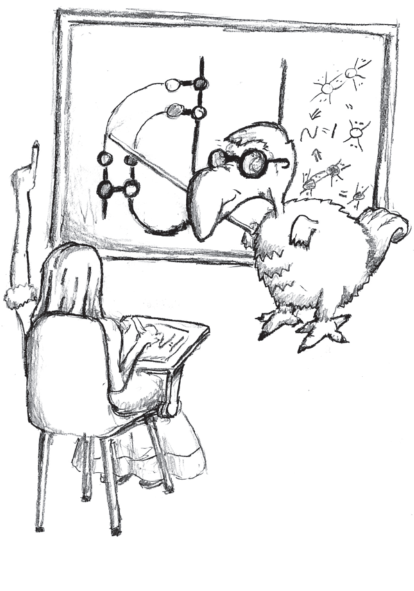
10

Quantum Theory: The Full Picture

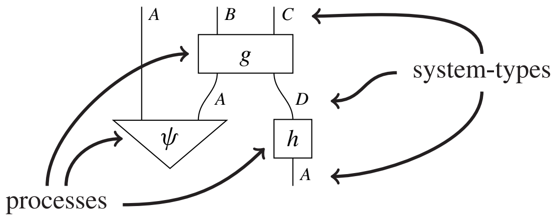
> Philosophy [i.e. physics] is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.
>
> –*Galileo Galilei, Il Saggiatore, 1623*

In the previous chapters we constructed diagrammatic representations of the key ingredients of quantum theory and related them to the usual quantum formalism in terms of Hilbert spaces and linear maps. However, now it is time to forget about the latter and do pure quantum picturalism! Since it has quite some time to get to this point, in this chapter we give the whole quantum story, as a tale of diagrams and diagrams only.



10.1 The Diagrams

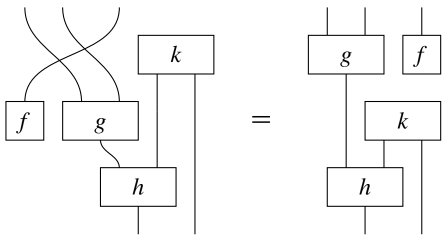
Diagrams consist of boxes and wires, which represent *processes* and *systems*, respectively:



The golden rule of diagrams is:

Only connectivity matters!

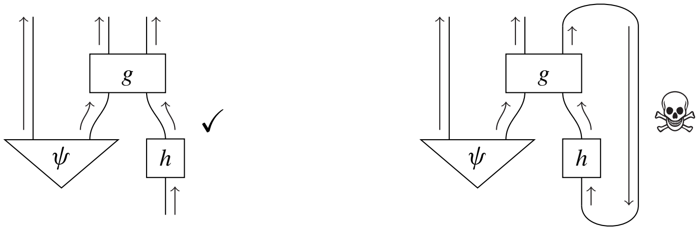
That is, two diagrams are equal whenever they contain the same boxes, connected in the same way, regardless of how we write them on the page:



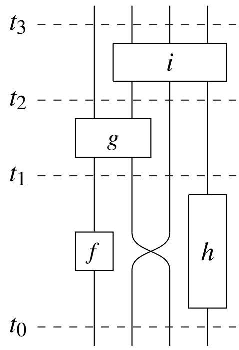
Underlying the story of this book is the story of an evolving diagrammatic language. In Section 2.2.1, we organised this evolution into layers of increasing expressiveness. Now we know all about these layers.

10.1.1 Circuit Diagrams

Circuit diagrams are diagrams that contain no directed cycles:

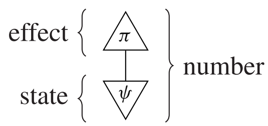


These diagrams are characterised by the fact that they give a clear notion of future and past. They can always be organised into time steps:



though not necessarily in a unique way.

Processes with no inputs are called *states*, and processes with no outputs are called *effects*. When a state hits an effect, a number pops out:



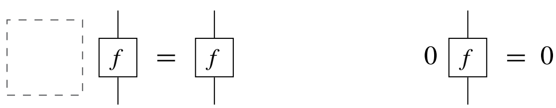
(10.1)

which we can interpret as the probability (or sometimes just the possibility) that π will happen, given state ψ. This is called the *generalised Born rule*.

In general, a number is just a process with no inputs or outputs:

which we usually write simply as λ. We always have one special number around, the empty diagram, a.k.a. 1:

Sometimes, we also have 0, the number that ‘eats everything’. Just as you would expect from these two numbers, 1 ‘times’ something is that thing again, and 0 ‘times’ something is 0:

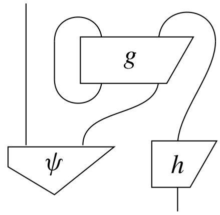


(10.2)

See also *circuits* in Section 3.2; *states, effects,* and *numbers* in Section 3.4.1; and *zero* in Section 3.4.2.

10.1.2 String Diagrams

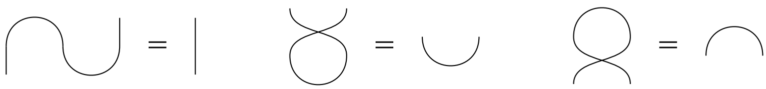
String diagrams do away with the ban on directed cycles and even allow inputs to be connected to inputs and outputs to outputs:



Equivalently, these come from circuit diagrams by appending special processes called:

cups :=  and *caps* := 

which satisfy the *yanking equations*:

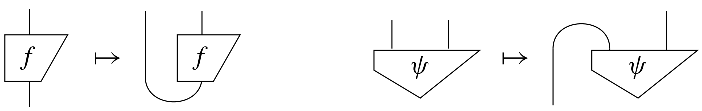


(10.3)

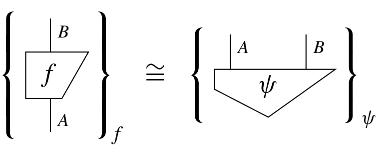
The existence of cups and caps explicitly witness *non-separability* in the following sense:

For any (non-trivial) system, the cup/cap is never ⊗-separable.

However, rather than dwelling on what we can't do with cups/caps (namely, separate them), it's much more interesting to see what we can do! They allow us to encode processes and bipartite states and to go back without losing anything:

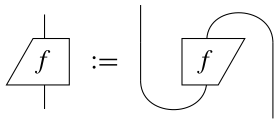


This *process–state duality* yields an isomorphism:



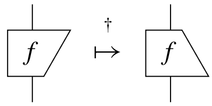
(10.4)

Cups and caps also induce a 180° rotation of boxes, called the *transpose*:

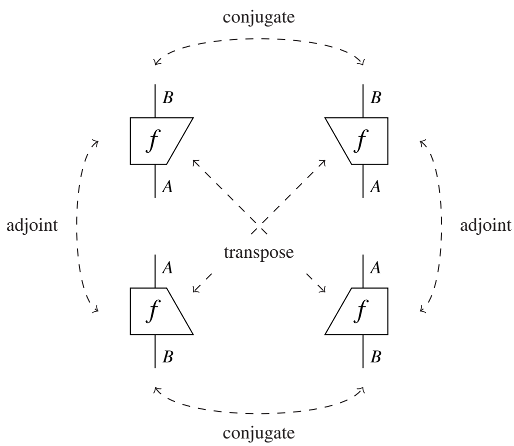


(10.5)

On the other hand, *adjoints* reflect boxes vertically:

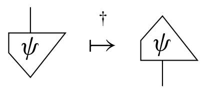


Combining these two operations gives us four incarnations for any box:

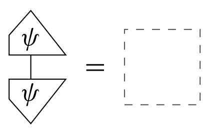


(10.6)

In particular, the adjoint of a state is the effect that tests for that state:

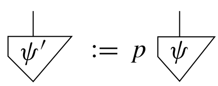


We expect testing a state for itself to give 1:



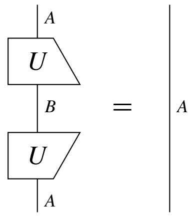
(10.7)

so by default, we expect states to be *normalised*. However, unnormalised states do naturally occur when expressing non-determinism; for example:



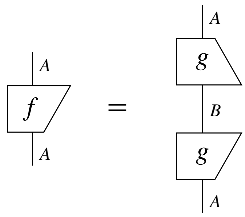
can be interpreted as 'ψ happens with probability p'.

Adjoints allow for defining special processes, for example, *isometries*:



(10.8)

unitaries, which are two-sided isometries, and *positive processes* f, i.e. those for which there exists some other process g such that:

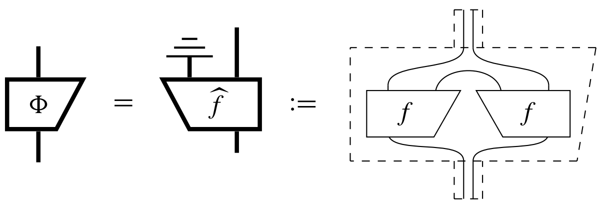


(10.9)

See also *(non)separability* in Section 4.1.1; *process–state duality* in Section 4.1.2; *transpose* in Section 4.2.1; *adjoints* in Section 4.3.1; *conjugates* in Section 4.3.2; *isometries and unitaries* in Section 4.3.4; and *positive processes* in Section 4.3.5.

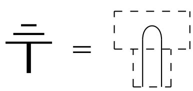
10.1.3 Doubled Diagrams

Doubled diagrams are diagrams of the form:



(10.10)

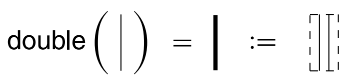
where we interpret the effect:



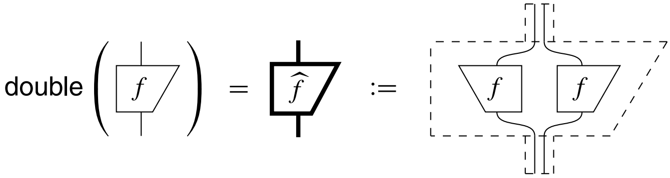
(10.11)

as *discarding* some outputs of a process.

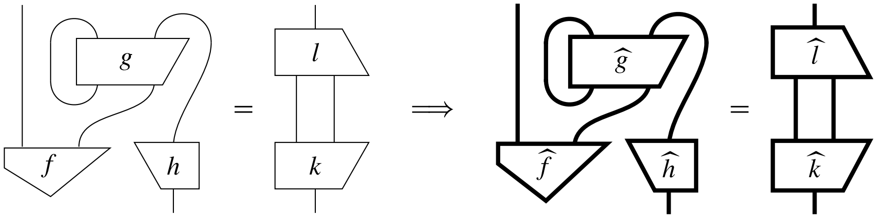
They arise from a two-step construction. First, we double systems:



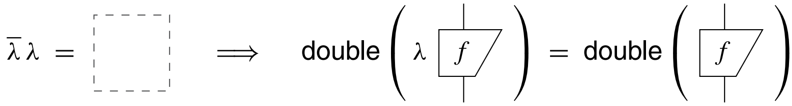
and processes:



This operation preserves diagrams, and hence equations between diagrams:

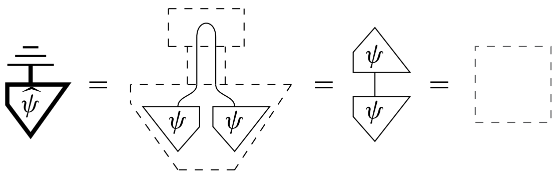


Therefore, anything we can prove using diagrams of single processes also holds doubled. The converse, that any equation between doubled processes also holds non-doubled, is almost true. The only thing that doesn't survive the transition are certain numbers, called *global phases*:



but since these won't have any effect on probabilities, good riddance!

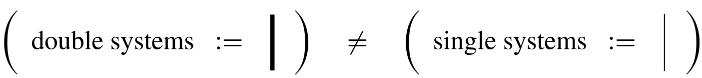
So if working with doubled processes is the same as just working with single processes (up to a global phase), why bother doubling at all? The crucial feature of doubling is that it makes room for something new. The second step in the doubling construction is we adjoin discarding, which represents the act of literally throwing a system away (or simply ignoring it). It works by connecting the two copies of a normalised state together, letting them annihilate:



(10.12)

Since discarding doesn't arise from doubling something, it is called *impure*. More generally, by discarding outputs of *pure* (i.e. doubled) processes, we obtain lots of other impure processes of the form (10.10).

By doubling all wires, we have also created a vacancy for a different type of system, which we can represent as plain old single wires:

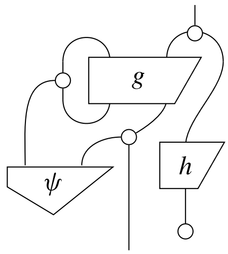


So how do these two types of systems interact? Via *spiders*.

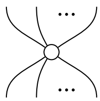
See also *doubling* in Section 6.1; *global phases* in Section 6.1.2; *discarding* in Section 6.2.1; *doubled process theory* in Section 6.2.4; and *classical wires* in Section 8.1.

10.1.4 Spider Diagrams

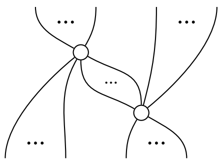
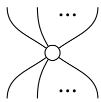
Spider diagrams consist of boxes and 'generalised wires', which are allowed to connect any number of inputs to any number of outputs:



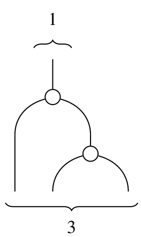
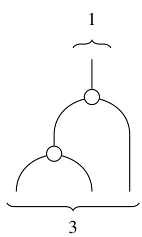
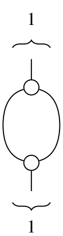
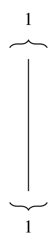
Again these can be equivalently presented as circuit diagrams by appending special processes called:

spiders :=  (10.13)

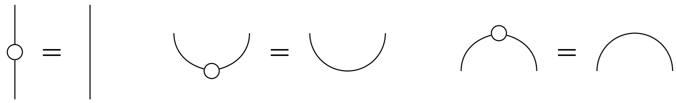
The only rule that governs them is that adjacent spiders fuse together:

 =  (10.14)

In particular, it follows that any two <u>connected</u> diagrams made up of spiders with the same number of inputs and outputs are equal:

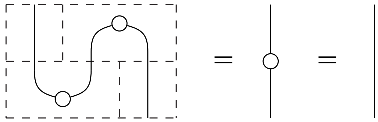
 =   = 

Given that two-legged spiders are just ‘plain old’ wires:



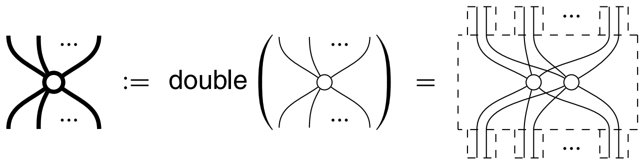
(10.15)

spider diagrams subsume string diagrams, and spider fusion generalises the yanking equations, e.g.:



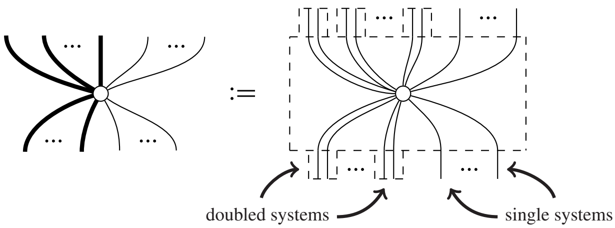
(10.16)

Doubling gives us two extra species of spiders for free. We obtain new pure processes by doubling the whole spider:



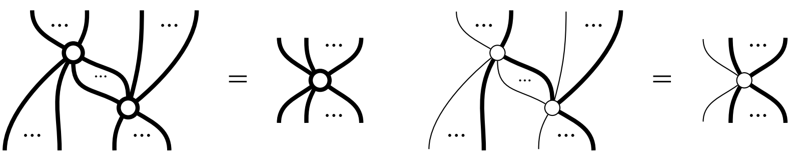
(10.17)

We also obtain *bastard spiders* by interpreting some pairs of legs together as doubled systems (a.k.a. *folding*), while leaving others single:



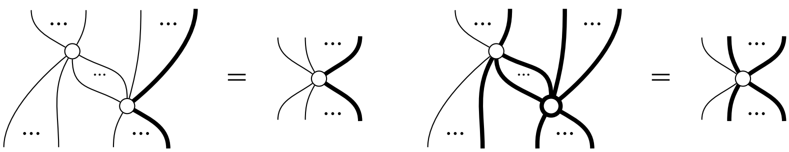
(10.18)

These satisfy their own fusion laws:



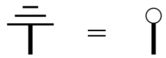
(10.19)

and additionally, bastard spiders can fuse with the other two kinds of spiders:



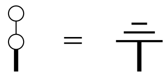
(10.20)

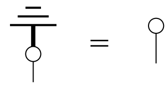
Discarding is itself a bastard spider:



(10.21)

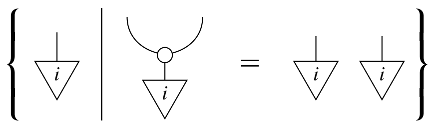
whence:





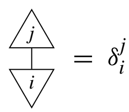
(10.22)

To every family of spiders, we can associate *copiable states*:



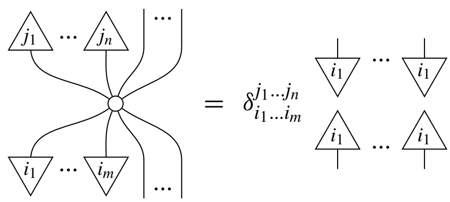
(10.23)

which are moreover orthonormal:



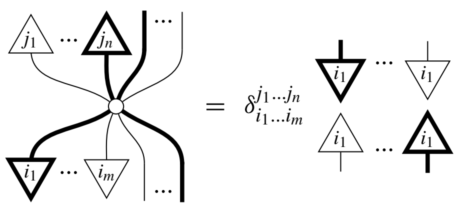
(10.24)

From this and spider fusion, we get a *generalised copy rule*:



(10.25)

By doubling or folding parts of this equation, quantum and bastard versions also emerge, e.g.:

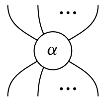
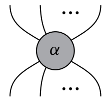


(10.26)

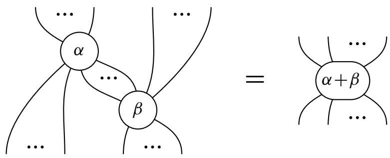
See also *spiders* in Section 8.2.3; *quantum and bastard spiders* in Section 8.3.3; and *copiable (a.k.a. ONB) states* in Section 8.2.2.

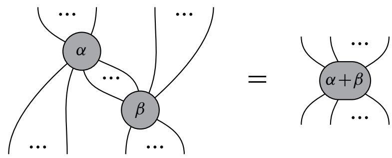
10.1.5 ZX-Diagrams

ZX-diagrams consist entirely of two kinds of *phase spiders*:

Z-spiders :=  *X-spiders* := 

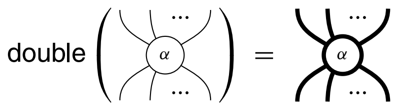
where $\alpha \in [0, 2\pi)$. When phase spiders of the same colour meet, they fuse and their *phases* add (modulo 2π):





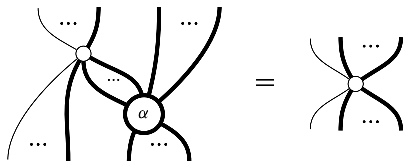
ZX-diagrams are the richest diagrams encountered in this book. They have two important ingredients: the diagrammatic structure of spiders and the group structure of the phases. In fact, the second arises from the first!

Like their undecorated cousins, phase spiders too can be doubled:



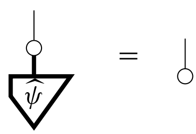
(10.27)

When a phase spider meets a bastard spider, i.e. when it comes into contact with the single-world, the phase is destroyed:



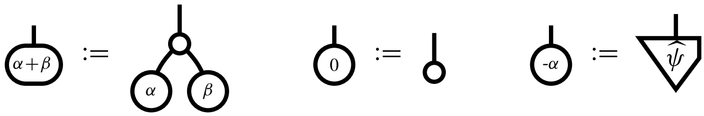
(10.28)

This property of not surviving the passage from double to single totally characterises phases. To see this, consider all states where:

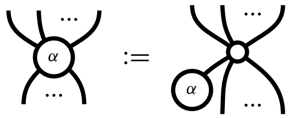


Then, as if by magic, a commutative *phase group* emerges. Setting:

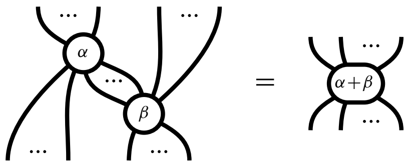
we have:



Then, by letting:

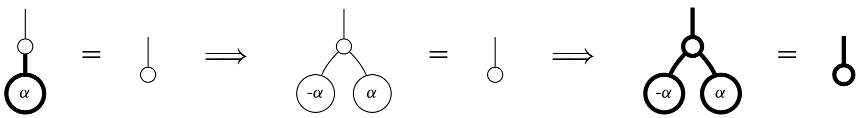


we recover all phase spiders and their associated fusion law:

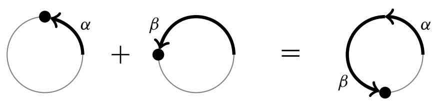


(10.29)

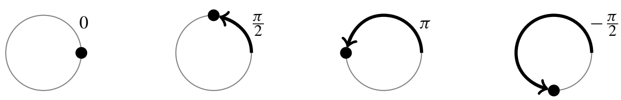
Actually, it wasn't so magic. The property of 'not surviving the passage from double to single' is precisely what does the non-trivial work, namely giving this group its inverses:



In the case of ZX-diagrams, this emergent group is the circle group $U(1)$:



and in the case of *Clifford ZX-diagrams*, we restrict just to a four-element subgroup $\mathbb{Z}_4$ of $U(1)$:



In either case, we might think of these as rotations of a sphere of some kind. But let's not get ahead of ourselves.

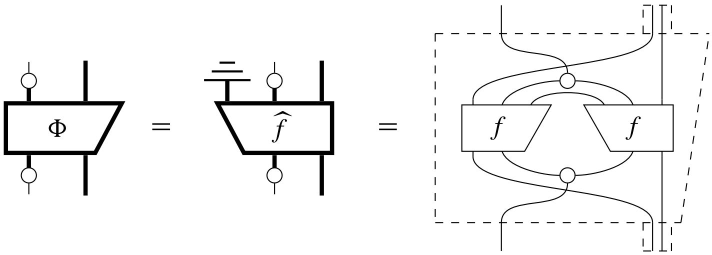
See also *phase spiders* in Section 9.1.2; *phase group* in Section 9.1.4; *ZX-diagrams* in Section 9.4.1; and *Clifford diagrams* in Section 9.4.2.

10.2 The Processes

A *process theory* is a collection of processes that make sense to plug together. In other words, it is *closed under forming diagrams*. Here's a process theory we really like, called *quantum theory*:

There are two kinds of systems, quantum and classical systems:

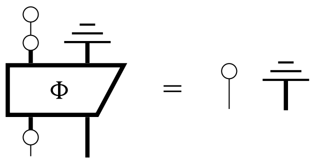
The processes that may be realised (not necessarily with certainty) are:



(10.30)

where the f-labeled box is made up of phase spiders:

Processes that can be realised with certainty furthermore obey *causality*:



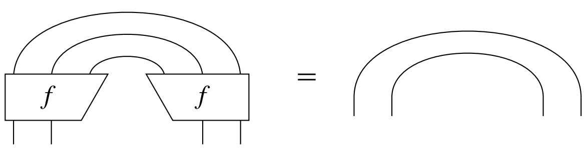
(10.31)

10.2.1 Causality

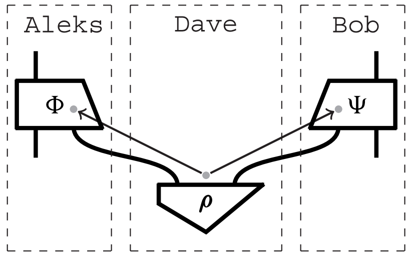
Causality is an extremely important postulate for quantum theory, which nevertheless has an extremely simple interpretation:

> *If the output of a process is discarded, it may as well have never happened.*

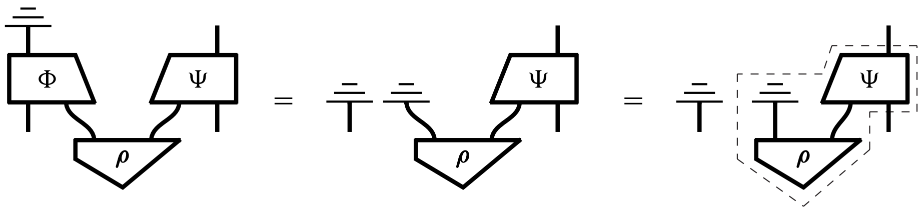
For a quantum process (10.30) causality is equivalent to f being an *isometry*:



It guarantees that quantum theory is compatible with special relativity; i.e. it is *non-signalling*. Non-signalling says that the flow of information must respect the *causal structure*. So, if Aleks and Bob are very far apart, but possibly share some correlation from the past:



then it is impossible for Aleks to communicate directly to Bob. This is witnessed by the fact that if Bob doesn't know the output of Aleks' process:

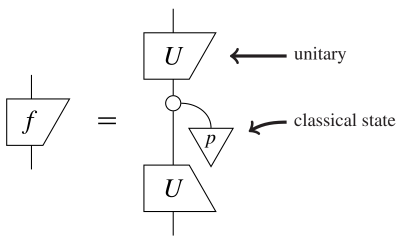


then he can't possibly know the input, either.

See also *causality* in Section 6.2.5; *causal structure* in Section 6.3.1; and *non-signalling* in Section 6.3.2.

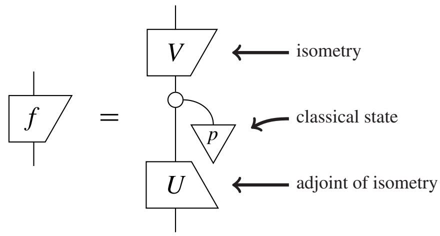
10.2.2 Process Decomposition and No-Broadcasting

A process theory has *spectral decompositions* if any positive process can be written in this form:



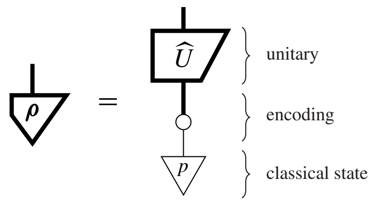
(10.32)

A process theory furthermore has *singular value decompositions* if any process can be decomposed as:

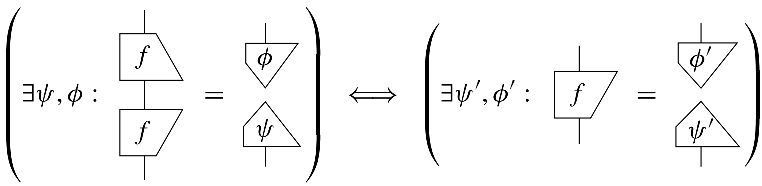


(10.33)

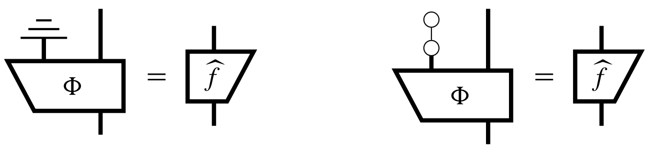
As a consequence of 10.32, quantum states $\boldsymbol{\rho}$ encode classical states:



From the form (10.32), using properties of spiders, it also follows that:

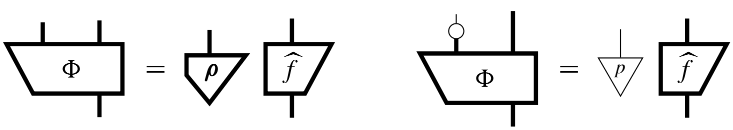


From this we can show that if the *reduced map* of a cq-map is pure:



(10.34)

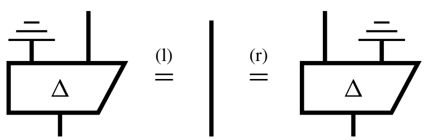
then the process Φ separates as follows:



(10.35)

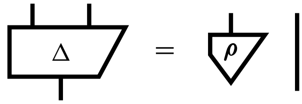
for (causal) states $\boldsymbol{\rho}$ and p.

From these separation results, we immediately arrive at *no-broadcasting*; that is, there exists no quantum process such that:



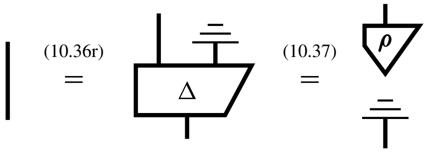
(10.36)

The identity is pure, so if such a process existed, it would separate as:



(10.37)

which yields a contradiction:



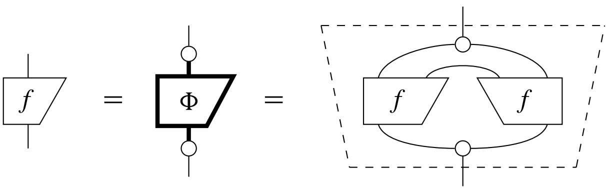
See also *spectral theorem* in Section 5.3.3.1; *spectral and singular value decompositions* in Section 8.2.5; and *no-broadcasting* in Section 6.2.8.

10.2.3 Examples

The processes in quantum theory that may be released non-deterministically are called **cq-maps**, whereas causal cq-maps are called simply **quantum processes**. In this section, we'll give some important examples.

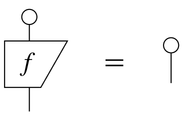
10.2.3.1 Classical Maps

Classical maps are cq-maps with no quantum inputs or outputs:



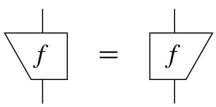
(10.38)

and *classical processes* are causal classical maps:



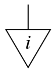
(10.39)

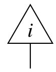
Consequently, classical maps are automatically self-conjugate:



(10.40)

The simplest examples of classical maps are ○-copiable states and effects, which represent:

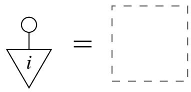
classical values := 

classical tests := 

A system that admits just two such values/tests, 0 and 1, is called a *bit*.

Examples of classical processes include:

- classical values, because:

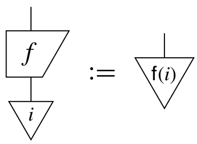


(10.41)

- *function maps*, i.e. processes that encode a function:

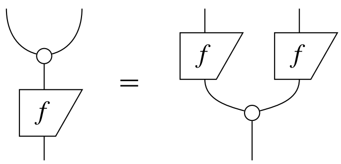
$$\mathsf{f} : \{1, \ldots, m\} \to \{1, \ldots, n\}$$

via:



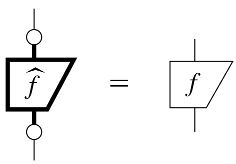
(10.42)

which are characterised by this equation:



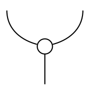
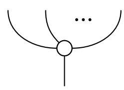
(10.43)

and consequently satisfy:

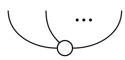


(10.44)

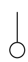
- ○-spiders with precisely one input:

delete := 
copy := 
n-copy := 

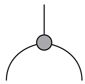
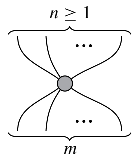
- ○-spiders with no inputs (after renormalising):

perfect correlation := $\frac{1}{D}$ 

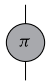
and as a special case:

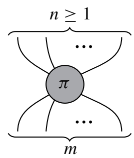
uniform distribution := $\frac{1}{D}$ 

- ● -spiders with more than one output (after renormalising):

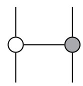
XOR-gate $:=\ \sqrt{2}$  *even-parity* $\approx$ 

- the same, but with π phases:

NOT-gate $:=$ 

odd-parity $\approx$ 

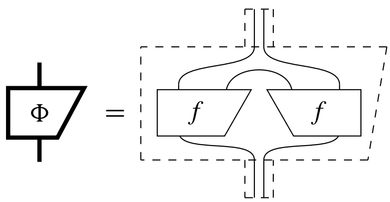
- and combinations of the above, e.g.:

CNOT-gate $:=\ \sqrt{2}$ 

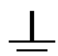
The last three examples are given in the special case of bits, but also generalise to other classical systems.

10.2.3.2 *Quantum Maps*

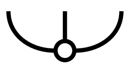
Quantum maps are cq-maps with no classical inputs or outputs:



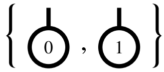
Some examples of *quantum states* are:

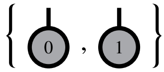
maximally mixed state $:=\ \frac{1}{D}$ 

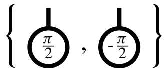
Bell state $:=\ \frac{1}{D}$ 

GHZ state $:=\ \frac{1}{D}$ 

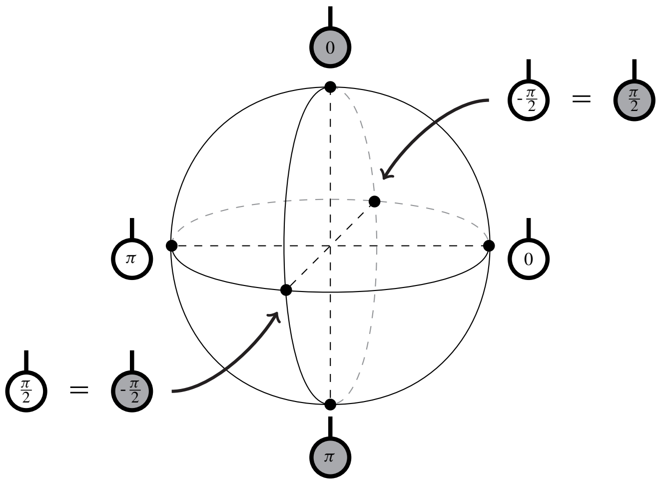
Restricting to *qubits*, i.e. two-dimensional quantum systems, we have:

Z-basis states :=


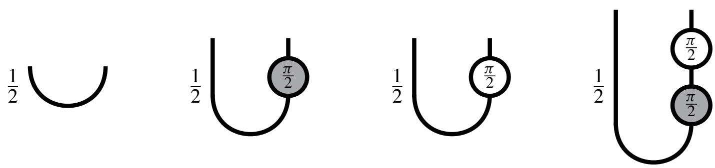
X-basis states :=


Y-basis states :=


which can be depicted on the *Bloch sphere*:

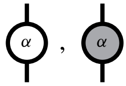
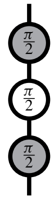


On a pair of qubits, the Bell state extends to the *Bell basis*:

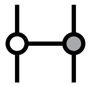
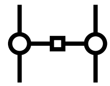


which is expressed in terms of the Bell state and the four *Bell maps*:

These are all unitary quantum processes, or *quantum gates*. Other important single-qubit quantum gates are:

phase gates :=  *H-gates* =  := 

and two-qubit quantum gates are:

CNOT-gates := D  *CZ-gates* := D 

10.2.3.3 Classical-Quantum Interactions

Many classical-quantum interactions arise as special cases of bastard spiders:

- *encode* and *measure* for all colours:

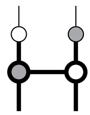
the corresponding *non-demolition measurements*:

and *decoherence*:

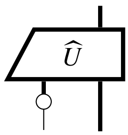
More general *demolition* and *non-demolition ONB measurements* arise from combining bastard spiders with a unitary $\widehat{U}$:

Important examples are the *Y-measurements*:

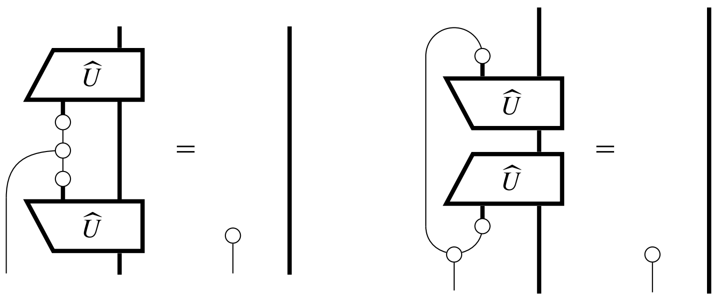
and the non-separable *Bell measurement*:



While measurements extract classical data from a quantum system, *controlled unitaries*:

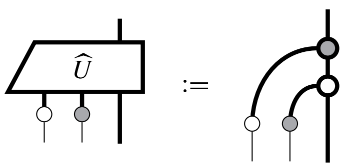


use classical data to change a quantum system. They are characterised by the following equations:



(10.45)

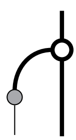
An example is the *correction* used in quantum teleportation:



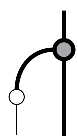
(10.46)

which further decomposes as:

Z-correction :=



X-correction :=



(10.47)

So, that gives us a pretty good collection of parts. Now let's plug them together and turn this thing on!

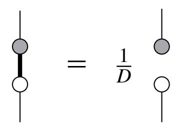
See also *classical maps and function maps* in Section 8.2.1; *classical logic gates* in Section 5.3.4; *parity maps* in Section 9.3.5; *maximally mixed state* in Section 6.2.2; *Bloch sphere* in Section 6.1.2; *Bell maps/basis* in Section 5.3.6; *phase gates* in Section 9.1.5; *H-gate* in Section 5.3.5; *measure and encode* in Section 8.1.3; *decoherence* in Section 8.3.2; *ONB-measurements* in Section 8.4.1; *controlled unitaries* in Section 8.4.2; and *Bell measurement/correction* in Section 9.2.7.

10.3 The Laws

We now turn to the most important laws governing quantum processes. We can already get a bit of mileage out of the fact that spiders of the same colour fuse together. However, things start to get really interesting when spiders of different colours start to fight.

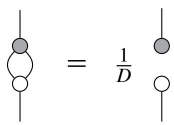
10.3.1 Complementarity

Spiders are *complementary* if:



(10.48)

or, equivalently, if:

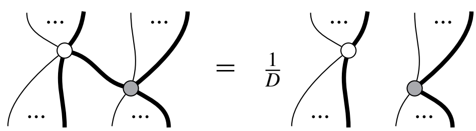


(10.49)

or in words:

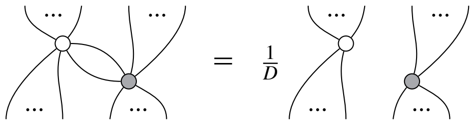
(encode in ○) THEN (measure in ●) = (no data flow)

From the simple form (10.48), we can derive lots of other equations. Notably, a version for bastard spiders:



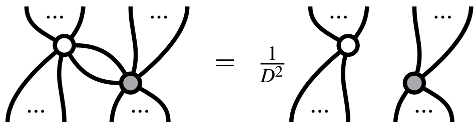
(10.50)

which unfolded yields:



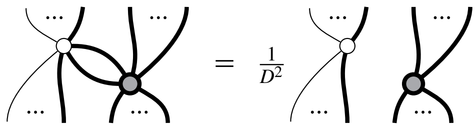
(10.51)

which doubled gives:



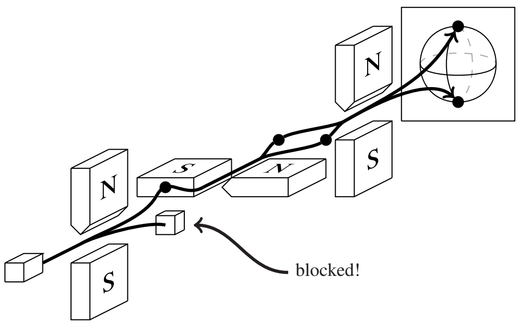
(10.52)

and which combined with the bastard/quantum spider fusion (10.20) yields yet another variation:

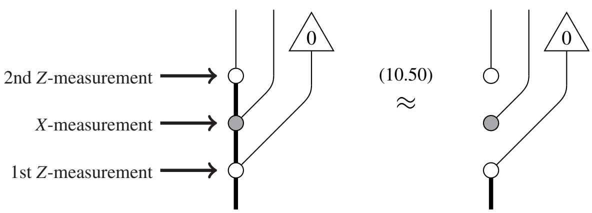


(10.53)

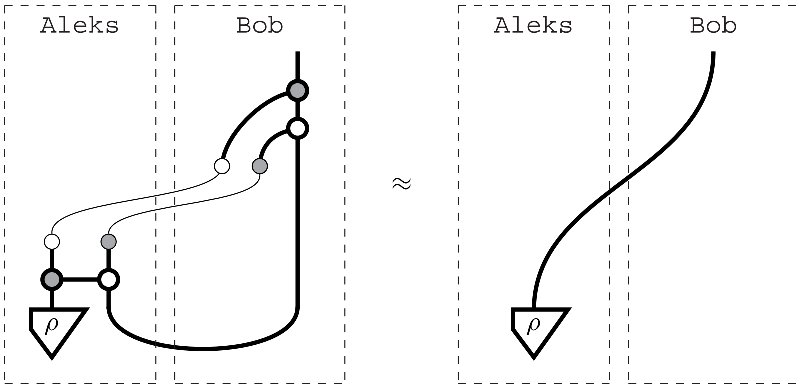
Complementarity is kind of a big deal. For example, it explains the behaviour of the Stern–Gerlach device:

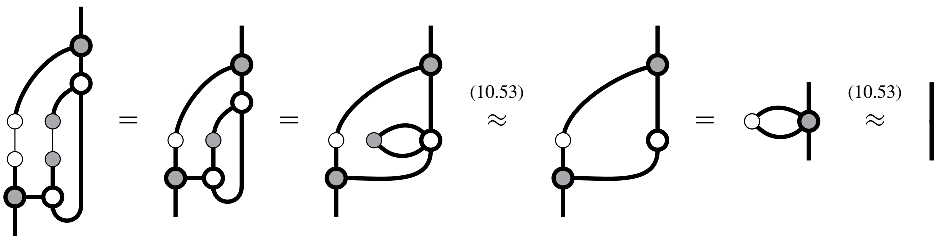


as:



it provides the diagrammatic magic for quantum teleportation:





and it induces basic properties of classical and quantum gates:

(10.54)

For complementary spiders, the copiable states of one colour are, up to a number, phase states for the other colour:

(10.55)

Hence complementarity gives us some new equations, the *κ-copy rules*:

(10.56)

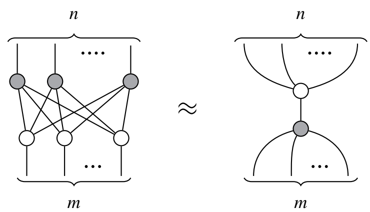
Up to a global phase, these copiable phase states pass right through phase gates of the other colour:

(10.57)

See also *complementarity* in Section 9.2.1; *Stern–Gerlach* in Section 9.2.5; and *teleportation via complementarity* in Section 9.2.7.

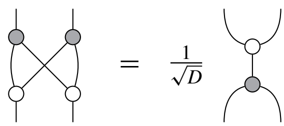
10.3.2 Strong Complementarity

Spiders are *strongly complementary* if:

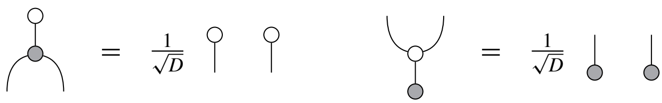


(10.58)

Equivalently, strong complementarity can be expressed as three rules:

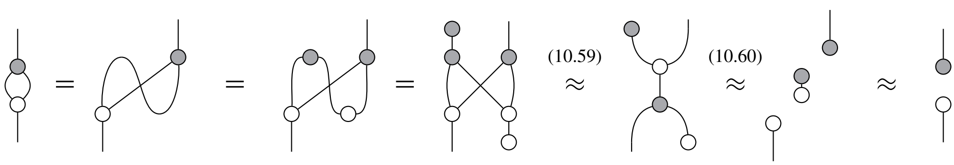


(10.59)

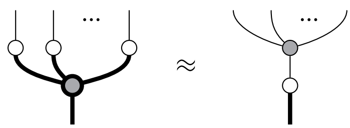


(10.60)

While these rules don't have a (single) natural interpretation as in the case of complementarity, they do imply complementarity:

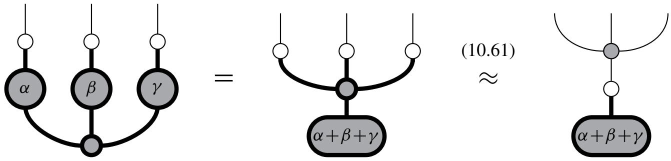


One way of interpreting equation (10.58) is to treat some parts of the equation as doubled, e.g.:



(10.61)

This consequence of strong complementarity is used to show that correlations for the GHZ state take a particular form:



which will play a crucial role in establishing *quantum non-locality*.

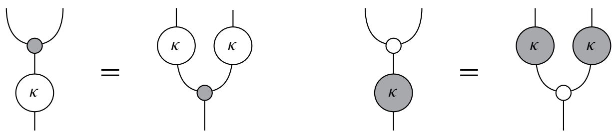
It also furnishes a plethora of new equations concerning copiable phases, stemming from the fact that, for strongly complementary spiders, the group-sum of two copiable phases is again a copiable phase. So, copiable phases actually form a subgroup of the phase group, called the *classical subgroup*:

So, in addition to the κ-copy rules (10.56), strong complementarity furthermore implies:

- that the unit is a copiable state:

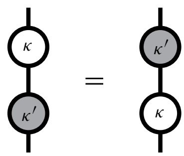
(10.62)

- the copiability of the corresponding phase gates:



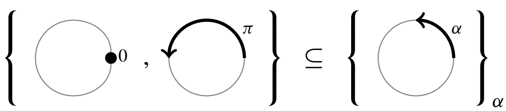
(10.63)

- and the commutation of classical phase gates, up to a global phase:



(10.64)

For ZX-diagrams, the classical subgroup is:



because:

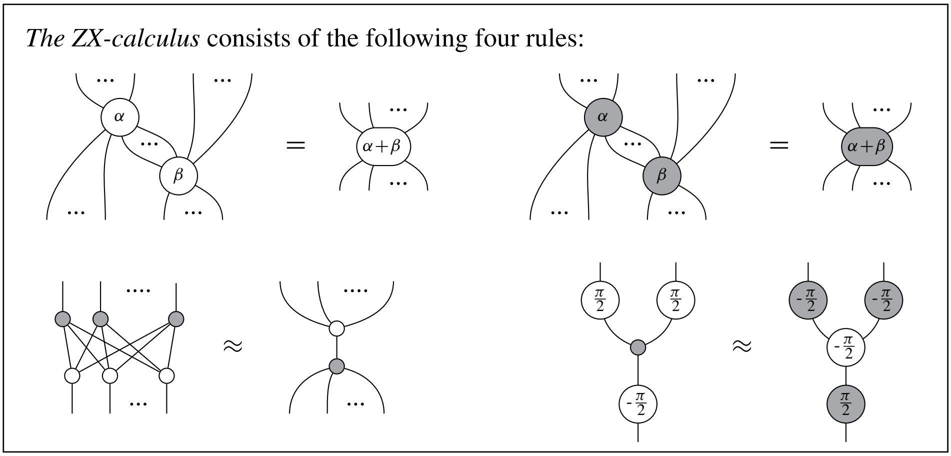
(10.65)

and similarly, with the colours reversed. But you don't have to take our word for it, you can prove it using the *ZX-calculus*!

See also *strong complementarity* in Section 9.3; *generalised form and doubling* in Section 9.3.3; and *classical subgroup* in Section 9.3.4.

10.3.3 ZX-Calculus

The ZX-calculus consists of the following four rules:

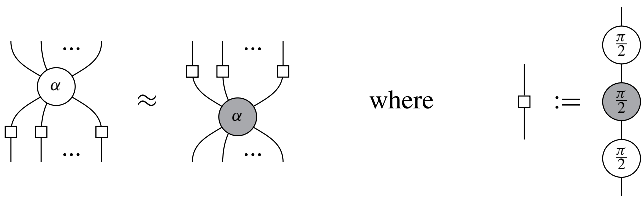


The ZX-calculus is a complete graphical calculus for Clifford ZX-diagrams. That is, if an equation holds between two ZX-diagrams, it is provable in the ZX-calculus. It consists of three kinds of rules, which tell us:

1. how spiders of the same colour combine,
2. how spiders of different colours can commute past each other, and
3. how to convert spiders of one colour into another.

The first kind of rule is phase spider fusion, whereas the second comes from strong complementarity. Hence, the first two kinds of rules hold for arbitrary strongly complementary spiders. On the other hand, the third kind of rule, the *Y-rule*, tells us something specifically about qubits and the Bloch sphere, namely that the Y-basis states can be copied in two equivalent ways.

The rules above are the most succinct way to give the calculus we know. However, a convenient, alternative presentation replaces the Y-rule with the *colour-change rule*:



(10.66)

From the ZX-calculus, we can also derive:

- the 0-*copy* rule:

(10.67)

and the π-*copy* rule:

(10.68)

which confirm that $\{0, \pi\}$ indeed forms the classical subgroup:

(10.69)

- the π-*commute* rule:

(10.70)

- the *phase eliminate* rule:

(10.71)

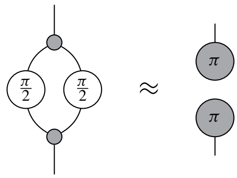
as well as its π-*eliminate* counterpart, specialising (10.57):

(10.72)

- Together (10.70) and (10.71) yield:

(10.73)

- the $\frac{\pi}{2}$-*supplementarity* rule:



(10.74)

- two equivalent expressions of the Y-basis:

(10.75)

- equations between bastard spiders, obtained from (un)folding:

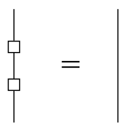
(10.76)

(10.77)

(10.78)

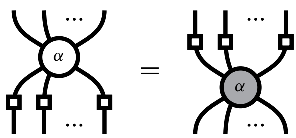
(10.79)

- self-inverseness of the H-gate:

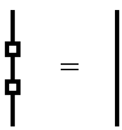


(10.80)

Many of these $\approx$-equations hold up to a global phase, so doubling yields:



(10.81)



(10.82)

(10.83)

(10.84)

(10.85)

(10.86)

(10.87)

So, from the humble four equations of the ZX-calculus, we can actually derive much more. Of course, we didn't do this just because it was fun (actually, it was kind of fun). In the next four chapters, we will see how these equations, along with the graphical presentation of quantum theory in general, can be exploited in quantum foundations, quantum computation, quantum resource theories and automating proofs in all of the above.

See also *ZX-calculus* in Section 9.4 and the *colour-change rule* in Section 9.4.4.

10.4 Historical Notes and References

While we have already given all the relevant references for the development of quantum picturalism in the previous chapters, here we give a (somewhat idiosyncratic) timeline of the events leading up to here. The use of diagrams to discuss quantum teleportation first appeared in Coecke (2003, 2014a). This paper was not published until 2014. From then onwards, the further refinement of the diagrammatic language was tightly intertwined with the development of the corresponding category-theoretic axiomatics. Part of the reason for this is that diagrams look a bit silly, and if a paper contains no hard mathematics there is no chance to get the work published in a prestigious venue. The categorical axiomatisation of the aforementioned paper was given in Abramsky and Coecke (2004). Basically, all that was used here were string diagrams. Of course, Penrose had already been drawing string diagrams since the 1970s (Penrose, 1971). However, they weren't used to describe quantum features such as quantum teleportation, for the simple reason that they weren't known to anyone yet. Penrose himself even said that 'the notation seems to be of value mainly for private calculations because it cannot be printed in the normal way', in a book coincidentally also published by Cambridge University Press (Penrose, 1984).

Doubling popped up quite soon after the 2004 paper, despite the fact that the relevant papers only went to press a bit later. In Coecke (2007) doubling was proposed in order to get the correct Born rule, and independently Selinger (2007) moreover adjoined impure processes. Also, the asymmetric boxes that allow one to clearly distinguish adjoints, transpose, and conjugate were introduced in Selinger (2007). The idea of a discarding process was put forward in Coecke and Perdrix (2010).

Spiders required an evolutionary process starting in 2006 and spanning 6 years (Coecke and Pavlovic, 2007; Coecke and Paquette, 2008; Coecke et al., 2010a, 2012) in order to adequately account for classical systems and processes. Completeness of spider diagrams is in Kissinger (2014b). In contrast, phases and (strong) complementarity in terms of spiders popped out all at once (Coecke and Duncan, 2008, 2011). Completeness is still an unfinished story, but the strongest current results for ZX-diagrams appear in Backens (2014a,b).

Causality, although it plays a very central role in this book, was the last one to enter the picture. Its importance became clear from the information-theoretic axiomatization of Chiribella et al. (2010, 2011).