Lambda Calculus

Now you can bring a computer to your tests!

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Table of Contents

1 Introduction

2 One Argument

3 Booleans

What is lambda calculus?
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- Created by Alonzo Church

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What is lambda calculus?

Introduction

- Created by Alonzo Church
- A way of representing pure mathematical functions
- Can represent any computer program
- Equivalent to Turing machines

In math class, we would define a function that accepts an argument x and outputs $x + 1$ as so:

$$f(x)$$

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If we wanted to find $4 + 1$, we could do this:

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You can think of λ as f, and . as $=$.

Table of Contents

1 Introduction

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How would we define a function that outputs $x + y$ in math class?

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$$f(x, y) = x + y$$

Currying
One Argument

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Why one argument?

- Very simple

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One Argument

- Very simple
- Very powerful

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One Argument

- Very simple
- Very powerful
- Functions can only have one variable

But I'm lazy
One Argument

Me too!

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Also, we assume that we evaluate a function with "multiple" arguments starting with the leftmost parameter.

This is stupid. It just makes everything harder.

One Argument

you're right!

For those examples, you're right!

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One Argument

For those examples, you're right!
Let's get to the fun stuff now!

Table of Contents

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Quote

"Any program can be written in lambda calculus."

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"Any program can be written in lambda calculus."

— Me, 5 minutes ago

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"Any program can be written in lambda calculus."

— Me, 5 minutes ago

So, let's bring on the Booleans!

Definition (TRUE)

$$\mathrm{TRUE} = \lambda xy.x$$

TRUE and FALSE

Booleans

Definition (TRUE)

$$\text{TRUE} = \lambda xy.x$$

Definition (FALSE)

$$\text{FALSE} = \lambda xy.y$$

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So TRUE returns the first value, and FALSE returns the second.

TRUE and FALSE

Definition (TRUE)

$$\text{TRUE} = \lambda xy.x$$

Definition (FALSE)

$$\text{FALSE} = \lambda xy.y$$

So TRUE returns the first value, and FALSE returns the second.

We will use TRUE and FALSE as shorthand for these definitions.

Definition (NOT)

$$\mathrm{NOT} = \lambda b.b(\mathrm{FALSE}\ \mathrm{TRUE})$$

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NOT TRUE

$$(\lambda b.b(\text{FALSE TRUE}))\ \text{TRUE} =$$

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$$\begin{aligned}(\lambda b.b(\mathrm{FALSE}\ \mathrm{TRUE}))\ \mathrm{TRUE} &= \mathrm{TRUE}(\mathrm{FALSE}\ \mathrm{TRUE}) \\ &= \lambda xy.x(\mathrm{FALSE}\ \mathrm{TRUE})\end{aligned}$$

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$$\text{AND} = (\lambda pq.p)(q\ p)$$

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AND(TRUE FALSE)

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Definition (AND)

$$\text{IFTHENELSE} = (\lambda btf.b)(t\ \ f)$$

Where b is a Boolean, t is the value to return if $b = \text{TRUE}$, and f is the value to return if $b = \text{FALSE}$.

IFTHENELSE(TRUE "Math is great" "Math is kool")

$$((\lambda btf.b)(t\ \ f))\ (\text{TRUE "Math is kool" "Math is lit"})$$