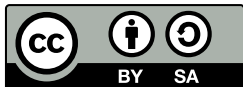


Cryptography

Vincent Macri



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2 Primes



Quick review

Modular Arithmetic

We define the **mod** operator as being the remainder when dividing two numbers. That is:

$$a \bmod b = \text{the remainder of } a \div b$$

In some programming languages, modulo is written as **%** or **rem**. Use whichever notation you are most comfortable with.

Examples

$$4 \bmod 2 = 0$$

$$7 \bmod 3 = 1$$

$$5 \bmod 2 = 1$$

$$9 \bmod 5 = 4$$

The definition of modulo (mod for short) is a bit trickier with negative numbers. It also doesn't matter for today, as we're only looking at mod with positive numbers.



We will also introduce a new notation, which is more of a shortcut.
If b divides a with no remainder, then we will write $b \mid a$.

More formally:

$$b \mid a \equiv a \bmod b = 0$$

Or:

$$b \mid a \iff a = bc$$

Where a , b , and c are positive integers.



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1 Modular Arithmetic

2 Primes



What is a prime number?

Primes

A **prime number** is a positive integer that is only divisible by 1 and itself.

Examples

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$$

If an integer greater than 1 is not prime, it is called a **composite number**.

1 is special, and is called the **unit number**



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Proof 1 is not prime.

In the past, some mathematicians said that 1 is prime. All of them are dead now.

$$\therefore 1 \notin \mathbb{P}$$



The largest prime number

Primes

The largest known prime number¹ is:

$$M_{77\,232\,917} = 2^{77\,232\,917} - 1$$

If you were to print this number out, it would be 6055 pages long!

This prime was discovered by Jonathan Pace on December 26, 2017 after 6 days of continuous computer computations. The discovery was published on January 3, 2018.

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What's special and useful about Mersenne primes? Not much.

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How many primes are there?

Primes

Is the number of primes finite?



How many primes are there?

Primes

Is the number of primes finite?

No! There are infinite prime numbers!

This was proved thousands of years ago by Euclid.



Proof of infinite primes

Primes

Assume the list of primes is finite, and there are only n prime numbers. We will call our list of prime numbers P .

$$P = \{p_1, p_2, \dots, p_{n-1}, p_n\}$$

Where p_k is the k th prime number.



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Where p_k is the k th prime number.

Now, let m be the product of all numbers in P plus 1.

$$m = (p_1 \times p_2 \times \dots \times p_{n-1} \times p_n) + 1 = \left(\sum_{i=1}^n p_i \right) + 1$$

m is either prime or not prime. Let's look at both cases.



Proof of infinite primes: m is prime

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First, let's consider the case that m is prime.



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If m is prime, our original list is incomplete, and there are more prime numbers than we listed.



Proof of infinite primes: m is not prime

Primes

If m is not prime, then it must be divisible by a prime number. Notice that m cannot be divisible by any numbers in P , as they would not divide a number that is a multiple of themselves plus 1.



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For example:

$$P = \{2, 3, 5, 7, 11, 13\}$$

$$m = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30\,031$$



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