Lambda Calculus

Now you can bring a computer to your tests!

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William Lyon Mackenzie C.I. Math Club

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- Created by Alonzo Church

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- A way of representing pure mathematical functions

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- Can represent any computer program

What is lambda calculus?
Introduction

- Created by Alonzo Church
- A way of representing pure mathematical functions
- Can represent any computer program
- Equivalent to Turing machines

In math class, we would define a function that accepts an argument x and outputs $x + 1$ as so:

$$f(x)$$

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If we wanted to find $4+1$, we could do this:

$$f(4) = 4 + 1 = 5$$

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In lambda calculus, we apply a value to a function like this:

$$(\lambda x.x + 1)4 = 4 + 1 = 5$$

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You can think of λ as f, and . as $=$.

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$$f(x, y) = x + y$$

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Currying
One Argument

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Why one argument?
One Argument

- Very simple

Why one argument?
One Argument

- Very simple
- Very powerful

- Very simple
- Very powerful
- Functions can only have one variable

Me too!

But I'm lazy

One Argument

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We have some shortcuts to help us write down lambda calculus expressions, but it's important to remember what they represent, without the shortcuts.

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Can be abbreviated as:

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Also, we assume that we evaluate a function with "multiple" arguments starting with the leftmost parameter.

This is stupid. It just makes everything harder.

you're right!

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This is stupid. It just makes everything harder.

For those examples, you're right!
Let's get to the fun stuff now!

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Quote

"Any program can be written in lambda calculus."

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— Me, 5 minutes ago

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"Any program can be written in lambda calculus."

— Me, 5 minutes ago

So, let's bring on the Booleans!

Definition (TRUE)

$$\mathrm{TRUE} = \lambda xy.x$$

TRUE and FALSE

Booleans

Definition (TRUE)

$$\text{TRUE} = \lambda xy.x$$

Definition (FALSE)

$$\text{FALSE} = \lambda xy.y$$

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So TRUE returns the first value, and FALSE returns the second.

Definition (TRUE)

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Definition (FALSE)

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So TRUE returns the first value, and FALSE returns the second.

We will use TRUE and FALSE as shorthand for these definitions.

Definition (NOT)

$$\text{NOT} = \lambda b.b(\text{FALSE TRUE})$$

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NOT TRUE

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Definition (AND)

$$\text{AND} = (\lambda pq.p)(q\ p)$$

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AND(TRUE FALSE)

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