Lambda Calculus
Now you can bring a computer to your tests!

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William Lyon Mackenzie C.I. Math Club

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Table of Contents

- Created by Alonzo Church

- Created by Alonzo Church
- A way of representing pure mathematical functions

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- Can represent any computer program

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- Can represent any computer program
- Equivalent to Turing machines

In math class, we would define a function that accepts an argument x and outputs $x + 1$ as so:

$$f(x)$$

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If we wanted to find $4 + 1$, we could do this:

$$f(4) = 4 + 1 = 5$$

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In lambda calculus, we apply a value to a function like this:

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You can think of λ as f, and . as $=$.

Table of Contents

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- Very simple

- Very simple
- Very powerful

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- Very powerful
- Functions can only have one variable

Me too!

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We have some shortcuts to help us write down lambda calculus expressions, but it's important to remember what they represent, without the shortcuts.

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Also, we assume that we evaluate a function with "multiple" arguments starting with the leftmost parameter.

you're right!

For those examples, you're right!

For those examples, you're right!
Let's get to the fun stuff now!

Table of Contents

Quote

"Any program can be written in lambda calculus."

> ### Quote
>
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> — Me, 5 minutes ago

Quote

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— Me, 5 minutes ago

So, let's bring on the Booleans!

Definition (TRUE)

$$\text{TRUE} = \lambda xy.x$$

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Definition (FALSE)

$$\text{FALSE} = \lambda xy.y$$

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So TRUE returns the first value, and FALSE returns the second.

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So TRUE returns the first value, and FALSE returns the second.

We will use TRUE and FALSE as shorthand for these definitions.

Definition (NOT)

$$NOT = \lambda b.b(\text{FALSE TRUE})$$

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NOT TRUE

$$(\lambda b.b(FALSE\ TRUE))\ TRUE =$$

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$$(\lambda b.b(FALSE\ TRUE))\ TRUE = TRUE(FALSE\ TRUE)$$

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NOT TRUE

$$
\begin{aligned}
(\lambda b.b(\text{FALSE TRUE})) \text{ TRUE} &= \text{TRUE}(\text{FALSE TRUE}) \\
&= \lambda xy.x(\text{FALSE TRUE}) \\
&= \text{FALSE}
\end{aligned}
$$

Definition (AND)

$$\text{AND} = (\lambda pq.p)(q\ p)$$

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AND(TRUE FALSE)

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AND(TRUE FALSE)

$$((\lambda pq.p)(q\ p))\ (\text{TRUE FALSE})$$
$$= \text{TRUE(FALSE TRUE)}$$
$$= \text{FALSE}$$

Definition (IFTHENELSE)

$$\text{IFTHENELSE} = (\lambda b t f.b)(t\ f)$$

Where b is a Boolean, t is the value to return if $b = \text{TRUE}$, and f is the value to return if $b = \text{FALSE}$.

IFTHENELSE(TRUE "Math is great" "Math is kool")

$$((\lambda b t f.b)(t\ f))\,(\text{TRUE "Math is kool" "Math is lit"})$$

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