Complex Numbers and the Riemann Hypothesis

How to win $1 000 000

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What is an Imaginary Number?

Complex Numbers

Usually, we work with *real* numbers.

Real include all integers, rational numbers, and irrational numbers.

e.g. $1,\ 6.9,\ \pi,\ e,\ \sqrt{123}$

Definition of i

i is defined as $\sqrt{-1}$.

$\therefore \mathrm{i}^2 = -1; (-\mathrm{i})^2 = -1$

What is a Complex Number?

Definition of a complex number

A complex number is the sum of a real and imaginary number.

e.g. $1 + \mathrm{i},\ \pi + ei,\ 0 + \mathrm{i},\ 1 + 0\mathrm{i}$, etc

Complex numbers have interesting properties, including how they and, multiply, and exponentiate.

You can add and subtract complex numbers like you would add polynomials (combine like terms).

Example 1

$$(5 + 3i) + (6 + 4i) = 11 + 7i$$

Example 2

$$(3 + 6i) - (6 - 4i) = -3 + 2i$$

For some complex number $a + bi$, its **conjugate** is $a - bi$.
The conjugate of a complex number $z = a + bi$ is denoted with $\overline{z}$.

Example

What is the conjugate of $3 + 5\text{i}$?

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Example

What is the conjugate of $3 + 5\mathrm{i}$?

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For some complex number $a + bi$, its **magnitude** is $a^2 + b^2$.
The magnitude of a complex number $z = a + bi$ is denoted with $|z|$.

Example

What is the magnitude of $3 + 5\mathrm{i}$?

Conjugates and Magnitudes
Complex Numbers

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$$3 - 5\mathrm{i}$$

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The magnitude of a complex number $z = a + bi$ is denoted with $|z|$.

Example

What is the magnitude of $3 + 5\mathrm{i}$?

$3^2 + 5^2$
$= 34$

You can multiply complex numbers like you would multiply binomials (using FOIL).

Example

If $a = 5 + 3\mathrm{i}$ and $b = 6 + 4\mathrm{i}$, then

$$
\begin{aligned}
& (5 + 3\mathrm{i}) \times (6 + 4\mathrm{i}) \\
&= (5 \times 6) + (5 \times 4\mathrm{i}) + (3\mathrm{i} \times 6) + (3\mathrm{i} \times 4\mathrm{i}) \\
&= 30 + 20\mathrm{i} + 18\mathrm{i} + 12\mathrm{i}^2 \\
&= 30 + 38\mathrm{i} - 12 \\
&= 18 + 38\mathrm{i}
\end{aligned}
$$

A complex number multiplied by its congugate always gives its magnitude.

Example

$$
\begin{aligned}
&(5+3\mathrm{i}) \times (5-3\mathrm{i})\\
&= 5^2 - (3\mathrm{i})^2\\
&= 34
\end{aligned}
$$

To divide complex numbers, make the denominator into a real number by multiplying top and bottom by its congugate.

Example

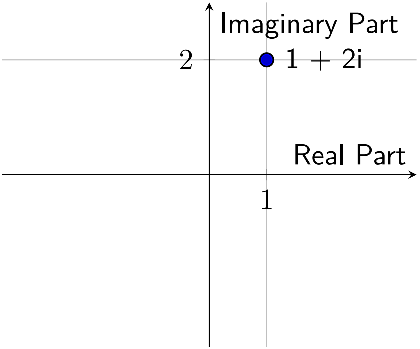
$$\frac{1+2i}{2-3i}$$
$$= \frac{(1+2i)(2+3i)}{(2-3i)(2+3i)}$$
$$= \frac{2+3i+4i+6i^2}{2^2-(3i)^2}$$
$$= \frac{-4+7i}{13}$$
$$= \frac{-4}{13} + \frac{7}{13}i$$

The general formula for dividing complex numbers a by b is:

Formula

$$\frac{a \times \overline{b}}{|b|}$$

Complex points can be visualized on the complex plane.



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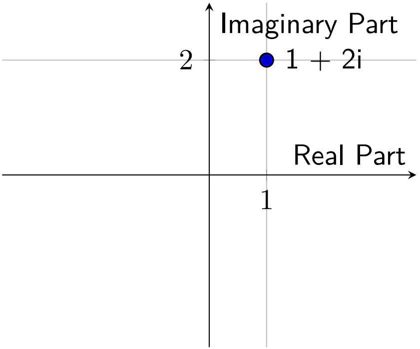

The **magnitude** of the number is the distance of the point from the origin.

The **argument** is the polar angle (angle counter-clockwise from the x-axis in the positive direction) of the point.

Converting $a + bi$ form to and from magnitude-argument (**polar**) form requires some trigonometry.

$a + bi$ form to polar form

Magnitude $= |z| = a^2 + b^2$
Argument $= arg(z) = atan2(a, b)$

$$atan2(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

(LaTeXcode stolen from Wikipedia)
The atan2 formula is derived from CAST rule.

Converting $a + bi$ form to and from magnitude-argument (**polar**) form requires some trigonometry.

Polar form to $a + bi$ form

Where $Re(z)$ is the real part of the complex number and $Im(z)$ is the imaginary part of the complex number
and $\theta = \arg(z)$,

$$Re(z) = |z| \times \cos(\theta)$$

$$Im(z) = |z| \times \sin(\theta)$$

Given a complex number in polar form, it can also be written in a closed-form expression (without converting back to $a + bi$).

Euler's Formula

For some complex number z:
$z = |z| \times e^{arg(z) \times \mathrm{i}}$

Anecdote: Within a certain set of people, whenever someone says "Euler's Formula" or "Euler's Theorem", another person always asks "which one?". It occurred to be that while this was not just a joke; we actually need clarification because we have at some point or another mentioned this formula, Euler's Formula about planar graphs, and the Euler-Fermat Theorem.

Exponentiating Complex Numbers

Complex Numbers

De Moivre's Formula gives us a useful way of exponentiating complex numbers in polar form.

Statement

For some complex number z and integer n, if $y = z^n$,

$$y| = |z|^n$$

$$arg(y) = arg(z) \times n$$

Less formally, a complex number raised to the n^{th} power has its magnitude raised to the n^{th} power and its argument multiplied by n.

This can be trivially proven with Euler's Formula.

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1 Complex Numbers

2 The Riemann Zeta Function

Definition (The Riemann zeta function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$$

$$s = \sigma + \mathrm{i}t$$

The formula

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$\zeta(s)$ is symmetric across the vertical line $\sigma = -\frac{1}{2}$.

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The sum of the natural numbers is related to, but not equal to, $-\frac{1}{12}$.