Sequences and Series

Your soon-to-be new best friend ♡

Samantha Unger

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What is Sigma notation?

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Fun fact: Sigma is the 18th letter of the Greek alphabet, and is transliterated as "s".

Using Sigma notation
Summations

$$\sum_{n=1}^{4} n$$

The Sigma notation consists of several components. In the example above, we have:

n under Sigma Index of summation. Some people use i, k, or x

1 First value of n (can be anything)

4 Term we end on

n after Sigma Formula for each turn

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n after Sigma Formula for each turn

$$\sum_{n=1}^{4} n = 1 + 2 + 3 + 4 = 10$$

Example
Summations

Let's try some quick maths!
We'll do this one together.
There's more on your worksheet.

Let's talk about why these work:

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Multiplying by a constant:

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Adding/subtracting:

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

Summation shortcuts ;)

Summing 1 equals n

$$\sum_{k=1}^{n} 1 = n$$

Summing the constant c equals $c \times n$

$$\sum_{k=1}^{n} c = nc$$

A shortcut when summing k (we will develop this later)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

These ones are pretty cool and useful:

A shortcut when summing k^2

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

A shortcut when summing k^3

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Try the super cool practice word problem on your sheet!

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What is an arithmetic sequence?

Arithmetic Sequences

- A sequence of numbers where there is a constant difference between successive terms
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We can define the n^{th} term with either of the following formulas:

$$a_n = a_1 + (n-1)d \qquad a_n = a_m + (n-m)d$$

Where we have:

a_n the n^{th} term

n the term number

d the constant difference

m the m^{th} term.

a_1 the first term of the series if we start counting from 1

How can we easily sum a finite arithmetic series?

[1] This story may or may not be true, but I choose to believe it anyway.

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- Let me tell you about my main man Gauss...[1]
- Pair up your values and divide by 2!

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Sum of finite arithmetic series

Arithmetic Sequences

How can we easily sum a finite arithmetic series?

- Let me tell you about my main man Gauss...[1]
- Pair up your values and divide by 2!

$$
\begin{array}{rrrrrrrrrrrrr}
 & 1 & + & 2 & + & 3 & +\dots & + & 98 & + & 99 & + & 100 \\
+ & 100 & + & 99 & + & 98 & +\dots & + & 3 & + & 2 & + & 1 \\
\hline
 & 101 & + & 101 & + & 101 & +\dots & + & 101 & + & 101 & + & 101
\end{array}
$$

This is 101×100, which we know is 10100.

But we added up the numbers twice, so we need to divide by 2.

$$10100 \div 2 = 5050 \qquad \therefore \sum_{n=1}^{100} n = 5050$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

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What is a geometric sequence?
Geometric Sequences

- Follows a pattern where each term in found by multiplying the previous term by a constant called the common ratio
- Examples: $3, 6, 12, 24, 48$ or $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

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We can define the n^{th} term with any of the following formulas:

$$a_n = a_{n-1} \times r \qquad a_n = a_1 \times r^{n-1}$$

Where we have:

a_n	the n^{th} term
a_{n-1}	the $(n-1)^{\text{th}}$ (previous) term
a_1	the first term of the series
n	the term number
r	the common ratio

$$\begin{array}{rclllllllllll} S_n & = & a & + & ar & + & ar^2 & + & ar^3 & + \dots & + & ar^{n-1} & & \\ rS_n & = & & + & ar & + & ar^2 & + & ar^3 & + \dots & + & ar^{n-1} & + & ar^n \end{array}$$

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ S_n(1-r) &= a(1-r^n) \\ \therefore S_n &= a\left(\frac{1-r^n}{1-r}\right) \end{aligned}$$

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What if I told you that the sum of some **infinite** series wasn't infinite?

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What if I told you that we can solve this? Can you sense the excitement?

$$\sum_{k=1}^{\infty} 3\left(\frac{1}{2}\right)^{k-1}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

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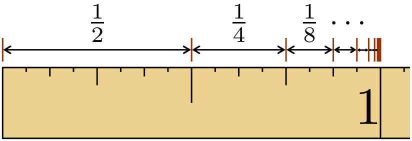
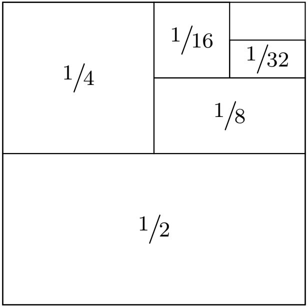
Here are two ways to visualize this:





What does that sum approach?

The Mind-blowing Part

From the previous visualizations it's clear that:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

Can we prove this with algebra?

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Can we prove this with algebra?

Yes!

Proof.

We will call the whole sum S: $S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

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Solving algebraically

The Mind-blowing Part

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Now, **subtract** them!
All the terms from $\frac{1}{4}$ onwards cancel out.

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And we get: $S - \frac{S}{2} = \frac{1}{2}$.

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Simplify: $\frac{S}{2} = \frac{1}{2}$.

So: $S = 1$. $\square$

We'll talk about it in a minute!

$$S_\infty = \sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}$$

But first, let's try using this formula for the example in the previous slide!

For this to work, the ratio r has to be greater than -1 and less than 1.

More formally:

$$-1 \leq r \leq 1$$

$$|r| < 1$$

When can we use this formula?

The Mind-blowing Part

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$$|r| < 1$$

But why?

This is not a geometric series. Does it have a finite sum anyways?

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

Let's look at why not:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$$

$$(1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}\right)+\dots$$

$$(1)+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}\right)+\dots$$

In each case, the top values are equal to or greater than the bottom ones.

Let's add up the bottom groups:

$$(1)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\dots=\infty$$

So our original series must also be infinite.

Convergent series: An infinite series for which the sequence of partial sums converges.

Divergent series: An infinite series for which the sequence of the partial sums of the series does not have a finite limit.

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Sigma Notation	Converges if	Diverges if
$S = \sum_{n=1}^{\infty}(t_1 + d(n-1))$	Never	Always

Properties of geometric series

Some fun properties

Sigma Notation	Converges if	Diverges if
$S = \sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r\rvert < 1$ with $S = \frac{a}{1-r}$	$\lvert r\rvert \geq 1$

Properties of harmonic series

Some fun properties

Sigma Notation	Converges if	Diverges if
$S = \sum_{n=1}^{\infty} \frac{1}{n}$	Never	Always

Properties of p-series

Some fun properties

Sigma Notation	Converges if	Diverges if
$S = \sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$

Properties of p-series

Some fun properties

Sigma Notation	Converges if	Diverges if
$S = \sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$

This is called the Basel Problem and it brought fame to Euler when he was 28 years old!

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$