Probability

Vincent Macri

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A lot of games have an element of chance. For example, knowing the probabilities of landing on different spaces in Monopoly will tell you which properties will make the most money.

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A quick overview of sets
Introduction

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- The cardinality of a set A, written as $n(A)$ is the number of elements in A. For example, $n(A) = 3$.

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Event: An event is a subset of the sample space. We will call this set E.

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If every possible outcome of an experiment has the same chance of occurring, then we can easily calculate the probability of an event (subset of the sample space) occurring.

The probability function, $p(E)$ gives us the probability of an event occurring.

$$p(E) = \frac{n(E)}{n(S)}$$

Probability is usually written as a number in the range $[0, 1]$.

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Basics

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Factorial

Basics

The factorial operation is the product of all integers from 1 to n.

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Definition

$$n! = 1 \times 2 \times \cdots \times (n-1) \times n$$

$0! = 1$, because otherwise you would break a lot of math.

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Example

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

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Problem
Math Elections

The student body of Mackenzie is voting on their 3 favourite parts of math at the school.

They have 5 options: calculus, vectors, $\LaTeX$, Moodle, and math club.

Everyone votes for their favourite part, second favourite part, and third favourite part of math at the school.

These are three separate votes, taken one after another.

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3. Whatever options placed first or second cannot place third, so we are left with 3 options for third place.

This means there are $5 \times 4 \times 3 = 60$ possible combinations.

We can develop the formula for this with factorials:

$$5 \times 4 \times 3 = 5 \times 4 \times 3 \times \frac{2!}{2!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = \frac{5!}{2!} = 60$$

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First	**Second**	**Third**
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If we do not care about ordering, these 6 options become 1 option.

This means that there are $\frac{60}{3!} = 10$ options for favourite parts of math at Mackenzie when we ignore the ordering.

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Number of ordered options

Generalizations

We can calculate the number of ordered subsets of size k from a set of cardinality n with the following formula:

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This formula is also written as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

And is read as "n choose k".

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Problem

Yahtzee, a game popular with my senior relatives, involves rolling 5 dice. In this game, you get a Yahtzee, which is worth a lot of points, if you roll 5 of a kind.

All of the dice have six sides.

Question

What is the probability of rolling a Yahtzee?

First, we will calculate the number of combinations of dice that can be rolled:

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We know that the set of Yahtzee rolls is:

$$\begin{aligned} Y = \{&(1,1,1,1,1),(2,2,2,2,2),(3,3,3,3,3),\\ &(4,4,4,4,4),(5,5,5,5,5),(6,6,6,6,6)\} \end{aligned}$$

Calculations
Yahtzee

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$$p(Y) = \frac{n(Y)}{n(R)} = \frac{6}{7776} = \frac{1}{1296} \doteq 0.00077 = 0.077\%$$

$\therefore$ the chance of rolling a Yahtzee is about 0.077%.

Question 2

How likely is it to roll four of a kind when rolling 5 dice?

We are looking at the chance of rolling exactly four of a kind. 5 of a kind does not count.

Quadruples solution

Yahtzee

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But we have five dice, so we must multiply this by the chance of rolling something other than one, and we must multiply by the number of different ways to arrange 4 ones and 1 other die:

$$\left(\frac{1}{6}\right)^4 \times \frac{5}{6} \times 5 = \frac{25}{7776}$$

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The chance of rolling quadruple ones is the same as rolling quadruple twos, threes, etc.. So:

$$\left(\frac{1}{6}\right)^4 \times \frac{5}{6} \times 5 \times 6 = \frac{25}{1296} \doteq 0.0193$$

Question 3

How likely is it to roll three of a kind when rolling 5 dice?

We are looking at the chance of rolling exactly three of a kind. 4 or 5 of a kind do not count. We will count a 2 of a kind and a 3 of a kind as a triple.

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The main difference is that we don't know how many ways we can order three of a kind with two other numbers. With 4 of a kind, this was trivial. It is not longer trivial with three of a kind.

$$\left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2 \times c \times 6$$

This is essentially the same formula we used last time, but replacing the number of combinations with the variable c. We must use reasoning to figure out c.

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Triples solution

Yahtzee

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$$c = \binom{5}{3} = 10$$

$$\left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2 \times 10 \times 6 = \frac{125}{648} \doteq 0.193$$