Advanced Maths ACSEE

Past Paper Questions and Answers by Topic

Contents

Form VI

11.0 Coordinate Geometry II

12.0 Vectors

13.0 Hyperbolic Function

14.0 Statistics

15.0 Probability

16.0 Complex Numbers

17.0 Differential Equations

18.0 Numerical Methods

11.0 Coordinate Geometry II
2021 PAST PAPERS - 2

8\. (a) Express $x^2+y^2=2x+2y$ in polar form.

(b) Find the equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ joining the points whose eccentric angles are θ and ϕ.

(c) Show that $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, hence find the equation of the tangent line at point P on the given hyperbola.

(d) Show whether the equation of a normal to the parabola $y^2=4ax$ at point (x_1, y_1) is $(x-x_1)y_1+2a(y-y_1)=0$.

(e) (i) Change the polar equation $r^2(b^2\cos^2\theta+a^2\sin^2\theta)=a^2b^2$ into the Cartesian equation.

(ii) Draw the graph of $r=2(1+\cos\theta)$.

8a

Given $x^2+y^2=2x+2y$ in polar form will be

$r^2 = 2r\cos\theta + 2r\sin\theta$

$r^2 = 2r(\cos\theta + \sin\theta)$

$\therefore\ r^2 = 2r(\cos\theta + \sin\theta)$ Ans

$r = 2(\cos\theta + \sin\theta)$ Ans

b. Solution.

Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The parametric equations of an ellipse will be $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$

The slope of the chord will be

$m = \frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta}$

The equation will be

8b

$$\frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$\frac{b(\sin\phi - \sin\theta)}{a(\cos\phi - \cos\theta)} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$\frac{b\left(2\cos\tfrac{1}{2}(\phi+\theta)\sin\tfrac{1}{2}(\phi-\theta)\right)}{a\left(-2\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\phi-\theta)\right)} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$-\frac{b}{a}\,\frac{\cos\tfrac{1}{2}(\theta+\phi)}{\sin\tfrac{1}{2}(\theta+\phi)} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$-xb\cos\tfrac{1}{2}(\theta+\phi) + ab\cos\theta\cos\tfrac{1}{2}(\theta+\phi) = ya\sin\tfrac{1}{2}(\theta+\phi) - ab\sin\tfrac{1}{2}(\theta+\phi)\sin\theta$$

$\cancel{ab}$

$$ab\left(\cos\theta\cos\tfrac{1}{2}(\theta+\phi) + \sin\tfrac{1}{2}(\theta+\phi)\sin\theta\right) = xb\cos\tfrac{1}{2}(\theta+\phi) + ya\sin\tfrac{1}{2}(\theta+\phi)$$

$$ab\left(\cos(\theta - \tfrac{1}{2}\theta - \phi/2)\right) = xb\cos\tfrac{1}{2}(\theta+\phi) + ya\sin\tfrac{1}{2}(\theta+\phi)$$

$$ab\cos(\tfrac{1}{2}\theta - \phi/2) = xb\cos\tfrac{1}{2}(\theta+\phi) + ay\sin\tfrac{1}{2}(\theta+\phi)$$

The equation of the chord will be

$\therefore$

$$xb\cos\tfrac{1}{2}(\theta+\phi) + ay\sin\tfrac{1}{2}(\theta+\phi) = ab\cos\tfrac{1}{2}(\theta-\phi) \quad \text{Ans}$$

8c

To show that $(a\sec\theta, b\tan\theta)$ lies on the hyperbola is the same as showing

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Using the L.H.S of the equation above

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(a\sec\theta)^2}{a^2} - \frac{(b\tan\theta)^2}{b^2}$$

$$= \frac{a^2\sec^2\theta}{a^2} - \frac{b^2\tan^2\theta}{b^2}$$

$$= \sec^2\theta - \tan^2\theta$$

From the trigonometry identity $\sec^2\theta - \tan^2\theta = 1$

Hence $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Hence shown) Ans

To obtain the equation of the tangent at P from P ($a\sec\theta$, $b\tan\theta$)

$x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b\sec^2\theta$

$\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right)$

$= b\sec^2\theta \cdot \frac{1}{a\sec\theta\tan\theta}$

$\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$

$= \frac{b\,1}{a\cos\theta} \times \frac{\cos\theta}{\sin\theta}$

$\frac{dy}{dx} = \frac{b}{a\sin\theta}$

but equation of tangent

$\frac{b}{a\sin\theta} = \frac{y - b\tan\theta}{x - a\sec\theta}$

$bx - ab\sec\theta = ay\sin\theta - ab\tan\theta\sin\theta$

$bx - ab\left(\frac{1}{\cos\theta}\right) = ay\sin\theta - ab\left(\frac{\sin\theta}{\cos\theta}\right)\sin\theta$

$bx\cos\theta - ab = ay\sin\theta\cos\theta - ab\sin^2\theta$

$bx\cos\theta - ay\sin\theta\cos\theta = ab - ab\sin^2\theta$

$bx\cos\theta - ay\sin\theta\cos\theta = ab\cos^2\theta$

$bx - ay\sin\theta = ab\cos\theta$

$\therefore$ The equation of tangent is

$bx - ay\sin\theta = ab\cos\theta$ Ans

d. Solution.

from $y^2 = 4a\,x$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

The slope of normal.

$m_1 m_2 = -1$

8d. $m_2 = \frac{-1}{m_1}$

$= -1\left(\frac{y}{2a}\right)$

$= \frac{-y}{2a}$

$m_{normal} = \frac{-y_1}{2a}$

The equation will be

$\frac{-y_1}{2a} = \frac{y - y_1}{x - x_1}$

$xy_1 - x_1y_1 = -2ay + 2ay_1$

$xy_1 - x_1y_1 + 2ay - 2ay_1 = 0$

$(x - x_1)y_1 + 2a(y - y_1) = 0$ Hence shown

$\therefore (x - x_1)y_1 + 2a(y - y_1) = 0$ Hence shown

e. from

$r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$

$b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$

$b^2(r\cos\theta)^2 + a^2(r\sin\theta)^2 = a^2b^2$

but $x = r\cos\theta$ and $y = r\sin\theta$

$b^2x^2 + a^2y^2 = a^2b^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$\therefore$ The cartesian equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Ans

8a

ii) Required to draw

$r = 2(1 + \cos\theta)$

The given curve is symmetrical about the initial line $\theta = 0°$

At tangency point

$r = 0$

$0 = 2(1 + \cos\theta)$

$0 = 1 + \cos\theta$

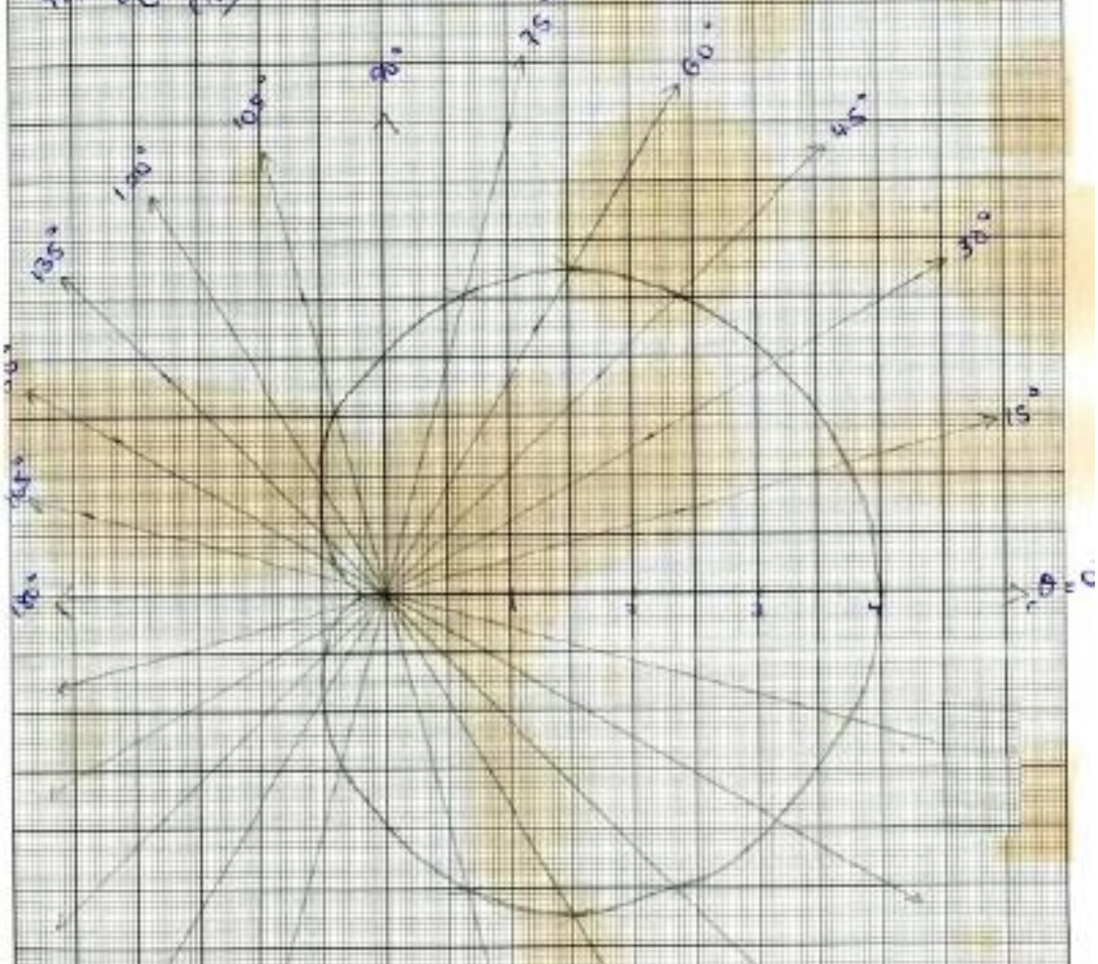
$\cos\theta = -1$

$\theta = \cos^{-1}(-1)$

$\theta = 180^\circ$

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
r	4	3.9	3.7	3.4	3	2.5	2	1.5	1	0.6	0.3	0.06	0

Qn 8e (ii)



2020 PAST PAPERS - 2

8. (a) Find the equation of a tangent to the ellipse $4x^2+y^2=6$ at $\left(\frac{1}{2},\sqrt{5}\right)$ in the form $ax+by+c=0$.

(b) The points $P\left(at_1^2,2at_1\right)$ and $Q\left(at_2^2,2at_2\right)$ lie on the parabola $y^2=4ax$. The tangents at the points P and Q intersect at R. Find the coordinates of R.

(c) Convert the following polar equations into Cartesian equations:

(i) $r^2=4\sin 2\theta$.

(ii) $r=3(1+\cos\theta)$.

(d) A curve is defined by the parametric equations $x=t^2$ and $y=\frac{2}{t}$ where $t\neq 0$. Show that the equation of the normal at the point $Q\left(p^2,\frac{2}{p^2}\right)$ is $p^4x-py+2=p^6$.

8a $4x^2+y^2=6$.

Differentiating to obtain slope.

$$8x+2y\frac{dy}{dx}=0.$$

$$2y\frac{dy}{dx}=-8x.$$

$$y\frac{dy}{dx}=-4x.$$

$$\frac{dy}{dx}=\frac{-4x}{y}.$$

$$\text{Slope}=\frac{-4\times \frac{1}{2}}{\sqrt{5}}$$

$$=\frac{-2}{\sqrt{5}}.$$

$$M=\frac{y-y_0}{x-x_0}.$$

$$\frac{-2}{\sqrt{5}}=\frac{y-\sqrt{5}}{x-\frac{1}{2}}$$

$$\sqrt{5}\left(y-\sqrt{5}\right)=-2\left(x-\tfrac{1}{2}\right)$$

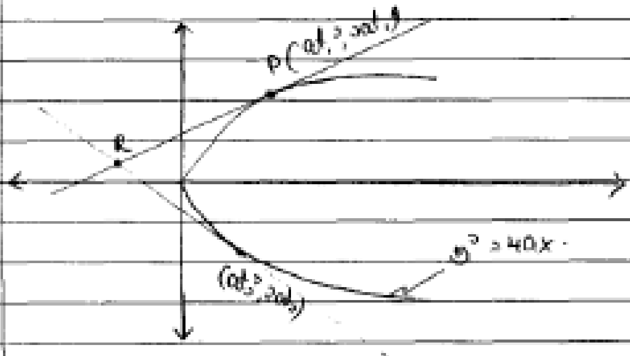
$$\sqrt{5}y-5=-2x+1$$

$$2x+\sqrt{5}y-5-1=0.$$

$$2x+\sqrt{5}y-6=0.$$

Equation of tangent is $2x+\sqrt{5}y-6=0$.

10b



Equation of tangent

$y^2 = 4ax$

$2y \frac{dy}{dx} = 4a$

$y \frac{dy}{dx} = 2a$

$\frac{dy}{dx} = \frac{2a}{y}$

$\frac{dy}{dx} = \frac{2a}{2at_1}$

$m = \frac{1}{t_1}$

$\frac{1}{t_1} = \frac{y - 2at_1}{x - at_1^2}$

$x - at_1^2 = t_1 y - 2at_1^2$

$x - at_1^2 + 2at_1^2 - t_1 y = 0$

$x + at_1^2 - t_1 y = 0$ . . . Equation of tangent i

10b 2nd tangent

$m = \frac{1}{t_2}$

$\frac{1}{t_2} = \frac{y - 2at_2}{x - at_2^2}$

$x - at_2^2 = yt_2 - 2at_2^2$

$x - yt_2 - at_2^2 + 2at_2^2 = 0$

$x - yt_2 + at_2^2 = 0$ ii

Since the two tangents intersect
then insert equation ii to i

$x = yt_2 - at_2^2$

$yt_2 - at_2^2 + at_1^2 - t_1 y = 0$

$yt_2 - at_2^2 + at_1^2 - t_1 y = 0$

$y(t_2 - t_1) = at_2^2 - at_1^2$

$y(t_2 - t_1) = a(t_2^2 - t_1^2)$

$y(t_2 - t_1) = a(t_2 - t_1)(t_2 + t_1)$

$y = a(t_1 + t_2)$

	but $x = at_2 \cdot at_2^2$	
	$x = (at_1 + at_2)t_2 - at_2^2$	
	$x = at_1t_2 + at_2^2 - at_2^2$	
	$x = at_1t_2$	
	thus point $R = (at_1t_2, a(t_1+t_2))$	

2019 PAST PAPERS - 2

8. (a) Show that the equation of a tangent to parabola $y^2 = 4ax$ at point (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

(b) Find the perpendicular distance of a point $(10,10)$ from the tangent to the curve $4x^2 + 9y^2 = 25$ at $(-18,1)$.

(c) Show that the equation $16x^2 + 25y^2 - 64x + 150y - 111 = 0$ is an equation of ellipse.

(d) (i) Show that $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ when $c^2 = a^2m^2 - b^2$.

(ii) Determine the equation of a tangent line to hyperbola $5x^2 - 4y^2 = 1$ if the slope of the tangent line is -2.

(e) (i) Transform the equation $x^2 + y^2 + 4x = 2\sqrt{x^2 + y^2}$ into a polar equation.

(ii) Draw the graph of the polar equation obtained in (i) above in the interval $0 \le \theta \le 2\pi$.

8	(a)	$y^2 = 4ax$
		$2y \cdot \frac{dy}{dx} = 4a$
		$\frac{dy}{dx} = \frac{4a}{2y}$

$$dy/dx = \frac{2a}{y}$$

slope of tangent $= \frac{2a}{y}$

at point (x_1, y_1)

slope of tangent $= \frac{2a}{y_1}$

Eqn of tangent:

$\Rightarrow$ slope $= \frac{\Delta y}{\Delta x}$

$$\frac{2a}{y_1} = \frac{y - y_1}{x - x_1}$$

$$2a(x - x_1) = y_1 y - y_1^2$$

$$2ax - 2ax_1 = y_1 y - y_1^2$$

but $y_1^2 = 4ax_1$

$$2ax - 2ax_1 = y_1 y - 4ax_1$$

$$2ax - 2ax_1 + 4ax_1 = y_1 y$$

$y_1 y = 2a(x + x_1)$ Hence shown.

(b) $4x^2 + 9y^2 = 25$

$$8x + 18y\,dy/dx = 0$$

$$18y\,dy/dx = -8x$$

$$dy/dx = \frac{-8x}{18y}$$

$$dy/dx = \frac{-4x}{9y}$$

slope of tangent at point $(-18, 1)$

$$M_T = \frac{-4(-18)}{9(1)}$$

$$M_T = 72/9$$

$M_T = 8$

Eqn of tangent:

$$\text{slope} = \Delta y / \Delta x$$

$$\frac{8}{1} = \frac{y-1}{x+18}$$

$$y - 1 = 8x + 144$$

$$8x - y + 144 + 1 = 0$$

$$8x - y + 145 = 0$$

Eqn of tangent:

$$8x - y + 145 = 0$$

Perpendicular distance $= \left| \frac{ax + by + c}{\sqrt{a^2+b^2}} \right|$

$$= \frac{|8x - y + 145|}{\sqrt{64+1}}$$

$$= \frac{|8x - y + 145|}{\sqrt{65}}$$

at point (10,10)

$$= \frac{|8(10) - (10) + 145|}{\sqrt{65}}$$

$$= 215/\sqrt{65}$$

$\therefore$ Perpendicular distance $= 215/\sqrt{65}$ units

$$16x^2 - 64x + 25y^2 + 150y = 111$$

Completing the square

$$16(x^2 - 4x) + 25(y^2 + 6y) = 111$$

$$111 = 16(x^2 - 4x + (4/2)^2 - (4/2)^2) + 25(y^2 + 6y + (6/2)^2 \text{ [illegible]}$$

$$16((x-2)^2 - 4) + 25((y+3)^2 - 9) = 111$$

$$16(x-2)^2 - 64 + 25(y+3)^2 - 225 = 111$$

$$16(x-2)^2 + 25(y+3)^2 = 111 + 64 + 225$$

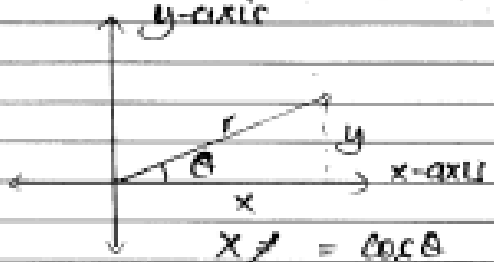
$$16(x-2)^2 + 25(y+3)^2 = 400$$

Divide by 400

$$\frac{16(x-2)^2}{400} + \frac{25(y+3)^2}{400} = 1$$

$$\frac{(x-2)^2}{25} + \frac{(y+3)^2}{16} = 1$$ shown

6 (c) i) $x^2 + y^2 + 4x + 2\sqrt{x^2+y^2}$



$$\frac{x}{r} = \cos\theta$$

$$x = r\cos\theta$$

but also; $r = \sqrt{x^2+y^2}$

$$r^2 = x^2 + y^2$$

$$r^2 + 4r\cos\theta = 2r$$

$$r^2 + 4r\cos\theta - 2r = 0$$

$$r^2 + r(4\cos\theta - 2) = 0$$

$$r(r + (4\cos\theta - 2)) = 0$$

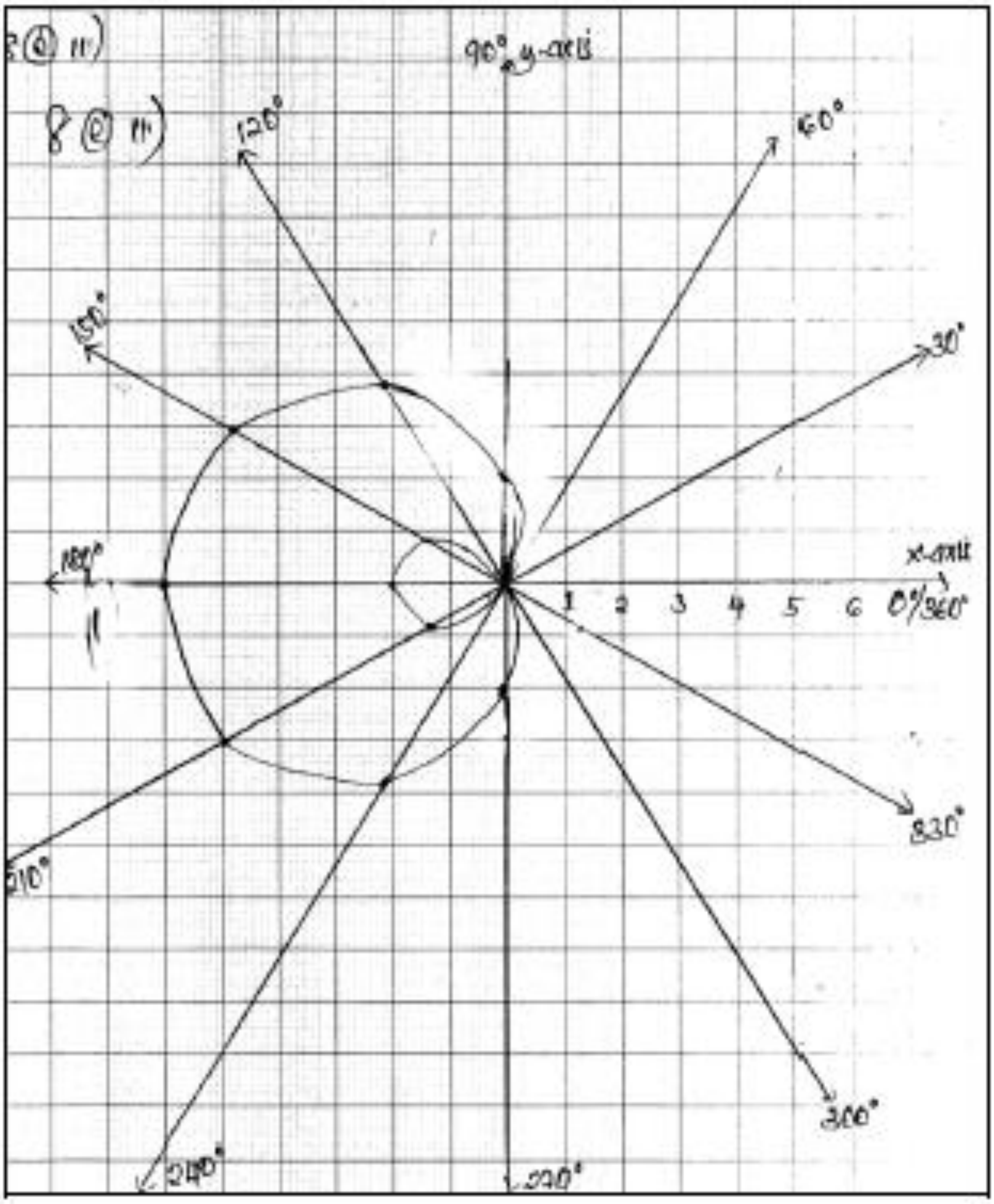
$\therefore$ $r = 0$

and $r = 2 - 4\cos\theta$

$\therefore$ $x^2+y^2+4x = 2\sqrt{x^2+y^2}$ in polar form is

$$r = 2 - 4\cos\theta$$

	θ	0°	30°	60°	90°	120°	150°	180°	210°
	r=2-4cosθ	-2	-1.46	0	2	4	5.46	6	5.46
		240°	270°	300°	330°	360°			
		4	2	0	-1.46	-2			



2018 PAST PAPERS - 2

8. (a) Show that the point $B(5,-5)$ lies on the parabola $y^2 = 5x$ and find the equation of the normal to the parabola at the point B in the form $y = mx + c$.

(b) If $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, find c in terms of a, b, m.

(c) (i) Find the rectangular equation of $r = 12(1 + \sin\theta)$.

(ii) Sketch the graph of $r = \sin 2\theta$ for $0 \le \theta \le \pi$.

08 Soln

$y^2 = 5x$ ——— (i)

point $= B(5,-5)$

if point B satisfy equation (i) on substituting it's value on (i) then lies on that parabola.

Consider L.H.S

$= y^2$ but $B(5,-5)$

$= (-5)^2$

$= 25$

Consider R.H.S

$= 5x$ $B(5,-5)$

$= 5 \times 5$

$= 25$

(i) Since R.H.S = L.H.S = 25 then point B lies on the parabola $y^2 = 5x$

gradient of tangent at B.

$y^2 = 5x$

$2y\frac{dy}{dx} = 5.$

$\frac{dy}{dx} = \frac{5}{2y}$ at $(5,-5)$

$\frac{dy}{dx} = \frac{5}{2 \times -5}$

$\frac{dy}{dx} = -\frac{1}{2}$

But tangent is perpendicular to normal then m_2 (slope of a normal)

$m_1 m_2 = -1$

$m_2 = \frac{-1}{-1/2} = 2.$

then, equation of a normal

From

$(y-y_1) = m(x-x_1)$

$(y+5) = 2(x-5)$

$y+5 = 2x-10$

$y = 2x-10-5$

$y = 2x-15$,

$\therefore$ equation of the normal to the parabola is given as $y = 2x-15$

02 (b) soln.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ——— (i) ellipse

$y = mx + c$ – – (2) tangent to ellipse

Substitute (2) into (i)

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$b^2x^2 + a^2(mx+c)^2 = a^2b^2$

$b^2x^2 + a^2(m^2x^2+2mcx+c^2) = a^2b^2$

$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$

$(b^2+a^2m^2)x^2 + 2a^2mcx + a^2(c^2-b^2) = 0$

But

equation (2) is a tangent then it touches ellipse at one point, hence equation above is perfect square

$b^2 = 4ac$

$(2a^2mc)^2 = 4a^2(b^2+a^2m^2)(c^2-b^2)$

$4a^4m^2c^2 = 4a^2(b^2+a^2m^2)(c^2-b^2)$

$a^2m^2c^2 = (b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2)$

$b^2c^2 = b^4 + a^2m^2b^2$

$c^2 = b^2 + a^2m^2$.

$c = \pm\sqrt{b^2+a^2m^2}$

$\therefore$ the value of c in terms of a, b, m is given as $c = \pm\sqrt{b^2+a^2m^2}$.

$\sqrt{x^2+y^2} = \frac{12\left(\sqrt{x^2+y^2}+y\right)}{\sqrt{x^2+y^2}}$

$(x^2+y^2) = 12\left(\sqrt{x^2+y^2}+y\right)$.

$\left[(x^2+y^2) - 12y\right]^2 = \left[12\sqrt{x^2+y^2}\right]^2$

$$(x^2+y^2)^2 - 24(x^2+y^2)y + 144y^2 = 144(x^2+y^2)$$

$$(x^2+y^2)^2 - 24(x^2+y^2)y = 144x^2.$$

$$x^4 + 2x^2y^2 + y^4 - 24x^2y - 24y^3 - 144x^2 = 0.$$

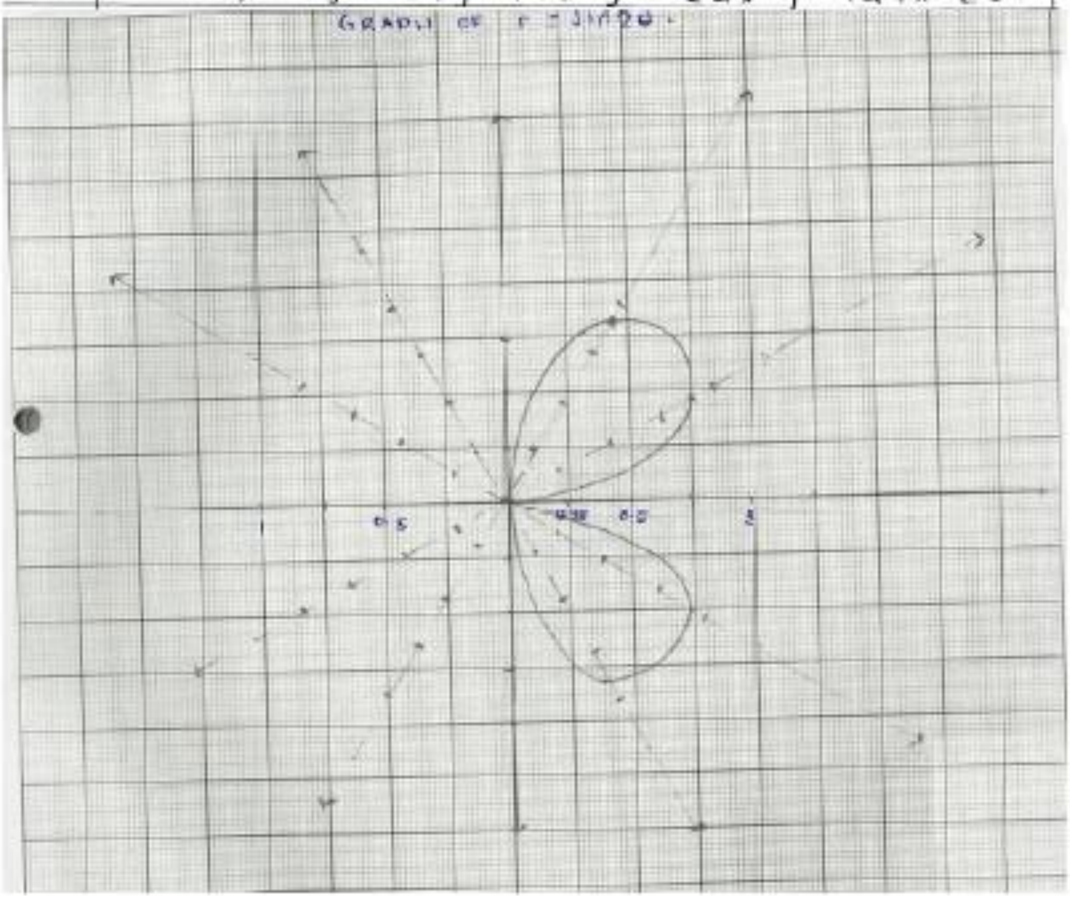
$$(x^2+y^2)^2 - 24(x^2+y^2)y = 144x^2.$$

$$x^4 + 2x^2y^2 + y^4 - 24x^2y - 24y^3 - 144x^2 = 0.$$

$$x^4 + y^4 - 24y^3 + 2x^2y^2 - 24x^2y - 144x^2 = 0.$$

$\therefore$ rectangular equation is given as

$$x^4 + y^4 - 24y^3 + 2x^2y^2 - 24x^2y - 144x^2 = 0$$



2017 PAST PAPERS - 2

8. (a) (i) The ellipse has its foci at the points $(-1, 0)$ and $(7, 0)$ when its ecentricity is $\frac{1}{2}$. Find its Cartesian equation.

(ii) Convert $y^2 = 4a(a - x)$ into polar equation.

(iii) Use the equation $y = 2x^2 - 6x + 4$ to determine its directrix and the focus.

(b) A cable used to support a swinging bridge approximates the shape of a parabola. Determine the equation of a parabola if the length of the bridge is 100m and the vertical distance from where the cable is attached to the bridge to the lowest point of the cable is 20m.

(c) (i) Define the term hyperbola.

(ii) Show that the latus rectum of the equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

(d) Sketch the graph of $r = 2 + 4\cos t$.

(20 marks)

8 a iii

$y = 2x^2 - 6x + 4$

$y = 2[x^2 - 3x] + 4$

$y = 2(x - 3x + (3/2)^2] + 4$

$y - 4 = 2(x - 3/2)^2$

$y - 4 + 9/2$

$y + 1/2 = 2(x - 3/2)^2$

$(x - 3/2)^2 = \frac{1}{2}[y + 1/2]$

8 a (iii)

$(x - 3/2)^2 = \frac{1}{2}[y + 1/2]$

Compare $(x - p)^2 = 4a(y - q)$

$p = 3/2 \quad q = -1/2$

$4a = 1/2$

$a = 1/8$

directric $y = (-a + q)$

$y = (-1/8 + -1/2)$

$y = -5/8$

directrix $y = \frac{-5}{8}$

Focus $(0, a) + (p, q)$

$p, (a + q)$

$\frac{3}{2}, \left(\frac{1}{8} + \frac{-1}{2}\right)$

Focus $\left(\frac{3}{2}, \frac{-3}{8}\right)$

$\therefore$ Focus is $\left(\frac{3}{2}, \frac{-3}{8}\right)$ directry $y = \left(\frac{-5}{8}\right)$

Extract 18.2, a sample of the best solution from one of the candidates who determined the directrix and the focus from the equation $y = 2x^2 - 6x + 4$.

2016 PAST PAPERS - 2

(20 marks)

8\. (a) (i) If $9y^2 - 54y - 25x^2 + 200x - 544 = 0$ is the hyperbola equation; find the center, the vertices, the foci and the equation of the asymptotes.

(ii) Convert $x^2 + y^2 = 4x$ into polar equation.

(iii) Convert $(1, -1)$ into polar coordinates.

(b) Find the equation of the tangent and normal at $P(a\cos\alpha, b\sin\alpha)$ to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

(c) (i) Define a conic section.

(ii) A man running a race- course notes that the sum of the distances from the two flag posts to him is always 10 meters. If the distance between the flag posts is 8 meters, find the equation of the path traced by the man.

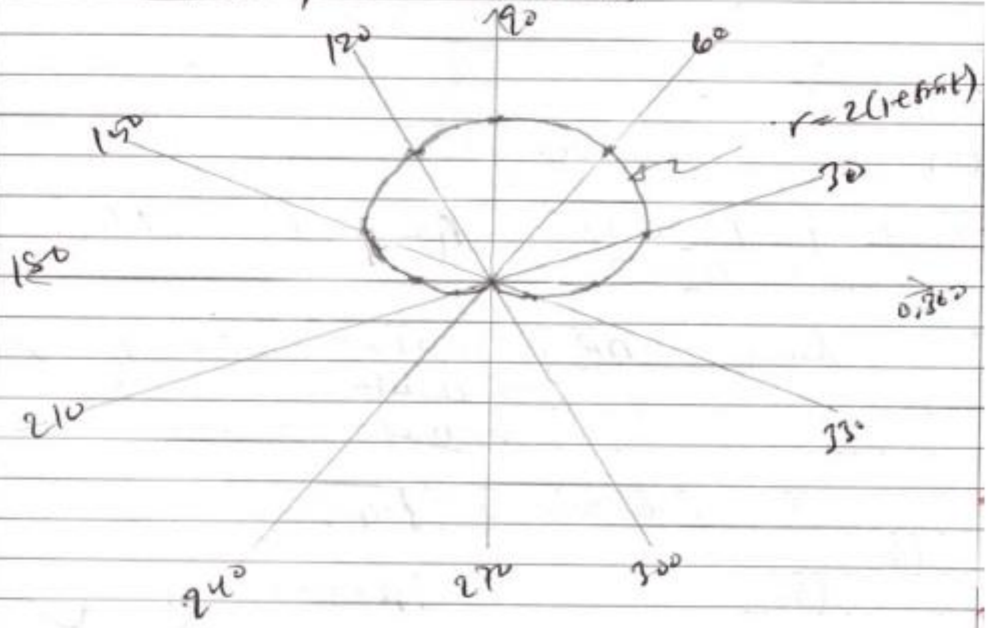
(d) Sketch the graph of $r = 2(1 + \sin t)$. **(20 marks)**

8 (b) Soln

Consider the table below.

t	0	30°	60°	90°	120	150	180	210	270
$r=2(1+\sin t)$	2	3	3.7	4	3.7	3	2	1	0
300	330	360	240						
0.3	1	2	0.3						

Sketch of $r = 2(1+\sin t)$



In Extract 18.1, the candidate prepared a table of values that enabled him/her to correctly sketch the graph of the polar curve.

2015 PAST PAPERS - 2

(20 marks)

8. (a) (i) Find the equation of the tangent to the ellipse $9x^2 + 25y^2 - 18x - 100y - 116 = 0$ at $(1, 5)$.

(ii) The normal at the point $p(4\cos\theta, 3\sin\theta)$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ meets the x-axis and y-axis at A and B respectively. Show that the locus of the midpoint of AB is an ellipse with the same eccentricity as the given ellipse.

(b) (i) Determine the polar equation of $x^2 + y^2 - 2x - 3y = 0$.

(ii) Draw the graph of the polar equation obtained in part b (i) above.

(c) (i) Show that the line $3x - y + 1 = 0$ touches the parabola $y^2 = 12x$.

(ii) If $ax + by + c = 0$ touches the parabola $x^2 - 8y = 0$, find an equation connecting a, b, and c.

(c)(i) to show that; $3x - y + 1 = 0$ touches the parabola $y^2 = 12x$ ---- (*)

8(c)(i) $y = 3x + 1$ --- (1) insert equation (1) into equation (*)

$(3x+1)^2 = 12x$

$\Rightarrow 9x^2 + 6x + 1 = 12x$

$9x^2 - 6x + 1 = 0$

for $b^2 - 4ac = ?$

$= (-6)^2 - 4(9)(1)$

$= 36 - 36$

$= 0$

$\therefore$ Since $b^2 - 4ac = 0$; then the line $3x - y + 1 = 0$ touches the parabola $y^2 = 12x$

8 (c)(ii) Given that $ax + by + c = 0$ touches the parabola $x^2 - 8y = 0$

$\Rightarrow x^2 = 8y$ --- (1)

also $by = -ax - c$

$y = -\frac{a}{b}x - \frac{c}{b}$ (2)

insert equation (2) into equation (1)

$x^2 = 8\left(\frac{-ax - c}{b}\right)$

$\Rightarrow bx^2 = -8ax - 8c$

$bx^2 + 8a \cdot x + 8c = 0$

Because it touches the parabola

$b^2 - 4ac = 0 \Rightarrow b^2 = 4ac \Rightarrow$

$(8a)^2 = 4(b)(8c) \Rightarrow \frac{64a^2}{32} = \frac{32bc}{32}$

$\Rightarrow 2a^2 = bc$ (is the required equation)

Extract 18.1 portrays response of a candidate who answered the question 8 (c) correctly. He/she verified that the linear equation touches the parabola and managed to find the equation connecting the constants a, b and c.

12.0 Vectors

2021 PAST PAPERS – 2

3. (a) If $\underline{a} = \underline{i} - \underline{j} + 2\underline{k}$ and $\underline{b} = \underline{i} + \underline{j}$, find the unit vector orthogonal to both $\underline{a}$ and $\underline{b}$.

(b) The position vectors of the points A and B are $2\underline{i} + 3\underline{j} - \underline{k}$ and $-\underline{i} + 5\underline{j} + 7\underline{k}$ respectively. If C divides $\overline{AB}$ internally in the ratio $2:1$, find the position vector of point C.

(c) Using the cosine rule, show that in the triangle ABC, $\underline{c} = \underline{b}\cos A + \underline{a}\cos B$.

3 a) given

$\underline{a} = \underline{i} - \underline{j} + 2\underline{k}$

$\underline{b} = \underline{i} + \underline{j} + 0\underline{k}$

the unit vector orthogonal to both a and b

let ~~the~~ - the vector be $\underline{c}$

$\underline{c} = \underline{a} \times \underline{b}$

$$\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$\underline{c} = \underline{i}(0-2) - \underline{j}(0-2) + \underline{k}(1+1)$

$\underline{c} = -2\underline{i} + 2\underline{j} + 2\underline{k}$

the unit vector $\underline{c}$

$$= \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{\sqrt{4+4+4}}$$

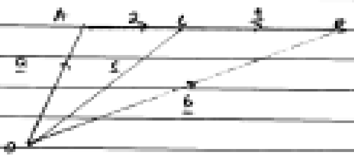
$$= \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{\sqrt{12}} = \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{2\sqrt{3}}$$

3 a) $\hat{c} = \frac{-1}{\sqrt{3}}\underline{i} + \frac{1\underline{j}}{\sqrt{3}} + \frac{\underline{k}}{\sqrt{3}}$

$\therefore$ The unit vector orthogonal to given a and b

is $\frac{-1}{\sqrt{3}}\underline{i} + \frac{1}{\sqrt{3}}\underline{j} + \frac{1}{\sqrt{3}}\underline{k}$

3 b)



$$\frac{\overline{AC}}{\overline{CB}} = \frac{2}{1}$$

$$\frac{\vec{OC} - \vec{OA}}{\vec{OB} - \vec{OC}} = \frac{2}{1}$$

$2\vec{OB} - 2\vec{OC} = \vec{OC} - \vec{OA}$

2020 PAST PAPERS – 2

3. (a) Find the unit vector perpendicular to both vectors $\underline{a}+\underline{b}$ and $\underline{a}-\underline{b}$ where $\underline{a}=3\underline{i}+2\underline{j}+2\underline{k}$ and $\underline{b}=\underline{i}+2\underline{j}-2\underline{k}$.

(b) The area of a parallelogram is $5\sqrt{6}$ units. If the adjacent sides of the parallelogram are $\underline{i}-2\underline{j}+\lambda\underline{k}$ and $2\underline{i}+\underline{j}-4\underline{k}$ respectively, find the positive value of λ.

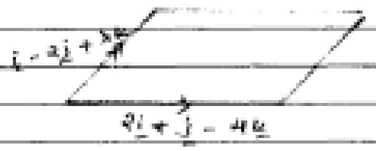
(c) A particle is moving so that at any instant its velocity $\underline{v}$ is given by $\underline{v}=3t\underline{i}-4\underline{j}+t^2\underline{k}$. If the particle is at point $P(1,0,1)$ when $t=0$, find;

(i) the displacement vector when $t=2$.

(ii) the magnitude of the acceleration when $t=2$.

b)

Area $= 5\sqrt{6}$



Let $\underline{a} = 2i + j - 4k$

$\underline{b} = i - 2j + \lambda k$

$A = |\underline{a} \times \underline{b}|$

$$\underline{a}\times\underline{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & -4 \\ 1 & -2 & \lambda \end{vmatrix}$$

$\underline{a} \times \underline{b} = i(\lambda - 8) - j(2\lambda + 4) + k(-4 - 1)$

$= (\lambda - 8)\underline{i} - (2\lambda + 4)\underline{j} - 5\underline{k}$

3. b)

$|\underline{a} \times \underline{b}| = \sqrt{(\lambda - 8)^2 + (2\lambda + 4)^2 + 25}$

$5\sqrt{6} = \sqrt{(\lambda - 8)^2 + (2\lambda + 4)^2 + 25}$

~~256~~

$150 = (\lambda - 8)^2 + (2\lambda + 4)^2 + 25$

$150 = \lambda^2 - \cancel{16\lambda} + 64 + 4\lambda^2 + \cancel{16\lambda} + 16 + 25$

$150 = 5\lambda^2 + 105$

$5\lambda^2 = 150 - 105$

$5\lambda^2 = 45$

$\lambda^2 = 9$

$\lambda = \pm\sqrt{9}$

$\lambda = \pm 3 \quad \therefore \lambda = 3.$

$\therefore$ The value of $+\lambda$ is 3

c) $\underline{v} = 3t\underline{i} - 4\underline{j} + t^2\underline{k}$

$P(1, 0, 1) \quad t = 0$

i) $\underline{v} = \frac{dr}{dt}$

$\frac{dr}{dt} = 3t\underline{i} - 4\underline{j} + t^2\underline{k}$

$\int dr = \int (3t\underline{i} - 4\underline{j} + t^2\underline{k})\,dt$

$r = \frac{3t^2}{2}\underline{i} - 4t\underline{j} + \frac{t^3}{3}\underline{k} + c$

$t = 0 \quad r = i + k$

$\underline{i} + \underline{k} = c$

$\therefore r(t) = \frac{3t^2}{2}\underline{i} - 4t\underline{j} + \frac{t^3}{3}\underline{k} + \underline{i} + \underline{k}$

$\underline{r} = \left(\frac{3}{2}t^2 + 1\right)\underline{i} - 4t\underline{j} + \left(\frac{t^3}{3} + 1\right)\underline{k} \quad t = 2$

$\underline{r}(2) = \left(\frac{3}{2}(2)^2 + 1\right)\underline{i} - 4(2)\underline{j} + \left(\frac{2^3}{3} + 1\right)\underline{k}$

3. c)

i) $\underline{r}(2) = 7\underline{i} - 8\underline{j} + \frac{11}{3}\underline{k}$

$\therefore$ Displacement vector when $t = 2$ $\underline{r} = 7\underline{i} - 8\underline{j} + \frac{11}{3}\underline{k}$

ii)

$\underline{v} = 3t\underline{i} - 4\underline{j} + t^2\underline{k}$

$a = \frac{dv}{dt}$

Differentiate $\underline{v}$ wrt t

$\frac{dv}{dt} = 3\underline{i} + 2t\underline{k}$

$\underline{a} = 3\underline{i} + 2t\underline{k}$

when $t = 2$

$\underline{a}(2) = 3\underline{i} + 2(2)\underline{k}$

$\underline{a}_{(2)} = 3\underline{i} + 4\underline{k}$

$|\underline{a}| = \sqrt{3^2 + 4^2}$

$|\underline{a}| = \sqrt{25}$

$|\underline{a}| = 5$ units $/s^2$

$\therefore$ Magnitude of acceleration when $t = 2$ is 5 unit$/s^2$.

2019 PAST PAPERS – 2

3. (a) If $\underline{a} = 2\underline{i} + 3\underline{j} + 4\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 2\underline{k}$, find the following:

   (i) The projection of $\underline{a}$ onto $\underline{b}$.

   (ii) The angle between vectors $\underline{a}$ and $\underline{b}$.

   (iii) The unit vector of $\underline{a} \times \underline{b}$.

   (b) The point K has position vector $3\underline{i} + 2\underline{j} - 5\underline{k}$ and a point L has position vector $\underline{i} + 3\underline{j} + 2\underline{k}$. Find the position vector of a point M which divides $\overrightarrow{KL}$ in the ratio of $4:3$.

   (c) A displacement vector is given by $\underline{r} = a\underline{i}\cos nt + b\underline{j}\sin nt$ where a and b are arbitrary constants. Find the corresponding velocity and acceleration when $t = 0$.

3 (a) (i)

$$\underline{a} \cdot \underline{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$= 4 + 3 + 8$$

$$= 15$$

$$|\underline{b}| = \sqrt{2^2 + 1^2 + 2^2}$$

$$= 3$$

$$\text{Proj}_b a = \frac{15}{3}$$

$$= 5$$

3 (a) (ii) Solution;

From:

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta$$

where

$$\underline{a} \cdot \underline{b} = 15$$

$$|\underline{b}| = 3$$

$$|\underline{a}| = \sqrt{2^2 + 3^2 + 4^2}$$

$$= \sqrt{29}$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{15}{3\sqrt{29}}$$

3a)(ii) $\cos\theta = 0.92847$

$$\theta = \cos^{-1}(0.92847)$$

$$\theta = 21.80^\circ$$

3 (a) (iii) Solution:

$$\text{Unit vector} = \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{|\underset{\sim}{a} \times \underset{\sim}{b}|}$$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} i & j & k \\ 2 & 3 & 4 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\underset{\sim}{a} \times \underset{\sim}{b} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$$

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = \sqrt{2^2 + 4^2 + (-4)^2}$$

$$= 6$$

$$\text{Unit vector} = \frac{2\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}}{6}$$

$$= \frac{1}{3}\underset{\sim}{i} + \frac{2}{3}\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$$

3 (b) Solution:

By internal division method,

$$\underset{\sim}{P} = n\lambda \quad \frac{n\underset{\sim}{a} + m\underset{\sim}{b}}{m+n} \qquad m:n = 4:3$$

3 (b) $\underset{\sim}{a} = 3i + 2j - 5k$

$\underset{\sim}{b} = i + 3j + 2k$

let

Vector $\underset{\sim}{M}$ be $\underset{\sim}{P}$

$$\underset{\sim}{P} = \frac{3(3\underset{\sim}{i} + 2\underset{\sim}{j} - 5k) + 4(\underset{\sim}{i} + 3\underset{\sim}{j} + 2k)}{3+4}$$

$$= \frac{9\underset{\sim}{i} + 6\underset{\sim}{j} - 15\underset{\sim}{k} + 4\underset{\sim}{i} + 12\underset{\sim}{j} + 8\underset{\sim}{k}}{7}$$

$$\underset{\sim}{M} = \frac{13\underset{\sim}{i} + 18\underset{\sim}{j} - 7\underset{\sim}{k}}{7}$$

By external division,

$$\underline{r} = \frac{n\underline{a} - m\underline{b}}{m-n}$$

$$\underline{r} = \frac{3(3\underline{i} + 2\underline{j} - 5\underline{k}) + 4(\underline{i} + 3\underline{j} + 2\underline{k})}{4-3}$$

$$M = 5\underline{i} - 6\underline{j} - 23\underline{k}$$

3 (c) Solution:

$$r = ai\cos nt + bj\sin \omega t$$

$$\frac{dr}{dt} = -ani\sin nt + b\omega j\cos \omega t$$

$$\frac{d^2r}{dt^2} = -an^2 i\cos nt - b\omega^2 j\sin \omega t$$

When $t = 0$,

For,

$$\frac{dr}{dt} = -ani\sin 0 + b\omega j\cos 0$$

$$\frac{dr}{dt} = 0 + b\omega \underline{j}$$

$$\text{Velocity} = (b\omega \underline{j})\ m/s$$

For Acceleration,

$$\frac{d^2r}{dt^2} = -an^2 i\cos n(0) - b\omega^2 j\sin 0$$

$$= -an^2\underline{i}\ m/s^2$$

2018 PAST PAPERS – 2

3. (a) (i) Find the work done in moving an object along a straight line from $(3,2,-1)$ to $(2,-1,4)$ in a force field given by $F = 4\underline{i} - 3\underline{j} + 2\underline{k}$.

(ii) If $\underline{a} = (3t+1)\underline{i} - \underline{j} - \underline{k}$ is perpendicular to $\underline{b} = (t+3)\underline{i} - 3\underline{j} - 2\underline{k}$, find the possible values of the constant t.

(b) The vertices of a quadrilateral are A (5, 2, 0), B (2, 6, 1), C (2, 4, 7) and D (5, 0, 6).

(i) Show that the quadrilateral is a parallelogram.

(ii) Find the actual area of the parallelogram in (b) (i).

(c) The vertices A, B and C of the triangle are at the points with position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively. Show that the area of the triangle is equal to $\frac{1}{2}|\underline{a}\times\underline{b}+\underline{b}\times\underline{c}+\underline{c}\times\underline{a}|$ square units.

3 (a) (i) $W = F \cdot d.$

$$d = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$$

$d = -i - 3j + 5k.$

$F = 4i - 3j + 2k.$

Work done $= F \cdot d.$

$$F \cdot d = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} = -4 + 9 + 10$$

$= 15$ units

Work done $= 15$ units.

3 (a) (ii) $a = (3t+1)i - j - k.$

$b = (t+3)i - 3j - 2k.$

$a \cdot b = |a|\,|b| \cos\theta.$

since a and b are perpendicular angle between them (θ) is $90°$.

$a \cdot b = |a||b| \cos 90°.$

$a \cdot b = 0.$

$$a \cdot b = \begin{pmatrix} 3t+1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} t+3 \\ -3 \\ -2 \end{pmatrix} = 0.$$

$(3t+1)(t+3) + 3 + 2 = 0$

$(3t+1)(t+3) + 5 = 0.$

$(3t+1)(t+3) = -5.$

$3t^2 + 9t + t + 3 = -5.$

$3t^2 + 10t + 3 = -5.$

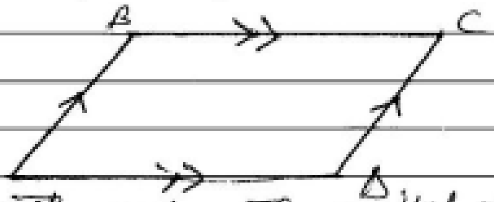
$3t^2 + 10t + 3 + 5 = 0.$

$3t^2 + 10t + 8 = 0$

$t = -4/3 \qquad t = -2.$

$\therefore \quad t = -4/3$ and $t = -2.$

3 (b)(i) condition for parallelogram.



$\overrightarrow{AB}$ parallel to $\overrightarrow{CD}$ and $\overrightarrow{BC}$ parallel to $\overrightarrow{AD}$.

and $|AB| = |CD|$ also $|BC| = |AD|$.

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}. \quad |AB| = \sqrt{26}.$$

$$\overrightarrow{BC} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix} \quad |BC| = \sqrt{40}.$$

$$\overrightarrow{DC} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}. \quad |DC| = \sqrt{26}.$$

$$\overrightarrow{AD} = \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix}. \quad |AD| = \sqrt{40}.$$

condition for parallel vector is angle between them is zero.

$\overrightarrow{AB} \cdot \overrightarrow{DC} = |AB||DC| \cos\theta.$

$$\cos\theta = \frac{\overrightarrow{AD} \cdot \overrightarrow{DC}}{|AB||DC|}.$$

$$\cos\theta = \frac{\begin{pmatrix}+3\\4\\1\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}}{(\sqrt{26})(\sqrt{26})} \Rightarrow \frac{9+16+1}{26}$$

$\cos\theta = 1$

$\theta = \cos^{-1} 1 = 0$

for $\vec{BC}\cdot\vec{AD} = |BC||AD| \cos\theta$

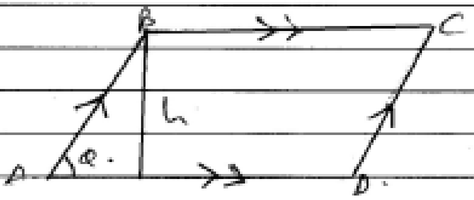
$$\cos\theta = \frac{\vec{BC}\cdot\vec{AD}}{|BC||AD|} \Rightarrow \frac{\begin{pmatrix}0\\2\\6\end{pmatrix}\begin{pmatrix}0\\2\\6\end{pmatrix}}{(\sqrt{40})(\sqrt{40})}$$

$$\cos\theta = \frac{0+4+36}{40} = \frac{40}{40} = 1$$

$\theta = \cos^{-1} 1 = 0^\circ$

since ~~their~~ magnitudes are equal and angle between the ~~sides~~ parallel sides is zero it is a parallelogram.

3(b) (ii)



$\text{Area} = |\vec{AD}| \times h$ $\quad \sin\theta = h / |\vec{AB}|$

$h = |\vec{AB}| \sin\theta$

$\text{Area} = |\vec{AD}|\,|\vec{AB}| \sin\theta$

but $|AD \times AB| = |\vec{AD}||\vec{AB}| \sin\theta$

$\text{Area} = |\vec{AD} \times \vec{AB}|$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix}$$

$$= i(24+2) - j(-18-0) + k(6-0)$$

$$\vec{AB} \times \vec{AB} = 26i + 18j + 6k.$$

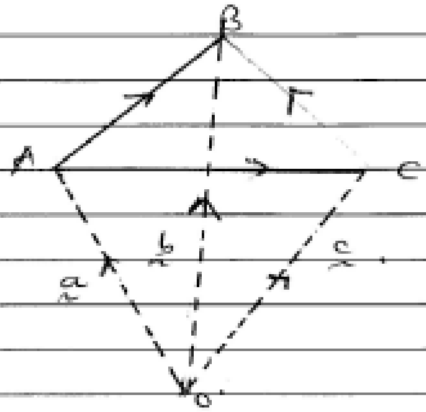
$$\text{area} = |AB \times AD| = \sqrt{(26)^2 + (18)^2 + (6)^2}$$

$$= \sqrt{1036}$$

$= 32.186$ square units.

Area $= \sqrt{1036}$ or 32.186 square units.

3(c).



$$\text{Area} = \tfrac{1}{2}(\vec{AB} \times \vec{AC}) \sin\theta.$$

$$\vec{OA} + \vec{AB} = \vec{OB}.$$

$$\vec{AB} = \vec{OB} - \vec{OA}.$$

$$\vec{AB} = b - a.$$

$$\vec{OA} + \vec{AC} = \vec{OC}.$$

$$\vec{AC} = \vec{OC} - \vec{OA}.$$

$$\vec{AC} = c - a.$$

$$\text{Area} = \tfrac{1}{2}(b-a)\times(c-a) \sin\theta.$$

$$\Rightarrow \tfrac{1}{2}\left((b\times c) - (b\times a) - (a\times c) + (a\times a)\right)\sin\theta.$$

$$= \tfrac{1}{2}(b\times c) - (b\times a) - (a\times c) \sin\theta..$$

	since $-(b \times a) = (a \times b)$.
	$-(a \times c) = (c \times a)$.
	Area $= \frac{1}{2} \lvert a \times b + b \times c + c \times a \rvert$

2017 PAST PAPERS – 2

3. (a) (i) If $\underline{a} = 3\underline{i} - 5\underline{j} - 2\underline{k}$ and $\underline{b} = 7\underline{i} + \underline{j} - 2\underline{k}$ are non-zero vectors. Find the projection of $\underline{a}$ onto $\underline{b}$.

(ii) Use vectors to prove the sine rule.

(b) If θ is the angle between two unit vectors $\underline{a}$ and $\underline{b}$ show that $\frac{1}{2}\lvert \underline{a} + \underline{b} \rvert = \cos\left(\frac{\theta}{2}\right)$.

(c) (i) If $G(t) = e^t\underline{i} + \cos t\underline{j} + t\underline{k}$, find $\frac{d}{dt}[(\sin t)G(t)]$.

(ii) Integrate the vector $e^t\underline{i} + 2t\underline{j} + \ln t\underline{k}$ with respect to t.

(d) Two vectors $\underline{a}$ and $\underline{b}$ have the same magnitude and an angle between them is $60°$. If their scalar product is $\frac{1}{2}$, find their magnitude. **(15 marks)**

3(b)	$\lvert \underline{a} + \underline{b} \rvert^2 = \lvert \underline{a} \rvert^2 + \lvert \underline{b} \rvert^2 + 2\lvert \underline{a} \rvert\lvert \underline{b} \rvert \cos\theta$
	For unit vectors
	$\lvert \underline{a} \rvert = \lvert \underline{b} \rvert = 1$.
	$\lvert \underline{a} + \underline{b} \rvert^2 = 1 + 1 + 2\cos\theta$
	$\lvert \underline{a} + \underline{b} \rvert^2 = 2 + 2\cos\theta$.
	But $\cos 2\theta = \cos^2\theta - \sin^2\theta$
	$\cos\theta = \cos^2\theta/2 - \sin^2\theta/2$
	$\lvert \underline{a} + \underline{b} \rvert^2 = 2 + 2\cos^2\theta/2 - 2\sin^2\theta/2$
	$\lvert \underline{a} + \underline{b} \rvert^2 = 2 - 2\sin^2\theta/2 + 2\cos^2\theta/2$
	$\lvert \underline{a} + \underline{b} \rvert^2 = 2(1 - \sin^2\theta/2) + 2\cos^2\theta/2$
	But $\cos^2\theta/2 = 1 - \sin^2\theta/2$.
	$\lvert \underline{a} + \underline{b} \rvert^2 = 2\cos^2\theta/2 + 2\cos^2\theta/2$
	$\lvert \underline{a} + \underline{b} \rvert^2 = 4\cos^2\theta/2$
	$\lvert \underline{a} + \underline{b} \rvert = 2\cos\theta/2$
	$\frac{1}{2}\lvert \underline{a} + \underline{b} \rvert = \cos(\theta/2)$
	∴ Hence Shown

Extract 13.1 shows that the candidate proved the given equation by using the dot product and trigonometric identities.

2016 PAST PAPERS – 2

3. (a) If the position vector $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are defined by $\overrightarrow{OA} = 2\underline{i} - \underline{j} + 3\underline{k}$, $\overrightarrow{OB} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ and $\overrightarrow{OC} = -\underline{i} + 3\underline{j} - 2\underline{k}$;

(i) Determine the cross product $\overrightarrow{AB} \times \overrightarrow{BC}$.

(ii) Find the exact value of the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(b) If $\underline{a} = 3\underline{i} + 2\underline{j} + 9\underline{k}$ and $\underline{b} = \underline{i} + \lambda\underline{j} + 3\underline{k}$. Find

(i) The value of λ so that $\underline{a} + \underline{b}$ is perpendicular to $\underline{a} - \underline{b}$.

(ii) The projection of $\underline{a}$ onto $\underline{b}$ and leave the answer in surd form.

(c) Derive the cosine's rule using the vectors $\underline{u}$ and $\underline{v}$. **(15 marks)**

3. (a) (i) Soln

Given;

$\overrightarrow{OA} = 2i - j + 3k$

$\overrightarrow{OB} = 3i + 2j - 4k$

$\overrightarrow{OC} = -i + 3j - 2k$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (3i + 2j - 4k) - (2i - j + 3k)$

$= i + 3j - 7k$

$\overrightarrow{AB} = i + 3j - 7k$

Also;

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\overrightarrow{BC} = (-i + 3j - 2k) - (3i + 2j - 4k)$

$\overrightarrow{BC} = -4i + j + 2k$

3 (a) (i) $\overrightarrow{AB} = i + 3j - 7k$

$\overrightarrow{BC} = -4i + j + 2k$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ 1 & 3 & -7 \\ -4 & 1 & 2 \end{vmatrix}$$

$= i(6 + 7) - j(2 - 28) + k(1 + 12)$

$= 13i - j(-26) + 13k$

$= 13i + 26j + 13k$

$\overrightarrow{AB} \times \overrightarrow{BC} = 13i + 26j + 13k$

(ii) Soln:

From:

$a \cdot b = |a||b| \cos\theta$

$\vec{AB} \cdot \vec{BC} = |\vec{AB}||\vec{BC}| \cos\theta$

$|\vec{AB}| = \sqrt{x^2+y^2+z^2}$

$|\vec{AB}| = \sqrt{1+9+49}$

$|\vec{AB}| = \sqrt{59}$

Also $|\vec{BC}| = \sqrt{16+1+4}$

$= \sqrt{21} = \sqrt{21}$

3 (a)(ii) $\vec{AB} \cdot \vec{BC} = -4+3-14$

$\vec{AB} \cdot \vec{BC} = 3-18$

$\vec{AB} \cdot \vec{BC} = -15$

From:

$\vec{AB} \cdot \vec{BC} = |\vec{AB}||\vec{BC}| \cos\theta$

$-15 = \sqrt{59}\sqrt{21} \cos\theta$

$\frac{-15}{\sqrt{1239}} = \cos\theta$

$\theta = \cos^{-1}\left[\frac{-15}{\sqrt{1239}}\right]$

$\theta = 115.22°$

∴ The exact value of angle between $\vec{AB}$ and $\vec{BC}$ is $115.22°$

(b) (i) Soln

Given's

$\underset{\sim}{a} = 3i + 2j + 9k$

$\underset{\sim}{b} = i + \lambda j + 3k$

$\underset{\sim}{a} + \underset{\sim}{b} = 4i + (2+\lambda)j + 12k$

$\underset{\sim}{a} - \underset{\sim}{b} = 2i + (2-\lambda)j + 6k$

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

3 (b) (i)

$(a+b) \cdot (a-b) = 0$

$8 + (2+\lambda)(2-\lambda) + (12)(6) = 0$

$8 + 4 - \lambda^2 + 72 = 0$

$-\lambda^2 = -84$

$\lambda^2 = 84$

$\lambda = \sqrt{84}$

$\therefore$ The value of λ is $\sqrt{84}$

(b) (ii) $\underset{\sim}{a} = 3i + 2j + 9k$

$\underset{\sim}{b} = i + \sqrt{84}\, j + 3k$

$$Proj_a^b = a \cdot \frac{b}{|b|}$$

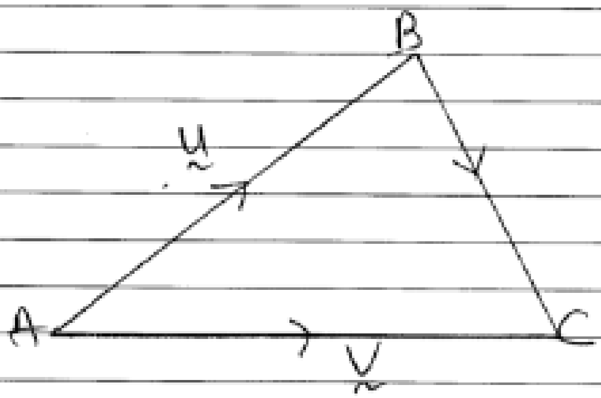
$$= (3i + 2j + 9k) \cdot \left(\frac{i + \sqrt{84}\, j + 3k}{\sqrt{1 + 84 + 9}} \right)$$

$$= (3i + 2j + 9k) \cdot \left(\frac{i + \sqrt{84}\, j + 3k}{\sqrt{94}} \right)$$

$$= \frac{3}{\sqrt{94}} + \frac{2\sqrt{84}}{\sqrt{94}} + \frac{9(3)}{\sqrt{94}}$$

$$= \frac{30 + 2\sqrt{84}}{\sqrt{94}} = \frac{\sqrt{94}\,(30 + 2\sqrt{84})}{94}$$

(c) Consider the figure below



let: Vector BC = c where c is the resultant vector ($\vec{BC}$)

$\vec{AB} + \vec{BC} = \vec{AC}$

$\underset{\sim}{u} + \vec{BC} = \underset{\sim}{v}$

$\vec{BC} = \underset{\sim}{v} - \underset{\sim}{u}$

Dot by $\vec{BC}$ both sides

$\vec{BC} \cdot \vec{BC} = (\underset{\sim}{v} - \underset{\sim}{u}) \cdot \vec{BC}$

$\vec{BC}^2 = (\underset{\sim}{v} - \underset{\sim}{u})(\underset{\sim}{v} - \underset{\sim}{u})$

3 (c)

$\vec{BC}^2 = \underset{\sim}{v}^2 - 2\underset{\sim}{u}\underset{\sim}{v} + \underset{\sim}{u}^2$

Recall;

$a \cdot b = |a||b| \cos\alpha$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos A$

$\vec{BC}^2 = \underset{\sim}{v}^2 - 2|\underset{\sim}{u}||\underset{\sim}{v}| \cos A + \underset{\sim}{u}^2$

$\vec{BC}^2 = \underset{\sim}{u}^2 + \underset{\sim}{v}^2 - 2|\underset{\sim}{u}||\underset{\sim}{v}| \cos A$

$\vec{BC}^2 = \underset{\sim}{u}^2 + \underset{\sim}{v}^2 - 2|\underset{\sim}{u}||\underset{\sim}{v}| \cos A$ is a cosine rule.

In Extract 13.1, the candidate was able to apply the knowledge of vectors to prove the cosine rule and answer other parts of this question.

2015 PAST PAPERS – 2

3. (a) (i) Find the expression for the work done used to move a 15kg baby from $\underline{r} = u_1 \underline{i} + u_2 \underline{j}$ to $\underline{p} = v_1 \underline{i} + v_2 \underline{j}$ and hence deduce the actual work done when (u_1, u_2) and (v_1, v_2) are $(-1, 7)$ and $(2, 3)$ respectively (use $g = 9.8$).

(ii) Two forces of 40N and 60N act on a point in a plane. If the angle between the force vectors is $30°$, find the magnitude and direction of the resultant force relative to the 60N force (write the magnitude of the resultant force to two significant figures and the angle to the nearest degree).

(b) The vertices of a triangle are A (1, 1, 2), B (3, 2, -1) and C (-4, 1, 3). Use your knowledge on vectors to find the area of triangle ABC.

(c) A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$ and $z = 2\sin 3t$ where t is time. If its position vector is $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$;

(i) Determine its velocity and acceleration at any time t.

(ii) Find the magnitude of the velocity and acceleration at time $t = 0$.

3 a) i) Consider vector distance, $\vec{d}$

$\underset{\sim}{d} = p - r$

$= v_1 i + v_2 j - (u_1 i + u_2 j)$

$= (v_1 - u_1) i + (v_2 - u_2) j$

Work done $= |F|\,|d| \cos\theta$

put $\theta = 0$

$|F| = 15 \times 9.8\ N$

$= 147 N$

$|d| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$

$\therefore$ Work done $= 147\sqrt{(v_1 - u_1)^2 + (v_2 - u_1)^2}$ Joules

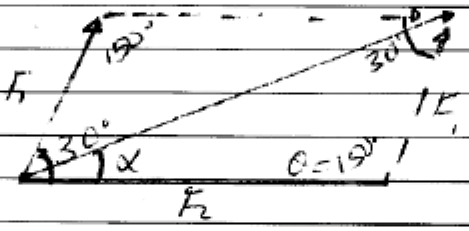
$\Rightarrow (u_1, u_2) = (-1, 7)$

$(v_1, v_2) = (2, 3)$

Work done $= 147 \sqrt{(2 - -1)^2 + (3 - 7)^2}$

$= 735$ Joules

2 a(ii).



$\alpha = ?$

$\theta = 150°$

3 a(ii) By using cosine rule

Resultant force, $F = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\theta}$

$= \sqrt{40^2 + 60^2 - 2(40)(60)\cos 150}$

$= 97\ N$

direction, by using sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \alpha}{F_1} = \frac{\sin \theta}{F}$$

$$\sin \alpha = \frac{F_1 \sin\theta}{F}$$

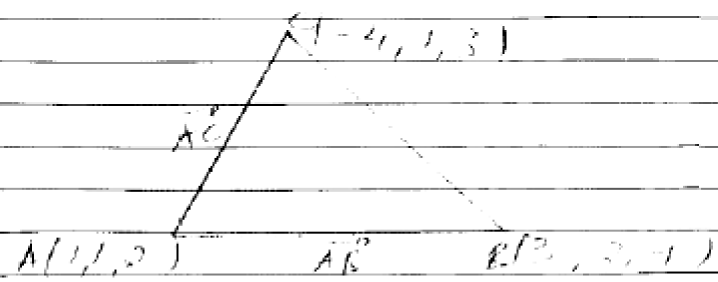
$$\alpha = \sin^{-1}\left(\frac{F_1 \sin\theta}{F}\right)$$

$$= \sin^{-1}\left(\frac{40 \sin 150}{97}\right)$$

$$= 12°$$

∴ Magnitude is 97N and direction is 12° relative to the force

2 b)



3(b). Consider $\overrightarrow{AC} = (-4-1)i + (1-1)j + (3-2)k$
$= -5i + k$

also $\overrightarrow{AB} = (3-1)i + (2-1)j + (-1-2)k$
$= 2i + j - 3k$

$\Rightarrow$ area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

Consider $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -5 & 0 & 1 \\ 2 & 1 & -3 \end{vmatrix}$

$= i\begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix} - j\begin{vmatrix} -5 & 1 \\ 2 & -3 \end{vmatrix} + k\begin{vmatrix} -5 & 0 \\ 2 & 1 \end{vmatrix}$

$= -i - j(15-2) + k(-5-0)$

$= -i - 13j - 5k$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-1)^2 + (-13)^2 + (-5)^2}$

$= 13.964$

Area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \times 13.964$

$= 6.982$

$\therefore$ Area $= 6.982$ square units

3(a).
Consider
$x = e^{-t}$
$\frac{dx}{dt} = -e^{-t}$

3 d. also $y = 2\cos 3t$

$$\frac{dy}{dt} = -6\sin 3t$$

also

$$z = 2\sin 3t$$

$$\frac{dz}{dt} = 6\cos 3t$$

d. $r = xi + yj + zk$

velocity, $\frac{dr}{dt} = i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}$

velocity, $v = i(-e^{-t}) + j(-6\sin 3t) + k(6\cos 3t)$

$$v = -e^{-t}i - 6\sin 3t\, j + 6\cos 3t\, k$$

also

acceleration $a = \frac{dv}{dt}$

$$\frac{dv}{dt} = e^{-t}i - 18\cos 3t\, j - 18\sin 3t\, k$$

$\therefore$ velocity $= -e^{-t}i - 6\sin 3t\, j + 6\cos 3t\, k$

acceleration $= e^{-t}i - 18\cos 3t\, j - 18\sin 3t\, k$

3 d. (ii)

$$|v| = \sqrt{(-e^{-t})^2 + (-6\sin 3t)^2 + (6\cos 3t)^2}$$

$t = 0$

$$|v| = \sqrt{(-e^0)^2 + (6\sin 3(0))^2 + (6\cos 3(0))^2}$$

3 d. (ii)

$$|v| = \sqrt{37}$$

$$= 6.083 \text{ units}$$

Magnitude of acceleration

$$|a| = \sqrt{(e^{-t})^2 + (-18\cos 3t)^2 + (-18\sin 3t)^2}$$

at $t = 0$

$$|a| = \sqrt{(e^{-0})^2 + (-18\cos 3(0))^2 + (-18\sin 3(0))^2}$$

$$|a| = 18.028 \text{ units}$$

$\therefore$ Magnitude for velocity is 6.083 units, acceleration is 18.028 units

Extract 13.1 is a sample solution from one of the candidates who was able to answer the given question correctly. He/she applied the concept of vector and differentiation rules to find the required work done, acceleration and velocity.

13.0 Hyperbolic Function

2021 PAST PAPERS

2. (a) Express $4\cosh\theta + 5\sinh\theta$ in the form $R\sinh(\theta+\alpha)$ and hence find the values of R and $\tanh\alpha$.

(b) Show that $\cosh^{-1}x = \ln\left\{x+\sqrt{x^2-1}\right\}$.

(c) If $y = \dfrac{\cosh 2x}{1+\sinh 2x}$ then find $\dfrac{dy}{dx}$.

u^6 $6u$
$6(x^2-x+4)(2x-1)$
$6(25-5+4)(10-1)$

2 (a) soln

Given $4\cosh\theta + 5\sinh\theta = R\sinh(\theta+\alpha)$

$4\cosh\theta + 5\sinh\theta = R\sinh\alpha\cosh\theta + R\sinh\theta\cosh\alpha$

by comparison

$R\sinh\alpha = 4$ — (i)

$R\cosh\alpha = 5$. — (ii)

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 25-16$

$R^2 = 9$

$R = 3$.

divide eqn (i) and (ii)

$\tanh\alpha = \frac{4}{5}$.

$\alpha = \tanh^{-1}\left(\frac{4}{5}\right)$.

$\alpha = 1.0986$. (Approx)

$\therefore 4\cosh\theta + 5\sinh\theta = 3\sinh\left[\theta + \tanh^{-1}\left(\frac{4}{5}\right)\right]$

$R = 3$

$\tanh\alpha = \frac{4}{5}$

2 (b) soln'

To show

$$\cosh^{-1}x = \ln(x + \sqrt{x^2-1}).$$

let $y = \cosh^{-1}x$

$\cosh y = x$

by definition'

$$\cosh y = \frac{e^y + e^{-y}}{2}$$

$$e^y + e^{-y} = 2x$$

$$e^{2y} - 2xe^y + 1 = 0$$

on solving

$$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

neglect -ve value

$$e^y = x \pm \sqrt{x^2-1}$$

$$\ln e^y = \ln(x + \sqrt{x^2-1})$$

$$y = \ln(x + \sqrt{x^2-1})$$

$\cosh^{-1}x = \ln(x + \sqrt{x^2-1})$ Hence shown'

2 (c) soln

Given

$$y = \frac{\cosh 2x}{1 + \sinh 2x}$$

by quotient rule

$$\frac{d(v/u)}{dx} = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{u^2}.$$

$$\frac{dy}{dx} = \frac{(1+\sinh 2x)(2\sinh 2x) - \cosh 2x(2\cosh 2x)}{(1+\sinh 2x)^2}$$

$$\frac{dy}{dx} = \frac{2\sinh 2x - 2(-\sinh^2 2x + \cosh^2 2x).}{(1+\sinh 2x)^2}$$

$$\frac{dy}{dx} = \frac{2\sinh 2x - 2(\cosh^2 2x - \sinh^2 2x)}{(1+\sinh 2x)^2}$$

$$\frac{dy}{dx} = \frac{2(\sinh 2x - 1)}{(1+\sinh 2x)^2}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2(\sinh 2x - 1)}{(1+\sinh 2x)^2}$$

2020 PAST PAPERS

2. (a) Show that $(\cosh A - \cosh B)^2 - (\sinh A - \sinh B)^2 = -4\sinh^2\left(\frac{A-B}{2}\right)$.

(b) Use the second derivative test to identify the nature of the stationary point of the function $f(t) = \cos 2t - 4\sinh t$.

2.(a) Considering L.H.S

$(\cosh A - \cosh B)^2 - (\sinh A - \sinh B)^2$

Using factor Formula

$$\Rightarrow \left[-2\sinh\frac{A+B}{2}\sinh\frac{A-B}{2}\right]^2 - \left[2\sinh\frac{A-B}{2}\cosh\frac{A+B}{2}\right]^2$$

$$= 4\left[\sinh^2\frac{A+B}{2}\sinh^2\frac{A-B}{2}\right] - 4\left[\sinh^2\frac{A-B}{2}\cosh^2\frac{A+B}{2}\right]$$

$$= 4\left[\sinh^2\frac{A+B}{2}\sinh^2\frac{A-B}{2} - \sinh^2\frac{A-B}{2}\cosh^2\frac{A+B}{2}\right]$$

$$= 4\left[\sinh^2\frac{A-B}{2}\left(\sinh^2\frac{A+B}{2} - \cosh^2\frac{A+B}{2}\right)\right]$$

but $\cosh^2 A - \sinh^2 A = 1$

$\sinh^2 A - \cosh^2 A = -1$

$$= 4\left[\sinh^2\frac{A-B}{2}(-1)\right]$$

$$= -4\sinh^2\frac{A-B}{2}$$

$$\therefore (\cosh A - \cosh B)^2 - (\sinh A - \sinh A)^2 = -4\sinh^2\frac{A-B}{2}$$

hence shown.

2019 PAST PAPERS

2. (a) Solve the equation $\operatorname{cosech}^{-1}(x) + \ln x - \ln 3 = 0$.

(b) Given that $\sinh x = \tan\theta$, prove that $x = \ln(\sec\theta + \tan\theta)$.

(c) Use the hyperbolic functions substitution to find $\int \frac{1}{\sqrt{(x^2+8x+25)}}\,dx$.

2 a) $= \ln\left[\left(\frac{1+\sqrt{1+x^2}}{x}\right)(x) \div 3\right] = 0$

$= \ln\left(\frac{1+\sqrt{1+x^2}}{3}\right) = 0$

$= e^{\ln\left(\frac{1+\sqrt{1+x^2}}{3}\right)} = e^0$

$= \frac{1+\sqrt{1+x^2}}{3} = 1$

$= 1+\sqrt{1+x^2} = 3$

$= \sqrt{1+x^2} = 3-1$

$\sqrt{1+x^2} = 2$

Squaring both sides

$1+x^2 = 4$

$x^2 = 4-1$

$x^2 = 3$

$x = \sqrt{3}$

$\therefore \underline{X = \sqrt{3}}$.

b) $\sinh x = \tan\theta$

Required to prove $x = \ln(\sec\theta + \tan\theta)$.

soln

$\sinh x = \tan\theta$.

$\left(\frac{e^x - e^{-x}}{2}\right) = \tan\theta$.

$e^x - e^{-x} = 2\tan\theta$.

$e^{2x} - 1 = 2e^x \tan\theta$.

$e^{2x} - 2\tan\theta\, e^x - 1 = 0$

By quadratic equation

$e^x = \frac{2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$

$= \frac{2\tan\theta \pm 2\sqrt{\tan^2 + 1}}{2}$

2 b) $e^x = \tan\theta \pm \sqrt{\tan^2\theta + 1}$

But from,

$\tan^2\theta + 1 = \sec^2\theta$

$\sec\theta = \sqrt{\tan^2\theta + 1}$

$\Rightarrow e^x = \tan\theta \pm \sec\theta$.

$e^x = \tan\theta + \sec\theta$

Applying ln both sides

$\ln e^x = \ln[\tan\theta + \sec\theta]$.

$x = \ln[\sec\theta + \tan\theta]$,

$\therefore$ $\underline{X = \ln[\sec\theta + \tan\theta]}$. Hence Proved.

c) Given $\int \frac{1}{\sqrt{x^2+8x+25}} dx$

$= \int \frac{1}{\sqrt{(x+4)^2 - 16 + 25}} dx$

$= \int \frac{1}{\sqrt{(x+4)^2 + 9}} dx$

$= \int \frac{1}{\sqrt{9 + (x+4)^2}} dx$

Let

$(x+4) = 3\sinh\theta$.

$dx = 3\cosh\theta \, d\theta$.

$\theta = \sinh^{-1}\left(\frac{x+4}{3}\right)$

Now,

$\int \frac{3\cosh\theta \, d\theta}{\sqrt{9 + 9\sinh^2\theta}}$

$= \int \frac{3\cosh\theta \, d\theta}{3\sqrt{1+\sinh^2\theta}}$.

But $\sqrt{1+\sinh^2\theta} = \cosh\theta$.

2 c) $\Rightarrow \int \frac{\cosh\theta \, d\theta}{\cosh\theta}$

$= \int d\theta$

$= \theta + c$

But $\theta = \sinh^{-1}\left(\frac{x+4}{3}\right)$.

$\therefore \int \frac{1}{\sqrt{x^2+8x+25}} dx = \sinh^{-1}\left(\frac{x+4}{3}\right) + c$

2018 PAST PAPERS

2. (a) Differentiate $\cosh^6 x$ with respect to x.

(b) Solve for x in the equation $3\cosh x + \sinh x = \frac{9}{2}$.

(c) Prove whether or not $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ for values of x.

2a. To find $f'(x)$ of $\cosh^6 x$.

$\cosh^6 x = (\cosh x)^6$.

By chain rule.

let $\cosh x = u$.

$\sinh x = \frac{dy}{dx}$.

$u^6 = y$.

$\frac{dy}{du} = 6u^5$

By chain rule.

$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$.

$\frac{dy}{dx} = 6u^5 \times \cancel{\sinh}$

$\frac{dy}{dx} = 6u^5 \sinh x$.

but $u = \cosh x$.

$\frac{dy}{dx} = 6(\cosh x)^5 \sinh x$.

$\frac{dy}{dx} = 6\sinh x \cosh^5 x$.

$\therefore f'(x) = 6\sinh x \cosh^5 x$.

2b. $3\cosh x + \sinh x = 9/2$.

$6\cosh x + 2\sinh x = 9$.

$6\left(\frac{e^x + e^{-x}}{2}\right) + 2\left(\frac{e^x - e^{-x}}{2}\right) = 9$.

$3(e^x + e^{-x}) + (e^x - e^{-x}) = 9$.

$3e^x + 3e^{-x} + e^x - e^{-x} = 9$.

$4e^x + 2e^{-x} = 9$

2b.

$$4e^{x}+2e^{-x}=9$$

$$4e^{x}+\frac{2}{e^{x}}=9$$

$$4e^{2x}+2=9e^{x}$$

$$4e^{2x}-9e^{x}+2=0$$

$$e^{x}=\frac{-b\pm\sqrt{b^{2}-4ac}}{2a}$$

$$e^{x}=\frac{9\pm\sqrt{81-4(4\times2)}}{2\times4}$$

$$e^{x}=\frac{9\pm\sqrt{81-32}}{8}$$

$$e^{x}=\frac{9\pm\sqrt{49}}{8}$$

$$e^{x}=\frac{9\pm7}{8}$$

$$e^{x}=\frac{16}{8} \text{ or } e^{x}=2/8$$

$$e^{x}=2 \text{ or } e^{x}=1/4$$

apply ln

$$x=\ln 2 \text{ or } x=\ln(1/4)$$

$$\therefore x=0.6931 \text{ or } x=-1.3863$$

2c. Required to prove for $\sinh^{-1}x=\ln(x+\sqrt{x^{2}+1})$

let $\sinh^{-1}x=y$.

$$x=\sinh y$$

$$x=\frac{e^{y}-e^{-y}}{2}$$

$$e^{2y}-2xe^{y}-1=0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$e^y = \frac{\cancel{2}x \pm \cancel{2}\sqrt{x^2+1}}{\cancel{2}}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Apply ln.

$$y = \ln\left(x \pm \sqrt{x^2+1}\right).$$

but $y = \sinh^{-1} x$.

$$\sinh^{-1} x = \ln\left(x \pm \sqrt{x^2+1}\right).$$

but their is no ln of (-ve) number then.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$ hence proved

2017 PAST PAPERS

2. (a) If $x = \ln\left\{\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}$, find e^x and e^{-x} and hence show that $\sinh x = \tan\theta$.

(b) If $a\cosh x + b\sinh x = c$, show that the value of $x = \ln\left(\frac{c \pm \sqrt{c^2 + b^2 - a^2}}{a+b}\right)$.

(c) Use the appropriate hyperbolic substitution to evaluate $\int_{0.1}^{0.8} \sqrt{x^2 + 4x + 3}\, dx$.

---

2. (b) Given:

$$a\cosh x + b\sinh x = c$$

Then:

$$\frac{a}{2}\left(e^x + e^{-x}\right) + \frac{b}{2}\left(e^x - e^{-x}\right) = c$$

$$ae^x + ae^{-x} + be^x - be^{-x} = 2c$$

$$(a+b)e^x + (a-b)e^{-x} = 2c$$

$$(a+b)e^{2x} + (a-b) = 2ce^x$$

$$(a+b)e^{2x} - 2ce^x + (a-b) = 0$$

Quadratic in e^x

$$e^x = \frac{2c \pm \sqrt{4c^2 - 4(a+b)(a-b)}}{2(a+b)}$$

$$e^x = \frac{2c \pm \sqrt{4c^2 - 4(a^2 - b^2)}}{2(a+b)}$$

$$e^x = \frac{2c \pm 2\sqrt{c^2 - a^2 + b^2}}{2(a+b)}$$

$$e^x = \frac{c \pm \sqrt{c^2 + b^2 - a^2}}{a+b}$$

$$\therefore \quad x = \ln\left(\frac{c \pm \sqrt{c^2 + b^2 - a^2}}{a+b}\right)$$ Hence shown.

Extract 2.1 shows that candidate was able to define the hyperbolic function correctly and expressed x as required.

2016 PAST PAPERS

2. (a) If $t = \tanh\frac{x}{2}$, express $\sinh x$ and $\cosh x$ in terms of t.

(b) Express $\sinh^{-1} x - \ln x$ in terms of natural logarithms; hence, find the limit as $x \to \infty$.

(c) Evaluate $\int_{\text{[illegible]}}^{\text{[illegible]}} \frac{1}{\sqrt{(4x^2 - 8x + 7)}}\,dx$ correct to four decimal places.

2 a) Given: $t = \tanh\frac{x}{2}$

Required: $\sinh x$ and $\cosh x$ in terms of t.

$$\sinh x = \sinh\left(\tfrac{x}{2} + \tfrac{x}{2}\right)$$
$$= 2\sinh\tfrac{x}{2}\cosh\tfrac{x}{2}$$
$$= \frac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{1}$$

Dividing by $\cosh^2\frac{x}{2}$ both to the numerator and denominator:

$$\sinh x = \frac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\cosh^2\frac{x}{2}}$$

2 a)

$$\sinh x = \frac{2\tanh\frac{x}{2}}{\operatorname{sech}^2\frac{x}{2}}$$

But $\operatorname{sech}^2\frac{x}{2} = 1 - \tanh^2\frac{x}{2}$

$$\sinh x = \frac{2\tanh\frac{x}{2}}{1 - \tanh^2\frac{x}{2}}$$
$$= \frac{2t}{1-t^2}$$

$$\therefore \sinh x = \frac{2t}{1-t^2}$$

$$\cosh x = \cosh\left(\tfrac{x}{2} + \tfrac{x}{2}\right)$$
$$= \cosh^2\tfrac{x}{2} + \sinh^2\tfrac{x}{2}$$
$$= \frac{\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}}{1}$$

Dividing by $\cosh^2\frac{x}{2}$ to both the numerator and denominator:

$$\cosh x = \frac{\dfrac{\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}}}{\dfrac{1}{\cosh^2\frac{x}{2}}}$$
$$= \frac{1 + \tanh^2\frac{x}{2}}{\operatorname{sech}^2\frac{x}{2}}$$

$$= \frac{1 + \tanh^2 x/2}{1 - \tanh^2 x/2}$$

$$= \frac{1+t^2}{1-t^2}$$

$$\therefore \cosh x = \frac{1+t^2}{1-t^2}$$

In Extract 2.1, the candidate was able to apply the hyperbolic function identities and double angle formulae to express $\cosh x$ and $\sinh x$ in terms of t.

2015 PAST PAPERS

2. (a) (i) Express $4\cosh\theta + 5\sinh\theta$ in the form $r\sinh(\theta+\alpha)$ giving the values of r and $\tanh\alpha$.

(ii) Prove that $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$.

(b) (i) Show that $\frac{1}{2}\sinh(2\ln x)\cosh(2\ln x) = \frac{1}{8x^4}(x^8 - 1)$.

(ii) Find the possible values of $\sinh x$ if $\begin{vmatrix} \cosh x & -\sinh x \\ \sinh x & \cosh x \end{vmatrix} = 2$. (Leave your answer in surd form)

(c) Sketch the graph of $y = \sinh^{-1} x$ and state its domain and range.

2 (a) $4\cosh\theta + 5\sinh\theta$ in form of $r\sinh(\theta+\alpha)$

Expansion.

$r[\sinh\theta\cosh\alpha + \cosh\theta\sinh\alpha]$.

Comparing

$r\sinh\theta\cosh\alpha + r\sinh\alpha\cosh\theta = 4\cosh\theta + 5\sinh\theta$

$R^2\sinh^2\alpha = 4^2$ ——— (i)

$R^2\cosh^2\alpha = 5^2$ ——— (ii)

$(\cosh^2\theta - \sinh^2\theta)r^2 = 5^2 - 4^2$

$r^2 = 5^2 - 4^2$

$r = \sqrt{25-16}$

$r = 3.$

$r = 3$

then from $r\sinh\alpha = 4$

$3\sinh\alpha = 4$

$\sinh\alpha = 4/3$

$\alpha = \sinh^{-1}(4/3)$

$\alpha = 1.0986$

$\therefore$ In form of $r\sinh(\theta+\alpha) = 3\sinh(\theta + 1.0986)$

$\cdot \tan\alpha = \left(\frac{4}{5}\right) = 0.8$

$\therefore r = 3$ and $\tanh\alpha = 0.8$

(ii)

$\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$

let $\cosh^{-1}x = y$

$\cosh y = x$

$\frac{e^y + e^{-y}}{2} = x$

$e^y + e^{-y} = 2x$

$e^{2y} + e^{-y+y} = 2xe^y$

$e^{2y} + 1 = 2xe^y$

$e^{2y} - 2xe^y + 1 = 0$

$e^y = \frac{2x \pm \sqrt{4(x)^2 - 4}}{2}$

$e^y = \frac{2x \pm 2\sqrt{x^2-1}}{2}$

$e^y = \frac{2x \mp 2\sqrt{x^2-1}}{2}$

$e^y = x \mp \sqrt{x^2-1}$

Introduction

$y \ln e = \ln(x \pm \sqrt{x^2 - 1})$

But $-$ve of logarithm log not exist then

$y = \ln(x + \sqrt{x^2 - 1})$

$y = \ln(x + \sqrt{x^2 - 1})$

proved

(b) (i) $\frac{1}{2} \sinh(2\ln x)\cosh(2\ln x) = \frac{1}{8x^4}(x^8 - 1)$

$\frac{1}{2}\sinh(\ln x^2)\cosh(\ln x^2)$.

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$

$\frac{1}{2}\left[\frac{e^{\ln x^2} - e^{-\ln x^2}}{2}\right]\left[\frac{e^{\ln x^2} + e^{-\ln x^2}}{2}\right]$

$\frac{1}{8}\left[e^{\ln x^2} - e^{-\ln x^2}\right]\left[e^{\ln x^2} + e^{-\ln x^2}\right]$

$\frac{1}{8}\left[x^2 - \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2}\right)$

$\frac{1}{8}\left[x^4 + 1 - 1 + (-\frac{1}{x^4})\right]$

$\frac{1}{8}\left[x^4 + (-\frac{1}{x^4})\right]$ factor out $\frac{1}{x^4}$.

$\frac{1}{8}\left[x^8 - 1\right]\frac{1}{x^4}$

$\frac{1}{8x^4}\left[x^8 - 1\right]$

$= \frac{1}{8x^4}\left[x^8 - 1\right]$

proved

(ii) $\begin{vmatrix} \cosh x & -\sinh x \\ \sinh x & \cosh x \end{vmatrix}$

$\cosh^2 x + \sinh^2 x = 2.$

but

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x = 1 + \sinh^2 x.$

$1 + \sinh^2 x + \sinh^2 x = 2$

$2\sinh^2 x = 2 - 1$

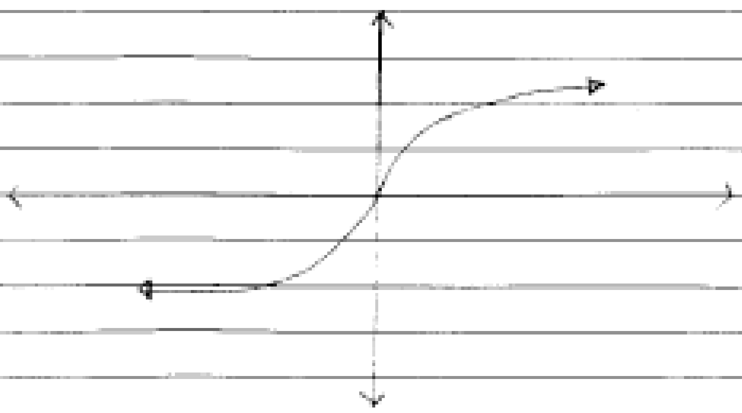
$2\sinh^2 x = 1$

$\sqrt{\sinh^2 x} = \sqrt{\tfrac{1}{2}}$

$\sinh x = \pm\sqrt{\tfrac{1}{2}}$

$\sinh x = \sqrt{\tfrac{1}{2}}$

(c) A graph of $y = \sinh^{-1} x$.



Domain : {All real numbers}

Range : {All real numbers}

Extract 2.1 shows the work of one of the best responses of the candidate who recalled and applied hyperbolic identities, performed computations and sketched the graph of hyperbolic inverse functions correctly. Moreover, he/she converted the inverse hyperbolic functions into logarithmic functions correctly.

14.0 Statistics

2021 PAST PAPERS

4. (a) If the standard deviation of the numbers $x_1, x_2, x_3, \ldots\ldots, x_n$ is 10, find the standard deviation of $2x_1+1, 2x_2+1, \ldots\ldots, 2x_n+1$.

(b) A classroom teacher measures the lengths of 50 students to the nearest centimeter. The results are summarized in the following table:

Length (cm)	31 – 35	36 – 40	41 – 45	46 – 50	51 - 55	56 – 60
Frequency (f)	3	6	17	10	9	5

(i) Calculate the first and third quartiles correct to two decimal places,

(ii) Calculate the 70^{th} percentile correct to one decimal place.

4(a)

$$\sigma_{new}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(2x_i+1 + -\left(\frac{2}{n}(x_1+x_2+x_3 \ldots + x_n)+1\right)\right)^2$$

$$\sigma_{new}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(2x_i+1-2\bar{x}-1\right)^2$$

$$\sigma_{new}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(2x_i-2\bar{x}\right)^2$$

$$\sigma_{new}^2 = \frac{1}{n}2^2\left(\sum_{i=1}^{n}(x_i-\bar{x})\right)$$

but

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}) = 100.$$

$$\delta_{new}^2 = 2^2 \times 100$$

$$\delta_{new} = \sqrt{4 \times 100}$$

$$\delta_{new} = 20$$

$\therefore$ The standard deviation of $x_1, x_2, x_3 \dots x_n$ is 10 and $2x_1 + 1, 2x_2 + 1, 2x_n + 1$ is 20

4(b) FREQUENCY DISTRIBUTION TABLE

Length	f	Cum.f	Real limits	x
31 – 35	3	3	30.5 – 35.5	33
36 – 40	6	9	35.5 – 40.5	38
41 – 45	17	26	40.5 – 45.5	43
46 – 50	10	36	45.5 – 50.5	48
51 – 55	9	45	50.5 – 55.5	53
56 – 60	5	50	55.5 – 60.5	58
	N=50			

4(b) (ii)

Position of 70^{th} percentile

$$= \left(\frac{nN}{100}\right)^{th}$$

$$= \left(\frac{70 \times 50}{100}\right)^{th}$$

$$= 35$$

Percentile class $\Rightarrow$ 46 – 50, $L = 45.5$

$$P_n = L_n + \left(\frac{\frac{nN}{100} - fb}{fw}\right)c$$

$fb = 26$ $fw = 10$

$$P_{70^{th}} = 45.5 + \left(\frac{35 - 26}{10}\right) \times 5$$

$P_{70^{th}} = 50.0$ correct to one decimal place

2020 PAST PAPERS

4. (a) Show that $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$.

(b) The following table shows the masses in gram of a sample of potatoes:

Mass (g)	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89	90 - 99
Frequency	2	14	21	73	42	13	9	4	2

(i) Using the coding method and the assumed mean $A = 54.5$, find the arithmetic mean.

(ii) Use the mean obtained in (b) (i) to find the variance and standard deviation correct to 2 decimal places.

(iii) Compute the 80 percentile correctly to three decimal places.

4. (a) Given:

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x^2 - n\bar{x}^2$$

From L·H·S:

$$\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Expand $(x_i - \bar{x})^2$

$$= ((x_i) - \bar{x})(x_i - \bar{x})$$

$$x_i^2 - 2\bar{x}x_i + \bar{x}^2$$

$$\sum_{i=1}^{n}(x_i^2 - 2\bar{x}x_i + \bar{x}^2)$$

$$\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n}\bar{x}x_i + \sum_{i=1}^{n}\bar{x}^2$$

but Mean $= \sum_{i=1}^{n} x_i = \bar{x}$

$$\sum_{i=1}^{n} x_i^2 - 2\left(\sum_{i=1}^{n}\bar{x}\right)\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\bar{x}^2$$

$$\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n}\bar{x}\cdot\bar{x} + \sum_{i=1}^{n}\bar{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 + \left(-2\sum_{i=1}^{n}\bar{x}^2 + \sum_{i=1}^{n}\bar{x}^2\right),$$

$$= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n}\bar{x}^2 \quad \text{let} = \text{since } \sum_{i=1}^{n} = n:$$

$$= \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$

Hence shown since L·H·S = R·H·S:

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2.$$

4. (b) Given: $A = 54.5$ $c = 10$:

Mass	X	f	CM.	$d = X - A$	$u = d/c$	u^2	fu	fu^2
10 - 19	14.5	2	2	-40	-4	16	-8	32.
20 - 29	24.5	14	16	-30	-3	9.	-42	126
30 - 39	34.5	21	37	-20	-2	4	-42	84
40 - 49	44.5	73	110	-10	-1	1.	-73	73
50 - 59	54.5	42	152	0	0	0	0	0
60 - 69	64.5	13.	165	10	1	1	13	13
70 - 79	74.5	9	174	20	2	4	18	36
80 - 89	84.5	4	178	30	3	9.	12.	36
90 - 99.	94.5	2	180	40	4.	16.	8	32

$N = 180$ $\quad \Sigma fu = -114.$ $\quad \Sigma fu^2 = 432.$

(i) Arithmetic Mean:

$$\bar{X} = A + c\frac{\Sigma fu}{N}$$

$$\bar{X} = 54.5 + 10\left(\frac{-114.}{180}\right)$$

$$\bar{X} = 48.1666667.$$

(ii) $$\delta^2 = c^2\left(\frac{\Sigma fu^2}{N} - \left(\frac{\Sigma fu}{N}\right)^2\right).$$

$$\text{Variance} = 10^2\left(\frac{432}{180} - \left(\frac{-114}{180}\right)^2\right).$$

Variance = 199.88889 into 2 decimal place:

Variance = 199.89.

Standard deviation:

$$\delta = c\sqrt{\left(\frac{\Sigma fu^2}{N}\right) - \left(\frac{\Sigma fu}{N}\right)^2}$$

$$\delta = 10\sqrt{\left(\frac{4B2}{180}\right) - \left(\frac{-114}{180}\right)^2}$$

$$= 10\sqrt{1.9988889}$$

$$= 14.1382$$

Into 2 decimals

$\delta = 14.14$

Standard deviation $= 14.14$

(iii) 80 percentile

$$\frac{80}{100}\Sigma fx$$

$$= \frac{80}{100} \times 180$$

$$= 144$$

From: $= L + \left(\frac{\frac{x}{100}\Sigma fx - \Sigma fb}{fw}\right)c$

$$= 49.5 + \left(\frac{144 - 110}{42}\right)10$$

$$= 49.5 + 8.09524$$

$$= 57.595$$

$\therefore$ 80 percentile $= 57.595$

2019 PAST PAPERS

4. (a) The sum of 20 numbers is 320 and the sum of the squares of these numbers is 5840.
   (i) Calculate the mean and standard deviation of 20 numbers.
   (ii) If one number is added to the 20 numbers so that the mean is unchanged, find this number and show whether the standard deviation will change or not.

(b) A watchman at Mlimani city shopping centre recorded the length of time to the nearest minute that a sample of 131 cars was parked in their car park. The results were as follow:

Time (minutes)	5 – 10	11 – 16	17 – 22	23 – 28	29 – 34	35 - 40
Frequency	15	28	37	26	18	7

(i) Calculate the median time correct to four significant figures.
(ii) By using the coding method and the assumed mean $A = 19.5$, calculate the mean in two decimal places.

4.a) Solution

Sum of 20 number $= 320$, So $n = 20$,

$\Sigma x = 320$, $\Sigma x^2 = 5840$

(i) Mean,

$\bar{x} = \frac{\Sigma x}{N} = \frac{320}{20} = 16$

Standard deviation,

Using the Formula, $\delta = \sqrt{\frac{\Sigma x^2}{N} - \left(\frac{\Sigma x}{N}\right)^2}$

$\delta = \sqrt{\frac{5840}{20} - \left(\frac{320}{20}\right)^2}$

$\delta = 6$

$\therefore$ Mean is 16 and Standard deviation is 6

(ii) Let y be the number added to 20 numbers

then, $N = 21$, $\Sigma x = 320 + y$

Since the mean is unchanged, the $\bar{x} = 16$

So, From Mean $\bar{x} = \frac{\sum x}{N}$

$$16 = \frac{320 + y}{21}$$

$$320 + y = 336$$

$$y = 336 - 320 = 16$$

The number that is added is 16

+ Standard deviation is calculate using the relation,

$$\delta = \sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2}$$

Here, $\sum x^2 = 5840 + 16^2 = 6096$

$\sum x = 336$, $N = 21$

then $\delta = \sqrt{\frac{6096}{21} - \left(\frac{336}{21}\right)^2}$

4a) (ii) Gives, $\delta = \sqrt{34.285}$

$$\delta = 5.8554$$

$\therefore$ The standard deviation will change from 6 to 5.8554

(b) Solution

The Frequency Distribution table.

Time (min)	f	X	Cf	$y = \frac{x - 19.5}{6}$	fy
5 – 10	15	7.5	15	−2	−30
11 – 16	28	13.5	43	−1	−28
17 – 22	37	19.5	80	0	0
23 – 28	26	25.5	106	1	26
29 – 34	18	31.5	124	2	36
35 – 40	7	37.5	131	3	21

then, $N = 131$ $\sum fy = 25$

(i) Median,

Median is given by, Median $= L + \left(\frac{N/2 - nb}{nw}\right) i$

Here, L = Is obtained from class interval 17 – 22

then, $\frac{N}{2} = 65.5$, $L = 16.5$, $nb = 43$, $nw = 37$

and i - interval $= 6$

then, Median $= 16.5 + \left(\frac{65.5 - 43}{37}\right) \times 6$

$$= 20.14864865 \simeq 2.015 \times 10^1$$

$\therefore$ Median is 20.15 to four significant figure

(ii) Using Coding method, $A = 19.5$

Mean by Coding method is $\bar{x} = A + C\left(\frac{\sum fy}{N}\right)$

Here $A = 19.5$, $C = 6$, $\sum fy = 25$, $N = 131$

then

4b) (ii) $\bar{x} = 19.5 + 6 \times \left(\frac{25}{131}\right)$

$$\bar{x} = 20.64503817 \simeq 20.65$$

$\therefore$ Mean $\bar{x} = 20.65$ to two decimal places

2018 PAST PAPERS

4. The following frequency distribution table represents a certain class of 100 students:

Class	1 - 10	11 - 20	21 - 30	31 - 40	41 - 50	51 - 60	61 - 70	71 - 80	81 - 90	91 - 100
Frequency	$t - 2$	1	20	$t + 2$	t	$t + 3$	23	11	$t + 4$	13

(a) Determine the value of t.

(b) Find the following measures of central tendency and dispersion correct to two decimal places:

(i) mean,

(ii) standard deviation,

(iii) mean deviation,

(iv) median.

4 a) Given $N = 100$

$100 = t - 2 + 1 + 20 + t + 2 + t + t + 3 +$
$23 + 11 + t + 4 + 13$

$100 = 5t + 75$

$25 = 5t$

$\frac{25}{5} = \frac{5t}{5}$

$t = 5$

4 b)

class	f	Cum F	x	fx	$f\lvert x-\bar{x}\rvert$	x^2	fx^2
1 - 10	3	3	5.5	16.5	157.797	30.25	90.75
11 - 20	1	4	15.5	15.5	42.599	240.25	240.25
21 - 30	20	24	25.5	510	651.98	650.25	13,005
31 - 40	~~8~~7	31	35.5	248.5	158.193	1260.25	8821.75
41 - 50	5	36	45.5	227.5	62.995	2070.25	10351.25
51 - 60	8	44	55.5	444	20.792	3080.25	24,642
61 - 70	23	67	65.5	1506.4	170.223	4290.25	98,675.75
71 - 80	11	78	75.5	830.5	191.411	5700.25	62,702.75
81 - 90	9	87	85.5	769.5	246.609	7310.25	65,792.25
91 - 100	13	100	95.5	1241.5	486.233	9120.25	118,563.25

$\Sigma fx = 5809.9$ $\Sigma fx^2 = 402{,}885$

i) Mean

$$\bar{x} = \frac{\Sigma fx}{N} = \frac{5809.9}{100}$$

$$\bar{x} = 58.099 \approx 58.10$$

$$\text{Mean } \bar{x} = 58.10$$

ii) Standard deviation

$$\delta^2 = \frac{\Sigma f(x-\bar{x})^2}{N}$$

$$\delta^2 = \frac{\Sigma fx^2}{N} - \bar{x}^2$$

$$\delta^2 = \frac{402885}{100} - 58.099^2$$

$$\delta^2 = 653.356$$

$$\delta^2 = \sqrt{653.356}$$

$$\delta = 25.56$$

∴ Standard deviation = 25.56

4(b) iii) Mean deviation

$$= \frac{\sum f|x - \bar{x}|}{N}$$

$$= \frac{2188.812}{100}$$

$$= 21.88812 \simeq 21.89$$

$\therefore$ Mean deviation $= 21.89$

4b (iv) Median.

$$\text{Median} = L + \left(\frac{N/2 - Fb}{Fm}\right) i$$

L = Lower real limit of Median class
Fb = Summation of frequency below Median frequency
Fm = frequency at Median class
i = Class Interval

$i = 10$, $\frac{N}{2} = \frac{100}{2} = 50$

Class = 61 – 70 , $Fb = 44$
$L = 60.5$, $Fm = 23$

$$\therefore \text{Median} = 60.5 + \left(\frac{50 - 44}{23}\right) 10$$

$$= 63.10869 \simeq 63.11$$

$\therefore$ Median $= 63.11$

4(b) iii) Mean deviation

$$= \frac{\sum f|x - \bar{x}|}{N}$$

$$= \frac{2188.812}{100}$$

$$= 21.88812 \approx 21.89$$

$\therefore$ Mean deviation $= 21.89$

4b (iv) Median.

$$\text{Median} = l + \left(\frac{N/2 - Fb}{Fm}\right) i$$

l = Lower real limit of Median class
Fb = Summation of frequency below Median frequency
Fm = Frequency at Median class
i = Class Interval

$i = 10$, $\frac{N}{2} = \frac{100}{2} = 50$

Class = 61 - 70 , Fb = 44
l = 60.5 , Fm = 23

$$\therefore \text{Median} = 60.5 + \left(\frac{50 - 44}{23}\right) 10$$

$$= 63.10869 \approx 63.11$$

$\therefore$ Median = 63.11

2017 PAST PAPERS

4. The following table shows distribution of marks in a matriculation examination of communication skills:

Marks	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	8	12	18	25	40	28	31	30	8

(i) Given that the assumed mean is 75.5, use the coding method to find the average marks.

(ii) Determine the lower quartile of the distribution.

(iii) Calculate the 75th percentile correct to four significant figures.

4. Frequency distribution table

Marks	f	$u=\left(\frac{x-75.5}{10}\right)$	fu	x
11 − 20	8	−6	−48	15.5
21 − 30	12	−5	−60	25.5
31 − 40	18	−4	−72	35.5
41 − 50	25	−3	−75	45.5
51 − 60	40	−2	−80	55.5
61 − 70	28	−1	−28	65.5
71 − 80	31	0	0	75.5
81 − 90	30	1	30	85.5
91 − 100	8	2	16	95.5
	$\Sigma f = 200$		$\Sigma fu = -317$	

4. (i) Average Marks, $\bar{x} = A + \frac{c\,\Sigma fu}{\Sigma f}$.

$A = 75.5$, $c = 10$, $\Sigma fu = -317$, $\Sigma f = 200$

$\bar{x} = 75.5 - \left(\frac{10 \times 317}{200}\right)$

$= 59.65$

$\therefore$ Average Marks $= 59.65$

4. (ii) Lower Quartile, $Q_1 = L_{Q_1} + \left(\frac{N/4 - f_{b_{Q_1}}}{f_{Q_1}}\right)c$

$N/4 = \frac{200}{4} = 50$

Class of Lower Quartile $= 41 - 50$.

$$L_{Q_1} = 40.5 \quad fb_{Q_1} = 38, \; f_{Q_1} = 25, \; c = 10$$

$$Q_1 = 40.5 + \left(\frac{50 - 38}{25}\right) 10$$

$$= 45.3$$

$\therefore$ Lower Quartile $= 45.3$

(iii) $$75^{th} \text{ Percentile} = L_{P_{75}} + \left(\frac{75/100 N - fb_{75}}{f_{P_{75}}}\right) \times C$$

$$75/100 N = 75/100 \times 200 = 150$$

Class containing $P_{75} = 71 - 80$

$$L_{P_{75}} = 70.5, \; fb_{P_{75}} = 131, \; f_{P_{75}} = 31, \; c = 10$$

$$P_{75} = 70.5 + \left(\frac{150 - 131}{31}\right) 10$$

$$= 76.62903226$$

$$\approx 76.63$$

75^{th} Percentile $= 76.63$ (4 sign. figures).

Extract 4.1, shows that the candidate was systematic in applying the required formulae.

2016 PAST PAPERS

4. (a) The frequency distribution of a variable X is classified in equal intervals of size C. The frequency in a class is denoted by f and the total frequencies is N. If the data is coded into a variable u by means of the relation $\bar{x} = a + C\bar{u}$, where X takes the central values of the class intervals, show that the standard deviation δ of the distribution is given by $\delta^2 = C^2\left(\frac{\sum fu^2}{N} - \left(\frac{\sum fu}{N}\right)^2\right)$

(b) The average heights of 20 boys and 30 girls are 160 cm and 155 cm respectively. If the corresponding standard deviations of boys and girls are 4 cm and 3.5 cm, find the standard deviation of the whole group.

(c) The following table shows the length of 100 earth worms in millimetres:

Length(mm)	95 - 109	110 - 124	125 - 139	140 - 154	155 - 169	170 - 184	185 - 199	200 - 214
Number of worms	2	8	17	26	14	16	6	1

Obtain the semi-interquartile range correct to two significant figures.

04. a/. From

$$\sigma = \sqrt{\frac{\sum f(x-\bar{x})^2}{N}}$$

but $N = \sum f$

Thus

$$\sigma = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}$$

$$\sigma = \sqrt{\frac{\sum f(x^2 - 2x\bar{x} + \bar{x}^2)}{\sum f}}$$

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} + \frac{\sum f\bar{x}^2}{\sum f} - 2\bar{x}\frac{\sum fx}{\sum f}}$$

04. a/.

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} + \bar{x}^2 - 2\bar{x}^2}$$

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

but $\bar{x} = \left(\frac{\sum fx}{\sum f}\right)$

Thus

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

but $\bar{x} = a + c\bar{u}$

Thus $x = a + cu$.

Hence: $$\sigma = \sqrt{\frac{\sum f(a+cu)^2}{\sum f} - \left(\frac{\sum f(a+cu)}{\sum f}\right)^2}$$

$$\sigma = \sqrt{\frac{\sum f(a^2 + 2acu + c^2u^2)}{\sum f} - \left(\frac{\sum fa}{\sum f} + \frac{\sum fcu}{\sum f}\right)^2}$$

but a = constant and c = constant

Thus $\frac{\sum fa}{\sum f} = a$ and $\frac{\sum fcu}{\sum f} = c\frac{\sum fu}{\sum f}$

$$\sigma = \sqrt{a^2\frac{\sum f}{\sum f} + 2ac\frac{\sum fu}{\sum f} + c^2\frac{\sum fu^2}{\sum f} - \left(a + c\frac{\sum fu}{\sum f}\right)^2}$$

$$\sigma = \sqrt{a^2 + 2ac\frac{\sum fu}{\sum f} + c^2\frac{\sum fu^2}{\sum f} - a^2 - 2ac\frac{\sum fu}{\sum f} - c^2\left(\frac{\sum fu}{\sum f}\right)^2}$$

4. a/ Then

$$\sigma = \sqrt{c^2 \frac{\Sigma fu^2}{\Sigma f} - c^2\left(\frac{\Sigma fu}{\Sigma f}\right)^2}$$

but $\Sigma f = N$

Then

$$\sigma = c\sqrt{\frac{\Sigma fu^2}{N} - \left(\frac{\Sigma fu}{N}\right)^2}$$

$$\sigma^2 = c^2\left(\frac{\Sigma fu^2}{N} - \left(\frac{\Sigma fu}{N}\right)^2\right)$$

Hence shown.

Extract 4.2 shows that the candidate was able to derive the formula for standard deviation using the coding method.

2015 PAST PAPERS

4. Kamunonge cooperative farm with 20 branches each recorded one among the following sales of wheat last month: 6.1, 11.0, 22.3, 30.8, 37.5, 34.5, 29.4, 10.9, 13, 5.4, 3.2, 15.6, 22.6, 21.3, 20.5, 31.3, 47.9, 46.3, 41.4 and 48.2. Group the data into class intervals 0 – 10, 10 – 20, etc. and determine:

(a) (i) Mode of the data correct to 4 significant figures.

(ii) Median of the data.

(iii) The standard deviation correct to 4 significant figures.

(b) The lower and upper quartiles.

	Class interval	f	cf	x	f(x)	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
4.	0 – 10	4	4	5	20	−20.5	420.25	1681
	10 – 20	3	7	15	45	−10.5	110.25	330.75
	20 – 30	5	12	25	125	−0.5	0.25	1.25
	30 – 40	4	16	35	140	9.5	90.25	361
	40 – 50	4	20	45	180	19.5	380.25	1521
				$\Sigma fx: 510$			$\Sigma f(x-\bar{x})^2 = 3895$	

$$\bar{x} = \frac{\Sigma fx}{N}$$

$$\bar{x} = \frac{510}{20}$$

$$= 25.5$$

a)ii) Median $= L + \left(\frac{N/2 - nb}{nw}\right)c$

Class interval is 20–30.

$L = 20 \quad nb = 7 \quad nw = 5 \quad c = 10 \quad N = 20$

Median $= 20 + \left(\frac{20/2 - 7}{5}\right)10$

$= 20 + (10 - 7)2$

$= 20 + 6$

$= 26.$

i) Mode $= L + \left(\frac{t_1}{t_1 + t_2}\right)c$

Modal class (20–30)

$L = 20 \quad t_1 = 2 \quad t_2 = 1 \quad c = 10.$

Mode $= 20 + \left(\frac{2}{2+1}\right)10$

$= 20 + (2/3)\,10$

$= 20 + 6.667$

4a)ii) Mode $= 2.667 \times 10^1$

iii) Standard Deviation $= \sqrt{\frac{\Sigma F(x - \bar{x})^2}{N}}$

$= \sqrt{\frac{3895}{20}}$

$= \sqrt{194.75}$

$= 1.396 \times 10^1.$

b). lower quartiles (Q_1)

$Q_1 = L + \left(\frac{N/4 - nb}{nw}\right)c$

$Q_1 = \frac{N}{4} = \frac{20}{4} = 5$

Class interval (10–20)

$Q \neq L = 10 \quad c = 10 \quad nb = 4 \quad nw = 3 \quad N/4 = 5.$

$Q_1 = L + \left(\frac{N/4 - nb}{nw}\right)c$

$= 10 + \left(\frac{5 - 4}{3}\right) 10$

$= 10 + 3.33$

$= \underline{13.33}$

The upper quartile Q_3.

$Q_3 = L + \left(\frac{3/4 N - nb}{nw}\right) c$

$Q_3 = \frac{3N}{4} = \frac{3 \times 20}{4} = 3 \times 5 = 15.$

4. b) Class interval of $(30 - 40)$

$L = 30 \quad c = 10 \quad nb = 12 \quad nw = 4. \quad 3N/4 = 15$

$Q_3 = L + \left(\frac{3N/4 - nb}{nw}\right) c$

$= 30 + \left(\frac{15 - 12}{4}\right) 10$

$= 30 + \frac{3}{4} \times 10$

$= 30 + 7.5$

$= 37.5.$

Extract 4.1 shows the work of one of the candidates who performed well in question 4. He/she used the appropriate formulas.

15.0 Probability

2021 PAST PAPERS - 2

1. (a) If A and B are such that $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{4}$ and $P(A\cup B)=\frac{1}{2}$, calculate;
   (i) $P(A\cap B')$.
   (ii) $P(A/B')$.
   (b) Two dices are thrown simultaneously.
   (i) List the sample space for this event.
   (ii) Find the probability that the sum of the numbers obtained on the dice is neither a multiple of 2 nor a multiple of 3.
   (c) If X is binomially distributed, the probability that the event will happen exactly x times in n trials is given by the function $P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$. Establish the validity of the Poisson approximation to the binomial distribution.

1 (a) Soln

$P(A) = 1/3 \quad P(B) = 1/4 \quad P(A\cup B) = 1/2$

$P(A\cap B')$

from,

$P(A) = P(A\cap B) + P(A\cap B')$

$P(A\cap B') = P(A) - P(A\cap B)$ — (i)

from $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$1/2 = 1/3 + 1/4 - P(A\cap B)$

$\therefore P(A\cap B) = 1/12$

1 (a) Substituting in eqn (i)

$P(A\cap B') = 1/3 - 1/12$

$= 1/4$

Hence,

$P(A\cap B') = 1/4$

1 (a) (i) Soln

$P(A/B')$

$$P(A/B') = \frac{P(A \cap B')}{P(B')}$$

$P(A \cap B') = 1/4$

$P(B') = 1 - P(B)$

$P(B') = 1 - 1/4 = 3/4$

$$P(A/B') = \frac{1/4}{3/4} = 1/3$$

Hence

$P(A/B') = 1/3$

1 (b) Soln

(i) let sample space be S

from

Dice 1/2	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

Thus

$S = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (1,2),$
$(2,2), (3,2), (4,2), (5,2), (6,2), (1,3), (2,3),$
$(3,3), (4,3), (5,3), (6,3), (1,4), (2,4), (3,4),$
$(4,4), (5,4), (6,4), (1,5), (2,5), (3,5), (4,5),$
$(5,5), (6,5), (1,6), (2,6), (3,6), (4,6), (5,6),$
$(6,6)\}$

(ii) Soln

Let multiple of 2 be A

Multiple of 3 be B

~~P(Neither A nor B)~~ = P(A∪B)'

Solving for (A∪B)

A = {(1,1), (3,1), (5,1), (2,2), (4,2), (6,2), (1,3), (3,3), (5,3), (2,4), (4,4), (6,4), (1,5), (3,5), (5,5), (2,6), (4,6), (6,6)}

1(b) (ii)

B = {(2,1), (5,1), (1,2), (4,2), (3,3), (6,3), (2,4), (5,4), (1,5), (4,5), (3,6), (6,6)}

A∪B = {(1,1), (3,1), (5,1), (2,2), (4,2), (6,2), (1,3), (3,3), (5,3), (2,4), (4,4), (6,4), (1,5), (3,5), (5,5), (2,6), (4,6), (6,6), (2,1), ~~(5,1)~~, (1,2), ~~(4,2)~~, ~~(3,3)~~, (6,3), ~~(2,4)~~, (5,4), ~~(1,5)~~, (4,5), (3,6), ~~(6,6)~~}

$n(A \cup B) = 24$

$n(S) = 36$

$P(A \cup B) = \frac{n(A \cup B)}{n(S)}$

$= 24/36$

$P(A \cup B) = 2/3$

$P(A \cup B)' = 1 - P(A \cup B) \quad [1 - 2/3]$

$= 1 - 2/3 = 1/3$

$= 1/3$

Hence Probability $= 1/3$

1. (c) Soln

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(x) = {}^nC_x p^x (1-p)^{n-x}$$

$$P(x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

$$P(x) = \frac{n(n-1)(n-2)(n-3)(n-4)\cdots(n-x+1)(n-x)!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

$(\lambda = \text{Mean})$

also, $\lambda = np$ $\quad p = \lambda/n$

$$P(x) = \frac{n(n-1)(n-2)(n-3)\cdots(n-x+1)\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}}{x!}$$

$$P(x) = \frac{n\,n(1-1/n)\,n(1-2/n)\,n(1-3/n)\cdots n\left(1+\frac{1-x}{n}\right)\lambda^x \left(1-\frac{\lambda}{n}\right)^n}{x!\; n^x \left(1-\lambda/n\right)^x}$$

$$P(x) = \frac{n^x(1-1/n)(1-2/n)(1-3/n)\cdots\left(1+\frac{1-x}{n}\right)\lambda^x \left(1-\lambda/n\right)^n}{x!\, n^x \left(1-\lambda/n\right)^x}$$

as $n \longrightarrow \infty$

$$P(x) = \frac{(1-1/\infty)(1-2/\infty)(1-3/\infty)\cdots\left(1+\frac{1-x}{\infty}\right)\lambda^x \left(1-\lambda/n\right)^n}{x!\,(1-\lambda/\infty)^x}$$

then $\left(1/\infty \equiv 0\right)$

1(c) $$P(x) = \frac{1(1)(1)(1)\cdots(1)\,\lambda^x \left(1+\frac{-\lambda}{n}\right)^n}{(1)\,x!} \quad \text{— (i)}$$

but

from,

$$\left(1+\frac{c}{x}\right)^x$$

as $x \longrightarrow \infty$

$$\left(1+\frac{r}{x}\right)^x \equiv e^r$$
$x \to \infty$

So,
$$\left(1+\frac{-\lambda}{n}\right)^n \equiv e^{-\lambda}$$
as $n \to \infty$ on substituting to eqn (2)

thus

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{where } [\lambda = \text{MEAN}]$$

Hence,

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad (\text{shown})$$

(Validity of Poisson approximation to binomial distribution)

2020 PAST PAPERS - 2

1. (a) Eggs are packed in boxes of 500. On the average 0.7% of the eggs are found to be broken. Find the probability that in a box of 500 eggs;
   (i) exactly 3 eggs are broken.
   (ii) at least two eggs are broken.
   (Write your answers in four significant figures)

   (b) As an experiment, a temporary roundabout is constructed at the crossroads. The time, X in minutes, which vehicles have to wait before entering the roundabout is a random variable having the following probability density function:
   $$f(x)=\begin{cases}0.8-0.32x, 0\le x\le 2.5\\ 0, \text{otherwise}\end{cases}$$
   Find the mean waiting time for vehicles and standard deviation for the distribution.

   (c) The mean weight of 600 male villagers in a certain village is 79.7 kg and the standard deviation is 6 kg. Assuming that the weights are normally distributed, find how many villagers weigh more than 90 kg.

   (d) How many possible combinations of six questions are there in an examination paper consisting of a total of eight questions?

1 (c) Soln.

Given, mean (μ) = ~~600~~ 79.7kg

Standard Deviation (σ) = 6kg.

Required, Number of weight more than 90kg

i.e. $P(X > 90) = ?$

From,

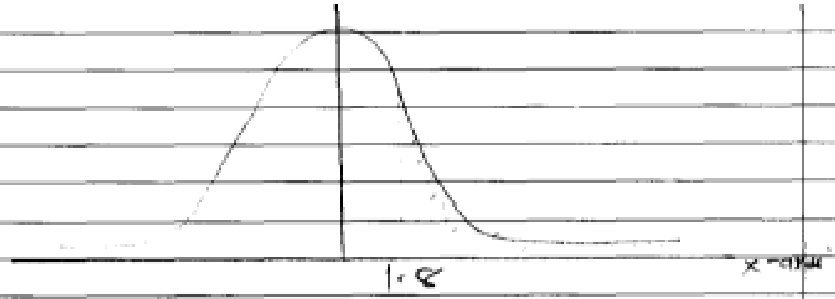
$$z = \frac{x - \mu}{\sigma}$$

For continuity.

$$P(X > 90.5) = P\left(z > \frac{90.5 - 79.7}{6}\right)$$

$$P(X > 90.5) = P(z > 1.8)$$

Consider y-axis.



$P(Z > 1.8) = P(1.8)$

$= 0.03592$

I (c) Number of villages.

$= 0.03593 \times 600$

$=$ 22 villages

∴ Number of villages weigh more than 90kg = 22 villages

I (d) Soln

Given,

$n = 8$

$r = 6$

From $^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$

$^{8}C_{6} = \frac{8!}{(8-6)!\,6!}$

$^{8}C_{6} = \frac{8!}{2!\cdot 6!}$

$= \frac{8 \times 7 \times 6!}{2 \times 6!}$

$= 56/2 = 28$

∴ There are 28 combination of six questions.

2019 PAST PAPERS - 2

5. (a) (i) Show that ${}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}$.

(ii) A machine produces a total of 10,000 nails a day which on average 5% are defective. Find the probability that out of 500 nails chosen at random 10 will be defective.

(b) (i) Find the probability that in four tosses of a fair die a 2 appears at most once.

(ii) The mean weight of 400 female pupils at a certain school is 65 kg and the standard deviation is 5 kg. Assuming that the weights are normally distributed, find how many pupils weigh between 50 and 67 kg.

(c) A random variable X has the probability density function

$$f(x)=\begin{cases} px, & \text{for } 0 \le x < 2 \\ p(4-x), & \text{for } 2 \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find the value of the constant p.

(ii) Sketch the graph of $f(x)$.

(iii) Evaluate $P\left(\frac{1}{2} \le X \le \frac{5}{2}\right)$.

There is no answer for this question

2018 PAST PAPERS - 2

6. (a) If two independent events are A and B such that $P(A) = 2$ and $P(B) = 0.4$, determine

(i) $P(\text{not A and B})$.

(ii) $P(\text{A or B})$.

(b) Kalihose's family consists of mother, father and their ten children. The family is invited to send a group of 4 representatives to a wedding. Find the number of ways in which the group can be formed, if it must contain;

(i) Both parents.

(ii) One parent only.

(iii) None of the parents.

(c) The density function of a continuous random variable X is given by

$$f(x)=\begin{cases} kx & 0 \le x \le 1 \\ k(2-x) & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find

(i) The value of the constant k,

(ii) $E(X)$,

(iii) $P\left(\frac{1}{2} \le x \le 1\frac{1}{2}\right)$.

(d) If X follows binomial distribution with mean 4 and variance 2, find $P(|x-4| \le 2)$ and write your answer in four significant figures.

6a Given $P(A) = 2$ and $P(B) = 0 \cdot 4$

(i) If two events are independent, means, $P(A \cap B) = P(A) \times P(B)$

then, $P(\text{not } A \text{ and } B) = P(A' \cap B)$

$= P(A') \times P(B)$

$= [1 - P(A)] \times P(B)$

$= [1 - 2] \times 0 \cdot 4$

$= -0 \cdot 4$

(ii) $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 2 + 0 \cdot 4 - (2 \times 0 \cdot 4)$

$= 1 \cdot 6$

$\therefore$ The probability in (i) is negative because we are given $P(A) = 2$.

but from axioms of probability it say that Probability of any event is in interval $0 \leq P(E) \leq 1$

and for (ii), The probability is greater than 1 because $P(A) = 2$ which is not possible.

b Total number of parents is 2.

and total number of children is 10

(i) For both parents, we use the concept of combination,

$2C2 \times 10C_2 = 45$ ways

$\therefore$ 45 ways for both parents.

6b (ii) For one parent only,

$$2C_1 \times 10C_3 = 240 \text{ ways}$$

∴ There are 240 ways for one parent only

(iii) For none of the parents,

$$2C_0 \times 10C_4 = 210 \text{ ways.}$$

∴ There 210 ways for none of the parents.

c Given,

$$f(x) = \begin{cases} kx, & 0 \leq x \leq 1 \\ k(2-x), & 1 \leq x \leq 2 \\ 0, & \text{other wise} \end{cases}$$

(i) From $\int_{-\infty}^{\infty} f(x)dx = 1$

then, $\int_0^1 kx\,dx + \int_1^2 k(2-x)\,dx = 1$

$$\left(\frac{kx^2}{2}\right)\Big|_0^1 + \left(k\left(2x - \frac{x^2}{2}\right)\right)\Big|_1^2 = 1$$

$$\frac{k}{2} + k(4-2) - k\left(2 - \frac{1}{2}\right) = 1$$

$$\frac{k}{2} + 2k - \frac{3}{2}k = 1$$

$$2k - 1 = 1$$

$$2k = 2$$

$$k = 1$$

∴ The value of the k is 1

6c (ii) $E(X) = \int_0^2 x f(x) dx$

then,

$$E(X) = \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx$$

$$E(X) = \left(\frac{x^3}{3}\right)\Big|_0^1 + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2$$

$$E(X) = \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)$$

$$E(X) = \frac{1}{3} + \frac{4}{3} - \frac{2}{3}$$

$$E(X) = 1$$

(iii) $P\left(\frac{1}{2} \le x \le 1\frac{1}{2}\right) = \int_{1/2}^1 x\,dx + \int_1^{3/2} (2 - x) dx$

$$= \left(\frac{x^2}{2}\right)\Big|_{1/2}^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^{3/2}$$

$$= \left(\frac{1}{2} - \frac{1}{8}\right) + \left(3 - \frac{9}{8}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{3}{4}$$

d Given, Mean $= np = 4$

and Variance $= npq = 2$

then make the two equations,

$np = 4$ —(i)

$npq = 2$ —(ii)

Make subject n in the equation (i)

$n = \frac{4}{p}$, then substitute to the equation

6d $\frac{4}{p} \times p \times q = 2$

$4q = 2$

$q = \frac{2}{4}$

$q = 0 \cdot 5$

then, $p = 0 \cdot 5$

Now, $P(|X-4| \leq 2) = P(-2 \leq X-4 \leq 2)$

$= P(-2+4 \leq X-4+4 \leq 2+4)$

$= P(2 \leq x \leq 6)$

$= P(2) + P(3) + P(4) + P(5) + P(6)$

From formular of probability in binomial distribution,

$P(X=x) = \binom{n}{x} p^x q^{n-x}$

but $np = 4$

$n = \frac{4}{0 \cdot 5}$

$n = 8$

$P(X=2) = \binom{8}{2}(0 \cdot 5)^2 (0 \cdot 5)^{8-2} = 0 \cdot 1094$

$P(X=3) = \binom{8}{3}(0 \cdot 5)^3 (0 \cdot 5)^5 = 0 \cdot 2188$

$P(X=4) = \binom{8}{4}(0 \cdot 5)^4 (0 \cdot 5)^4 = 0 \cdot 2734$

$P(X=5) = \binom{8}{5}(0 \cdot 5)^5 (0 \cdot 5)^3 = 0 \cdot 2188$

$P(X=6) = \binom{8}{6}(0 \cdot 5)^6 (0 \cdot 5)^2 = 0 \cdot 1094$

If we add them,

$P(|X-4| \leq 2) = 0 \cdot 9297$ to 4 significant figures.

2017 PAST PAPERS - 2

6. (a) Define the following terms and write one example for each term:

   (i) Continuous random variable.

   (ii) Discrete random variable

   (iii) Probability density function.

   (b) (i) A group of students consist of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has at least a boy and a girl?

   (ii) If $P(A)=\frac{1}{4}$, $P(A/B)=\frac{1}{2}$ and $P(B/A)=\frac{2}{3}$. Verify whether A *and* B are independent events or are mutually exclusive events.

   (c) Rehema and Seni play a game in which Rehema should win 8 games for every 7 games won by Seni. Prove that if they play three games, the probability that Rehema wins at least two games is approximately to 0.55.

   (d) In a family, the boy tells a lie in 30 percent cases and the girl in 35 percent cases. Find the probability that both contradict each other on the same fact.

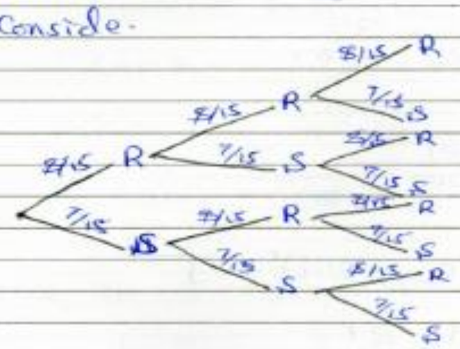
6 c) Soln.

$n(S) = 8+7 = 15$ games.

$P(\text{Rehema}) = 8/15$

$P(\text{Seni}) = 7/15$.

Conside.



$n(E) = \{RRR, RRS, RSR, SRR, \}$

$= \left(\frac{8}{15}\times\frac{8}{15}\times\frac{8}{15}\right)+\left(\frac{8}{15}\times\frac{8}{15}\times\frac{7}{15}\right)+\left(\frac{8}{15}\times\frac{7}{15}\times\frac{8}{15}\right)+\left(\frac{7}{15}\times\frac{8}{15}\times\frac{8}{15}\right)$

$\frac{512}{3375}+\frac{448}{3375}+\frac{448}{3375}+\frac{448}{3375}$

$= 0.549925$

$\simeq 0.55$

Hence, proved.

In Extract 16.2, the candidate was able to use a tree diagram in answering part (c) correctly.

2016 PAST PAPERS - 2

6. (a) The probability that a keyboard picked at random from the assembly line in a factory will be defective is 0.01. If a sample of three is to be selected:

(i) Construct the probability distribution of the defective keyboards.

(ii) Find the mean and standard deviation (Give your answers correct to 2 decimal places).

(b) The bag R contains 5 red and 3 green balls and bag P contains 3 red and 5 green balls. If one ball is drawn from bag R and two from bag P, find the probability that out of three balls drawn two are red and one is green.

(c) The random variable X has a probability distribution $P(x)$ of the following form, where k is a certain number.

$$P(X)=\begin{cases} k & \text{if } x=0 \\ 2k & \text{if } x=1 \\ 3k & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases}$$

(i) Determine the value of k.

(ii) Find $P(x<2)$, $P(x\le 2)$ and $P(x\ge 2)$.

(d) (i) If X is a discrete random variable where $E(X)$ is the expected value of X, show that $E(Ax+b)=aE(x)+b$ where a and b are constants.

(ii) The modern seeds of a certain crop have the probability of germinating 0.9. If six seeds are sown, what is the probability of at most 5 seeds are germinating?

(20 marks)

6 (a)(i)

X	0	1	2	3
$P(X=x)$	0.970	0.029	2.97×10^{-4}	1×10^{-6}

$P(X=x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$.

$p = 0.01$, $q = 0.99$.

(ii) Soln

$p = 0.01$ $q = 1 - 0.01 = 0.99$.
$n = 3$

Mean $= np = 0.01 \times 3 = 0.03$
$= \cancel{0.03}$
$= 0.03$.

Standard deviation $= \sqrt{npq}$

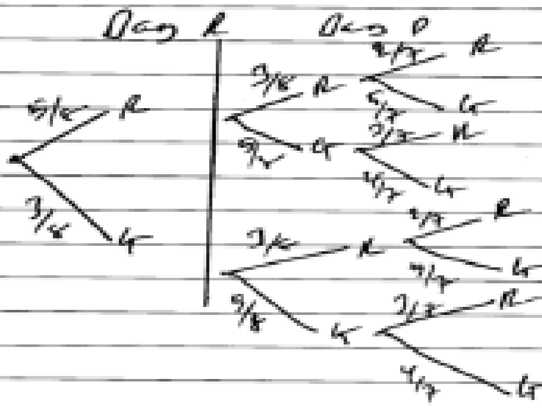
$= \sqrt{3 \times 0.01 \times 0.99}$

Standard deviation $= 0.172 \approx \underline{\underline{0.17}}$.

6 (b). Soln

Bag R
5 - red
3 - Green

Bag P
7 - red
5 - green.



Probability that two are Red and one is green
$R_1R_2G_3$, $R_1G_2R_3$, $G_1R_2R_3$

$\frac{5}{8} \times \frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{3}{8} \times \frac{3}{7}$

$= \frac{75}{448} + \frac{75}{448} + \frac{15}{448} = \frac{75+75+15}{448}$

$= \frac{165}{448} = \frac{3}{8}$

Probability that two are Red and one is green
$5 \quad \frac{3}{8}$

6 (c) Soln

$$P(x) = \begin{cases} k & \text{if } x = 0 \\ 2k & \text{if } x = 1 \\ 3k & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

(i) The value of k

$P(x) = k \quad \text{if } x = 0$
$P(x) = 2k \quad \text{if } x = 1$
$P(x) = 3k \quad \text{if } x = 2$

Take $k + 2k + 3k = 1$
$6k = 1$
$k = \frac{1}{6}$.

$$P(x) = \begin{cases} \frac{1}{6} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 2 \end{cases}$$

(ii) $P(X<2) = \frac{1}{6} + \frac{1}{3}$

$= \frac{1+2}{6} = \frac{3}{6} = \frac{1}{2}$

$P(X<2) = \frac{1}{2}$

6 (c) (ii)

$P(X\leq 2) = \frac{1}{6} + \frac{1}{3} + \frac{1}{2}$

$P(X\leq 2) = 1$

$P(X\geq 2) = \frac{1}{2}$

(d) (i) Soln

$E(ax+b) = \sum (ax+b)P(x)$

$= \sum axP(x) + \sum bP(x)$

$= a\sum xP(x) + b\sum P(x)$

but $\sum P(x) = 1$

$= aE(x) + b(1)$

$= aE(x) + b$

(ii) Soln

Probability of germinating, $P = 0.9$

$q = 1 - 0.9 = 0.1$

6 (a)(ii).

$n = 6.$

Required $P(X \leq 5)$.

$P(X \leq 5) = 1 - P(X = 6)$

$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$P(X \leq 5) = 1 - {}^{6}C_{6}(0.9)^{6}(0.1)^{0}$

$= 1 - 0.531441$

$= 0.468559.$

The probability of at most 5 seeds are germinating

0.468559

In Extract 16.2, the candidate demonstrated good understanding of the basic concepts of probability and used them correctly in answering this question.

2015 PAST PAPERS - 2

6\. (a) A school needs 10 prefects out of which 5 are supposed to be girls and 5 are to be boys. If 5 boys are to be selected from a group of 8 boys and 5 girls from 9 girls; in how many different ways can the 10 prefects be selected.

(b) Three athletes from Tanzania will participate in an International Coca Cola marathon race next year. If the probabilities to complete the marathon are 0.9, 0.7 and 0.6 respectively, find the probability that at least two of them will complete the marathon.

(c) If a random variable X has probability density function

$$f(x) = \begin{cases} \frac{|x|}{8} & \text{for } -2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the

(i) expected value of X.

(ii) Standard Deviation of X.

(iii) Variance of X.

(d) If $P(A \cup B) = 80\%$ and $P(A \cup \bar{B}) = 70\%$ determine $P(A)$.

(20 marks)

6a. Solution.

By concept of combination.

	Requirements	Available
Boys	5	8
Girls.	5	9.

No of ways of selecting 5 boys is given by

$^{8}C_{5}$

No of ways of selecting 5 girls is given by

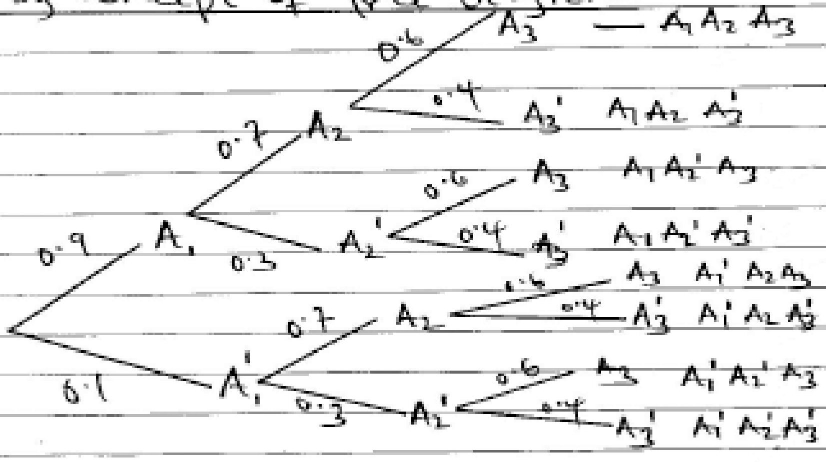
$^{9}C_{5}$.

Total no of ways.

$^{8}C_{5} \times {}^{9}C_{5}$

$= 7056$ ways.

(b) By concept of tree diagram.



where

A_1 – athlete one will compete

A_1' – athlete one wont compete

A_2 – athlete two will compete

~~where~~ A_2' – athlete two wont compete

A_3 – athlete 3 to compete

A_3' – athlete not to compete

(i) Probability of atleast two.

$= P(A_1A_2A_3) + P(A_1A_2A_3') + P(A_1A_2'A_3) + P(A_1'A_2A_3)$

$= 0.9 \times 0.7 \times 0.6 + 0.9 \times 0.7 \times 0.4 + 0.9 \times 0.3 \times 0.6 + 0.1 \times 0.7 \times 0.6$

$=$ ~~0.786~~ 0.834

(c) $f(x) = \begin{cases} \frac{|x|}{8} & \text{for } -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

~~Find mode~~

solution.

This can be rewritten as

$f(x) = \begin{cases} -x/8 & \text{for } -2 \le x \le 0 \\ x/8 & \text{for } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

was to satisfy.

(i) Expectation $E(x)$

$$E(x) = \int_a^b x f(x)\, dx$$

$$E(x) = \int_{-2}^{0} \frac{-x^2}{8}\, dx + \int_0^4 \frac{x^2}{8}\, dx$$

$$= \underline{2.33333.}$$

(ii) Standard deviation of x

$$S.D = \sqrt{var(x)}$$

$$var(x) = \int x^2 f(x)\, dx - (\bar{x})^2$$

$$= \int_{-2}^{0} \frac{-x \cdot x^2}{8}\, dx + \int_0^4 \frac{x^2 \cdot x}{8} - (2.3333)^2$$

$$= 0.5 + 8 - (2.3333)^2$$

$$var = 3.0557$$

$$S.D = \sqrt{var(x)}$$

$$S.D = \underline{1.748.}$$

(iii) $var(x)$

$$var(x) = \int x^2 f(x)\, dx - (E(x))^2$$

$$= \int_{-2}^{0} \frac{-x^3}{8}\, dx + \int_0^4 \frac{x^3}{8}\, dx - (2.3333)^2$$

$= 0.5 + 8 - (2.3333)^2$

$= \underline{2.05555 71111}$

d) $P(A \cup B) = 80\%$

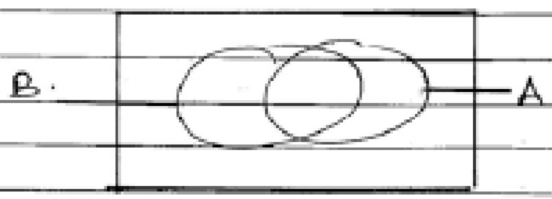
$P(A \cup B') = 70\%$

From probability table derived from venn diagram.

A — A'

B — A∩B   A'∩B

B' —



	A	A'
B	$A \cap B$	$A' \cap B$
B'	$A \cap B'$	$A' \cap B'$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$+ \; P(A \cup B') = P(A) + P(B') - P(A \cap B')$

$P(A \cup B) + P(A \cup B') = 2(P(A)) + 100 - P(A \cap B) - P(A \cap B')$

$80\% + 70\% = 2P(A) + 100\% - P(A \cap B) - P(A \cap B')$

Now

$P(A) = P(A \cap B) + P(A \cap B')$

$80\% + 70\% = 2P(A) + 100\% - [P(A \cap B) + P(A \cap B')]$

$80\% + 70\% = 2P(A) + 100\% - P(A)$

$150\% = 2P(A) + 100\% - P(A)$

$P(A) = \underline{50\%}$

In Extract 16.2, the candidate demonstrated high level of understanding of the basic concepts of probability.

16.0 Complex Numbers

2021 PAST PAPERS - 2

4. (a) If $2x+10yi-4y=-12+5i$, find the values of x and y.

(b) Express $(\cos\theta+i\sin\theta)^{-n}$ in the form $a+ib$.

(c) Given that $z=x+iy$, express the complex number $\frac{z+i}{iz+2}$ in polynomial form and hence find the resulting complex number when $z=1+2i$.

4 a) $2x+10yi-4y=-12+5i$

$(2x-4y)+10yi=-12+5i$

$2x-4y=-12 \quad \cdots \text{i)}$

$10y=5 \quad \cdots \text{ii)}$

$y=\frac{5}{10}$

$y=\frac{1}{2}$

$2x-4y=-12$

$2x-4\left(\frac{1}{2}\right)=-12$

$2x-2=-12$

$2x=-12+2$

$2x=-10$

$x=-5$

$\therefore$ The value of x is -5 and y is $\frac{1}{2}$.

b) $(\cos\theta+i\sin\theta)^{-n}$

$=\cos(-n)\theta+i\sin(-n)\theta$

$=\cos n\theta-i\sin n\theta$

c) $z=x+iy$

$\frac{z+i}{iz+2}=\frac{x+iy+i}{i(x+iy)+2}$

$=\frac{x+i(y+1)}{ix+i^2y+2}$

$=\frac{x+i(y+1)}{ix-y+2}$

$=\frac{x+i(y+1)}{(2-y)+ix}$

4c) $=\frac{x+i(y+1)}{(2-y)+ix}\times\frac{(2-y)-ix}{(2-y)-ix}$

$=\frac{x(2-y)-ix^2+i(2-y)(y+1)-i^2x(y+1)}{(2-y)^2-(ix)^2}$

$=\frac{x(2-y)-ix^2+i(2-y)(y+1)+x(y+1)}{(2-y)^2+x^2}$

$$= \frac{\left(x(2-y)+x(y+1)\right) + i\left((2-y)(y+1) - x^2\right)}{(2-y)^2 + x^2}$$

$$= \frac{(2x - xy + xy + x) + i(2y + 2 - y^2 - y - x^2)}{(2-y)^2 + x^2}$$

$$= \frac{3x + i(y - y^2 - x^2 + 2)}{(2-y)^2 + x^2}$$

$$= \frac{3x}{(2-y)^2 + x^2} + \frac{i(y - y^2 - x^2 + 2)}{(2-y)^2 + x^2}$$

$$\therefore \frac{z+i}{iz+2} = \frac{3x}{(2-y)^2 + x^2} + \frac{i(y - y^2 - x^2 + 2)}{(2-y)^2 + x^2}$$

When $z = 1 + 2i$

$x = 1,\ y = 2$

$$\frac{z+i}{iz+2} = \frac{3(1)}{(2-2)^2 + 1^2} + \frac{i(2 - 2^2 - 1^2 + 2)}{(2-2)^2 + 1^2}$$

$$= \frac{3}{1} + \frac{i(-1)}{1}$$

$$= 3 - i$$

$$\therefore \frac{z+i}{iz+2} = 3 - i$$

2020 PAST PAPERS - 2

4. (a) If $z = a + ib$, prove that $z.\bar{z}$ is a real number for all complex number z.

(b) Given that $z = \cos\theta + i\sin\theta$, express $\cos^4\theta$ as the sum of cosines of multiple of θ.

(c) If $z = \cos\alpha + i\sin\alpha$, show that $\frac{1}{1+z} = \frac{1}{2}\left(1 - i\tan\left(\frac{\alpha}{2}\right)\right)$.

b) $z = \cos\theta + i\sin\theta$ required $\cos^4\theta$ in term of multiple of θ.

Recall that $z = \cos\theta + i\sin\theta$

$1/z = \cos\theta - i\sin\theta$.

So: $z + 1/z = 2\cos\theta$.

Also for $z^n + 1/z^n = 2\cos n\theta$.

From $z + 1/z = 2\cos\theta$ by

$= (z + 1/z)^4 = (2\cos\theta)^4$

$= z^4 + 4z^2 + 6 + 4/z^2 + 1/z^4 = 16\cos^2\theta$.

$= \left(z^4 + 1/z^4\right) + 4\left(z^2 + 1/z^2\right) + 6 = 16\cos^2\theta$.

but $z^4 + 1/z^4 = 2\cos 4\theta$

$z^2 + 1/z^2 = 2\cos 4\theta$

$= 2\cos 4\theta + 4(2\cos 2\theta) + 6 = 16\cos^4\theta$

$= 2\cos 4\theta + 8\cos 2\theta + 6 = 16\cos^2\theta$

4 b) $= 2\cos 4\theta + 8\cos 2\theta + 6 = 16\cos^4\theta$

$= \frac{2(\cos 4\theta + 4\cos 2\theta + 3)}{16} = \frac{16\cos^4\theta}{16}$

$\therefore \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

2019 PAST PAPERS - 2

1. (a) Express the complex number $\left(\frac{1+i}{1-i}\right)^8 + \left(\frac{\sqrt{3}}{1-i}\right)^4$ in the form $a+ib$.

(b) Show that $[r(\cos\theta + i\sin\theta)]^n = r^n e^{in\theta}$.

(c) If the point P represents the complex number $z = x+iy$ on the Argand diagram, describe the locus of P if $|z-i| = 3|z+i|$.

$e^{i\theta} = \cos\theta + i\sin\theta$

Times by r both sides

$re^{i\theta} = r[\cos\theta + i\sin\theta]$

raise to power n both sides

$(re^{i\theta})^n = [r[\cos\theta + i\sin\theta]]^n$

$r^n e^{in\theta} = [r[\cos\theta + i\sin\theta]]^n$

Hence shown

c). sub $z = x+iy$

then

$|z-i| = 3|z+i|$

$|x+iy-i| = 3|x+iy+i|$

$|x+i(y-1)| = 3|x+i(y+1)|$

1. c)

$\sqrt{x^2+(y-1)^2} = 3\sqrt{x^2+(y+1)^2}$

Square both sides

$x^2+(y-1)^2 = 9(x^2+(y+1)^2)$

$x^2+y^2-2y+1 = 9(x^2+y^2+2y+1)$

$x^2+y^2-2y+1 = 9x^2+9y^2+18y+9$

$8x^2+8y^2+20y+8=0$

$2x^2+2y^2+5y+2=0$ --- locus.

on comparison with

$x^2+y^2+\frac{5}{2}y+1=0$

and $x^2+y^2+2gx+2fy+c=0$

then

$2g=0$ $\quad 2f=\frac{5}{2}$ $\quad c=+1$

$g=0$ $\quad f=\frac{5}{4}$

and

radius $(r) = \sqrt{g^2+f^2-c}$

$= \sqrt{0^2+(\frac{5}{4})^2-(+1)}$

$r = 3/4$

centre $(-g,-f) = (-0, -\frac{5}{4})$

So the locus is circle of equation $x^2+y^2+\frac{5}{2}y+1=0$ whose centre is $(0,-\frac{5}{4})$ and radius of $3/4$ units circle.

$c(0,-\frac{5}{4})$

2018 PAST PAPERS - 2

1. (a) Use Demoivre's theorem to prove that $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$.

(b) (i) The equation $6 - z^2 = 8i - (2+4i)z$ has roots z_1 and z_2. If $z_1 = 3 + i$; find the other root z_2 in form $a + ib$.

(ii) Express $\frac{6}{x^2 - 2x + 10}$ in partial fractions with complex linear denominators.

(c) If z is any complex number, such that $z = r(\cos n\theta + i\sin n\theta)$, use mathematical induction to prove that $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all positive integers n.

(d) If $|z+1| = 2|z-1|$, prove that z lies on a circle whose radius is $\frac{4}{3}$.

Q1. (a) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

from Demoivre's theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2$
$+ 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$
$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta + -10i\cos^2\theta\sin^3\theta$
$+ i\sin^5\theta + 5\cos\theta\sin^4\theta$
$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta -$
$10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Q1. then :

$\cos 5\theta + i\sin 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i($
$5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating imaginary part

$$\frac{\sin 5\theta}{\sin\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\sin\theta}$$

$$= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 16\cos^4\theta - 12\cos^2\theta + 1$$

Hence

$$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$$

(b) (i) $6 - z^2 = 8i - (2+4i)z$ roots z_1 and z_2

$z^2 - (2+4i)z + 8i - 6 = 0$

$z^2 - (2+4i)z + (8i-6) = 0$

$z^2 - (\text{sum of roots})z + (\text{product of roots}) = 0$

$z^2 - (z_1+z_2)z + z_1z_2 = 0$

$z_1 + z_2 = 2+4i$

$z_1z_2 = 8i - 6$

let $z_2 = x+iy$, $z_1 = 3+i$

$3+i + x+iy = 2+4i$

$3+x + (1+y)i = 2+4i$

$3+x = 2$, $1+y = 4$

$x = 2-3$ $\quad$ $y = 4-1$

$x = -1$ $\quad$ $y = 3$

$z_2 = -1+3i$

$z_1z_2 = (-1+3i)(3+i)$

Q1. (b) (i) $z_1z_2 = -3 - i + 9i + 3i^2$

$z_1z_2 = -3 - i + 9i - 3$

$z_1z_2 = 8i - 6$ Hence shown

$\therefore$ $z_2 = \underline{-1+3i}$

where $a = -1$, $b = 3$

(ii) $$\frac{6}{x^2-2x+10} = \frac{6}{x-(1+3i))(1-3i+x)}$$

$$\frac{6}{(x-(1+3i))}\cdot\frac{1}{x-(1-3i)} = \frac{A}{(x-1-3i)} + \frac{B}{(x-1+3i)}$$

$$\frac{6}{x^2-2x+10} = \frac{A(x-1+3i) + B(x-1-3i)}{(x-1-3i)(x-1+3i)}$$

$$6 = A(x-1+3i) + B(x-1-3i)$$

put $x = 1-3i$

$$6 = A(1-3i-1+3i) + B(1-3i-1+3i)$$

$$6 = A(0) + B(-6i)$$

$$1 = -Bi$$

$$B = \frac{1}{-i}$$

$$\underline{B = 1i}$$

put $x = 1+3i$

$$6 = A(1+3i-1+3i) + B(1+3i-1-3i)$$

$$6 = 6(Ai) + B(0)$$

$$A = \frac{1}{i}$$

$$A = -1i$$

then into partial fractions.

Q1. (b) (ii)

$$\frac{6}{x^2-2x+10} = \frac{-1i}{(x-1-3i)} + \frac{1i}{(x-1+3i)}$$

(c) By mathematical induction

$z^n = r^n(\cos n\theta + i\sin n\theta)$

1. proof for $n=1$

$z^1 = r^1(\cos 1\theta + i\sin 1\theta)$

$z = r(\cos\theta + i\sin\theta)$

Hence it is true for $n=1$.

2. Assume it is true for $n=k$

$z^k = r^k(\cos k\theta + i\sin k\theta)$ ——— (1)

3. proof for $n=k+1$

$z^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

consider r.H.S

$z^{k+1} = z^k \cdot z^1$

$= r^k(\cos k\theta + i\sin k\theta)(r(\cos\theta + i\sin\theta))$

$= r^k \cdot r[\cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta]$

$= r^k \cdot r[\cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta]$

$= r^k \cdot r\ [(\cos k\theta \cos\theta - \sin k\theta \sin\theta) + i(\sin k\theta \cos\theta +$
$\cos k\theta \sin\theta)]$

$= r^k \cdot r\ [\cos(k\theta + \theta) + i\sin(k\theta + \theta)]$

$= r^k \cdot r\ [\cos(k+1)\theta + i\sin(k+1)\theta]$

$= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$ L.H.S

Hence proved!

2017 PAST PAPERS - 2

1. (a) Use the Demoivre's theorem to find the value of $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}$.

(b) Show that $[r(\cos\theta + i\sin\theta)]^n = r^n e^{in\theta}$ and hence find in form of $re^{i\theta}$ all complex numbers z, such that $z^3 = \frac{5+i}{2+3i}$.

(c) (i) Solve the equation $x^4 + 1 = 0$ and leave the roots in radical form.

(ii) If $w = \frac{z+2}{2}$ and $|z| = 4$, find the locus of the w. **(15 marks)**

1(c) (i) Soln

$x^4 + 1 = 0$

$x^4 = -1$

$x^4 = -1 + 0i$

$x^4 = \cos\pi + i\sin\pi$

$x^4 = \cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k)$

$x = \cos\left(\frac{\pi + 2\pi k}{4}\right) + i\sin\left(\frac{\pi + 2\pi k}{4}\right)$

If $k = 0$

$x = \cos\frac{\pi}{4} + i\sin\pi/4 = \frac{\sqrt{2}}{2}(1+i)$

If $k = 1$

$x = \cos\frac{3\pi}{4} + i\sin 3\pi/4 = \frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

If $k = 2$

$x = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

If $k = 3$

$x = \cos 7\pi/4 + i\sin 7\pi/4 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

In Extract 11.1, the candidate was able to evaluate the modulus and argument of the complex number and used the formula to get the roots.

2016 PAST PAPERS - 2

1. (a) (i) Use the De moivre's theorem to find the value of $(1+i)^5$.

(ii) Use the mathematical induction to prove that
$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + \sin n\theta)$.

(b) (i) If $\arg\left(\frac{z-1}{z+i}\right) = \frac{\pi}{4}$ and $z = x+iy$, find the locus of the point representing z in an argand diagram.

(ii) Solve the following system of equations where z and w are complex numbers.
$$\begin{cases} iz - w = 2i \\ iz + iw = i \end{cases}$$

(c) One of the roots of the equation $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$ is $1+i$. Find the other roots.

(15 marks)

(ii) $r(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + \sin n\theta)$

let $n = 1$

$r(\cos\theta + i\sin\theta)^1 = r(\cos\theta + i\sin\theta)$

true

let $n = k$

$= r^k(\cos k\theta + i\sin k\theta)$

$\therefore$ true

let $n = k+1$

$(r(\cos\theta + i\sin\theta))^{k+1} = (r(\cos\theta + i\sin\theta))^k \times r(\cos\theta + i\sin\theta)$

$= r^k(\cos k\theta + i\sin k\theta) \cdot r(\cos\theta + i\sin\theta)$

$= r^k \cdot r(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= r^k \cdot r[\cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta]$

$= r^{(k+1)}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin\theta\cos k\theta + \sin k\theta\cos\theta)]$

$= r^{(k+1)}[\cos(k\theta+\theta) + i\sin(k\theta+\theta)]$

$\therefore = r^{(k+1)}[\cos(k+1)\theta + i\sin(k+1)\theta]$

$\therefore$ True

In Extract 11.1, the candidate managed to prove the given identity by mathematical induction, showing that he/she had adequate knowledge on the tested concept.

2015 PAST PAPERS - 2

Answer all questions in this section.

1. (a) Use the Euler formula for exponentials $z = e^{i\theta}$, to show that
$$\frac{1}{2}\left(z+\frac{1}{z}\right)=1-\frac{\theta^2}{2}-\frac{i\theta^3}{6}+\frac{\theta^4}{24}+\dots$$

(b) Given that one root of the equation $z^4+z^3+3z^2+z+2=0$ is i, find the other roots.

(c) (i) If $z_1 = 1+i\sqrt{3}$ and $z_2=\sqrt{3}+i$, find the modulus and the principle argument of $z_1 z_2$.

(ii) If z is a complex number find the equation of locus represented by $|z-(2-i)|=|z-(3+2i)|$.

(d) If $z=\sqrt{3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ and $z^2 w = 21\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right]$, express w in modulus argument form.

1(a) Given $z = e^{i\theta} \Rightarrow \frac{1}{z} = \bar{z} = e^{-i\theta}$

$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ $\quad \because i^2 = -1,\ i^3 = -i,\ i^4 = 1$

$\therefore z = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$

$\therefore z = e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{6} + \frac{\theta^4}{24} + \dots$

$\bar{z} = \frac{1}{z} = e^{-i\theta} = 1 - i\theta + \frac{(-i\theta)^2}{2!} + \frac{(-i\theta)^3}{3!} + \frac{(-i\theta)^4}{4!}$

$\therefore \frac{1}{z} = e^{-i\theta} = 1 - i\theta - \frac{\theta^2}{2} + \frac{i\theta^3}{6} + \frac{\theta^4}{24} + \dots$

$$\therefore \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}\left(\begin{array}{l} 1 + \cancel{i\theta} - \theta^2/2 - \cancel{i\theta^3/6} + \theta^4/24 + \dots \\ + 1 - \cancel{i\theta} - \theta^2/2 + \cancel{i\theta^3/6} + \frac{\theta^4}{24} + \dots \end{array}\right)$$

$$\therefore \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}\left(2 - \frac{2\theta^2}{2} + \frac{2\theta^4}{24} + \dots\right)$$

$$\therefore \frac{1}{2}\left(z+\frac{1}{z}\right) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \dots$$

(hence shown)

1(b)

$$\begin{array}{r|l} & z^2 + z + 2 \\ \hline z^2+1 & z^4 + z^3 + 3z^2 + z + 2 \\ & -(z^4 + z^2) \\ \hline & z^3 + 2z^2 + z + 2 \\ & -(z^3 + z) \\ \hline & 2z^2 + 2 \\ & -(2z^2+2) \\ \hline & - \; - \; - \end{array}$$

$\therefore z^2+z+2$ is also a factor

Let $z^2+z+2=0$

$$z = \frac{-1 \pm \sqrt{1^2 - 4(2)}}{2}$$

$$z = \frac{-1 \pm \sqrt{-7}}{2}$$

$$z = \frac{-1 \pm i\sqrt{7}}{2}$$

$\therefore$ The other roots are $-i$, $\frac{-1+i\sqrt{7}}{2}$ and $\frac{-1-i\sqrt{7}}{2}$.

1(c) (i) $z_1 z_2 = (1+i\sqrt{3})(\sqrt{3}+i)$.

$z_1 = 1 + i\sqrt{3}$ ~~argu~~

$\Rightarrow \arg(z_1) = \theta_1 = \tan^{-1}(\sqrt{3}) = 60°$

modulus $z_1 = r_1 = \sqrt{1+(\sqrt{3})^2} = 2$.

$z_2 = \sqrt{3}+i$

$\theta_2 = \arg(z_2) = \tan^{-1}(1/\sqrt{3}) = 30°$

$|z_2| = \sqrt{\sqrt{3}^2+1} = 2$

1(b) $z_1 z_2 = 4(\cos 90° + i\sin 90°)$

$\therefore |z_1 z_2| = 4$ (modulus of z_1z_2)

and $\arg(z_1z_2)$ = principal argument of $z_1z_2 = \underline{90°}$.

1(c) (ii) Let $z = x+iy$,

$|z-(2-i)| = |z-(3+2i)|$

$\Rightarrow |x+iy-2+i| = |x+iy-3-2i|$

$\Rightarrow |(x-2)+i(y+1)| = |(x-3)+i(y-2)|$.

$\Rightarrow \sqrt{(x-2)^2+(y+1)^2} = \sqrt{(x-3)^2+(y-2)^2}$

$\Rightarrow (x-2)^2+(y+1)^2 = (x-3)^2+(y-2)^2$

$\Rightarrow \cancel{x^2}$

$(x-2)^2-(x-3)^2 = (y-2)^2-(y+1)^2$

$\Rightarrow (x-2+x-3)(\cancel{x}-2-\cancel{x}+3) = (y-2+y+1)(\cancel{y}-2-\cancel{y}-1)$

$\Rightarrow (2x-5)(1) = (2y-1)(-3)$

$\Rightarrow 2x-5 = -6y+3$

$-6y = 2x-8$

$y = -\frac{x}{3}+\frac{4}{3}$

$\therefore$ The locus is straight line with slope, $-\frac{1}{3}$ and y-intercept at $= \frac{4}{3}$

1(d) Given

$z = \sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$z^2 w = 21\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$$

$$\because z^2 = \left(\sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^2$$

$$z^2 = 3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$\therefore w = \frac{21\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]}{7\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\frac{2\pi}{3}\right)}$$

$$\cancel{\therefore w =}$$

$$\therefore w = 7\left[\cos\left(-\frac{2\pi}{3} - \frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3} - \frac{2\pi}{3}\right)\right]$$

$$\therefore w = 7\left[\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right]$$

Extract 11.1 illustrates a candidate's solution in which he/she was able to apply the Euler's formula to deal with series involving complex numbers, found the roots of the given complex polynomial function and determined the modulus and the principle argument of the complex equation. In part (d), the candidate managed to express correctly the complex numbers in modulus – argument form.

17.0 Differential Equations

2021 PAST PAPERS - 2

7. (a) Form a differential equation whose solution is $x = \tan(Ay)$.

(b) Solve the differential equation $\frac{d^2\theta}{dt^2} - 4\frac{d\theta}{dt} + 4\theta = \frac{3}{7}$.

(c) A biologist is researching the population of a specie. She tries a number of different models for the rate of growth of the population and solves them to compare with observed data. Her first model is $\frac{dn}{dt} = kn\left(1 - \frac{n}{a}\right)$ where n is the population at time t years, k is a constant and a is the maximum population sustainable by the environment. Given that $k = 0.2$, $a = 100000$ and the initial population is 30000;

(i) find the general solution of the differential equation.

(ii) estimate the population after 5 years to 2 significant figurers.

7. a)

$x = \tan(Ay)$

$\tan^{-1} x = Ay$

$A = \frac{1}{y}\tan^{-1} x$

$A = \frac{\tan^{-1} x}{y}$

differentiate both sides.

$\frac{d}{dx}\left(\frac{\tan^{-1}x}{y}\right) = \frac{d(A)}{dx}$

$$\frac{y\frac{d(\tan^{-1}x)}{dx} - \tan^{-1}x \cdot \frac{dy}{dx}}{y^2} = 0$$

$y\frac{d(\tan^{-1}x)}{dx} - \tan^{-1}x \cdot \frac{dy}{dx} = 0$

but $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$

$$\frac{y}{1+x^2} - \frac{dy}{dx}\tan^{-1}x = 0$$

$$\therefore (\tan^{-1}x)\frac{dy}{dx} = \frac{y}{1+x^2}$$

7 b) given D.E

$$\frac{d^2\theta}{dt^2} - 4\frac{d\theta}{dt} + 4\theta = 3/7$$

for complementary function (θ_c)

when; $\frac{d^2\theta}{dt^2} - 4\frac{d\theta}{dt} + 4\theta = 0 \longrightarrow$

let $\theta_c = e^{mt}$

$$\theta_c' = me^{mt}$$

$$\theta_c'' = m^2e^{mt}$$

substitute in the D.E

$$m^2e^{mt} - 4(me^{mt}) + 4e^{mt} = 0$$

$$e^{mt}(m^2 - 4m + 4) = 0$$

$$e^{mt} \neq 0, \quad m^2 - 4m + 4 = 0 \text{ (AQE)}$$

$$m^2 - 4m + 4 = 0$$

$$m = 2$$

$$\therefore \theta_c = e^{2t}(At + B)$$

for particular Integral (θ_p)

let $\theta_p = c$

$$\theta_p' = 0$$

$$\theta_p'' = 0$$

substitute in the D.E

$$0 - 4(0) + 4c = 3/7$$

$$4c = 3/7$$

$$c = 3/28$$

$$\therefore \theta_p = 3/28$$

for general solution,

$$\theta = \theta_c + \theta_p$$

$$\therefore \theta = e^{2t}(At + B) + 3/28$$

7. c) is given

$$\frac{dn}{dt} = kn\left(1 - \frac{n}{a}\right)$$

$$\frac{dn}{n\left(1 - \frac{n}{a}\right)} = k\,dt$$

Integrate both sides

$$\int_{n_0}^{n} \frac{dn}{n(1-n/a)} = \int_0^t k\,dt$$

$$\int_{n_0}^{n} \frac{dn}{n(1-n/a)} = kt$$

consider

$$\frac{1}{n(1-n/a)} = \frac{A}{n} + \frac{B}{1-n/a}$$

$$1 = A(1-n/a) + Bn$$

when $n=0$

7 (c)(i) $A=1$

when $n=a$

$$1 = A(1-a/a) + aB$$

$$B = 1/a$$

$$\frac{1}{n(1-n/a)} = \frac{1}{n} + \frac{1}{a(1-n/a)}$$

$$\frac{1}{n(1-n/a)} = \frac{1}{n} + \frac{1}{a-n}$$

$$\int \frac{dn}{n(1-n/a)} = \int \frac{dn}{n} + \int \frac{dn}{a-n}$$ (without limits)

$$\int \frac{dn}{n(1-n/a)} = \ln(n) + -\ln(a-n)$$

with limits

$$= \ln n + -\ln(a-n)\Big|_{n_0}^{n}$$

INDEX NUMBER | Examiners use only

7. (c)(i) $= \ln n - \ln(a-n)\Big|_{n_0}^{n} = kt$

$$[\ln n - \ln(a-n)] - [\ln n_0 - \ln(a-n_0)] = kt$$

$$\ln\left(\frac{n}{a-n}\right) - \ln\left(\frac{n_0}{a-n_0}\right) = kt$$

$$\ln\left(\frac{n}{a-n} \times \frac{a-n_0}{n_0}\right) = kt$$

given $a = 100000$

$n_0 = 30000$

$k = 0.2$

$$\ln\left(\frac{n}{100000-n} \times \frac{100000-30000}{30000}\right) = 0.2t$$

$$\ln\left(\frac{7n}{300000-3n}\right) = 0.2t$$

General solution of the differential equation

$$\therefore \ln\left(\frac{7n}{300000-3n}\right) = 0.2t$$

7 (c) (ii) Population after $t = 5$ years

from D.E

$$\ln\left(\frac{7n}{300000-3n}\right) = 0.2t$$

$$\frac{7n}{300000-3n} = e^{0.2(5)}$$

$7n = 2.718(300000 - 3n)$

$15.155n = 2.718 \times 300000$

$n = 53810.15262$

$\therefore$ population after 5 years;

$n \approx 54000$

2020 PAST PAPERS - 2

7. (a) Solve the differential equation $y\dfrac{d^2y}{dx^2} + 25 = \left(\dfrac{dy}{dx}\right)^2$ given that $\dfrac{dy}{dx} = 4$ when $y = 1$, and $y = \dfrac{5}{3}$ when $x = 0$.

(b) Solve the differential equation $xy^2 + x^2y\dfrac{dy}{dx} = \sec^2 2x$.

(c) The rate at which atoms in a mass of a radioactive material are disintegrating is proportional to the number of atoms (N) present at any time t. If N_0 is the number of atoms present at time $t = 0$, solve the differential equation that represents this information.

(d) If half of the original mass disintegrates in 152 days, find the constant of proportionality for the solution obtained in (c). (Give your answer to three significant figures).

07. a) $y\frac{d^2y}{dx^2} + 25 = \left(\frac{dy}{dx}\right)^2$

let $p = \frac{dy}{dx}$

$\frac{dp}{dx} = \frac{d^2y}{dx^2}$

$y\frac{dp}{dx} = p^2 - 25$

but $\frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx}$

$\frac{dp}{dx} = p\frac{dp}{dy}$

$yp\frac{dp}{dy} = p^2 - 25$

$\frac{p\,dp}{p^2-25} = \frac{dy}{y}$

Intergrating

$\frac{1}{2}\ln(p^2-25) = \ln y + \ln A$

$\ln\sqrt{p^2-25} = \ln Ay$

$\sqrt{p^2-25} = Ay$

$p^2 - 25 = (Ay)^2$

$p^2 = (Ay)^2 + 25$

07. a) $p^2 - 25 = A^2y^2$

$\left(\frac{dy}{dx}\right)^2 - 25 = A^2y^2$

$(4)^2 - 25 = By^2$ $\quad\because A^2 = B$

$-9 = B(1)$

$$\left(\frac{dy}{dt}\right)^2 = 25 - 9y^2$$

$$\frac{dy}{dt} = \sqrt{25-9y^2}$$

$$\frac{dy}{\sqrt{25-9y^2}} = dx$$

let $9y^2 = 25\sin^2\theta$

$3y = 5\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{3y}{5}\right)$

$3\frac{dy}{d\theta} = 5\cos\theta \quad , dy = \frac{5}{3}\cos\theta\, d\theta$

$$\frac{5/3 \cos\theta\, d\theta}{5\sqrt{1-\sin^2\theta}} = dx$$

$$\frac{1}{3}\int d\theta = \int dx$$

$$\frac{1}{3}\theta = x + A$$

Q7. a)

$$\frac{1}{3}\theta = x + A$$

$$\frac{1}{3}\left|\sin^{-1}\frac{3y}{5}\right| = x + A$$

putting $y = 5/3$ and $x = 0$

$$\frac{1}{3}\left|\sin^{-1}(1)\right| = A$$

$$\frac{\pi}{6} = A$$

$$\sin^{-1}\frac{3y}{5} = 3x + \frac{\pi}{6} \times 3$$

$$\frac{3y}{5} = \sin\left(3x + \frac{\pi}{2}\right)$$

$$\frac{3y}{5} = \cos 3x$$

$$\therefore \quad y = \frac{5}{3}\cos 3x$$

2019 PAST PAPERS - 2

7. (a) Form a differential equation whose general solution is given by $x = e^{2t}(A + Bt)$ where A and B are constants.

(b) (i) Show that $y = 2 - \cos x$ is a particular integral of the differential equation $\frac{d^2y}{dx^2} + 4y = 8 - 3\cos x$ and find the general solution.

(ii) Find the particular solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 8 - 3\cos x$ such that when $x = 0$, $y = 1.5$ and $\frac{dy}{dx} = 0$.

(c) A rumour is spreading through a large city at a rate which is proportional to the product of the fractions of those who heard it and of those who have not heard it, so that x is the fraction of those who heard it after time t.

(i) If initially a fraction c has heard the rumour, show that $x = \frac{c}{c + (1-c)e^{-kt}}$.

(ii) If 10% have heard the rumour at noon and another 10% by 3:00 pm, find x as a function of t. What further population would you expect to have heard it by 6:00 pm?

7. (a) $x = e^{2t}(A + Bt)$

$(e^{-2t})x = e^{2t}(A + Bt)(e^{-2t})$

$xe^{-2t} = A + Bt$

$e^{-2t}\,dx/dt + -2xe^{-2t} = 0 + B$

$e^{-2t}\,dx/dt + -2xe^{-2t} = B$

$0 = -2e^{-2t}\,dx/dt + e^{-2t}\,d^2x/dt^2 + 4xe^{-2t} - 2e^{-2t}\frac{dx}{dt}$

$0 = e^{-2t}\frac{d^2x}{dt^2} - 4e^{-2t}\frac{dx}{dt} + 4xe^{-2t}$

$0 = e^{-2t}\left(\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4\right)$

$0 = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4$

7. (b) (i) $y = 2 - \cos x$

$\frac{d^2y}{dx^2} + 4y = 8 - 3\cos x$

$y = 2 - \cos x$ ——— (i)

$\frac{dy}{dx} = \sin x$

and $\frac{d^2y}{dx^2} = \cos x$ ——— (ii)

Substituting (i) and (ii) into

$$\frac{d^2y}{dx^2} + 4y = 8 - 3\cos x$$

$$(\cos x) + 4(2 - \cos x) = 8 - 3\cos x$$

$$\cos x + (4 \times 2) - 4\cos x = 8 - 3\cos x$$

$$8 + \cos x - 4\cos x = 8 - 3\cos x$$

$$8 - 3\cos x = 8 - 3\cos x$$

LHS = RHS.

$\therefore$ & $y_p = 2 - \cos x$ is a particular solution.

7.

$$\frac{d^2y}{dx^2} + 4y = 8 - 3\cos x$$

Complementary solution;

Consider $\frac{d^2y}{dx^2} + 4y = 0$

7. (b) (i) let $y = e^{px}$

$dy/dx = pe^{px}$

$d^2y/dx^2 = p^2e^{px}$

Substituting in $\frac{d^2y}{dx^2} + 4y = 0$

$$p^2e^{px} + 4e^{px} = 0$$

$$(p^2 + 4)e^{px} = 0$$

$$p^2 + 4 = 0$$

$$p^2 = -4$$

$$p = \pm 2i, \quad p = 0 \pm 2i$$

now $y_c = Ae^{\alpha}(\cos\beta\theta + \sin\beta\theta)$

where α – real part $= 0$

β – imag. part $= 2$.

$$y_c = Ae^0(\cos 2x + \sin 2x)$$

$$y_c = A(\cos 2x + \sin 2x).$$

General solution, $y = y_c + y_p$

$$y = A(\cos 2x + \sin 2x) + 2 - \cos x$$

(b) (ii) $\frac{d^2y}{dx^2} + 4y = 8 - 3\cos x$

particular solution.

$$y = A(\cos 2x + \sin 2x) + 2 - \cos x$$

$y = 1.5$, $x = 0$ and $dy/dx = 0$

$$\cancel{y}\ 1.5 = A(\cos 2(0) + \sin 2(0)) + 2 - \cos(0)$$

$$1.5 = A(\cos 0 + \sin 0) + 2 - \cos 0$$

$$1.5 = A(1 + 0) + 2 - 1$$

$$1.5 = A + 1$$

$$1.5 - 1 = A$$

$$A = 0.5$$

Particular solution

$$y = \frac{1}{2}(\cos 2x + \sin 2x) + 2 - \cos x$$

7. (c) Given x – fraction of those who heard the rumour

now $(1-x)$ is the fraction of those who didn't hear.

7. (c) but rate, $r \propto x(1-x)$

(i) $r = kx(1-x)$

$$r = k(x - x^2)$$

but $r = dx/dt$

$$dx/dt = k(x - x^2)$$

$$\int_{x_0}^{x_f} \frac{dx}{x - x^2} = \int_0^t k\,dt$$

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

$$\therefore \frac{1}{x(1-x)} = \frac{A(1-x) + B(x)}{x(1-x)}$$

$$\therefore 1 = A(1-x) + Bx$$

$$1 + 0x = A + x(B - A)$$

On comparing the two sides

$$A = 1, \quad B = A = 1$$

now $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$

7. (c) $$\int_{x_0}^{x_f} \frac{dx}{x} + \int_{x_0}^{x_f} \frac{dx}{1-x} = \int_0^t k\,dt$$

$$\ln x \Big|_{x_0}^{x_f} + \Big[-\ln(1-x)\Big]_{x_0}^{x_f} = kt \Big|_0^t$$

$$\ln\left(\frac{x_f}{x_0}\right) - \ln\left(\frac{1-x_f}{1-x_0}\right) = kt$$

$$\ln\left(\frac{x_f}{x_0}\right) + \ln\left(\frac{1-x_0}{1-x_f}\right) = kt$$

$$\ln\left[\frac{x_f}{x_0}\left(\frac{1-x_0}{1-x_f}\right)\right] = kt$$

$$\frac{x_f}{x_0}\left(\frac{1-x_0}{1-x_f}\right) = e^{kt}$$

but $x_f = x$

and $x_0 = c$

$$\frac{x}{c}\left(\frac{1-c}{1-x}\right) = e^{kt}$$

$\frac{9}{x}$ $\quad \frac{c}{x}\left(\frac{1-x}{1-c}\right) = e^{-kt}$

$$\frac{c}{x}(1-x) = (1-c)e^{-kt}$$

$$c - cx = x(1-c)e^{-kt}$$

$$c = x\left(c + (1-c)e^{-kt}\right)$$

$$x = \frac{c}{c + (1-c)e^{-kt}}$$ Shown!

7. (c) (ii)

Given $c = 10\% = 0.1$

$t = 3$ hrs

~~x~~ $x = 20\% = 0.2$

from

$$x = \frac{c}{c + (1-c)e^{-kt}}$$

$$0.2 \; x = \frac{0.1}{0.1 + (1-0.1)e^{-k(3)}}$$

$$0.2(0.1 + 0.9e^{-3k}) = 0.1$$

$$0.1 + 0.9e^{-3k} = \frac{0.1}{0.2}$$

$$e^{-3k} = \frac{\left(\frac{0.1}{0.2} - 0.1\right)}{0.9}$$

$$e^{-3k} = \frac{4}{9}$$

$$e^{3k} = \frac{9}{4}$$

$$3k = \ln(9/4)$$

$$k = \frac{1}{3}\ln 9/4$$

x at at 6:00 pm

$$x = \frac{c}{c + (1-c)e^{-kt}}$$

7. (c) (ii) where $t = 3$

$c = 20\% = 0.2$

$$x = \frac{0.2}{0.2 + (1-0.2)e^{-\frac{3}{3}\ln 9/4}}$$

$x = 0.36$

36% will have hear the rumour by 6:00 pm

2018 PAST PAPERS - 2

7. (a) Form the differential equation by eliminating arbitrary constants, in the equation $Ax^2 + By^2 = 1$.

(b) Solve $(1+y^2)dx = (\tan^{-1} y - x)dy$.

(c) The rate of decrease of the temperature of a body is proportional to the difference between the temperature of the body and that of the surrounding air. If water at temperature 100°C cools in 20 minutes to 78°C in a room of temperature 25°C, find the temperature of water after 30 minutes correctly to two decimal places.

(d) Find the general equation for the equation $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 2\sin x$ given that $y = 1$, $\frac{dy}{dx} = 0$ when $x = 0$.

7(c). Given that.

Rate of temperature fall $\propto$ excess temp.

or $$-\frac{d\theta}{dt} \propto (\theta - \theta_s)$$

or $$-\frac{d\theta}{dt} = k(\theta - \theta_s)$$

θ_s = Surrounding air temperature

k = constant.

$$\frac{d\theta}{\theta - \theta_s} = -k\,dt$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\theta - \theta_s} = -k\int_0^t dt$$

$$\ln\left(\frac{\theta - \theta_s}{\theta_0 - \theta_s}\right) = -kt.$$

$$k = -\frac{1}{t}\ln\left(\frac{\theta - \theta_s}{\theta_0 - \theta_s}\right).$$

Given that, $\theta_0 = 100°C$, $t = 20$ min

$\theta = 78°C$, $\theta_s = 25°C$

$$k = \frac{1}{20}\ln\left(\frac{78 - 25}{100 - 25}\right)$$

$$k = -\frac{1}{20}\ln\left(53/75\right) \quad \text{——— (1)}$$

7(c). $\ln\left(\frac{\theta-\theta_s}{\theta_0-\theta}\right) = -kt$

when, $t = 30$ min $\theta = ?$,

$$\theta = \theta_s + (\theta_0 - \theta_s)e^{-kt}$$

$$\theta = 25 + (\theta_0 - 25)e^{-kt}$$

30 minutes after the water was initially at $\theta_0 = 100°c$,

$$\theta = 25 + (100-25)e^{+(\frac{1}{20}\ln(\frac{50}{75})\times 30)}$$

$$\theta = \underline{69.55°C}$$

Therefore, 30 minutes after the water was originally at 100°c, it cooled to $\underline{\underline{69.55°c}}$.

7(d) Given $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 2\sin x$

AUXILLARY QUADRATIC EQUATION

A.E,

$m^2 - 7m + 6 = 0$

$m = 6$ or $m = 1$

Hence the Complementary function solution is

$$y_c = Ae^{6x} + Be^{x} \text{ ——— (i)}$$

For particular integral,

let $y_p = A\sin x + B\cos x$.

A and B = Constants,

$y_p' = A\cos x - B\sin x$

$y_p'' = -A\sin x - B\cos x$.

Insertions into D.E.

$-A\sin x - B\cos x - 7A\cos x + 7B\sin x + 6A\sin x + 6B\cos x = 2\sin x$.

$(-A + 7B + 6A)\sin x + (-B - 7A + 6B)\cos x = 2\sin x$

$(5A + 7B)\sin x + (-7A + 5B)\cos x = 2\sin x$.

Equating corresponding coefficients

$5A + 7B = 2$

$-7A + 5B = 0$.

Hence $A = \frac{5}{37}$ $B = \frac{7}{37}$.

$$y_p = \frac{5}{37}\sin x + \frac{7}{37}\cos x \text{ ——— (ii)}$$

Hence the General solution is

$y = y_c + y_p$

$y = Ae^{6x} + Be^{x} + \frac{5}{37}\sin x + \frac{7}{37}\cos x.$

7(d) When $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$

$1 = A + B + \frac{7}{37}$

$A + B = \frac{30}{37}$ ——— (iii).

$\frac{dy}{dx} = 6Ae^{6x} + Be^{x} + \frac{5}{37}\cos x - \frac{7}{37}\sin x$

Eqn x 2.

$$0 = 6A + B + 5/37 .$$

$$6A + B = -5/37 \quad \text{—(iv)}$$

Solving (iii) and (iv)

$$A = -7/37 \qquad B = 1.$$

Hence the solution is.

$$y = -\frac{7}{37}e^{6x} + e^{x} + \frac{5}{37}\sin x + \frac{7}{37}\cos x$$

solution is

$$y = -\frac{7}{37}e^{6x} + e^{x} + \frac{5}{37}\sin x + \frac{7}{37}\cos x.$$

2017 PAST PAPERS - 2

7. (a) (i) Solve the differential equation $\frac{r\tan\theta}{a^2 - r^2}\frac{dr}{d\theta} = 1$ given that $r = 0$ when $\theta = \frac{\pi}{4}$.

(ii) Verify that $y = 10\sin 3x + 9\cos 3x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 9y = 0$ if $y = 0, \frac{dy}{dx} = 0$ when $x = 0$.

(b) The population of a certain country doubles in 15 years. In how many years will it be six times under the assumption that the rate of increase is proportional to the number of inhabitants?

(c) Find the particular solution of the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cos x$.

(d) Form a differential equation whose general solution is $y = Ae^{mx} + Be^{-mx}$ where A, B and m are constants.

(20 marks)

7 c $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = \cos x$

$y_{C.F} = m^2 + 3m + 2 = 0$

$m_1 = -1$

$m_2 = -2$

$y_{C.F} = Ae^{m_1 x} + Be^{m_2 x}$

$y_{C.F} = Ae^{-x} + Be^{-2x}$

$y_{P.I} = A\cos x + B\sin x$

$y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$-A\cos x - B\sin x + 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = \cos x$

$-A\cos x - B\sin x - 3A\sin x + 3B\cos x + 2A\cos x + 2B\sin x = \cos x$

$-A + 3B + 2A = 1.$

$-B - 3A + 2B = 0$

$A + 3B = 1.$

$-3A + B = 0$

$A = 0.1$

$B = 0.3$

$y_{P.I} = \frac{1}{10}\cos x + \frac{3}{10}\sin x$

$y = y_{C.F} + y_{P.I}$

$y = Ae^{-x} + Be^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x$

Extract 17.1, shows how one of the candidates' solved the differential equation in part (c) correctly.

2016 PAST PAPERS - 2

7. (a) (i) If $x(1-y)\frac{dy}{dx}+2y=0$ and $y=2$ when $x=e$, show that $x^2ye^{-y}=2$.

(ii) Solve the differential equation $(2x-y)\frac{dy}{dx}=2x-y+2$ given that $y=1$ when $x=2$.

(b) Form a differential equation whose solution is the function $y=Ae^{2x}+Be^{-3x}$ where A and B are arbitrary constants.

(c) A tank contains a solution of salt in water. Initially the tank contains 1000 litres of water with 10 kg of salt dissolved in it. The mixture is poured off at a rate of 20 litres per minute, and simultaneously pure water is added at a rate of 20litres per minute. All the time the tank is stirred to keep the mixture uniform.

(i) Find the mass of the salt in the tank after 5 minutes.

(ii) How long the mass of the salt in the tank falls to 5kg?

(d) Find the general solution of the differential equation $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=10e^{-2x}$.

(d) given :

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10e^{-2x}$$

For C.F ,

taking the a.q.e.

$$m^2 - 4m + 3 = 0$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

07. (a) From

$y_{PI} = ke^{-2x}$

$y_{PI} = \frac{2}{3}e^{-2x}$ - - - - - - - - (ii)

Thus;

$y_{soln} = y_{CF} + y_{PI}$

$y = Ae^{3x} + Be^{x} + \frac{2}{3}e^{-2x}$

Hence;

The general solution of the given D.E is

$y = Ae^{3x} + Be^{x} + \frac{2}{3}e^{-2x}$

In Extract 17.1, the candidate worked out correctly the complementary function and the particular solution and added them together to obtain the required general solution.

2015 PAST PAPERS - 2

7. (a) (i) Form the first order differential equation which represent the family of the curve $x^2 + y^2 - 2kx = 0$.

(ii) Find the particular solution of the differential equation $x\frac{dy}{dx} = x + y$, given that $y = -1$ when $x = 1$.

(b) Solve the initial value problem $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$, $y(0) = 4$ and $\frac{dy}{dx} = -5$ when $x = 0$.

(c) The rate of decrease of temperature of water is direct proportional to the difference between temperature of water and that of the medium. If water at a temperature of $100C^\circ$ cools in 10 minutes to $80C^\circ$ in a room temperature of $25C^\circ$ find:

(i) temperature of water after 20 minutes.

(ii) the time when the temperature is $40C^\circ$.

(Any approximation in calculations must be presented in 5 decimal places)

(d) If $\frac{dy}{dx} = \frac{x^2y}{x^3+1}$, find the equation of the solution curve which passes through (2, 3).

(20 marks)

7a)i) $x^2 + y^2 - 2kx = 0$

$2x + 2y\,\frac{dy}{dx} - 2k = 0$

$2y\,\frac{dy}{dx} = 2k - 2x$

$\frac{dy}{dx} = \frac{k - x}{y}$

but $k = \frac{x^2+y^2}{2x}$

$\frac{dy}{dx} = \frac{\frac{x^2+y^2}{2x} - x}{y}$

$y\,\frac{dy}{dx} = \frac{x^2+y^2-2x^2}{2x}$

7a)i) $y\,\frac{dy}{dx} = \frac{y^2-x^2}{2x}$

$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$

7a)ii) $x\frac{dy}{dx} = x + y$

$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x}$

let $u = y/x$

$y = ux$

$\frac{dy}{dx} = u + x\,\frac{dy}{dx}$

$u + x\frac{dy}{dx} = 1 + u$

$x\,\frac{dy}{dx} = 1$

$\int du = \int \frac{dx}{x}$

$\frac{y}{x} = \ln x + c$

$\cancel{\frac{y}{x}} = \ln x + c$

$y = x\ln x + cx$

$x = 1,\ y = -1$

$-1 = 1(\ln 1) + (+1)c$

$-1 = c$

$\therefore$ The particular solution is $\underline{\underline{y = x\ln x - x}}$

7b). $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = xe^x$

$m^2 - 2m + 1 = 0$

$m = \frac{2 \pm \sqrt{4-4}}{2}$

$m = \frac{2 \pm 0}{2}$

$m = 1$

$y_c = e^x(Ax + B)$

Let $y_p = ax^3e^x$

$y'_p = ax^3e^x + ae^x(3x^2)$

$= ax^3e^x + 3ax^2e^x$

7b). $y''_p = 3ax^2e^x + ax^3 + 3ax^2e^x + 3ae^x(2x)$

$= 3ax^2e^x + ax^3 + 3ax^2e^x + 6axe^x$

$= 6ax^2e^x + ax^3 + 6axe^x$

$\cancel{6ax^2e^x} + ax^3 + 6axe^x - 2ax^3e^x - \cancel{6ax^2e^x} + ax^3e^x = xe^x$

$6ax = x$

$6a = 1$

$a = 1/6$

$\therefore y_p = ax^3e^x$

$= \frac{1}{6}x^3e^x$

$\therefore$ The ~~po~~ general solution is $\underline{y = e^x(Ax + B)} + \frac{1}{6}x^3e^x$

$y = e^x(Ax + B) + \frac{1}{6}x^3e^x$

$y(0) = 4$

$4 = B$

$dy/dx = e^x(A) + e^x(Ax+B) + \frac{1}{6}x^3e^x + \frac{3}{6}x^2e^x$

$-5 = A + B$

$-5 = A + 4$

$A = -5 - 4$

$A = -9$

$\therefore$ the particular solution is

$$\underline{y = e^x(-9x + 4) + \frac{1}{6}x^3e^x}$$

7a) $-\frac{d\theta}{dt} \propto \theta - \theta_e$

where θ_e - temperature of the medium

$\frac{d\theta}{dt} = -k(\theta - \theta_e)$

$\int \frac{d\theta}{\theta - \theta_e} = \int -k \, dt$

$\ln|\theta - \theta_e| = -kt + c$

$\theta - \theta_e = Ae^{-kt}$

at $t = 0$, $\theta = 100°C$, $\theta_e = 25°C$

$100 - 25 = Ae^0$

$75 = A$

$\theta - \theta_e = 75e^{-kt}$

At $t = 10$, $\theta = 80$

$80 - 25 = 75e^{-10k}$

$\frac{55}{75} = e^{-10k}$

$-10k = \ln(55/75)$

$k = -\frac{1}{10}\ln(55/75)$

At $t = 20$, $\theta = ?$

$\theta - 25 = 75e^{-(-\frac{1}{10}\ln(55/75) \times 20)}$

$\theta - 25 = 75e^{2\ln(55/75)}$

i) $\underline{\theta = 65.33333 \ °C}$

ii) At $t = ?$ θ

$\theta - \theta_e = 75e^{-kt}$

$40 - 25 = 75e^{-(-\frac{1}{10}\ln(55/75) \times t)}$

$\frac{1}{5} = e^{-t \times 0.031015492}$

$\ln 1/5 = -0.031015492t$

$t = 51.89142$

$\therefore$ The time will be 51.89142 minutes

7d) $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3+1}$

$(x^3+1)\,dy = x^2 y\,dx.$

$\displaystyle\int \frac{dy}{y} = \int \frac{x^2}{x^3+1}\,dx$

$\displaystyle\int \frac{dy}{y} = \int \frac{x^2}{t} \cdot \frac{dt}{3x^2}$ let $x^3+1 = t$, $3x^2\,dx = dt$

$\displaystyle\int \frac{dy}{y} = \frac{1}{3}\int \frac{1}{t}\,dt$

$\ln y = \frac{1}{3}\ln t + c$

$\ln y = \frac{1}{3}\ln(x^3+1) + c$

$\ln y = \frac{1}{3}\ln(x^3+1) + c$

$3\ln y = \ln A(x^3+1)$

$y^3 = A(x^3+1)$

at $\overset{x}{(2},\overset{y}{3)}$

$27 = A(9)$

$A = 3.$

$\therefore$ The ~~ge~~ particular solution is $y^3 = 3(x^3+1)$

Extract 17.1 illustrates responses of a candidate who answered the question correctly. He/she used the basic concepts of differential equations to formulate and solve the given equations.

18.0 Numerical Methods

2021 PAST PAPERS

7. (a) Use the trapezium rule with 5 ordinates to find an approximate value for $\int_0^1 \frac{2}{1+x^2}dx$ correctly to 4 decimal places.

(b) Use Simpson's rule with 5 ordinates to find an approximation for $\int_0^1 \frac{2}{1+x^2}dx$ correct to four decimal places.

(c) Find the value of the integral $\int_0^1 \frac{2}{1+x^2}dx$ correct to four decimal places.

(d) Compare the actual value in part (c) with the approximated values obtained in part (a) and (b).

7 (a) Given.

$$\int_0^1 \frac{2}{1+x^2}dx$$

For 5 ordinates = 4 strips

$$h = \frac{1-0}{4} = 0.25.$$

x	0	0·25	0·5	0·75	1
$y=\frac{2}{1+x^2}$	2	1·88235	1·6	1·28	1
Ordinates	y_0	y_1	y_2	y_3	y_n

From Trapezium Rule.

$$\int_0^1 \frac{2}{1+x^2}dx = \frac{h}{2}\left[y_0+y_n+2\Sigma\text{all ordinates}\right].$$

$$= \frac{0.25}{2}\left[2+1+2(1.88235+1.6+1.28)\right].$$

$$= 1.5656.$$

(b) $\int_0^1 \frac{2}{1+x^2}dx$,

For 5 ordinates, Strips = 4.

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x	0	0·25	0·5	0·75	1
$y=\frac{2}{1+x^2}$	2	1·88235	1·6	1·28	1
Ordinates	y_0	y_1	y_2	y_3	y_n

For Trapezium rule:

$$\int_0^1 \frac{2}{1+x^2}dx = \frac{h}{3}\left[y_0+y_n+2\Sigma\text{even ordinates}+4\Sigma\text{odd ordinates}\right]$$

use only

7 (b) $$\int_0^1 \frac{2}{1+x^2}dx = \frac{0.25}{3}\left[2+1+2(1.6)+4(1.88235+1.28)\right],$$

$$= 1.5708$$

$\therefore \int_0^1 \frac{2}{1+x^2} dx = 1.5708$.

(c) $\int_0^1 \frac{2}{1+x^2} dx$.

$= 2\int_0^1 \frac{1}{1+x^2} dx$.

let $1+x^2 = 1+\tan^2\theta$.

$x = \tan\theta$.

$\frac{dx}{d\theta} = \sec^2\theta$.

$dx = \sec^2\theta \, d\theta$.

$= 2\int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{1+\tan^2\theta}$

$= 2\int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{\sec^2\theta}$

$= 2\theta \Big]_0^{\pi/4}$.

$= \frac{\pi}{2} - 0$.

$= 1.5708$.

$\therefore \int_0^1 \frac{2\,dx}{1+x^2} = 1.5708$.

7 (d) Comparison.

Between the actual value with approximated value from Trapezium rule.

$\Delta y = |y - y'|$

$= |1.5708 - 1.5656|$

$= 0.0052$.

Between actual value with approximated value from Simpson rule.

$\Delta y = |y - y'|$

$\Delta y = |1.5708 - 1.5708|$.

	$\Delta y = 0$.	
	$\therefore$ This implies that the value obtained by Simpson's	
	rule in part (b) is more accurate than the one	
	obtained in part (a) by trapezium rule.	

2020 PAST PAPERS

7. (a) By using the trapezium rule with 5 ordinates, find an approximate value for $\int_0^4 x\sqrt{9+x^2}\,dx$ correct to three decimal places.

(b) Use Simpson's rule with 5 ordinates to find an approximation for $\int_0^4 x\sqrt{9+x^2}\,dx$ correct to three decimal places.

(c) Find the value of integral $\int_0^4 x\sqrt{9+x^2}\,dx$.

(d) Which of the two methods in (a) and (b) gives a better approximation of $\int_0^4 x\sqrt{9+x^2}\,dx$.

7a	x	0	1	2	3	4
	$y = x\sqrt{9+x^2}$	0	3.16228	7.21110	12.72792	20.
		y_1	y_2	y_3	y_4	y_5

$$A = d/2\left(y_1 + y_5 + 2(y_2 + y_3 + y_4)\right).$$

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{2}\left((0+20) + 2(3.16228 + 7.21110 + 12.72792)\right)$$

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{2}\left(20 + 46.2026\right).$$

$$\int_0^4 x\sqrt{9+x^2}\,dx = 33.1013$$

$$\int_0^4 x\sqrt{9+x^2}\,dx. = 33.101$$

7b Simpson's rule states that

$$A = d/3\left(y_1 + y_n + 4(\text{even ordinates}) + 2(\text{odd ordinates})\right).$$

$$A = d/3\left(y_1 + y_5 + 4(y_2 + y_4) + 2y_3\right)$$

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{3}\left((0+20) + 4(3.16228 + 12.72792) + 2(7.21110)\right)$$

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{3}\left(20 + 63.5608 + 14.4222\right)$$

$$\int_0^4 x\sqrt{9+x^2}\,dx = 32.661$$

7c.	$\int_0^4 x\sqrt{9+x^2}\,dx$	
	~~let $u = 9+x^2$~~	
	~~$du = 2x\,dx$~~	
	$\frac{du}{2} = x\,dx$	
	$= \int_0^4 (\sqrt{u})\frac{du}{2}$	
	$= \frac{1}{2}\int(\sqrt{u})\,du = \frac{1}{2}\int u^{1/2}\,du$	
7c.	$= \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right) + c$	
	$= \frac{u^{3/2}}{3} + c$	
	But $u = 9+x^2$	
	$= \frac{1}{3}(9+x^2)^{3/2}\Big]_0^4$	
	$= \frac{1}{3}\left[(25)^{3/2} - (9)^{3/2}\right]$	
	$= \frac{1}{3}(125 - 27)$	
	$\doteq 32.667$	
	$\int_0^4 x\sqrt{9+x^2}\,dx = 32.667$	
7d.	Absolute error $= \lvert$ True value $-$ Measured value $\rvert$	
	Absolute error involved in using Trapezium rule	
	Absolute error $= \lvert 32.667 - 33.101 \rvert = 0.434$.	
	In Simpson's rule	
	Absolute error $= \lvert 32.667 - 32.661 \rvert = 0.006$	
	Since $0.006 < 0.434$.	
	Hence Simpson's rule gives better approximation	

2019 PAST PAPERS

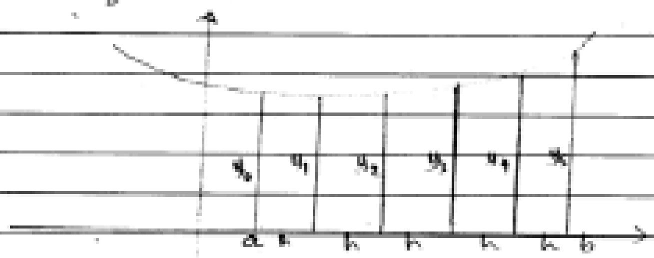
7. (a) The value $A = \int_a^b f(x)\,dx$ represents the area under the graph of $y = f(x)$ between $x = a$ and $x = b$. Derive the trapezium rule with 6 ordinates to find an approximation of $A = \int_a^b f(x)\,dx$.

(b) Using the trapezium rule obtained in (a) (ii), approximate $\int_1^7 \frac{x^3}{1+x^4}\,dx$, correct to three decimal places.

(c) Evaluate the actual integral of $\int_1^7 \frac{x^3}{1+x^4}\,dx$ and then calculate the relative error in the approximation obtained in (b). (Give your answers correct to three decimal places)

Deriving trapezium rule.

7 (a).



6 ordinates are $y_0 + y_1 + y_2 + y_3 + y_4 + y_5$

to find area of trapezium

$A_1 = \frac{1}{2}(y_0 + y_1)h.$

$A_2 = \frac{1}{2}(y_1 + y_2)h.$

$A_3 = \frac{1}{2}(y_2 + y_3)h.$

$A_4 = \frac{1}{2}(y_3 + y_4)h.$

$A_5 = \frac{1}{2}(y_4 + y_5)h.$

$A = A_1 + A_2 + A_3 + A_4 + A_5.$

$A = \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + y_3 + y_4 + y_4 + y_5).$

$A = \frac{h}{2}(y_0 + y_5 + 2\sum \text{remaining ordinates}).$

7 (b). $\int_1^7 \frac{x^3}{1+x^4}\,dx.$

$A = \int_1^7 \frac{x^3}{1+x^4}\,dx = \frac{h}{2}(y_0 + y_5 + 2\sum \text{remaining ordinates})$

$$h = \frac{b-a}{n} = \frac{7-1}{5} = \frac{6}{5}$$

x	1	2.2	3.4	4.6	5.8	7
y	0.5	0.4525935	0.2894183	0.2160909	0.1727762	0.142798
	y_0	y_1	y_2	y_3	y_4	y_5

$$\int_1^7 \frac{x^3}{1+x^4}\,dx = \frac{1.2}{2}\left(0.5 + 0.142798 + 2(0.4525936 + 0.2894187 + 0.2160907 + 0.1727769)\right)$$

$$\int_1^7 \frac{x^3}{1+x^4}\,dx = \underline{1.726}$$

7c

$$\int_1^7 \frac{x^3}{1+x^4}\,dx$$

consider.

$$\int \frac{x^3}{1+x^4}\,dx.$$

Let $u = 1+x^4$.

$du = 4x^3\,dx$

$$\int \frac{x^3}{u} \cdot \frac{du}{4x^3}.$$

$$\frac{1}{4}\int \frac{du}{u}$$

$$\frac{1}{4}\ln u. + c.$$

$$\frac{1}{4}\ln(1+x^4)\Big|_1^7$$

$$\frac{\ln(2402)}{4} - \frac{\ln 2}{4}$$

$= 1.77273$

actual value is 1.773.

$$\text{relative error} = \frac{\text{Absolute error}}{\text{True value.}}$$

$$\text{relative error} = \frac{|1.773 - 1.726|}{1.773}$$

$\therefore$ Relative error is 0.027.

2018 PAST PAPERS

7. (a) Approximate the value of $\int_3^7 \frac{1}{x-2}dx$ correct to four decimal places by using;

(i) The trapezoidal rule with five ordinates and

(ii) The Simpson's rule with five ordinates.

(b) Evaluate the exact value of the integral $\int_3^7 \frac{1}{x-2}dx$ and compare your answer with those found in part (a).

7(a).

x	3	4	5	6	7.
y.	1	0.5	0.3333	0.25	0.2
	y_1	y_2	y_3	y_4	y_n.

a (i) By Trapezoidal rule.

$$A = \frac{d.}{2}\left(y_1 + y_n + 2\sum \text{remaining ordinates}\right)$$

$$d = \frac{7-3.}{\text{number of strips} = 4} = \frac{4.}{4} = 1.$$

$d = 1.$

$$A = \frac{1.}{2}\left(1 + 0.2 + 2\left(0.5 + 0.3333 + 0.25\right)\right)$$

$$A = \frac{1}{2}\left(3.3666\right) =$$

$A = 1.6833.$

$$\therefore \int_3^{7.} \frac{1.}{x-2} = 1.6833.$$

a (ii) By Simpson. rule.

$$A = \frac{d}{3}\left(y_1 + y_n + 2\sum \text{odd ordinates} + 4\sum \text{even. ordinates}\right)$$

$$A = \frac{1}{3}\left(1 + 0.2 + 2\left(0.3333\right) + 4\left(0.5 + 0.25\right)\right)$$

$A = \frac{1}{3}\left(1.2 + 0.6666 + 3\right).$

$= \frac{1}{3}\left(4.8666\right).$

$= ~~b~~ \; 1.6222.$

$\therefore \int_3^7 \frac{1}{x-2} = 1.6222.$

(b). $\int_3^7 \frac{1}{x-2}\,dx$ let $u = x-2.$ $\frac{du}{dx} = 1.$

$\int_3^7 \frac{1}{u}\,du = \left[\ln u\right]_3^7 = \left[\ln(x-2)\right]_3^7$

$= \left[\ln(5) - \ln 1\right] = 1.6094.$

$\therefore$ when Trapezoidal rule was used. the difference between Exact value and that obtained by Trapezoidal rule was $1.6833 - 1.6094 = 0.0739.$

$\therefore$ When simpson rule was used the difference between Exact value and that obtained by simpson rule was $1.6222 - 1.6094 = 0.0128.$

Since deviation of the value obtained by simpson rule from Exact value is less than the ~~difference~~ deviation of Trapezoidal rule.

Simpson rule is more accurate than Trapezoidal rule.

2017 PAST PAPERS

7. (a) Show that the Newton Raphson Formula of finding the roots of the equation $12x^3 + 4x^2 - 15x - 4 = 0$ is $x_{n+1} = \frac{(24x_n + 4)x_n^2 + 4}{(36x_n + 8)x_n - 15}$ and use this formula to find the roots of $12x^3 + 4x^2 - 15x - 4 = 0$ correct to three decimal places.

(b) Approximate the area under the curve $y = \frac{1}{x-2}$ between $x = 2$ and $x = 3$ with six ordinates by:

(i) Trapezoidal rule,

(ii) Simpson rule.

(c) Which among the rules in 7(b) gives a better approximation to the area?

7(a) Given $12x^3 + 4x^2 - 15x - 4 = 0$

let $f(x_n) = 12x_n^3 + 4x_n^2 - 15x_n - 4$

$f'(x_n) = 36x_n^2 + 8x_n - 15$

from Newton Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{12x_n^3 + 4x_n^2 - 15x_n - 4}{36x_n^2 + 8x_n - 15}\right)$$

7(a) Again let $x_0 = -0.25$

From $x_{n+1} = \frac{(24x_n + 4)x_n^2 + 4}{(36x_n + 8)x_n - 15}$

1st iteration

$$x_1 = \frac{(24(-0.25) + 4)(-0.25)^2 + 4}{(36(-0.25) + 8)(-0.25) - 15}$$

$x_1 = -0.263$

2nd iteration

$$x_2 = \frac{24(-0.263) + 4)(-0.263)^2 + 4}{(36(-0.263) + 8)(-0.263) - 15}$$

$x_2 = -0.263$

$\therefore$ The roots of $12x^3 + 4x^2 - 15x - 4 = 0$ are 1.192, -1.162 and -0.263

In Extract 7.1, the candidate demonstrated a good understanding of how to find the roots of a polynomial by using the Newton-Raphson formula.

2016 PAST PAPERS

7. (a) (i) Write down four sources of errors in numerical computations.

(ii) If x_{n+1} is a better approximation to a root of the equation $f(x_n)=0$. Derive the Newton-Raphson method for the function $f(x_n)$.

(b) Use the Newton-Raphson method obtained in (a) (ii) to derive the secant formula and comment why would you want to use it instead of the Newton-Raphson method.

(c) Using the secant method obtained in (b) with $x_1 = 2$ and $x_2 = 3$ perform three iterations to approximate the root of $x^2 - 2x - 1 = 0$ and hence compute the absolute error correct to four decimal places.

7 c) Soln.

Given:

$x_1 = 2,$

$x_2 = 3$

$x^2 - 2x - 1 = 0$

$f(x) = x^2 - 2x - 1.$

from Secant method;

$$x_{n+2} = x_n - \left(\frac{x_n - x_{n+1}}{f x_n - f x_{n+1}}\right) f x_n.$$

1st iteration, $n = 1$.

$$x_3 = x_1 - \left(\frac{x_1 - x_2}{f x_1 - f x_2}\right) f(x_1)$$

but; $f(x) = x^2 - 2x - 1$

for $x_1 = 2$;

$f(x_1) = 2^2 - 2(2) + 1$

$f(x_1) = 2 \cdot -1$

$\therefore \quad (2, -1)$

for $x_2 = 3$; $\quad f(x_2) = 3^2 - 2(3) + 1$

$f(x_2) = 4 \cdot 2.$

thus;

$$x_3 = x_1 - \left(\frac{x_1 - x_2}{fx_1 - fx_2}\right) f(x_1)$$

$$= 2 \cdot 2 - \left(\frac{2 - 3}{-1 - 2}\right)(-1).$$

$$x_3 = 2.3333.$$

2nd iteration; n = 2.

$$x_4 = x_2 - \left(\frac{x_2 - x_3}{fx_2 - fx_3}\right) f(x_2).$$

$x_2 = 3, \quad f(x_2) = 2.$

$x_3 = 2.333 \quad f(x_3) = -0.2222.$

thus;

$$x_4 = x_2 - \left(\frac{x_2 - x_3}{fx_2 - fx_3}\right) fx_2.$$

$$= 3 - \left(\frac{3 - 2.3333}{2 - {}^-0.2222}\right)(2)$$

$$x_4 = 2.40000.$$

3rd iteration: , n = 3

$$x_5 = x_3 - \left(\frac{x_3 - x_4}{fx_3 - fx_4}\right) fx_3.$$

$x_3 = 2.3333, \quad f(x_3) = -0.2222.$

$x_4 = 2.4 \quad\quad f(x_4) = -0.04.$

$$x_5 = 2.3333 - \left(\frac{2.3333 - 2.4}{-0.2222 - {}^{-}0.04}\right)(-0.2222)$$

$\underline{x_5 = 2.4146}$

$\therefore$ <u>The approximate root is 2.4146.</u>

from; $x^2 - 2x - 1 = 0.$

on solving;

$x = 2.4142.$

$\therefore$ Absolute error $= |2.4142 - 2.4146|.$

$= 0.0004.$

Extract 7.2 shows that the candidate was familiar with the tested concepts of Numerical Methods.

2015 PAST PAPERS

7. (a) Starting with $x_0 = -1$, approximate the root of $f(x) = x + e^x$ in four iterations using the Newton-Raphson method. All your iterations should be presented in five significant figures.

(b) (i) Apply both Simpson's and Trapezium rule with eleven ordinates to find an approximate value of $\int_0^1 \sin(1 + \sqrt{x})dx$. Give your answers correct to four decimal places.

(ii) Why does Simpson's rule said to be more efficient than trapezium?

7a) N-R's formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = x + e^x$$

$$f(x_n) = x_n + e^{x_n}$$

$$f'(x_n) = 1 + e^{x_n}$$

$$x_{n+1} = x_n - \left(\frac{x_n + e^{x_n}}{1 + e^{x_n}}\right)$$

$$= \frac{x_n + x_n e^{x_n} - x_n - e^{x_n}}{1 + e^{x_n}}$$

$$x_{n+1} = \frac{(x_n - 1)e^{x_n}}{1 + e^{x_n}}$$

7a) When $x_0 = -1$

for 1st iteration $n = 0$

$$x_1 = \frac{(x_0 - 1)e^{x_0}}{1 + e^{x_0}}$$

$$= \frac{(-1-1)e^{-1}}{1 + e^{-1}}$$

$$\therefore x_1 = -0.53788$$

2nd iteration $n = 1$

$$x_2 = \frac{(x_1 - 1)e^{x_1}}{1 + e^{x_1}}$$

$$x_2 = \frac{(-0.53788 - 1)e^{-0.53788}}{1 + e^{-0.53788}}$$

$$x_2 = -0.56699$$

3rd iteration $n = 2$

$$x_3 = \frac{(x_2 - 1)e^{x_2}}{1 + e^{x_2}}$$

$$= \frac{(-0.56699 - 1)e^{-0.56699}}{1 + e^{-0.56699}}$$

$$x_3 = -0.56714$$

4th iteration $n=3$

$$x_4 = \frac{(x_3 - 1)e^{x_3}}{1 + e^{x_3}}$$

$$= \frac{(-0.56714 - 1)e^{-0.56714}}{1 + e^{-0.56714}}$$

$$x_4 = \underline{-0.56714}$$

$\therefore$ approximate root of $f(x) = x + e^x$

$= -0.56714$

7b)

i) $N = 11$

$n = 11 - 1$

$n = 10$

$a = 0,\ b = 2$

$$h = \frac{b-a}{n}$$

$$= \frac{2-0}{10}$$

$$= 2/10$$

$\therefore h = 0.2$

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$y = \sin(1+\sqrt{x^2})$	0.8415	0.9924	0.9981	0.9793	0.9481	0.9093	0.8655	0.8183

1.6	1.8	2.0
0.7686	0.7173	0.6649
y_8	y_9	y_n

Using Simpson's rule

$$\int f(x) = \frac{h}{3}\left(y_0 + y_n + 4\Sigma odd + 2\Sigma even \right)$$

$$= \frac{h}{3}\left(y_0 + y_n + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8) \right)$$

$$= \frac{0.2}{3}\left(0.8415 + 0.6649 + 4(0.9924 + 0.9793 + 0.9093 + 0.8183 + 0.7173) + 2(0.9981 + 0.9481 + 0.8655 + 0.7686)\right)$$

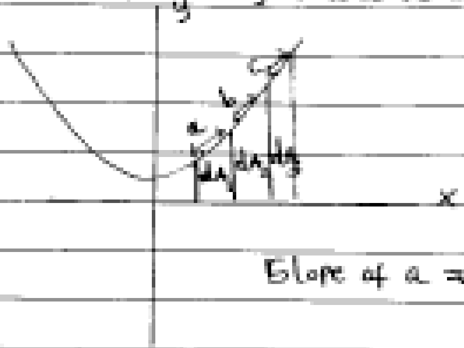
$= 1.75556$

$= 1.7556$ (4 d.p)

Using Trapezoidal

$$\int f(x) = \frac{h}{2}\left(y_0 + y_n + 2\Sigma \text{ other ordinates} \right)$$

	$= \frac{0.2}{2}(0.8415 + 0.6649 + 2(0.9924 + 0.9981$
	$+ 0.9793 + 0.9481 + 0.9093 + 0.8655 + 0.8$

7b) i	$+ 0.7486 + 0.7173))$
	$\int f(x) = 1.75002.$
	$= 1.7500$ (4 d.p)
(ii)	Because it uses more divisions of the interval ordinates and since the functions are curves the closer the intervals the more accurate the area below it. (as gradient of straight line is more accurate)
	
	Slope of a ≈ b ≈ c
	∴ the addition becomes more accurate as a straight line is found between 2 closer points.

Extract 7.2 shows a sample response from a script of a candidate who scored full marks. This candidate demonstrated competence and high level of skills in the topic of Numerical Methods.