Introduction to the Fourier Transform

Wave Voodoo

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- Even/Odd Functions
- Periodic Functions
- Trigonometry
- Complex Numbers (and $e^{i\theta}$)
- Integration and IBP

- Decomposition of a periodic function into sines and cosines
- Expressed as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right) \right)$$

- or

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L} + \phi_n\right)$$

- or

$$\sum_{n=-\infty}^{\infty} c_n e^{i \frac{\pi n x}{L}}$$

What is a Fourier Transform?

Introduction

- Transformation of a function of time to a function of frequency
- Expressed as
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x\xi} \mathrm{d}x$$
- Exists both as *continuous* and *discrete* FT
- Today we will be looking at the former only

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$$f_e(x)\cos\left(\frac{\pi m x}{L}\right) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi m x}{L}\right)$$

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$$\int_{-L}^{L} f_e(x)\cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x = \int_{-L}^{L}\sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x$$

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$$\int_{-L}^{L} f_e(x) \cos\left(\frac{\pi m x}{L}\right) \mathrm{d}x = \int_{-L}^{L} \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) \cos\left(\frac{\pi m x}{L}\right) \mathrm{d}x$$

$$= \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{\pi n x}{L}\right) \cos\left(\frac{\pi m x}{L}\right) \mathrm{d}x$$

$$= \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \frac{1}{2} \left(\cos\left((m+n)\frac{\pi x}{L}\right) + \cos\left((m-n)\frac{\pi x}{L}\right)\right) \mathrm{d}x$$

- Consider the integral $\int_{-L}^{L} \cos\left((m+n)\frac{\pi x}{L}\right) \mathrm{d}x$
-
$$\frac{L}{m+n}\sin\left((m+n)\frac{\pi L}{L}\right) - \frac{L}{m+n}\sin\left((m+n)\frac{-\pi L}{L}\right)$$
-
$$= \frac{L}{m+n}\left(\sin((m+n)\pi) + \sin((m+n)\pi)\right) = 0$$
- And so we have:

$$\int_{-L}^{L} f_e(x)\cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x = \frac{1}{2}\sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left((m-n)\frac{\pi x}{L}\right)\mathrm{d}x$$

- For the reasons outlined last slide, $\int_{-L}^{L} \cos\left(\frac{(m-n)\pi x}{L}\right) \mathrm{d}x$ *usually* equals 0
- However, if $m \equiv n$:
- $\int_{-L}^{L} \cos\left(\frac{(n-n)\pi x}{L}\right) \mathrm{d}x = \int_{-L}^{L} 1 \mathrm{d}x = 2L$
- EVERY other value of m will yield a result of 0

$$\begin{aligned}
\int_{-L}^{L} f_e(x)\cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x &= \frac{1}{2}\sum_{n=0}^{\infty}\int_{-L}^{L}\cos\left(\frac{(m-n)\pi x}{L}\right)\mathrm{d}x \\
&= \frac{1}{2}\left(0+0+\cdots+0+0+2La_m+0+0\ldots\right) \\
&= La_m \\
a_m &= \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x \\
\therefore a_n &= \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\left(\frac{\pi n x}{L}\right)
\end{aligned}$$

Special Case: $n \equiv 0$

Fourier Series

$$
\begin{aligned}
f_e(x) &= \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) \\
\int_{-L}^{L} f_e(x)\mathrm{d}x &= \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{\pi n x}{L}\right)\mathrm{d}x \\
&= 2La_0 + 0 + 0 + 0 \ldots \\
\therefore a_0 &= \frac{1}{2L}\int_{-L}^{L} f_e(x)\mathrm{d}x
\end{aligned}
$$

- b_n can be obtained just as easily, as:

$$b_n = \frac{1}{L}\int_{-L}^{L} f_o(x)\sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x \;\; (b_0 = 0)$$

- A function neither even nor odd can be obtained via combination
- We can now express any periodic function as sines and cosines![1]
-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{\pi n x}{L}\right) + b_n\sin\left(\frac{\pi n x}{L}\right)\right)$$

[1] Restrictions may apply

Exponential Form!

Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{L}}$$

Exponential Form!

Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{L}}$$

$$f(x)e^{-i\frac{\pi m x}{L}} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{L}} \times e^{-i\frac{\pi m x}{L}}$$

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Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{L}}$$

$$f(x)e^{-i\frac{\pi m x}{L}} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{L}} \times e^{-i\frac{\pi m x}{L}}$$

$$\int_{-L}^{L} f(x)e^{-i\frac{\pi m x}{L}}\mathrm{d}x = \sum_{n=-\infty}^{\infty} c_n \int_{L}^{L} e^{i(n-m)\frac{\pi x}{L}}\mathrm{d}x$$

$$
\begin{aligned}
(n \not\equiv m) \;\; \int_{-L}^{L} e^{i(n-m)\frac{\pi x}{L}} \mathrm{d}x = & \int_{-L}^{L} \cos\left((n-m)\frac{\pi x}{L}\right) \mathrm{d}x \\
& + i \int_{-L}^{L} \sin\left((n-m)\frac{\pi x}{L}\right) \mathrm{d}x \\
= & \frac{L}{\pi(n-m)} \left(\sin((n-m)\pi) - \sin(-(n-m)\pi)\right) \\
& + i \frac{L}{\pi(n-m)} \left(\cos((n-m)\pi) - \cos(-(n-m)\pi)\right) = 0
\end{aligned}
$$

OR $(n \equiv m)$

$$
\begin{aligned}
& = \int_{-L}^{L} \cos(0)\, \mathrm{d}x + i \int_{-L}^{L} \sin(0) \mathrm{d}x \\
& = 2L
\end{aligned}
$$

$$\int_{-L}^{L} f(x) e^{-i\frac{\pi m x}{L}} \mathrm{d}x = 2Lc_m$$

$$c_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{\pi m x}{L}} \mathrm{d}x$$

$$\therefore c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{\pi n x}{L}} \mathrm{d}x$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x + \sum_{n=1}^{\infty} \frac{1}{L}\int_{-L}^{L}\left(f(x)\cos\left(\frac{\pi n x}{L}\right)\mathrm{d}x\right)\cos\left(\frac{\pi n x}{L}\right)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{L}\int_{-L}^{L}\left(f(x)\sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x\right)\sin\left(\frac{\pi n x}{L}\right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L}\left(f(x)e^{-i\frac{\pi n x}{L}}\mathrm{d}x\right)e^{i\frac{\pi n x}{L}}$$

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- Let $\xi = \frac{n}{2L}$ (The frequency of any term in the sequence), and extend L to ∞.
- Now,
$$c_n = \frac{1}{2L}\int_{-\infty}^{\infty} f(x)e^{-i2\pi x\xi}\mathrm{d}x$$
- Just let $2Lc_n = \hat{f}(\xi)$, and that's our Fourier Transform!
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi x\xi}\mathrm{d}x$$

- If $\xi = \frac{n}{2L}$, then let $\Delta\xi = \frac{1}{2L}$

$$\lim_{L->\infty} f(x) = \lim_{L->\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2L} \hat{f}(\xi) e^{i2\pi x\xi}$$
$$= \lim_{L->\infty} \sum_{n=-\infty}^{\infty} \Delta\xi \hat{f}(\xi) e^{i2\pi x\xi}$$

- But $\lim_{L->\infty} \frac{1}{2L} = 0$, and $\lim_{n->\pm\infty} \frac{n}{2L} = \pm\infty$

$$f(x) = \lim_{\Delta\xi->0} \sum_{\xi=-\infty}^{\infty} \Delta\xi \hat{f}(\xi) e^{i2\pi x\xi}$$
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi x\xi} \mathrm{d}\xi$$

Disclaimer

- The Fourier Series only applies to *piecewise continuous* functions
- The Fourier Transform is usually impossible to apply properly to functions over an infinite domain
- (Unless you're willing to teach yourself about the Dirac δ function)