A puzzle

Alice and **Bob** are trying to send each other a message without **Eve the Eavesdropper** being able to read it.

There are some positive integers in each column. **Only Alice** can access the numbers in her column. **Only Bob** can access the numbers in his column. **Anyone** can access the numbers in the public column.

Alice	Public	Bob
a	$g \quad n$	b
a is between 1 and n.	g is a small prime number. n is a **very** big number.	b is between 1 and n.

If Eve can read anything Alice and Bob send to each other, how can Alice and Bob both know a number without Eve knowing that number as well?

Cryptography

Vincent Macri

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We define the mod operator as being the remainder when dividing two numbers. That is:

$$a \bmod b = \text{the remainder of } a \div b$$

In some programming languages, modulo is written as % or rem. Use whichever notation you are most comfortable with.

Examples

$$4 \bmod 2 = 0 \qquad 7 \bmod 3 = 1$$

$$5 \bmod 2 = 1 \qquad 9 \bmod 5 = 4$$

The definition of modulo (mod for short) is a bit trickier with negative numbers. It also doesn't matter for today, as we're only looking at mod with positive numbers.

While we could do long division to find the remainder when we want to calculate modulo, I prefer to use this formula:

$$a \bmod b = a - b\left\lfloor \frac{a}{b} \right\rfloor$$

Where $\lfloor x \rfloor$ is the floor function, which rounds a number **down** to an integer.

Calculating modulo

Modular Arithmetic

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Where $\lfloor x \rfloor$ is the floor function, which rounds a number **down** to an integer.

There are other method, but I think this one is the hardest to mess up. Use whatever method you are most comfortable with.

How do we do division in modular arithmetic?

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Division is the inverse of multiplication.

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Inverse of modular multiplication
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It turns out a has an inverse in $\mod n$ if and only if a and n are coprime.

$$3 \times 7 \mod 20 = 1$$

Here, 7 is the inverse of 3, and 3 is the inverse of 7.

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2 does **not** have an inverse modulo 4.

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There is no integer value for b that satisfies this equation.

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2 does **not** have an inverse modulo 4.

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There is no integer value for b that satisfies this equation.

We will soon learn the algorithm to find the inverse to modular multiplication.

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What is a prime number?

Primes

A **prime number** is a positive integer that is only divisible by 1 and itself.

Examples

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$$

If an integer greater than 1 is not prime, it is called a **composite number**.

1 is special, and is called the **unit number**

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1 is special, and is called the **unit number**

Proof 1 is not prime.

In the past, some mathematicians said that 1 is prime. All of them are dead now.

$$\therefore 1 \notin \mathbb{P}$$

$\square$

The largest known prime number[1] is:

$$M_{77\,232\,917} = 2^{77\,232\,917} - 1$$

If you were to print this number out, it would be 6055 pages long!

This prime was discovered by Jonathan Pace on December 26, 2017 after 6 days of continuous computer computations. The discovery was published on January 3, 2018.

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The largest prime number

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What's special and useful about Mersenne primes? Not much.

[1] As of January 5th, 2018

How many primes are there?
Primes

Is the number of primes finite?

Is the number of primes finite?

No! There are infinite prime numbers!
This was proved thousands of years ago by Euclid.

Assume the list of primes is finite, and there are only n prime numbers. We will call our list of prime numbers P.

$$P = \{p_1, p_2, \dots, p_{n-1}, p_n\}$$

Where p_k is the kth prime number.

Proof of infinite primes
Primes

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$$P = \{p_1, p_2, \ldots, p_{n-1}, p_n\}$$

Where p_k is the kth prime number.

Now, let m be the product of all numbers in P plus 1.

$$m = (p_1 \times p_2 \times \cdots \times p_{n-1} \times p_n) + 1 = \left(\sum_{i=1}^{n} p_i \right) + 1$$

m is either prime or not prime. Let's look at both cases.

Proof of infinite primes: m is prime

First, let's consider the case that m is prime.

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Note that m is not in our original list, P.

If m is prime, our original list is incomplete, and there are more prime numbers than we listed.

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If m is not prime, then it must be divisible by a prime number. Notice that m cannot be divisible by any numbers in P, as they would not divide a number that is a multiple of themselves plus 1.

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$$m = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30\,031$$

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Here, we can see that since $30\,031$ is a multiple plus 1 of every number in P, no numbers in P will divide it.

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Here, we can see that since $30\,031$ is a multiple plus 1 of every number in P, no numbers in P will divide it. But if $30\,031$ is not prime, then it divisible by a prime number, so there must be some prime numbers missing from our original list. $30\,031$ is divisible by 59 and 509, so these numbers are missing from our list.

How do we check if a number is prime?

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How do we check if a number is prime **quickly**?

How do we check if a number is prime?

How do we check if a number is prime **quickly**?

With a **very** fast computer. Algorithms exist (some of which run in polynomial time) but they are **very** slow.
Here, “quickly” means the computer will finish before we die.

This fits in the margins

Primes

Theorem (Fermat's Little Theorem)

Let p be a prime number, and a an integer that does not divide p.

Then:

$$a^{p-1} \bmod p = 1$$

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Theorem (Euler-Fermat Generalization)

Fermat's Little Theorem can be generalized as:

$$a^{\phi(n)} \bmod n = 1$$

Where $\phi(n)$ is Euler's totient function, which gives us the number of integers less than or equal to n that are coprime to n.

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For an extra challenge, prove the **Euler-Fermat Generalization** using **Fermat's Little Theorem**.

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We will now introduce a new notation, which is more of a shortcut. If b divides a with no remainder, then we will write $b \mid a$.
More formally:

$$b \mid a \equiv a \bmod b = 0$$

Or:

$$b \mid a \iff a = bc$$

Where a, b, and c and positive integers.
If $b \mid a$, then b is a factor of a.

Theorem (The Unique Factorization Theorem)

Every positive integer has a unique representation as a product of prime numbers.
That is, for all numbers $n \in \mathbb{Z}^+$:

$$n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$$

Where p_i is prime, and a_i is a positive integer.

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Example (180)

$$180 = 2^2 \times 3^2 \times 5$$

$$180 = 2 \times 2 \times 3 \times 3 \times 5$$

How do we factor a number?

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How do we factor a **large** number?

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How do we factor a **large** number?

Try this one:

RSA-2048

251959084756578934940271832400483985714292821262040320277771378360436620207075955562640185258807
844069182906412495150821892985591491761845028084891200728449926873928072877767359714183472702618
963750149718246911650776133798590957000973304597488084284017974291006424586918171951187461215151
726546322822168699875491824224336372590851418654620435767984233871847744479207399342365848238242
811981638150106748104516603773060562016196762561338441436038339044149526344321901146575444541784
240209246165157233507787077498171257724679629263863563732899121548314381678998850404453640235273
81951378636564391212010397122822120720357

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963750149718246911650776133798590957000973304597488084284017974291006424586918171951187461215151
726546322822168699875491824224336372590851418654620435767984233871847744479207399342365848238242
811981638150106748104516603773060562016196762561338441436038339044149526344321901146575444541784
240209246165157233507787077498171257724679629263863563732899121548314381678998850404453640235273
81951378636564391212010397122822120720357

This number has two factors. Nobody knows what they are. There was a $\$200\,000$ prize to factor this number. People had over 15 years to factor it, but nobody was able to before the contest period ended.

The greatest common divisor (GCD) of two numbers is the largest number that divides both numbers.

What is the greatest common divisor?
Factoring

The greatest common divisor (GCD) of two numbers is the largest number that divides both numbers.

More formally:

$$x = \gcd(a, b) \mid a, b \in \mathbb{Z}$$

Where x is the largest number such that:

$$x \mid a \land x \mid b$$

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Where x is the largest number such that:

$$x \,|\, a \wedge x \,|\, b$$

If two numbers are coprime, their $\gcd$ is 1.

How do we find the greatest common divisor?

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We could list all the factors, and the biggest one would be the $\gcd$. But, as we saw above, factoring is very hard. Is there a better way?

How do we find the greatest common divisor?

We could list all the factors, and the biggest one would be the $\gcd$. But, as we saw above, factoring is very hard. Is there a better way?

Of course. Otherwise I wouldn't ask.

Thousands of years ago, Euclid came up with an algorithm to find the $\gcd$.

Euclidean algorithm

To find $\gcd(a, b)$, do the following:

1. Let $r_0 = a$, $r_1 = b$, and $i = 1$.
2. If $r_i = 0$ then $\gcd(a, b) = r_i$.
3. Write $r_{i-1} = q_i r_i + r_{i+1}$ and increment i by 1.
   Here, $r_{i+1} = r_i \bmod r_{i-1}$.
4. Go back to step 2.

The extended Euclidean algorithm is essentially the Euclidean algorithm in reverse.
We use substitution while working backwards.
It allows us to find two integers, x and y, that satisfy:

$$\gcd(a, b) = ax + by$$

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When $\gcd(a, b) = 1$, $x = a^{-1} \mod b$, where a^{-1} is the inverse of a in modulus b.

Find the inverse of 3 in modulus 26.

$$
\begin{aligned}
26 &= (8)3 + 2 \\
3 &= (1)2 + 1 \\
2 &= (2)1 + 0
\end{aligned}
$$

Extended Euclidean algorithm example

Factoring

Find the inverse of 3 in modulus 26.

$$\begin{aligned} 26 &= (8)3 + 2 \\ 3 &= (1)2 + 1 \\ 2 &= (2)1 + 0 \end{aligned}$$

$$1 = 3 - (1)2 = 3 - (26 - (8)3)$$

$$1 = (9)3 - 26$$

Find the inverse of 3 in modulus 26.

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$$3 = (1)2 + 1$$
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$$1 = (9)3 - 26$$

And so the inverse of 3 in $\bmod 26$ is 9.
We can verify this:

$$(3 \times 9) \bmod 26 = 1$$

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What are factors used for?
Introduction to Cryptography

Factorization of numbers is very useful in cryptography.
The reason for this is that factoring large numbers takes a **very** long time, but the maths for checking factorization are quick.
We can use this to develop a way to encode messages so they can only be read by certain people. This is called cryptography.

Simply put, cryptography is the study of ways to encrypt messages.

Encryption is when you transform a message so that it cannot easily be read by someone without a key. Encryption is like a lock, but instead of locking your house, it locks information.

The use of encryption goes back thousands of years.

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For example, with $n = 3$:
`Plaintext message: ‘Crypto is fun!’`
`Encrypted message: ‘Fubswr lv ixq!’`
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Obviously, this isn't very secure. Why?

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Today, we will learn how to encrypt messages using the RSA system.

While RSA has not been broken by anyone, there are systems that are considered to be **more** secure because they provide something called "perfect forward secrecy". However, a lot of these systems are similar to, or even use RSA.

What is RSA?

What is RSA?

- Public keys

What is RSA?

- Public keys
- Private keys

What is RSA?

- Public keys
- Private keys
- Padlocks

RSA overview

RSA

What is RSA?

- Public keys
- Private keys
- Padlocks
- Factoring is hard

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Alice and Bob.

Key generation
RSA

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4. For the final step, Alice computes d to be the inverse of e. $e \times d \mod \phi(n) = 1 \mid 0 < d < \phi(n)$

Alice's public keypair is (n, e).
Alice's private key is d.

Bob wants to send a message to Alice. He has turned his message into a number m, such that $m < n$ (if $m \geq n$, then Bob will split the message up into multiple short messages).
If m and n are not coprime, then one could easily factorize n, thus breaking the encryption.

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Bob then downloads Alice's public key from somewhere he trusts and calculates the **ciphertext**, c:

$$c = m^e \bmod n$$

And he sends the encrypted message, c to Alice.

To decrypt the message, Alice calculates calculates r, which is equal to m, Bob's message:

$$r = c^d \bmod n$$

Decrypting

RSA

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How do we know that $r = m$?

Theorem (Euler's theorem)

When a and n are coprime positive integers:

$$a^{\phi(n)} \bmod n = 1$$

Proof that $r = m$

Remember that $(m^e)^d = m^{ed}$. So:

$$r = m^{ed} \bmod n$$

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We know that $ed = 1 \bmod \phi(n)$ because we chose e and d to have that property.
This means that an integer, q, exists such that:

$$ed = q\phi(n) + 1$$

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So:

$$\begin{aligned} m^{ed} &= m^{q\phi(n)+1} \mod n \\ &= \left(m^{\phi(n)}\right)^q m \mod n \\ &= 1^q m \mod n \\ &= m \mod n \end{aligned}$$

$$\therefore r = m^{ed} = m$$

Board example
RSA

Let's do an example on the board with the following numbers:

$$p = 7$$

$$q = 11$$

$$n = 77$$

$$\phi(n) = (6)(10) = 60$$

$$e = 17$$