Euclid Preparation 2

Analytic Geometry

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The standard form for a line with slope $-\frac{A}{B}$ and intercepts $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$

Formula

$$Ax + By + C = 0$$

Formula 2
Toolkit

The equation of the line with slope m through (x_0, y_0)

Formula

$$y - y_0 = m(x - x_0)$$

The equation of the line with intercepts at $(a, 0)$ and $(0, b)$

Formula

$$\frac{x}{a} + \frac{y}{b} = 1$$

The formula for the midpoint M of $A(x_1, y_1)$ and $B(x_2, y_2)$

Formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The distance D between the points $A(x_1, y_1)$ and $B(x_2, y_2)$

Formula

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The distance D between the line $Ax + By + C = 0$ and the point (x_0, y_0)

Formula

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Formula 7
Toolkit

The area of the triangle $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$

Formula

$$A = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|$$

Formula 8
Toolkit

The equation of the circle with centre (h, k) and radius r

Formula

$$(x - h)^2 + (y - k)^2 = r^2$$

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Problem

If the line $2x - 3y - 6 = 0$ is reflected in the line $y = -x$, find the equation of the image line.

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Analytic geometry problem 1

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When reflecting in $y = -x$, we swap x and y, and change the sign.

So, we get:

$$(3, 0) \to (0, -3)$$

$$(0, -2) \to (2, 0)$$

Solution

Next, we find an equation for a line which goes through $(0, -3)$ and $(2, 0)$.

Analytic geometry problem 1 solution continued

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$$3x - 2y - 6 = 0$$

And that is our line.

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Let $P = (p, 0)$.
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$$A = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|$$

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$$|-22-6p|=60$$

Solution

$$|-22-6p| = 60$$

Analytic geometry problem 2 solution continued

Solution

$$
\begin{aligned}
|-22 - 6p| &= 60 \\
p &= \frac{19}{3}, -\frac{41}{3}
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Analytic geometry problem 2 solution continued

Problems

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$$\begin{aligned} |-22 - 6p| &= 60 \\ p &= \frac{19}{3}, -\frac{41}{3} \end{aligned}$$

So the points are $\left(\frac{19}{3}, 0\right)$ and $\left(-\frac{41}{3}, 0\right)$.

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$$x^2 + y^2 - 4 = x^2 + y^2 - 6x + 2$$
$$x = 1$$

Solution

$$x = 1$$

If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at $(1, \pm\sqrt{3})$.

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If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at $(1, \pm\sqrt{3})$.

So, the length of the entire chord is $2\sqrt{3}$.

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Let the x-intercept be k.

What does this make the y-intercept? $2k$.

So the equation of the line can be written as:

$$2x + y - 2k = 0$$

Solution

We can use the formula for distance from a point to a line:

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Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

$$A = \frac{1}{2} \times k \times 2k = k^2 = (\pm\sqrt{5})^2 = 5$$