James Tanton's Exploding Dots

Vincent Macri

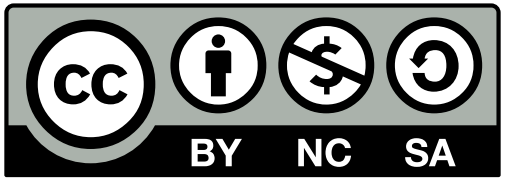
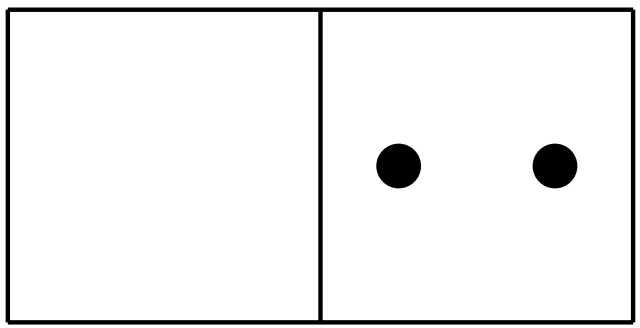


Table of Contents

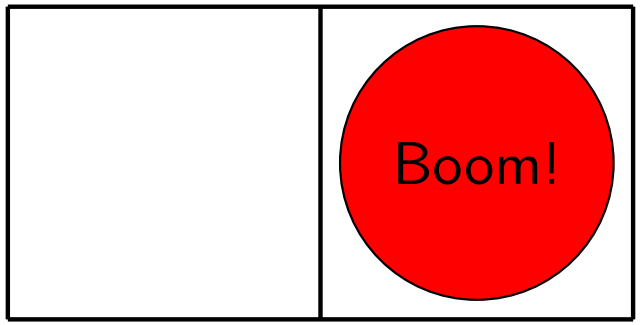
Mechania

The $2 \leftarrow 1$ machine



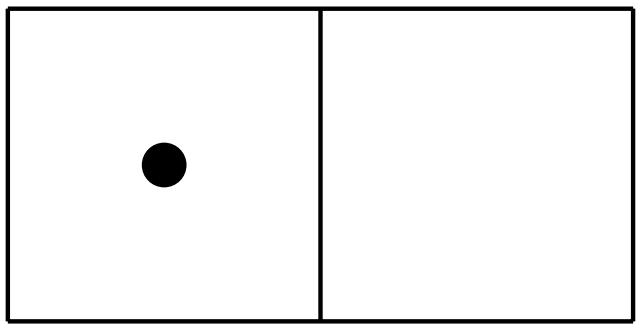
Mechania

The $2 \leftarrow 1$ machine



Mechania

The $2 \leftarrow 1$ machine



Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1}$$

$$2 \xrightarrow{2\leftarrow 1}$$

$$3 \xrightarrow{2\leftarrow 1}$$

$$4 \xrightarrow{2\leftarrow 1}$$

$$5 \xrightarrow{2\leftarrow 1}$$

$$13 \xrightarrow{2\leftarrow 1}$$

Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$

$$2 \xrightarrow{2\leftarrow 1}$$

$$3 \xrightarrow{2\leftarrow 1}$$

$$4 \xrightarrow{2\leftarrow 1}$$

$$5 \xrightarrow{2\leftarrow 1}$$

$$13 \xrightarrow{2\leftarrow 1}$$

Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$

$$2 \xrightarrow{2\leftarrow 1} 10$$

$$3 \xrightarrow{2\leftarrow 1}$$

$$4 \xrightarrow{2\leftarrow 1}$$

$$5 \xrightarrow{2\leftarrow 1}$$

$$13 \xrightarrow{2\leftarrow 1}$$

Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$

$$2 \xrightarrow{2\leftarrow 1} 10$$

$$3 \xrightarrow{2\leftarrow 1} 11$$

$$4 \xrightarrow{2\leftarrow 1}$$

$$5 \xrightarrow{2\leftarrow 1}$$

$$13 \xrightarrow{2\leftarrow 1}$$

Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$

$$2 \xrightarrow{2\leftarrow 1} 10$$

$$3 \xrightarrow{2\leftarrow 1} 11$$

$$4 \xrightarrow{2\leftarrow 1} 100$$

$$5 \xrightarrow{2\leftarrow 1}$$

$$13 \xrightarrow{2\leftarrow 1}$$

Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$

$$2 \xrightarrow{2\leftarrow 1} 10$$

$$3 \xrightarrow{2\leftarrow 1} 11$$

$$4 \xrightarrow{2\leftarrow 1} 100$$

$$5 \xrightarrow{2\leftarrow 1} 101$$

$$13 \xrightarrow{2\leftarrow 1}$$

Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$

$$2 \xrightarrow{2\leftarrow 1} 10$$

$$3 \xrightarrow{2\leftarrow 1} 11$$

$$4 \xrightarrow{2\leftarrow 1} 100$$

$$5 \xrightarrow{2\leftarrow 1} 101$$

$$13 \xrightarrow{2\leftarrow 1} 1101$$

What if. . .

What if we had a $3 \leftarrow 1$ machine?

What if. . .

What if we had a $10 \leftarrow 1$ machine?

Table of Contents

Insighto

What are these machines?

These machines are another way of handling arithmetic, but they work in any base.

A $b \leftarrow 1$ machine handles numbers in base b.

We'll use a $10 \leftarrow 1$ machine for now.

Table of Contents

Arithmos
What next?

We can count with these machines, but what else can we do?

Arithmos
What next?

We can count with these machines, but what else can we do?

What's the first thing you learn to do with numbers after counting?

Addition!

Arithmos
Examples

What is $234 + 125$?

Arithmos
Examples

What is $234 + 125$?

$$\begin{array}{cccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

Arithmos
Examples

What is $234 + 125$?

$$\begin{array}{cccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

What is $234 + 187$?

Arithmos
Examples

What is $234 + 125$?

$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

What is $234 + 187$?

$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 8 & 7 \\ \hline & 3 & 11 & 11 \end{array}$$

Arithmos
Examples

What is $234 + 125$?

$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

What is $234 + 187$?

$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 8 & 7 \\ \hline & 3 & 11 & 11 \end{array}$$

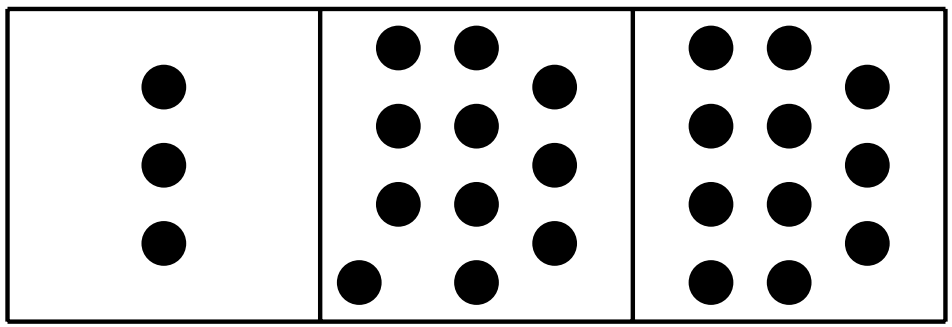
Three hundred eleventy eleven!

But now society thinks I'm weird. Let's fix that.

Arithmos

Explode the dots!

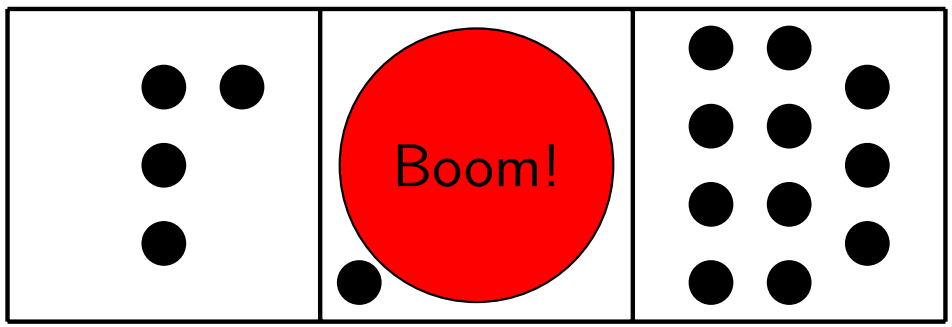
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

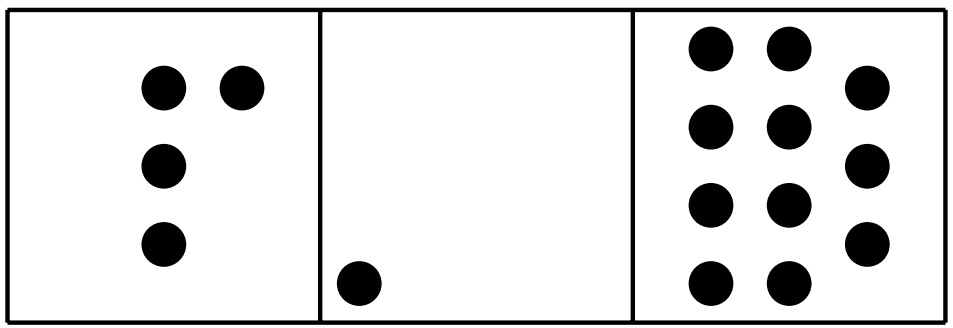
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

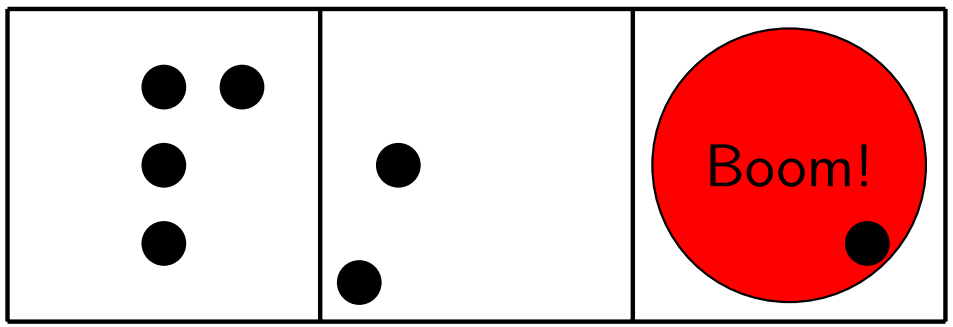
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

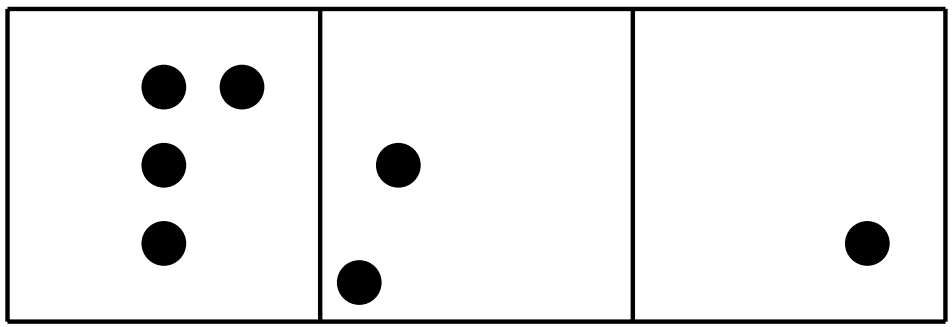
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



$$421$$

What do we learn after addition?

Multiplication!

What is 2876×3?

$$\begin{array}{cccc} 2 & 8 & 7 & 6 \\ \times & & & 3 \\ \hline 6 & 24 & 21 & 18 \end{array}$$

Let's fix this one together for society.

6	24	21	18

Table of Contents

Antidotia
Subtraction

Theorem 1

Subtraction does not exist.

Theorem 2

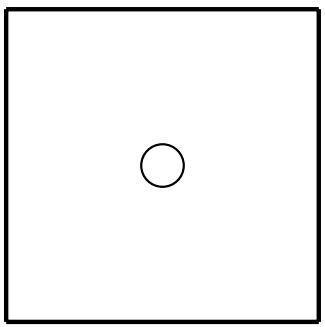
What we call subtraction is just the addition of negative numbers.

Or, subtraction is the addition of the opposite.

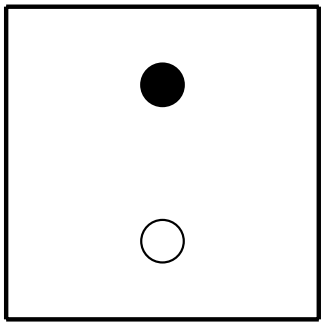
Antidotia
The antidot

The opposite of a dot is an antidot. I'll call these tods.

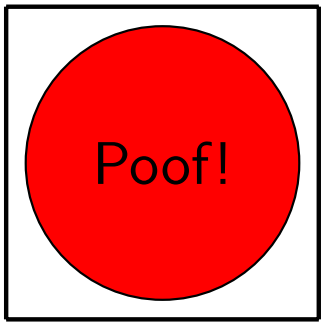
This is one tod in one of our machines:



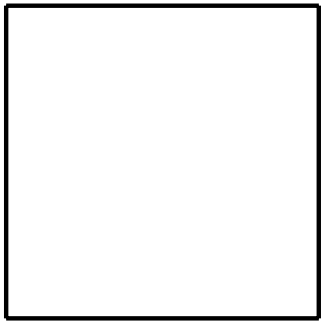
Antidotia
How do tods behave?



Antidotia
How do tods behave?



Antidotia
How do tods behave?



Antidotia
Examples

What is $564 - 123$?

Antidotia
Examples

What is $564 - 123$?

$$\begin{array}{rccc} & 5 & 6 & 4 \\ - & 1 & 2 & 3 \\ \hline & 4 & 4 & 1 \end{array}$$

Antidotia
Examples

What is $564 - 123$?

$$\begin{array}{rccc} & 5 & 6 & 4 \\ - & 1 & 2 & 3 \\ \hline & 4 & 4 & 1 \end{array}$$

What is $441 - 254$?

Antidotia
Examples

What is $564 - 123$?

$$\begin{array}{rccc} & 5 & 6 & 4 \\ - & 1 & 2 & 3 \\ \hline & 4 & 4 & 1 \end{array}$$

What is $441 - 254$?

$$\begin{array}{rccc} & 4 & 4 & 1 \\ - & 2 & 5 & 4 \\ \hline & 2 & -1 & -3 \end{array}$$

What is $564 - 123$?

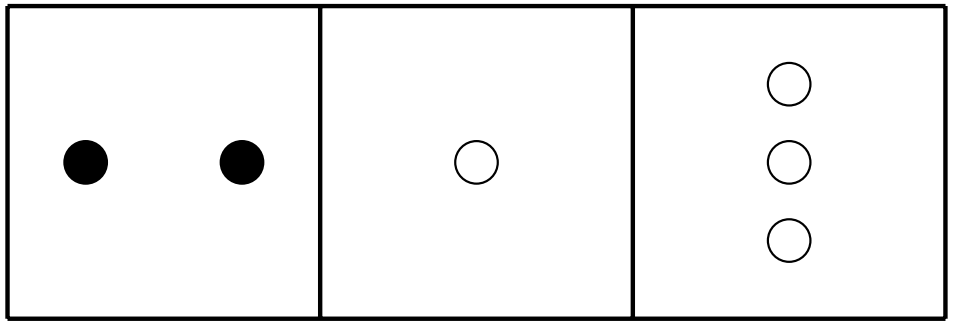
$$\begin{array}{rccc} & 5 & 6 & 4 \\ - & 1 & 2 & 3 \\ \hline & 4 & 4 & 1 \end{array}$$

What is $441 - 254$?

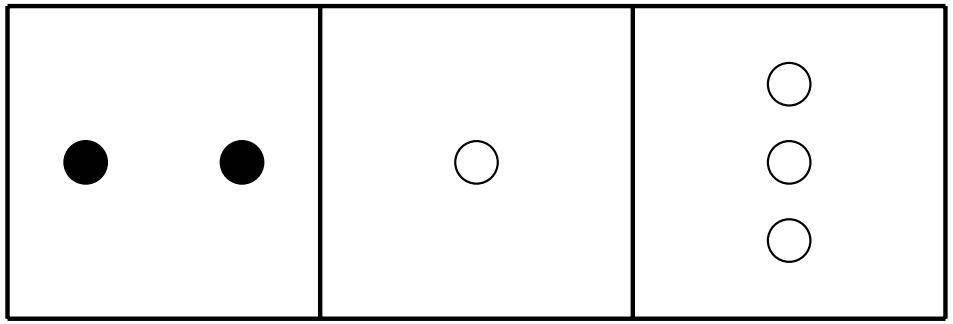
$$\begin{array}{rccc} & 4 & 4 & 1 \\ - & 2 & 5 & 4 \\ \hline & 2 & -1 & -3 \end{array}$$

Let's fix this together on the board for society's sake. We'll use the exploding dots method.

Antidotia
Another method

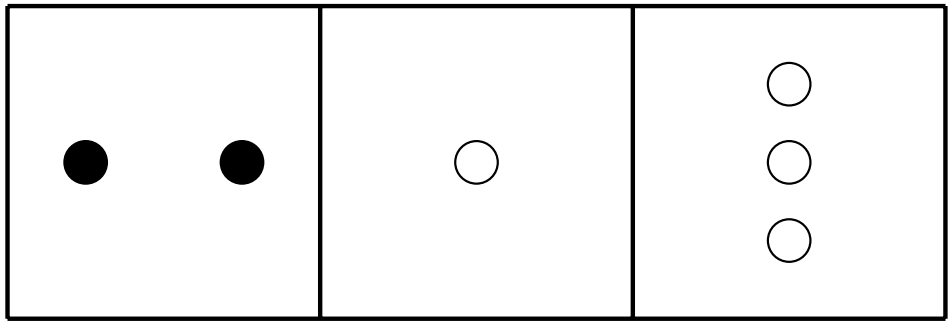


Antidotia
Another method



There is another way to fix this which is helpful for doing math mentally.

Antidotia
Another method

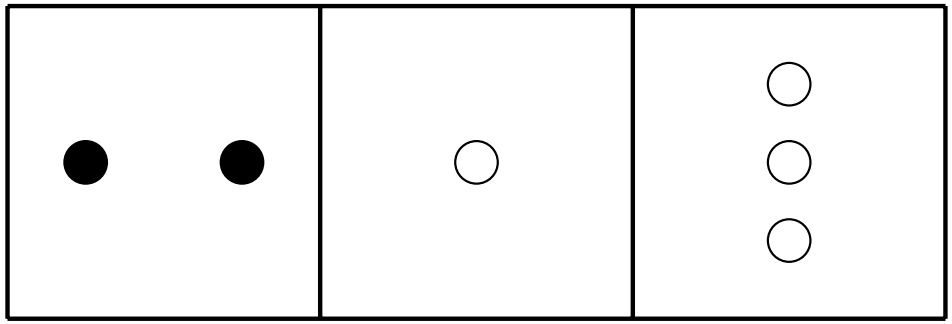


There is another way to fix this which is helpful for doing math mentally.

Let's look the place values:

$$200 + -10 + -3$$

Antidotia
Another method



There is another way to fix this which is helpful for doing math mentally.

Let's look the place values:

$$200 + -10 + -3$$

This is very easy to do mentally.

$$200 + -10 = 190 + -3$$

$$190 + -3 = 187$$

Table of Contents

Obelus
Division

Fun fact before we get started on division, did you know that the $\div$ sign is called an obelus?

Obelus

Long division method

What is $276 \div 12$?

Don't use a calculator!

Obelus

Long division method

What is $276 \div 12$?

Don't use a calculator!

$$
\begin{array}{r}
23 \\
12\overline{)\,276} \\
\underline{240} \\
36 \\
\underline{36} \\
0
\end{array}
$$

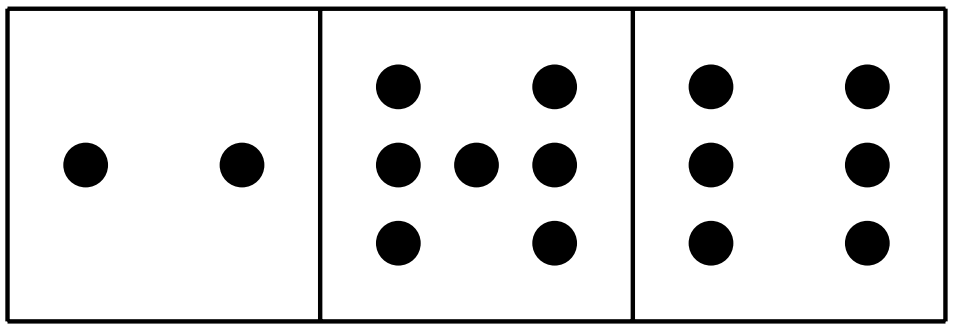
This is stupid and convoluted. Let's use exploding dots instead.

Obelus

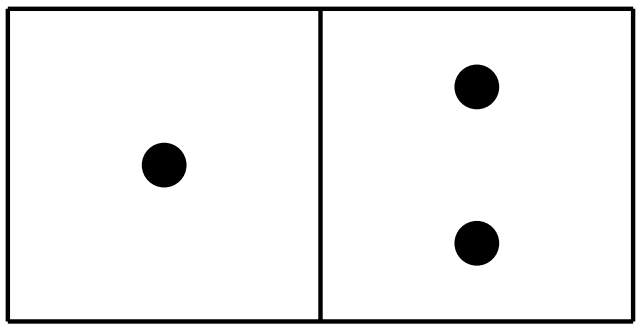
Exploding dots method

$$276 \div 12$$

276:



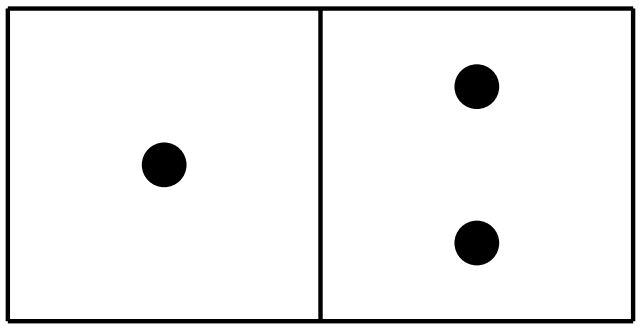
12:



Let's look for groups of 12 in 276.

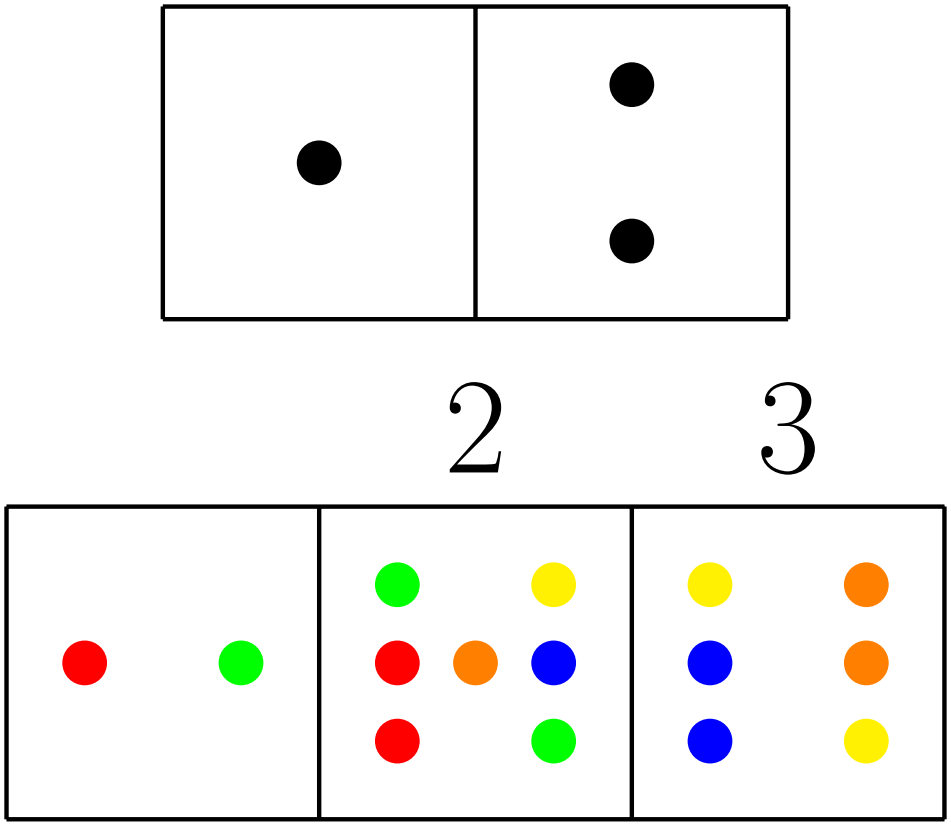
Obelus

Exploding dots solution



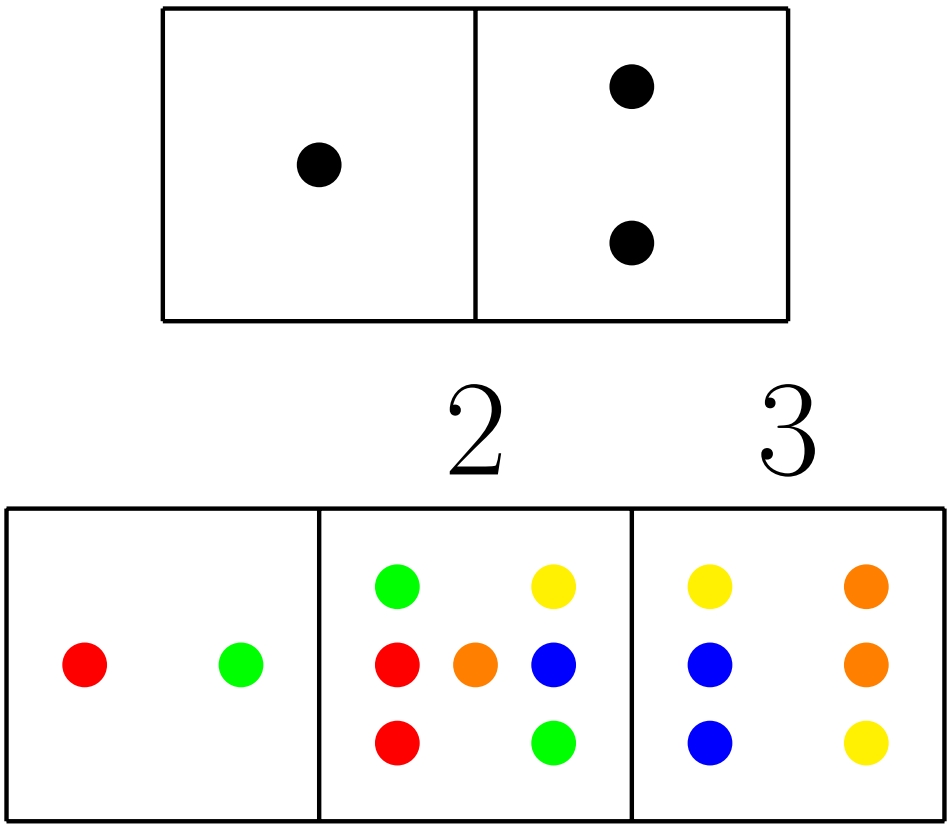
Obelus

Exploding dots solution



Obelus

Exploding dots solution



$$273 \div 12 = 23$$

Obelus
Examples together

What is $2783 \div 23$?

Obelus
Examples together

What is $2783 \div 23$? 121

Obelus

Examples together

What is $2783 \div 23$? 121

What is $2785 \div 23$?

Obelus

Examples together

What is $2783 \div 23$? 121

What is $2785 \div 23$? 121 R2 or $121 + \frac{2}{23}$

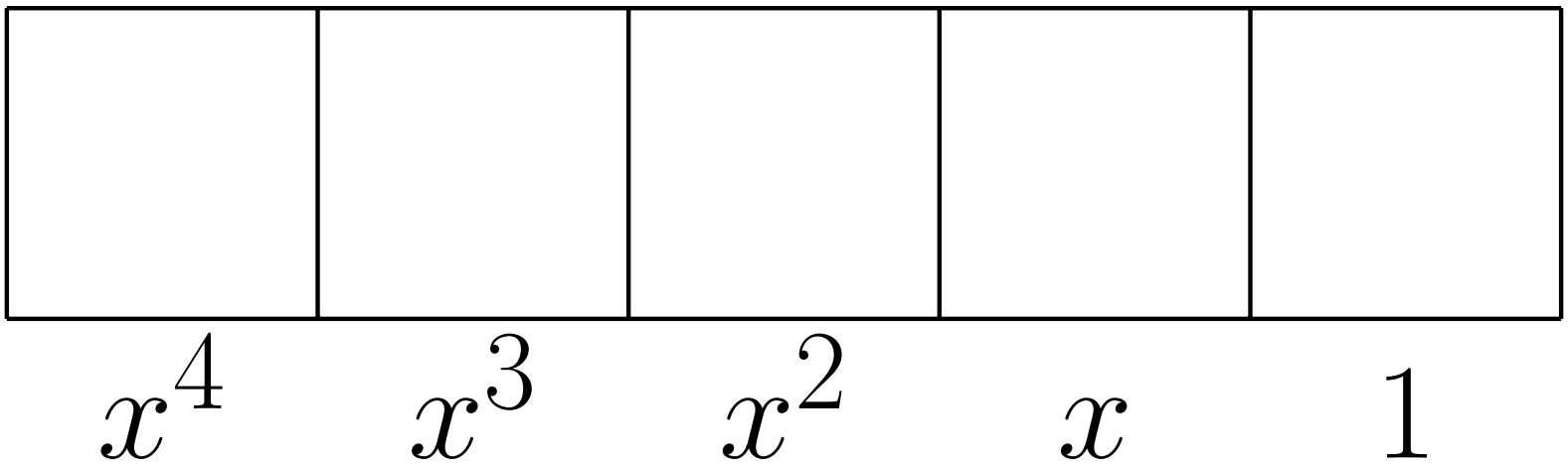
Table of Contents

What if. . .

What if we had a $1 \leftarrow x$ machine? What would it look like?

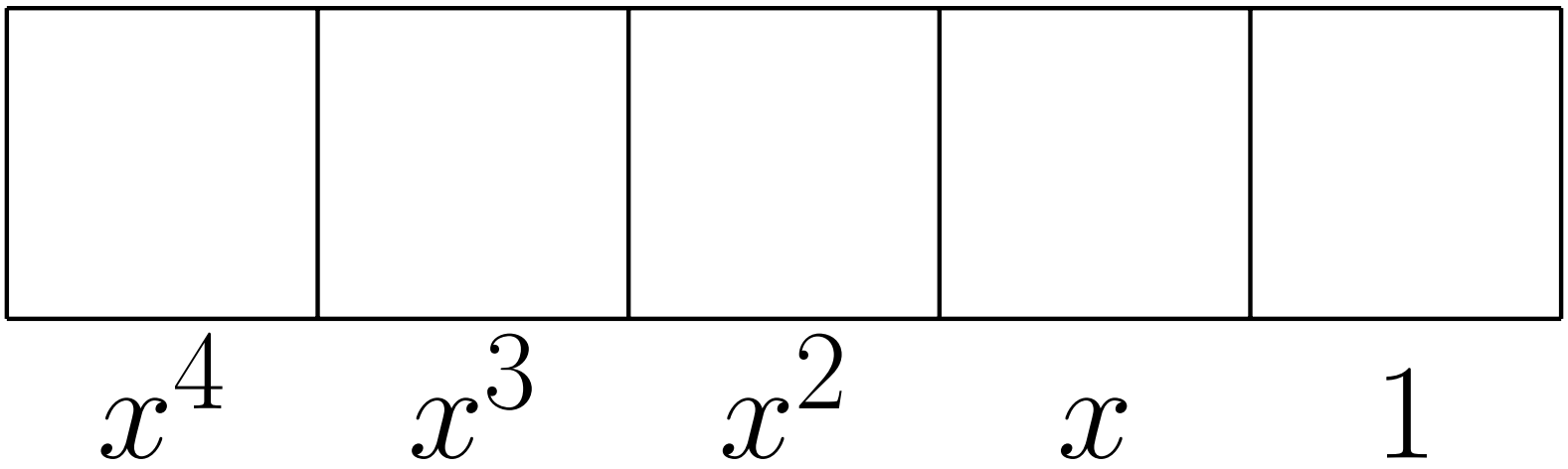
What if. . .

What if we had a $1 \leftarrow x$ machine? What would it look like?



What if. . .

What if we had a $1 \leftarrow x$ machine? What would it look like?



With this knowledge, we can now manipulate polynomials just like we do regular numbers.

Eks

Polynomial division

What is $\frac{2x^2 + 7x + 6}{x + 2}$?

Eks

Polynomial division

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

Eks

Polynomial division

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$?

Eks

Polynomial division

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$? x^2+2x+1

Eks

Polynomial division

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$? x^2+2x+1

What is $\dfrac{x^3-3x+2}{x+2}$?

Eks

Polynomial division

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$? x^2+2x+1

What is $\dfrac{x^3-3x+2}{x+2}$? x^2-2x+1

Table of Contents

Infinitia

Now it gets *really* fun

$$\frac{1}{1-x}$$

Using dots and boxes, we get what is knows at the geometric series formula:

Using dots and boxes, we get what is knows at the geometric series formula:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 \ldots$$

Using dots and boxes, we get what is knows at the geometric series formula:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 \ldots$$

In calculus, we would call this the Taylor series of $\frac{1}{1-x}$.

Infinitia

This is really cool

Using the dots and boxes method, find the Taylor series of:

$$\frac{1}{1 - x - x^2}$$

Congratulations!

We just went from kindergarten arithmetic to advanced calculus in 40 minutes!