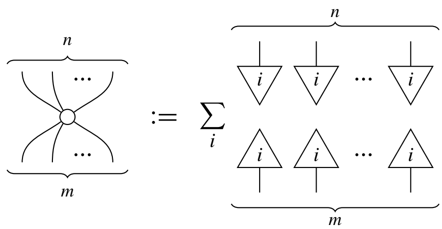
Imagine now a slowly growing crescendo in the background:

It would be natural to aim for all the equations that hold between these classical maps. How many would there be? Maybe hundreds? Maybe an infinite number? Now imagine a blast of a gong:



There is only one equation! Indeed, all the equations that we have have seen thus far are in fact instances of one and the same equation. To see this, one first needs to realise that all the classical maps that we have seen thus far are special cases of one family of classical maps, which we call spiders.

Definition 8.31 *Spiders* are linear maps of the form:



(8.37)

Intuitively, spiders force all of the inputs and outputs to be the same basis element. As such, sometimes it is helpful to think of them as a 'big Kronecker delta', which is exactly what we get if we compute the matrix of a spider:

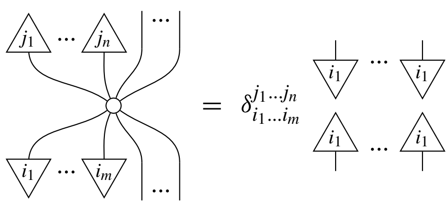
$$\delta_{i_1\ldots i_m}^{j_1\ldots j_n} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_m = j_1 = \cdots = j_n \\ 0 & \text{otherwise} \end{cases}$$

The usual Kronecker delta:

$$\delta_i^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

arises as a special case, as the matrix of the identity map.

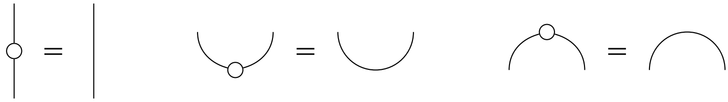
Exercise 8.32 Prove the *generalised copy rule* for spiders:



(8.38)

which, for example, generalises equation (8.21) as well as (8.27).

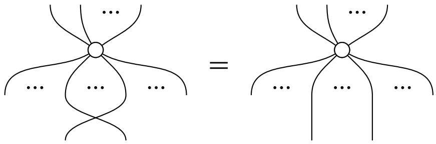
From the definition, we can conclude a couple of things about spiders. First, a spider with only two legs is just a wire:



(8.39)

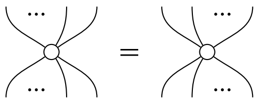
and second, spiders exhibit a lot of symmetry.

Proposition 8.33 All spiders are invariant under 'leg-swapping':



(8.40)

and conjugation (i.e. horizontal reflection):



(8.41)

Proof Both equations follow directly from (8.37). □

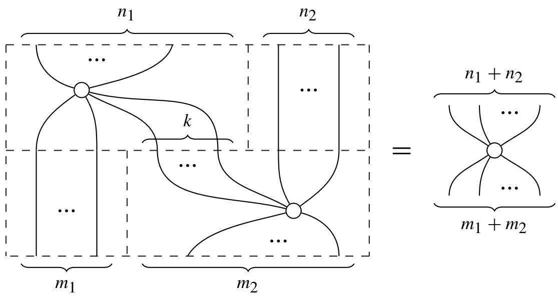
And now the grand finale ...

All of the equations for classical maps we have encountered so far can now be subsumed by one simple rule:

If two spiders touch, they fuse together.

More precisely, we have the following.

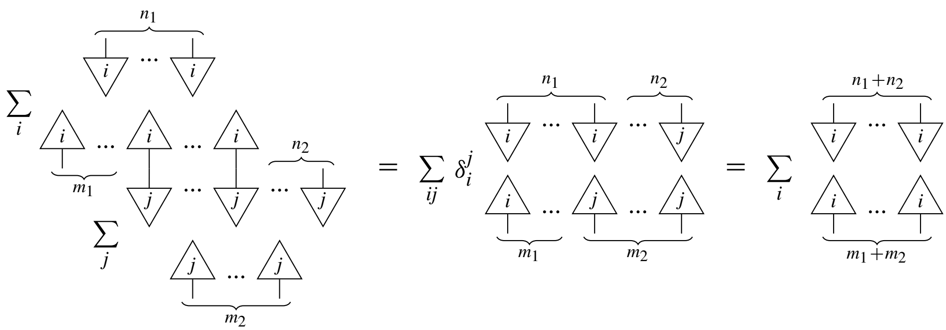
Theorem 8.34 Spiders compose as follows:



(8.42)

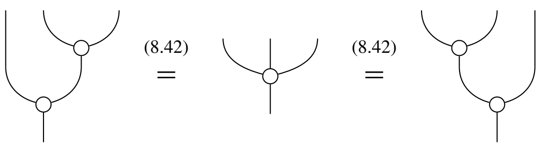
for $k \geq 1$.

Proof Unfolding the two spiders, we obtain:



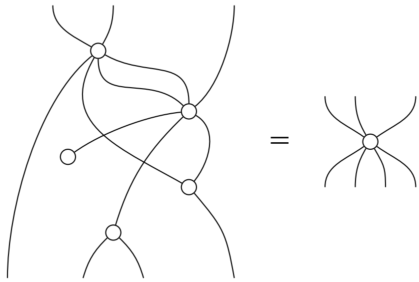
□

We can now prove any of the equations from the previous sections by just squashing the LHS and the RHS down to a single spider. For instance, this equation involving copying:



More generally, by (8.39) cups and caps are also spiders, so we can squash diagrams of spiders into one big spider.

Corollary 8.35 Any connected string diagram consisting only of spiders is equal to a single spider:

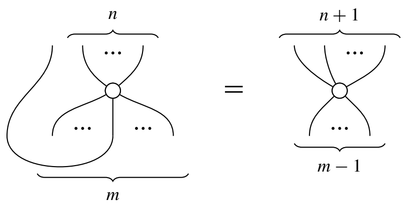


(8.43)

Hence, such a diagram is uniquely determined by its number of inputs and outputs, and we can establish equality simply by counting them.

We refer to this rule as *spider fusion*.

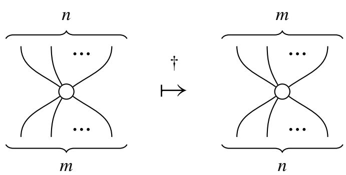
Corollary 8.36 Spiders satisfy 'leg flipping', that is, if we bend one of a spider's legs up or down, we get again a spider:



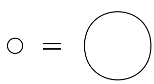
(8.44)

and hence the transpose of a spider is again a spider.

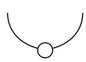
Of course, combining this with the fact that spiders are self-conjugate, it follows that taking the adjoint of a spider also yields a spider:



Exercise 8.37 Prove using just the properties of spiders that the spider with no legs equals the 'circle' (i.e. the dimension):



Exercise 8.38 As already pointed out in Remark 8.23, some spiders are not causal. For correlating to be causal it suffices to introduce a normalisation factor:

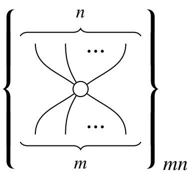
$\frac{1}{D}$ 

However, for comparing and matching no number would do the job, given that for $i \neq j$ we have:

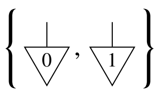
More generally, which spiders are causal, which can be made causal by normalising, and which cannot?

In Section 5.2.2 we saw how one can encode bit strings as basis states. We can also associate spiders with them.

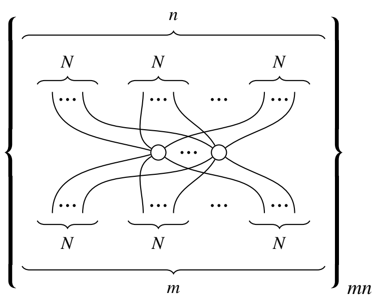
Exercise 8.39 The spiders:



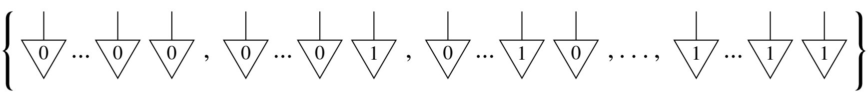
for the two-dimensional basis:



are associated with *bits*. Show that the following family of classical maps:



is also a family of spiders. Furthermore, show that it is associated with the ONB of *N-bitstrings*:



Remark 8.40 The equations:

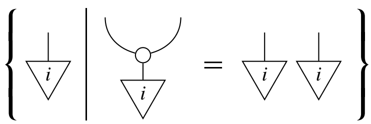
(8.45)

which are required, e.g. for 'leg-flipping', come from the fact that we have chosen a self-conjugate ONB (cf. Section 5.2.3). In a few (rare) cases, it is useful to define spiders for non-self-conjugate bases, in which case we should either drop these equations or fix them somehow (cf. Section* 8.6.3).

8.2.4 If It behaves like a Spider It Is One

Given the importance of spiders, we need to figure out how to recognise them. That is, how can we distinguish a real spider from, say, a dodo in a spider costume desperately trying to fight extinction?

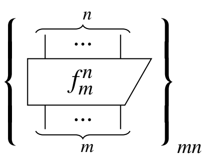
In Theorem 8.18 we showed that the 'copy spider' defines an ONB in terms of the states that it 'copies':



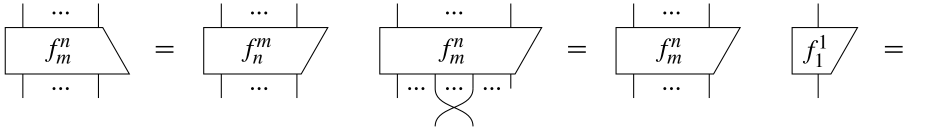
(8.46)

So consequently, a family of spiders as defined in Definition 8.31 always fixes an ONB. Surprisingly, the converse is also true, and this is a non-trivial result: any collection of linear maps that composes like a family of spiders is in fact a family of spiders:

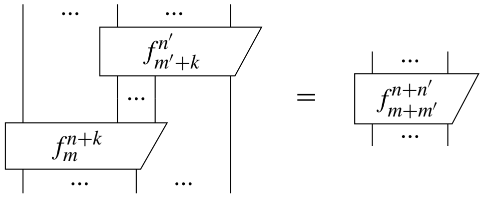
Theorem 8.41 Any collection of linear maps:



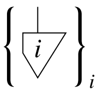
which satisfy:



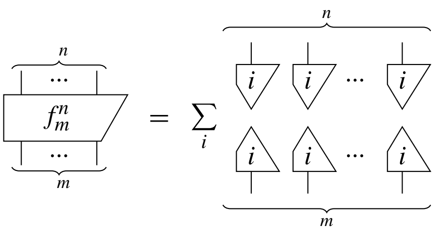
and compose as follows:



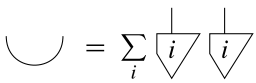
is a family of spiders. That is, there exists an ONB:



such that:

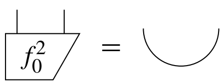


Since we are mostly interested in spiders for self-conjugate ONBs (cf. Remark 8.1) we will now specialise this result to this case. Back in Proposition 5.60 we saw that an ONB is self-conjugate if and only if:



We now know that the RHS is just a spider with two outputs, so we have the following.

Corollary 8.42 The collection of linear maps from Theorem 8.41 represents a self-conjugate ONB if and only if it additionally satisfies:



The proof of Theorem 8.41 uses some techniques from *representation theory* and goes beyond the scope of this book. We do give some indication of how it goes in Section* 8.6.1. But the punchline is that the equations that we identified as holding for spiders actually 'axiomatise' these spiders; that is, spiders and nothing but spiders can satisfy all of these. Therefore we call these the *spider equations*. An important consequence is the fact that now we can define ONBs, which at first seem totally undiagrammatic, using the purely diagrammatic concept of spiders.

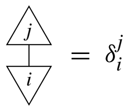
Corollary 8.43 An ONB can be defined totally in terms of diagram equations, namely, the spider equations.

Furthermore, since spiders make sense in any process theory, copiable states/effects do as well. Surprisingly, orthonormality is (nearly) automatic.

Exercise 8.44 Assuming that the numbers in a process theory satisfy:

$$\lambda^2 = \lambda \quad \Longrightarrow \quad \lambda \in \{0, 1\}$$

(which is true e.g. for real numbers, complex numbers, and booleans), show that the copiable states (8.46) for any family of spiders are always orthonormal:



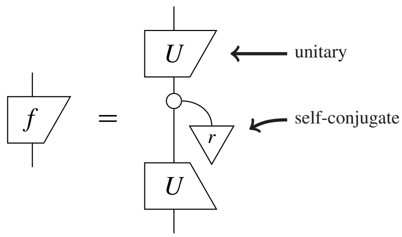
but they don't always form a basis.

Exercise* 8.45 Give a family of spiders in **relations** whose copiable states do <u>not</u> form an ONB.

8.2.5 All Linear Maps as Spiders + Isometries

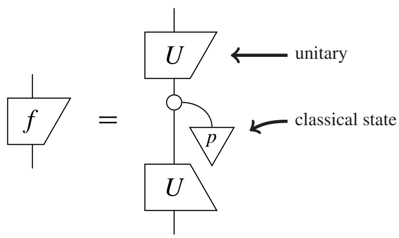
The spectral theorem (Theorem 5.71) lets one decompose self-adjoint linear maps in terms of ONBs. Now that we know that ONBs are really all about spiders, we can see that the spectral theorem means any self-adjoint linear map has a spider hiding inside.

Theorem 8.46 Any self-adjoint linear map f admits a *spectral decomposition*:



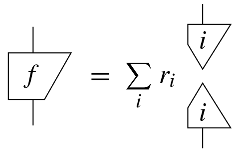
(8.47)

If f is moreover positive, then r is a classical state, so any positive f decomposes as:

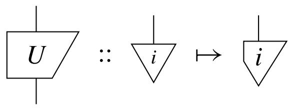


(8.48)

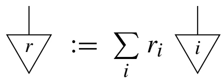
Proof By the spectral theorem f decomposes as follows:



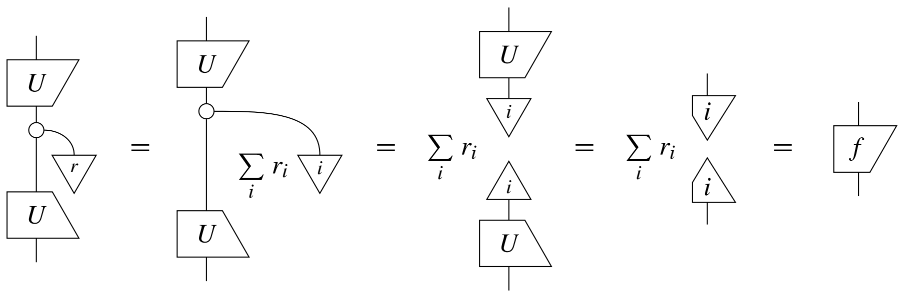
for some ONB and real numbers r_i. Then, let U be the unitary that sends the ONB associated with the spider to the ONB above:



and let:

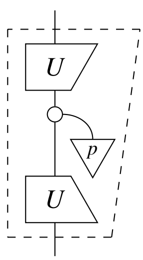


Then:



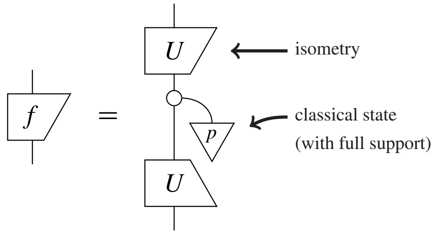
Also by Theorem 5.71, if f is positive, then the numbers r_i are positive. Hence r becomes a classical state. □

Decomposition (8.48) tells us what's 'inside' any positive linear map:



namely, nothing but spiders, unitaries, and classical states.

A slight variation of this decomposition allows us to express a positive linear map in terms of a classical state with full support (cf. Convention 8.29). This can be realised by making the system in the middle smaller and replacing the unitaries with isometries:

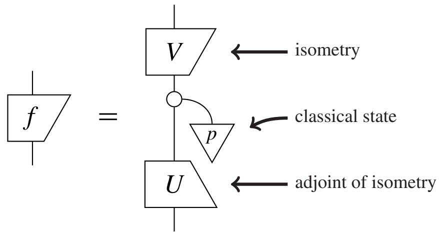


(8.49)

Exercise 8.47 Prove that any positive linear map decomposes as (8.49).

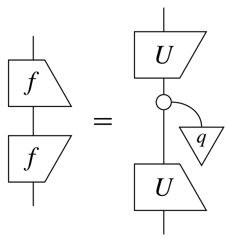
Furthermore, by relaxing the requirement that the isometry on the bottom and the top be the same, we can obtain a decomposition which applies to all linear maps:

Theorem 8.48 Any linear map f admits a *singular value decomposition*:



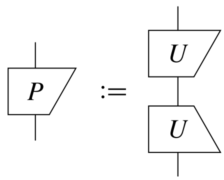
for some isometries U and V and a classical state p with full support.

Proof Since $f^\dagger \circ f$ is positive, we can rely on the spectral theorem to decompose it as in (8.49):

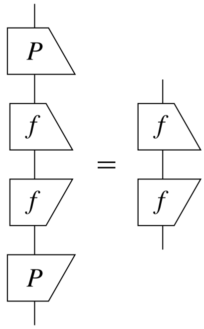


(8.50)

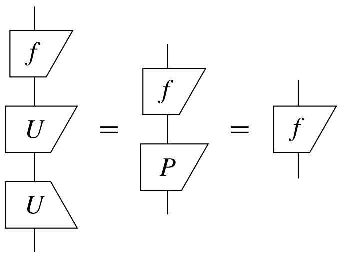
Now note that, since U is an isometry, the following is a projector (i.e. positive and idempotent, as in Definition 4.69):



Then, by the form of (8.50), it immediately follows that:

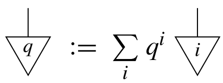


Hence, by Exercise 5.77 (which also follows directly from the spectral theorem), we have:



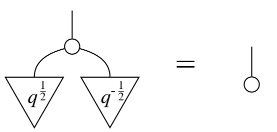
(8.51)

Now, for:

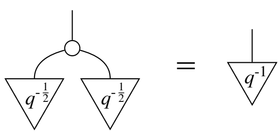


with all $q^i \neq 0$, we define two additional states:

which by (8.32) satisfy:

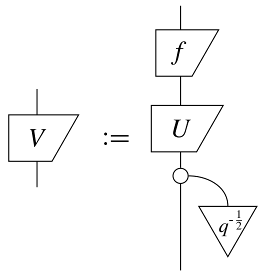


(8.52)



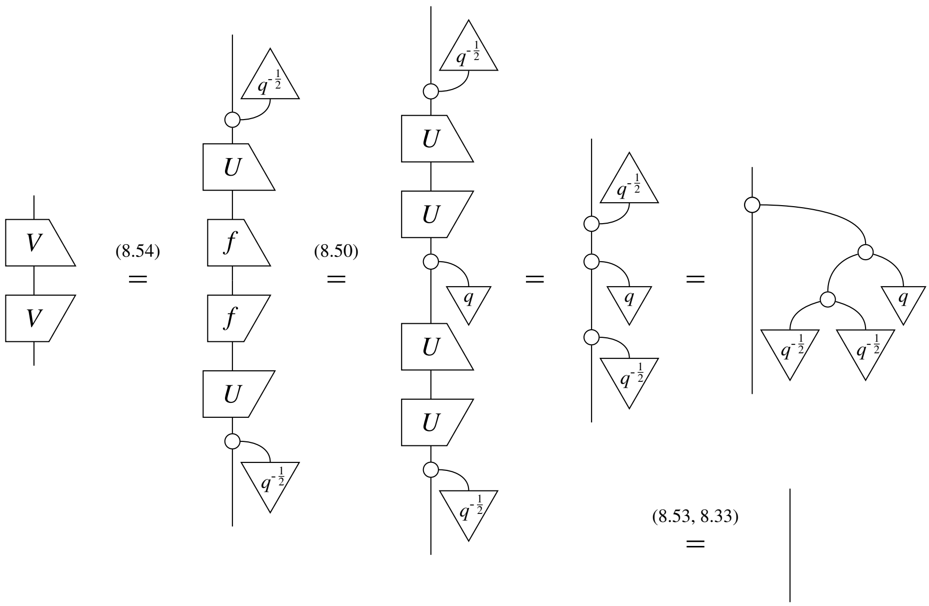
(8.53)

We can now show that:

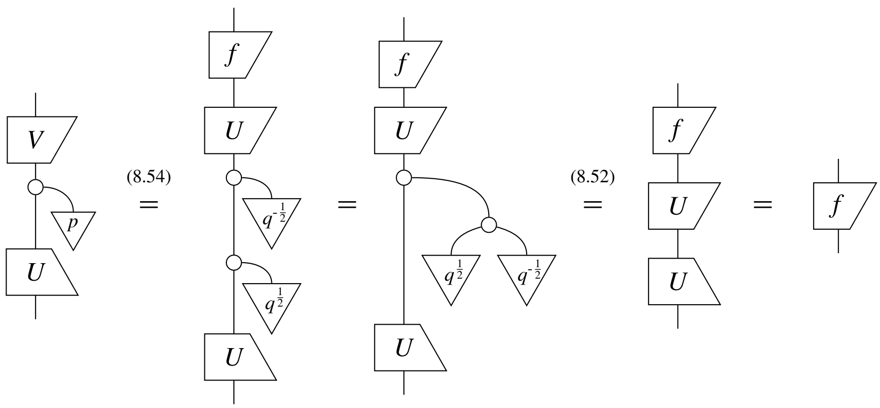


(8.54)

is an isometry:



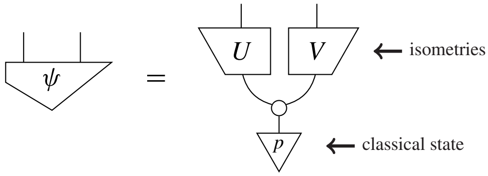
Letting $p := q^{\frac{1}{2}}$ in:



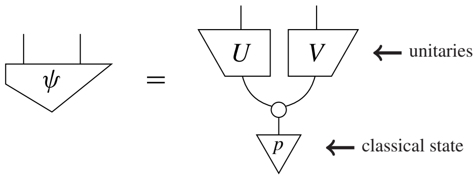
completes the proof. □

By bending the wire, we also discover what's 'inside' any bipartite state.

Corollary 8.49 Any bipartite state ψ decomposes as follows:



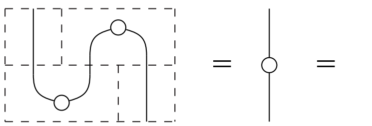
Exercise 8.50 Show that when the two output systems of a bipartite state ψ are the same, it can be decomposed as:



Remark 8.51 The 'sideways' version of the singular value decomposition for a bipartite state is often called the *Schmidt decomposition*.

8.2.6 Spider Diagrams and Completeness

In our definition of spiders we already singled out cups and caps as special cases of spiders, which we referred to as correlating and comparing. But of course, what defines cups and caps is the relationship between them, namely that if we compose them we get an identity. This is in fact a direct instance of the spider fusion rule (8.34):



This is kind of funny, to think about 'yanking wires' as a special case of 'fusing spiders', but that's exactly what we established here, that:

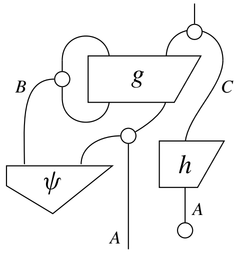
$$\boxed{\textit{Reasoning with string diagrams is an instance of reasoning with spiders!}}$$

Whereas a wire connects two ends together, a spider connects many ends together. Spider fusion is then all about connecting many things together by means of multiple spiders.

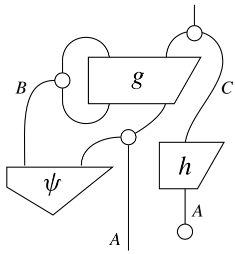
And just like we could treat string diagrams either as 'circuit diagrams + caps/cups' or as a new kind of diagram (cf. Theorem 4.19), we can do the same with diagrams containing spiders.

Definition 8.52 A *spider diagram* consists of boxes and wires that are allowed to connect any number of inputs and outputs together.

Diagrammatically, we represent these ‘multiwires’ as spiders, for example:

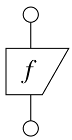


In the associated diagram formula (cf. Definitions 3.8 and 4.21), we can represent them just by repeating wire names as many times as we like:

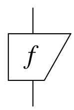


$$\longleftrightarrow \quad \psi^{B_1\check{A}_1} g_{B_1\check{A}_1}^{B_1\widehat{C}_1} h_{A_2}^{\widehat{C}_1} \tag{8.55}$$

For ‘multiwires’, it’s no longer clear which wire names correspond to inputs/outputs, so we mark inputs with a check $(\check{-})$ and outputs with a hat $(\widehat{-})$. This for example allows one to distinguish:



$\longleftrightarrow \quad f_{A_1}^{A_2}$ vs.



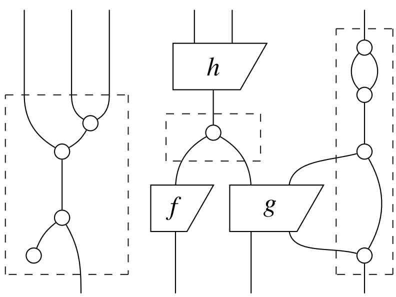
$\longleftrightarrow \quad f_{\check{A}_1}^{\widehat{A}_2}$

Theorem 8.53 The following two notions are equivalent:

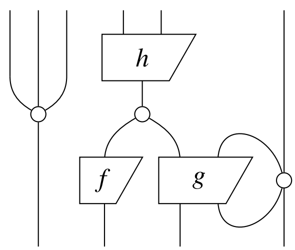
(i) spider diagrams and
(ii) circuit diagrams to which we adjoin spiders for each type

in the sense that (ii) can be unambiguously expressed as (i) and vice versa.

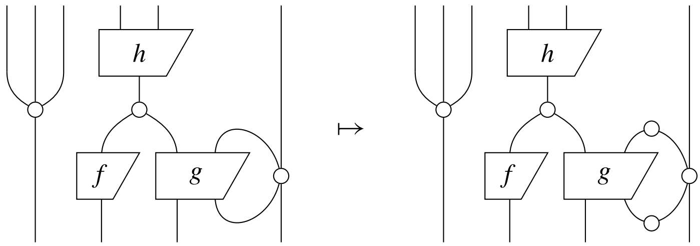
Proof We can translate a circuit diagram with spiders to a spider diagram just by fusing all of the connected spiders into single spiders:



$\mapsto$



In the opposite direction, if we replace cup- and cap-shaped wires in a spider diagram with the corresponding spiders, we obtain a circuit diagram with spiders:



□

Recall from Section 5.4.1 that string diagrams are complete for **linear maps**. That is, an equation between string diagrams holds for all Hilbert spaces and linear maps if and only if the string diagrams are the same.

It is always possible that by enriching the diagram language, we could break completeness. Since spider-diagram language is richer, we can write down more equations (i.e. equations involving spiders), but can we still prove them all? Thankfully, the answer is yes.

Theorem 8.54 Spider diagrams are complete for **linear maps**. That is, for any two spider diagrams D and E, the following are equivalent:

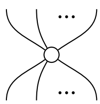
- $D = E$
- For all interpretations of D, E into **linear maps**, $[\![D]\!] = [\![E]\!]$.

We can thus make a statement analogous to the one about string diagrams:

An equation between spider diagrams holds for all Hilbert spaces and linear maps if and only if the spider diagrams are the same.

8.3 Quantum Maps from Spiders

Readers of the previous two sections might notice a suspicious similarity between the notation for spiders:



and for measuring and encoding:

This is of course no accident. In this section, we'll see how these are also a certain species of spider. We'll then take advantage of that fact to do some pretty cool stuff. In particular, this will let purely diagrammatic rules such as spider fusion do most of the hard work from now on.

8.3.1 Measuring and Encoding as Spiders

The whole point of cq-maps is to allow classical and quantum systems to interact. Using our paradigm:

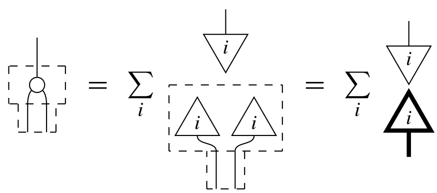
$$\frac{\text{classical}}{\text{quantum}} = \frac{\text{thin/single wires}}{\text{thick/double wires}}$$

we can express boxes whose inputs and outputs consist of both classical and quantum wires. We saw in Section 8.1.3 that quantum states can be turned into classical data, and vice versa, using measure and encode. Can we give these two processes a form that is as elegant as the spider form for classical processes? Yes we can!

We have already seen maps that have a pair of wires in and a single wire out and vice versa: the copying and matching spiders. However, rather than treating these as operations on classical data, we can treat a pair of wires as a single quantum wire, hence forming a bridge from classical to quantum:

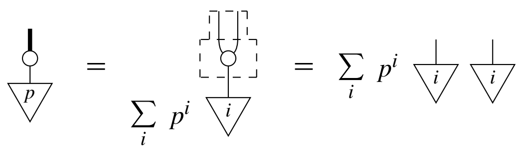
(8.56)

Unfolding the definition of spiders, we see this indeed gives us measuring:

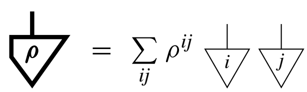


and taking the adjoint we obtain encoding.

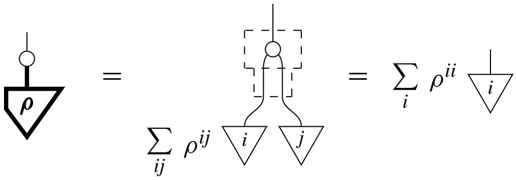
Hence, we can understand what these processes do in terms of spiders. For example, on ONB states, encoding unfolds as copying, so we obtain:



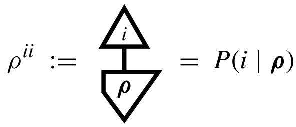
Similarly, measuring can be understood in terms of matching, so to see what measuring does to an arbitrary quantum state ρ:



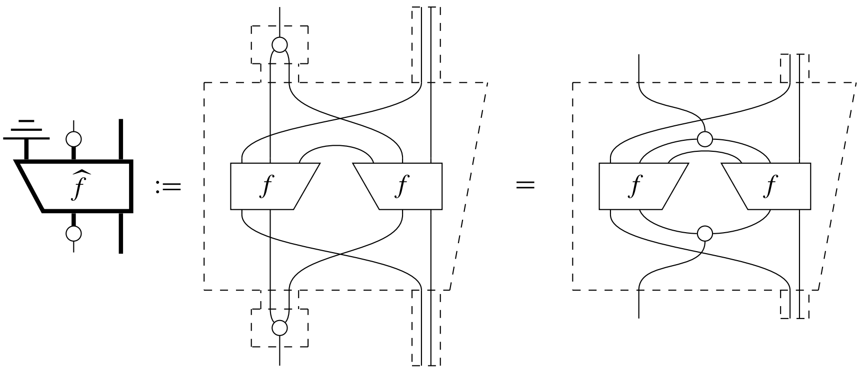
we can use the fact that measurement unfolds as matching:



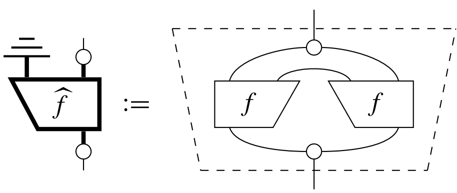
That is, all non-diagonal entries of the matrix of $\boldsymbol{\rho}$ in the product basis are gone, and the diagonal ones (which by Corollary 5.41 are all positive) are retained unaltered. As we already saw in Section 8.1.3, these diagonal entries are the probabilities for the ONB measurement according to the Born rule:



Corollary 8.6 stated that we can always express a cq-map in terms of a pure quantum map via measure, encode, and discarding. We can now give this generic form in terms of spiders:

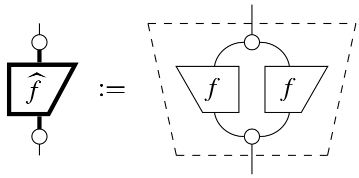


In the special case of classical maps we have:

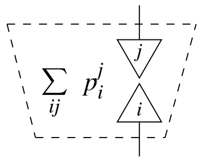


But in fact, this simplifies further.

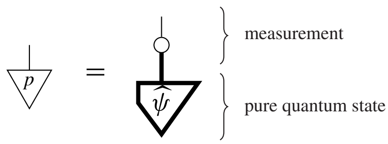
Exercise 8.55 Show that all classical maps are of the form:



Since this simplification relies on particular properties of linear maps, you will need to use the following form for classical maps:

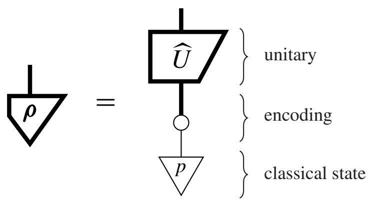


Clearly Exercise 8.55 does not extend to more general cq-maps, since this would imply in particular that all quantum maps are pure. As a special case of Exercise 8.55, we now know that we can always write a classical state p as an ONB-measurement of some pure quantum state:

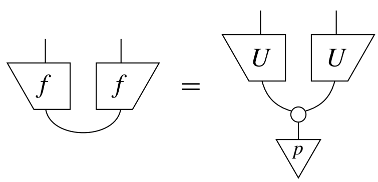


Moreover, now that we know that encoding arises from the copying spider, we can show that every quantum state arises from encoding a classical state, thanks to the diagrammatic form of the spectral theorem.

Proposition 8.56 Any quantum state $\boldsymbol{\rho}$ encodes a classical state as follows:



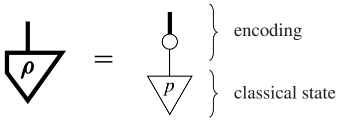
Proof Unfolding the equation above yields:



(8.57)

for some f. This is just the bent-over version of decomposition (8.48), which exists thanks to the spectral theorem. □

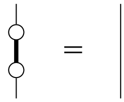
Remark 8.57 The presence of the unitary U is necessary since we opted for self-conjugate ONBs throughout this book. Otherwise, as we show in Section* 8.6.3, the decomposition can be simplified:



We can also exploit the fact that measuring and encoding both are spiders 'in disguise' to produce some more equations.

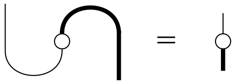
Proposition 8.58 We have:

1. Encoding followed by measuring is equal to doing nothing:



(8.58)

2. Encoding's transpose is measuring:



(8.59)

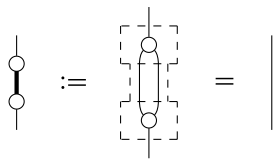
3. Measuring followed by deleting yields discarding:

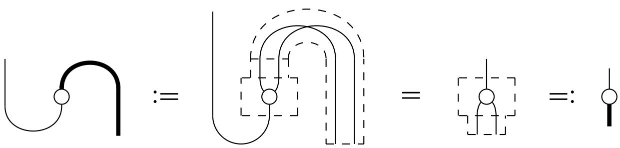
(8.60)

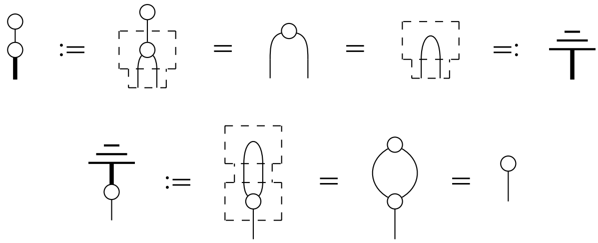
4. Encoding followed by discarding yields deleting:

(8.61)

Proof All of these equations follow from unfolding the doubled parts and applying spider fusion:



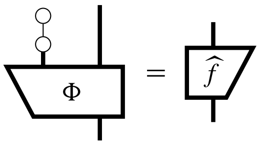




□

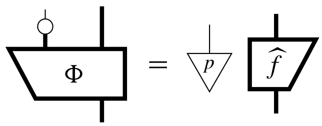
The fact that discarding decomposes as in the first equation of (8.60) allows us to produce a new version of Proposition 6.79, which said that a quantum map separates when the reduced map is pure. Besides discarding quantum outputs, this also applies to deleting classical outputs.

Proposition 8.59 If the *reduced map* of a cq-map is pure:



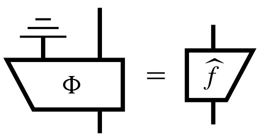
(8.62)

then the process Φ separates as follows:

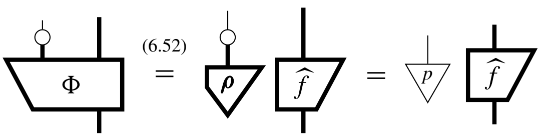


for some (causal) classical state p (a.k.a. probability distribution).

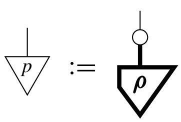
Proof By equation (8.60), equation (8.62) is equivalent to:



so by Proposition 6.79 we have:



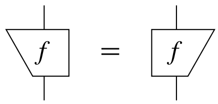
where we set:



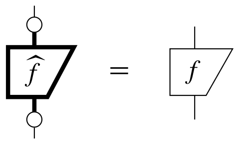
and which is causal by causality of measure and $\boldsymbol{\rho}$. □

Also, several things that were derived earlier using sums can now be derived diagrammatically, for example the following.

Exercise 8.60 Prove diagrammatically that classical maps are self-conjugate:



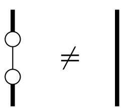
Use this result to show that for function maps we have:



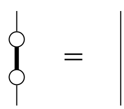
8.3.2 Decoherence

This section is dedicated to a very important (and infamous!) quantum process. We say 'infamous' because it has a bad habit of messing up the nice pure quantum states physicists try to prepare in their labs, which makes building things like quantum computers very difficult indeed.

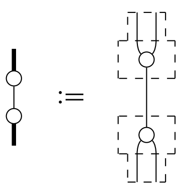
Equation (8.58) indicates that if we go from classical data to classical data via a quantum system, the classical system remains unchanged. However, measurement is only a one-sided inverse of encoding. That is, if we compose the maps in the opposite order, this most certainly does not leave the quantum system unchanged, due to the invasive nature of measurement, so:



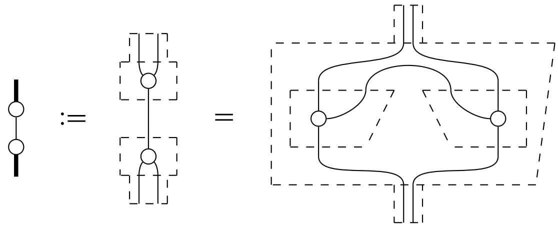
while



Definition 8.61 *Decoherence* relative to an ONB is the quantum process:



In fact, we already encountered this process in the proof of Proposition 8.5, which gave a normal form for cq-maps. There, we relied crucially on the fact that this process was a quantum map. Now that we know measuring and encoding are spiders, this follows just from spider fusion:

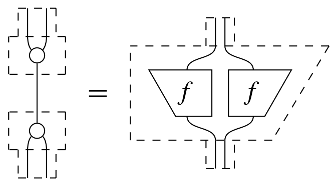


(8.63)

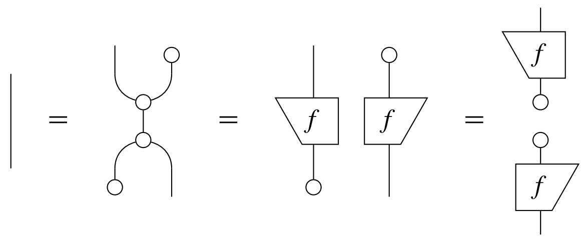
We still need to show that decoherence is not equal to the identity, but we can actually say something much stronger. Not only is it not equal to the identity, it is not even pure.

Proposition 8.62 Decoherence is not a pure quantum map.

Proof Suppose decoherence is a pure quantum map. Then, for some linear map f we would have:



But then, using spider fusion, we have:



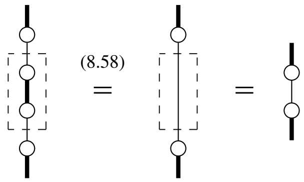
i.e. the identity is disconnected, so decoherence cannot be pure. □

When looking at the diagrammatic form of decoherence, the fact that it is not an identity should not come as a surprise. While the input and output are doubled, in the middle the quantum system seems to be 'squeezed' through a single (classical) wire. Hence, something that lives in two wires is forced into one wire (which will in general come with some data loss) before being injected back into two wires. In physical terms, this means that a quantum state is forced to become classical, and then quantum again.

On the other hand, once decoherence is applied to a quantum system, the damage is done. That is, a second application will leave everything unchanged.

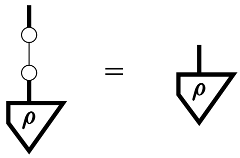
Lemma 8.63 Decoherence is a projector.

Proof Decoherence is clearly self-adjoint, and also idempotent:

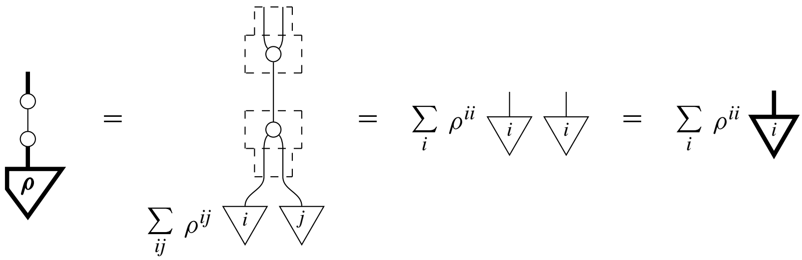


□

In particular, we can now identify a subset of *decoherent* states $\boldsymbol{\rho}$, i.e. those that are unaffected by (further) decoherence:

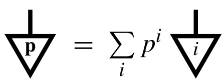


What are these states? Since decoherence is the composite of measure and encode, we already know from the previous section what it does to the matrix of a quantum state:



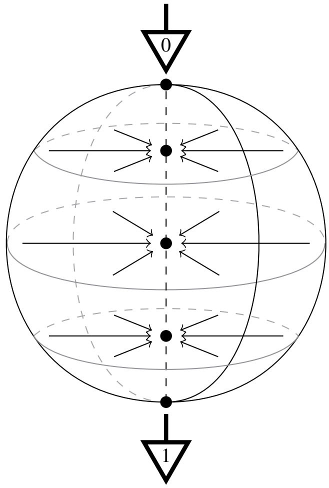
Hence it preserves exactly those states whose non-diagonal entries are all zero. This is stated equivalently.

Theorem 8.64 Decoherence preserves a quantum state $\boldsymbol{\rho}$ if and only if it encodes a probability distribution, i.e. is of the form:


(8.64)

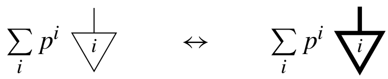
Exercise 8.65 When defining decoherence as a completely positive map (cf. Remark 6.50), what does it do to density matrices?

For qubits, the Bloch ball (see Section 6.2.7) provides a nice geometrical picture of what decoherence does to states, and which states are left unchanged by (further) decoherence. Decoherence projects every state onto the axis passing through the two ONB states:

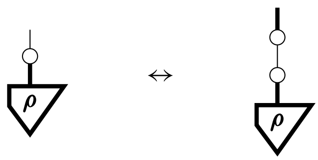


(8.65)

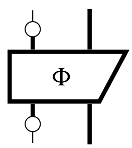
Recall the correspondence:



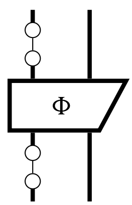
which in sum-free terms can be rewritten as:



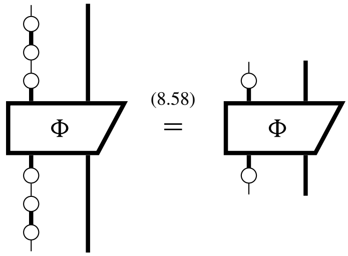
This correspondence says that a quantum output to which decoherence is applied behaves just like a classical output. In fact, this correspondence extends to arbitrary quantum processes. By plugging measuring and encoding maps at its classical inputs and outputs we can turn any cq-map:



into a quantum map:



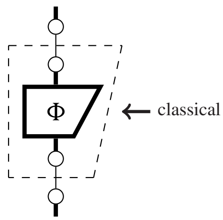
where the presence of a classical input/output of the cq-map we started from is now witnessed by the presence of decoherence. We also easily recover this cq-map by plugging encoding and measuring maps:



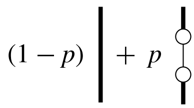
The moral of the story is:

Decoherence forces quantum systems to behave as classical systems.

For a quantum map, this can be directly seen by drawing a picture. When we compose decoherence with the inputs and outputs of any quantum process, at its core it becomes classical:



Since decoherence forces a quantum system to behave like a classical one it shouldn't come as a surprise that decoherence is no friend to anyone wishing to exploit quantum features to, for example, build a quantum computer. In practice, what often happens to quantum systems in the real world is that they undergo some *partial decoherence*:



where the longer we try to store a system, the greater the value of p becomes. The time it takes for p to become 1 is called the *decoherence time*, which typically is very small. One of the biggest challenges in building a quantum computer is to get this time to be as long as possible while still being able to interact with the quantum system in interesting ways.

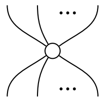
Exercise* 8.66 Characterise partial decoherence without using sums.

8.3.3 Classical, Quantum, and Bastard Spiders

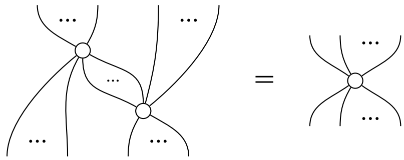
In this section, we will see how measure and encode arise as members of a new species of spider. In fact, all of the calculations in the rest of this book consist of letting different spider species meet and interact with each other. In this section, we perform a bit of spider taxonomy by distinguishing three types of spiders: classical spiders, quantum spiders, and (most interestingly) bastard spiders.

We already know what classical spiders are.

Definition 8.67 A *classical spider* is a spider with only thin legs:

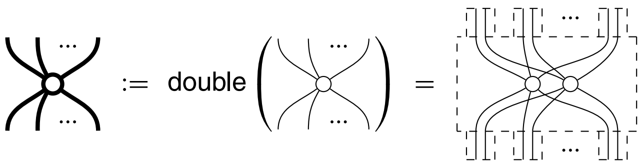


And we also know how they compose:



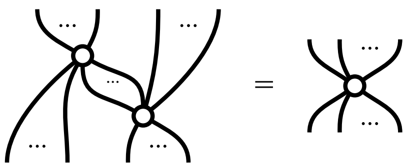
Now, since these classical spiders are linear maps, we can turn them into pure quantum maps by doubling them.

Definition 8.68 A *quantum spider* is a quantum map of the form:



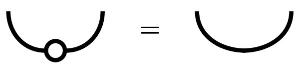
Since equations between linear maps carry over into the doubled world (cf. Corollary 6.16) we also know how they compose.

Corollary 8.69 Quantum spiders compose as follows:

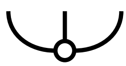


(8.66)

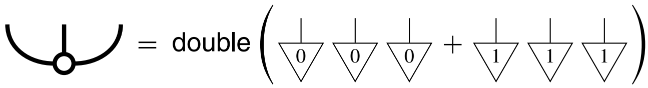
Example 8.70 (The GHZ state) Important examples of quantum spiders are quantum spider states. We have encountered one already: the two-system quantum spider state, which is just the *Bell state*:



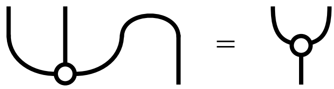
The three-system quantum spider state:



is called the *Greenberger–Horne–Zeilinger (GHZ) state* and has many important applications. For example, in Section 11.1 we will provide a proof of quantum non-locality based on it. Written in terms of an ONB, the qubit GHZ state is:

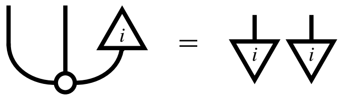


Since it is related to the (doubled) copying spider as follows:



by Theorem 8.18 it characterises a (doubled) ONB.

Simply by relying on copying it follows that if any of the three systems is in one of the basis states of this ONB, then so are the other two:



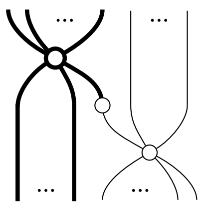
(8.67)

The other n-system quantum spider states:

are usually referred to as *generalised GHZ states*.

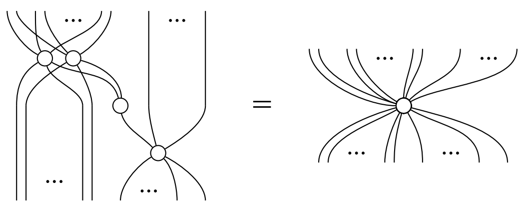
The third type of spiders contains both classical and quantum wires. We have already encountered two extremely important examples:

More generally, we can use measure and encode to connect any classical spider to any quantum spider:



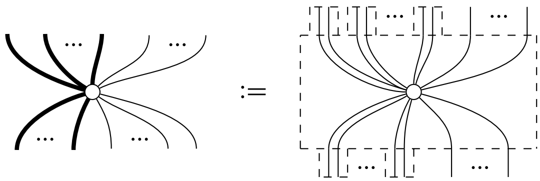
(8.68)

in order to obtain a whole family of classical-quantum hybrids. When unfolding the doubled spider we see that we can fuse all the dots together into a single dot:



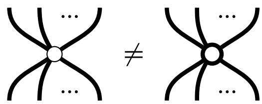
Therefore, the most general spiders that we can obtain using (8.68) are the following ones, which involve only a single (non-bold) dot.

Definition 8.71 A *bastard spider* is a cq-map of the form:

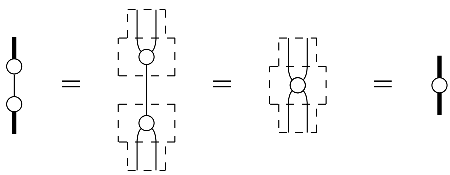


That is, they are the spiders obtained by interpreting some pairs of legs of a classical spider together as doubled systems (a.k.a. *folding*), while leaving others single.

Owing to the fact that it only consists of a single (non-bold) dot, a bastard spider with no quantum legs is the same as a classical spider, but a bastard spider with no classical legs is not a quantum spider:

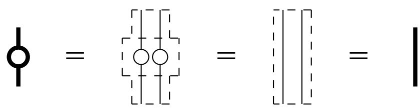


For example, decoherence is a bastard spider with one quantum input and one quantum output:

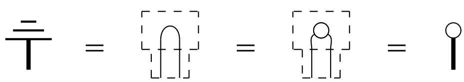


(8.69)

while the corresponding quantum spider is the identity:

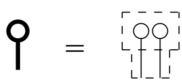


Discarding is also a bastard spider:

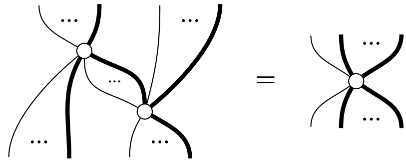


(8.70)

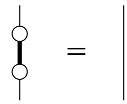
whereas the quantum spider with one quantum input is a pure (non-causal) quantum effect:



Of course, bastard spiders also fuse together:



However, the result may not always be a bastard spider, e.g.:

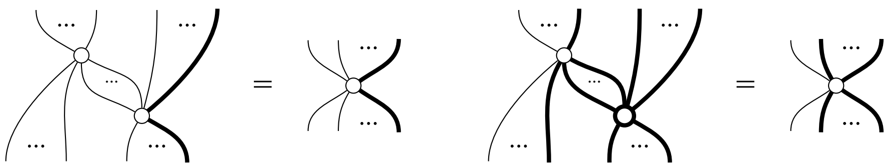


To understand bastard spider fusion, including fusion with classical and quantum spiders, one should think in terms of two species of spiders rather than three, namely:

- 'single-dot' spiders := classical spiders + bastard spiders
- 'double-dot' spiders := quantum spiders

The following is the resulting theorem.

Theorem 8.72 Any composition of spiders involving at least one single-dot spider yields a single-dot spider, for example:



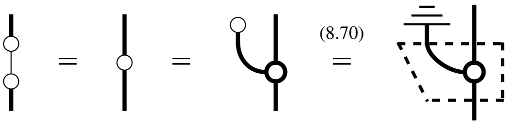
This can be put somewhat differently.

Corollary 8.73 Any connected diagram of classical, quantum, or bastard spiders must be equal to one of the following:

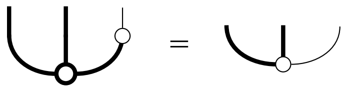
1. a quantum spider if it contains only double dots,
2. a classical spider if it has only classical inputs/outputs, or
3. a bastard spider otherwise.

The paradigmatic example of a single-dot spider that is not a classical spider is decoherence, which forces quantum systems to behave like classical systems. Theorem 8.72 generalises this fact to the world of spiders, where being single-dot can be interpreted as 'being infected by classicality'.

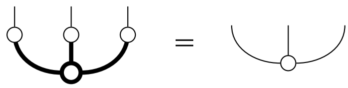
Example 8.74 In equation (8.63), we showed that decoherence is a quantum map. We can now see this proof as an instance of bastard spider fusion:



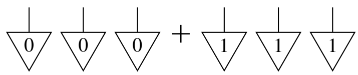
Example 8.75 When we measure one system of three systems in a GHZ state, we obtain a bastard spider:



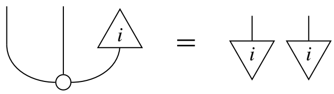
If we measure all three systems, we end up with a classical spider:



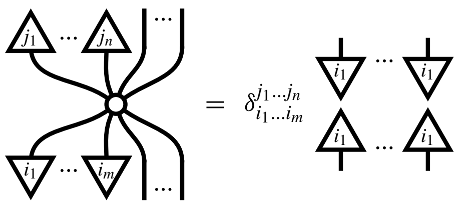
This classical spider describes the classical outcomes of the measurement. Writing it in an ONB in the case of a qubit:



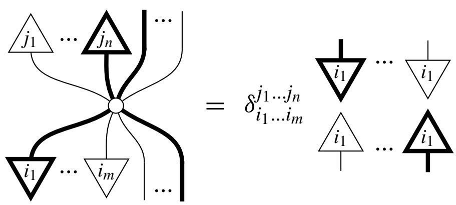
we see that in all three measurements we will obtain the same outcome, although this outcome may be either 0 or 1. This again follows from the generalised copy rule:



which also extends to quantum and bastard spiders:

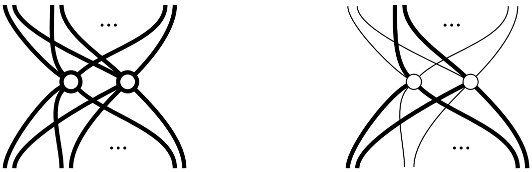


(8.71)



(8.72)

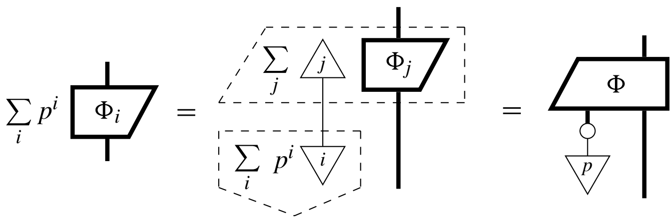
Also as with classical spiders, quantum/bastard spiders can be combined to form product quantum/bastard spiders on compound systems:



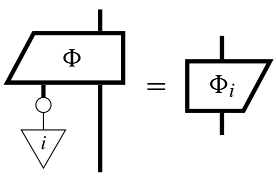
In the case of bastard spiders, we only allow classical legs to pair with classical legs and quantum legs with quantum legs. In particular, the measurement/encoding maps for compound systems are just the obvious things:

8.3.4 Mixing with Spiders

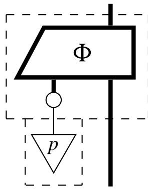
Spiders enable us to define mixing, for which we previously relied on sums, entirely diagrammatically. First, let's take a moment to recall what 'mixing' means for quantum processes. A mixture (cf. Definition 6.70) gives us a way to represent a situation where we have one of several possible quantum processes, but we aren't sure which one. In other words, there is some classical randomness associated with how we obtained this process, which can be expressed as some probability distribution fed into a controlled quantum process. In fact, we can easily show that every mixture arises this way:



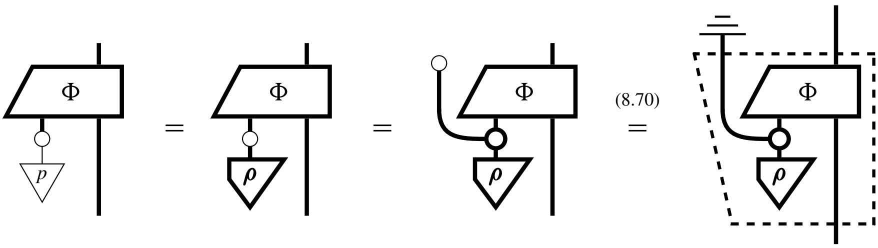
The quantum process with a classical input represents the things being mixed, and the probability distribution p represents the mixing itself. We can obtain the components of the mixture by means of ONB-states:



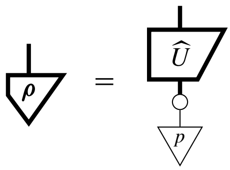
Let's now revisit some of the results from Chapter 6 on mixing. First, the fact that every mixture of causal quantum maps is again a causal quantum map (cf. Theorem 6.71) follows from the fact that composing causal cq-maps:



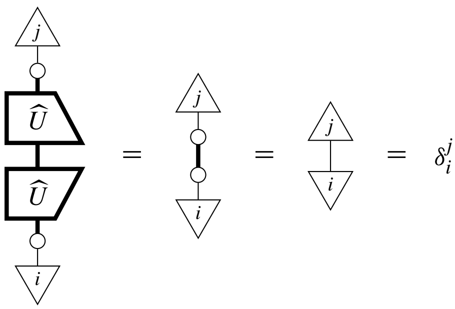
yields again a causal cq-map. In this case there are no classical inputs/outputs, so it is indeed a causal quantum map. The fact that every mixture can be interpreted in terms of discarding part of a system now follows from bastard spider fusion:



The fact that every causal quantum state can be regarded as a mixture of pure causal quantum states (Theorem 6.72) is simply the fact that any quantum state encodes a classical state (Proposition 8.56):

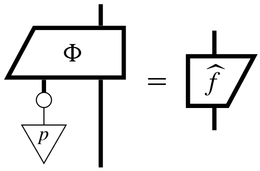


and the fact that these states can always be chosen to be orthonormal is already built in:

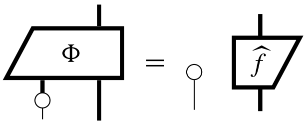


We'll now use this diagrammatic form of mixing to prove something we didn't prove yet, namely, that if the result of mixing quantum processes is pure, then all the processes that have been mixed must be pure, and in fact equal to the result of mixing. For this, we will assume that the probability distribution p used in the mixture has full support (cf. Convention 8.29), which makes sense, since a component with probability 0 contributes nothing to the mixture anyway.

Proposition 8.76 If a mixture is pure:

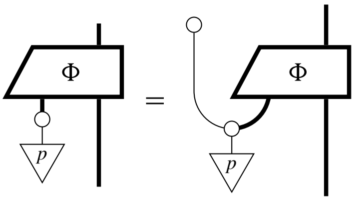


then:

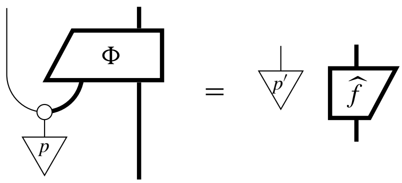


(8.73)

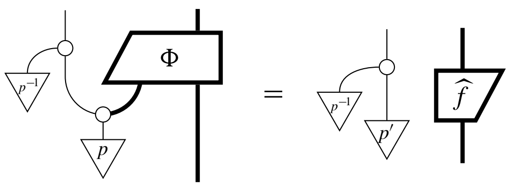
Proof First, we represent any mixture as a reduced cq-map:



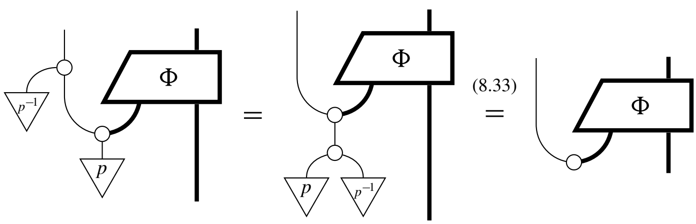
By Proposition 8.59 the cq-map must separate as follows:



We assume p has full support, so apply p^{-1} to both sides:

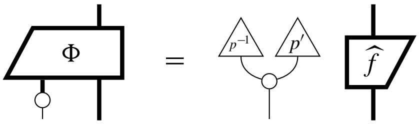


Using spider fusion the LHS becomes:



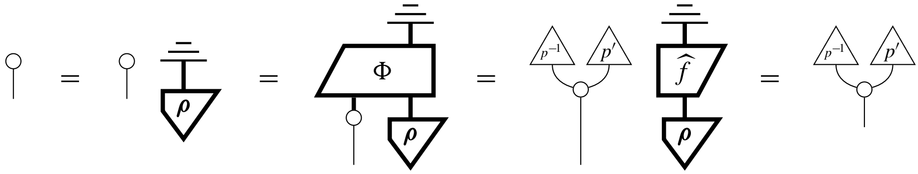
 https://doi.org/10.1017/9781316219317.009

so bending the classical wire down gives:



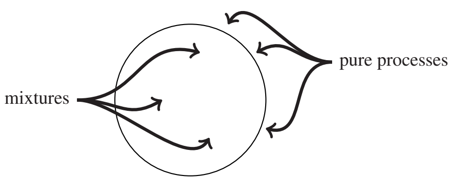
(8.74)

Finally, the fact that the classical effect in (8.74) must be deleting follows from causality of Φ (which also implies causality of $\widehat{f}$). This can be seen by plugging any causal state $\boldsymbol{\rho}$ into the quantum input and discarding the output:



□

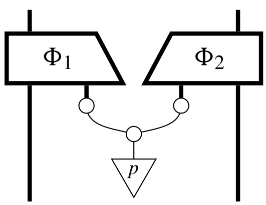
While this proof is entirely diagrammatic, it establishes a geometric property of the space of quantum states, namely that each pure state is 'extremal'; i.e. no pure state can be decomposed as a non-trivial mixture (a.k.a. convex combination). For example, in the Bloch ball we have:



8.3.5 Entanglement for Impure States

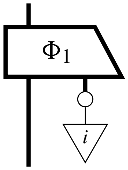
Mixing with spiders will in turn allow us to provide a full characterisation of quantum entanglement, which applies not only to pure states but to all quantum states. We will furthermore give a diagrammatic proof that this reduces to $\otimes$-separability for pure states.

The tricky part about entanglement for impure states is that, by means of mixing, we can obtain $\otimes$-non-separable processes even in cases where there is nothing like quantum entanglement going on. Suppose for example we have a mixture like this:

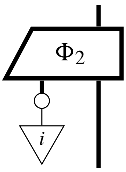


(8.75)

Now, we have two processes in a mixture, but rather than being controlled by two independent classical states, these processes are controlled by the same classical state, as indicated by the presence of a copying map. So, even though we don't know which process actually happens, we at least know that whenever:



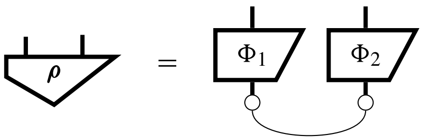
happens on the left,



must happen on the right. That is, they are *classically correlated*. While the diagram as a whole is connected, the part that connects the two components is entirely classical. Such a connection is of a completely different nature from, for example, the quantum cups that we have been exploiting for deriving all kinds of quantum features.

Thus, to properly define entanglement, we should say not only that a quantum state doesn't separate, but also that it is not merely connected by classical correlations.

Definition 8.77 A bipartite quantum state ρ is *entangled* if it cannot be written in the following form for some quantum maps Φ_1 and Φ_2:

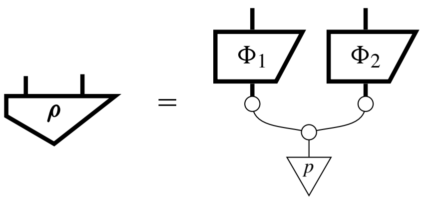


(8.76)

If a state is not entangled, then we call it *disentangled*.

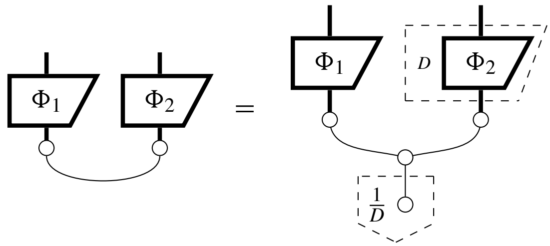
Even though we define disentangled states using the classical cup (i.e. 'perfect correlations'), we could equally well use any classical correlations, as we did in (8.75) above.

Proposition 8.78 A bipartite quantum state ρ is entangled if it cannot be written in the following form for some quantum maps Φ_1 and Φ_2 and probability distribution p:

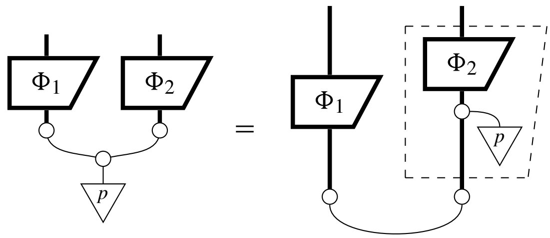


(8.77)

Proof If a state is disentangled as in Definition 8.77, then:

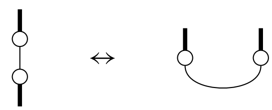


so it matches the form (8.77). Conversely, we have:



so it is indeed in the form (8.76). □

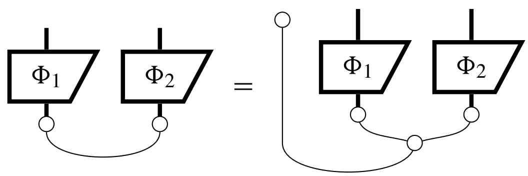
Example 8.79 Using process–state duality, decoherence gives rise to a classically correlated state of the form (8.76):



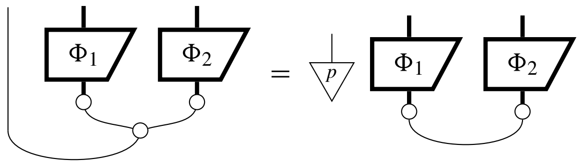
Back in Chapter 6, when we defined entanglement for pure states as ⊗-non-separability, we didn't know anything about classical wires. However, this still arises as a special case.

Proposition 8.80 If a pure bipartite quantum state is disentangled in the sense of Definition 8.77, then it is ⊗-separable.

Proof First, just as in the proof of Proposition 8.76, we represent a process that is assumed to be pure as a reduced process using spider fusion. In this case, we can represent any pure disentangled state as the following reduced state:

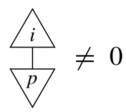


Since the bipartite state obtained from deleting a classical system is pure, by Proposition 8.59 the whole state must separate as follows:

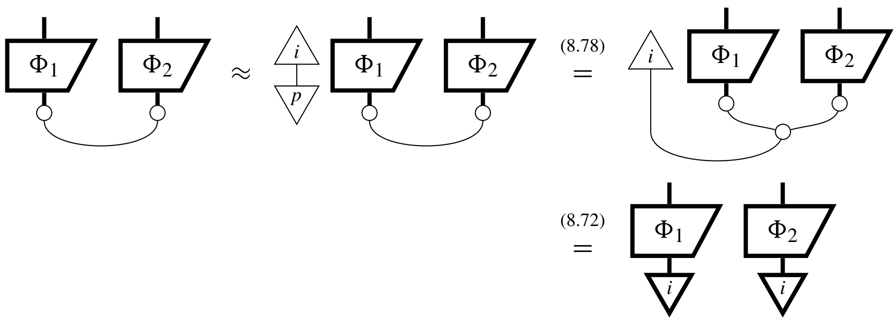


(8.78)

for some causal classical state p. So there exists at least one basis state i such that:



Hence we have:

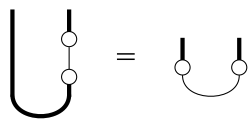


□

This yields the following fact as corollary.

Corollary 8.81 The quantum cup is entangled.

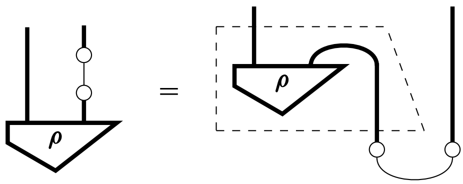
Having diagrammatic forms at hand for many quantum features, we can now easily investigate how these relate. For example, what happens if we apply decoherence to one of two systems in a Bell state? It disentangles:



This is an instance of a more general fact.

Theorem 8.82 Decoherence destroys entanglement; that is, if we apply decoherence to one of two systems in an entangled state, then it disentangles.

Proof We have:

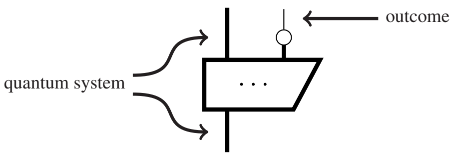


□

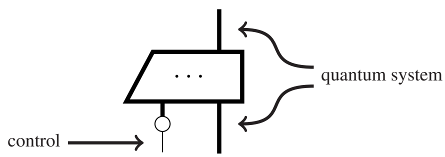
8.4 Measurements and Protocols with Spiders

We will now use spiders to give diagrammatic presentations of all of the families of quantum measurements and quantum protocols we have encountered so far, without all of the branches and sums.

Non-demolition measurements are cq-maps of this shape:



while controlled unitaries have a dual shape:

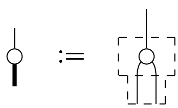


By wiring together maps like these, we will give graphical presentations of teleportation, dense coding, and entanglement swapping.

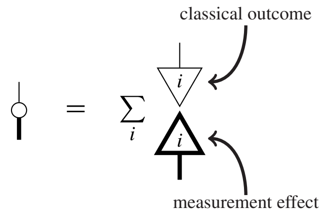
We top off this section by revisiting Naimark dilation. By translating it to the language of cq-maps and spiders, this now becomes a tautology.

8.4.1 ONB Measurements

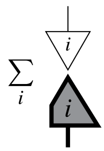
For the measurement:



we used the same basis for the measurement effects as we did for the classical outcomes:

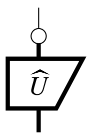


Of course, there is no reason that we should assume these bases are the same, so arbitrary (demolition) ONB-measurements are quantum processes of this form:



But by Proposition 7.4, we know that we can obtain measurements in any ONB by applying a unitary to some fixed ONB, so we have the following corollary.

Corollary 8.83 Every demolition ONB measurement is of the form:

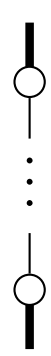


(8.79)

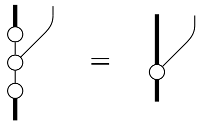
where $\widehat{U}$ is a unitary quantum process.

The simple measuring process:

is of course a special case, where $\widehat{U}$ is the identity. The measuring process also has a non-demolition counterpart, which leaves the quantum system intact but sends every state to an eigenstate of the measurement, depending on the measurement outcome. We can picture this as measuring, followed by encoding:

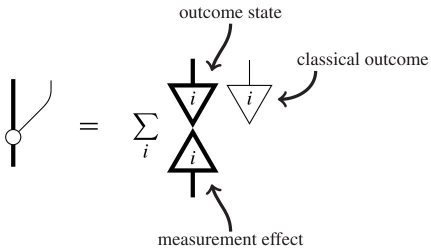


However, we should also get the classical value out at the end, so before feeding it in to encoding, we make a copy:

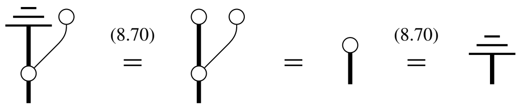


(8.80)

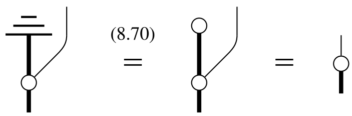
If we expand this as a sum, we see it captures the expression of a non-demolition ONB measurement as a cq-map:



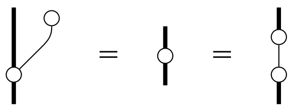
In particular, it is a bastard spider, and we can show causality by bastard spider fusion:



If we discard the quantum output, we obtain the demolition measurement we had before:



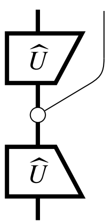
On the other hand, if we discard the classical output, we get decoherence:



(8.81)

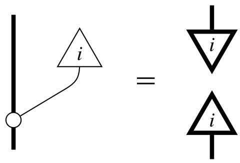
As in the demolition case, we express a general non-demolition ONB measurement in terms of bastard spiders and a unitary.

Proposition 8.84 Every non-demolition ONB measurement is of the form:



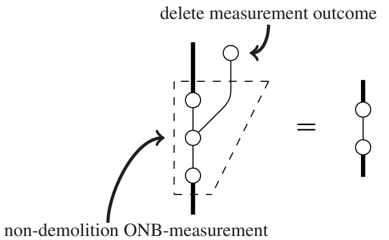
where $\widehat{U}$ is a unitary quantum process.

Exercise 8.85 First show that Exercise 8.32 extends to quantum spiders and bastard spiders. Then, use the following instance of this result:



to prove Proposition 8.84.

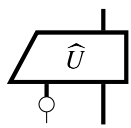
Remark 8.86 Equation (8.81) gives us another way to understand decoherence:



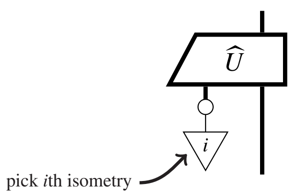
We made a big point in the previous chapter that quantum measurement is just some sort of quantum process. It represents an interaction between a quantum system and, well, us. A happy result of this interaction is that we get some information about the quantum state: a measurement outcome. Decoherence is what happens to a quantum state when some interaction with its environment causes the state the collapse, as if it had been measured. Unfortunately, it's a particularly bad kind of measurement, because it happens spontaneously and we don't even get to know the outcome!

8.4.2 Controlled Unitaries

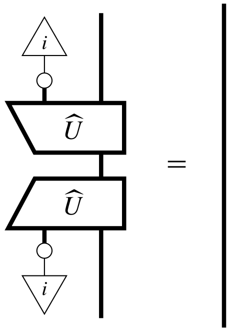
In order to diagrammatically present the protocols over the next few sections, we need to represent a controlled isometry as a single cq-map:



Since we can recover the individual isometries as follows:

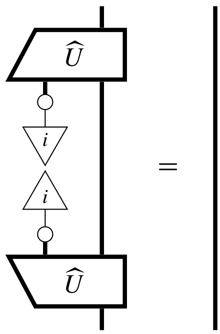


a controlled isometry is a cq-map such that, for all i:



(8.82)

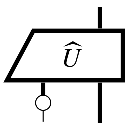
A controlled unitary then additionally satisfies:



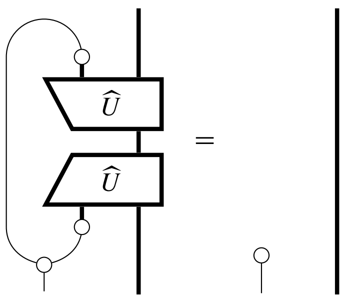
(8.83)

With the help of spiders, we can roll these indexed sets of equations into single equations.

Proposition 8.87 A cq-map:

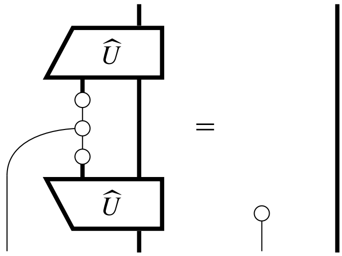


is a controlled isometry if and only if it satisfies:



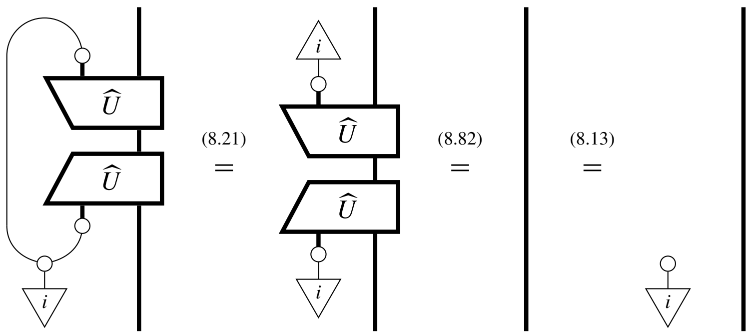
(8.84)

and is moreover a controlled unitary if and only if it additionally satisfies:

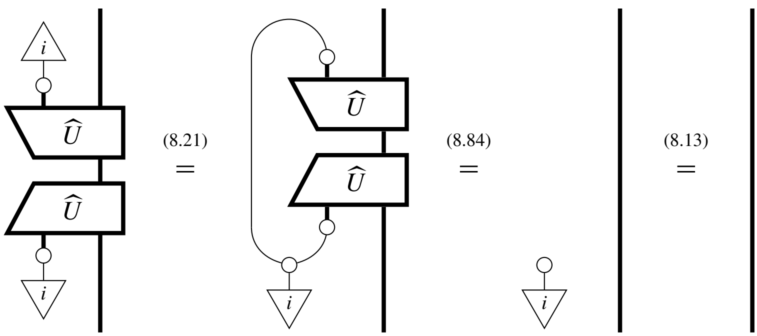


(8.85)

Proof We will show the equivalence of (8.82) and (8.84). First, assume (8.82) holds for all i. Then:



Since the LHS and RHS agree on all ONB states, (8.84) follows. Conversely, assuming (8.84), for all i we have:

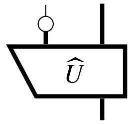


The equivalence of (8.83) and (8.85) follows similarly. □

Exercise 8.88 Complete the proof of Proposition 8.87. That is, extend it to the case of controlled unitaries.

We can also ask if a controlled isometry as a whole is itself an isometry. It turns out this is the case.

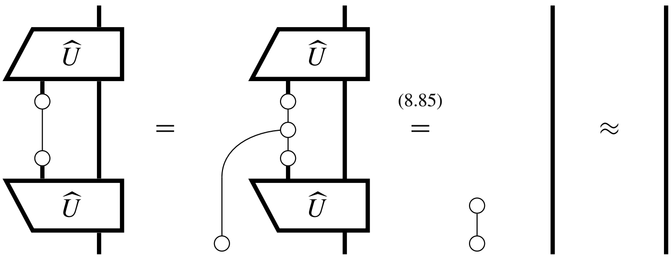
Proposition 8.89 A quantum process:



(8.86)

satisfying equation (8.85) is an isometry up to a number.

Proof We have:



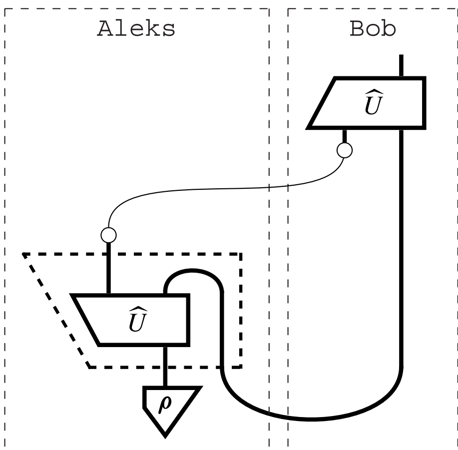
(8.87)

□

Exercise 8.90 Is (8.86) unitary if we also have (8.84)?

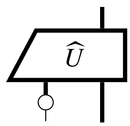
8.4.3 Teleportation

General teleportation can now be presented as follows:

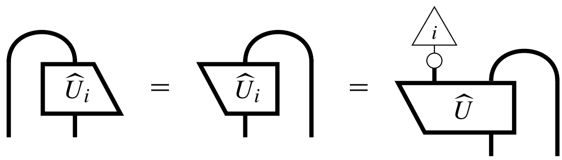


(8.88)

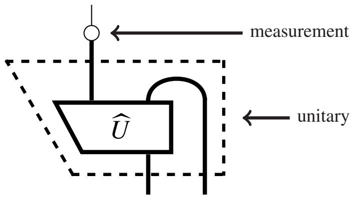
The key is that both:



and its adjoint occur in the diagram, which will enable us to invoke Proposition 8.89 to cancel them out. However, in each of those two occurrences the cq-map is playing a very different role. On the one hand, it is a controlled unitary obeying equations (8.84) and (8.85). On the other hand, it is used to construct an ONB measurement:

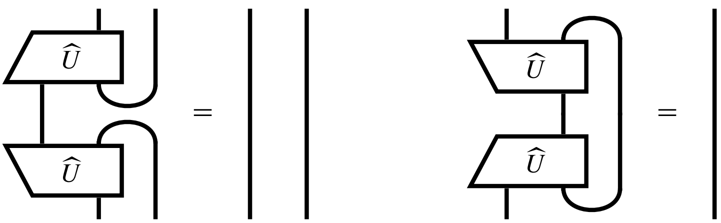


We now know, by Corollary 8.83, that we obtain an ONB measurement precisely when we have a unitary followed by the measure process:



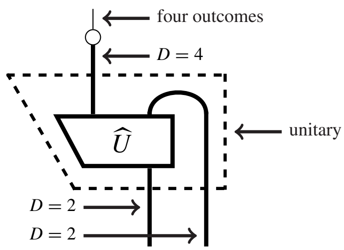
(8.89)

Unitarity of the marked map means that the following two equations hold:



(8.90)

In particular, if each of the input systems is D-dimensional, the output system must be D^2-dimensional for the above map to be unitary. So, if each input system is a qubit, the output must be four-dimensional:

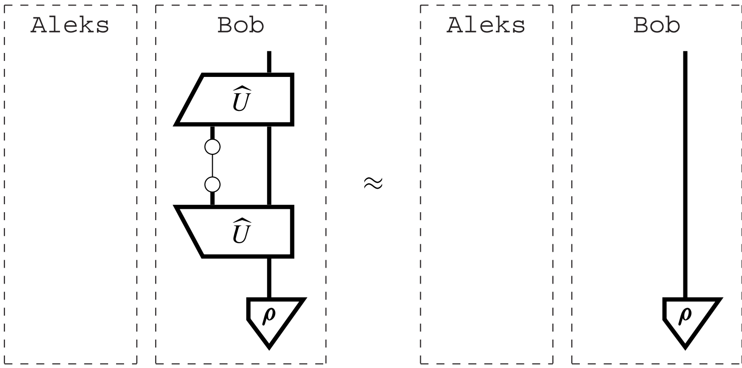


Remark 8.91 Note that unitarity has taken the place of equations (6.79) and (6.80), which we used in Section 6.4.6 to give a generalised teleportation protocol. In either case, these

guarantee an ONB of measurement effects. This is not necessary for teleportation to 'work'. For that, we only need (8.89) to be causal, and hence only the first of equations (8.90) needs to hold. This corresponds to having more measurement outcomes than strictly necessary; e.g. Aleks could perform a POVM measurement by flipping a coin and deciding to perform one of two ONB measurements.

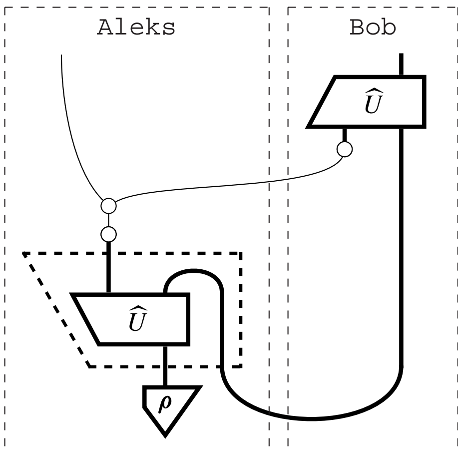
We can now show that (8.88) correctly implements teleportation:

(8.88) =



where the last step uses Proposition 8.89.

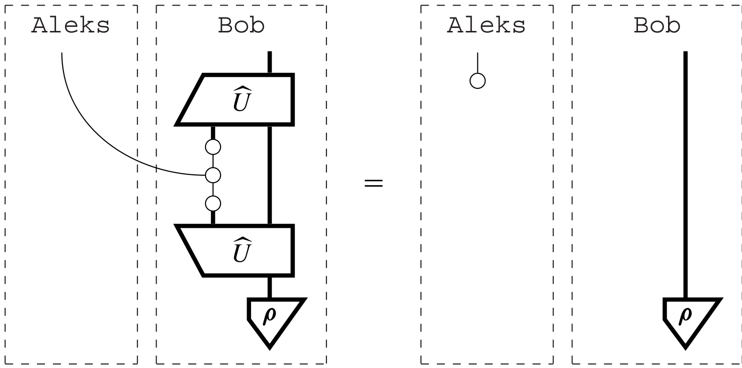
In case you didn't notice yet, there is something slightly weird about the manner in which we have described teleportation until now. Aleks' measurement outcome is used by Bob to do the appropriate correction, but then this measurement outcome seems to have been deleted from everyone's notebooks and memories. In reality, we expect to get some classical data out at the end, corresponding to the measurement outcomes Aleks got. We can fix this by making a copy of the classical data before it is 'consumed' by the controlled unitary:



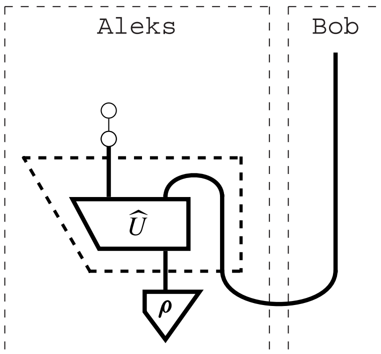
(8.91)

Interestingly, now we need the full power of equation (8.85) to prove correctness, rather than its reduced version as in (8.87):

(8.91) =



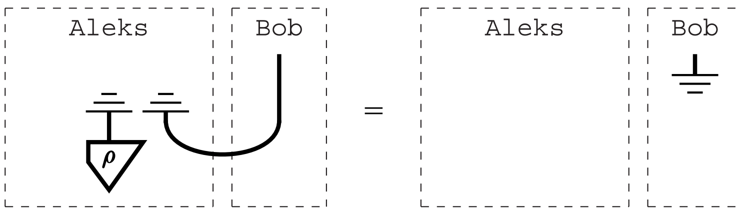
We can now also give a clearer (sum-free) picture of what happens if we delete the classical data immediately after the measurement and don't do any correction at all, as we did in Section 6.4.4. That would now look this:



(8.92)

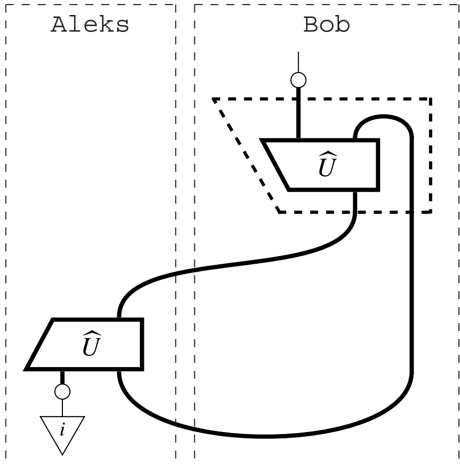
From causality for cq-maps it follows that we have:

(8.92) =



8.4.4 Dense coding

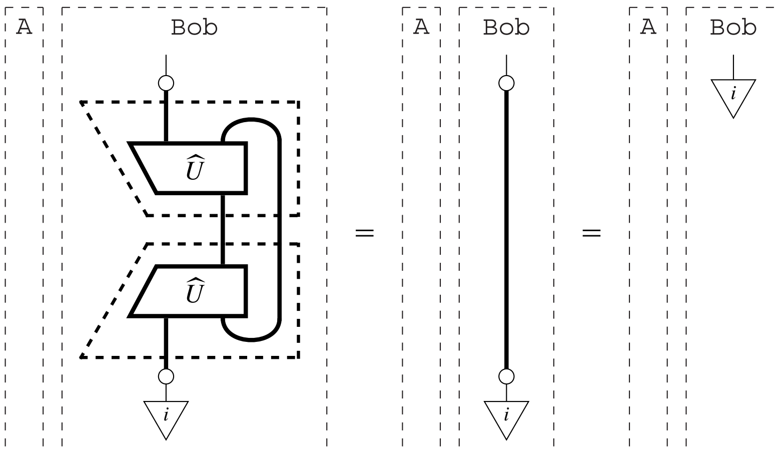
Recall from Section 8.1.2 that dense coding is the protocol where Aleks uses quantum systems to send classical data. He does this by using his classical data to perform a controlled unitary on half a Bell state, then sending his half to Bob, who can then recover Aleks' data by measuring both quantum systems together. As a diagram, dense coding looks like this:



(8.93)

which simplifies to:

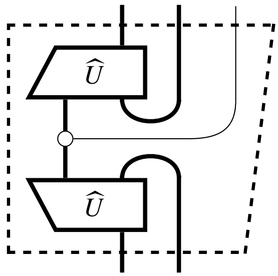
(8.92) =



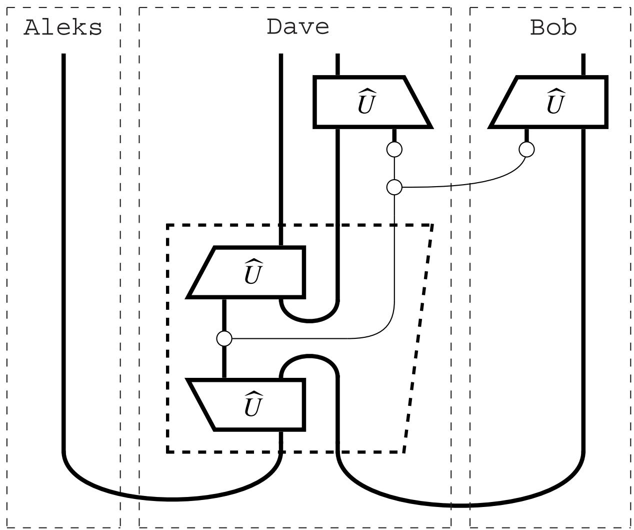
where the first step uses (8.90), i.e. unitarity of the marked boxes. So, in order to prove correctness we rely on an equation different from the one for teleportation, something that we already pointed out in Remark 8.2.

8.4.5 Entanglement Swapping

Recall from Section 7.2.4 that entanglement swapping is a protocol that swaps the entanglement among four quantum systems, by means of a non-demolition measurement on two of those systems. To realise the protocol, we start with Aleks and Bob each sharing a Bell state with a third party (in this case Dave the dodo). Dave then performs a non-demolition variant of the measurement we were using before:



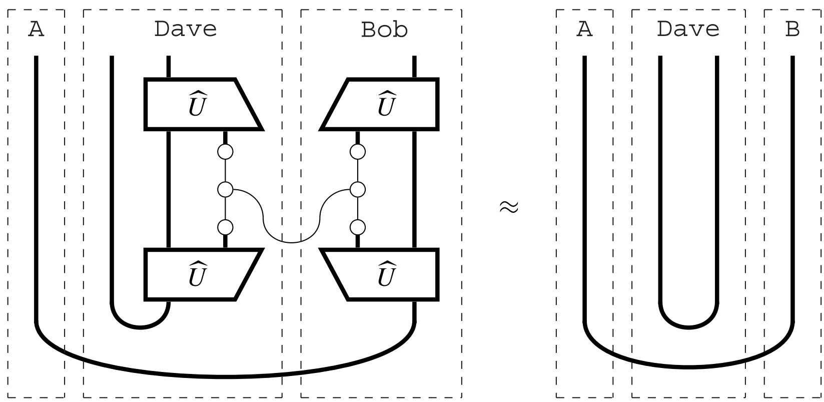
This is indeed a non-demolition ONB measurement by Proposition 8.84. The outcome of this measurement then needs to be copied to two controlled unitaries, which perform corrections. So, the whole protocol looks like this:



(8.94)

We can simplify by using bastard spider fusion and the controlled-isometry equations to eliminate all of the $\widehat{U}$ maps:

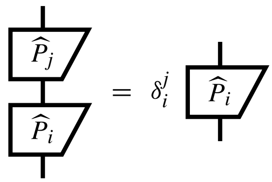
(8.94) $=$



where we use Proposition 8.89 twice.

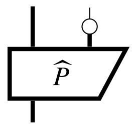
8.4.6 Von Neumann Measurements

In Section 7.3.1, we showed that a von Neumann measurement can be defined as a quantum process obeying the collapse postulate; that is, for all i, j:

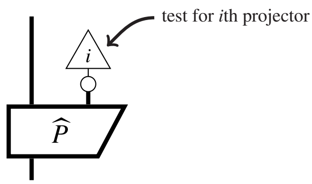


(8.95)

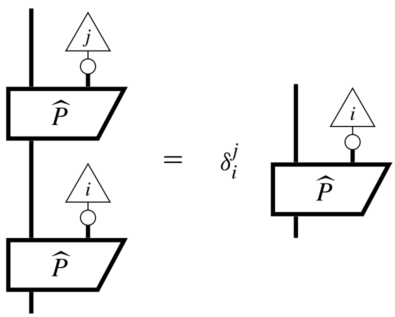
We can now write a von Neumann measurement as a single cq-map:



whose associated projectors can be recovered as follows:

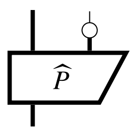


Hence (8.95) now becomes, for all i and j:

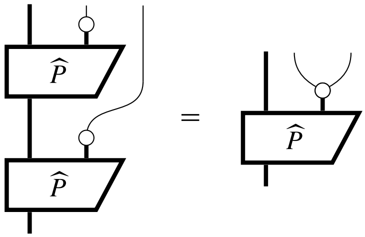


(8.96)

Proposition 8.92 A quantum process:

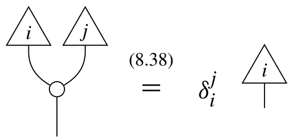


is a von Neumann measurement if and only if it satisfies:



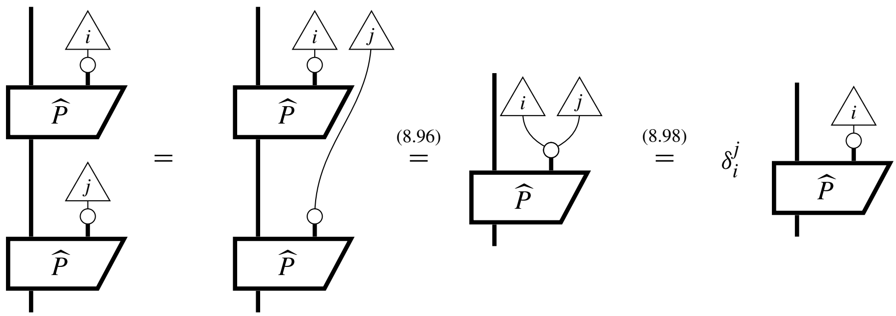
(8.97)

Proof First, assume (8.97), then using:



(8.98)

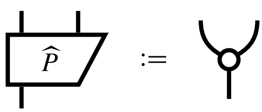
we obtain:



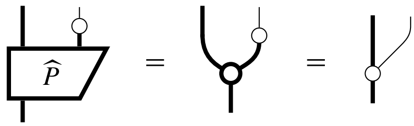
Conversely, by assuming (8.96), we can show similarly that the LHS and RHS of (8.97) agree on all classical ONB effects. Hence they are equal (cf. the proof of Proposition 8.87). □

Equation (8.97) has a direct operational reading. Von Neumann measurements have the property that if we measure once, then measure again, we should always get the same result the second time. Equation (8.97) captures this as follows: if we measure twice, we will get precisely the same output as if we measure once, then copy the measurement outcome.

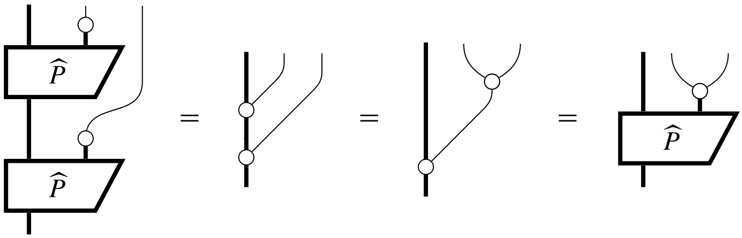
Example 8.93 The non-demolition ONB measurement given by (8.80) is a von Neumann measurement, letting:



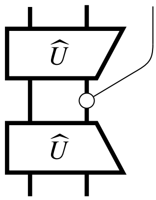
Then, using bastard spider fusion we have:



so, again using bastard spider fusion:

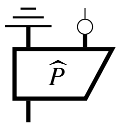


Exercise 8.94 Show that the more general form of a non-demolition ONB measurement, as presented in Proposition 8.84, is a von Neumann measurement and that, more generally, for any unitary $\widehat{U}$:



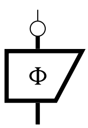
is a von Neumann measurement.

Just as before, we obtain demolition von Neumann measurements by discarding the quantum output of the associated non-demolition measurement:

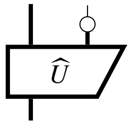


8.4.7 *POVMs and Naimark Dilation*

In the previous chapter, we defined a demolition POVM measurement as any quantum process consisting of effects. Thus, a 'demolition POVM measurement' is really just a generic process from a quantum to a classical system:



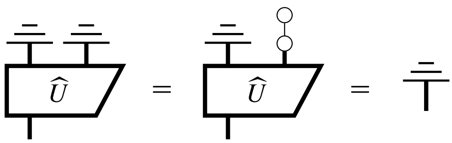
As we saw in Section 7.3.3, 'non-demolition POVM measurements' are just quantum processes where each of the branches is pure. As a cq-map, this becomes:



(8.99)

We showed in Section 7.3.3 that these are already generic enough to recover all demolition POVM measurements by discarding the quantum system.

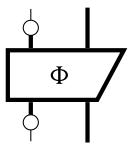
It is easy to see how causality for the cq-map (8.99) translates into causality for the underlying pure quantum map $\widehat{U}$:



(8.100)

Since $\widehat{U}$ is pure and causal, it is an isometry by Theorem 6.56.

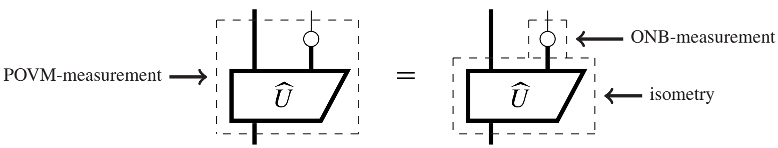
Exercise 8.95 Extend (8.100) to the case of arbitrary cq-maps. That is, show that for any quantum process:



(8.101)

we can always choose the quantum map Φ to be causal. Conversely, show that for any causal quantum map Φ, the associated cq-map (8.101) is causal.

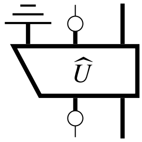
Let's now have another look at Naimark's dilation theorem (cf. Theorem 7.31), which states that any non-demolition POVM measurement can be expressed in terms of an isometry and an ONB measurement. The non-demolition POVM measurement (8.99) is a causal cq-map, hence $\widehat{U}$ is an isometry. Thanks to our representation of quantum processes as cq-maps:



there is nothing left to prove!

Combining this result with Stinespring dilation for causal quantum maps, we obtain a simple alternative presentation of quantum theory.

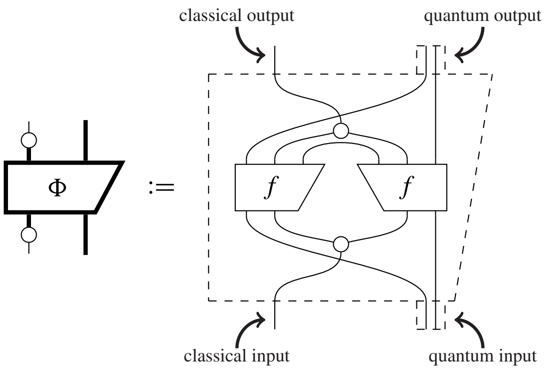
Theorem 8.96 Quantum processes are linear maps of the form:



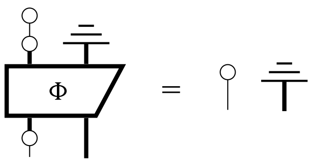
where $\widehat{U}$ is an isometry.

8.5 Summary: What to Remember

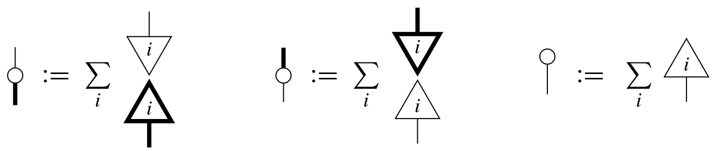
1. The theory of **quantum processes** (a.k.a. quantum theory) is the processes theory of *classical-quantum maps*:



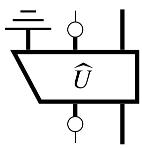
which are moreover *causal*:



where:

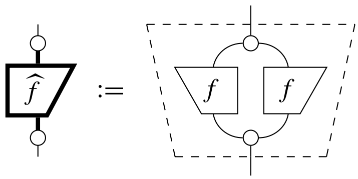


Equivalently, by Stinespring dilation, quantum processes are linear maps of the form:

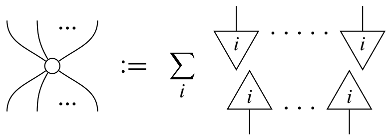


where $\widehat{U}$ is an isometry.

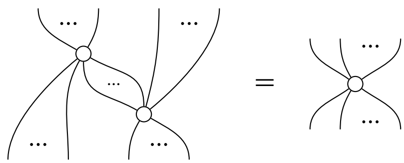
2. The theory of all **classical-quantum maps** admits string diagrams. It includes **quantum maps** and **classical maps** as sub-theories, where the latter consists of processes of the form:



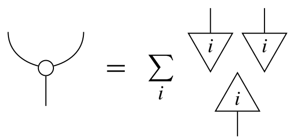
3. Particularly well-behaving classical maps are *classical spiders*:



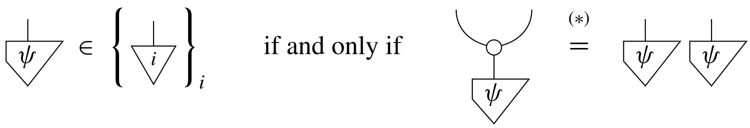
They compose as follows:



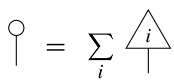
4. An example of a classical spider is *copying*:



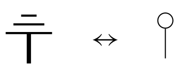
In particular, copying determines an ONB:



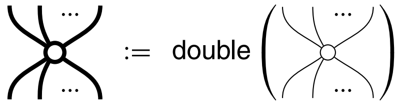
Another example is *deleting*:

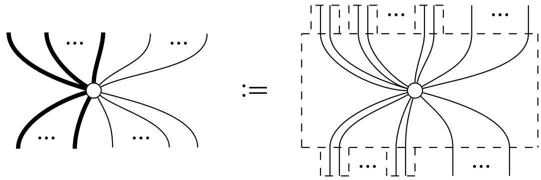


which, as we saw in **1** above, plays a key role in stating causality. Namely, it is the classical counterpart to discarding:

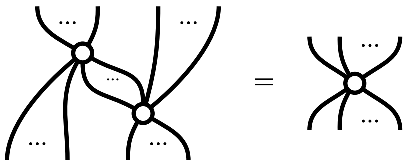


5. By doubling classical spiders or pairing certain legs:

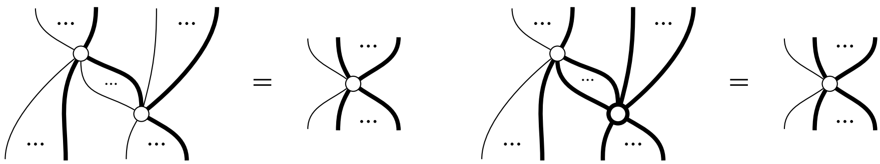




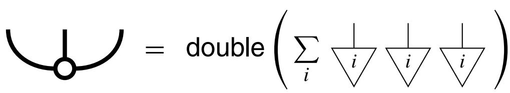
we obtain *quantum spiders* and *bastard spiders*, respectively. Quantum spiders compose just like classical spiders:



whereas any composition of spiders involving at least one single-dot spider yields again a single-dot spider:

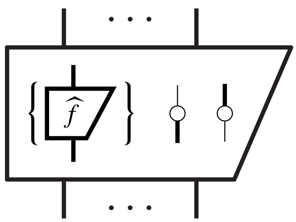


6. An example of a quantum spider is the *GHZ state*:

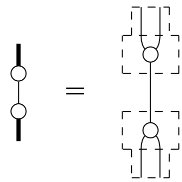


Examples of bastard spiders are *measure* and *encode*:

which we also already encountered in **1** above. In particular, all cq-maps can be obtained by composing pure quantum maps, measuring, and encoding:

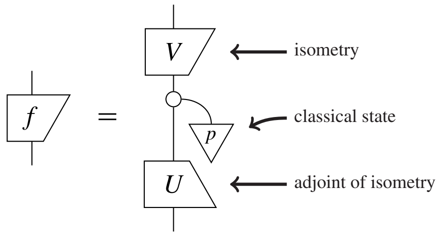


Another bastard spider is *decoherence*:

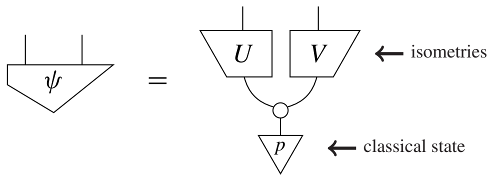


which models a quantum state degrading to a classical state.

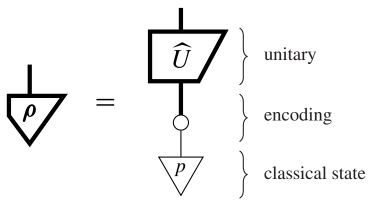
7. Any linear map f decomposes as:



for an appropriately chosen spiders, and bipartite states decompose as:

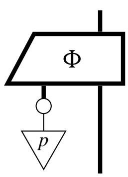


All quantum states $\boldsymbol{\rho}$ encode classical states, via:

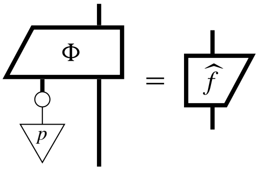


(8.102)

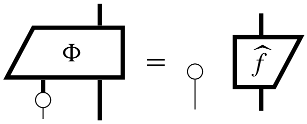
8. Mixing a set of processes of the same type by means of a probability distribution p can be represented as a cq map:



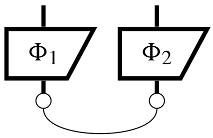
By (8.102) every causal quantum state can be regarded as a mixture of pure causal quantum states. If a mixture is pure:



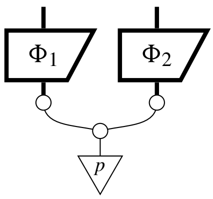
then:



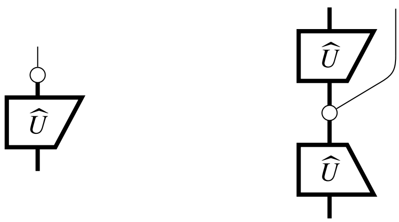
9. A bipartite quantum state is *entangled* if it <u>cannot</u> be written in the form:



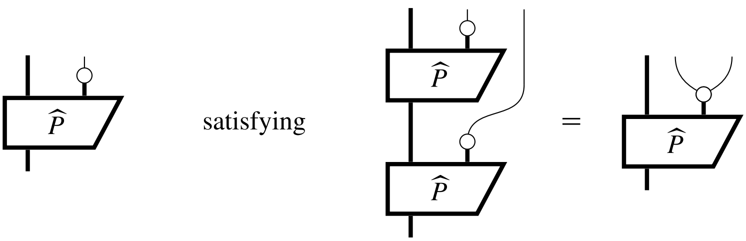
or, equivalently, if it <u>cannot</u> be written as a mixture as follows:



10. Demolition ONB measurements and non-demolition ONB measurements are of the following forms, respectively, for some unitary $\widehat{U}$:

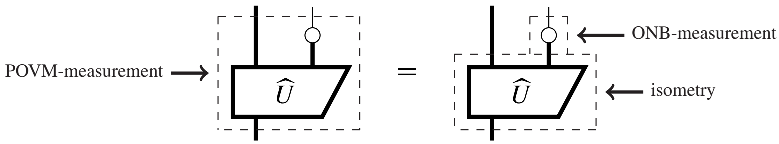


Non-demolition von Neumann measurements are quantum processes:

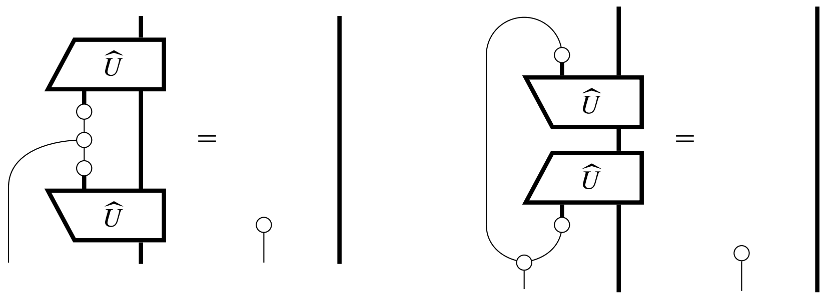


Demolition von Neumann measures are the same, but with the quantum output discarded. Demolition and non-demolition POVM measurements are quantum processes of the following forms:

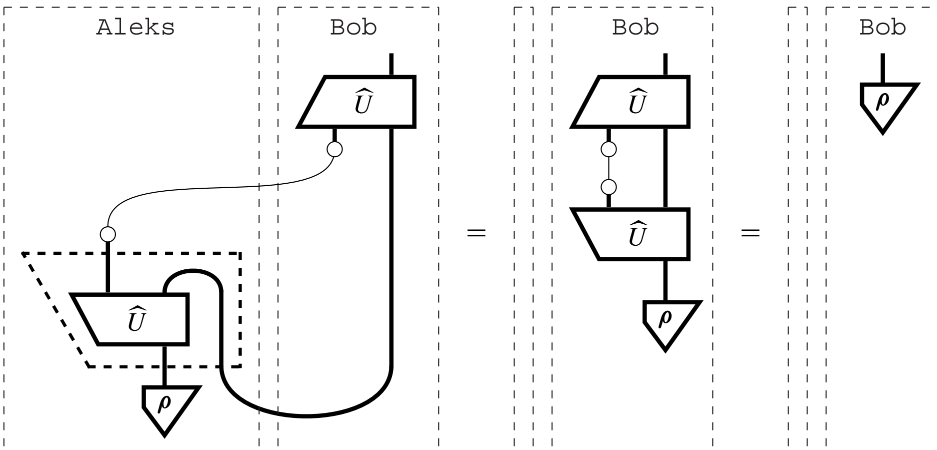
In particular, $\widehat{U}$ must be an isometry by causality, so Naimark dilation now boils down to two different readings of the same diagram:



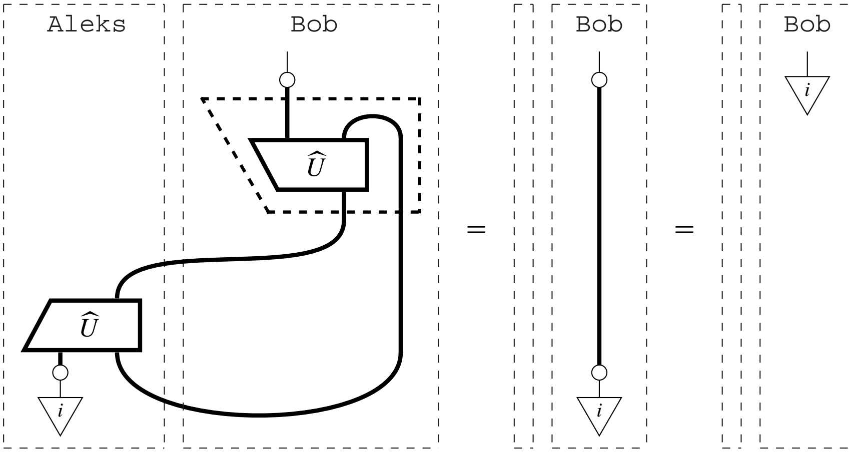
11. Controlled unitaries can be defined as follows:



This enables us to give totally diagrammatic presentations of quantum protocols such as *quantum teleportation*:



and *dense coding*:

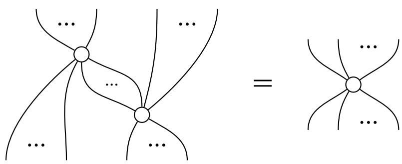


8.6 Advanced Material*

In this section, we elaborate a bit on what spiders are, from a more (co)algebraic perspective. In fact, if we drop the commutativity requirement, we discover how they become a graphical tool to study C*-algebras, which are a popular structure to study for algebraists and mathematical physicists. Then we briefly discuss how spiders for non-self-conjugate ONBs look. They turn out to have some hairs on their legs! And if that wasn't scary enough, we'll see in the last section that we've got spiders coming out of our mouths too!

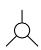
*8.6.1 Spiders Are Frobenius Algebras**

We introduced spiders as a species of creature with any number of legs that fuse when they 'shake legs':

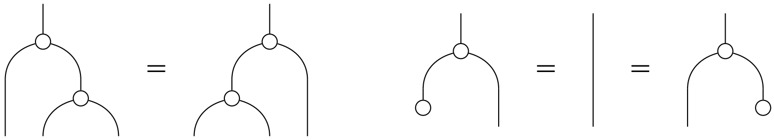


However, this is quite different from the usual way one defines these processes, and in fact, not the way we originally encountered them. Usually they are defined in terms of something that may be more familiar to mathematicians.

Definition 8.97 An *associative algebra* on a vector space V consists of a pair of linear maps:

 $: V \otimes V \to V$  $: \mathbb{C} \to V$

such that is associative and has unit :

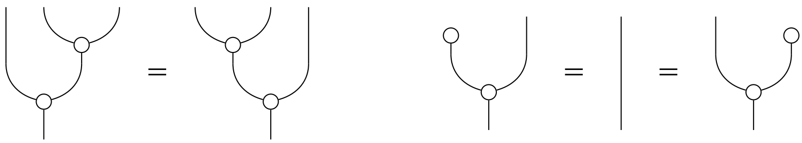


This is the same as having a multiplication operation on elements of V that is linear in both arguments, associative, and unital. However, the benefit of writing it this way is it's very easy just to turn everything upside-down! It's a common convention in category theory to call something a 'co-Thing', if it is a Thing with all of the maps turned upside-down.

Definition 8.98 A *coassociative coalgebra* on a vector space V consists of a pair of linear maps:

$$: V \to V \otimes V \qquad\qquad : V \to \mathbb{C}$$

such that is *coassociative* and has *counit* :



The most obvious way to get such a thing is to let V be a Hilbert space rather than just a vector space, and take the adjoint of an associative algebra:

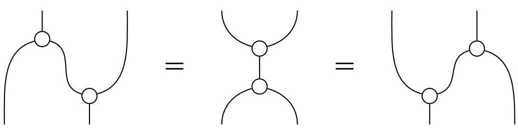
$$:= \left(\quad \right)^{\dagger} \qquad\qquad := \left(\quad \right)^{\dagger}$$

While *algebraic* structures may be quite familiar, *coalgebraic* structures might be less so. Perhaps one of the reasons for this is they tend not to be very interesting in process theories where $\otimes$ behaves like a Cartesian product. For example, if we replace **linear maps** with **functions** in Definition 8.98, the only coassociative coalgebras are the 'universal' copying functions:

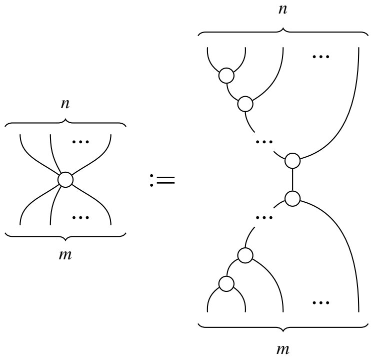
$$: X \to X \times X :: \; x \mapsto (x, x)$$

However, when $\otimes$ is non-Cartesian, as with **linear maps**, we have lots of interesting coalgebras, and, more importantly, we can define interesting structures that have both an algebraic and a coalgebraic part. Our key example is the following.

Definition 8.99 A *Frobenius algebra* consists of an associative algebra (,) and a coassociative coalgebra (,) that additionally satisfy the *Frobenius equations*:



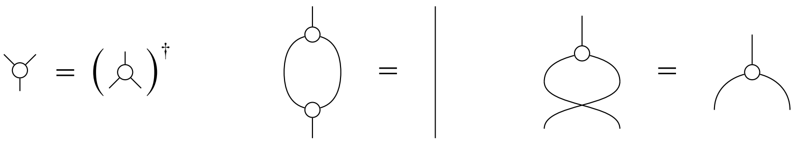
We can define 'spider-like' maps using a Frobenius algebra:



(8.103)

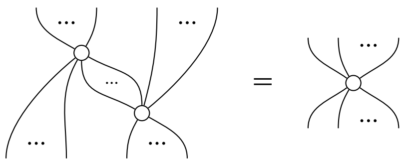
However, for these spiders to fuse as in Theorem 8.34 and for the species to be closed under taking adjoints we need a particularly 'special' kind of Frobenius algebra.

Definition 8.100 A dagger special commutative Frobenius algebra (†-SCFA) is a Frobenius algebra that additionally satisfies:

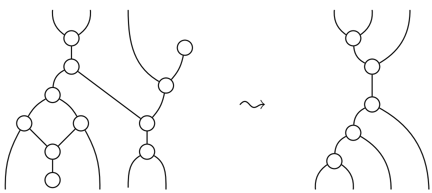


Evidently, the †-part of †-SCFA takes care of horizontal reflection, while the SC-part guarantees the following.

Proposition 8.101 Spiders defined as in (8.103) for a †-SCFA compose as:



Proof (sketch) It suffices to show that any connected diagram consisting of (, , ,) can be transformed into a canonical form, just using the Frobenius algebra equations:

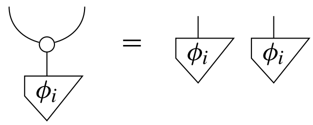


In particular, two connected spiders can be transformed into a canonical form, which will be one big spider. This can be shown by induction over the number of dots in the diagram. □

One thing that is notable is that none of the definitions in this section makes use of any vector space structure, so they actually make sense in any dagger symmetric monoidal category.

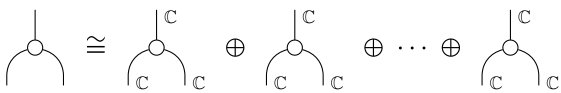
The algebraic presentation of spiders enables us to use standard theorems from algebra to prove that they always define an orthonormal basis.

Theorem 8.102 For any †-SCFA, the set of states $\{\phi_i\}_i$ such that:



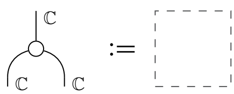
always forms an ONB. Thus, every family of spiders uniquely determines (and is uniquely determined by) an ONB.

Proof (sketch) The proof goes in two stages. First, one can show that the algebra part of a †-SCFA is always *semi-simple*. Semi-simple algebras (which we won't define here) are well understood, especially in finite dimensions, thanks to *Wedderburn's theorem*. This theorem implies in particular that any associative algebra that is semi-simple and *commutative* is actually isomorphic to a direct sum of copies of the trivial algebra on $\mathbb{C}$:

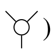
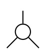


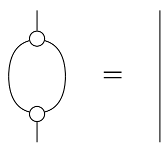
(8.104)

This 'trivial algebra' might look strange, since we don't usually draw wires for $\mathbb{C}$. It's actually just:



which is obviously associative and has as its unit also the empty diagram. The algebra (8.104) always has a basis of copyable states, which look like this:

Then, the fact that ()† =  suffices to show that any basis of copyable states must be orthogonal, and furthermore:

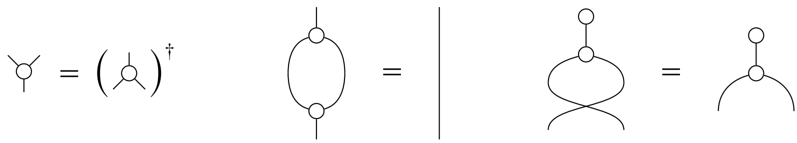


implies that it must consist of normalised states. □

*8.6.2 Non-commutative Spiders**

In Theorem 8.102 we saw that a †-SCFA fixes a unique ONB. So what happens if we drop some of the letters in '†-SCFA'? If we drop the S of 'special', then, rather than ONBs, we obtain orthogonal bases. If we drop the †, but retain the S, then we obtain arbitrary bases. That's quite interesting. However, if we start to mess around with the C of 'commutative', then even more interesting things happen. We won't just drop C but instead we'll replace it with something weaker.

Definition 8.103 A dagger special *symmetric* Frobenius algebra (†-SSFA) is a Frobenius algebra that additionally satisfies:



This definition looks almost the same as before, but note that the third equation not longer has an output. Despite this seemingly minor change, †-SSFAs are actually much more general than their commutative cousins.

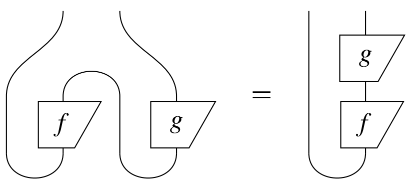
Theorem 8.104 Every finite-dimensional C*-algebra is isomorphic to a †-SSFA, and vice versa.

Thus, the (less familiar) notion of †-SSFA is actually just a diagrammatic version of the (more familiar) notion of C*-algebra. Crucially, this allows non-commutative, *quantum algebras*, in addition to the classical, commutative ones. The most important of these is the Frobenius algebra we associate with a double wire.

Exercise 8.105 Show that the following linear maps define a †-SSFA:

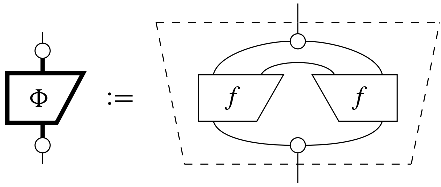
which we will call a *pants algebra*.

Applying map–state duality (and ignoring the $\frac{1}{\sqrt{D}}$), we can see that the pants algebra just takes a pair of linear maps and composes them:



These pants algebras are often called M_n in the literature, referring to the fact that they essentially perform matrix composition. Among all of the †-SSFAs, they play a very special role.

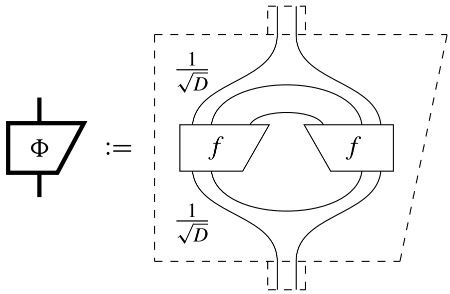
Recall that classical maps are cq-maps of the form:



(8.105)

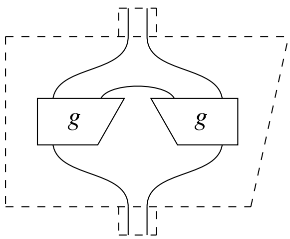
where the dots on the top and bottom are spiders (a.k.a. †-SCFAs). If we now generalise this to †-SSFAs, then we could, for instance, take these to be pants algebra. In that case, something very nice happens.

Proposition 8.106 A linear map Φ is a quantum map if and only if there exists some linear map *f* such that:

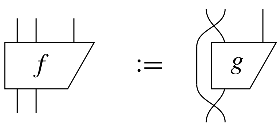


(8.106)

Proof The RHS of (8.106) is already in the form of a quantum map. Conversely, any quantum map takes the form:

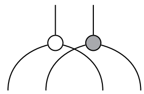


for some g. This can be put in the form of (8.106) by taking:



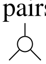
The numbers in (8.106) then cancel with the resulting circle. □

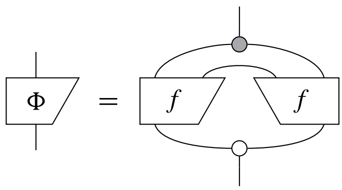
It may not be immediately obvious what we win here, given the equivalent form (8.106) of a quantum map is actually more complicated than the one we started with. However, now the condition of being a classical map and being a quantum map is always (8.105), where the only thing that varies is the algebra. In other words, we can treat classical and quantum types on the same footing. What's more, if we have two †-SSFAs on systems A and B:



gives us a †-SSFA on the system $A \otimes B$. So, we can additionally treat arbitrary $\otimes$-compositions of classical and quantum systems in the same way as our basic types by means of (8.105). All the information about which parts of the system are classical and which parts are quantum is then encoded in the resulting algebra.

Hence, it makes sense to think of not just a Hilbert space A as a type, but the pair $(A, \bigcirc)$ consisting of A and a †-SSFA on A. Using this as a guide, we define a new process theory from an old one.

Definition 8.107 The process theory **CP*[linear maps]** has as types pairs $(A, \bigcirc)$ consisting of a Hilbert space A and a †-SSFA on A, and processes from $(A,$  $)$ to $(B,$  $)$ are linear maps Φ from A to B of the form:



for some linear map f.

In fact, cq-maps are precisely those maps on **CP*[linear maps]** whose types are $\otimes$-compositions of classical and quantum systems, thus:

$$\textbf{cq-maps} \quad \subset \quad \textbf{CP*[linear maps]}$$

The full process theory of **CP*[linear maps]** contains 'fully classical' and 'fully quantum' systems, as well as some extra stuff in between:

This 'extra stuff' includes $\otimes$-compositions of classical and quantum systems, as well as more general semi-classical (or sometimes called 'super-selected') systems, which arise as direct sums of quantum systems. This general direct sum form for systems in **CP*[linear maps]** follows from Wedderburn's theorem, which was mentioned briefly in the proof of Theorem 8.102.

Note that the only thing we used about the process theory of **linear maps** to define **CP*[linear maps]** is that it admits string diagrams, so we can play this 'CP*' game with other process theories (or monoidal categories) and see what comes out. The results can sometimes be surprising! For example, **CP*[relations]** yields a process theory whose types are *groupoids* (i.e. certain mathematical objects that generalise groups) and whose maps are relations preserving some of the groupoid structure.

*8.6.3 Hairy Spiders**

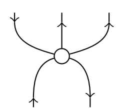
In Sections 8.2.4 and 8.3.1 we made reference to spiders for non-self-conjugate ONBs. So how would those look? In Remark 8.40, we said that one solution was simply to drop the equations:

(8.107)

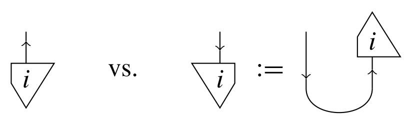
or 'fix' them to account for conjugation. In fact, we already done this for certain 'two-legged' spiders in Section* 4.6.2:

We can throw in dots:

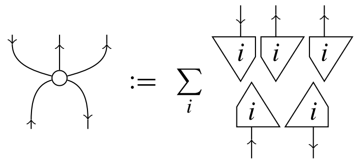
and add some extra legs:



In Section 4.6.2, we used arrows to distinguish a system A from its dual system A^*. This has the handy side effect that the conjugate of a basis state has a different type from the state itself:



and, similarly, the adjoint has a different type from the transpose. Hence the arrows on the legs of a 'hairy' spider tell use which basis states/effects should be conjugated:

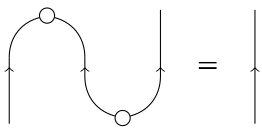


We have, by definition, fixed the cup/cap equations (8.107):

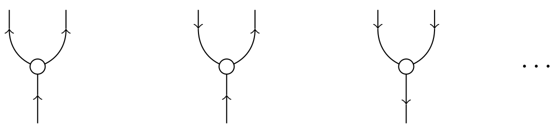
and whenever a spider has one ‘in-arrow’ leg and one ‘out-arrow’ leg it is a wire:

The rules of the game are now that spiders can only ‘shake legs’ when the orientations of the legs match. Note in particular that now a new kind of cups/caps arises:

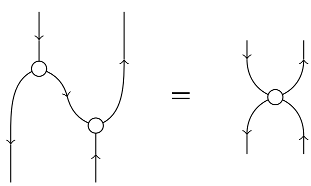
which still satisfy yanking equations:



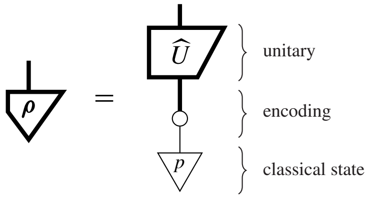
Similarly, there are several variations on our good old copying spider:



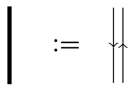
which can fuse with other hairy spiders, e.g.:



In Remark 8.57, we stated that the unitary in the decomposition:



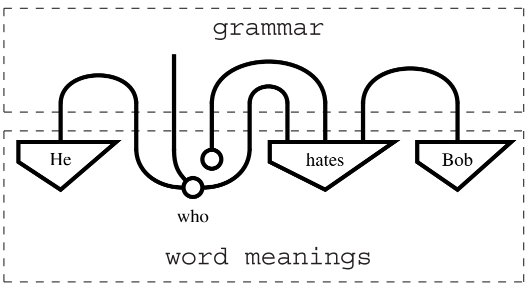
of any quantum state ρ was necessitated by the restriction to self-conjugate ONBs. Now, setting:



we can drop $\widehat{U}$ and obtain a perfect symmetry between classical-as-quantum and quantum-as-classical decompositions:

*8.6.4 Spiders as Words**

In Section* 6.6.3, we say that sentences can be represented using diagrams. Spiders play a role in this story as well, since they can be used to represent relative pronouns:



Here, the spider combines 'he' with 'hates Bob', in order to return whatever obeys these two properties, that is, being a (probably) human male, as well as hating Bob.

So, not only is this book is full of spiders, but in fact every book is!

8.7 Historical Notes and References

The diagrammatic representation of classical data was initiated in Coecke and Pavlovic (2007), in which spider-less but fully diagrammatic definitions of controlled unitaries, von Neumann measurements, and teleportation were given. The passage to spiders took place in Coecke and Paquette (2008). Much earlier, in Davies and Lewis (1970), the classical data produced by quantum measurements were also represented in terms of an ONB. However, no distinction was made between the spaces in which the classical data was represented and those in which the quantum systems were described. The two versus one wire paradigm and an early form of the resulting classical-quantum maps were introduced in Coecke et al. (2010a). Representing ONB-measurements as bastard spiders was first done in Coecke et al. (2012).

'Bastard' was the original name of the band Motörhead, but Lemmy changed it after being told that a band by the name of Bastard would never get a slot on Top of the Pops. This won't matter for our spiders given that Top of the Pops doesn't exist anymore. RIP Lemmy and Lil' Philthy.

These days, there are several other diagrammatic presentations of quantum theory on the market, e.g. Chiribella et al. (2010) and Hardy (2012). However, in these presentations, classical data is always treated non-diagrammatically. On the other hand, the fundamental importance of purification as a characteristic of quantum theory (cf. Section 8.2.2) was put forward in Chiribella et al. (2010, 2011). A study of the connection between distinguishability and copiability within process theories was done by Chiribella (2014).

Dense coding was first proposed in Bennett and Wiesner (1992), and its diagrammatic treatment appeared first in Coecke and Pavlovic (2007). The paper of Coecke and Paquette (2008) gives a diagrammatic proof of Naimark dilation, but that proof is unnecessarily complicated, as compared with the one that we presented here. The diagrammatic representation of Bayesian inversion of Example* 8.27 is taken from Coecke and Spekkens (2012).

The notion of a Frobenius algebra is due to Brauer and Nesbitt (1937) but was first presented in its modern, categorical form by Carboni and Walters (1987). The fact that (special) commutative Frobenius algebras have canonical forms (i.e. spiders) that 'fuse' together comes from a 'folk theorem' relating Frobenius algebras to geometrical objects called *cobordisms*. A standard reference is Kock (2004). An explicit proof of the 'spider' form for special commutative Frobenius algebras was given using distributive laws by Lack (2004), where what we call spiders are represented as cospans of functions between finite sets. The fact that spiders characterise bases is from Coecke et al. (2013c). Though the authors there rely on the spectral theory of C*-algebras to show that copiable states form a basis, our presentation relies on the (much older) classification theorem of Wedderburn (1906). Completeness for spider diagrams was given by Kissinger (2014b).

The connection between †-Frobenius algebras and C*-algebras (cf. Remark 8.104) is from Vicary (2011). The CP*-construction, a categorical construction that gave rise to the process theory **CP*[linear maps]** of Section 8.6.2, was given in Coecke et al. (2013a). An axiomatization of this construction, similar to the axiomatization of **quantum maps** of Section* 6.6.2, was done by Cunningham and Heunen (2015); the relationship between this process theory and quantum logic (cf. Section* 7.6.2) is discussed in Coecke et al. (2013b). Characterisations of classical and quantum systems in terms of information-theoretic constraints were investigated using the CP*-construction in Heunen and Kissinger (2016), generalising the C*-algebraic results of Clifton et al. (2003).

There is also a body of work that aims to classify spiders in **relations** rather than in **linear maps**. This is actually not so weird, given that Carboni and Walters (1987) introduced Frobenius algebras in order to axiomatise **relations**. This effort started with Coecke and Edwards (2011) where it was observed that there were some unexpected spiders, which don't arise from ONBs. After that Pavlovic (2009) classified all spiders in **relations**, and Heunen et al. (2012b) extended this to the case of non-commutative Frobenius algebras.

Within the context of natural language meaning, relative pronouns in terms of spiders appeared in Sadrzadeh et al. (2013, 2014). More recent work in this area that emphasises even more the structural connection with quantum theory can be found in Piedeleu et al. (2015), Balkir et al. (2016), and Bankova et al. (2016).

The last sentences of the introduction to this chapter and Section* 8.6.4 are in reference to David Wong's comedy horror novel *This Book Is Full of Spiders*.