Euclid Preparation 4

Proofs

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What is it?

Direct Proof

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It follows a set of logically true statements, which allow one to arrive from statement A to statement B.

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Proof by example is a sub-category of direct proofs, and is not covered in this lesson. It consists of finding a single case to a supposition in the form of “There exists N such that ...”.

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An integer N is odd if it can be expressed as $N = 2M + 1$ where M is also an integer. Sometimes, $2M - 1$ is used.

Example

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Odd Number Squared

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Given an odd integer N, prove that N^2 is also an odd integer.

If N is odd, it can be expressed as $2M - 1$.

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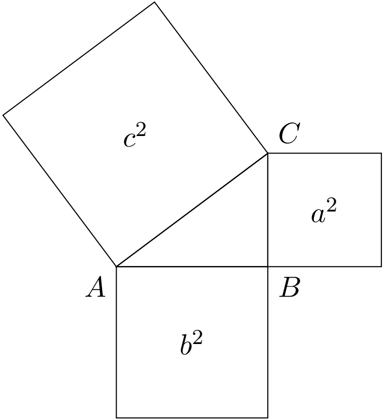
$$\begin{aligned} N^2 &= (2M-1)^2 \\ &= 4M^2 - 4M + 1 \\ &= 2(2M^2 - 4M) + 1 \end{aligned}$$

Therefore N^2 is also odd.

Example

Given a right angled triangle, the sum of the squares of the two legs is equal to the square of the hypotenuse.



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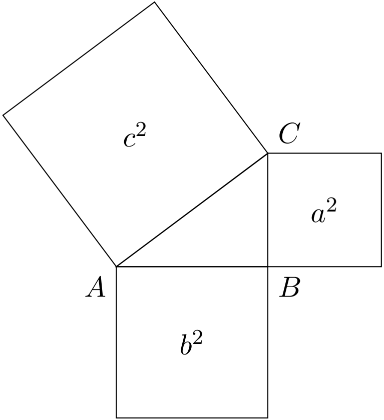
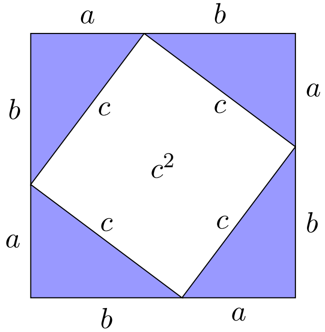


In Canada, this theorem is provided to students without proof, despite the fact that over 1000 different ones exist.

Pythagorean Theorem
Direct Proof



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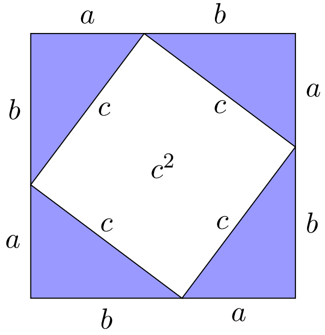

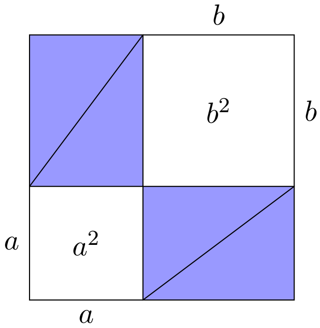

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A proof by contradiction is another common method of proving a statement.

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It makes an assumption, and then follows through a set of logical steps that show that if the assumption is true, the assumption must be false. This results in a contradiction, meaning the assumption is false.

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Proof by Contradiction

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It makes an assumption, and then follows through a set of logical steps that show that if the assumption is true, the assumption must be false. This results in a contradiction, meaning the assumption is false.

A similar style of proof is called “Proof by Contraposition”, which proves a statement “If A then B” by proving that “If not A, then not B”.

Odd Number Squared

Proof by Contradiction

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Suppose that N is not an odd integer. Thus, it can be expressed as $2M$.

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N^2 is not odd. Therefore, N *is not* not an odd integer, and must be odd.

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Irrationality of $\sqrt{2}$
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Example

Prove that $\sqrt{2}$ is irrational.

Let's assume that $\sqrt{2}$ is rational.

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Example

Prove that $\sqrt{2}$ is irrational.

Let's assume that $\sqrt{2}$ is rational.Then, it can be expressed as $\frac{a}{b}$ where a and b are integers and have no common factors. If they have common factors, divide both by the factor until the original condition is met. This has been shown to be possible for any two integers in an earlier lesson.

$$\sqrt{2} = \frac{a}{b}$$

Irrationality of $\sqrt{2}$
Proof by Contradiction

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Irrationality of $\sqrt{2}$

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\begin{aligned}
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This means that a is even, and can be expressed as $2c$.

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As before, the above statement implies that b is even.

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$$b^2 = 2c^2$$

As before, the above statement implies that b is even. However, if both a and b are even, then they share a common factor 2. This contradicts the original assumption. Therefore, $\sqrt{2}$ is irrational.

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It splits the question into all the possible cases, and then proves that the statement is true for all possible cases.

It can look very disorganized, especially if the cases are singular numbers instead of sets, but it is still a proof.

Odd Number Squared
Proof by Exhaustion

Example

Given an integer N, prove that N^2 must be even, or one greater than even.*

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Given an integer N, prove that N^2 must be even, or one greater than even.*

Separate the problem into two cases: N is odd, or N is even. Then, you will find that one case will yield an even number (namely N is even) and the other case will yield a number one greater than even (namely N is odd).

Example

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For this problem, we will make 3 cases: N is divisible by 3, N is one less than a number divisible by 3, and N is one greater than a number divisible by 3.

Case 1 (N is divisible by 3, and can be expressed as $3M$):

Case 1

Proof by Exhaustion

Case 1 (N is divisible by 3, and can be expressed as $3M$):

$$\begin{aligned} N^3 &= (3M)^3 \\ &= 27M^3 \\ &= 9(3M^3) \end{aligned}$$

Case 2 (N is one less than a number divisible by 3, and can be expressed as $3M - 1$):

Case 2 (N is one less than a number divisible by 3, and can be expressed as $3M - 1$):

$$\begin{aligned} N^3 &= (3M-1)^3 \\ &= 27M^3 - 27M^2 + 9M - 1 \\ &= 9(3M^3 - 9M^2 + M) - 1 \end{aligned}$$

Case 3 (N is one greater than a number divisible by 3, and can be expressed as $3M + 1$):

Case 3 (N is one greater than a number divisible by 3, and can be expressed as $3M+1$):

$$\begin{aligned} N^3 &= (3M+1)^3 \\ &= 27M^3 + 27M^2 + 9M + 1 \\ &= 9(3M^3 + 9M^2 + M) + 1 \end{aligned}$$

The three cases are *exhaustive* (cover all possibilities), and the result fits the conjecture. Therefore, the statement is true.

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It is also sometimes called “Proof by Ternary Exclusion”, when there are two cases, and no third one is possible.

Odd Number Squared

Nonconstructive Proof

Example

Given an integer N, prove that N^2 must be even, or one greater than even.*

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Nonconstructive Proof

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Given an integer N, prove that N^2 must be even, or one greater than even.*

This problem can be solved non-constructively. Simply noting that every number is either even, or one greater than even (odd), proves that this must be true. This is a nonconstructive proof because you proved the statement without finding the exact answer. A better example follows.

Example

Prove that it is possible to have an irrational number to the power of another irrational number result in a rational number.

Irrational to the Irrational

Nonconstructive Proof

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Prove that it is possible to have an irrational number to the power of another irrational number result in a rational number.

Recall that $\sqrt{2}$ is irrational. Let us consider two cases: $\sqrt{2}^{\sqrt{2}}$ and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

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If the first case is rational, then we have found an example to match the statement.

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If it is not rational, it must be irrational. This is the exhaustive part.

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If it is not rational, it must be irrational. This is the exhaustive part.However, simplifying the second case allows one to see that the result is 2.

One of the two cases must be an irrational to the power of an irrational resulting in a rational. However, the specific one is unknown. This is the nonconstructive part.

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Despite its name, the proof is still deductive, not inductive.

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$$\begin{aligned} N^2 &= 1^2 \\ &= 1 \end{aligned}$$

Odd Number Squared

Proof by Induction

Next, assume that it is true for all N. Now prove that it is true for any $N+2$. It cannot be for $N+1$, because an odd number plus 1 is even, while the proof asks for odd. Recall that if N is odd, it can be expressed as $2M+1$.

$$
\begin{aligned}
(N+2)^2 &= (2M+3)^2 \\
&= 2M^2 + 12M + 9 \\
&= 2(M^2 + 6M + 4) + 1
\end{aligned}
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Therefore, any $N+2$ is also odd.

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$$\begin{aligned}(N+2)^2 &= (2M+3)^2 \\ &= 2M^2 + 12M + 9 \\ &= 2(M^2 + 6M + 4) + 1\end{aligned}$$

Therefore, any $N + 2$ is also odd.This works because we showed that the statement is true for $N = 1$. From the above, we know that it is also true for $N + 2$, in this case 3. However, this implies that it is also true for 5, which in turn implies that it is true for 7, and so on until all odd integers are covered.

Example

Prove that $p(N) = 0 + 1 + 2 + 3 + ... + N - 1 + N = \frac{N(N+1)}{2}$.

First, check that it is true for $N = 0$.

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Example

Prove that $p(N) = 0 + 1 + 2 + 3 + ... + N - 1 + N = \frac{N(N+1)}{2}$.

First, check that it is true for $N = 0$.

$$\begin{aligned} 0 &= \frac{0(0+1)}{2} \\ &= 0 \end{aligned}$$

Assume it is true from p(N). Now prove that it is true for p(N+1).

What is being proven!

$$0 + 1 + 2 + .. + N + (N + 1) = \frac{(N+1)((N+1)+1)}{2}$$

$$0 + 1 + 2 + .. + N + (N + 1) = \frac{N(N + 1)}{2} + (N + 1)$$

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$$0 + 1 + 2 + .. + N + (N + 1) = \frac{(N+1)((N+1)+1)}{2}$$

$$\begin{aligned} 0 + 1 + 2 + .. + N + (N + 1) &= \frac{N(N+1)}{2} + (N + 1) \\ &= \frac{N(N+1) + 2(N+1)}{2} \end{aligned}$$

Inductive Portion

Proof by Induction

What is being proven!

$$0+1+2+..+N+(N+1)=\frac{(N+1)((N+1)+1)}{2}$$

$$\begin{aligned}
0+1+2+..+N+(N+1) &= \frac{N(N+1)}{2}+(N+1)\\
&= \frac{N(N+1)+2(N+1)}{2}\\
&= \frac{(N+2)(N+1)}{2}
\end{aligned}$$

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Which is what needed to be proven. $Q.E.D.$

Example

Prove that $4^N - 1$ is always divisible by 3 if N is a positive integer.

First, check that the statement is true for $N = 1$.

$$\begin{aligned} 4^N - 1 &= 4^1 - 1 \\ &= 3 \end{aligned}$$

Assume it is true for N, now prove for $N + 1$.

Example

Prove that $4^N - 1$ is always divisible by 3 if N is a positive integer.

First, check that the statement is true for $N = 1$.

$$\begin{aligned} 4^N - 1 &= 4^1 - 1 \\ &= 3 \end{aligned}$$

Assume it is true for N, now prove for $N + 1$.This time, we will take the difference between the statement for N and $N + 1$. Because the statement is assumed to be true for N, if their difference is divisible by 3, then it must also be divisible by 3 for $N + 1$.

$$4^{N+1} - 1 - (4^N - 1) = 4^{N+1} - 4^N$$

$$\begin{aligned} 4^{N+1} - 1 - (4^N - 1) &= 4^{N+1} - 4^N \\ &= 4^N(4^1 - 1) \end{aligned}$$

$$
\begin{aligned}
4^{N+1} - 1 - (4^N - 1) &= 4^{N+1} - 4^N \\
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&= 4^N(3)
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Thus, the statement is true.