PiXL INTERNATIONAL

Advanced Maths ACSEE

Past Paper Questions and Answers by Topic

Contents

Form V

1.0 Calculating Devices

2021 PAST PAPERS

1. (a) Use a non-programmable scientific calculator to compute:

(i) the value of $\sqrt[x]{\ln(\log x)^4 + e^{5x}\sin 3x}$ correct to 7 decimal places, for $x = 0.2$ radian.

(ii) the derivative of $\left(x^2 - x + 4\right)^6$ at $x = 5$.

(iii) the modulus of $\dfrac{(4+3i)\times(3+4i)}{(3+i)}$ in radian. (Give your answers to four significant figures).

$\dfrac{12+16i+9i+16i^2}{3+i}$ 12

(b) The weight x of 10 insects in mg are 1.20, 0.04, 1.40, 0.04, 0.716, 0.17, 1.20, 1.20, 2.40 and 3.00 respectively. Use a non-programmable calculator to compute $\sum x^2$ and δx correct to three decimal places.

1. (a) (i) $\sqrt[x]{\ln(\log x)^4 + e^{5x}\sin 3x} = 0.0000112$.

(ii) 429,981,696.

(iii) 7.906

(b) the Value of $\Sigma x^2 = 21.585$

the Value of $\delta x = 0.931$

2020 PAST PAPERS

1. (a) Use a non-programmable calculator to evaluate:

(i) $\dfrac{\tan 25^0 30' - \sqrt[5]{0.03e^{-3}}}{\ln 3.2 + 0.006e^{0.3}}$ correct to six significant figures.

(ii) $\sum_{n=4}^{7} \dfrac{2^{-n}(n!)}{\ln(0.3n)}$ correct to 3 decimal places.

(b) The population of Dar es Salaam city is modeled by the equation $P(t) = P_0 e^{\lambda t}$ where $\lambda = 0.034657$ per year. Use a non-programmable calculator to find the time t in years when the population in the city is three times the initial population P_0.

1a)	i. 1.74857×10^{-1}	
	ii. 89.686	
1b)	$P = P_0 e^{\lambda t}$	
	$3P_0 = P_0 e^{\lambda t}$	
	$3 = e^{\lambda t}$	
	Apply ln both sides	
	$\ln 3 = \ln e^{\lambda t}$	
	$\ln 3 = \lambda t \ln e$	
	$\ln e = 1$	
	$\ln 3 = \lambda t$	
	$t = \frac{\ln 3}{\lambda}$ given $\lambda = 0.034657$	
	$t = 31.7$ years	
	The time is 31.7 years.	

2019 PAST PAPERS

1. (a) By using a non-programmable calculator:

(i) Calculate $\log_e\left(e^4 + 2\ln 5\right) + \log 5$ (Give your answer correct to six decimal places).

(ii) Obtain the value of $\sqrt{\frac{(4.03)^3 \times (814765)^{0.5}}{\sqrt{5}}}$ correct to three significant figures.

(b) The monthly salaries in Tanzanian shillings for 20 employees of KNCU are 260,000.00, 170,000.00, 85,000.00, 505,000.00, 129,000.00, 89,000.00, 220,000.00, 157,000.00, 103,000.00, 480,000.00, 790,000.00, 600,000.00, 340,000.00, 144,000.00, 128,000.00, 90,000.00, 102,000.00, 185,000.00, 219,000.00 and 195,000.00. Use the statistical functions of the scientific calculator to calculate:

(i) the mean ($\bar{x}$) and

(ii) the standard deviation ($x\delta n$).

1	a) (i) 4.756253302 $\simeq$ 4.756253 (6 dps).
	(ii) 162.5448295 $\simeq$ 163 (3 sfs).
	b) (i) Mean ($\bar{x}$) = 249,550 Tshs. or Tsh 249,550.
	(ii) Standard Deviation = Tshs 190,709.3272.

2018 PAST PAPERS

1. (a) By using a non-programmable calculator , evaluate the following expressions correct to four decimal places:

(i) $\left(\sqrt{\dfrac{(8.621)(27.34)}{52.18+0.0724}}\right)^{\frac{3}{5}}$ and (ii) $\sum_{x=0}^{2} xe^{x}\log(x+1)^{\frac{1}{3}}$.

(b) Given that $a=14.2,\ b=12.6,\ c=8.4,\ T=(s(s-a)(s-b)(s-c))^{\frac{1}{2}}$ and $2s=a+b+c$. Use a non-programmable calculator to find the value of T correctly to four decimal places.

01 a) i) $\left(\sqrt{\dfrac{(8.621)(27.34)}{(52.18+0.0724)}}\right)^{3/5}$

$=\left(\sqrt{\dfrac{235.69814}{52.2524}}\right)^{3/5}$

$=\left(\sqrt{4.510761994}\right)^{3/5}$

$=(2.123855455)^{3/5}$

$=\underline{1.5714}$.

$\therefore \left(\sqrt{\dfrac{(8.621)(27.34)}{52.18+0.0724}}\right)^{3/5} = \underline{1.5714}$.

ii) $\sum_{x=0}^{2} xe^{x}\log(x+1)^{1/3} = 0e^{0}\log(0+1)^{1/3} + 1e^{1}\log(1+1)^{1/3} + 2e^{2}\log(2+1)^{1/3}$.

$= 0 + 2.718281828(0.100343331) +$

$= 0 + 0.272761455 + 2.350317145$

$= 2.6231$

$\therefore \sum_{x=0}^{2} xe^{x}\log(x+1)^{1/3} = \underline{2.6231}$.

b) $T=(s(s-a)(s-b)(s-c))^{1/2}$

$a=14.2,\ b=12.6,\ c=8.4,\ 2s=a+b+c$ that is $s=\frac{a+b+c}{2}$.

$T=((17.6)(17.6-14.2)(17.6-12.6)(17.6-8.4))^{1/2}$

$T=(17.6(3.4)(5)(9.2))^{1/2}$.

$T=(2,752.64)^{1/2}$

$\therefore$ The value of $T = \underline{52.4656}$.

2017 PAST PAPERS

1\. (a) By using a scientific calculator compute:

(i) $\dfrac{\sqrt{240}\times e^{\ln\frac{1}{3}}\sin 22^\circ}{\sqrt{\tan 17^\circ}\times 3^{4\ln 11}}$ correct to 3 significant figures,

(ii) $\ln\dfrac{\sqrt{98.2}\times(0.0076)^{-1}\times 10^{7}}{\tan\dfrac{\pi}{3}\times\cos^{3}\dfrac{\pi}{4}}$ correct to 6 significant figures,

(iii) $\sqrt{\dfrac{(0.485)^{e}+\tan^{-1}(1.54)e}{(62.54)^{4}\sin^{-1}(0.4561)}}$ correct to 4 decimal places.

(b) If $M^{d}=\dfrac{p}{\pi t^{2}}\left[\dfrac{4}{3}\ln\left(\dfrac{D}{d}\right)+\sqrt{\log p}\right]^{\frac{1}{3}}$, with the aid of a non programmable calculator evaluate D given that $P=1.6\times10^{3}$, $t=56\times10^{-2}$, $M=50.6\times10^{2}$ and $d=\lim\limits_{x\to\infty}\left(\dfrac{\cosh x}{e^{x}}\right)$ correct to four decimal places.

1 a) i) 0.0000928

ii) 23.7816.

iii) 0.0006

b). $M^{d}=\dfrac{P}{\pi t^{2}}\left(\dfrac{4}{3}\ln\left(\dfrac{D}{d}\right)+\sqrt{\log P}\right)^{1/3}$

$P=1.6\times10^{3}$

$t=56\times10^{-2}$

$M=50.6\times10^{2}$.

$d=\lim\limits_{x\to\infty}\left(\dfrac{\cosh x}{e^{x}}\right)$

$\cancel{ch}$ $\cosh x=\dfrac{e^{x}+e^{-x}}{2}$

$\cosh x=\dfrac{e^{2x}+1}{2e^{x}}$

$\dfrac{\cosh x}{e^{x}}=\dfrac{e^{2x}+1}{2e^{x}}\times\dfrac{1}{e^{x}}=\dfrac{e^{2x}+1}{2e^{2x}}=\dfrac{1}{2}+\dfrac{1}{2e^{2x}}$

$d=\lim\limits_{x\to\infty}\dfrac{1}{2}+\dfrac{1}{2e^{2(\infty)}}$

$=\dfrac{1}{2}+0$

$d=\dfrac{1}{2}$.

$$-\left(\frac{M^d \pi t^2}{P}\right)^3 - \sqrt{\log P'} = \frac{4}{3}\ln\left(\frac{D}{d}\right)$$

$$\frac{3}{4}\left[\left(\frac{M^d \pi t^2}{P}\right)^3 - \sqrt{\log P}\right) = \ln\left(\frac{D}{d}\right)$$

$$\frac{D}{d} = e^{\frac{3}{4}\left[\left(\frac{M^d \pi t^2}{P}\right)^3 - \sqrt{\log P'}\right]}$$

$$D = d \times \left(e^{3/4\left[\left(\frac{M^d \pi t^2}{P}\right)^3 - \sqrt{\log P}\right]}\right)$$

$$D = 0.1306$$

In Extract 1.1, the candidate's work demonstrates a good understanding of how to use a calculator.

2016 PAST PAPERS

1. (a) Using a scientific calculator find the following correct to four decimal places:

   (i) $\dfrac{\sqrt{(3.12\times\log 5)^3}}{\sqrt[3]{\left(\cos\frac{\pi}{9}+\sin 46^\circ\right)}}$.

   (ii) $\left[\dfrac{\sqrt{e^3\log_2 6\times\sinh^{-1}(0.6972)}}{(\ln 3.5)\times(\cos 64.5^\circ)\times(\tan 46^\circ)}\right]\times(0.6467)^3$.

   (b) A rat has a mass 30 grams at birth. It reaches maturity in 3 months. The rate of growth is modeled by the equation $\frac{dm}{dt}=120\,(2.1985t-3)^2$, where m grams is the mass of the rat, t months after birth. Use the scientific calculator to find the mass of the rat when fully grown.

01 (a) (i) 2.7204

(ii) 2.2695

01 (b) Given:

$$\frac{dm}{dt} = 120(2.1985t - 3)^2$$

To get mass of rat after 3 months the above equation is integrated within reasonable limits.

then:

by separating variables

$$dm = 120(2.1985t - 3)^2\,dt$$

applying $\int$ both sides

$$\int_{30}^{m} dm = 120\int_{0}^{3}(2.1985t - 3)^2\,dt$$

Where: m is final mass of rat at maturity.

by using scientific calculator:

$$m\Big|_{30}^{m} = 120(11.1411)$$

$$m - 30 = 1336.9344$$

$$\therefore m = 30 + 1336.9344$$

$$= 1366.93449 \simeq 1366.939$$

Extract 1.2 shows how the candidate was able to use a scientific calculator correctly to perform computations.

2015 PAST PAPERS

1. (a) Using a non-programmable calculator:

   (i) Calculate $\log_3\left(e^3+2\ln 5\right)+\log 5$ and write your answer to six decimal places.

   (ii) Obtain the value of $\sqrt{\frac{(4.03)^3\times(814765)^{0.5}}{\sqrt{5}}}$ to three significant figures.

   (iii) Find the value of $\left(\frac{{}^5C_3\times\ln 2}{\sqrt[3]{43}}\right)+\begin{vmatrix}2e & \ln 2\\ e & \ln 2\end{vmatrix}$ to four decimal places.

   (b) Evaluate $\sum_{y=1}^{4} e^{\ln y}\left(1+(y+1)\ln y\right)$ to four significant figures.

1	a) i) 4.756253
	ii) 163
	iii) 5.5917
	b) 22.34

Extract 1.1 shows the work of one of the candidates who was able to use a non-programmable calculator correctly in solving the given items.

2.0 Sets

2021 PAST PAPERS

5. (a) Use the appropriate laws to simplify the expression $(A \cap B') \cup (A' \cap B) \cup (A \cap B)$.

(b) An investigation of eating habits of 110 rabbits revealed that 50 rabbits eat rice, 43 eat maize, 45 eat banana, 12 eat rice and maize, 13 eat maize and banana, 15 eat banana and rice and 5 eat all three types of food. Summarize the given information on a Venn diagram.

(c) Use the Venn diagram obtained in part (b) to find the number of rabbits which eat;

(i) only one type of food,

(ii) banana and maize but not rice,

(iii) none of the food.

05 (a) $(A \cap B') \cup (A' \cap B) \cup (A \cap B)$

$A \cap (B' \cup B) \cup (A' \cap B)$ --- Distributive law

$A \cap (\mathcal{U}) \cup (A' \cap B)$ --- Complement law

$A \cup (A' \cap B)$ --- Identity law

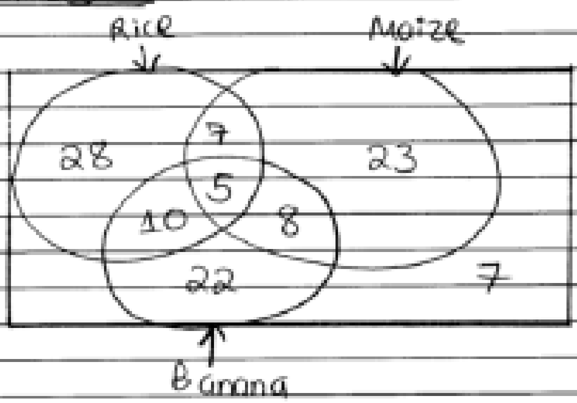
$(A \cup A') \cap (A \cup B)$ --- Distributive law

$\mathcal{U} \cap (A \cup B)$ --- Complement law

$A \cup B$ --- Identity law

$= \underline{A \cup B}$

(b)



(c) (i) 73 rabbits

(ii) 8 rabbits

(iii) 7 rabbits

2020 PAST PAPERS

5. (a) Use the appropriate laws of set to simplify $(A\cup B)'\cap(A\cap B)'$.

(b) The Malya social Training College Cultural group consists of 36 villagers, 25 of them participate in dancing, 28 participate in singing, while 26 among them participate in drama, 19 villagers dance and sing; 18 villagers dance and play drama and 15 participate in all three activities. If each villager participate in at least one of the activities, use Venn diagram to find the number of villagers;

(i) who are either dancing or playing drama,

(ii) who participate in at most two activities and

(iii) who neither play drama nor sing.

5 a) $(A\cup B)'\cap(A\cap B)'$ — given.

$= (A'\cap B')\cap(A'\cup B')$ — DeMorgans law

$= A'\cap(A'\cup B')\cap B'\cap(A'\cup B')$ — Distributive law

$= (A'\cup\phi)\cap(A'\cup B')\cap(B'\cup\phi)\cap(B'\cup A')$ — Identity law

$= A'\cup(\phi\cap B')\cap B'\cup(\phi\cap A')$ — Distributive law

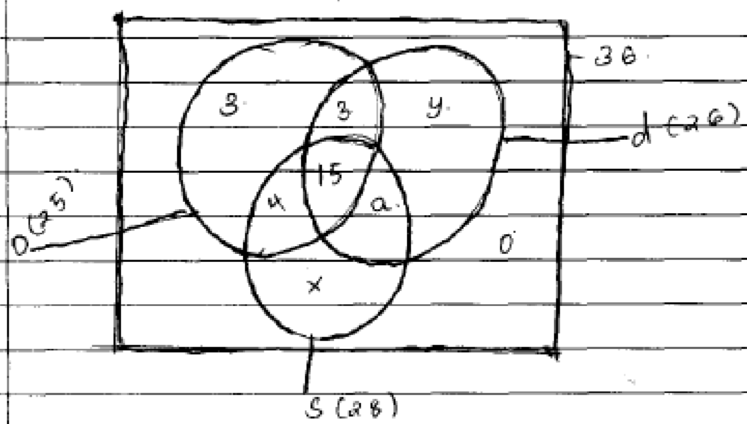
$= A'\cup\phi\cap B'\cup\phi$ — Identity law

$= A'\cap B'$ — Identity law.

b) Let D - represent dancing

S - represent singing

d - represent drama.



$3+3+15+4+x+a+y = 36$

$a+y+x = 11$ — (1)

$4+15+a+x = 28$

$a+x = 9$ — (2)

$3+15+a+y = 26$

$a+y = 8$ — (3)

On solving

$a = \cancel{8}\ 6$

$x = 3$

$y = 2$.

5 b) i) Who are either dancing or playing drama

$= 3+3+15+4+a+y$

$= \cancel{8}\ 25+6+2$

$= 33$ villagers

ii) who participate in atmost two activities

$= 3+3+4+a+x+y$

$= 3+3+4+6+3+2$

$= 21$ villagers.

iii) who neither play drama nor sing

$= 3$ villagers.

2019 PAST PAPERS

5. (a) Use set properties to prove that for any non-empty sets A and B, $(A \cap B') \cup (B \cap A') = (A \cup B) - (A \cap B)'$.

(b) A student at the Sokoine University of Agriculture made a study about the types of livestock in a nearby village. The student came up with the following findings: 82 villagers kept cattle, 110 villagers kept goats, 73 villagers kept pigs; 59 villagers kept cattle and goats, 53 kept goats and pigs; 32 kept cattle and pigs; 20 villagers kept all three types of livestock. If the village has 200 occupants, by using Venn diagram, find the number of villagers who kept;

(i) only one type of livestock,
(ii) only two types of livestock,
(iii) none of the livestock.

5a soln.

$(A \cap B') \cup (B \cap A') = (A \cup B) - (A \cap B)'$ --- given.

Consider L.H.S.

$(A \cap B') \cup (B \cap A')$ ---- given

$((A \cap B') \cup B) \cap ((A \cap B') \cup A')$ --- distributive law.

$[(A \cup B) \cap (B' \cup B)] \cap [(A \cup A') \cap (B' \cup A')]$ -- distributive law

$[(A \cup B) \cap U] \cap [U \cap (B' \cup A')]$ ---complement law.

$(A \cup B) \cap (B' \cup A')$ --- Identity law.

$(A \cup B) \cap (A' \cup B')$ --- Commutative law.

$(A \cup B) \cap (A \cap B)'$ --- De-Morgan's law.

$\underline{(A \cup B) - (A \cap B)'}$ --- from definition $A - B = A \cap B'$.

$(A \cap B') \cup (B \cap A') \neq (A \cup B) - (A \cap B)'$ and ~~not~~ is equal to $(A \cup B) - (A \cap B)$

5b soln.

let

B - people keeping goats.

A - villagers keeping cattles

C - villagers keeping pigs.

Data:

$n(A) = 82$ villagers.

$n(B) = 110$ villagers

$n(C) = 73$ villagers

$n(A \cap B) = 59$ villagers

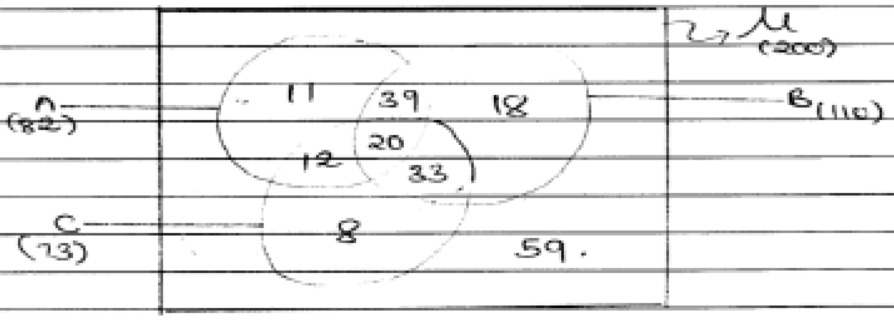
$n(B \cap C) = 53$ villagers

$n(A \cap C) = 32$ villagers

$n(A \cap B \cap C) = 20$ villagers

$\mathcal{U} = 200$ villagers.

By venn-diagram.



5b i) soln.

only one type of livestock

$= n(A)\text{only} + n(B)\text{only} + n(C)\text{only}.$

$= 11 + 18 + 8$

$= 37$ villagers.

$\therefore$ 37 villagers kept only one type of livestock.

5b ii) soln.

only two types of livestock.

$= 12 + 39 + 33.$

$= 84$ villagers

$\therefore$ 84 villagers kept only two types of livestock.

5b iii) soln.

none of the livestock; n(A∩B)

$n(A \cup B \cup C)' = 200 - [11 + 12 + 20 + 39 + 33 + 8 + 18]$

$= 200 - 141$

$= 59$ villagers.

$\therefore$ 59 villagers kept none of the livestock.

2018 PAST PAPERS

5. (a) Simplify $A-(A-B)$ using properties of sets.

(b) Shade set $A'\cap(B-C)$.

(c) In a survey of 500 movie viewers, 250 were listed as liking 'zecomedy', 200 as liking 'zembwela' and 85 were listed as liking both 'zecomedy' as well as 'zembwela'. Using the appropriate formula, find how many people were liking neither zecomedy' nor 'zembwela'.

5(a) Soln.

$A-(A-B)$

let $P = A-B$

$\Rightarrow P = A\cap B'$ – definition of $x-y = x\cap y'$

$\Rightarrow A-P = A\cap P'$ – definition of $x-y = x\cap y'$

$\Rightarrow A-(A-B) = A\cap(A\cap B')'$

$= A\cap(A'\cup B)$ – De morgan's law

$= (A\cap A')\cup(A\cap B)$ – Distributive law

$= \phi \cup (A\cap B)$ – Compliment law

$= A\cap B$ – Identity law

$\therefore A-(A-B) = A\cap B$

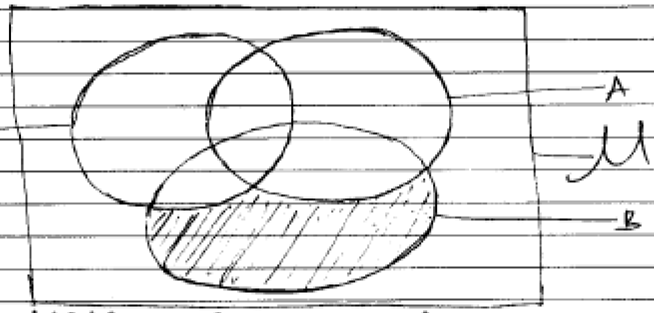
(b) Required: To shade $A'\cap(B-C)$

but

$B-C = B\cap C'$ – definition of $x-y = x\cap y'$

$\Rightarrow A'\cap(B-C) = A'\cap(B\cap C') = (A\cup C)'\cap B$

Consider a venn diagram for set A, B and C



$\therefore A'\cap(B-C)$ is shaded.

$n(B) = 200$

$n(A\cap B) = 85$

Required: $n(A\cup B)'$

$n(A\cup B) = n(A) + n(B) - n(A\cap B)$

$= 250 + 200 - 85$

$= 365$

$\Rightarrow n(A\cup B)' = n(U) - n(A\cup B)$

$= 500 - 365$

$= 135$ viewers

$\therefore$ People liking neither zecomedy nor zembela = 135 people.

2017 PAST PAPERS

5. (a) Use the laws of algebra to simplify:

(i) $[A \cap (B \cap C')] \cup C$.

(ii) $(X \cap Y') \cup (X \cap Y) \cup (Y \cap X')$.

(b) Out of a group of 17 girl guides and 15 boy scouts, 22 play handball, 16 play basketball, 12 of the boy scouts play handball, 11 of the boy scouts play basketball, 10 of the boy scouts play both and 3 of the girl play neither of the two.

(i) How many girls play both handball and basketball?

(ii) How many in the group play handball only and basketball only?

5. (a) Using laws of Algeba to simplify

(i) $[A \cap (B \cap C')] \cup C$

$(A \cup C) \cap [(B \cap C') \cup C]$ .. distributive law.

$A \cup C \cap [(B \cup C) \cap (C' \cup C)]$ distributive law

$A \cup C \cap [(B \cup C) \cap U]$ ... Complement law

$(A \cup C) \cap (B \cup C)$ .... Identity law.

$(A \cap B) \cup C$ .... distributive law.

5. (a)(ii) $(X \cap Y') \cup (X \cap Y) \cup (Y \cap X')$

$[(X \cap Y') \cup (X \cap Y)] \cup (Y \cap X')$ ... Associative law

$[X \cap (Y' \cup Y)] \cup (Y \cap X')$ ... distributive law

$(X \cap U) \cup (Y \cap X')$ ... complement law

$X \cup (Y \cap X')$ ... Identity law

$(X \cup Y) \cap (X \cup X')$ ... distributive law

$(X \cup Y) \cap U$ .... complement law

$X \cup Y$ .... Identity law.

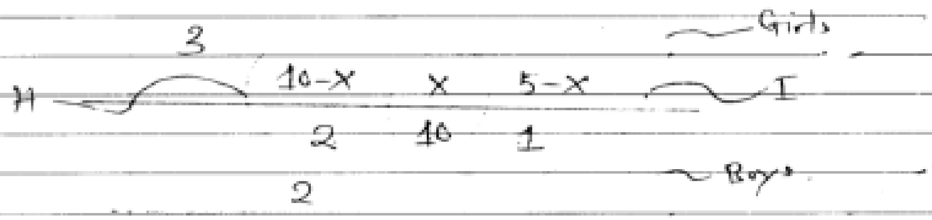
5. (b) $n(G) = 17$ .... Girls

$n(B) = 15$ .... Boys.

Let H be set of Handball game.

I be set of basketball game.

Into Venn diagram.



5. (b)(i) Girls play both Handball and basket ball.

$n(I \cup H) = n(G) - 3$

$(10 - x) + x + (5 - x) = 17 - 3$

$15 - x = 14$

$x = 1.$

$\therefore$ Number of Girls $= 1.$

5. (b)(ii) · Hand ball only.

$(10 - x) + 2 = 10 - 1 + 2$

$= 11.$

$\therefore$ 11 plays Handball only.

· Basketball only

$(5 - x) + 1 = (5 - 1) + 1$

$= 5$

$\therefore$ 5 play basketball only.

Extract 5.1, illustrates a correct solution by one of the candidates who applied laws of algebra of sets correctly.

2016 PAST PAPERS

5. (a) Use the laws of algebra of sets to;

(i) verify that $X \cup (X \cap Y) = X$.

(ii) simplify $\left[A \cap (A \cup B)'\right]'$.

(b) If A, B and C are three non-empty sets, use venn diagram to show whether $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

(c) A class contains 15 boys and 15 girls. A survey of the class showed that;

20 pupils were studying Geography,
14 pupils were studying Mathematics,
10 of the girls were studying Geography,
4 of the girls were studying Mathematics,
3 of the girls were studying both Geography and Mathematics,
3 of the boys were studying neither Geography nor Mathematics.

How many pupils were studying both Mathematics and Geography? (Use Venn Diagrams).

5 (a) i) $X \cup (X \cap Y) = X$

From L·H·S $X \cup (X \cap Y)$

$(X \cap \mu) \cup (X \cap Y)$ - By identity law

$X \cap (\mu \cup Y)$ - Distributive law

$X \cap \mu$ - Identity law

X - Identity law

Hence verified.

(ii) $[A \cap (A \cup B)']'$

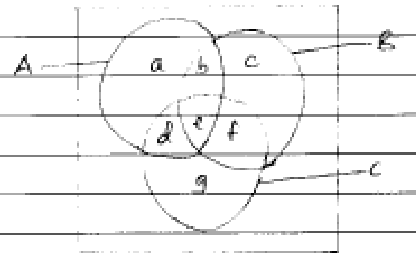
$A' \cup (A \cup B)$ - By demorgans law

$(A' \cup A) \cup B$ - Associative law

$\mu \cup B$ - Complement law

μ - Identity law

(b)



From the venn diagram: $n(A) = a + b + d + e$

$n(B) = b + c + e + f$

$n(C) = d + e + f + g$

$n(A \cap B) = b + e$

$n(A \cap C) = d + e$

$n(B \cap C) = e + f$

$n(A \cap B \cap C) = e$

$n(A \cup B \cup C) = a + b + c + d + e + f + g$

$= (a + b + d + e) + c + f + g$

$= n(A) + (b + c + e + f) + g - b - e$

$= n(A) + n(B) + (d + e + f + g) - b - e - d - e - f$

5 (b) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - (b + e) - d - e - f$

$= n(A) + n(B) + n(C) - n(A \cap B) - (d + e) - f$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - f - e + e$

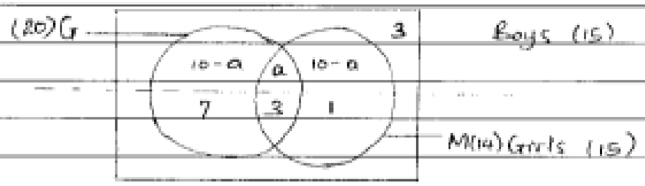
$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - (f + e) + e$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Hence shown that:

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

(c)



$15 - 3 = 12$ were studying geography or mathematics.

$10 - a + a + 10 - a = 12$

$20 - a = 12$

$a = 8$

$8 + 3 = 11$

$\therefore$ 11 students were studying both mathematics and geography.

In Extract 5.1, the candidate demonstrated good understanding of laws of Algebra. He/she managed to derive the formula for the number of elements of three sets in part (b) and used Venn diagram to solve the given problem in part (c) correctly.

2015 PAST PAPERS

5. (a) (i) Use a Venn diagram to show that $(A \cap B) \cup (A' \cap B) = B$.

(ii) Find the members of set R where $R = \left\{ x : \frac{x^2 - 9}{x^2 - 1} \leq 0,\ x \in \Re \right\}$.

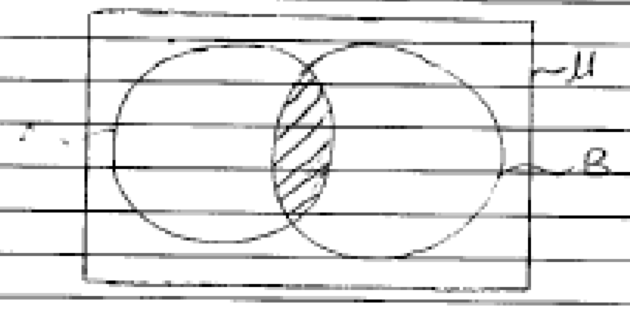
(b) Use the basic properties of set operations to simplify the following:

(i) $(A \cap B) \cup (A - B)$.

(ii) $\left[(A \cup B)' \cap (A \cap B) \right]'$.

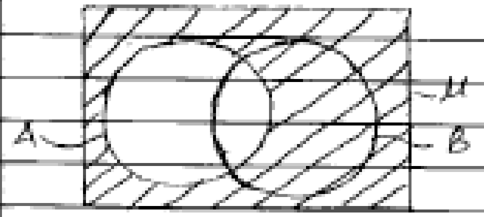
(c) In a bunch of twenty flowers, twelve are yellow and nine are red. If four of the flowers are neither yellow nor red, how many of the flowers are both yellow and red? (Use Venn diagram).

5ai) $A \cap B$

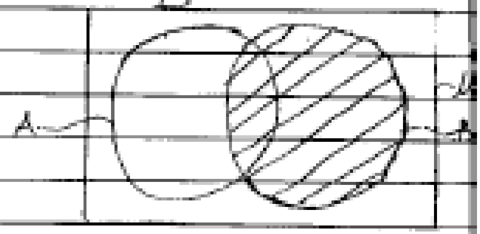


$A' \cap B$

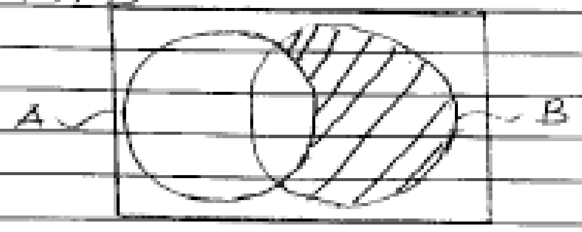
A'



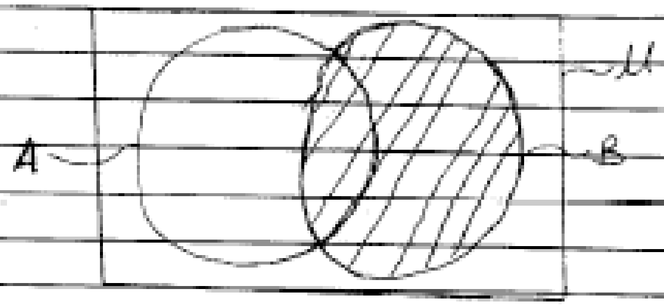
B



$A' \cap B$



5ai) $(A \cap B) \cup (A' \cap B)$



Hence shown $(A \cap B) \cup (A' \cap B) = B$

5a ii) $R = \left\{ x : \frac{x^2-9}{x^2-1} \le 0 , x \in R \right\}$

$x^2 - 9 = (x-3)(x+3)$

$x = 3 \quad , x = {}^-3$

$x^2 - 1 = (x-1)(x+1)$

$x = 1 , \quad x = {}^-1 .$

	$-\infty \le x \le -3$	$-3 \le x < -1$	$-1 \le x < 0$	$0 \le x < 1$	$1 < x \le 3$	$3 \le x < \infty$
$f(x)$	+	−	+	+	−	+

The values which bring $\frac{x^2-9}{x^2-1} \le 0$ lie between $^-3 \le x < -1$ and $1 < x \le 3$

$\therefore R = \{ x : {}^-3 \le x < {}^-1 ; 1 < x \le 3 \}$

5b i) $(A \cap B) \cup (A - B)$

$(A \cap B) \cup (A \cap B')$ set difference.

$A \cap (B \cup B')$ Distributive law

$A \cap U$ Compliment law

A Identity law.

5b ii) $[(A \cup B)' \cap (A \cap B)']'$

$(A \cup B) \cup (A \cap B)$ Demorgan's law

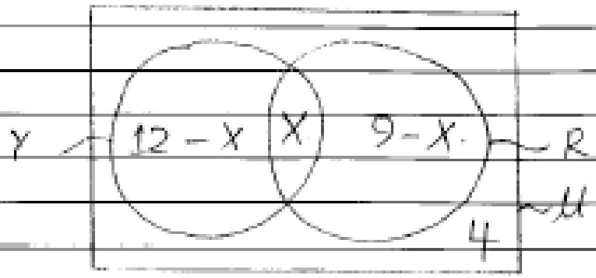
let $A \cup B = X$

$X \cup (A \cap B)$

$(X \cup A) \cap (X \cup B)$ Distributive law.

$(A \cup B \cup A) \cap (A \cup B \cup B)$

	$((A \cup A) \cup B) \cap (A \cup (B \cup B))$ Commutative law.
	$(A \cup B) \cap (A \cup B)$ Idempotent law.
	$A \cup B$ Idempotent law
5C	Let Yellow flowers be Y Red flowers be R



$12 - X + X + 9 - X + 4 = 20$

$21 - X + 4 = 20$

$25 - X = 20$

$X = 25 - 20$

$X = 5$

$\therefore$ 5 FLOWERS

Extract 5.1 illustrates a sample solution from a candidate who was able to show that $(A \cap B) \cup (A' \cap B) = B$ using Venn diagram, find the members of set R by inspection method, simplified the given set expressions and evaluated the number of flowers with both colors using Venn diagram correctly.

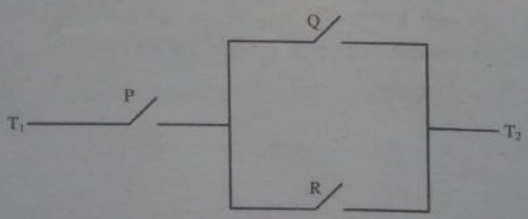
3.0 Logic

2021 PAST PAPERS - 2

2. (a) The contrapositive of the statement Y is given by $\sim(Q \wedge P) \to \sim P$. By using the laws of algebra of propositions, show that its inverse is a tautology.

(b) Test the validity of the argument whose conclusion is $\sim Q$ and premises are $P \to (\sim P \to Q)$, $Q \to \sim P$ and P.

(c) (i) Construct a truth table for the compound statement that corresponds to the following circuit:



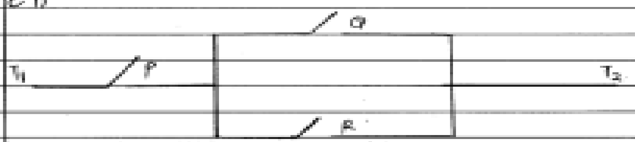
(ii) Draw a simple network diagram for the statement $(P \to Q) \wedge (P \vee Q)$.

2 b) $[(P \to (\sim P \to Q)) \wedge [(Q \to \sim P) \wedge P]] \to \sim Q$

				a	b	c	d	e	
P	Q	~P	~Q	~P→Q	P→Q	Q→~P	c∧P	b∧d	e→~Q
T	T	F	F	T	T	F	F	F	T
T	F	F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F	F	T
F	F	T	T	F	T	T	F	F	T

Since the last column of the truth table is a tautology, hence the arguement is valid.

c i)



$\equiv P \wedge (Q \vee R)$

P	Q	R	Q ∨ R	P ∧ (Q ∨ R)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

2 c ii)

$(P \to Q) \wedge (P \vee Q)$	Given
$\equiv (\sim P \vee Q) \wedge (P \vee Q)$	Implication law
$\equiv (Q \vee \sim P) \wedge (Q \vee P)$	Commutative law
$\equiv Q \vee (\sim P \wedge P)$	Distributive law
$\equiv Q \vee F$	Complementary law
$\equiv Q$	Identity law

2020 PAST PAPERS - 2

2\. (a) Use the laws of propositions of algebra to simplify $(p \wedge q) \vee [\sim r \wedge (q \wedge p)]$.

(b) Describe the following argument in symbolic form and test its validity by using a truth table:

"If he begs pardon then he will remain in school. Either he is punished or he does not remain in school. He will not be punished. Therefore, he did not beg pardon.

(c) Construct an electrical network for the proposition $(p \wedge q) \wedge [(r \vee s) \wedge w]$.

2a Solution

$(p \wedge q) \vee (\sim r \wedge (q \wedge p))$ ... Given

$(p \wedge q) \vee ((p \wedge q) \wedge \sim r)$ ... Commutative law

$[(p \wedge q) \wedge (T)] \vee [(p \wedge q) \wedge \sim r]$ ... Identity law

$(p \wedge q) \wedge [T \vee \sim r]$ ... Distributive law

$(p \wedge q) \wedge T$ ... Complement law

$p \wedge q$. Identity law

2b Solution

let statement

p - He begs pardon

q - He remain in school

r - He is punished

Compound statement

$$[(p \to q) \wedge (r \vee \sim q) \wedge \sim r] \to \sim p$$

Truth table

						a.	b	c	d	
p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \to q$	$r \vee \sim q$	$a \wedge b$	$c \wedge \sim r$	$d \to \sim p$
T	T	T	F	F	F	T	T	T	F	T
T	T	F	F	F	T	T	F	F	F	T
T	F	T	F	T	F	F	T	F	F	T
T	F	F	F	T	T	F	T	F	F	T
F	T	T	T	F	F	T	T	T	F	T
F	T	F	T	F	T	T	F	F	F	T
F	F	T	T	T	F	T	T	T	F	T
F	F	F	T	T	T	T	T	T	T	T

a : $p \to q$ b : $r \vee \sim q$ c : $[(p \to q) \wedge (r \vee \sim q)]$

d : $[(p \to q) \wedge (r \vee \sim q)] \wedge \sim r$

Since there are only T values in the last column of the truth table then the Argument is Valid

3c Electrical network for $(p \wedge q) \wedge [(r \vee s) \wedge w]$

2019 PAST PAPERS - 2

2. (a) Let P be "She is tall" and Q be "She is beautiful". Write the verbal representation of the following statements:

(i) $P \wedge Q$.

(ii) $P \wedge \sim Q$.

(iii) $\sim P \wedge \sim Q$

(iv) $\sim (P \vee \sim Q)$

(b) Using the laws of algebra of propositions simplify $[P \wedge (P \vee Q)] \vee [Q \wedge (P \vee Q)]$.

(c) (i) Find a simplified sentence having the following truth table:

p	q	r	
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

(ii) Draw a simple electric network that corresponds to the compound statement obtained in c (i).

2a. (i). $P \wedge Q$

$\equiv$ She is tall and beautiful.

(ii). $P \wedge \sim Q$

$\equiv$ She is tall but not beautiful

(iii). $\sim P \wedge \sim Q$

$\equiv$ She is neither tall nor beautiful.

(iv). $\sim (P \vee \sim Q)$

$\equiv$ It is not true that she is tall or not beautiful.

2b. Soln

$(P \wedge (P \vee Q)] \vee [Q \wedge [P \vee Q]]$

$\equiv [(P \vee F) \wedge (P \vee Q)] \vee [(F \vee Q) \wedge (P \vee Q)]$ Identity law

$\equiv [P \vee (F \wedge Q)] \vee [(F \wedge P) \vee Q]$ Distributive law

$\equiv [P \vee F] \vee [F \vee Q]$ Identity law

$\equiv P \vee Q$ Identity law

$[P \wedge (P \vee Q)] \vee [Q \wedge [P \vee Q]] \equiv P \vee Q$

2c. Soln:

let the sentence be M.

Considering F

$M \equiv (p \vee \sim q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

$M \equiv (p \vee \sim q \vee r) \wedge ((p \vee q) \vee \sim r) \wedge ((p \vee q) \vee r))$ Associative law

$\equiv (p \vee \sim q \vee r) \wedge ((p \vee q) \vee [\sim r \wedge r])$ Distributive law

$\equiv (p \vee \sim q \vee r) \wedge ((p \vee q) \vee F)$ Compliment law

$\equiv (p \vee \sim q \vee r) \wedge (p \vee q)$ Identity law

$\equiv (p \vee (\sim q \vee r)) \wedge (p \vee q)$ Associative law

$\equiv p \vee ((\sim q \vee r) \wedge q)$ Distributive law

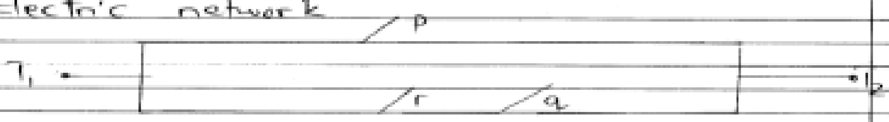
$\equiv p \vee ((\sim q \wedge q) \vee (r \wedge q))$ Distributive law

$\equiv p \vee (F \vee (r \wedge q))$ Compliment law

$\equiv p \vee (r \wedge q)$ Identity law

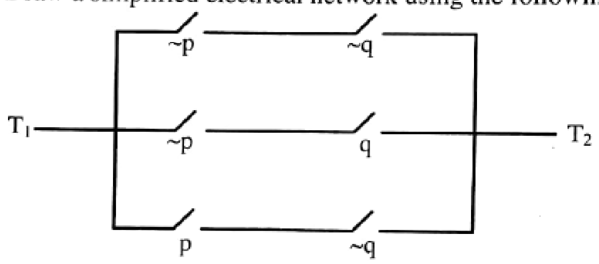
Statement, $M \equiv p \vee (r \wedge q)$

2c(ii) Electric network



2018 PAST PAPERS - 2

2. (a) (i) Draw a simplified electrical network using the following circuit.



(ii) Simplify the proposition $\sim((p \wedge q) \to (p \vee q))$ by using laws of algebra.

(b) Use the truth table to determine whether $[(p \to \sim q) \wedge (q \vee r) \wedge p] \to r$ is tautology.

(c) Use the laws of algebra to prove that the propositions $p \wedge (q \vee r)$ and $[p \to (q \vee \sim r)] \to (p \wedge q)$ are equivalent.

From the circuit

2(a) (i) $(\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$

Using law of algebra

$\sim p \wedge (\sim q \vee q) \vee (p \wedge \sim q)$ distributive law

$(\sim p \wedge T) \vee (p \wedge \sim q)$ complement law

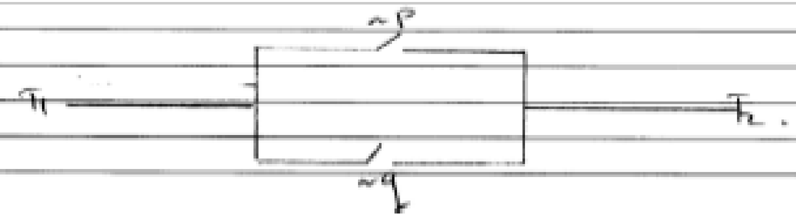
$\sim p \vee (p \wedge \sim q)$ Identity law

$(\sim p \vee p) \wedge (\sim p \vee \sim q)$ Distributive law

$T \wedge (\sim p \vee \sim q)$ complement law

$(\sim p \vee \sim q)$ Identity law

Simplified electrical network.



2a (ii) $\sim((p\wedge q)\rightarrow(p\vee q))$

$\sim(\sim(p\wedge q)\vee(p\vee q))$ $\quad p\rightarrow q\equiv\sim p\vee q$

$((p\wedge q)\wedge(\sim(p\vee q)))$ $\quad$ Demorgans law

$(p\wedge q)\wedge(\sim p\wedge\sim q)$ $\quad$ Demogans law

$p\wedge\sim p\wedge q\wedge\sim q$ $\quad$ Associative property

$F\wedge F$ $\quad$ Complement law

$\underline{F}$ $\quad$ Identity law

~~let~~
let $A=p\rightarrow\sim q$ $\quad B=q\vee r$ $\quad C=A\wedge B$ $\quad D=(C\wedge p)$

(b)

p	q	r	~q	A p→~q	B q∨r	C A∧B	D C∧P	D→r
T	T	T	F	F	T	F	F	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	F	F	T
F	T	T	F	T	T	T	F	T
F	T	F	F	T	T	T	F	T
F	F	T	T	T	T	T	F	T
F	F	F	T	T	F	F	F	T

The proposition is TAUTOLOGY since the last column ends with Truth value only.

2 (c) Using laws of algebra to prove $p\wedge(q\vee r)$ and $[p\rightarrow(q\vee\sim r)]\rightarrow(p\wedge q)$ are equivalent

Starting with
$p\wedge(q\vee r)$

Simplifying $[p\rightarrow(q\vee\sim r)]\rightarrow p\wedge q$

$[\sim p\vee(q\vee\sim r)]\rightarrow p\wedge q$ $\quad p\rightarrow q\equiv\sim p\vee q$,

$[p \wedge \sim(q \vee \sim r)] \vee p \wedge q$ Dermogan's law

$[p \wedge (\sim q \wedge r)] \vee p \wedge q$ Dermogan's law

$[p \wedge \sim q \wedge r] \vee [p \wedge q]$. Associative law

$p \wedge [(\sim q \wedge r) \vee q]$. Distributive law

$p \wedge [\sim q \vee q \wedge q \vee r]$ Distributive law

$p \wedge [T \wedge q \vee r]$ Complement law

$p \wedge (q \vee r)$ Identity law

Hence shown.

Hence $[p \to (q \vee \sim r)] \to p \wedge q$ is equivalent to $p \wedge (q \vee r)$

2017 PAST PAPERS - 2

2. (a) (i) Write the contrapositive of the inverse $p \to q$.

(ii) Use the truth table to verify that the statement $(p \vee q) \wedge ((\sim p) \wedge (\sim q))$ is a contradiction.

(b) (i) Use the laws of algebra of propositions to simplify the statement $q \vee (p \wedge \sim q) \vee (r \wedge q)$ and hence draw the corresponding simple electrical network.

(ii) Use the truth table to show that $p \leftrightarrow q$ logically implies $p \to q$.

(c) Without using the truth tables, prove that the proposition $[(p \to q) \wedge (\sim q)] \to \sim p$ is tautology. **(15 marks)**

(i) $p \rightarrow q$.

its converse $= q \rightarrow p$, inverse $\sim p \rightarrow \sim q$.

contrapositive of the inverse.

$q \rightarrow p$.

(ii): $(p \vee q) \wedge [(\sim p) \wedge (\sim q)]$.

let $p \vee q$ be A.

and $\sim p \wedge \sim q$ be B.

$(p \vee q) \wedge [\sim p \wedge \sim q]$ be C.

Truth table:

p	q	~p	~q	A	B	C
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

∴ since the last column of the truth table contain of only truth value false F then the statement is contradiction

Extract 12.1, shows the answer from the candidate who drew the truth table us
truth values correctly.

2016 PAST PAPERS - 2

2. (a) (i) Draw the simplified electrical circuit for the argument:
$[p \wedge (p \vee q)] \vee [q \wedge \sim (p \wedge q)]$

(ii) Use the truth table values only to show whether or not $p \leftrightarrow \sim q$ and $\sim (p \wedge q)$ are logically equivalent.

(b) (i) Use the truth table to test the validity of the following argument: If I am intelligent, then I will pass this examination. I am intelligent. Therefore I will pass this examination.

(ii) Write the converse, inverse and contrapositive of the statement 'If Mathematics is interesting then Biology is boring and tough'. **(15 marks)**

2a(ii) $P \leftrightarrow \sim q \equiv \sim (P \wedge q)$.

P	q	$\sim q$	$P \leftrightarrow \sim q$	$P \wedge q$	$\sim (P \wedge q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	F	T
F	F	T	F	F	T

$\therefore$ $P \leftrightarrow \sim q$ is not logically equivalent to $\sim (P \wedge q)$.

In Extract 12.1, the candidate was able to apply the basic skills on the topic of logic to verify the equivalence of the given symbolic statements.

2015 PAST PAPERS - 2

2. (a) (i) Prepare a truth table for the proposition $((q \to \sim p) \wedge ((p \vee r) \wedge q)) \to r$.

(ii) Determine the truth value and comment on the validity of the argument below using truth table.

$$\begin{array}{l} p \to (q \vee \sim r) \\ \underline{q \to (p \wedge r)} \\ [(q \vee \sim r) \wedge (p \wedge r)] \to r \end{array}$$

(b) Using the laws of algebra in logic:

(i) Determine the validity of the following argument: "If there is rain, the crops will grow well. If crops grow well, there is no famine. But there is famine. Therefore there is no rain.

(ii) Simplify the proposition $\sim ((p \vee q) \vee (\sim p \wedge q))$.

(c) Translate the following compound statements in symbolic notation using letters P, Q and R to stand for the statements:

(i) Either the manufactured drug is not fault and accepted by the Tanzania Food and Drug Authority (TFDA) or the drug is fault and is not accepted by the TFDA.

(ii) If Kapirima is a member of a social committee then the committee is strong. The committee is strong if and only if Kaprima's argument is accepted by other members. Therefore Kaprima's argument is not accepted and the committee is not strong. **(15 marks)**

2 (a) i/ Soln.

$$[(q \rightarrow \sim p) \wedge ((p \vee r) \wedge q)] \rightarrow r$$

p	q	r	$\sim p$	$q \xrightarrow{(a)} \sim p$	$p \vee r$	$\overset{(b)}{(p \vee r) \wedge q}$	$a \wedge b$	$(a \wedge b) \rightarrow r$
T	T	T	F	F	T	T	F	T
T	T	F	F	F	T	T	F	T
T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	T	T	T	F	F	T
F	F	F	T	T	F	F	F	T

Where a is $q \rightarrow \sim p$
b is $(p \vee r) \wedge q$

ii/ Soln.
Truth table.

$$[p \rightarrow (q \vee \sim r) \wedge (q \rightarrow p \wedge r)] \rightarrow ([(q \vee \sim r) \wedge (p \wedge r)] \rightarrow r)$$

p	q	r	$\overset{(a)}{q \vee \sim r}$	$p \xrightarrow{(c)} a$	$\overset{(b)}{p \wedge r}$	$q \xrightarrow{(d)} b$	$c \wedge d$	$a \overset{(e)}{\wedge} b$	$e \rightarrow r$	$f \rightarrow g$
T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F	F	T	T
T	F	T	F	F	T	T	F	F	T	T
T	F	F	T	T	F	T	T	F	T	T
F	T	T	T	T	F	F	F	F	T	T
F	T	F	T	T	F	F	F	F	T	T
F	F	T	F	T	F	T	T	F	T	T
F	F	F	T	T	F	T	T	F	T	T

−2− (a) ii/

∴ Since the proposition is tautology therefore the argument is valid

(b) ii/ $\sim[(p \vee q) \vee (\sim p \wedge q)]$

Soln

$(\sim p \wedge \sim q) \wedge (p \vee \sim q)$ Demorgans law

$(\sim p \wedge \sim q \wedge p) \vee (\sim p \wedge \sim q \wedge \sim q)$ Distributive law

$(F \wedge \sim q) \vee (\sim p \wedge \sim q)$ Complement law

$F \vee \sim p \wedge \sim q$ Identity law

$\sim p \wedge \sim q$ Identity law

i/ Soln.

Let p be there is rain

q be crops will grow well

r be there is famine

Argument

$(p \rightarrow q), (q \rightarrow \sim r), r \vdash \sim p$

Proposition

$[(p \to q) \wedge (q \to \sim r) \wedge r] \longrightarrow \sim p$

$\sim[(\sim p \vee q) \wedge (\sim q \vee \sim r) \wedge r] \vee \sim p$ Definition $p \to q \equiv \sim p \vee q$

-2- (b) i/

$[(\sim r \vee q) \wedge T] \vee [T \wedge (\sim p \vee \sim q)]$ Complement law

$(\sim r \vee q) \vee (\sim p \vee \sim q)$ Identity law

$(\sim r \vee \sim p) \vee (q \vee \sim q)$ Commutative law

$(\sim r \vee \sim p) \vee T$ Complement law

T Identity law

∴ Since the proposition is tautology, therefore the argument is valid.

(c) i/ Soln.

Let p be Manufactured drug is fault

Q be accepted by TFDA

$\cancel{\sim p \vee}\ ([\sim p \wedge Q] \vee [p \wedge \sim Q])$

∴ In Symbolic $[(\sim p \wedge Q) \vee (p \wedge \sim Q)]$

ii/ Soln

Let p kapirima is a member of social community

Q be community is strong

R kapirima's argument is accepted by other member.

In Symbolic

$(P \to Q),\ Q \leftrightarrow R \vdash \sim R \wedge \sim Q$

∴ $[(P \to Q) \wedge (Q \leftrightarrow R)] \longrightarrow (\sim R \wedge \sim Q)$

Extract 12.1 shows a sample solution of a candidate who was able to answer the given question correctly. He/she prepared a truth table using the appropriate laws of propositions of algebra, simplified the given statements, tested the validity of the given argument and expressed the given statements in symbolic form.

4.0 Coordinate Geometry I

2021 PAST PAPERS

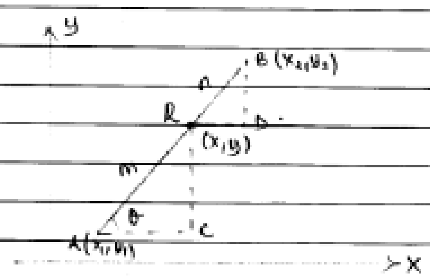
8. (a) (i) The point $R(x, y)$ divides a line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m:n$ internally. Derive the formula for the ratio theorem.

(ii) Use the ratio formula in (i) to find the coordinates of the point which divides the line joining the points $(5,-4)$ and $(-3,2)$ internally in the ratio 1:2.

(b) (i) If the triangle has three vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, derive the formula to find its area.

(ii) Use the formula obtained in (b) (i) to find the area of the triangle with the vertices $A(3,1)$, $B(2k,3k)$ and $C(k,2k)$. Hence show that the vertices are collinear when $k=-2$.

8. a. i.



from the graph.

$\sin\theta = \frac{CR}{AR} = \frac{DB}{BR}$

8. a. i. but $\frac{AR}{BR} = \frac{m}{n}$

$\sin\theta = \frac{y-y_1}{m} = \frac{y_2-y}{n}$

$ny - ny_1 = my_2 - my$

$my + ny = my_2 + ny_1$

$y(m+n) = my_2 + ny_1$

$y = \frac{my_2 + ny_1}{m+n}$

from the graph.

$\cos\theta = \frac{AC}{AR} = \frac{DR}{BR}$

$\frac{x - x_1}{m} = \frac{x_2 - x}{n}$

$nx - nx_1 = mx_2 - mx$

$mx + nx = mx_2 + nx_1$

$x(m+n) = mx_2 + nx_1$

$x = \frac{mx_2 + nx_1}{m+n}$

$\therefore$ The ratio theorem.

$$R(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right).$$

ii. let the point be (x, y)

from the ratio theorem.

$$x = \frac{mx_2 + nx_1}{m+n}.$$

$$= \frac{1(-3) + 2(5)}{1+2}.$$

$$= \frac{-3+10}{3}.$$

use only

8. a. ii. $x = 7/3$

$$y = \frac{my_2 + ny_1}{m+n}$$

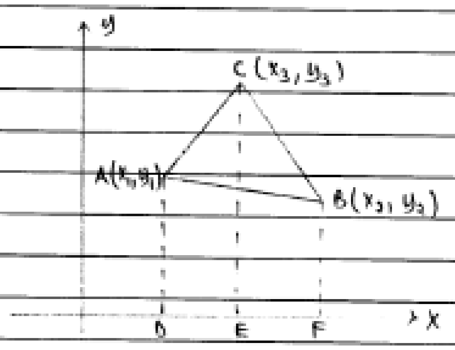
$$= \frac{1(2) + 2(-4)}{1+2}.$$

$$= \frac{2-8}{3}$$

$$= -2$$

$\therefore$ Point $(x, y) = (7/3, -2)$.

b. i.



Area of $\triangle ABC$ = area of ABCDEF − area of ABDEF.

area of ABCDEF = area of ACED + area of BCEF

$$= \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) + \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3).$$

area of ABDEF $= \tfrac{1}{2}(y_2 + y_1)(x_2 - x_1)$.

area of $\triangle ABC = \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) + \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3) - \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1)$

$$= \tfrac{1}{2}\left[y_1x_3 - \cancel{y_1x_1} + \cancel{y_3x_3} - y_3x_1 + \cancel{y_2x_2} - y_2x_3 + y_3x_2 - \cancel{y_3x_3} - y_1x_2 + \cancel{y_1x_1} - \cancel{y_2x_2} + y_2x_1 \right]$$

$= \frac{1}{2}(y_1x_3 - y_3x_1 - y_2x_3 + y_3x_2 - y_1x_2 + y_2x_1)$.

$= \frac{1}{2}(y_2x_1 - y_3x_1 + y_3x_2 - y_1x_2 + y_1x_3 - y_2x_3)$

$= \frac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]$

$= \frac{1}{2}[3(3k-2k) - 2k(1-2k) + k(1-3k)]$

$= \frac{1}{2}[3(k) - 2k + 4k^2 + k - 3k^2]$.

$= \frac{1}{2}(3k - 2k + k + k^2)$

$= \frac{1}{2}(k^2 + 2k)$.

$\therefore$ Area of triangle $= \frac{1}{2}(k^2 + 2k)$ square units.

For collinearity area of triangle $= 0$.

area $= \frac{1}{2}(k^2 + 2k)$

when $k = -2$.

area $= \frac{1}{2}[(-2)^2 + 2(-2)]$.

$= \frac{1}{2}(4 - 4)$

$= \frac{1}{2}(0)$

$= 0$.

$\therefore$ Since the area of the triangle is 0, then the vertices are collinear when $k = -2$.

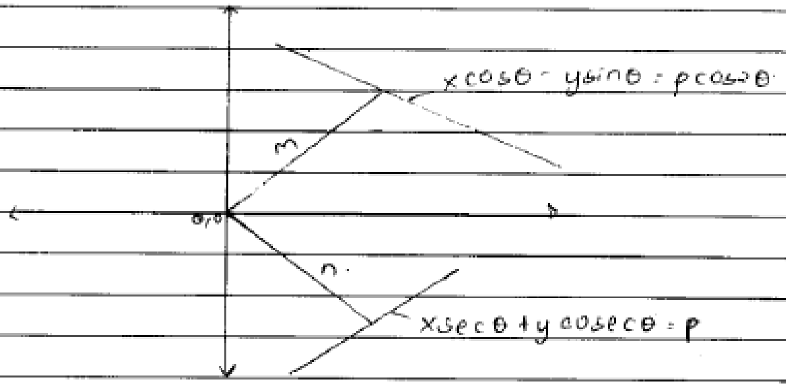
2020 PAST PAPERS

8. (a) If m and n are lengths of the perpendicular distance from the origin to the lines $x\cos\theta - y\sin\theta = p\cos 2\theta$ and $x\sec\theta + y\operatorname{cosec}\theta = p$ respectively, prove that $m^2 + 4n^2 = p^2$.

(b) Show that the bisector of the acute angle between $y = x + 1$ and the x-axis has the gradient of $-1 + \sqrt{2}$.

(c) A point P lies on the circle of radius 2 whose centre is at the origin. If A is the point $(4, 0)$. find the locus of a point which divides AP in the ratio $1:2$.

8



$m = \left| \dfrac{x\cos\theta - y\sin\theta - p\cos 2\theta}{\sqrt{\cos^2\theta + \sin^2\theta}} \right|$

but $(x, y) = (0,0)$; $\cos^2\theta + \sin^2\theta = 1$

so, $m = \left| \dfrac{-p\cos 2\theta}{1} \right|$

$n = \left| \dfrac{x\sec\theta + y\,\text{cosec}\,\theta - p}{\sqrt{\sec^2\theta + \text{cosec}^2\theta}} \right|$

but $(x, y) = (0,0)$

$\sec^2\theta + \text{cosec}^2\theta = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}$

$= \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta\sin^2\theta}$

$\sec^2\theta + \text{cosec}^2\theta = \dfrac{1}{\cos^2\theta\sin^2\theta}$

$n = \left| \dfrac{-p}{\sqrt{\dfrac{1}{\cos^2\theta \sin^2\theta}}} \right|$

$n = -(\cos\theta \sin\theta)\, p$

$m^2 = (p\cos 2\theta)^2$

$4n^2 = 4(\cos^2\theta \sin^2\theta)\, p^2$

8. $m^2 + 4n^2 = p^2(\cos 2\theta)^2 + p^2(4\cos^2\theta \sin^2\theta)$

$= p^2\left[(\cos 2\theta)^2 + (\sin 2\theta)^2\right]$

$= p^2$

$\therefore m^2 + 4n^2 = p^2$ — Hence prooved

b). $y = x + 1$

$y = 0$ — x-axis

from

$\left| \dfrac{Ax + By + C}{\sqrt{A^2 + B^2}} \right| = \left| \dfrac{ax + by + c}{\sqrt{a^2 + b^2}} \right|$

$\left| \dfrac{x - y + 1}{\sqrt{1^2 + (-1)^2}} \right| = \left| \dfrac{0x + y + 0}{\sqrt{0^2 + 1^2}} \right|$

$\dfrac{x - y + 1}{\sqrt{2}} = \pm(y)$

for acute angle we take the positive value.

so,

$x - y + 1 = y\sqrt{2}$

$y(\sqrt{2} + 1) = x + 1$

$y = \dfrac{x}{\sqrt{2} + 1} + 1$

$y = mx + c$

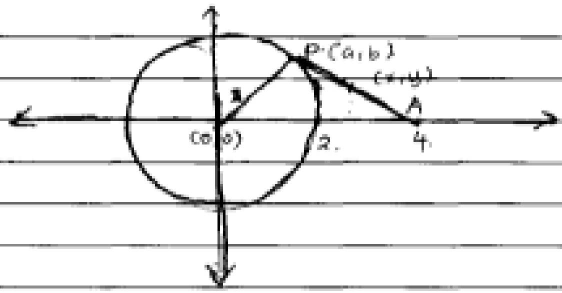
$m = \dfrac{1}{\sqrt{2} + 1}$

On rationalizing

$m = \dfrac{1(\sqrt{2} - 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \dfrac{\sqrt{2} - 1}{2 - 1} = \sqrt{2} - 1$

Slope, $m = -1 + \sqrt{2}$ Hence shown

8 (c)



from $(x,y) = \left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right]$

$m : n = 1 : 2$

$(x_1, y_1) = (4, 0)$

$(x_2, y_2) = (a, b)$

$x^2 + y^2 = 4^2$ — circle.

$a^2 + b^2 = 4.$ — ①

$(x,y) = \left(\frac{1(a) + 2(4)}{3}, \frac{1(b) + 2(0)}{3}\right)$

$(x,y) = \left(\frac{8 + a}{3}, \frac{b}{3}\right)$

$x = \frac{8 + a}{3}$ $\quad y = \frac{b}{3}$

$a = 3x - 8$ $\quad b = 3y$

into eqn ①

$(3x - 8)^2 + (3y)^2 = 4$

$(3x)^2 - 48x + 64 + 9y^2 = 4$

$x^2 + y^2 - \frac{16}{3}x + \frac{20}{3} = 0$

$\therefore$ The locus is $x^2 + y^2 - \frac{16}{3}x + \frac{20}{3} = 0$

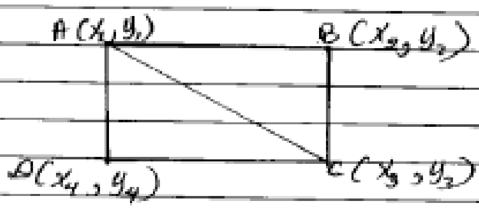
2019 PAST PAPERS

8. (a) The vertices of rectangle ABCD are given by points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$. Derive the formula to calculate the area of the rectangle.

(b) Use the formula obtained in part (a) to find the area of the rectangle whose vertices are the points $A(1, 1)$, $B(3, 5)$, $C(-2, 4)$ and $D(-1, -5)$.

(c) Show that the line $3x - 4y + 14 = 0$ is a tangent to a circle $x^2 + y^2 + 4x + 6y - 3 = 0$.

08 (a)



Consider triangle ACD

$$A_1 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}$$

$$A_1 = \frac{1}{2}\left[(x_1y_3 + x_3y_4 + x_4y_1) - (x_3y_1 + x_4y_3 + x_1y_4)\right]$$

$\cancel{A_1}$ Consider triangle ABC

$$A_2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$A_2 = \frac{1}{2}\left[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right]$$

Add $A_1 + A_2 = A$

$$A = \frac{1}{2}(x_1y_3 + x_3y_4 + x_4y_1 - x_3y_1 - x_4y_3 - x_1y_4) +$$
$$\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

$$A = \frac{1}{2}\left(x_1(y_3 - y_4 + y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)\right)$$

08 (a) $A = \frac{1}{2}\left[x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)\right]$

∴ Area of a rectangle derived.

(b) Given:-

$$A = \frac{1}{2}\left[(5+5) + 3(4-1) + -2(-5-5) + -1(1-4)\right]$$

$$A = \frac{1}{2}(10 + 9 + 20 + 3)$$

$A =$ 21 square units.

2018 PAST PAPERS

8. (a) If the circles $x^2+y^2-2y-8=0$ and $x^2+y^2-24x+hy=0$ cut orthogonally, determine the value of h.

(b) Find the equation of the normal line passing through point K(7, 4) to the circle whose equation is $x^2+y^2-4x-6y+9=0$.

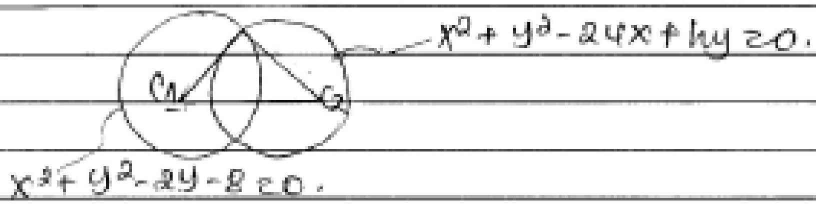
(c) Calculate the area of the triangle whose vertices are the points L(3, 5), M(4, 2) and N(6,3).

8. a)

soln.

Circle 1 : $x^2+y^2-2y-8=0$

circle 2: $x^2+y^2-24x+hy=0$.



For orthogonal circles.

$c_1+c_2=2(g_1g_2+f_1f_2)$.

From circle 1 : $x^2+y^2-2y-8=0$

$-2fy=-2y$

$2f_1=2$

$f_1=1$.

$g_1=0$.

$2f_1 = 2$

$f_1 = 1$

$g_1 = 0$

$c_1 = -8$

From circle 2: $x^2+y^2-24x+hy=0$

$-2gx = -24x$

$2g_2 = 24$

$g_2 = 12$

$f_2 =$

$-2fy = hy$

$-2f_2 = h$

$f_2 = -h/2$

$c_2 = 0$

From:

$c_1 + c_2 = 2(g_1g_2 + f_1f_2)$

$-8 + 0 = 2(0 \times 12 + (1 \times -h/2))$

8 (a) $-8 = 2(-h/2)$

$-8 = -h$

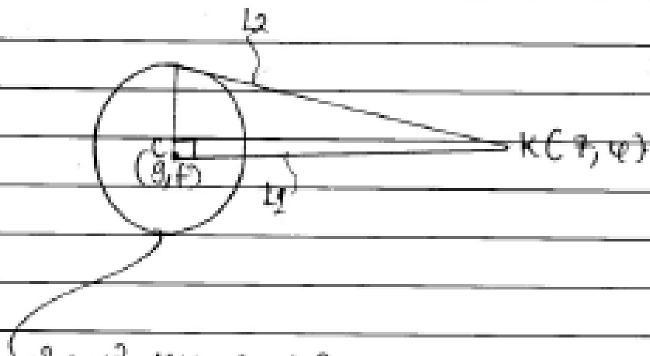
$8 = h$

$\therefore$ value of h is 8.

8 (b) Data

point K(7,4)

equation circle $= (x^2+y^2-4x-6y+9) = 0$



$x^2+y^2-4x-6y+9=0$

$-2fy = -6y$

$-f = -3$

$-f = -3.$

$f = 3.$

$-2gx = -4x$

$-2g = -4$

$g = 2.$

$m = \frac{\delta y}{\delta x}.$

$m_1 = \frac{\delta y}{\delta x}.$

points $C(2,3)$ and $R(7,4)$

$m_1 = \frac{4-3}{7-2}$

$m_1 = 1/5.$

8 (b). From linear equation.

$m = \frac{Y_2 - Y_1}{X_2 - X_1}$

$1/5 = \frac{Y-4}{X-7}$

$\frac{1}{5} = \frac{y-4}{x-7}$

$x - 7 = 5y - 20$

$5y - x = 20 - 7$

$5y - x = 13.$

$5y - x = 13$

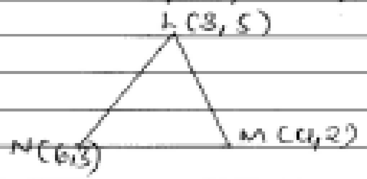
$x + 13 - 5y = 0$

$\underline{x - 5y + 13 = 0}$

$\therefore$ the equation of normal line is $x - 5y + 13 = 0$

8 (c). Required triangle's area.

points $L(3,5)$, $M(4,2)$ and $N(6,3)$



$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 3 & 5 \\ 4 & 2 \\ 6 & 3 \\ 3 & 5 \end{vmatrix}$$

8	(c)	$A = \frac{1}{2}\left[\left((4\times5)+(6\times2)+(3\times3)\right) - \left((3\times2)+(4\times3)+(6\times5)\right)\right]$
		$A = \frac{1}{2}\left[(20+12+9) - (6+12+30)\right]$
		$A = \frac{1}{2}\left[41 - 48\right]$
		$A = \frac{1}{2}\lvert -7 \rvert$
		$A = \frac{1}{2} \times 7$
		$A = 3.5$ square units
		$\therefore$ Area of triangle is 3.5 square units.

2017 PAST PAPERS

8. (a) Find the value of k such that $k(x^2 + y^2) + (y - 2x + 1)(y + 2x + 3) = 0$ is a circle. Hence obtain the centre and radius of the resulting circle.

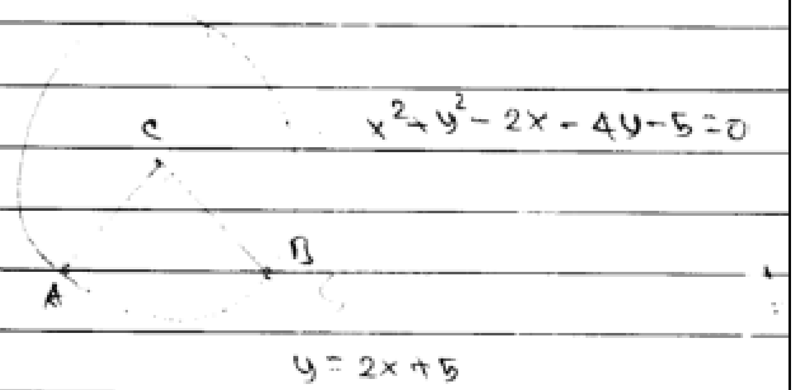
(b) The circle $x^2 + y^2 - 2x - 4y - 5 = 0$ has a centre C and is cut by the line $y = 2x + 5$ at A and B. Show that BC is perpendicular to AC and hence find the area of triangle ABC.

(c) Find the equation of the straight line which passes through the intersection of the lines $3x + 2y + 4 = 0$ and $x - y = 2$ and forms the triangle with the axes whose area is 8 square units.

b) Given the circle $x^2+y^2-2x-4y-5=0$

line $y=2x+5$

i.e



Soln.

$x^2+y^2-2x-4y-5=0$ —(i)

$y=2x+5$ —(ii)

Solving —(i) and —(ii) simultaneously

$x^2+(2x+5)^2-2x-4(2x+5)-5=0$

$x^2+4x^2+20x+25-2x-8x-20-5=0$

$5x^2+10x=0$

$5x(x+2)=0$

$x_1 = 0$ or $x_2 = -2$

Hence

$y = 2x + 5$

$y_1 = 2(0) + 5 = 5$

$y_2 = 2(-2) + 5 = 1$

Hence $A(0, 5)$ $B(-2, 1)$

~~For $\bar{AC}$~~ For C

$C = (-g, -f)$

From $x^2 + y^2 - 2x - 4y + 5 = 0$

$-2 = 2g$

$g = -1$

$-4 = 2f$

$f = -2$

i.e $C = (-g, -f)$

$= (1, 2)$

For $\bar{AC}$

slope, $m_1 = \frac{\Delta y}{\Delta x}$

$= \frac{5-2}{0-1} = \frac{3}{-1}$

$m_1 = -3$

For $\bar{BC}$

slope, $m_2 = \frac{\Delta y}{\Delta x}$

$= \frac{1-2}{-2-1} = \frac{-1}{-3}$

$m_2 = \frac{1}{3}$

Hence $m_1 m_2 = -3 \times \frac{1}{3}$

$m_1 m_2 = -1$

Since slope of $\bar{AC}$ (m_1) × slope $(\bar{BC})(m_2)$ = 1

Hence the lines are perpendicular

for Area

from

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 0 & 5 & 1 \\ -2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$A = \frac{1}{2}\left[0(1-1) - 5(-2-1) + 1(-4-1)\right]$$

$$= \frac{1}{2}(15 - 5) = \frac{10}{2}$$

$A =$ 5 square unit

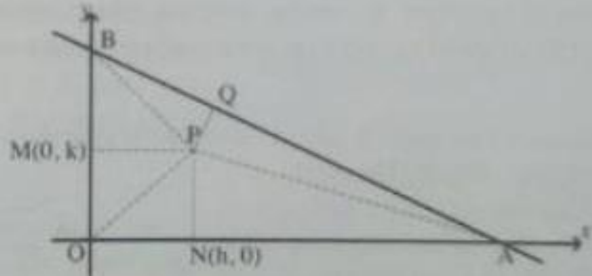
$\therefore$ The area of Δ ABC is 5 square unit.

In Extract 8.2, the candidate showed a good understanding of the concepts coordinate geometry in answering part (b) of the question.

2016 PAST PAPERS

correct to four decimal places.

8. (a) (i) The line $Ax + By + C = 0$ meets coordinates axes at A and B. If P is a point (h, k) and $PQ = p$ is the perpendicular distance to AB. Use the information given and the figure below to derive the perpendicular distance of the point P from the line AB.



(ii) The perpendicular distance from the point $(2, 5)$ to the line $ny = 2x - 4$ is $\sqrt{5}$. Find the value of n.

(b) Write down the equation to the bisector of the acute angle between the lines $3x + 4y = 1$ and $5x - 12y + 6 = 0$.

(c) Find the length of a tangent from the centre of the circle $x^2 + y^2 + 6x + 8y - 1 = 0$ to the circle $x^2 + y^2 - 2x + 4y - 3 = 0$.

8(a) Soln

distance $\overline{AB}$

$A = (-C/A, 0) \quad B = (0, -C/B)$

$$AB = \sqrt{(0 + C/B)^2 + (-C/A - 0)^2}$$

$$= \sqrt{\frac{C^2}{B^2} + \frac{C^2}{A^2}} = C\sqrt{\frac{A^2 + B^2}{A^2B^2}}$$

$$AB = \frac{C}{AB}\sqrt{A^2 + B^2}$$

$$\text{Area of } \Delta APB = \frac{1}{2}\begin{vmatrix} h & k & 1 \\ -c/A & 0 & 1 \\ 0 & -c/B & 1 \end{vmatrix}$$

$$A = \frac{1}{2}\left| h\left(\frac{c}{B}\right) - k\left(\frac{-c}{A}\right) + \frac{c^2}{AB} \right|$$

$$A = \frac{c}{2AB}(Ah + Bk + C) \text{ . — (i).}$$

8(a)

(i) Also.

$$A = \frac{1}{2} \cancel{bh} = \frac{1}{2} \overline{AB} p .$$

$$A = \frac{1}{2} \frac{c}{AB} \sqrt{A^2+B^2} \cdot p \cdot \quad \text{— (ii).}$$

eqn (i) = eqn (ii).

$$\frac{c}{2AB}(Ah+Bk+c) = \frac{c}{2AB}\sqrt{A^2+B^2}\ p$$

$$p = \left| \frac{Ah + Bk + C}{\sqrt{A^2+B^2}} \right| .$$

$$\therefore \text{ perpendicular distance is } p = \boxed{\left| \frac{Ah+Bk+C}{\sqrt{A^2+B^2}} \right|}$$

hence derived.

Extract 8.2 shows how the candidate derived the perpendicular distance formula in part (a) (i) correctly.

2015 PAST PAPERS

8. (a) (i) Sketch the diagram of the locus of points which move such that they are equidistant from two intersecting lines.

(ii) Find the equations of bisectors to two intersecting lines whose equations are $6x - 8y = -7$ and $4x + 3y = 12$.

(iii) Find the equation of the locus of points which is equidistant from the lines $y = 2x$ and $2x + 4y - 3 = 0$.

(b) Determine the distance of the point $(8, -6)$ from the line $2x + 5y + 34 = 0$.

8 a) ii) let the equations be

$$a_1x + b_1y + c_1 = 6x - 8y + 7$$

$$a_2x + b_2y + c_2 = 4x + 3y - 12$$

Formula for bisectors of lines.

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \left|\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

$$\frac{6x - 8y + 7}{\sqrt{6^2 + (-8)^2}} = \pm\left(\frac{4x + 3y - 12}{\sqrt{4^2 + 3^2}}\right)$$

$$\frac{6x - 8y + 7}{10} = \pm\left(\frac{4x + 3y - 12}{5}\right)$$

8 a) ii) $6x - 8y + 7 = \pm 2(4x + 3y - 12)$

$$6x - 8y + 7 = \pm(8x + 6y - 24)$$

either

$$6x - 8y + 7 = 8x + 6y - 24 \quad \text{or}$$

$$6x - 8y + 7 = -8x - 6y + 24$$

for

$$6x - 8y + 7 = 8x + 6y - 24$$

$$8x - 6x + 6y + 8y - 24 - 7 = 0$$

$$2x + 14y - 31 = 0$$

for

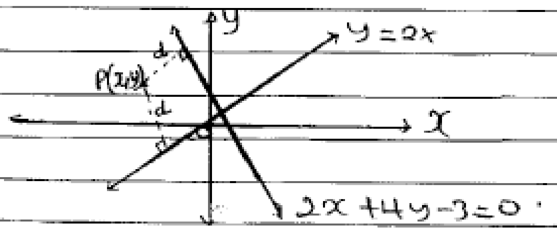
$$6x - 8y + 7 = -8x - 6y + 24$$

$$6x - 8y + 8x + 6y + 7 - 24 = 0$$

$$14x - 2y - 17 = 0$$

$\therefore$ The equations of bisectors are $2x + 14y - 31 = 0$ and $14x - 2y - 17 = 0$

8 (a) iii)



8 a) iii) from distance formula.

$$d = \left| \frac{ax + by + c}{\sqrt{a^2 + b^2}} \right|$$

for $y = 2x$

$2x - y = 0$

$$d = \frac{2x - y}{\sqrt{2^2 + (-1)^2}}$$

$$d = \frac{2x - y}{\sqrt{5}}$$

for $2x + 4y - 3$

$$d' = \frac{2x+4y-3}{\sqrt{2^2+4^2}}$$

$$d' = \frac{2x+4y-3}{\sqrt{20}}$$

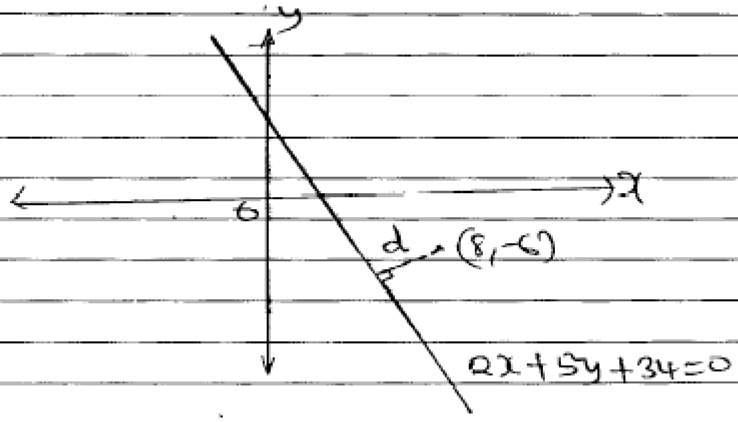
$$d' = \frac{2x+4y-3}{2\sqrt{5}}$$

but the locus of a point is such that

$$d = d'$$

$$\frac{2x-y}{\sqrt{5}} = \frac{2x+4y-3}{2\sqrt{5}}$$

8	a) iii) $2(2x-y) = 2x+4y-3$
	$4x-2y = 2x+4y-3$
	$4x-2x-2y-4y+3=0$
	$2x-6y+3=0$
	$\therefore$ The equation of locus of points is $2x-6y+3=0$
8	b)



from distance formula:

x:

$$d = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$$(x_1, y_1) = (8, -6)$$

$$d = \frac{2(8) + 5(-6) + 34}{\sqrt{2^2 + 5^2}}$$

8 b) $d = \frac{16 - 30 + 34}{\sqrt{29}}$

$d = \frac{20}{\sqrt{29}}$ units

$d = 3.714$ units

$\therefore$ distance of a point is 3.714 units

Extract 8.1 shows a candidate's work in which he /she was able to sketch the diagram of the locus and determined correctly the equations of bisectors of two intersecting lines as well as using the distance formula accurately.

5.0 Functions

2021 PAST PAPERS

6. (a) If $f(x)=\sqrt{x^2-9}$ and $g(x)=x-2$, find $f(g(x))$.

(b) Draw the graph of $f(g(x))$ obtained in part (a).

(c) Given that $y=\dfrac{x^2-9}{x-1}$;

(i) find the asymptotes of y,

(ii) draw the graph of y.

6 (a) soln

Given $f(x) = \sqrt{x^2-9}$.

$g(x) = x-2$.

Now

$f(g(x)) = ?$

$f(g(x)) = \sqrt{(x-2)^2 - 9}$.

$f(g(x)) = \sqrt{x^2-4x-5}$.

$\therefore \quad f(g(x)) = \sqrt{x^2-4x-5}$.

(b) Graph of $f(g(x))$.

vertical asymptote

$x^2-4x-5 = 0$

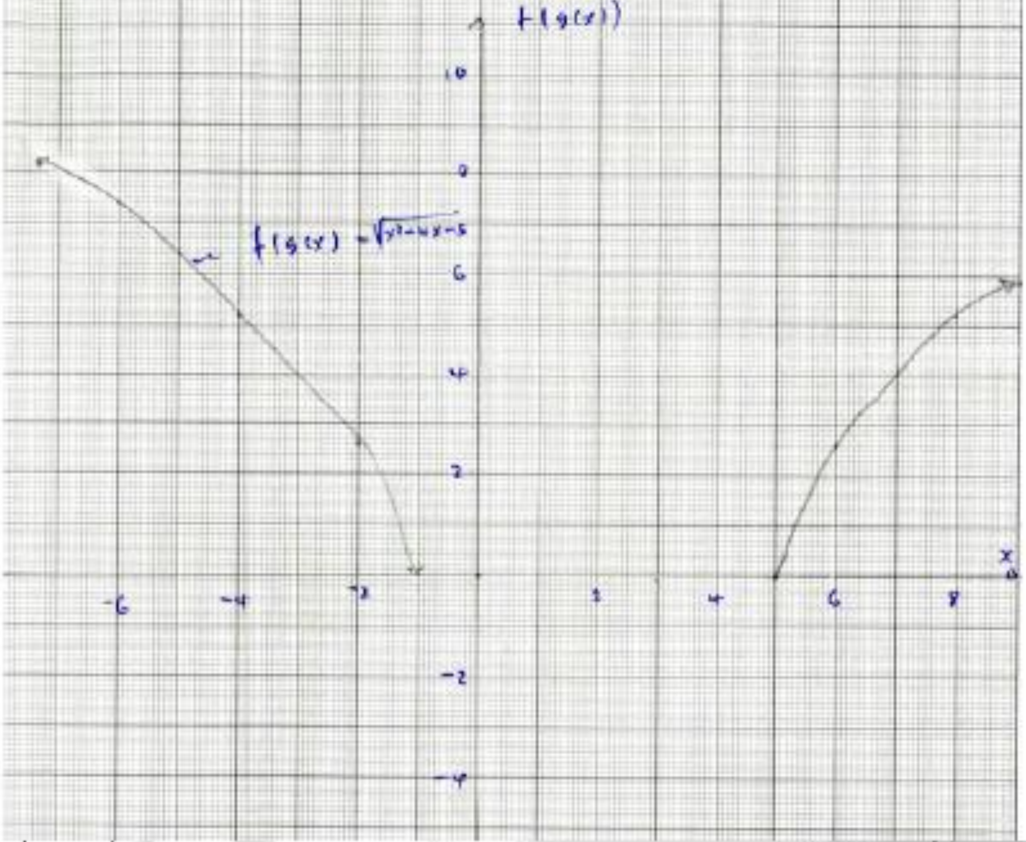
x-intercept $x = 5$ and $x = -1$

(5, 0) and (-1, 0)

Range = $\{y: y \in \mathbb{R};\ y \geq 0\}$

Domain = $\{x: x \in \mathbb{R};\ \text{except } -1 < x < 5\}$.

6 (b) GRAPH of $f(g(x))$



6	(c) soln	
	Given	
	$y = \dfrac{x^2-9}{x-1}$.	
	(i)	
	Vertical asymptote;	
	$x-1 = 0$	
	$\underline{x = 1}$	
	Slant asymptote;	
	$x+1$	
	$x-1 \overline{)\ x^2-9.}$	
	$\underline{x^2 - x}$	
	$x+9.$	
	$\underline{x-1}$	
	$-8.$	

$$y = x + 1 - \frac{3}{x-1}$$

$$x \to \infty$$

$\therefore$ Slant asymptote

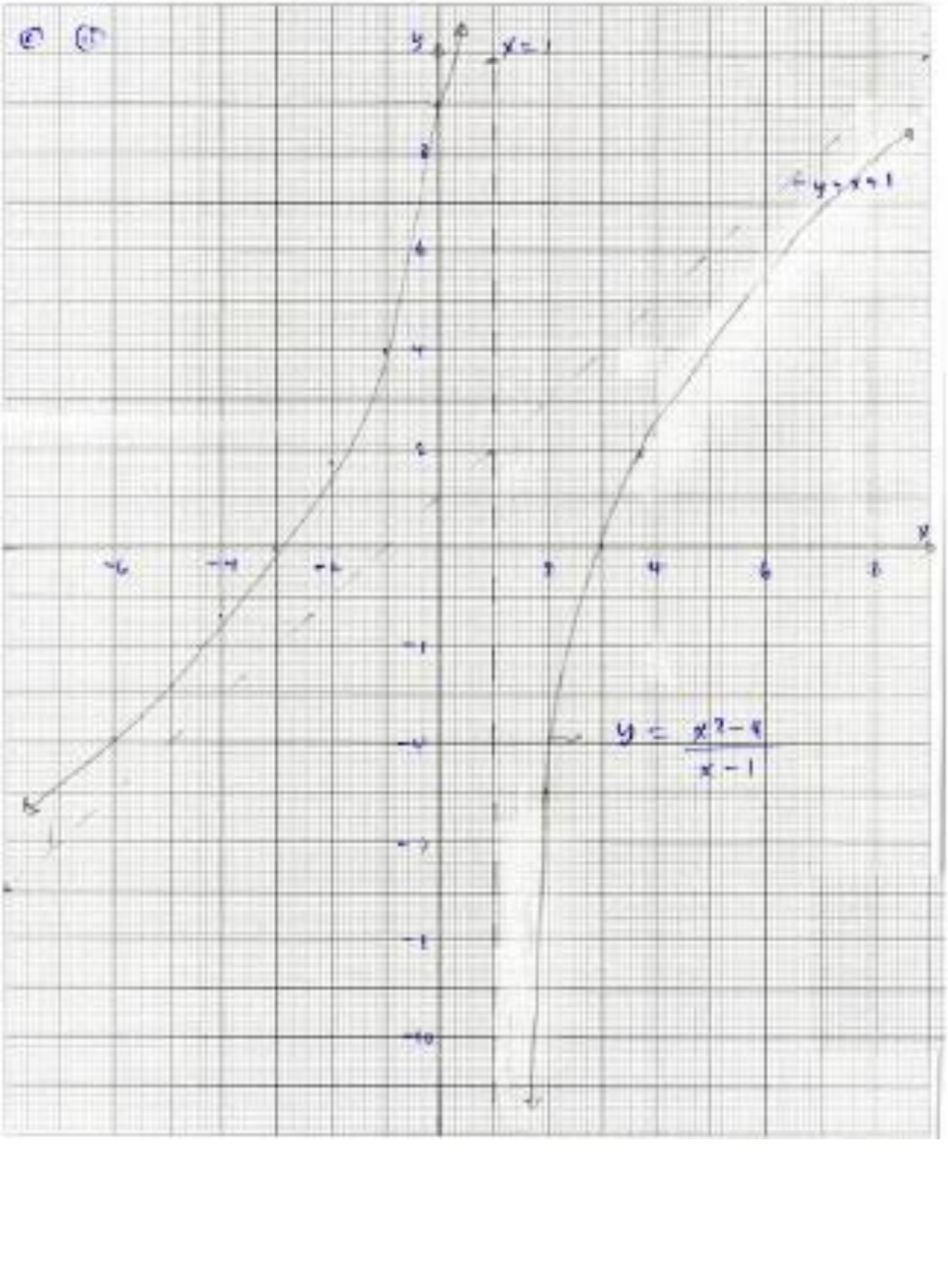
$y = x + 1$.

(ii) Graph of y:

Domain: $\{x : x \in \mathbb{R};$ except $x = 1\}$

Range: {All real number}.

(c) (i)



2020 PAST PAPERS

6. (a) (i) If $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$, find fog,

(ii) Copy and complete the following table of values,

x	-3	-2	-1	0	1	2
fog						

(iii) Use the table of values in (ii) to sketch the graph fog.

(b) If $y = \dfrac{x^2 - 2x - 3}{x^2 - 4}$;

(i) find the vertical and horizontal asymptotes,

(ii) sketch the graph of y.

6. (a) (i) $f(x) = x^2+1$

$g(x) = \sqrt{x-1}$

$fog\,x = f(g(x))$

$= (\sqrt{x-1})^2 + 1$

$= (x-1)+1$

$= x-1+1$

$\therefore f \cdot g = x$

(ii)

x	-3	-2	-1	0	1	2
fog	-3	-2	1	0	1	2

(iii) On the graph paper.

(b) $y = \dfrac{x^2 - 2x - 3}{x^2 - 4}$

(i) Horizontal Asymptote (H.A)

$y = \dfrac{\text{Leading Coefficient of Numerator polynomial}}{\text{Leading Coefficient of Denominator polynomial}}$

$y = \dfrac{1}{1}$

$\therefore$ Horizontal Asymptote is $y = 1$

Vertical Asymptote (V.A)

Let: $x^2 - 4 = 0$

$x^2 = 4$

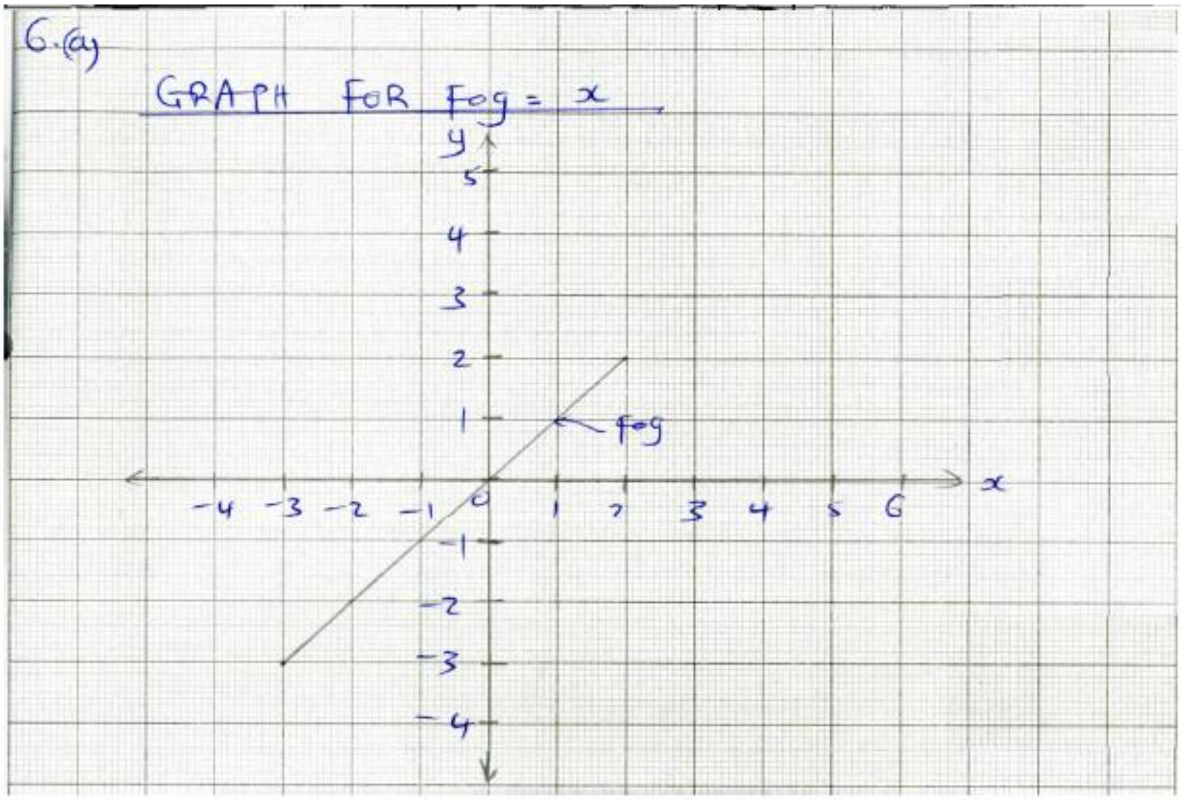
$x = \sqrt{4}$

$x = \pm 2$

$\therefore$ Vertical Asymptotes are $x = -2$ and $x = 2$

Let: $y = f(x) = \dfrac{x^2 - 2x - 3}{x^2 - 4}$

x	-3	-2	-1	0	1	2	3	4
y	2.4	∞	0	0.75	1.3	∞	0	0.4



2019 PAST PAPERS

6. (a) (i) Mention any two properties of $f(x) = b^x$.

(ii) Draw the graph of $f(x) = \left(\frac{1}{2}\right)^x$ for $-3 \le x \le 3$.

(b) Given that $y = \dfrac{x^2 - 2x - 3}{x^2 - 4}$

(i) Find the vertical and horizontal asymptotes.

(ii) Sketch the graph of y.

6(b)i Properties of $f(x) = 6^x$

- The domain of f(x) satisfies all real numbers for values of x.

ie Domain = $\{x : x \in \mathbb{R}\}$

- The Range of f(x) does not satisfy all real numbers for values of y.

ie

$$y = 6^x$$
$$\log y = \log 6^x$$
$$\log y = x \log 6$$
$$x = \frac{\log y}{\log 6}$$

$$\therefore y > 0$$

$$\therefore \text{Range} = \{y : y \in \mathbb{R} \text{ and } y > 0\}$$

→ If y = 0 or negative value, the function is not satisfied and it is undefined

6(b)ii Soln

Required:- Graph of $f(x) = \left(\frac{1}{2}\right)^x$ for $-3 \leq x \leq 3$

Table of values, $y = \left(\frac{1}{2}\right)^x$

x	−3	−2	−1	0	1	2	3
y	8	4	2	1	½	¼	⅛

Intercepts:

X - intercept , y = 0

6(a) ii.

$0 = \left(\frac{1}{2}\right)^x$

$\log 0 = x \log\left(\frac{1}{2}\right)$

$x =$ undefined

$\therefore$ No x-intercept

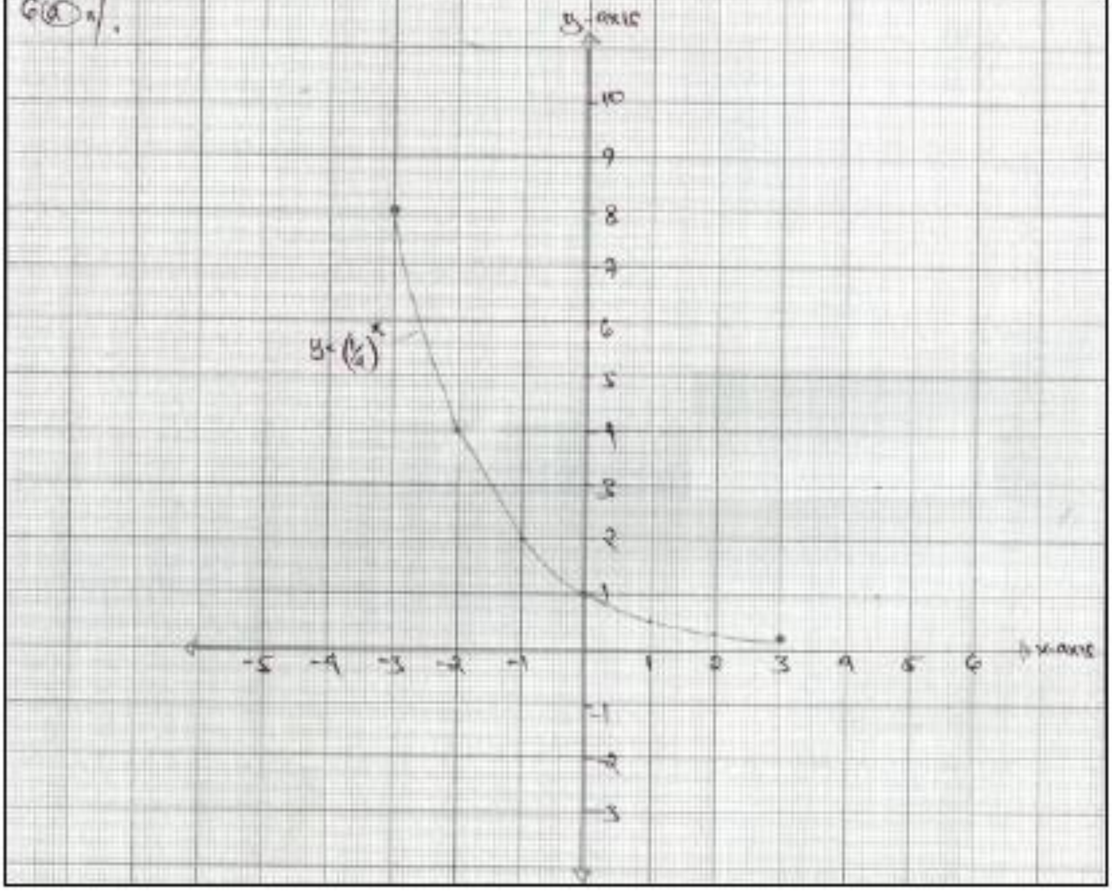
y-intercept, $x = 0$

$y = \left(\frac{1}{2}\right)^0$

$y = 1$

$\therefore$ y-intercept ; $(0, 1)$

6(a) iii.



6(b) Soln

Given $y = \dfrac{x^2 - 2x - 3}{x^2 - 4}$

i/. Vertical and Horizontal Asymptotes.

Vertical asymptotes:

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm\sqrt{4}$

$x = \pm 2$

$\therefore$ Vertical asymptotes: $x = 2$ and $x = -2$

Horizontal asymptotes

$y = \dfrac{x^2 - 2x - 3}{x^2 - 4}$

6(b)i $y = \dfrac{x^2/x^2 - 2x/x^2 - 3/x^2}{x^2/x^2 - 4/x^2}$

$y = \dfrac{1 - 2/x - 3/x^2}{1 - 4/x^2}$

As $x \to \infty$

$y = \dfrac{1 - 2/\infty - 3/\infty}{1 - 4/\infty}$

$y = \dfrac{1 - 0}{1 - 0}$

$y = 1$

$\therefore$ Horizontal asymptote: $y = 1$

6(b)ii To sketch graph of y

Intercepts

X-intercept, $y = 0$

$0 = \dfrac{x^2 - 2x - 3}{x^2 - 4}$

$x^2 - 2x - 3 = 0$

$x = 3$ and $x = -1$

$\therefore$ X-intercepts: $(3, 0)$ and $(-1, 0)$

Y-intercept, $x = 0$

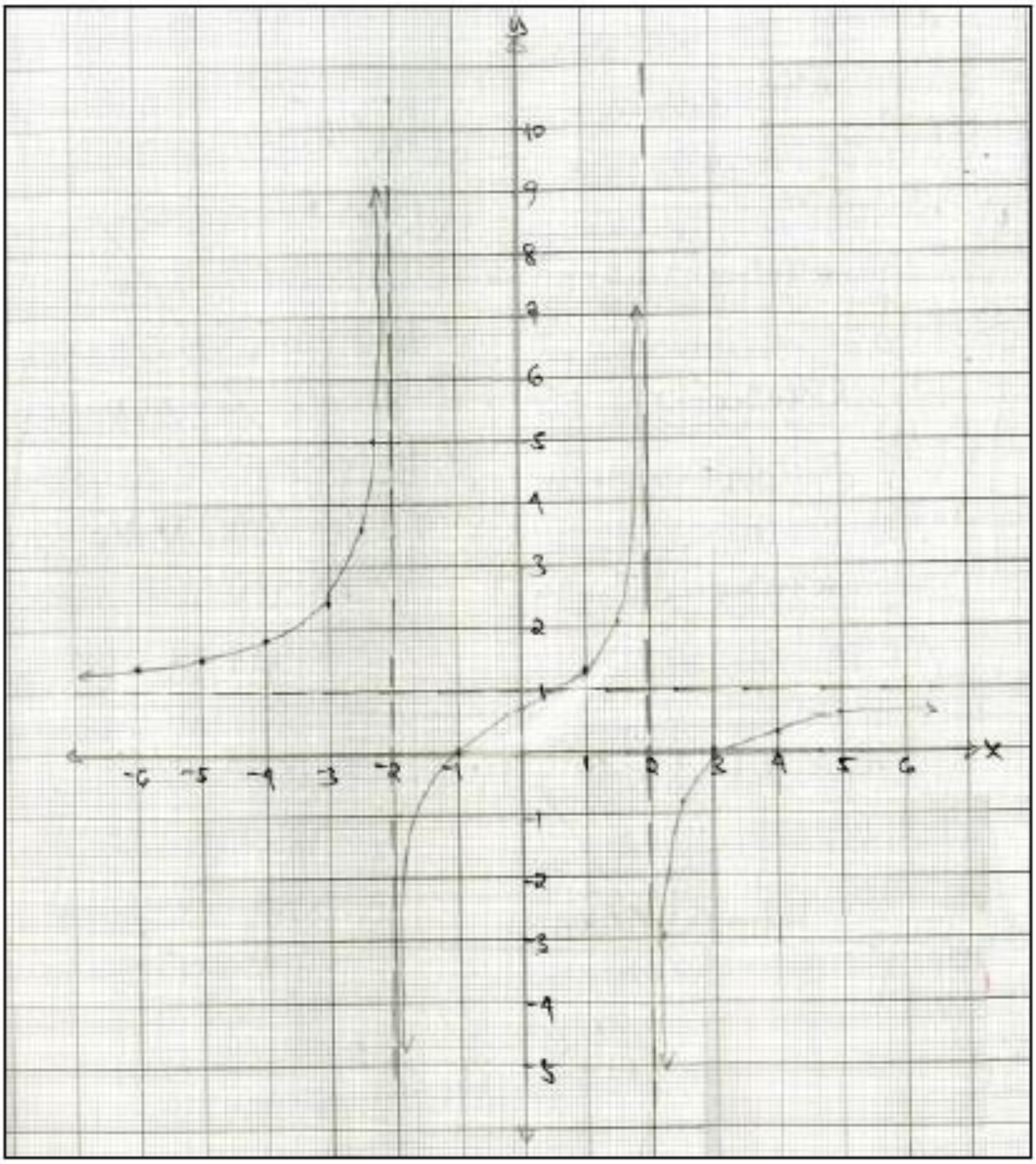
$y = \dfrac{0^2 - 2(0) - 3}{0^2 - 4}$

$y = 3/4$

$\therefore$ Y-intercept: $(0, 3/4)$

6(b)ii Table of values

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	1.4	1.5	1.8	2.4	∞	0	0.75	1.3	∞	0	0.4	0.6



y
10
9
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
x

2018 PAST PAPERS

6. (a) (i) Given the functions $f(t) = e^t$ and $g(t) = \ln t$. Show that $f \circ g(t) = gof(t)$.

(ii) If $f(t) = at$, $g(t) = bt^2 + 3$, $(fog)(2) = 35$ and $(fog)(3) = 75$, find the values of a and b.

(b) Given that, $f(x) = \dfrac{x^3}{1 - x^2}$

(i) Find horizontal and vertical asymptotes of $f(x)$.

(ii) Sketch the graph of $f(x)$.

(iii) State the domain and range of the function $f(x)$.

6 (a) (ii) $f(t) = at$, $g(t) = bt^2 + 3$

find fog

$fog = a(bt^2 + 3)$

$= abt^2 + 3a.$

but $fog(2) = 35$

$35 = ab(2)^2 + 3a.$

$35 = 4ab + 3a. \quad - - (i)$

also $fog(3) = 75$

$75 = ab(3)^2 + 3a.$

$75 = 9ab + 3a \quad - - (ii)$

Solve simultaneously

$$- \begin{cases} 9ab + 3a = 75 \\ 4ab + 3a = 35 \end{cases}$$

$5ab = 40$

$ab = 8.$

$a = 8/b$ take eqn (i)

$35 = 4\left(\frac{8}{b}\right) b + 3\left(\frac{8}{b}\right)$

$35 = 32 + \frac{24}{b}.$

$35b = 32b + 24$

$3b = 24$

$b = 8 \quad - - (iii)$

and $a = 8/b$

$a = 8/8 = 1 \quad - - (iv)$

$\therefore a = 1$ and $b = 8.$

6(b) $f(x) = \dfrac{x^3}{1-x^2}$.

i) Horizontal asymptotes.

$y = \dfrac{x^3}{1-x^2}$

$y = \dfrac{x^3/x^2}{1/x^2 - x^2/x^2}$

$y = \dfrac{x}{1/x^2 - 1}$

as $x \to \infty$

$y = \dfrac{x}{0-1}$

$\underline{y = -x}$

vertical asymptotes.

let D(denominator (x)) = 0

$1 - x^2 = 0$

$x^2 = 1$

$x = \pm 1$

$\underline{x = 1 \text{ or } x = -1}$

find x and y intercept

x-intercept, $y = 0$.

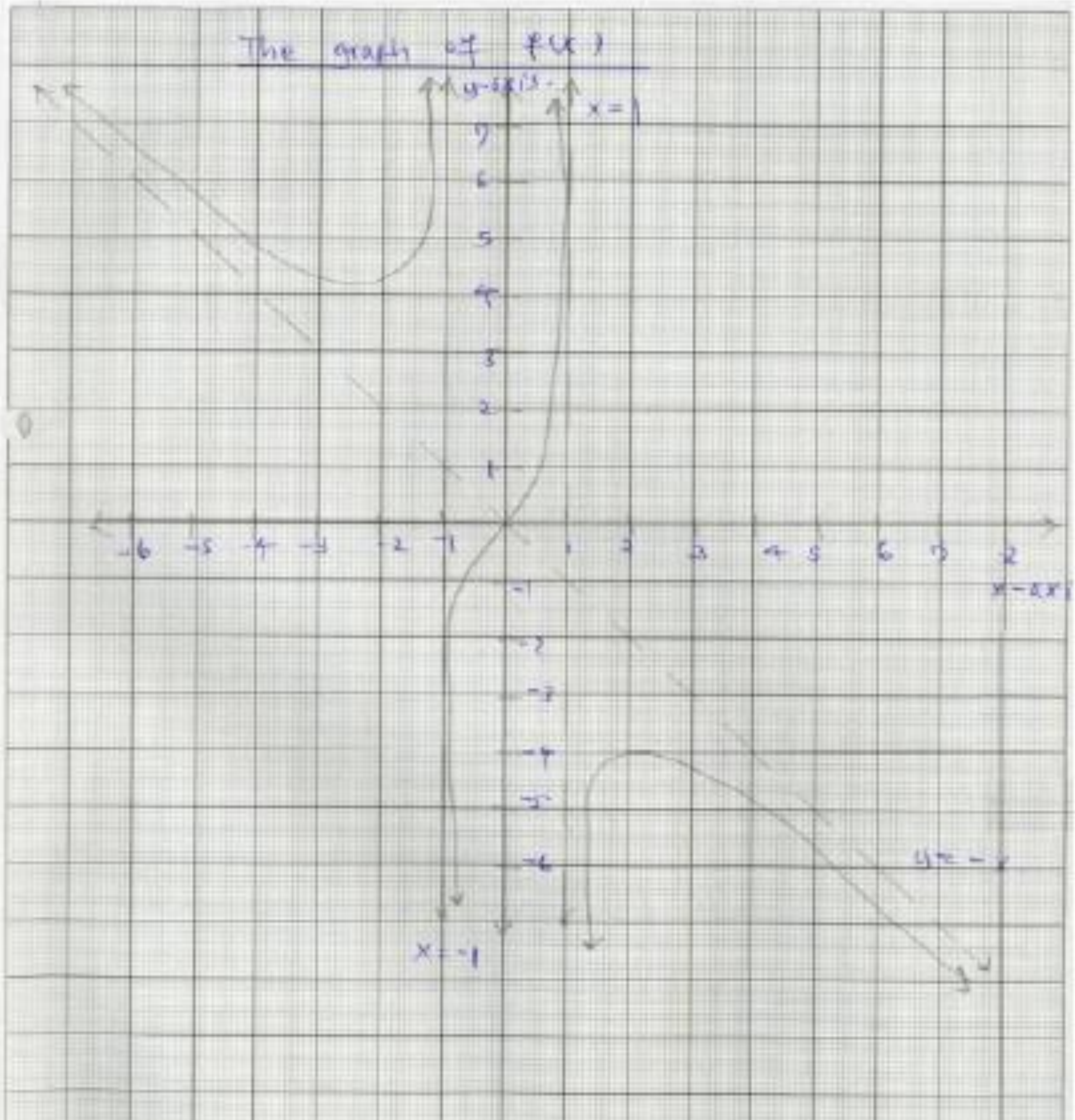
$0 = \dfrac{x^3}{1-x^2}$

$x = 0$ x-intercept, $(0,0)$

y-intercept, $x = 0$

$y = \dfrac{0}{1-0} = 0$.

y-intercept $(0,0)$



6 (b)	(i) A graph of f(x)
	(ii) Domain = {all real numbers, $x \neq 1$ and $x \neq -1$}
	Range = {all real numbers}.

2017 PAST PAPERS

6. (a) Draw the graph of $f(x)=x^3-3x^2-6x+8$ in the interval $[-5,6]$. Hence tell how $f(x)$ behaves for positively and negatively large values of x.

(b) Find $f\circ g(x)$ given that $f(x)=2x^2+1$ and $g(x)=\frac{4x}{x^2-2}$, hence

(i) Determine the vertical and horizontal asymptotes of $f\circ g(x)$.

(ii) Draw the graph of $f\circ g(x)$.

(iii) State the domain and range of $f\circ g(x)$.

6. (a) Given:

$$f(x) = x^3 - 3x^2 - 6x + 8$$

Interval $[-5, 6]$

Table of Value.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	-162	-80	-28	0	10	8	0	-8	-10	0	28	84

check on the graph:

Then,

as $x \rightarrow +\infty$

$f(x) \rightarrow +\infty$

also:

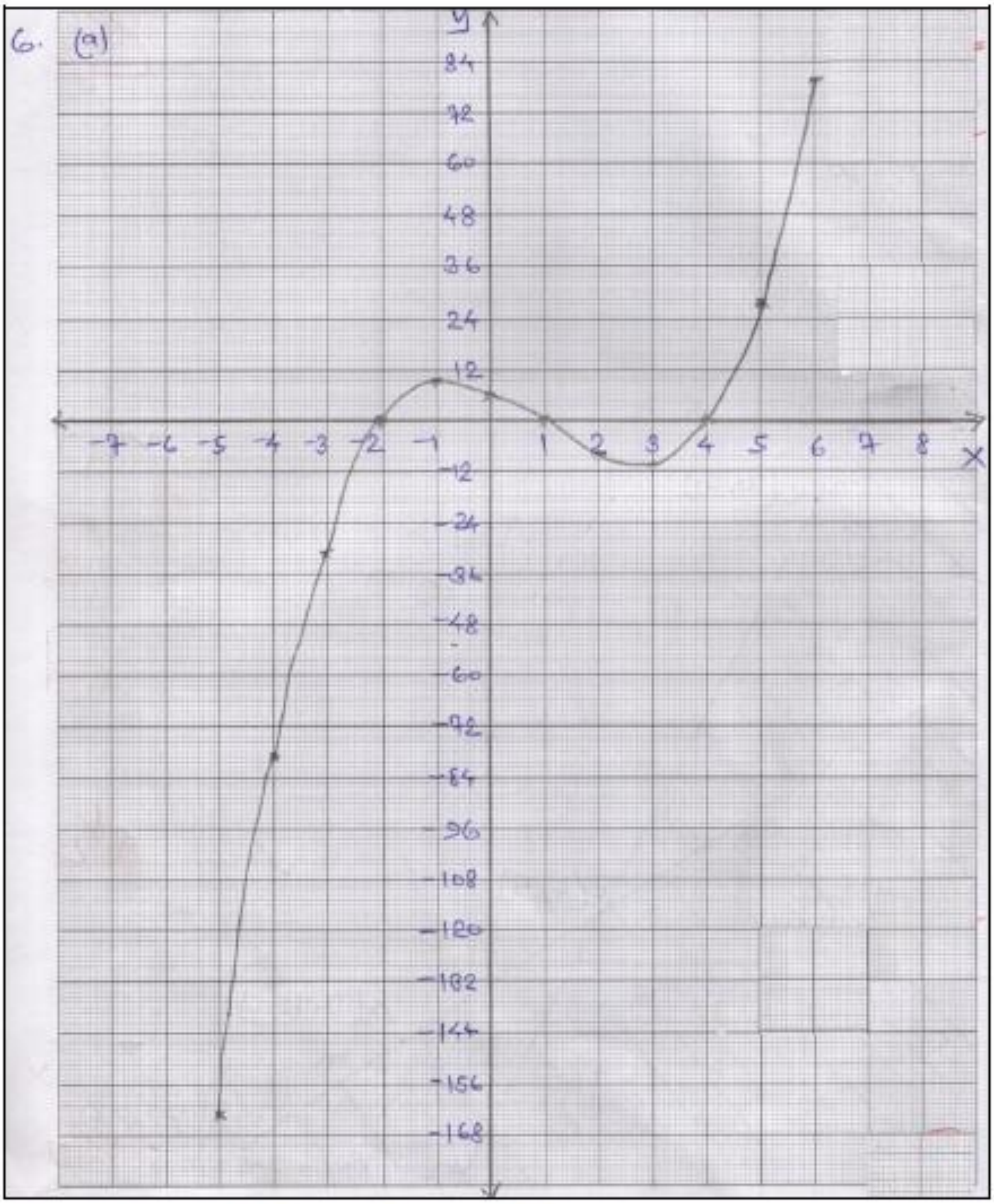
as $x \rightarrow -\infty$

$f(x) \rightarrow -\infty$

$\therefore$ for positively large value of x $f(x)$ approaches to positively large value

also

for negatively large value of x $f(x)$ approaches to negatively large value.



In Extract 6.1, the candidate successfully prepared a table of values and managed to draw the correct graph.

2016 PAST PAPERS

6. (a) Use the table of values to draw the graph of $f(x)=2+e^{2x}$ if $-3\le x\le 1.2$ and $g(x)=1-e^{x}$ if $-3\le x\le 2.7$ on the same xy plane.

(b) Given that, $f(x)=x+1-\frac{1}{x}$ and that $g(x)=\frac{1}{x}$;

(i) Write down the composite function $g\,o\,f(x)$ in its simplest form.

(ii) Find the value of x if $g\,o\,f(x)=f\,o\,g(x)$.

(c) Find the equation of the asymptotes of the curve $y=\frac{x^2+3}{x-1}$ and sketch the curve showing the coordinates of the turning points.

6 a) $f(x)=2+e^{2x}$ for $-3\le x\le 1.2$

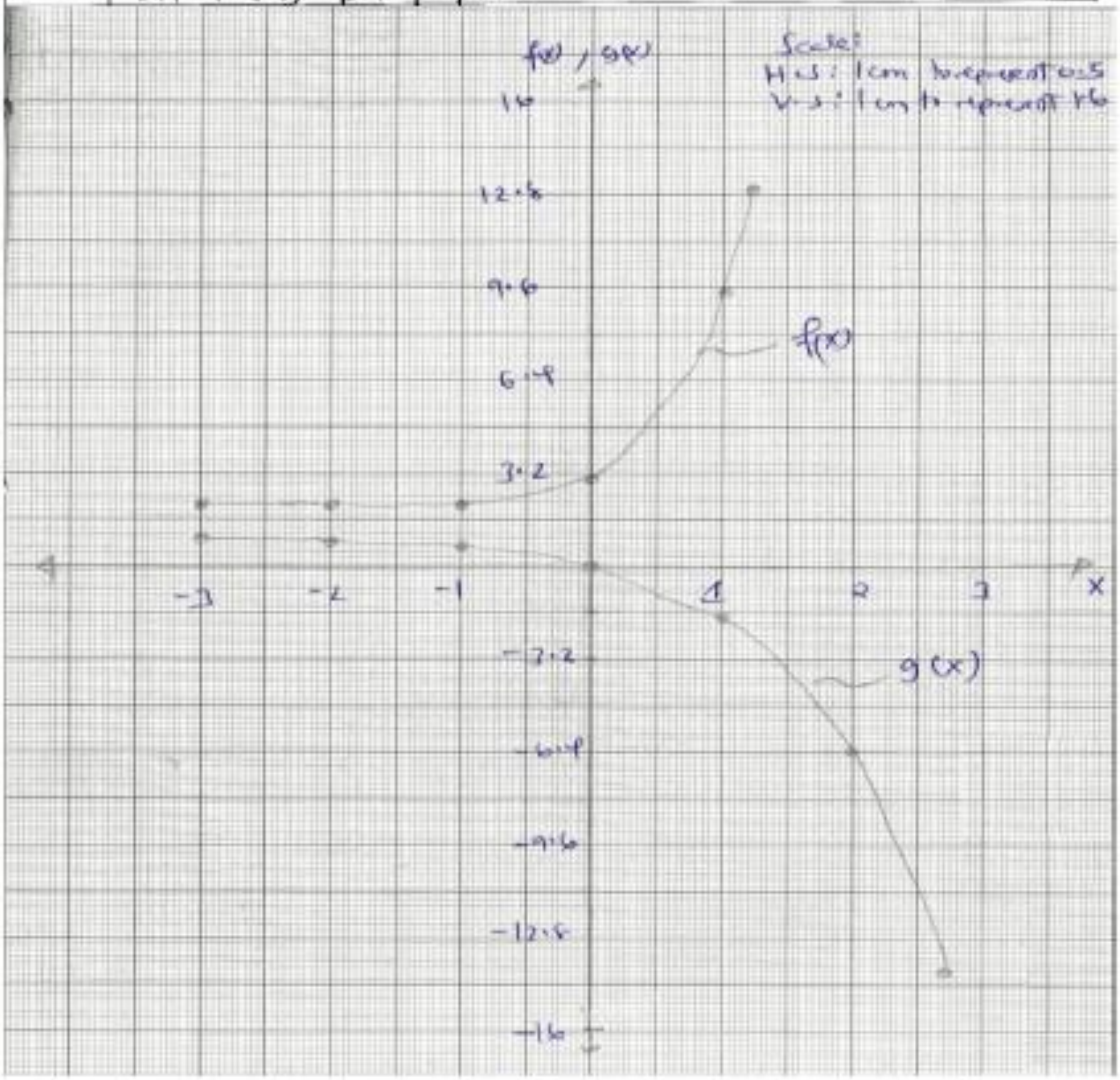
$g(x)=1-e^{x}$ if $-3\le x\le 2.7$

Table of values

X	−3	−2	−1	0	1	1.2
fx	2	2.02	2.14	3	9.39	13.02

X	−3	−2	−1	0	1	2	2.7
g(x)	0.95	0.86	0.63	0	−1.72	−6.39	−13.88

The graph of f(x) and g(x) against x are drawn on the graph paper.



In Extract 6.1, the candidate succeeded to use the tables of values to draw the graph in part (a).

2015 PAST PAPERS

6 (a) If $f : x \rightarrow 5x + 4$ and $g : x \rightarrow 6x - k$:

(i) Determine the value of k for which $f \circ g(x) = g \circ f(x)$.

(ii) Prove that $f \circ (f \circ f(x)) = 125x + 124$.

(b) Draw the graph of $\frac{2x^2}{x^2 - 9}$.

6. (a) $f(x) = 5x + 4$

$g(x) = 6x - k$

i/ $fog(x) = f(6x - k)$

$= 5(6x - k) + 4$

$= \cancel{30x - 30k}$

$= 30x - 5k + 4$

$gof(x) = g(5x + 4)$

$= 6(5x + 4) - k$

$= 30x + 24 - k$

$fog(x) = gof(x)$

$\cancel{30x} - 5k + 4 = \cancel{30x} + 24 - k$

$-5k + k = 24 - 4$

$-4k = 20$

$k = -5$

$\therefore \quad \underline{k = -5}$

ii) $f \circ (f \circ f(x)) = 125x + 124$

$f \circ f(x) = f(5x+4)$
$= 5(5x+4) + 4$
$= 25x + 20 + 4$
$= 25x + 24$

$f \circ (25x+24) = 5(25x+24) + 4$
$= 125x + 120 + 4$
$= 125x + 124$

$\therefore \; f \circ (f \circ f(x)) = 125x + 124$ hence proved

6. (b) $y = \dfrac{2x^3}{x^2 - 9}$

Vertical asymptotes

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

$x = 3, \; x = -3$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-15.6	-18.3	α	3.2	0.25	0	-0.25	-3.2	α	18.3	15.6

$$
\begin{array}{r|l}
 & 2x \\
x^2 - 9 & 2x^3 \\
 & -\;(2x^3 - 18x) \\
\hline
 & 18x
\end{array}
$$

$= 2x + \dfrac{18x}{x^2 - 9}$

$\dfrac{18x}{x^2 - 9}$

$\dfrac{18/x}{1 - 9/x} \quad x \to \alpha$

$= 0$

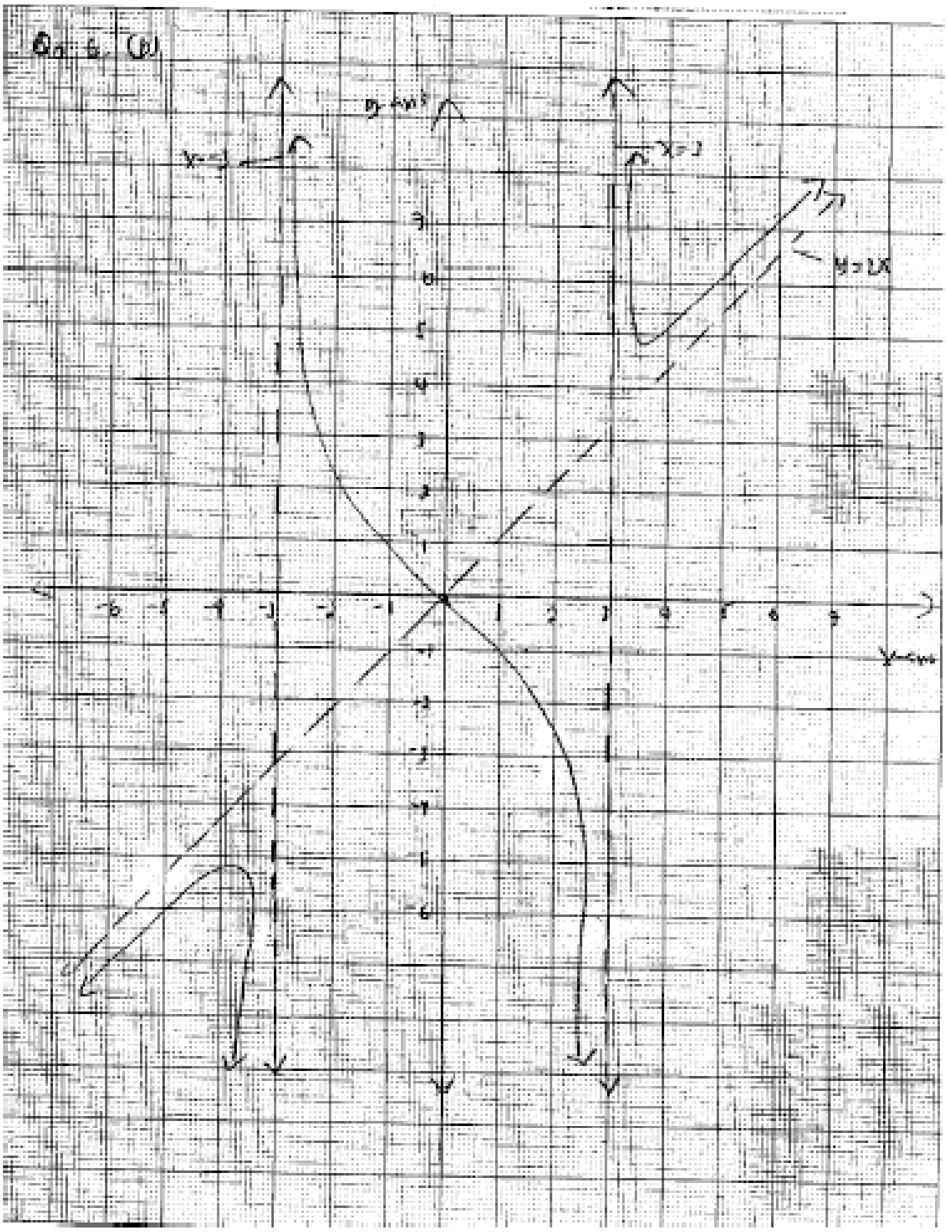
$y = 2x$

Slant asymptote $y = 2x$

Y-intercepts $x = 0$, $y = 0$ $(0, 0)$

X-intercepts $y = 0$ $x = 0$ $(0, 0)$

$\Rightarrow$ The graph is on the graph paper



Extract 6.1 shows the solution from a candidate who was able to determine the value of k from the given equation. The candidate was also able to prove that $f\circ(f\circ f(x)) = 125x + 124$. In part (b), he/she drew the graph of the rational function as required.

6.0 Algebra

2021 PAST PAPERS – 2

6. (a) By using the first five terms in the expansion of $(1+x)^n$, find the value of $(1.98)^{10}$ correct to three decimal places.

(b) The polynomial x^5+4x^2+ax+b leaves the remainder of $2x+3$ when it is divided by x^2-1. Use the remainder theorem to find the values of a and b.

(c) The roots of the quadratic equation $x^2+2mx+n=0$ differ by 2. Show that $m^2=1+n$.

(d) If $A=\begin{pmatrix} 4 & -1 & 1 \\ 0 & 0 & 2 \\ m & -1 & 1 \end{pmatrix}$ is singular, find the value of m.

(e) Use Cramer's rule to solve the following system of equations: $\begin{cases} 5x+6y+4z=5 \\ 7x-4y-3z=8 \\ 2x+3y+2z=2 \end{cases}$

6. (a) $(1+x)^n$

from

$$(1+x)^n = 1 + nx + \frac{n \cdot (n-1) x^2}{2!} + \frac{n \cdot (n-1)(n-2) x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!}$$

Required expansion of $(1.98)^{10}$.

$$(1.98)^{10} = (2 - 0.02)^{10}$$

$$= \left[2\left(1 - \frac{0.02}{2}\right)\right]^{10}$$

$$= 2^{10}(1 - 0.01)^{10}$$

$$2^{10}(1-0.01)^{10} = 2^{10}\left[1 + 10(-0.01) + \frac{10 \cdot (9)(-0.01)^2}{2!} + \frac{10 \cdot 9 \cdot 8 (-0.01)^3}{3!} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot (-0.01)^4}{4!}\right]$$

$$2^{10}(1-0.01)^{10} = 2^{10}\left[1 - 0.1 + 4.5 \times 10^{-3} - 1.2 \times 10^{-4} + 2.1 \times 10^{-6}\right]$$

$$= 2^{10}(0.9043821)$$

$$= 926.0872448$$

Correct to three decimal places.

$926.0872448 \simeq 926.087$

$\therefore (1.98)^{10} = 926.087$

(b). Given the polynomial

$x^5 + 4x^2 + ax + b$

remainder $2x+3$ when divided by x^2-1.

6. (b). Soln.

$$P(x^2-1) = x^5 + 4x^2 + ax + b = 2x + 3.$$

$$P(\pm 1) = x^5 + 4x^2 + ax + b = 2x + 3.$$

Start with positive 1.

$$P(1) = (1)^5 + 4(1)^2 + a(1) + b = 2(1) + 3$$

$$1 + 4 + a + b = 2 + 3$$

$$a + b + 5 = 5$$

$$a + b = 0 \quad \ldots\ldots \quad (i)$$

then negative -1.

$p(-1) = (-1)^3 + 4(-1)^2 + a(-1) + b = 2(-) + 3$

$-1 + 4 - a + b = -2 + 3$

$-a + b + 3 = 1$.

$-a + b = -2$. .... (ii)

Solve simultaneously the two eqn.

$a + b = 0$

$-a + b = -2$.

Hence $a = 1$ and $b = -1$.

$\therefore$ The value of a and b are 1 and -1

(c). Given the quadratic equation.

$x^2 + 2mx + n = 0$.

roots.

Sum of roots $= -b/a$

product of roots $= c/a$

6. (c) Sum $\alpha + \beta = -2m$ ..... (i)

$\alpha\beta = n$. ..... (ii)

but the roots differ by 2.

$\alpha - \beta = 2$. ..... (iii)

$\alpha = 2 + \beta$ ..... (iv)

Substitute eqn (iv) into eqn (i)

$2 + \beta + \beta = -2m$

$2 + 2\beta = -2m$.

$2\beta = -2m - 2$.

$\beta = -m - 1$. .... (v)

then.

Substitute eq (v) into eqn (iv)

$\alpha = 2 + (-m - 1)$

$\alpha = 2 - m - 1$

$\alpha = 1 - m$ .... (vi)

Sol from.

$\alpha . \beta = n$.

substitute the required information

$$(1-m)(-m-1) = n.$$
$$(-m-1+m^2+m) = n$$
$$-1+m^2 = n$$
$$m^2 = 1+n$$

$$m^2 = 1+n$$
Hence shown.

6 (d). Given,

$$A = \begin{pmatrix} 4 & -1 & 1 \\ 0 & 0 & 2 \\ m & -1 & 1 \end{pmatrix} \text{ is singular}$$

Since we know singular matrix is the one whose determinant is equal to zero.

Hence, we find determinant of element A.

$$\det A = 4\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} - -1\begin{vmatrix} 0 & 2 \\ m & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ m & -1 \end{vmatrix}$$

$$\det A = 8 - 2m + 0$$

but
$$\det A = 0$$
$$0 = 8 - 2m$$
$$m = 4.$$

∴ The value of m is 4.

(e). Soln.

By using Cramer's rule

Given.

$$5x + 6y + 4z = 5$$
$$7x - 4y - 3z = 8$$
$$2x + 3y + 2z = 2.$$

Matrix form.

$$\begin{pmatrix} 5 & 6 & 4 \\ 7 & -4 & -3 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 2 \end{pmatrix}$$

6 (b)
$$\begin{pmatrix} 5 & 6 & 4 \\ 7 & -4 & -3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 2 \end{pmatrix}$$

By cramer's rule.

$$\det(\delta) = 5\begin{vmatrix} -4 & -3 \\ 3 & 2 \end{vmatrix} - 6\begin{vmatrix} 7 & -3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 7 & -4 \\ 2 & 3 \end{vmatrix}$$

$= 5 - 120 + 116$

$= 1.$

about x.

$$\det(\delta x) = \begin{pmatrix} 5 & 6 & 4 \\ 8 & -4 & -3 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\det(\delta x) = 5\begin{vmatrix} -4 & -3 \\ 3 & 2 \end{vmatrix} - 6\begin{vmatrix} 8 & -3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 8 & -4 \\ 2 & 3 \end{vmatrix}$$

$\det(\delta x) = 1.$

about Y.

$$\begin{pmatrix} 5 & 5 & 4 \\ 7 & 8 & -3 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\det(\delta Y) = 5\begin{vmatrix} 8 & -3 \\ 2 & 2 \end{vmatrix} - 5\begin{vmatrix} 7 & -3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 7 & 8 \\ 2 & 2 \end{vmatrix}.$$

$\det(\delta Y) = 2.$

about Z =

$$\begin{pmatrix} 5 & 6 & 5 \\ 7 & -4 & 8 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\det(\delta z) = 5\begin{vmatrix} -4 & 8 \\ 3 & 2 \end{vmatrix} - 6\begin{vmatrix} 7 & 8 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 7 & -4 \\ 2 & 3 \end{vmatrix}$$

$\det(\delta z) = -3.$

6 (b) then.

$X = \dfrac{\delta x}{\delta} = \dfrac{1}{1}$

$x = 1$

$Y = \dfrac{\delta Y}{\delta} = \dfrac{2}{1}$

$Y = 2$

$Z = \dfrac{\delta z}{\delta} = \dfrac{-3}{1}$

$z = -3.$

--

$\therefore$ The value of x, y and z are $(1, 2, -3)$

2020 PAST PAPERS – 2

6. (a) Use the principles of mathematical induction to prove that $3^{2(n+1)} - 8n - 9$ is divisible by 8.

(b) Find the inverse of the matrix $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix}$.

(c) By using the inverse matrix obtained in (b), find the values of x, y and z in the simultaneous equations $\begin{cases} 3x - y + 2z = 11 \\ 2x + 3y + z = -1 \\ x + 2y - z = -6 \end{cases}$.

Qn 6: (a) Verification when $n = 1$

$= 3^{2(n+1)} - 8n - 9$

$= 3^{2(1+1)} - 8(1) - 9$

$= 81 - 8 - 9$

$= 64$

$= 8(8)$

It is true for $n = 1$

Assume it is true for $n = k$.

$\Rightarrow 3^{2(k+1)} - 8k - 9 = 8M$

$3^{2k} \cdot 3^2 - 8k - 9 = 8M$

$3^{2k} \cdot 3^2 = 8M + 8k + 9 \quad \text{--- (i)}$

Required to prove is when $n = k+1$

$= 3^{2(k+1+1)} - 8(k+1) - 9$

$= 3^{2(k+1+1)} - 8(k+1) - 9$

$= 3^{2(k+1)} \cdot 3^2 - 8(k+1) - 9$

but

$3^{2(k+1)} = 8M + 8k + 9$

$= (8M + 8k + 9) \cdot 3^2 - 8(k+1) - 9$

$= 9(8M + 8k + 9) - 8k - 8 - 9$

$= 72M + 72k + 81 - 8k - 17$

$= 72M + 64k - 64$

$= 8(9M + 8k - 8)$ it is true for $n = k+1$

$\therefore$ Hence it is divisible by 8.

2019 PAST PAPERS – 2

4 (a) (i) Express $\frac{1}{r(r+1)}$ in partial fractions.

(ii) From (a) (i) deduce the formula for $\sum_{r=1}^{n}\frac{1}{r(r+1)}$.

(b) A teacher bought pens, pencils and note books for her students. She bought 3 pens, 6 pencils and 3 note books in the first week; 1 pen, 2 pencils and 2 note books in the second week; as well as 4 pens 1 pencil and 4 note books in the third week. If she spent 3,000, 1,100 and 2,600 shillings in the first, second and third week respectively, use the inverse matrix method to find the price of each item.

(c) Use synthetic division to find the quotient and the remainder when $2x^4+3x^3-2x+5$ is divided by $x+5$.

4. (a) (i) Let: $\frac{1}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$

$1 = A(r+1) + Br$.

when $r = 0$.

$1 = A$

when $r = -1$

$1 = -B \Rightarrow B = -1$.

$\therefore \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$

4. (a) (ii) From: $\sum_{r=1}^{n}\frac{1}{r(r+1)}$

But $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$

$\therefore \sum_{r=1}^{n}\frac{1}{r(r+1)} = \sum_{r=1}^{n}\frac{1}{r} - \frac{1}{r+1}$

expanding the terms from 1 to n.

for $r = 1$: $1 - \cancel{1/2}$

$r = 2$: $+\cancel{1/2} - \cancel{1/3}$

$r = 3$: $+\cancel{1/3} - \cancel{1/4}$

$r = 4$: $+\cancel{1/4} - \cancel{1/5}$

⋮

$r = n-1$: $+\cancel{1/(n-1)} - \cancel{1/n}$

$r = n$: $+\cancel{1/n} - 1/(n+1)$

$$= 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1}$$

$$= \frac{n}{n+1}$$

4. (c) from: $x + 5 = 0$

$x = -5$

-5	2	3	0	-2	5
+	0	-10	35	-175	885
	2	-7	35	-177	890

$\therefore$ when $2x^4 + 3x^3 - 2x + 5$ is divided by $x + 5$, the quotient is $2x^3 - 7x^2 + 35x - 177$ and remainder is 890

2018 PAST PAPERS – 2

4. (a) Express $\dfrac{3x+1}{(x+1)(x^2+2x+3)}$ in partial fractions.

(b) (i) If $a^x = \left(\dfrac{a}{k}\right)^y = k^m$ where $a \neq 1$; show that $y = \dfrac{mx}{m-x}$.

(ii) If $x^y = y^{2x}$ and $y^2 = x^3$, solve for x and y.

(c) Expand $\sqrt{1+x}$ as far as the term in x^3 and use the result to obtain the value of $\sqrt{16.08}$ correct to six decimal places.

4(a)

$$\frac{3x+1}{(x+1)(x^2+2x+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+3}.$$

$$3x+1 = A(x^2+2x+3) + (Bx+C)(x+1)$$

$$3x+1 = Ax^2 + 2Ax + 3A + Bx^2 + Cx + Bx + C.$$

$$3x+1 = (A+B)x^2 + (2A+B+C)x + (3A+C)$$

Then on comparing

$A + B = 0$

$2A + B + C = 3$

$3A + C = 1$

On solving

$A = -1$ $B = 1$ and $C = 4$

$$\therefore \frac{3x+1}{(x+1)(x^2+2x+3)} = \frac{x+4}{x^2+2x+3} + \frac{-1}{x+1}$$

$$\therefore \frac{3x+1}{(x+1)(x^2+2x+3)} = \frac{x+4}{x^2+2x+3} - \frac{1}{x+1}$$ in partial fraction

(b) $a^x = \left(\frac{a}{k}\right)^y$ also $a^x = k^m$ $\left(\frac{a}{k}\right)^y = k^m$

Taking $a^x = \frac{a^y}{k^y}$ $k^y = a^{y-x}$

$\ln(k)^y = \ln(a^{y-x})$

$y \ln k = (y-x)\ln a$ —— (i)

Again:

$a^y = k^m \cdot k^y$

$a^y = k^{m+y}$

$\ln a^y = m+y \ln k$

$y \ln a = (m+y)\ln k$ —— (ii)

Considering eqn (i) $\ln k = \frac{(y-x)\ln a}{y}$; $\ln a = \frac{y \ln k}{y-x}$

Using eqn (ii) $y \ln a = (m+y)\ln k$

$y \ln a = (m+y)\ln k$ but $\ln a = \frac{y \ln k}{y-x}$

$y\left[\frac{y \ln k}{y-x}\right] = (m+y)\ln k$

$y^2 \ln k = (y-x)(m+y)\ln k$ divide by $\ln k$

$y^2 = (y-x)(m+y)$

$y^2 = y^2 - yx - mx + my$

$y^2 - y^2 = my - yx - mx$

$mx = my - yx$

$mx = y[m-x]$

$y = \frac{mx}{m-x}$ shown.

c (ii) $x^y = y^{2x}$ and $y^2 = x^3$

$x^y = (y^2)^x$ —— (i)

but $y^2 = x^3$

$x^y = (x^3)^x$

$x^y = x^{3x}$

$y = 3x$ —— (ii)

$x^y = y^{2x}$ and $y^2 = x^3$

Introducing natural logarithm throughout

$\ln x^y = \ln y^{2x}$

$y \ln x = 2x \ln y$

$y \ln x = 2x \ln y$ —— (iii)

and $y^2 = x^3$

$\ln y^2 = \ln x^3$

$$2\ln y = 3\ln x.$$
$$\ln x = \frac{2}{3}\ln y. \quad \text{---(IV)}$$

Substituting eqn (III) in eqn (IV)
$$y\left[\frac{2}{3}\ln y\right] = 2x\ln y.$$

$$\frac{2y}{3}\ln y = 2x\ln y.$$
$$\frac{y}{3} = x.$$
$$3x = y.$$
$$\ln y = \frac{3}{2}\ln x.$$
$$y\ln x = \frac{3}{2}(x)\cdot 2\ln x.$$
$$y = 3x.$$
Taking $y^2 = x^3.$
but $y = 3x.$
$$(3x)^2 = x^3.$$
$$\frac{9x^2}{x^2} = \frac{x^3}{x^2}$$
$$x = 9.$$
Since $x = 9$
$$y^2 = x^3.$$
$$y^2 = (9)^3.$$
$$\sqrt{y^2} = \sqrt{3^6}$$
$$y = 27$$
$\therefore$ $x = 9$ and $y = 27.$
The value of x is 9 and the value of y is 27.

(c) $\sqrt{1+x} = (1+x)^{1/2}$
from $(1+x)^n = 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3$

but $n = \frac{1}{2}$
$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{x^2}{2} + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{x^3}{6}.$$
$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$
Required $\sqrt{16.08}$
$$\sqrt{16.08} = \sqrt{16+0.08} = \sqrt{16\left(1+\frac{0.08}{16}\right)}$$
$$= 4\left[1+\frac{1}{200}\right]^{1/2}.$$

$\therefore \sqrt{16.08} = 4\left[1+x\right]^{1/2}$ when $x = 1/200$.

$$\sqrt{16.08} = 4\left[1 + \frac{1}{2}\left(\frac{1}{200}\right) + \left[\left(\frac{1}{200}\right)^2 \cdot \frac{-1}{8}\right] + \left[\frac{1}{16}\left(\frac{1}{200}\right)^3\right]\right]$$

$$\sqrt{16.08} = 4\left[1 + \frac{1}{400} - \frac{1}{320\,000} + \frac{1}{16(200)^3}\right]$$

$$= 4\left[1.002496883\right]$$

$$= 4.009987531$$

Correct to six decimal places $= 4.009988$

$$\sqrt{16.08} = 4.009988$$

2017 PAST PAPERS – 2

4. (a) (i) Solve the equation $\log_3 x - 3 + \log_x 9 = 0$.

(ii) The equations $x^2 + 9x + 2 = 0$ and $x^2 + kx + 5 = 0$ have common root. Find the quadratic equation giving two actual possible values of k.

(b) Find the sum of the series $\frac{5}{1\times2\times3} + \frac{8}{2\times3\times4} + \frac{11}{3\times4\times5} + \ldots + \frac{3n+2}{n(n+1)(n+2)}$, hence find $\sum_{r=1}^{\infty} \frac{3r+2}{r(r+1)(r+2)}$.

(c) If $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, find the value of $A^{-1}B$. **(15 marks)**

4 (a) (i) required to solve equation

$\log_3 x - 3 + \log_x 9 = 0$

then

$\log_3 x + \log_x 9 = 3$.

$\frac{\log x}{\log 3} + \frac{\log 9}{\log x} = 3$.

$\frac{\log x}{\log 3} + \frac{2\log 3}{\log x} = 3$.

$\frac{\log x . \log x + \log 3 . 2\log 3}{\log 3 . \log x} = 3$.

$(\log x)^2 + 2(\log 3)^2 = 3 \log x \log 3$.

$(\log x)^2 - 3 \log x \log 3 + 2(\log 3)^2 = 0$

$(\log x)^2 - (3 \log 3) \log x + 2(\log 3)^2 = 0$

Let $\log x = y$

$y^2 - 3\log 3 y + 2(\log 3)^2$.

on solving we get

$y = \frac{3\log 3 \pm \sqrt{(3\log 3)^2 - 4(2(\log 3)^2)}}{2}$

$y = \frac{3\log 3 \pm \sqrt{9(\log 3)^2 - 8(\log 3)^2}}{2}$

$y = \frac{3\log 3 \pm \sqrt{(\log 3)^2}}{2}$

$y = \frac{3\log 3 \pm \log 3}{2}$

4(a)(i) then $y = \frac{3\log 3 \pm \log 3}{2}$

for +ve

$y = \frac{3\log 3 + \log 3}{2} = 2\log 3$.

for -ve

$y = \frac{3\log 3 - \log 3}{2}$

$y = \log 3^2$

but $y = \log x$.

$y = \log x = 2\log 3$ or $y = \log x = \log 3$.

$\log x = \log 3^2$ or $y = \log x = \log 3$

$x = 9$ $\quad$ $x = 3$.

$\therefore$ the value of x is 3 or 9.

In Extract 14.2, the candidate showed the important steps to solve the logarithmic equation and successfully arrived to the final correct answer.

2016 PAST PAPERS – 2

4. (a) (i) Find the value of a if the 17^{th} and 18^{th} terms of the expansion $(2+a)^{50}$ are equal.

(ii) The roots of the equation $x^3 + px^2 + qx + 30 = 0$ are in the ratio 2:3:5. Find the value of p and q.

(b) (i) State the principle of Mathematical Induction as it is used in mathematics.

(ii) Use the principle of mathematical induction to prove that $\sum_{r=1}^{n} 3r - 1 = \frac{n}{2}(3n+1)$

(15 marks)

Q4. (a) (i) given

$$(2+a)^{50}$$

From:

term, $U_{r+1} = {}^{n}C_r\, a^{n-r}\, b^{r}$.

For

$17 = r+1 \Rightarrow r = 16$

$18 = r+1 \Rightarrow r = 17$.

$$U_{17} = {}^{50}C_{16} \cdot (2)^{50-16} (a)^{16}$$

$$U_{18} = {}^{50}C_{17} \cdot (2)^{50-17} (a)^{17}.$$

but.

$$U_{17} = U_{18}$$

$${}^{50}C_{16} \cdot 2^{34} \cdot a^{16} = {}^{50}C_{17} \cdot 2^{33} \cdot a^{17}.$$

$$a = \frac{{}^{50}C_{16} \cdot 2^{34}}{{}^{50}C_{17} \cdot 2^{33}}$$

$$a = 2 \times \frac{1}{2}$$

$a = 1$

Hence,

The value of $a = 1$

(ii) For the equation $x^3 + px^2 + qx + 30 = 0$

$\alpha : \beta : \gamma = 2 : 3 : 5$

$\alpha/\beta = 2/3$ and $\beta/\gamma = 3/5$

04. (a)(ii)
$\begin{cases} \alpha + \beta + \gamma = -p \\ \alpha\beta + \alpha\gamma + \beta\gamma = q \\ \alpha\beta\gamma = -30 \end{cases}$ From the cubic equ given

Now: $\frac{\alpha}{\beta} = \frac{2}{3} \Rightarrow \alpha = \frac{2}{3}\beta$

$\frac{\beta}{\gamma} = \frac{3}{5} \Rightarrow \gamma = \frac{5}{3}\beta$

From:

$\Rightarrow \alpha + \beta + \gamma = -p$

$\frac{2}{3}\beta + \beta + \frac{5}{3}\beta = -p$

$\frac{10}{3}\beta = -p$

$\beta = -\frac{3}{10}p$ ----- (i)

$\Rightarrow$ Again:

$\alpha\beta\gamma = -30$

$\left(\frac{2}{3}\beta\right)(\beta)\left(\frac{5}{3}\beta\right) = -30$

$\frac{10}{9}\beta^3 = -30$

$\beta^3 = -27$

$\beta^3 = (-3)^3$

$\beta = -3$

~~then~~ But

$\beta = -\frac{3}{10} p$

$p = -\frac{10}{3} \times -3 = 10$

04. (a) (ii) $\beta\left(\frac{2}{3}\beta\right) + \left(\frac{2}{3}\beta\right)\left(\frac{5}{3}\beta\right) + \beta\left(\frac{5}{3}\beta\right) = q$

$q = \beta^2\left[\frac{2}{3} + \frac{10}{9} + \frac{5}{3}\right]$

$q = \frac{31}{9} \times \beta^2$ but $\beta = -3$

$q = 31$

Hence,

The value of $p = 10$ and $q = 31$.

Extract 14.2 shows that the candidate had adequate knowledge on the concepts of finding coefficients of the terms in the expansion $(x+y)^n$ and roots of cubic equation.

2015 PAST PAPERS – 2

(15 marks)

4. (a) Solve the following system of equations by using Cramer's rule.

$$2x + y - z = 3$$
$$x - y + z = 0$$
$$x + 2y + z = -3$$

(b) (i) Use binomial expansion of $\left(1 - \frac{1}{50}\right)^{\frac{1}{2}}$ to find the value of $\sqrt{2}$ correct to seven significant figures.

(ii) Decompose $\frac{2}{4n^2 - 1}$ into partial fractions and hence find $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

(c) One of the zeros of the polynomial function $f(x) = x^4 - (2+h)x^3 + (2h-5)x^2 + (5h+6)x - 6h$ is obtained when $h = 1$. Find the value of the constants p, q and r when $f(x) = (1 - 2x + x^2)(px^2 - qx - r)$.

(d) Given the simultaneous equations $\begin{cases} 3^x - 2^y = 0 \\ x + y - 1 = 0 \end{cases}$ show that $y = \log_6 3$.

4 a) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

let $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

$|A| = 2\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$

$|A| = -6 + -3$

$|A| = -9$

deleting column of x

$\begin{pmatrix} 3 & 1 & -1 \\ 0 & -1 & 1 \\ -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} \cancel{x} \\ \cancel{y} \\ \cancel{z} \end{pmatrix} = Ax$

$|Ax| = 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix} + -\begin{vmatrix} 0 & -1 \\ -3 & 2 \end{vmatrix}$

4 a) $|Ax| = -9$ $\qquad x = \frac{|Ax|}{|A|} = \frac{-9}{-9}$

$x = 1$

Deleting column of y

$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -3 & 1 \end{pmatrix} = Ay$

$|Ay| = 2\begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix}$

$|Ay| = 9$ $\qquad y = \frac{|Ay|}{|A|}$

$y = \frac{9}{-9} = -1$

Deleting Column of z

$$\begin{pmatrix}2 & 1 & 3\\ 1 & -1 & 0\\ 1 & 2 & -3\end{pmatrix} = A_z$$

$$|A_z| = 2\begin{vmatrix}-1 & 0\\ 2 & -3\end{vmatrix} - \begin{vmatrix}1 & 0\\ 1 & -3\end{vmatrix} + 3\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix}$$

$$|A_z| = 18$$

$$z = \frac{|A_z|}{|A|} = \frac{18}{-9} = -2$$

$\therefore x = 1, \quad y = -1$ and $z = -2$

b(i) From:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

let $x = -1/50$ & $n = 1/2$

4b(i) $$(1+(-x))^{1/2} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-x)^3$$

$$(1-x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{3}{48}x^3$$

$$(1-x)^{1/2} = 1 - x/2 - x^2/8 - \frac{1}{16}x^3$$

Putting $x = 1/50$

$$\left(1 - 1/50\right)^{1/2} = 0.9899495$$

$$\sqrt{\frac{49}{50}} = 0.9899495$$

$$\frac{7}{\sqrt{25\times 2}} = 0.9899495$$

$$\frac{7}{5\sqrt{2}} = 0.9899495 \qquad \sqrt{2} = \frac{7}{5\times 0.9899495}$$

$$\sqrt{2} \simeq 1.414214$$

4b(ii) $$\frac{2}{(2n)^2 - 1^2} = \frac{2}{(2n-1)(2n+1)}$$

$$\frac{2}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$\frac{2}{(2n-1)(2n+1)} = \frac{(2n+1)A + (2n-1)B}{(2n-1)(2n+1)}$$

$$2 = (2n+1)A + (2n-1)B$$

If $n = 1/2$

$2 = 2A \qquad A = 1$

If $n = -1/2$

$2 = -2B \qquad B = -1$

4b(ii) $\therefore \dfrac{2}{4n^2-1} = \dfrac{1}{2n-1} - \dfrac{1}{2n+1}$ (In partial fraction)

Now:

$$\sum_{1}^{n} \left(\frac{2}{4n^2-1}\right)$$

Suppose if we find $\displaystyle\sum_{m=1}^{m=n} \frac{2}{4m^2-1}$

$$\frac{1}{2m-1} - \frac{1}{2m+1}$$

$m = 1 \qquad \dfrac{1}{1} - \dfrac{1}{3}$

$m = 2 \qquad \dfrac{1}{3} - \dfrac{1}{5}$

$m = 3 \qquad \dfrac{1}{5} - \dfrac{1}{7}$

$\vdots$

$m = n-1 \qquad \dfrac{1}{2n-3} - \dfrac{1}{2n-1}$ +

$m = n \qquad \dfrac{1}{2n-1} - \dfrac{1}{2n+1}$

$$S_n = 1 - \frac{1}{2n+1}$$

$$S_n = \frac{2n+1-1}{2n+1}$$

$$S_n = \frac{2n}{2n+1}$$

$$\therefore \sum_{n=1}^{n} \frac{2}{4n^2-1} = \frac{2n}{2n+1}$$

4 c) $f(x) = x^4 - (2+h)x^3 + (2h-5)x^2 + (5h+6)x - 6h$

$h = 1 \quad f(x) = 0$

$0 = x^4 - 3x^3 - 3x^2 + 11x - 6$

factors; $x = 1$

$x = 3$

$(x-1)(x-3) = x^2 - 3x - x + 3$

$x^2 - 4x + 3$

$$\begin{array}{r} x^2 + x - 2 \\ x^2-4x+3 \overline{\big) x^4 - 3x^3 - 3x^2 + 11x - 6} \\ \underline{x^4 - 4x^3 + 3x^2} \\ x^3 - 6x^2 + 11x - 6 \\ \underline{x^3 - 4x^2 + 3x} \\ -2x^2 + 8x - 6 \\ \underline{-2x^2 + 8x - 6} \\ 0 + 0 \end{array}$$

$f(x) = (x^2 - 4x + 3)(x^2 + x - 2)$

Expanding $(1 - 2x + x^2)(px^2 - qx - r) = f(x)$

$f(x) = (1 - 2x + x^2)px^2 - qx(1 - 2x + x^2) - r(1 - 2x + x^2)$

$f(x) = px^2 - 2px^3 + px^4 - qx + 2qx^2 - qx^3 - r + 2rx - rx^2$

$f(x) = px^4 - (2p+q)x^3 + ((p-r)+2q)x^2 + (2r-q)x - r$

Equating with

$f(x) = x^4 - 3x^3 - 3x^2 + 11x - 6$

$-r = -6 \qquad r = 6$

$2r - q = 11 \qquad q = 2r - 11 = 1$

$px^4 = x^4 \qquad p = 1$

$\therefore \; p = 1, \; q = 1$ and $r = 6$

4 d) $3^x - 2^y = 0$

$x + y = 1$

$3^x = 2^y$

$x \log 3 = y \log 2$

$x = \frac{y \log 2}{\log 3}$

$x + y = 1$

$\frac{y \log 2}{\log 3} + y = 1$

$$y\left(\frac{\log 2+\log 3}{\log 3}\right)=1$$

$$y=\frac{\log 3}{\log 2+\log 3}$$

$$y=\frac{\log 3}{\log(2\times 3)}=\frac{\log 3}{\log 6}$$

$$y=\log_6 3$$

Hence shown.

Extract 14.1 is a good solution of a candidate who applied correctly knowledge about determinants to get the solution of the given system of equations, used the binomial expansion to get the correct value of $\sqrt{2}$ and applied correctly knowledge of algebra to compute the required polynomial.

7.0 Trigonometry

2021 PAST PAPERS - 2

5. (a) If A, B and C are angles of a right angled triangle such that $\cos A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, find the value of $\tan 2A$, $\cos(A+B)$ and $\operatorname{cosec}(A-B)$ in the form $\frac{x}{y}$.

(b) (i) Show that $\cot\left(x+\frac{\pi}{2}\right) - \tan\left(x-\frac{\pi}{2}\right) = \frac{2\cos 2x}{\sin 2x}$.

(ii) Solve the equation $4\cos 2\theta - 2\cos\theta + 3 = 0$, for $0^\circ \le \theta \le 360^\circ$.

(c) Express $\cos^4\theta$ in terms of cosines multiples of θ.

5 a) $\cos A = 3/5 = A/H$

but $H^2 = O^2 + A^2$ -- pythagoras theorem

$\therefore H^2 = O^2 + A^2$

$5^2 = O^2 + 3^2$

$\therefore O = 4$

$\therefore \sin A = O/H = 4/5$

$\tan A = O/A = 4/3$.

$\cos B = 5/13 = A/H$

but $H^2 = O^2 + A^2$ -- pythagoras theorem.

$\therefore 13^2 = 5^2 + O^2$

$\therefore O = 12$

$\therefore \sin B = O/H = 12/13$

$\tan B = O/A = 12/5$

from $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$\tan 2A = \frac{2(4/3)}{1-(4/3)^2}$

$= -24/7$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\therefore = (3/5)(5/13) - (4/5)(12/13)$

$= -33/65$

$\operatorname{cosec}(A-B) = \frac{1}{\sin(A-B)}$

$\sin(A-B) = \sin A\cos B - \sin B\cos A$

$= (4/5)(5/13) - (12/13)(3/5)$

$= -16/65$

$\therefore \operatorname{cosec}(A-B) = -65/16$

$\therefore \tan 2A = -24/7$

$\cos(A+B) = \underline{-33/65}$

$\operatorname{cosec}(A-B) = \underline{\underline{-65/16}}$

5 b) i) $\cot(x + \pi/2) - \tan(x - \pi/2) = \dfrac{2\cos 2x}{\sin 2x}$

proof

$= \tan(\pi/2 - (x + \pi/2)) - \tan(x - \pi/2)$

$= \tan(-x) - \tan(x - \pi/2)$

$= -\tan x - \tan(x - \pi/2)$

$= -\tan x + \tan(\pi/2 - x)$

$= -\tan x + \cot x$

$= \cot x - \tan x$

$= \dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}$

$= \dfrac{\cos^2 x - \sin^2 x}{\sin x\cos x}$

$= \dfrac{\cos 2x}{\frac{1}{2}\sin 2x}$

$= \dfrac{2\cos 2x}{\sin 2x}$

hence shown.

ii) $4\cos 2\theta - 2\cos\theta + 3 = 0$

$4(2\cos^2\theta - 1) - 2\cos\theta + 3 = 0$

$8\cos^2\theta - 4 - 2\cos\theta + 3 = 0$

$8\cos^2\theta - 2\cos\theta - 1 = 0$

$\cos\theta = \cancel{-2\pm}\ \dfrac{2 \pm \sqrt{(-2)^2 - 4(-1)(8)}}{2(8)}$

$\cos\theta = \dfrac{2 \pm \sqrt{4 + 32}}{16}$

$\cos\theta = \dfrac{2 \pm 6}{16}$

$\cos\theta = 8/16$ or $-4/16$

$\cos\theta = 1/2$ or $-1/4$

$\theta = \cos^{-1}(1/2)$ or $\cos^{-1}(-1/4)$

5b) ii) for $\theta = \cos^{-1}(1/2)$

$\theta = 60°, 300°$

for $\theta = \cos^{-1}(-1/4)$

$\theta = (180° - 75.52°)$ and $(180° + 75.52°)$

$\theta = 104.48°$ and $255.52°$

$\therefore \theta = \underline{60°, 104.48°, 255.52° \text{ and } 300°}$

c) $\cos^4\theta = ?$

from $\cos 2\theta = 2\cos^2\theta - 1$

and $\cos 4\theta = 2\cos^2 2\theta - 1$, we get

$\therefore \cos 4\theta = 2(2\cos^2\theta - 1)^2 - 1$

$\cos 4\theta = 2(4\cos^4\theta + 1 - 4\cos^2\theta) - 1$

$\cos 4\theta = 8\cos^4\theta + 2 - 8\cos^2\theta - 1$

$\cos 4\theta = 8\cos^4\theta + 2 - 8\cos^2\theta - 1$

$8\cos^4\theta = \cos 4\theta - 1 + 8\cos^2\theta$

$8\cos^4\theta = \cos 4\theta - 1 + 4(2\cos^2\theta)$

$8\cos^4\theta = \cos 4\theta - 1 + 4(\cos 2\theta + 1)$

$\cos^4\theta = 1/8\,(\cos 4\theta - 1 + 4\cos 2\theta + 4)$

$= 1/8\,(\cos 4\theta + 4\cos 2\theta + 3)$

$\therefore \cos^4\theta = \underline{1/8\,(\cos 4\theta + 4\cos 2\theta + 3)}$

2020 PAST PAPERS - 2

5. (a) (i) Simplify the expression $\frac{1}{\sqrt{x^2-a^2}}$ where $x = a\,\mathrm{cosec}\,\theta$.

(ii) Prove that $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\,\mathrm{cosec}\,\theta$.

(b) (i) Express $2\cos\theta+5\sin\theta$ in the form of $R\sin(\theta-\alpha)$.

(ii) If $\cos\alpha-\cos\beta=m$ and $\sin\alpha-\sin\beta=n$, express $\cos(\alpha-\beta)$ in terms of m and n.

(c) Use t - substitution to find the general solution of the equation $3\cos\theta-4\sin\theta+1=0$.

(b) (i) $R\sin(\theta-\alpha) = 2\cos\theta + 5\sin\theta$

$\Rightarrow R\sin\theta\cos\alpha - R\sin\alpha\cos\theta = 2\cos\theta + 5\sin\theta$

Comparing; $R\cos\alpha = 5$ and

$R\sin\alpha = -2$

squaring and adding;

$R^2(\sin^2\alpha+\cos^2\alpha) = 5^2 + (-2)^2$

5 (b) (i) $\Rightarrow R^2 = 29$ $[\because \sin^2\alpha+\cos^2\alpha = 1]$

$\Rightarrow R = \pm\sqrt{29}$

Dividing; $\Rightarrow \tan\alpha = -2/5$

$\Rightarrow \alpha = \tan^{-1}(-2/5)$

$= -21.8°$

$\therefore 2\cos\theta + 5\sin\theta = \pm\sqrt{29}\sin(\theta + 21.8°)$

(ii) Given; $m = \cos\alpha - \cos\beta$

$n = \sin\alpha - \sin\beta$

squaring and adding;

$\Rightarrow m^2+n^2 = (\sin^2\alpha+\cos^2\alpha) + \cancel{\sin^2\alpha}$

$(\sin^2\beta+\cos^2\beta) - 2\cos\alpha\cos\beta$

$- 2\sin\alpha\sin\beta$

$= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$[\because \sin^2\alpha + \cos^2\alpha = 1]$

$= 2 - 2\cos(\alpha-\beta)$

$\Rightarrow 2\cos(\alpha-\beta) = 2 - (m^2+n^2)$

$\therefore \cos(\alpha-\beta) = 1 - \left(\frac{m^2+n^2}{2}\right).$

2019 PAST PAPERS - 2

(a) Use factor formulae to show that $\sin 5\alpha + \sin 2\alpha - \sin\alpha = \sin 2\alpha(2co3\alpha + 1)$.

(b) Simplify the expression $\frac{1+\sin\phi}{5+3\tan\phi-4\cos\phi}$ using small angles approximation up to the term containing ϕ^2.

(c) Prove that $\cos\beta(\tan\beta+3)(3\tan\beta+1) = 3\sec\beta = 10\sin\beta$.

(d) Find the greatest and least value of the function $\frac{1}{4\sin x - 3\cos x + 6}$.

5 a) $\sin 5\alpha + \sin 2\alpha - \sin\alpha$

$= \sin 5\alpha - \sin\alpha + \sin 2\alpha$

$= 2\sin\left(\frac{5\alpha-\alpha}{2}\right)\cos\left(\frac{5\alpha+\alpha}{2}\right) + \sin 2\alpha$

$= 2\sin 2\alpha\cos 3\alpha + \sin 2\alpha$

$= \sin 2\alpha(2\cos 3\alpha + 1)$

$\therefore \sin 5\alpha + \sin 2\alpha - \sin\alpha = \sin 2\alpha(2\cos 3\alpha + 1)$

hence shown

b) $\frac{1+\sin\phi}{5+3\tan\phi-4\cos\phi}$

as $\phi \to 0$, $\sin\phi \approx \phi$

$\tan\phi \approx \phi$

$\cos\phi \approx 1 - \frac{1}{2}\phi^2$

5 b) then

$\frac{1+\sin\phi}{5+3\tan\phi-4\cos\phi} \approx \frac{1+\phi}{5+3\phi-4(1-\frac{1}{2}\phi^2)}$

$\approx \frac{1+\phi}{5+3\phi-4+2\phi^2}$

$\approx \frac{1+\phi}{2\phi^2+3\phi+1}$

$\approx \frac{1+\phi}{(\phi+1)(2\phi+1)}$

$\approx \frac{1}{1+2\phi}$

$\approx (1+2\phi)^{-1}$

by using binomial theorem

$\frac{1+\sin\phi}{5+3\tan\phi-4\cos\phi} \approx 1 - 2\phi + \frac{2(2\phi)^2}{2} + \ldots$

$\approx 1 - 2\phi + 4\phi^2 + \ldots$

we neglect the higher powers

then

$$\frac{1+\sin\phi}{5+3\tan\phi-4\cos\phi} \approx 1-2\phi+4\phi^2$$

$$\therefore \frac{1+\sin\phi}{5+3\tan\phi-4\cos\phi} \approx 1-2\phi+4\phi^2$$

5 c) $\cos\beta(\tan\beta+3)(3\tan\beta+1)$

$= \cos\beta(3\tan^2\beta+\tan\beta+9\tan\beta+3)$

$= \cos\beta(3(\tan^2\beta+1)+10\tan\beta)$

but $1+\tan^2\beta = \sec^2\beta$

$= \cos\beta(3\sec^2\beta+10\tan\beta)$

$= 3\sec\beta + \frac{10\sin\beta}{\cos\beta}(\cos\beta)$

$= 3\sec\beta + 10\sin\beta$

$\therefore \cos\beta(\tan\beta+3)(3\tan\beta+1) = 3\sec\beta + 10\sin\beta$

But the question was incorrect in typing since it was typed as prove that
$\cos\beta(\tan\beta+3)(3\tan\beta+1) = 3\sec\beta - 10\tan\beta$
instead of
$\cos\beta(\tan\beta+3)(3\tan\beta+1) = 3\sec\beta + 10\tan\beta$
hence proved

d) Given the function

$$f(x) = \frac{1}{4\sin x - 3\cos x + 6}$$

let

$4\sin x - 3\cos x = R\sin(x+\alpha)$

$= R\sin x\cos\alpha - R\cos x\sin\alpha$

by comparison

$$\begin{cases} R\cos\alpha = 4 \\ R\sin\alpha = 3 \end{cases}$$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 16+9$

6	a) $R^2(\cos^2\alpha + \sin^2\alpha) = 25$
	$R^2 = 25$
	$R = 5$
	$\frac{R\sin\alpha}{R\cos\alpha} = 3/4$
	$\alpha = \tan^{-1}(3/4)$
	$\alpha = 36.87°$
	then
	$4\sin x - 3\cos x = 5\sin(x - 36.87°)$
	then
	$f(x) = \frac{1}{5\sin(x - 36.87°) + 6}$
	the greatest value of $f(x)$ is obtained when $\sin(x - 36.87°) = -1$
	then
	$f(x) = \frac{1}{-5+6} = 1$
	the least value is obtained when $\sin(x - 36.87°) = 1$.
	$f(x) = \frac{1}{5+6} = 1/11$
	∴ The greatest value of the function is 1 and the least value is $\frac{1}{11}$

2018 PAST PAPERS - 2

5. (a) (i) If $2\cos\theta = \frac{1}{x} + x$, prove that $2\cos3\theta = x^3 + \frac{1}{x^3}$.

(ii) Use t-formula to solve the equation $5\cos\alpha - 2\sin\alpha = 2$ for $-180^\circ \le \alpha \le 180^\circ$.

(b) (i) If $\theta = \frac{\pi}{8}$ and $\tan2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ show that $\tan\left(\frac{1}{8}\pi\right) = \sqrt{2} - 1$.

(ii) Given $a\sin\theta + b\cos\theta = c$, show that $a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 - c^2}$.

(c) (i) Simplify the expression $\tan^{-1} x + \tan^{-1}\left(\frac{1-x}{1+x}\right)$.

(ii) Find all the values of θ which satisfy the equation $\cos x\theta + \cos(x+2)\theta = \cos\theta$.

(d) If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, show that $\frac{a+b+c}{abc} = 1$.

5 a/ i/ $2\cos\theta = \frac{1}{x} + x.$

$\cos3\theta = \cos(2\theta + \theta)$

$\cos3\theta = \cos2\theta\cos\theta - \sin2\theta\sin\theta$

$= (\cos^2\theta + \cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$\cos3\theta = 4\cos^3\theta - 3\cos\theta$

$2\cos3\theta = 8\cos^3\theta - 6\cos\theta$

$2\cos3\theta = (2\cos\theta)^3 - 3(2\cos\theta).$

$= \left(\frac{1}{x} + x\right)^3 - 3\left(x + \frac{1}{x}\right).$

$= \left(\frac{1}{x}\right)^3 + 3\left(\frac{1}{x}\right)^2 x + 3x^2\left(\frac{1}{x}\right) + x^3 - 3\left(x + \frac{1}{x}\right)$

$= \frac{1}{x^3} + \frac{3}{x} + 3x + x^3 - 3x - \frac{3}{x}.$

$2\cos3\theta = \frac{1}{x^3} + x^3.$

Hence proved.

ii) From t formula.

$$\cos\alpha = \frac{1-t^2}{1+t^2} \quad \text{where } t = \tan\alpha/2$$

$$\sin\alpha = \frac{2t}{1+t^2}$$

$$5\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) = 2$$

$$5(1-t^2) - 4t = 2(1+t^2).$$

$$5 - 5t^2 - 4t = 2 + 2t^2$$

$$7t^2 + 4t - 3 = 0$$

Solving quadratically

$t = 0.428571428$ or $t = -1.$

$$\tan\alpha/2 = 0.428571428.$$

$$\alpha/2 = \tan^{-1}(0.428571428) = 23.2°$$

from General soln

$$\theta = 180n + \alpha.$$

$$\frac{\alpha}{2} = 180n + 23.2$$

$$\alpha = 360n + 46.4° \quad \text{where } n = 0, 1, 2 \ldots$$

$\alpha = 46.4°, 406.4°$

Also $\frac{\alpha}{2} = \tan^{-1}(-1). = -45$

$$\frac{\alpha}{2} = 180n + (-45)$$

$$\alpha = 360n - 90$$

$\alpha = -90, 270$

$\therefore \alpha = \cancel{46}\ -90°, 46.4°.$

5 b/ i) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

Let $\theta = \pi/8$.

$$\tan(\pi/4) = \frac{2\tan(\pi/8)}{1-\tan^2(\pi/8)} = 1$$

$$1-\tan^2(\pi/8) = 2\tan(\pi/8)$$

$$\tan^2(\pi/8) + 2\tan(\pi/8) - 1 = 0$$

$$\tan(\pi/8) = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2}$$

$$\tan(\pi/8) = \frac{-2 \pm \sqrt{8}}{2}$$

$$\tan(\pi/8) = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$\tan(\pi/8) = -1 \pm \sqrt{2}.$$

But since $\pi/8$ is in first quadrant, Hence has two values

Then

$$\tan(\pi/8) = \sqrt{2} - 1.$$

Hence proved

5 b/ ii) $(a\sin\theta + b\cos\theta)^2 = c^2$

$$a^2\sin^2\theta + 2ab\sin\theta\cos\theta + b^2\cos^2\theta = c^2$$

$$a^2(1-\cos^2\theta) + 2ab\sin\theta\cos\theta + b^2(1-\sin^2\theta) = c^2$$

$$a^2 - a^2\cos^2\theta + 2ab\sin\theta\cos\theta + b^2 - b^2\sin^2\theta = c^2$$

$$a^2\cos^2\theta + 2ab\cos\theta\sin\theta + b^2\sin^2\theta = a^2+b^2-c^2$$

$a\cos\theta(a\cos\theta -$

$$a^2\cos^2\theta - ab\cos\theta\sin\theta - ab\sin\theta\cos\theta + b^2\sin^2\theta = a^2+b^2-c^2$$

$$a\cos\theta(a\cos\theta - b\sin\theta) - b\sin\theta(a\cos\theta - b\sin\theta) = a^2+b^2-c^2$$

$$(a\cos\theta - b\sin\theta)(a\cos\theta - b\sin\theta) = a^2+b^2-c^2$$

$$(a\cos\theta - b\sin\theta)^2 = a^2+b^2-c^2$$

Hence

$$a\cos\theta - b\sin\theta = \pm\sqrt{a^2+b^2-c^2}$$

Proved

c) Let $\tan^{-1}(x) + \tan^{-1}\left(\frac{1-x}{1+x}\right) = c$.

Let $\tan^{-1}x = A \qquad \tan A = x.$

$\tan^{-1}\left(\frac{1-x}{1+x}\right) = B. \qquad \tan B = \frac{1-x}{1+x}$

$$A + B = c$$

$$\tan(A+B) = \tan c$$

$$\tan c = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan c = \frac{x + \left(\frac{1-x}{1+x}\right)}{1 - x\left(\frac{1-x}{1+x}\right)}$$

$$\tan c = \frac{x + x^2 + 1 - x}{1 + x - x + x^2}$$

5 c) i)

$$\tan c = \frac{1+x^2}{1+x^2}.$$

$$\tan c = 1.$$

$$c = \tan^{-1}(1).$$

$$c = \pi/4.$$

Hence

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1-x}{1+x}\right) = \pi/4.$$

5 c) ii)

$$\cos x\theta + \cos(x+2)\theta = \cos\theta$$

By factor formula.

$$2\cos\left(\frac{x\theta + x\theta + 2\theta}{2}\right)\cos\left(\frac{x\theta - x\theta - 2\theta}{2}\right) = \cos\theta.$$

$$2\cos(x\theta + \theta)\cos\theta = \cos\theta$$

$$2\cos(x\theta+\theta)\cos\theta - \cos\theta = 0$$

$$\cos\theta\left(2\cos(x+1)\theta - 1\right) = 0$$

Hence

$$\cos\theta = 0. \quad \text{and} \quad \cos(x+1)\theta = \tfrac{1}{2}$$

15 c/

$$\theta = \cos^{-1}(0) = 90$$

$$\theta = 360^\circ n \pm 90$$

$$\theta = 360^\circ n \pm 90 \text{ --- } i$$

Also $\cos(x+1)\theta = \tfrac{1}{2}$

$$(x+1)\theta = \cos^{-1}(\tfrac{1}{2}).$$

$$(x+1)\theta = 60^\circ$$

$$(x+1)\theta = 360^\circ n \pm 60^\circ$$

$$\theta = \frac{1}{(x+1)}\left(360^\circ n \pm 60^\circ\right).$$

$\therefore$ All values of θ are

$$\theta = 360^\circ n \pm 90^\circ \text{ and } \theta = \frac{1}{(x+1)}(360^\circ n \pm 60^\circ).$$

where $n = 0, 1, 2, 3, 4 \ldots$

5 d/

$$\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi.$$

$$\tan^{-1}a + \tan^{-1}b = \pi - \tan^{-1}c.$$

Apply tan both sides

$$\tan(\tan^{-1}a + \tan^{-1}b) = \tan(\pi - \tan^{-1}c).$$

$$\frac{\tan(\tan^{-1}a) + \tan(\tan^{-1}b)}{1 - \tan(\tan^{-1}(a))\tan(\tan^{-1}b)} = \frac{\tan\pi - \tan(\tan^{-1}c)}{1 - \tan\pi\tan(\tan^{-1}c)}$$

$$\frac{a+b}{1-ab} = \frac{0-c}{1-0(c)}$$

$$\frac{a+b}{1-ab} = \frac{-c}{1.}$$

$$a+b = -c(1-ab)$$
$$a+b = -c+abc$$
$$a+b+c = abc$$

$$\frac{a+b+c}{abc} = 1$$

$\therefore \frac{a+b+c}{abc} = 1$ Hence shown.

2017 PAST PAPERS - 2

5. (a) (i) Use trigonometric identities to prove that $16\sin^5\theta - 21\sin^3\theta + 5\sin\theta = \sin 5\theta$.

(ii) If $x\sec\theta + y\tan\theta = 3$ and $x\tan\theta + y\sec\theta = 2$, eliminate θ from the equations.

(b) (i) If $\tan\theta = \frac{4}{3}$ and $0° \le \theta \le 360°$, find without using tables the value of $\tan\left(\frac{1}{2}\theta\right)$.

(ii) Show that $\frac{\cos 3x - \cos 5x}{4\sin 2x \cos 2x} = \sin x$.

(c) Given that $\tan^{-1}A + \tan^{-1}B + \tan^{-1}C = \pi$, verify that $A + B + C = ABC$.

(d) Express the sum of $\sec x$ and $\tan x$ as the tangent of $\left(\frac{\pi}{4} + \frac{x}{2}\right)$ and hence find in surd form the value of $\tan\frac{\pi}{12}$.

(20 marks)

5. (b) (i) Given:

$$\tan \alpha = \frac{4}{3}$$

$$0^\circ \leq \alpha \leq 360^\circ$$

Then:

$$\tan\left(\tfrac{1}{2}\alpha\right)$$

from:

$$\tan \alpha = \tan\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)$$

$$\tan \alpha = \frac{\tan \frac{\alpha}{2} + \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$$

$$\tan \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$$

Then:

$$\frac{4}{3} = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$$

$$4\left(1 - \tan^2 \frac{\alpha}{2}\right) = 6 \tan \frac{\alpha}{2}$$

$$4 - 4\tan^2 \frac{\alpha}{2} = 6 \tan \frac{\alpha}{2}$$

$$4\tan^2 \frac{\alpha}{2} + 6 \tan \frac{\alpha}{2} - 4 = 0$$

Quadratic in $\tan \frac{\alpha}{2}$.

$$\tan \frac{\alpha}{2} = \frac{-6 \pm \sqrt{36 - 4(4)(-4)}}{2(4)}$$

$$\tan \frac{\alpha}{2} = \frac{-6 \pm 10}{8}$$

$$\tan \frac{\alpha}{2} = \frac{1}{2} \text{ or } -2$$

In Extract 15.2 the candidate correctly evaluated the value of $\tan\left(\frac{1}{2}\theta\right)$ without using tables.

2016 PAST PAPERS - 2

5. (a) (i) Solve the equation $\tan^{-1}\left(\frac{x-1}{x+2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$ and leave the answer in surd form.

(ii) Prove the identity $\frac{1+\sin x}{1-\sin x}\equiv(\tan x+\sec x)^2$.

(b) If $2\sin\theta+\cos\theta=1$, use t-formula to find the value of θ in the interval $0°\le\theta\le180°$.

(c) (i) Show that $\frac{\cos\theta+\cos2\theta+\cos3\theta+\cos4\theta}{\sin\theta+\sin2\theta+\sin3\theta+\sin4\theta}=\cot\left(\frac{5\theta}{2}\right)$.

(ii) Verify that $\frac{\sin(A+B+C)+\sin(A-B-C)}{\cos(A+B+C)-\cos(A-B-C)}=\frac{\tan B\tan C-1}{\tan B+\tan C}$.

(d) Express $3\sin\theta-4\cos\theta$ in the form $R\sin(\theta-\alpha)$ giving values of R and α. **(20 marks)**

5 c) i) $\frac{\cos\theta+\cos2\theta+\cos3\theta+\cos4\theta}{\sin\theta+\sin2\theta+\sin3\theta+\sin4\theta}=\cot\left(\frac{5\theta}{2}\right)$

Soln

Consider L.H.S

$$=\frac{\cos\theta+\cos4\theta+\cos2\theta+\cos3\theta}{\sin\theta+\sin4\theta+\sin2\theta+\sin3\theta}.$$

$= \cancel{2\cos\frac{5\theta}{2}}$ By factor formula.

$$=\frac{2\cos\frac{5\theta}{2}\cos\frac{3\theta}{2}+2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}}{2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}+2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}}.$$

5 c) i) $$=\frac{2\cos\frac{5\theta}{2}\left(\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}\right)}{2\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}\right)}.$$

$$=\frac{\cos\frac{5\theta}{2}}{\sin\frac{5\theta}{2}}$$

$$=\cot\frac{5\theta}{2}.$$

Hence Shown L.H.S = R.H.S

5 c) ii) $\dfrac{\sin(A+B+C)+\sin(A-B-C)}{\cos(A+B+C)-\cos(A-B-C)} = \dfrac{\tan B\tan C-1}{\tan B+\tan C}$

Consider L.H.S.

$$= \frac{\sin(A+B+C)+\sin(A-B-C)}{\cos(A+B+C)-\cos(A-B-C)}$$

By factor formulae

$$= \frac{2\sin\left(\frac{A+B+C+A-B-C}{2}\right)\cos\left(\frac{A+B+C-(A-B-C)}{2}\right)}{-2\sin\left(\frac{A+B+C+A-B-C}{2}\right)\sin\left(\frac{A+B+C-(A-B-C)}{2}\right)}$$

$$= \frac{2\sin A\cos(B+C)}{-2\sin A\sin(B+C)}$$

5 c) ii) $= \dfrac{2\sin A\cos(B+C)}{-2\sin A\sin(B+C)}$

$$= \frac{\cos(B+C)}{-\sin(B+C)}$$

$$= \frac{\cos B\cos C+\sin B\sin C}{-(\sin B\cos C+\sin C\cos B)}$$

$$= \frac{\sin B\sin C-\cos B\cos C}{\sin B\cos C+\sin C\cos B}$$

Divide by $\cos B\cos C$ throughout each term in numerator and denominator

$$= \frac{\frac{\sin B\sin C}{\cos B\cos C}-\frac{\cos B\cos C}{\cos B\cos C}}{\frac{\sin B\cos C}{\cos B\cos C}+\frac{\sin C\cos B}{\cos B\cos C}}$$

$$= \frac{\frac{\sin B \sin C}{\cos B \cos C} - \frac{\cos B \cos C}{\cos B \cos C}}{\frac{\sin B \cos C}{\cos B \cos C} + \frac{\sin C \cos B}{\cos B \cos C}}$$

$$= \frac{\tan B \tan C - 1}{\tan B + \tan C}$$

Hence verified L.H.S = R.H.S.

Extract 15.1 shows the solution of a candidate who managed to answer part (c) (i) and (ii) correctly.

2015 PAST PAPERS - 2

5. (a) Solve the trigonometric equation $\sec^2\theta + \tan\theta - 1 = 0$ for $0° \le \theta \le 360°$.

(b) Factorize completely the trigonometric expression $\cos\alpha - \cos 3\alpha - \cos 5\alpha + \cos 7\alpha$.

(c) (i) Verify that $\frac{\cos^2 t - 3\cos t + 2}{\sin^2 t} = \frac{2 - \cos t}{1 + \cos t}$

(ii) Prove that $\frac{\sin 3A \sin 6A + \sin A \sin 2A}{\sin 3A \cos 6A + \sin A \cos 2A} = \tan 5A$

(d) Show that $\frac{\sin^2 x + 2\sin x + 1}{\cos^2 x} = \frac{\cos^2 x}{1 - 2\sin x + \sin^2 x}$

5a) $\cancel{1} + \tan^2\theta + \tan\theta - \cancel{1} = 0$

$\tan\theta(\tan\theta + 1) = 0$

$\tan\theta = 0 \quad , \quad \tan\theta = -1$

$\theta = 180°n \quad , \quad \theta = 180°n - 45°$

for $n = 0, 1, 2 \ldots$

$\theta = 0°, 180°, 135°, 315°, 360°$

$\therefore \theta = 0°, 135°, 180°, 315°$ and $360°$

b) $= \cos\alpha - \cos 3\alpha - \cos 5\alpha + \cos 7\alpha$

$= \cos 7\alpha + \cos\alpha - (\cos 5\alpha + \cos 3\alpha)$

$= \cancel{2\cos 4\alpha \sin 3\alpha - (2\sin}$

$= 2\cos 4\alpha \cos 3\alpha - (2\cos 4\alpha \cos\alpha)$

$= 2\cos 4\alpha [\cos 3\alpha - \cos\alpha]$

$= 2\cos 4x [-2\sin 2x \sin x]$

$\therefore = -4\cos 4\alpha \sin 2\alpha \sin\alpha$

5 c i) Consider the Left hand Side

$$= \frac{(\cos t - 2)(\cos t - 1)}{\sin^2 t}$$

$$= \frac{(\cos t - 2)(\cos t - 1)}{1 - \cos^2 t}$$

$$= \frac{(\cos t - 2)(\cos t - 1)}{(1 - \cos t)(1 + \cos t)}$$

$$= \frac{-(\cos t - 2)\cancel{(1 - \cos t)}}{\cancel{(1 - \cos t)}(1 + \cos t)}$$

$$= \frac{-\cos t + 2}{1 + \cos t}$$

$$= \frac{2 - \cos t}{1 + \cos t}$$ hence verified

ii) Consider the Left hand side

$$= \frac{-\frac{1}{2}(\cos 9A - \cos 3A) + (-\frac{1}{2})(\cos 3A - \cos A)}{\frac{1}{2}[\sin 9A + \sin(-3A)] + \frac{1}{2}(\sin 3A - \sin A)}$$

$$= \frac{-\frac{1}{2}[\cos 9A - \cancel{\cos 3A} + \cancel{\cos 3A} - \cos A]}{\frac{1}{2}[\sin 9A - \cancel{\sin 3A} + \cancel{\sin 3A} - \sin A]}$$

$$= \frac{-[\cos 9A - \cos A]}{\sin 9A - \sin A}$$

$$= \frac{+[+\cancel{2}\sin 5A \cancel{\sin 4A}]}{\cancel{2}\cos 5A \cancel{\sin 4A}}$$

$$= \frac{\sin 5A}{\cos 5A}$$

$= \tan 5A$ hence proved

ii) Consider the Left hand side

$$= \frac{-\frac{1}{2}(\cos 9A - \cos 3A) + (-\frac{1}{2})(\cos 3A - \cos A)}{\frac{1}{2}[\sin 9A + \sin(-3A)] + \frac{1}{2}(\sin 3A - \sin A)}$$

$$= \frac{-\frac{1}{2}[\cos 9A - \cancel{\cos 3A} + \cancel{\cos 3A} - \cos A]}{\frac{1}{2}[\sin 9A - \cancel{\sin 3A} + \cancel{\sin 3A} - \sin A]}$$

$$= \frac{-[\cos 9A - \cos A]}{\sin 9A - \sin A}$$

$$= \frac{+[+\cancel{2}\sin 5A \cancel{\sin 4A}]}{\cancel{2}\cos 5A \cancel{\sin 4A}}$$

$$= \frac{\sin 5A}{\cos 5A}$$

$$= \tan 5A$$ hence proved

5 d) Consider the left hand side

$$= \frac{\sin^2 x + 2\sin x + 1}{\cos^2 x}$$

$$= \frac{(\sin x + 1)(\sin x + 1)}{\cos^2 x}$$

$$= \frac{(\sin x + 1)(\sin x + 1)}{1 - \sin^2 x}$$

$$= \frac{\cancel{(\sin x + 1)}(\sin x + 1)}{\cancel{(1 + \sin x)}(1 - \sin x)}$$

$$= \frac{\sin x + 1}{1 - \sin x} \cdot \frac{1 - \sin x}{1 - \sin x}$$

$$= \frac{1 - \sin^2 x}{(1 - \sin x)^2}$$

$$= \frac{\cos^2 x}{1 - 2\sin x + \sin^2 x}$$

hence shown.

Extract 15.1 shows the solution of a candidate who managed to answer the given question correctly. He/she was able to apply trigonometric identities to find the angles and used the factor formulas to factorize expressions and proved the equations.

8.0 Linear Programming

2021 PAST PAPERS

3. (a) A farmer needs 10 kg of SA and 15 kg of CAN. He can buy bags containing 2 kg of SA and 1 kg of CAN or he can buy tins containing 1 kg of SA and 3 kg of CAN. If the cost of each bag and tin are 20/= and 50/= respectively, write down four inequalities representing the problem.

6(16)9

(b) Mr. Chapakazi has two storage depots. He stores 200 tons of rice at depot 1 and 300 tons at depot 2. The rice has to be sent to three marketing centres A, B and C. The demands at A, B and C are 150, 150 and 200 tons respectively. The transportation cost per ton from the deposits to each market center is shown in the following table:

Deposits	Market Centre		
	A	B	C
Depot 1	50	100	70
Depot 2	80	150	40

How many tons of rice should be sent from the depots to each marketing centre so that the transportation cost is minimum?

Q3. (a) let

'x' be the number of bags bought

'y' be the number of tins bought

Constraints

$2x + y \geq 10$

$x + 3y \geq 15$

$x \geq 0$

$y \geq 0$

objective function

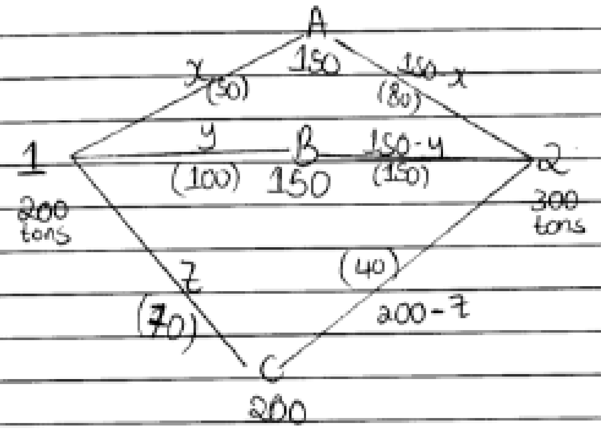
$f(x,y) = 20x + 50y$

Q3(b) let

"x" be the tons of rices from Depot 1 to A

"y" be the tons of rices from Depot 1 to B

"z" be the tons of rices from Depot 1 to C



Constraints

$x \geq 0$ --- (i)

$y \geq 0$ --- (ii)

$z \geq 0$

but

$x + y + z = 200$

$z = 200 - (x+y)$

$200 - (x+y) \geq 0$

$x + y \leq 200$ --- (iii)

$x \leq 150$ --- (iv)

$y \leq 150$ --- (v)

$x + y \geq 0$ --- (vi)

Objective function

$f(x,y) = 50x + 100y + 70z + 80(150-x) + 150(150-y) + 40(200-z)$

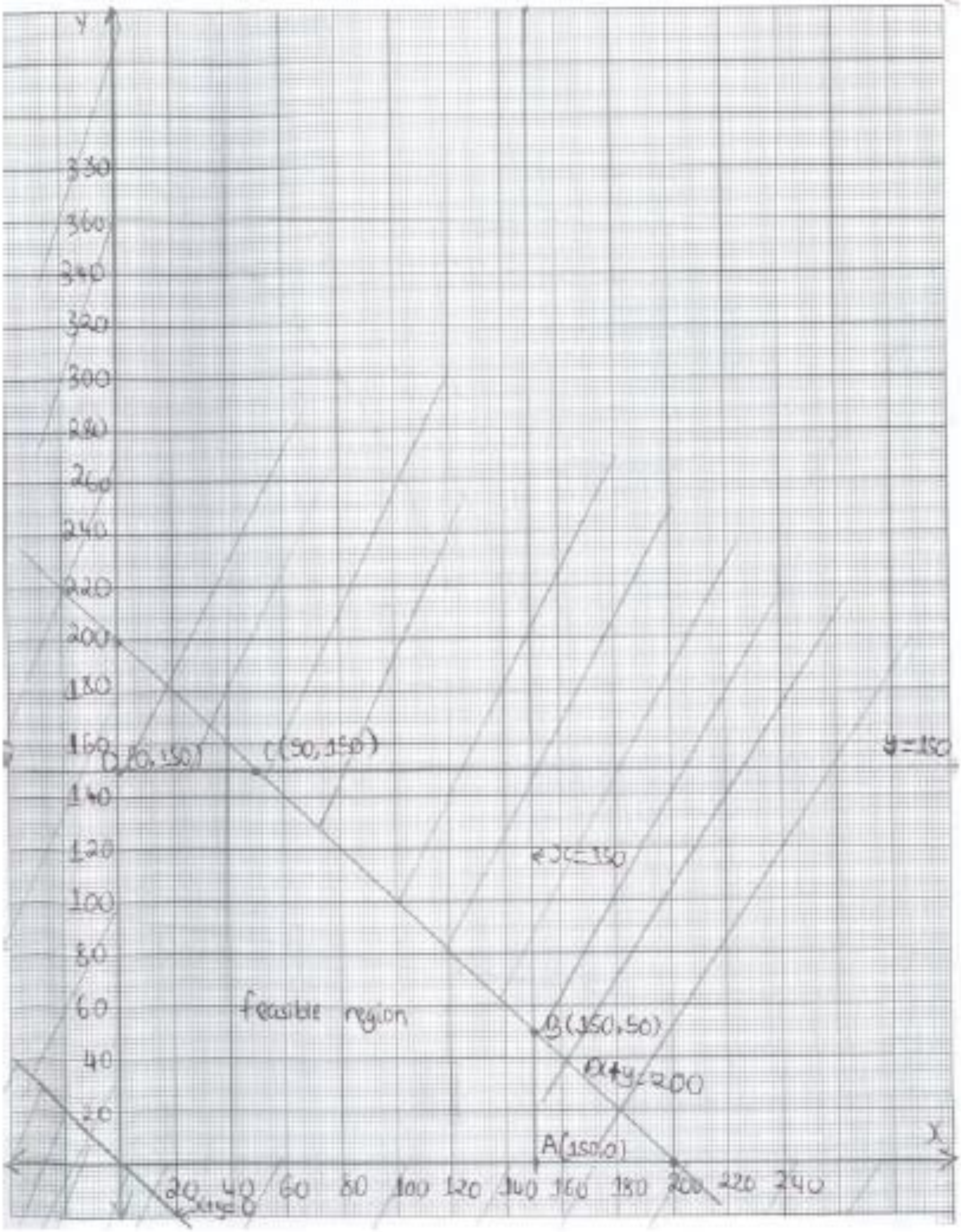
$= -30x - 50y + 30z + 42500$

$= -30x - 50y + 30(200 - x - y) + 42500$

$f = -60x - 80y + 48500$

Corner points	$f = -60x - 80y + 48500$
A(150,0)	39500
B(150,50)	35500
C(50,150)	33500
D(0,150)	36500

∴ 50 tons should be sent from Depot 1 to A, 150 tons from Depot 1 to B, 100 tons from Depot 2 to A and 200 from Depot 2 to C to make a minimum cost of 335001=



2020 PAST PAPERS

3. (a) A farm stocks two types of local brews called Kibuku and Lubisi, both of which are produced in cans of the same size. He wishes to order fresh supplies and finds that he has room for up to 1,500 cans. He know that Lubisi is more popular and so proposes to order at least thrice as many cans of Lubisi as Kibuku. He wishes, however, to have at least 120 cans of Kibuku and at most 950 cans of Lubisi. The profit on a can of Kibuku is sh. 3,000 and a can of Lubisi is sh. 4,000. Taking x to be the number of cans of Kibuku and y to be the number of cans of Lubisi which he orders, formulate this as a linear programming problem.

(b) A cooperative society has two storage depots for storing beans. The storage capacity at depot 1 and 2 is 200 and 300 tons of beans respectively. The tons of beans has to be sent to three marketing centres X, Y and Z. The demand of beans at X, Y and Z is 150, 150 and 200 tons respectively. The following table shows the cost of transport in Tshs. per ton from each depot to each marketing centre:

	To		
From	X	Y	Z
Depot 1	10,000	20,000	14,000
Depot 2	16,000	30,000	8,000

How many tons of beans should be sent from each depot to each of the marketing centre?

Objective function

$f(x,y) = 3000x + 4000y$ (Maximize)

linear programming Model

$f(x,y) = 3000x + 4000y$ subjected to

i) $x + y \leq 1500$

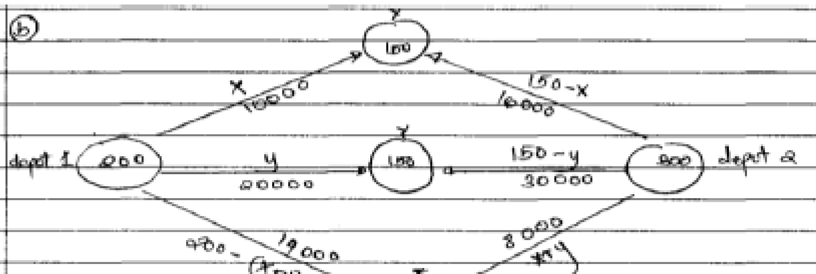
ii) $y - 3x \geq 0$

iii) $x \geq 100$

iv) $y \leq 950$

v) $x \geq 0$ and vi) $y \geq 0$.

Q3 (b)



iii) $200 - (x+y) \geq 0$

$x + y \leq 200$

iv) $x+y \geq 0$

Non constraints

v) $x \geq 0$

vi) $y \geq 0$

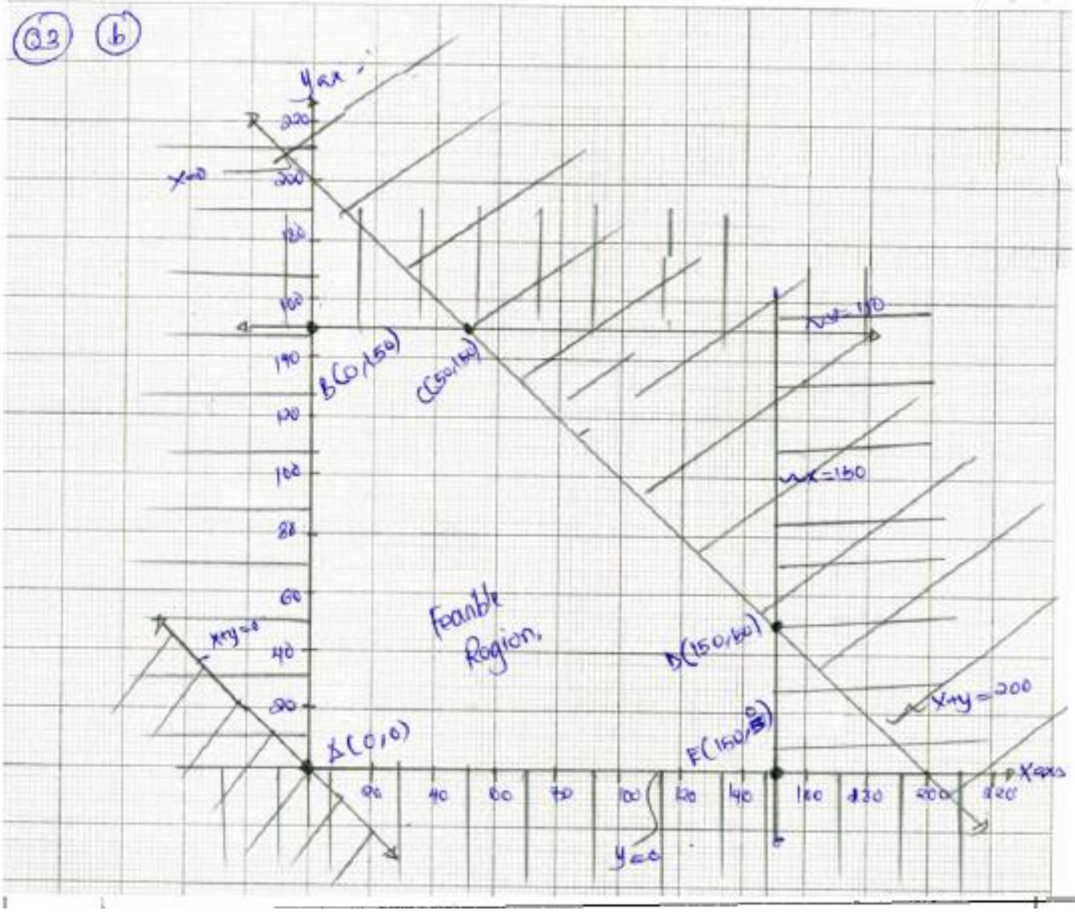
Q3 (b) Objective function

$f(x,y) = 10,000x + 20,000y + 2,800,000 - 14000x$

$- 14000y + 8000x + 8000y + 4,500,000 - 30000y$

$+ 2,400,000 - 16000x$

$f(x,y) = -12,000x - 16000y + 9700,000$



corner points	$f(x,y) = -12000x - 16000y + 9,700000$
A(0, 0)	9700000
B(0, 150)	7,300,000
C(50, 150)	6,700,000
D(150, 50)	7,100,000
E(150, 0)	7,900,000

Q2 (b) The transportation of tons should be as follows.

	X	Y	Z
Depot 1	50	150	0
Depot 2	100	0	200

2019 PAST PAPERS

3. Mr. Masumbuko has two traditional stores A and B for storing groundnuts. He stored 80 bags in A and 70 bags in B. Two customers C and D placed orders for 35 and 60 bags respectively. The transport costs per bag from each store are summarized in the following table:

From	To	
	C	D
A	8	12
B	10	13

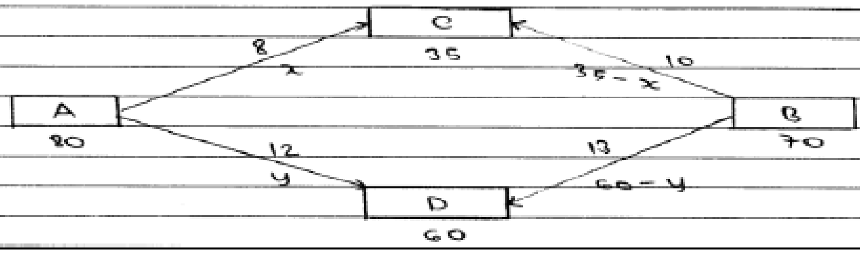
(a) How many bags of groundnuts should the farmer deliver to each customer in order to minimize the transportation cost?

(b) Determine the minimum cost of transport.

3. (a) Let x be number of bags transported from A to C

y be the number of bags transported from A to D.

Consider:



Constraints:

$35 - x \geq 0$

$x \leq 35$ - - - - - (i)

$60 - y \geq 0$

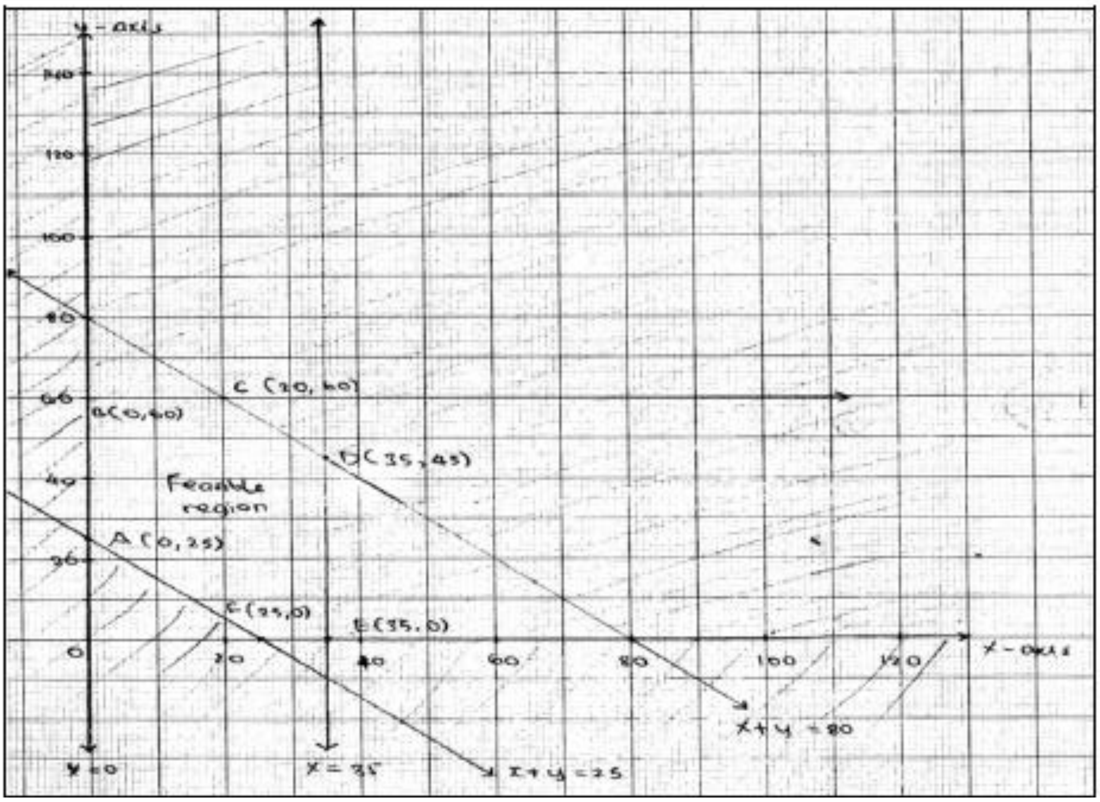
$y \leq 60$ - - - - - (ii)

$x + y \leq 80$ - - - - - (iii)

$(35 - x) + (60 - y) \leq 70$

$x + y \geq 25$ - - - (iv)

$x, y \geq 0$ - - - - - (v)



3 (a) Objective function:

$$f(x,y) = 8x + 12y + 10(35 - x) + 13(60 - y)$$

$$= 8x + 12y + 350 - 10x + 780 - 13y$$

$$= 1130 - 2x - y$$

Equations:

$x + y = 25$

x	0	25
y	25	0

$x + y = 80$

x	0	80
y	80	0

TABLE OF VALUES:

Corner points	$f(x,y) = 1130 - 2x - y$	Values
A(0, 25)	$1130 - 2(0) - (25)$	1,105/=
B(0, 60)	$1130 - 2(0) - (60)$	1,070/=
C(20, 60)	$1130 - 2(20) - 60$	1,030/=
D(35, 45)	$1130 - 2(35) - 45$	1,015/=
E(35, 0)	$1130 - 2(35) - 0$	1,060/=
F(25, 0)	$1130 - 2(25) - 0$	1,080/=

Optimal points are (35, 45).

∴ He should transport as follows;

From	To C	To D
A	35	45
B	0	15

3(b) ∴ The minimum cost of transport is 1,015/=

2018 PAST PAPERS

3. Mama Lishe has 140, 80 and 130 units of ingredients A, B and C respectively. A piece of bread requires 1, 1 and 2 units of A, B, C respectively. A pancake requires 5, 2 and 1 units of A, B and C respectively.

   (a) Taking x and y to be the number of pieces of bread and pancakes respectively, write down three inequalities which satisfy these conditions.

   (b) Draw a graph which shows a region representing possible values of x and y.

   (c) If the price for a piece of bread is 300/= and a pancake is 500/=, how many of each snacks should she bake in order to maximize her gross income?

   (d) What would be her gross income?

3 a) Taking x be the number of pieces of bread
Taking y be the number of pieces of pancakes

Inequalities satisfying the conditions are:-

$$x + 5y \leq 140$$
$$x + 2y \leq 80$$
$$2x + y \leq 130$$

b) For $x + 5y \leq 140$

x	y
0	28
140	0

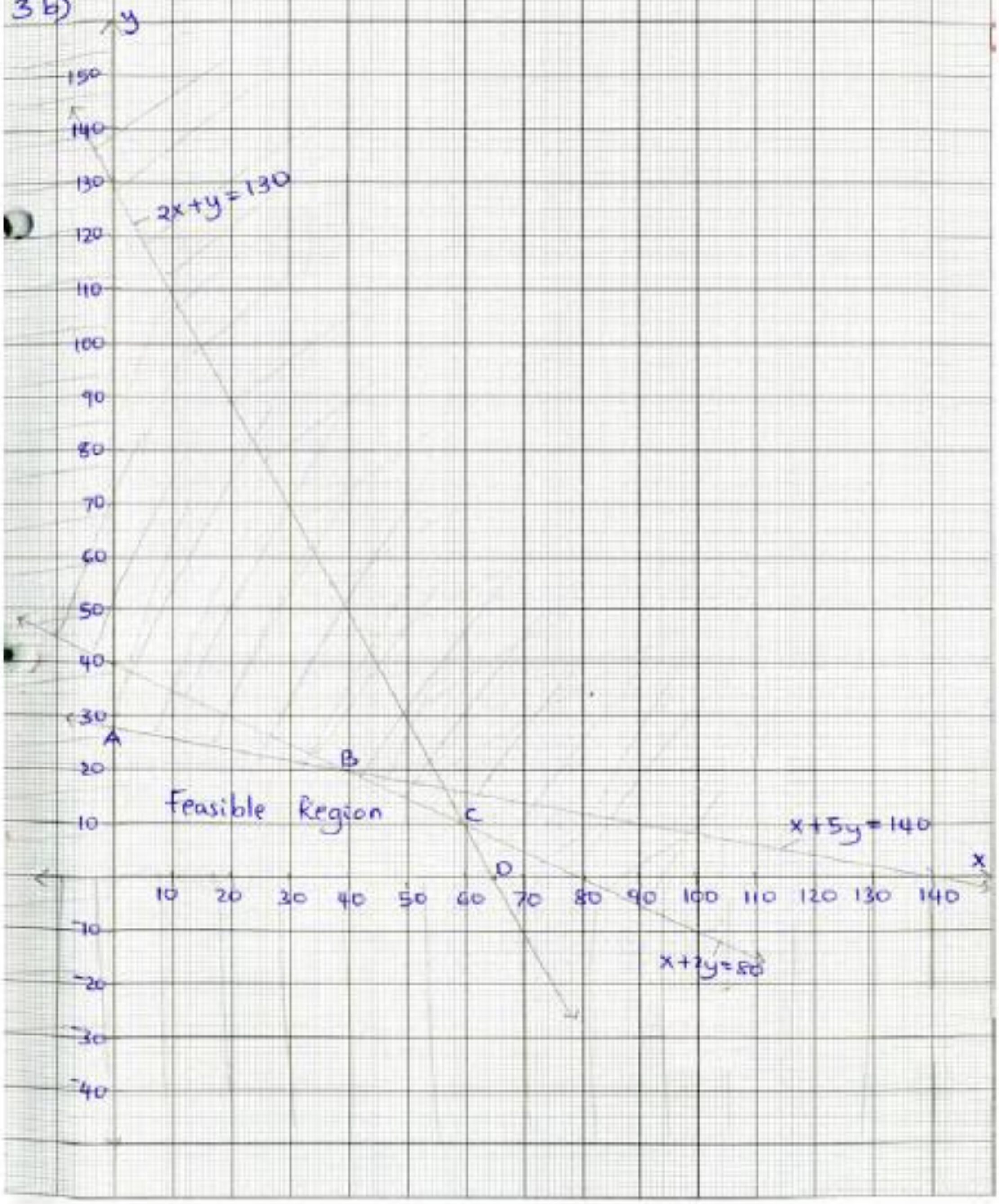
For $x + 2y \leq 80$

x	y
0	40
80	0

For $2x + y \leq 130$

x	y
0	130
65	0

3 b)


y
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
-10
-20
-30
-40
2x+y=130
A
B
C
D
Feasible Region
x+5y=140
x+2y=80
x
10
20
30
40
50
60
70
80
90
100
110
120
130
140

3 c) Objective function

$f(x,y) = Z = 300x + 500y$

Corner point	Corresponding values $Z = 300x + 500y$
A(0, 28)	14000
B(40, 20)	22000
C(60, 10)	23000
D(65, 0)	19500

In order to maximize her gross income, she should bake 60 breads and 10 pancake

d) Her gross income = 23000/=

2017 PAST PAPERS

3. Following an illness, a patient is required to take pills containing minerals and vitamins. The contents and costs of two types of pills, Felgood and Getbetta, together with the patient's daily requirement, are shown in the following table:

	Mineral	Vitamin	Cost
Feelgood	80 mg	4 mg	3,000/=
Getbetta	20 mg	3 mg	1,500/=
Daily requirement	420 mg	31 mg	

If the daily prescription contains x Feelgood pills and y Getbetta pills, find the cheapest way of prescribing the pills and the cost.

3. Programming summary.

	Mineral	Vitamin	Cost
Feel good	80	4	3000/=
Get better	20	3	1500/=
Daily requirement	420	31	

Given:

x feelgood pills

y get better pills.

Constraints

$80x + 20y \geq 420$ — (i)

$4x + 3y \geq 31$ — (ii)

$x \geq 0$ — (iii)

$y \geq 0$ — (iv)

3. Objective function:

$f(x,y) = 3000x + 1500y$

Minimizing cost.

Linear equations:

$80x + 20y = 420$ —— (i)

x	0	5.25
y	21	0

$4x + 3y = 31$ —— (ii)

x	0	7.75
y	10.3	0

$x = 0$ —— (iii)

$y = 0$ —— (iv)

check on the graph:

Then: From the graph

Corner points	$f(x,y) = 3000x + 1500y$
A(7.75, 0)	23250/=
B(4, 5)	19500/=
C(0, 21)	31500/=

Optimal solution 19500/=
optimal point B (4, 5)

∴ The daily prescription should contain 4 feelgood pills and 5 Get better pills at a cheapest cost of 19,500/=.

In Extract 3.1, the candidate was able to formulate the correct constraints as well as the objective function.

2016 PAST PAPERS

3. (a) Mr. Mutu takes two types of vitamin pills. He must have at least 16 units of vitamin A, 5 units of vitamin B and 20 units of vitamin C. He can choose between pill M which contains 8 units of A, 1 unit of B and 2 units of C; and pill N which contains 2 units of A, 1 unit of B and 7 units of C. Pill M costs 150 shillings and pill N costs 300 shillings. How many pills of each type should he buy in order to minimize the cost?

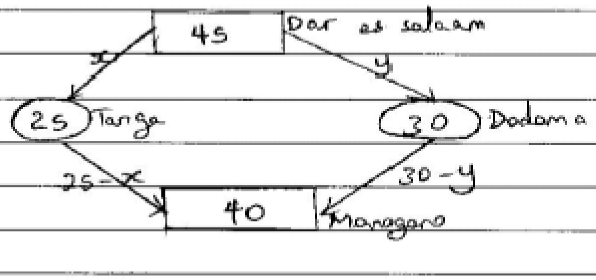
(b) A TV dealer has stores in Dar es Salaam and Morogoro and retailers in Tanga and Dodoma. The stores have a stock of 45 TV and 40 TV sets respectively while the requirements of the retailers are 25 and 30 TV sets respectively. If the cost of transporting a TV set from Dar es Salaam to Tanga is Tsh 5,000/= and from Dar es Salaam to Dodoma is Tsh 9,000/=, from Morogoro to Dodoma is Tsh 3,000/= and Morogoro to Tanga is Tsh 6,000/=;

(i) How should the TV dealer supply the requested TV sets at minimum cost?

(ii) What is the minimum cost?

3(b) (i) Let x be the number of TV sets supplied from Dar to Tanga and y be from Dar to Dodoma.

From \ To	Tanga	Dodoma
Dar es salaam	5,000/=	9000/=
Morogoro	6000/=	3,000/=



Objective function:

$C = 5000x + 9000y + 150{,}000 - 6000x + 90000 - 3000y$

$C = 6000y - 1000x + 240{,}000$

Constraints:

$x \geq 0$

$y \geq 0$

$25 - x \geq 0 \Rightarrow x \leq 25$

$30 - y \geq 0 \Rightarrow y \leq 30$

$x + y \leq 45$

$25 - x + 30 - y \leq 40 \Rightarrow x + y \geq 15$

Corner points	$C = 6000y - 1000x + 240{,}000$
A (0, 30)	420,000/=
B (0, 15)	330,000/=
C (15, 0)	225,000/=
D (25, 0)	215,000/=
E (25, 20)	335,000/=
F (15, 30)	405,000/=

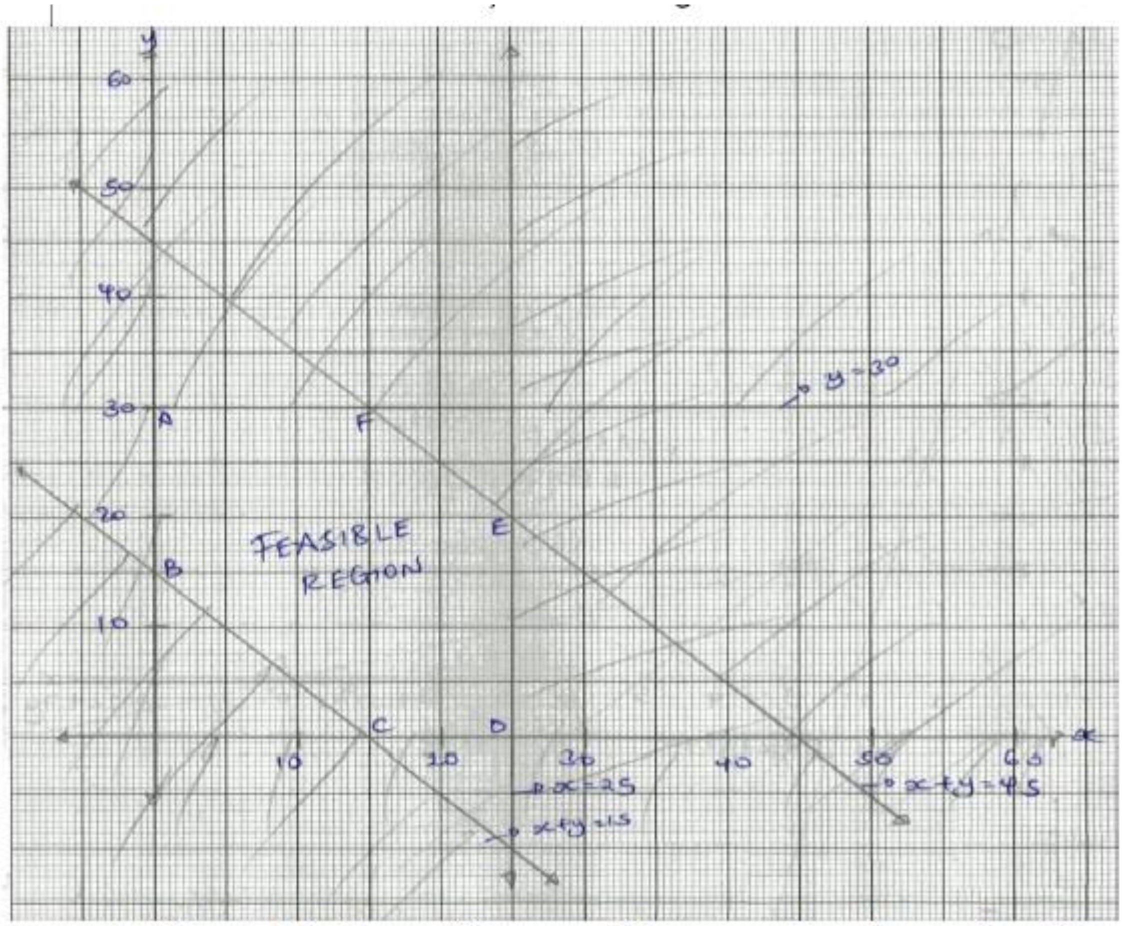
(b) (i) Optimum point is (25, 0)

∴ TV sets supplied from Dares salaam to Tanga are 25

from Dar es salaam to Dodoma are 0

from Morogoro to Tanga are 0

and from Morogoro to Dodoma are 30



In Extract 3.1, the candidate was able to transform the word problem mathematically, represent the inequality graphically and find the minimum cost correctly.

2015 PAST PAPERS

3\. (a) A company owns two mines. Mine A produces 1 ton of high grade ore, 3 tons of medium grade ore and 5 tons of low grade ore each day; and mine B produces 2 tons of each of the three grades of ore each day. The company needs 80 tons of high grade ore, 160 tons of medium grade ore and 200 tons of low grade ore. How many days should each mine be operated if it costs shs 200,000/= per day to operate each mine?

(b) A sugar company ships sugar from two origins S_1 and S_2 to three market centers M_1, M_2 and M_3. The table showing the available tons of sugar and the required tons together with the unit transportation cost in shillings is shown below:

	M_1	M_2	M_3	Available
S_1	20	10	5	220
S_2	10	25	30	100
Requirement	120	80	120	

(i) Use the given information in the table to formulate the objective cost function Z to be minimized.

(ii) Write down all equalities and inequalities of the transportation problem.

(iii) Verify whether the transportation problem in 3(b) is a balanced one or not. Use X_{ij}'s to denote the amount transported from source i to destination j.

3\. a)

	Mine A	Mine B	Minimum
H.G.O	1	2	80
MGO	3	2	160
LGO	5	2	200

Objective function

let x represent number of days for mine A and y for mine B

$f(x,y) = 200{,}000x + 200{,}000y$ — ①

Inequalities	Equations
$x + 2y \geq 80$	$x+2y=80$ $(0,40)$ $(80,0)$
$3x + 2y \geq 160$	$3x+2y=160$ $(0,80)$ $(\frac{160}{3}, 0)$
$5x + 2y \geq 200$	$5x+2y=200$ $(0,100)$ $(40,0)$
$x \geq 0$	$x=0$
$y \geq 0$	$y=0$

Corner points

A $(0,100)$ $f(A) = 20,000,000/=$

B $(20,50)$ $f(B) = 14,000,000/=$

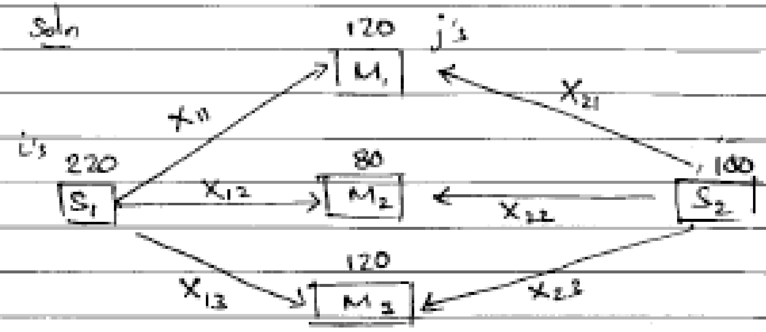
C $(40,20)$ $f(C) = 12,000,000/=$

D $(80,0)$ $f(D) = 16,000,000/=$

C is the optimal point $x = 40$ $y = 20$

3. a) Mine A should be operated for 40 days and mine B for 20 days.

b) Soln



i) Objective cost function

$f(x) = 20X_{11} + 10X_{12} + 5X_{13} + 10X_{21} + 25X_{22} + 30X_{23}$

$f(x) = Z$

$Z = 20X_{11} + 10X_{12} + 5X_{13} + 10X_{21} + 25X_{22} + 30X_{23}$

ii) Equalities.

$X_{11} + X_{12} + X_{13} = 220$

$X_{11} + X_{21} = 120$

$X_{21} + X_{22} + X_{23} = 100$

$X_{12} + X_{22} = 80$ and $X_{13} + X_{23} = 120$

<u>Inequalities</u>

$X_{11} \leq 120$ $\quad$ $X_{11} \geq 0$

$X_{12} \leq 80$ $\quad$ $X_{12} \geq 0$

$X_{13} \leq 120$ $\quad$ $X_{13} \geq 0$

$X_{21} \leq 120$ $\quad$ $X_{21} \geq 0$

$X_{22} \leq 80$ $\quad$ $X_{22} \geq 0$

$X_{23} \leq 120$ $\quad$ $X_{23} \geq 0$

b) iii) for a balanced problem

Sum of supply (sugar available) = Sum of demand

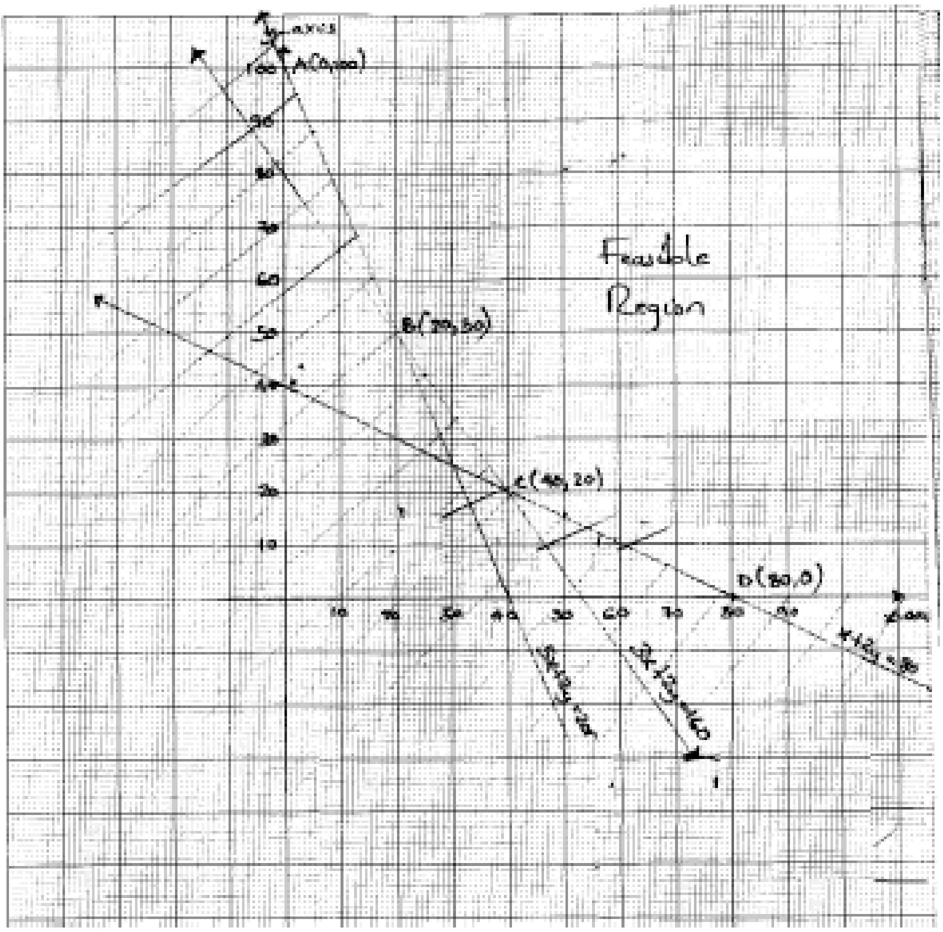
from the problem

Available sugar = 220 + 100 = 320 units

Demand (requirement) = 120 + 80 + 120 = 320 units

Hence the transportation problem is a balanced one.

3. a)



Extract 3.1 shows one of the best responses where the candidate was able to formulate the correct objective function, used the correct inequalities for the constraints, drew correct graphs, identified the feasible region and was able to solve the transportation problem using the notation that was given X_{ij} correctly.

9.0 Differentiation

2021 PAST PAPERS

10. (a) Show that there is a solution to the equation $x^3 - 6x^2 + 9x + 1 = 0$ between $x = -1$ and $x = 0$. Without using the table of values sketch the curve given by $y = x^3 - 6x^2 + 9x + 1$.

(b) Use Taylor's theorem to expand $(x+h)^{\frac{1}{2}}$ in ascending powers of h up to the term containing h^3. Hence obtain the value of $\sqrt{10}$ giving your answer correct to five decimal places.

10 (a) Given

$f(x)$ $x^3 - 6x^2 + 9x + 1 = 0$

For a function to have a solution between point A and point B therefore

$f(x_A) f(x_B) = -$ value

Since

$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 1$

$f(-1) = -15$

$f(0) = (0)^3 - 6(0) + 9(0) + 1$

$= 1$

~~$f(-1) \times f(0) = -15 \times 1 = -15$~~

Since $f(-1) \cdot f(0) = -$value here

The function has a solution between $x = -1$ and $x = 0$.

Given $y = x^3 - 6x^2 + 9x + 1$

For maximum and minimum values. $dy/dx = 0$

$dy/dx = 3x^2 - 12x + 9$

$3x^2 - 12x + 9 = 0$.

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3$ and $x = 1$

at $x = 3, y = 1$ at $x = 1, y = 5$

for 2nd derivative.

$d^2y/dx^2 = 6x - 12$ testing the points

$(x=3)$ gives $+ve$ $(x=1)$ give $-ve$

hence, $x=3$ (minimum)

$(x=1)$ (maximum

~~09~~ 10 ⓑ The roots

if $f(x) = 0$

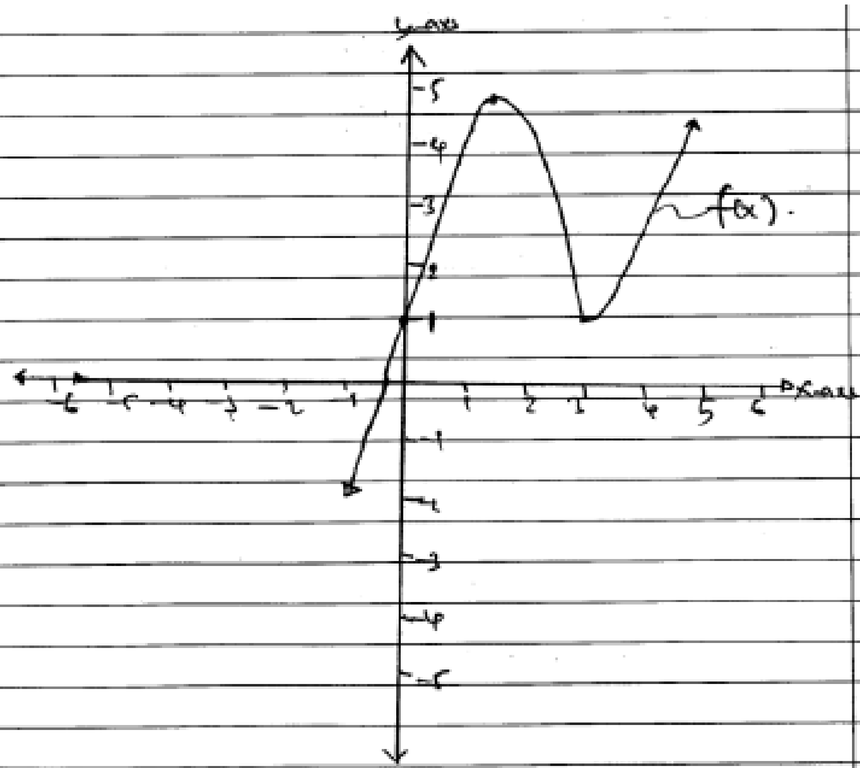
$x^3 - 6x^2 + 9x + 1 = 0$

On solving for roots

$x_1 = -0.1038$ (other roots are complex roots)

hence graph function x-axis only once.

ⓒ Sketching the graph Also, for $x = 0$ $f(x) = 1$



INDEX NUMBER.......................... use only

b) (b) $(x+h)^{\frac{1}{2}}$

$a = x.$

Taylor theorem

$$f(a+h) = f(a) + \frac{h f'(a)}{1!} + \frac{h^2 f''(a)}{2!} + \frac{h^3 f'''(a)}{3!}$$

$f(x) = x^{\frac{1}{2}}$ $\quad f(x).$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$

$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}} = -\frac{1}{4\sqrt{x^3}}$

$f'''(x) = \frac{3}{8} x^{-\frac{5}{2}} = \frac{+3}{8\sqrt{x^5}}.$

Hence

$$(x+h)^{\frac{1}{2}} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{4\sqrt{x^3}(2!)} + \frac{3h^3}{8\sqrt{x^5}(3!)}$$

$$(x+h)^{\frac{1}{2}} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8\sqrt{x^3}} + \frac{3h^3}{48\sqrt{x^5}}$$

for $\sqrt{10}$ let $x = 9$ $h = 1$

$$(9+1)^{1/2} = \sqrt{9} + \frac{1}{2\sqrt{9}} - \frac{1}{8\sqrt{9^3}} + \frac{3}{48\sqrt{9^5}}$$

$$= 3 + \frac{1}{6} - \frac{1}{216} + \frac{1}{3888}$$

$$(10)^{1/2} = 3.16229$$

2020 PAST PAPERS

10. (a) Find the derivative of x^n from first principles.

(b) Use Taylor's theorem to expand $\cos\left(\frac{\pi}{6}+h\right)$ in ascending powers of h up to the term containing h^3.

(c) If $x+y=10$, find the least possible value of x^2+y^2.

Qn 10 (a) solution.

Derivative of x^n from the first principles.

from the formula.

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

where by

$f(x) = x^n$.

$f(x+h) = (x+h)^n$.

$$f'(x) = \lim_{h\to 0}\left(\frac{(x+h)^n - x^n}{h}\right) \quad \text{——} \ (1)$$

from Binomial expansion.

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}h^2}{2!} + \frac{n(n-1)(n-2)x^{n-3}h^3}{3!}$$

Substitute expansion into equation (1).

$$f'(x) = \lim_{h\to 0}\left(\frac{x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}h^2}{2!} + \frac{n(n-1)(n-2)x^{n-3}h^3}{3!} - x^n}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{nx^{n-1}h + \frac{n(n-1)x^{n-2}h^2}{2!} + \frac{n(n-1)(n-2)x^{n-3}h^3}{3!}}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)x^{n-2}h}{2!} + \frac{n(n-1)(n-2)x^{n-3}h^2}{3!}\right)$$

$$f'(x) = \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)x^{n-2}(0)}{2!} + \frac{n(n-1)(n-2)x^{n-3}(0^2)}{3!}\right)$$

$\therefore f'(x) = nx^{n-1}$

$\therefore$ Derivatives of $x^n = nx^{n-1}$.

Qn 10 (b) solution.

use Taylor theorem to expand $\cos(\pi/6 + h)$ to h^3.

from Taylors theorem below.

$$f(x+h) = f(h) + f'(h)x + \frac{f''(h)x^2}{2!} + \frac{f'''(h)x^3}{3!} + \frac{f^{iv}(h)x^4}{4!}$$

$f(h) = \cos h$

let $\cos\left(\frac{\pi}{6}+h\right) = \cos(x+h)$

$f(h) = \cos h$, $f(x) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

$f'(h) = -\sin h$, $f'(x) = -\sin\frac{\pi}{6} = -\frac{1}{2}$.

$f''(h) = -\cos h$, $f''(x) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$.

$f'''(h) = \sin h$, $f'''(x) = \sin\frac{\pi}{6} = \frac{1}{2}$.

from..

$$f(x+h) = f(h) + f'(h)x + \frac{f''(h)x^2}{2!} + \frac{f'''(h)x^3}{3!}$$

$$\cos\left(\frac{\pi}{6}+h\right) = \frac{\sqrt{3}}{2} + \left(\frac{-1}{2}\right)x + \left(-\frac{\sqrt{3}}{2}\right)\frac{x^2}{2!} + \left(\frac{1}{2}\right)\frac{x^3}{3!}$$

$$\cos\left(\frac{\pi}{6}+h\right) = \frac{\sqrt{3}}{2} - \frac{x}{2} - \frac{\sqrt{3}\cdot x^2}{4} + \frac{x^3}{12}.$$

$$\cos\left(\frac{\pi}{6}+h\right) = \frac{6\sqrt{3} - 6x - 3\sqrt{3}x^2 + x^3}{12}$$

Qn10. (c) Solution.

$x+y = 10$. ——— equation (i).

Least possible value of x^2+y^2.

from.

$(x+y)^2 = x^2+y^2+2xy$.

$x^2+y^2 = (x+y)^2 - 2xy$

$x^2+y^2 = (x+y)^2 - 2xy$

from equation (i).

$x+y = 10$.

$x^2+y^2 = (10)^2 - 2xy$.

$x^2+y^2 = 100 - 2xy$.

$y = (10-x)$

$x^2+y^2 = 100 - 2x(10-x)$

$x^2+y^2 = 100 - 20x + 2x^2$

$x^2+y^2 = 2x^2 - 20x + 100$.

$\frac{dy}{dx}(x^2+y^2) = 4x - 20$

from $\frac{dy}{dx}(x^2+y^2) = 0$.

$4x - 20 = 0$.

$\frac{4x}{4} = \frac{20}{4}$

$x = 5.$

$\frac{d^2}{dx^2}(x^2+y^2) = 4.$

From $(x^2+y^2) = 2x^2 - 20x + 100.$

$(x^2+y^2) = 2(5)^2 - 20(5) + 100$

$(x^2+y^2) = 50$

$\therefore$ The values possible is $(5, 50)$

2019 PAST PAPERS

10. (a) If $y = \left(\frac{1-x^2}{1+x^2}\right)^n$, show that $(1-x^4)\frac{dy}{dx} + 4nxy = 0$.

(b) If the minimum value of $f(x) = 2x^3 + 3x^2 - 12x + k$ is one-tenth of its maximum value, find the value of k.

(c) (i) If $f(x, y) = x^3y + e^{xy^2}$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(ii) If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right)$, find $\frac{\partial^2 z}{\partial x \partial y}$ at $(1,1)$.

10 a) $y = \frac{(1-x^2)^n}{(1+x^2)^n}$

$\ln y = n \ln \left[\frac{1-x^2}{1+x^2}\right]$

$\ln y = n \ln(1-x^2) - n\ln(1+x^2)$

$\frac{1}{y} \, dy/dx = \frac{-2xn}{1-x^2} - \frac{2xn}{1+x^2}$

$\frac{1}{y} \frac{dy}{dx} = \frac{-2xn(1+x^2) - 2xn(1-x^2)}{1-x^4}$

$\frac{1}{y} \frac{dy}{dx} = \frac{-2xn - 2x^3 n - 2xn + 2x^3 n}{1-x^4}$

$\frac{1}{y} \, dy/dx = \frac{-4xn}{1-x^4}$

$dy/dx = \frac{-4nxy}{1-x^4}$

$(1-x^4) \, dy/dx + 4nxy = 0.$

10 iv) $z = x^2 \tan^{-1}(y/x)$

$\partial z/\partial y$ = required where x is constant

let $u = x^2$

$\partial u/\partial y = 0$

$v = \tan^{-1}(y/x)$

$\tan v = y/x$

$\sec^2 v \, \partial v/\partial y = 1/x$

$\partial v/\partial y = \frac{1}{x(1+y^2/x^2)}$

$\partial v/\partial y = \frac{x}{x^2+y^2}$

$\partial z/\partial y = u \, \partial v/\partial y + v \, \partial u/\partial y$

$\partial z/\partial y = x^2 \cdot \left[\frac{x}{x^2+y^2}\right]$

10

$$\frac{\partial z}{\partial y} = \frac{x^3}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial(x^3)}{\partial x} - (x^3)\frac{\partial(x^2+y^2)}{\partial x}}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2+y^2)\cdot 3x^2 - (x^3)\cdot(2x)}{(x^2+y^2)^2}$$

at $(1,1)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(1+1)3 - (1\times 2)}{(1+1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 1.$$

2018 PAST PAPERS

10. (a) Given the curve $x\sin y + y\cos x = 2$. Find $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ and $y = \pi$.

(b) Use the second derivative test to investigate the stationary values of the function $f(x) = 2x^2 - 8x + 5$.

(c) Differentiate $f(x) = \frac{1}{2}\cos 3x$ from first principles.

10	a)	$x \sin y + y \cos x = 2$
		$x \cos y \frac{dy}{dx} + \sin y(1) + {}^{-}y \sin x + \cos x \frac{dy}{dx} = 0$
		$x \cos y \frac{dy}{dx} + \cos x \frac{dy}{dx} = y \sin x - \sin y$
		$\frac{dy}{dx}(x \cos y + \cos x) = y \sin x - \sin y$
		$\frac{dy}{dx} = \frac{y \sin x - \sin y}{x \cos y + \cos x}$
		when $x = \pi/2$ and $y = \pi$
		$\frac{dy}{dx} = \frac{\pi \sin \pi/2 - \sin \pi}{\pi/2 \cos \pi + \cos \pi/2}$
		$\frac{dy}{dx} = \frac{\pi(1) - 0}{\pi/2(-1) + 0}$
		$\frac{dy}{dx} = \frac{\pi}{-\pi/2}$
		$\frac{dy}{dx} = -2$
	b)	$f(x) = 2x^2 - 8x + 5$
		let $y = f(x)$
		then,
		$\frac{dy}{dx} = 4x - 8$
		For stationary point $\frac{dy}{dx} = 0$
		$0 = 4x - 8$
		$4x = 8$
		$x = 2$
		$y = 2(2)^2 - 8(2) + 5$
		$y = -3$
		$\therefore$ Stationary point $= (2, -3)$

10	b)	$\frac{dy}{dx} = 4x - 8$
		$\frac{d^2y}{dx^2} = 4$
		$\frac{d^2y}{dx^2}$ is positive then the point is minimum

2017 PAST PAPERS

10. (a) If $x\sqrt{1+y}+y\sqrt{1+x}=0$, prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$.

(b) Given that $f=\sin xy$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

(c) Using Taylor's theorem, expand $\sin\left(\frac{\pi}{6}+h\right)$ in ascending power of h up to the h^4 term and hence evaluate $\sin 31°$ correct to three decimal places.

10. (a) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

Soln:

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

Square both sides

$$x^2(1+y) = y^2(1+x)$$

$$x^2 + x^2y = y^2 + y^2x$$

$$x^2 - y^2 = y^2x - x^2y$$

$$(x-y)(x+y) = xy(y-x)$$

$$(x-y)(x+y) = -xy(x-y)$$

$$x+y = -xy$$

$$x = -xy - y$$

$$x = -y(x+1)$$

$$-y = \frac{x}{1+x}$$

$$y = \frac{-x}{1+x}$$

$$y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = -\left[\frac{(1+x)(1) - x(1)}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = -\left[\frac{1+x-x}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

Hence proved !!

In Extract 10.1, the candidate showed correct steps of finding the derivative of the given equation.

2016 PAST PAPERS

10. (a) Find the derivative of $\frac{1}{x}+\cos 3x$ from first principle.

(b) Use the Taylor theorem to obtain the series expansion for $\cos\left(x+\frac{\pi}{3}\right)$ stating terms including that in x^3. Hence obtain a value for $\cos 61°$ giving your answer correct to five decimal places.

(c) Show whether the line $2x-y=0$ and the curve $4x^2-4xy+y^2 4x-8y+10=0$ intersect at a right angle.

(d) A two variable function f is defined by $z=f(x,y)=x^2+xy+y^2$; find $\frac{\partial z}{\partial y}$ at (1,1,1)

10. (a) Soln

Given

$y=\frac{1}{x}+\cos 3x.$

Then

$y+\Delta y=\frac{1}{(x+\Delta x)}+\cos 3(x+\Delta x).$

$\therefore \Delta y=\frac{1}{(x+\Delta x)}+\cos(3x+\Delta x)-y$

$\therefore \Delta y=\frac{1}{(x+\Delta x)}+\cos(3x+3\Delta x)-\frac{1}{x}-\cos 3x$

$\Delta y=\frac{x-x+\Delta x}{x(x+\Delta x}-2\sin\left(\frac{3\Delta x}{2}\right)\sin\left(3x+\frac{3\Delta x}{2}\right)$

$\Delta y=\frac{-\Delta x}{x(x+\Delta x)}-2\sin\left(\frac{3\Delta x}{2}\right)\sin\left(3x+\frac{3\Delta x}{2}\right)$

but $\sin\Delta x \simeq \tan\Delta x \simeq \Delta x$

also

$\therefore \frac{\Delta y}{\Delta x}=\frac{-1}{x(x+\Delta x)}-2\left(\frac{3}{2}\right)\sin\left(3x+\frac{\Delta x}{2}\right)$

then since Δx = very small

$\frac{dy}{dx}\simeq\frac{\Delta y}{\Delta x}$ — $\Delta x\to 0$

$\therefore \frac{dy}{dx}=\frac{-1}{x(x)}-3\sin(3x+0)$

$=\frac{-1}{x^2}-3\sin 3x$

$\therefore dy/dx=-1/x^2-3\sin 3x$

Extract 10.2 shows how well the candidate differentiated the given function from first principles. He/she applied the factor formula $\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ on $\cos(3x+3\Delta x)-\cos 3x$ to get $\frac{dy}{dx}=-\frac{1}{x^2}-3\sin 3x$.

2015 PAST PAPERS

10. (a) If $y = (1+2t)^3$ and $x = t^2$, find $\frac{dy}{dx}$.

(b) Find $\frac{d}{dx}\left(\tan\sqrt{6x^3+2}\right)$.

(c) (i) If $U = x^2 e^{\frac{y}{x}}$, find dU.

(ii) Show that $(3x^2y - 2y^2)dx + (x^3 - 4xy + 6y^2)dy$ can be written as an exact differential equation of a function $\phi(x, y)$ and find this function.

10.a) $y = (1+2t)^3$

$\frac{dy}{dt} = 3(1+2t)^2(2)$

$\frac{dy}{dt} = 6(1+2t)^2$

$x = t^3$

$\frac{dx}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

10a) $\frac{dy}{dx} = 6(1+2t)^2 \times \frac{1}{3t^2}$

$\frac{dy}{dx} = \frac{2(1+2t)^2}{t^2}$

10b). $\frac{d}{dx}\left(\tan\sqrt{6x^3+2}\right)$

let $y = \tan\sqrt{6x^3+2}$

let $u = \sqrt{6x^3+2}$

$u^2 = 6x^3+2$

$2u\frac{du}{dx} = 18x^2$

$\frac{du}{dx} = \frac{9x^2}{u} = \frac{9x^2}{\sqrt{6x^3+2}}$

$y = \tan u$

$\frac{dy}{dx} = \sec^2 u \frac{du}{dx}$

$\frac{dy}{dx} = \sec^2\left(\sqrt{6x^3+2}\right) \cdot \frac{9x^2}{\sqrt{6x^3+2}}$

$\therefore \frac{dy}{dx} = \frac{9x^2\sec^2\left(\sqrt{6x^3+2}\right)}{\sqrt{6x^3+2}}$

10c) i) $u = x^2 e^{y/x}$

let $t = e^{y/x}$

$\ln t = \frac{y}{x} = yx^{-1}$

$\frac{1}{t}\frac{dt}{dx} = y\left(\frac{-1}{x^2}\right) + \frac{1}{x}\frac{dy}{dx}$

10(c)(i) $\frac{dt}{dx} = \left(\frac{-y}{x^2} + \frac{1}{x}\frac{dy}{dx}\right)e^{y/x}$

$u = x^2 t$

$\frac{du}{dx} = x^2\frac{dt}{dx} + 2xt$

$\frac{du}{dx} = x^2\left(\frac{-y}{x^2} + \frac{1}{x}\frac{dy}{dx}\right)e^{y/x} + 2xe^{y/x}$

$\therefore\ du = \left(x^2e^{y/x}\left(\frac{-y}{x^2} + \frac{1}{x}\frac{dy}{dx}\right) + 2xe^{y/x}\right)dx$

10(c)(ii) $(3x^2y - 2y^2)dx + (x^3 - 4xy + 6y^2)dy$

$\frac{\delta M}{\delta y} = 3x^2 - 4y$ $\qquad$ $\frac{\delta N}{\delta x} = 3x^2 - 4y$

$\frac{dP}{dx} = 3x^2y - 2y^2$ $\qquad$ $\frac{dP}{dy} = x^3 - 4xy + 6y^2$

$\int dP = \int 3x^2y - 2y^2\,dx$ $\qquad$ $\int dP = \int x^3 - 4xy + 6y^2\,dy$

$P(x,y) = \frac{3x^3y}{3} - 2xy^2 + A(y)$ $\qquad$ $P(x,y) = x^3y - \frac{4xy^2}{2} + \frac{6y^3}{3} + B(x)$

$\frac{3x^3y}{3} - 2xy^2 + A(y) = x^3y - 2xy^2 + 2y^3 + B(x)$

$\therefore\ A(y) = 2y^3$

$B(x) = 0$

$\therefore\ x^3y - 2xy^2 + 2y^3 = c$

$\therefore$ Since $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x} = 3x^2 - 4y$, therefore the equation can be written as an exact differential equation.

$\therefore$ The function is $x^3y - 2xy^2 + 2y^3 = c$.

In Extract 10.1 the candidate demonstrated competence in using appropriate techniques of differentiation; that is substitution and the chain rule.

10.0 Integration

2021 PAST PAPERS

9\. (a) Find $\int x^3 \cos x \, dx$.

(b) Prove that $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx = \frac{\pi}{4} - \frac{1}{2}$.

use only

9 (a)

soln'

Given'

$$\int x^3 \cos x \, dx.$$

Integration by part (ILATE)

from

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}.$$

$$U = x^3. \Rightarrow \frac{du}{dx} = 3x^2$$

$$\frac{dv}{dx} = \cos x \qquad v = \sin x.$$

Now

$$\int x^3 \cos x \, dx = x^3 \sin x - 3\int x^2 \sin x \, dx.$$

$$= x^3 \sin x - 3\int x^2 \sin x \, dx \quad \text{--- (1)}$$

Also for

$$\int x^2 \sin x \, dx = -x^2\cos x + 2\int x\cos x \, dx$$

$$\int x\cos x \, dx = x\sin x - \int \sin x \, dx$$

$$= x\sin x + \cos x + C.$$

Then

$$\int x^3 \cos x \, dx = x^3 \sin x - 3\left[-x^2\cos x + 2(x\sin x + \cos x)\right]$$

$$\therefore \int x^3\cos x \, dx = x^3 \sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$$

9 (b) soln

To prove

$$\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}} \, dx = \frac{\pi}{4} - \frac{1}{2}.$$

consider L.H.S

$$= \int_0^1 \frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}} \, dx.$$

let $x^2 = \cos 2\theta$

$$2x \, dx = -2\sin 2\theta \cdot d\theta$$

$$x \, dx = -\sin 2\theta \cdot d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \; -\sin 2\theta \; d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{\cos^2\theta + \sin^2\theta - \cos^2\theta + \sin^2\theta}{\cos^2\theta + \sin^2\theta + \cos^2\theta - \sin^2\theta}} \; (-\sin 2\theta) \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \cdot (-\sin 2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} \frac{\sin\theta}{\cos\theta} \cdot (-\sin 2\theta)\, d\theta$$

use only

9 (b)

$$= \int_{\frac{\pi}{4}}^{0} \frac{\sin\theta}{\cos\theta} (-2\sin\theta\cos\theta)\, d\theta$$

$$-\int_a^b f(x)\, dx = \int_b^a f(x)\, dx.$$

$$= \int_0^{\pi/4} 2\sin^2\theta \, d\theta.$$

but $2\sin^2\theta = 1 - \cos 2\theta$

$$= \int_0^{\pi/4} (1 - \cos 2\theta)\, d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\right] - \left[0 - 0\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx = \frac{\pi}{4} - \frac{1}{2}$$

Hence proved

2020 PAST PAPERS

9. (a) Find the integral $\int \frac{\sin x}{1+\cos x} dx$.

(b) Evaluate the integral $\int_1^{e^2} \ln x \, dx$.

(c) Find the length of the arc of the curve given by the parametric equations $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$.

9a) $\int \frac{\sin x \, dx}{1+\cos x}$ Soln,

$\int \frac{\sin x \, dx}{1+\cos x}$

let $1+\cos x$ be u

$du/dx = -\sin x$

$dx = -du/\sin x$

$= \int \frac{\sin x}{u} \cdot \frac{-du}{\sin x}$

$-\int \frac{du}{u}$

$= -\ln u + c$, but $u = 1+\cos x$

$= -\ln(1+\cos x) + c$

b) $\int_1^{e^2} \ln x \, dx$ Soln,

from integration by parts, ILATE

$\int_1^{e^2} x^0 \ln x \, dx$

let $\ln x$ be u,

$du/dx = 1/x$, $du = dx/x$

$\int dv = \int x^0 dx$

$V = x$

$\int u\,dv = uv - \int v\,du$

$\int x^0 \ln x \, dx = x\ln x - \int x \frac{dx}{x}$

$\int \ln x \, dx = x\ln x - \int dx$

$\int \ln x \, dx = x\ln x - x$

$$\int_1^{e^2} \ln x\,dx = \left(e^2 \ln e^2 - e^2\right) - \left((1)\ln(1) - (1)\right)$$

$$\int_1^{e^2} \ln x\,dx = 2e^2 - e^2 + 1$$

$$\int_1^{e^2} \ln x\,dx = e^2(2-1) + 1$$

$$\int_1^{e^2} \ln x\,dx = \underline{e^2 + 1}$$

9c) Soln;

$$S = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$x = a(\cos\theta + \theta\sin\theta)$$

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$$

$$\frac{dx}{d\theta} = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$$

$$\frac{dy}{d\theta} = a\theta\sin\theta$$

$$S = \int \sqrt{(a\theta\cos\theta)^2 + (a\theta\sin\theta)^2}\,.\,d\theta$$

$$S = \int \sqrt{(a\theta)^2[\cos^2\theta + \sin^2\theta]}\,.\,d\theta$$

$$S = \int \sqrt{(a\theta)^2}\,d\theta$$

$$S = \int_0^{2\pi} a\theta\, d\theta$$

$$S = a\int_0^{2\pi} \theta\, d\theta$$

$$a\left[\frac{\theta^2}{2}\right]_0^{2\pi}$$

$$a\frac{(2\pi)^2}{2} = \frac{a4\pi^2}{2} = a2\pi^2$$

$$\therefore \quad S = a2\pi^2$$

2019 PAST PAPERS

9. (a) If $I_n = \int \sec^n x\,dx$, obtain a reduction formula for I_n in terms of I_{n-2} and use it to integrate $\int \sec^5 x\,dx$.

(b) Find the length of the arc given by $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ between $\theta = 0$ and $\theta = 2\pi$.

9. a/ Given

$I_n = \int \sec^n x\,dx.$

$I_n = \int \sec^{n-2} x \cdot \sec^2 x \,dx.$

Let

$u = (\sec x)^{n-2}.$

$\frac{du}{dx} = n-2\,(\sec x)^{n-3} \cdot \sec x \tan x.$

$\frac{du}{dx} = n-2\ \sec^{n-2} x\ \tan x.$

$du = (n-2)\sec^{n-2} x \tan x\ dx.$

$dv = \sec^2 x\ dx.$

$\int dv = \int \sec^2 x\ dx.$

$v = \tan x$

From

$\int u\,dv = uv - \int v\,du.$

$\int \sec^n x\,dx = \sec^{n-2} x \cdot \tan x - (n-2)\int \tan x \cdot \tan x \sec^{n-2} x\,dx$

$I_n = \sec^{n-2} x \tan x - (n-2)\int \tan^2 x \sec^{n-2} x\,dx$

but $\tan^2 x = \sec^2 x - 1.$

9. a/ $I_n = \sec^{n-2} x \tan x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x\,dx.$

$I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^n x - \sec^{n-2} x\,dx.$

$I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$

$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$(1 + n - 2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$I_n = \frac{1}{n-1}\left(\sec^{n-2} x \tan x + (n-2) I_{n-2}\right)$

Given $n = 5$:

$$\int \sec^5 x \cdot dx = \frac{1}{5-4}\left(\sec^3 x \tan x + 3\int \sec^3 x\, dx\right)$$

$$\int \sec^5 x \, dx = \frac{1}{4}\left(\sec^3 x \tan x + 3\int \sec^3 x\, dx\right)$$

consider $\int \sec^3 x \, dx$

$$\int \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \cdot dx$$

let $u = \sec x$. and $dv = \sec^2 x \cdot dx$

$$\frac{du}{dx} = \sec x \tan x. \qquad \int dv = \int \sec^2 x \cdot dx$$

$$v = \tan x$$

but

$$\int u\,dv = uv - \int v\,du.$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx.$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int (\sec^2 x - 1)\sec x \, dx.$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx.$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

$$\cancel{\int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right)}.$$

Now.

$$\int \sec^5 x \, dx = \frac{1}{4}\left(\sec^3 x \tan x + \frac{3 \sec x \tan x}{2} + \frac{3}{2}\ln|\sec x + \tan x|\right.$$

9. b/ $x = a\cos\theta + a\theta\sin\theta.$

$$\frac{dx}{d\theta} = -a\sin\theta + a\theta\cos\theta + a\sin\theta.$$

$$\frac{dx}{d\theta} = a\theta\cos\theta.$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\cos\theta - a\left(-\theta\sin\theta - \cos\theta\right).$$

$$\frac{dy}{d\theta} = a\cos\theta + a\theta\sin\theta - a\cos\theta.$$

$$\frac{dy}{d\theta} = a\theta\sin\theta.$$

from

$$l = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta.$$

$$l = \int_0^{2\pi} \sqrt{(a\theta)^2(\sin^2\theta + \cos^2\theta)} \cdot d\theta.$$

$$l = \int_0^{2\pi} \sqrt{a^2\theta^2}\, d\theta \quad \text{since } \cos^2\theta + \sin^2\theta = 1,$$

$$l = \int_0^{2\pi} a\theta\, d\theta.$$

$$l = a\int_0^{2\pi} \theta\, d\theta.$$

$$l = a\left[\frac{\theta^2}{2}\right]_0^{2\pi}$$

$$l = \frac{a}{2}\left[4\pi^2 - 0\right]$$

$$l = 2a\pi^2 \text{ units.}$$

$\therefore$ length of an arc is $2a\pi^2$ units

2018 PAST PAPERS

9. (a) Find $\int \frac{x-2}{(x^2+2)(x+1)}dx$.

(b) Evaluate $\int_0^{\frac{5}{3}\pi} \frac{\tan x + \sin x}{\cos x}dx$.

(c) (i) If A and B are any two points on the graph of $y = f(x)$, derive the arc length formula for the curve AB from $x = a$ to $x = b$.

(ii) Find the length of a curve $y = \frac{3}{4}x$ from $x = 0$ to $x = 4$.

9. (a) Let $I = \int \frac{x-2}{(x^2+2)(x+1)} dx$

Let $\frac{x-2}{(x^2+2)(x+1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x+1}$

$\Rightarrow x-2 = (Ax+B)(x+1) + C(x^2+2)$

9. $x-2 = (Ax+B)(x+1) + C(x^2+2)$

put $x=-1$, $-2 = 0 + 3C$

$\Rightarrow C = -1$

put $x=0$, $-2 = B + 2C$

$\Rightarrow -2 = B - 2$

$\Rightarrow B = -2+2 = 0$.

Comparing coefficients of x^2

$$0 = A + C \Rightarrow A = -C = 1$$

$$\therefore \frac{x-2}{(x^2+2)(x+1)} = \frac{x}{x^2+2} - \frac{1}{x+1}$$

$$I = \int \left(\frac{x}{x^2+2} - \frac{1}{x+1} \right) dx$$

$$I = \frac{1}{2} \int \frac{2x}{x^2+2} \, dx - \int \frac{1}{x+1} \, dx$$

$$I = \frac{1}{2} \ln (x^2+2) - \ln (x+1) + C$$

$$\therefore \int \frac{x-2}{(x^2+2)(x+1)} \, dx = \frac{1}{2} \ln (x^2+2) - \ln (x+1) + C$$

(b) let $I = \displaystyle\int_0^{5/3 \pi} \frac{\tan x + \sin x}{\cos x} \, dx$

$$I = \int_0^{5/3 \pi} \left(\frac{\tan x}{\cos x} + \frac{\sin x}{\cos x} \right) dx$$

$$I = \int_0^{5/3 \pi} \left(\sec x \tan x + \frac{\sin x}{\cos x} \right) dx$$

9. $$I = \Big[\sec x - \ln (\cos x) \Big]_0^{5/3 \pi}$$

$$I = \left(2 - \ln \frac{1}{2} \right) - (1 - \ln 1)$$

$$I = (2 + \ln 2) - (1 - 0)$$

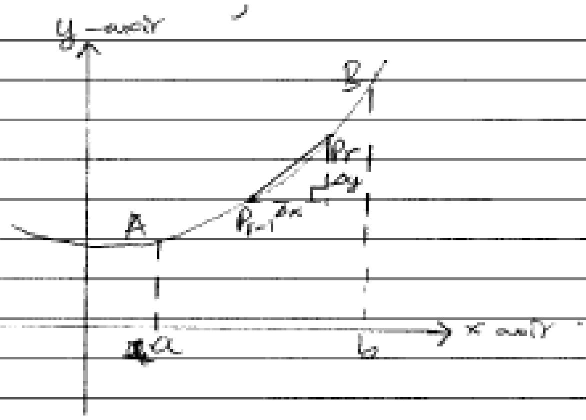
$$I = 1 + \ln 2 \simeq 1.6931$$

$$I = 1.6931$$

$$\therefore \int_0^{5/3 \pi} \left(\frac{\tan x + \sin x}{\cos x} \right) dx = \underline{1.6931}$$

(c) (i) consider the diagram below

y-axis



Suppose ~~aan~~ an arc AB is divided into small large number of ~~so~~ chords with very small length

Then length of arc AB $\longrightarrow$ Summation of length of chords

9. (c) (i) from the figure above by using pythagoras theorem length of chord $P_{r-1}P_r$ is given by $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$\Rightarrow P_{r-1}P_r^{\,2} = (\Delta x)^2 + (\Delta y)^2$$

$$\Delta r \quad P_{r-1}P_r^{\,2} = \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right](\Delta x)^2$$

$$\Rightarrow P_{r-1}P_r = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\cdot \Delta x$$

as $\Delta x \to 0$, $\Delta y/\Delta x \to dy/dx$.

length arc ~~a~~ AB is given by summation of these small chords from $x = a$ to $x = b$. This is done by integration (since $\Delta x \to 0$).

$\therefore$ length arc AB $(S) = \displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

but $\dfrac{dy}{dx} = f'(x)$

$$\Rightarrow S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

(ii) $y = \frac{3}{4}x \Rightarrow \frac{dy}{dx} = \frac{3}{4}$.

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{3}{4}\right)^2} = \sqrt{\frac{4^2+3^2}{4^2}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{5}{4}$$

9. (a)(ii)

$$\therefore S = \int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^4 \frac{5}{4}\, dx$$

$$S = \left[\frac{5}{4}x\right]_0^4$$

$$S = 5 \text{ units}$$

$\therefore$ length of the curve is 5 units

2017 PAST PAPERS

9. (a) Evaluate $I_{ab} = \int \sin ax \cos bx \, dx$ if $a \neq b$ and use it to find the value of n in $\int_0^n \sin 3x \cos 2x \, dx = \frac{3-\sqrt{3}}{5}$.

(b) Find the length of arc of the semi-cubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

9 (b)

Soln

$$y^2 = x^3$$

$$y = x^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x.$$

$$S = \int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

$$S = \int_1^4 \sqrt{1+\frac{9}{4}x}\, dx.$$

let $u = 1+\frac{9}{4}x$

$$\frac{du}{dx} = \frac{9}{4} \qquad \frac{4\,du}{9} = dx.$$

$$S = \int_1^4 \frac{4}{9} \cdot u^{1/2}\, du$$

$$S = \int_1^4 \frac{4}{9} u^{1/2} du = \frac{4}{9} \cdot \frac{2}{3}\left[u^{3/2}\right]_1^4$$

$$S = \frac{8}{27}\left[\sqrt{\left(1+\frac{9}{4}x\right)^3}\right]_1^4$$

$$S = \frac{8}{27}\left[\sqrt{1000} - \sqrt{2197/64}\right]$$

$$S = \frac{8}{27} \qquad S = 7.634 \text{ units}$$

$\therefore$ The length of an arc is 7.634 units

In Extract 9.2 a candidate was able to find the length of the arc as required.

2016 PAST PAPERS

9. (a) (i) Show whether $\int \frac{f'(x)}{f(x)}dx = \ln A f(x)$, where A is a constant.

(ii) Find $\int \cos 2x \cos 4x \cos 6x \, dx$.

(b) Evaluate $\int_0^{\frac{\pi}{2}} x \sin x \cos x \, dx$.

(c) Find the area of the region bounded by the curve $y = 3x^2 - 2x + 1$, the lines $x + 1 = 0$, $x - 2 = 0$ and $y = 0$.

(d) The area between the curve $3x^2 + y^2 = 9$ and the y – axis from $y = -3$ to $y = 3$ is rotated about the y-axis. Find the volume of the solid generated.

9(b) $\int_0^{\pi/2} x \sin x \cos x \, dx$.

$\int_0^{\pi/2} x \cdot \frac{2 \sin x \cos x}{2} \, dx$

$\frac{1}{2}\int_0^{\pi/2} x \cdot \sin 2x \, dx$.

$u = x \qquad \frac{dv}{dx} = \sin 2x$.

$\frac{du}{dx} = 1 \qquad v = \frac{-\cos 2x}{2}$.

$I = uv - \int v \frac{du}{dx} \, dx$

$I = x \cdot -\frac{\cos 2x}{2} - \int \frac{-\cos 2x}{2} \, dx$

$\frac{1}{2}\left(-\frac{x \cos 2x}{2} + \int \frac{\cos 2x}{2} \, dx \right)$

$\frac{1}{2}\left(-\frac{x}{2}\cos 2x + \frac{1}{2} \cdot \frac{\sin 2x}{2} \right) \Big|_0^{\pi/2}$.

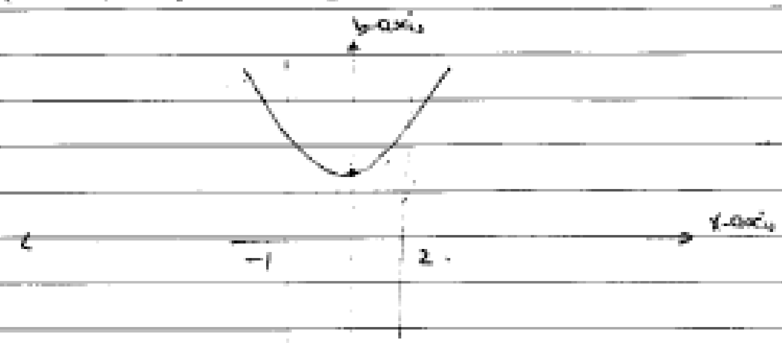
$-\frac{x \cos 2x}{4} + \frac{\sin 2x}{4} \Big|_0^{\pi/2}$.

$\left(\frac{-\frac{\pi}{2} \cos \pi}{4} + \frac{\sin \pi}{4} \right) - \left(0 + \frac{\sin 0}{4} \right)$

$\frac{\pi}{8} + 0 \quad = \frac{\pi}{8}$

9(c) $y = 3x^2 - 2x + 1$

$x = -1 \quad x = 2 \quad y = 0$



$A = \int y\,dx$

$A = \int_{-1}^{0} 3x^2 - 2x + 1 \; dx + \int_{0}^{2} (3x^2 - 2x + 1)\,dx$

$A = \frac{3x^3}{3} - \frac{2x^2}{2} + x \Big|_{-1}^{0} + \frac{3x^3}{3} - \frac{2x^2}{2} + x \Big|_{0}^{2}$

$A = 0 - (-1 - 1 - 1) + (8 - 4 + 2) - (0)$

$A = (3) + (6)$

$A = 9$ units square

Extract 9.1 shows that the candidate was able to evaluate the definite integral of trigonometrical in part (b). The candidate also managed to find the area bounded by the given curve and the lines in part (c).

2015 PAST PAPERS

9. (a) Integrate $\int \sec^3 x \, dx$.

(b) Evaluate $\int_{-1}^{0} \left(\frac{2x+3}{x^2+2x+4} \right)$

(c) Find the area of the region bounded by the graphs of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ (leave your answer in surd form).

9(a)

$$\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx$$

$$u = \sec x \longrightarrow u' = \sec x \tan x$$

$$v' = \sec^2 x \longrightarrow v = \tan x.$$

$$\int uv' = uv - \int u'v.$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x(\cancel{1+}\sec^2 x) \, dx.$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \int \frac{\sec x \, (\sec x + \tan x)}{\sec x + \tan x} dx$$

$$\therefore \int \sec^3 x \, dx = \frac{\sec x \tan x + \ln(\sec x + \tan x)}{2} + C$$

9(b)

$$\int_{-1}^{0} \frac{2x+3}{x^2+2x+4} dx$$

Let $2x+3 = A(2x+2) + B.$

$2x+3 = 2Ax + 2A + B.$

$\therefore \; 2A = 2$

$A = 1.$

$3 = 2A + B$

$\therefore B = 1.$

$$\int_{-1}^{0} \frac{2x+3}{x^2+2x+4} dx = \int_{-1}^{0} \frac{2x+2+1}{x^2+2x+4} dx$$

$$= \int_{-1}^{0} \frac{2x+2}{x^2+2x+4} dx + \int_{-1}^{0} \frac{dx}{x^2+2x+4}$$

$$= \left[\ln(x^2+2x+4) \right]_{-1}^{0} + \int_{-1}^{0} \frac{dx}{x^2+2x+1+3}$$

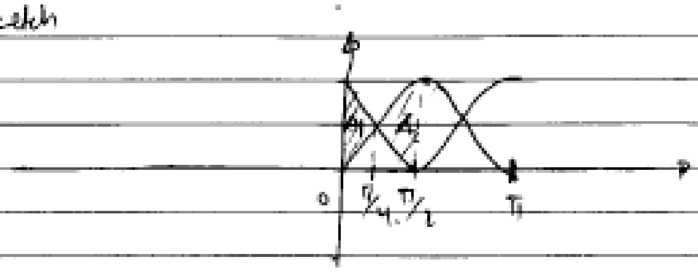
$$= \left(\ln(x^2+2x+4) \right]_{-1}^{0} + \frac{1}{3} \int_{-1}^{0} \frac{dx}{1 + \left(\frac{x+1}{\sqrt{3}} \right)^2}$$

$$= \left[\ln(x^2+2x+4) + \frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{x+1}{\sqrt{2}} \right) \right]_{-1}^{0}$$

$$\therefore \int_{-1}^{0} \frac{2x+3}{x^2+2x+4} = \ln(0+4) + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \ln\left(-1^2-2+4\right) - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{0}{\sqrt{3}}\right)$$

$$= \underline{0.58998}$$

9(c) Sketch



$$\text{Area} = A_1 + A_2.$$

$$\text{Area} = \left|\int_0^{\pi/4}(\sin x - \cos x)\right| + \left|\int_{\pi/4}^{\pi/2}(\sin x - \cos x)\right|$$

$$= \left|\left[-\cos x - \sin x\right]_0^{\pi/4}\right| + \left|\left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2}\right|$$

$$= \left|\left(-\cos\pi/4 - \sin\pi/4 + \cos 0 - \sin 0\right)\right| + \left(-\cos\pi/2 - \sin\pi/2 + \cos\pi/4 + \sin\pi/4\right)$$

$$= \left|-\sqrt{2}/2 - \sqrt{2}/2 + 1\right| + |$$

$$= |1-\sqrt{2}| + |\sqrt{2}-1|$$

$$= \sqrt{2}-1 + \sqrt{2}-1$$

$$\text{Area} = 2\sqrt{2}-2.$$

$$\therefore \underline{\text{Area bounded} = 2\sqrt{2}-2}$$

Extract 9.2 shows how the candidate had applied correctly the techniques of integration. The candidate also managed to evaluate the area bounded by the given curves obtained after drawing the graphs of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$.