Complex Numbers and the Riemann Hypothesis

How to win \$1 000 000

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Usually, we work with *real* numbers.

Real include all integers, rational numbers, and irrational numbers.

e.g. $1,\ 6.9,\ \pi,\ e,\ \sqrt{123}$

Definition of i

i is defined as $\sqrt{-1}$.

$\therefore \mathrm{i}^2 = -1; (-\mathrm{i})^2 = -1$

Definition of a complex number

A complex number is the sum of a real and imaginary number.

e.g. $1 + \mathrm{i},\ \pi + ei,\ 0 + \mathrm{i},\ 1 + 0\mathrm{i}$, etc

Complex numbers have interesting properties, including how they and, multiply, and exponentiate.

Adding and Subtracting Complex Numbers

You can add and subtract complex numbers like you would add polynomials (combine like terms).

Example 1

$$(5 + 3i) + (6 + 4i) = 11 + 7i$$

Example 2

$$(3 + 6i) - (6 - 4i) = -3 + 2i$$

For some complex number $a + bi$, its **conjugate** is $a - bi$.
The conjugate of a complex number $z = a + bi$ is denoted with $\overline{z}$.

Example

What is the conjugate of $3 + 5\mathrm{i}$?

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$$3 - 5\mathrm{i}$$

Conjugates and Magnitudes
Complex Numbers

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Example

What is the conjugate of $3 + 5\mathrm{i}$?

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For some complex number $a + bi$, its **magnitude** is $a^2 + b^2$.
The magnitude of a complex number $z = a + bi$ is denoted with $|z|$.

Example

What is the magnitude of $3 + 5\mathrm{i}$?

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Example

What is the magnitude of $3 + 5\mathrm{i}$?

$3^2 + 5^2$
$= 34$

You can multiply complex numbers like you would multiply binomials (using FOIL).

Example

If $a = 5 + 3\mathrm{i}$ and $b = 6 + 4\mathrm{i}$, then

$$\begin{aligned} & (5 + 3\mathrm{i}) \times (6 + 4\mathrm{i}) \\ &= (5 \times 6) + (5 \times 4\mathrm{i}) + (3\mathrm{i} \times 6) + (3\mathrm{i} \times 4\mathrm{i}) \\ &= 30 + 20\mathrm{i} + 18\mathrm{i} + 12\mathrm{i}^2 \\ &= 30 + 38\mathrm{i} - 12 \\ &= 18 + 38\mathrm{i} \end{aligned}$$

A complex number multiplied by its conjugate always gives its magnitude.

Example

$$
\begin{aligned}
&(5+3\mathrm{i})\times(5-3\mathrm{i})\\
&=5^2-(3\mathrm{i})^2\\
&=34
\end{aligned}
$$

To divide complex numbers, make the denominator into a real number by multiplying top and bottom by its conjugate.

Example

$$\frac{1+2\mathrm{i}}{2-3\mathrm{i}}$$

$$= \frac{(1+2\mathrm{i})(2+3\mathrm{i})}{(2-3\mathrm{i})(2+3\mathrm{i})}$$

$$= \frac{2+3\mathrm{i}+4\mathrm{i}+6\mathrm{i}^2}{2^2-(3\mathrm{i})^2}$$

$$= \frac{-4+7\mathrm{i}}{13}$$

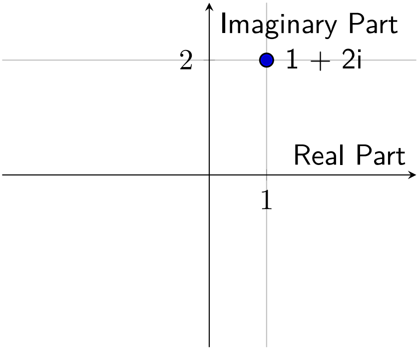
$$= \frac{-4}{13}+\frac{7}{13}\mathrm{i}$$

The general formula for dividing complex numbers a by b is:

Formula

$$\frac{a \times \overline{b}}{|b|}$$

Complex points can be visualized on the complex plane.



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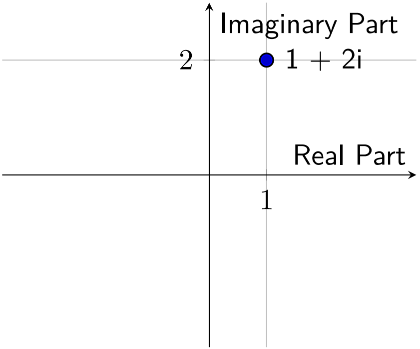

The **magnitude** of the number is the distance of the point from the origin.
The **argument** is the polar angle (angle counter-clockwise from the x-axis in the positive direction) of the point.

Converting $a + bi$ form to and from magnitude-argument (**polar**) form requires some trigonometry.

$a + bi$ form to polar form

Magnitude $= |z| = a^2 + b^2$
Argument $= arg(z) = atan2(a, b)$

$$atan2(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

(LaTeXcode stolen from Wikipedia)
The atan2 formula is derived from CAST rule.

Converting $a + bi$ form to and from magnitude-argument (**polar**) form requires some trigonometry.

Polar form to $a + bi$ form

Where $Re(z)$ is the real part of the complex number and $Im(z)$ is the imaginary part of the complex number
and $\theta = \arg(z)$,

$$Re(z) = |z| \times \cos(\theta)$$

$$Im(z) = |z| \times \sin(\theta)$$

Given a complex number in polar form, it can also be written in a closed-form expression (without converting back to $a + bi$).

Euler's Formula

For some complex number z:
$z = |z| \times e^{arg(z) \times \mathrm{i}}$

Anecdote: Within a certain set of people, whenever someone says "Euler's Formula" or "Euler's Theorem", another person always asks "which one?". It occurred to be that while this was not just a joke; we actually need clarification because we have at some point or another mentioned this formula, Euler's Formula about planar graphs, and the Euler-Fermat Theorem.

Exponentiation with Complex Numbers

Complex Numbers

De Moivre's Formula gives us a useful way of exponentiating complex numbers in polar form.

Statement

For some complex number z and integer n, if $y = z^n$,

$$y| = |z|^n$$

$$arg(y) = arg(z) \times n$$

Less formally, a complex number raised to the n^{th} power has its magnitude raised to the n^{th} power and its argument multiplied by n.

This can be trivially proven with Euler's Formula.

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Definition (The Riemann zeta function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \, \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$$

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There is another definition of the function for $\sigma \leq 1$, which we will get to later.

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The sum of naturals is **related to**, but **not equal to**, $-\frac{1}{12}$.

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Definition (The Clay Mathematics Institute)

"The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

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We will not be solving the Riemann hypothesis today.

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What are trivial zeros?

Trivial zeros

Trivial zeros are those that occur when $\mathrm{Re}(s)$ is a negative even number and $\mathrm{Im}(s) = 0$.

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Formally:
Let T be the set of negative even integers:

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Then:

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Then:

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These zeros are trivial because mathematicians understand them.

Why are they trivial?

Trivial zeros

For nontrivial reasons, the Riemann zeta function can also be defined as:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \, \{s \in \mathbb{C} \mid s \neq 1\}$$

Proof.

For all $n \in \mathbb{N}$:

$$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\sin\left(\frac{\pi \times -2n}{2}\right)\Gamma(1+2n)\zeta(1+2n)$$

Negative evens are zero

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A video
Extra Content

https://youtu.be/sD0NjbwqlYw?t=9m20s