Euclid Preparation 2

Analytic Geometry

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Workshop Overview

Part I:

1. Toolkit
2. Problems

Part I

Analytic Geometry

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Formula 1
Toolkit

The standard form for a line with slope $-\frac{A}{B}$ and intercepts $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$

Formula

$$Ax + By + C = 0$$

The equation of the line with slope m through (x_0, y_0)

Formula

$$y - y_0 = m(x - x_0)$$

The equation of the line with intercepts at $(a, 0)$ and $(0, b)$

Formula

$$\frac{x}{a} + \frac{y}{b} = 1$$

The formula for the midpoint M of $A(x_1, y_1)$ and $B(x_2, y_2)$

Formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Formula 5
Toolkit

The distance D between the points $A(x_1, y_1)$ and $B(x_2, y_2)$

Formula

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The distance D between the line $Ax + By + C = 0$ and the point (x_0, y_0)

Formula

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Formula 7
Toolkit

The area of the triangle $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$

Formula

$$A = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|$$

The equation of the circle with centre (h, k) and radius r

Formula

$$(x-h)^2+(y-k)^2=r^2$$

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Problem

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When reflecting in $y = -x$, we swap x and y, and change the sign.

So, we get:

$$(3, 0) \to (0, -3)$$

$$(0, -2) \to (2, 0)$$

Solution

Next, we find an equation for a line which goes through $(0, -3)$ and $(2, 0)$.

Analytic geometry problem 1 solution continued

Problems

Solution

Next, we find an equation for a line which goes through $(0, -3)$ and $(2, 0)$.

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$$\frac{x}{2} + \frac{y}{-3} = 1$$

Solution

Next, we find an equation for a line which goes through $(0, -3)$ and $(2, 0)$.

$$\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \frac{x}{2} + \frac{y}{-3} &= 1 \\ 3x - 2y - 6 &= 0 \end{aligned}$$

Solution

Next, we find an equation for a line which goes through $(0, -3)$ and $(2, 0)$.

$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{2} + \frac{y}{-3} = 1$$
$$3x - 2y - 6 = 0$$

And that is our line.

Problem

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Solution

Let $P = (p, 0)$.
Then, we use the formula for area:

$$A = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|$$

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$$A = \frac{1}{2}|33 + 0 + 5p - 55 - 11p - 0|$$

Problem

If $A(3,5)$ and $B(11,11)$ are fixed points, find the point(s) P on the x-axis such that the area of the triangle ABP equals 30.

Solution

Let $P=(p,0)$.
Then, we use the formula for area:

$$A=\frac{1}{2}|x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_3y_2-x_1y_3|$$

$$A=\frac{1}{2}|33+0+5p-55-11p-0|$$

$$|-22-6p|=60$$

Solution

$$|-22 - 6p| = 60$$

Solution

$$\begin{aligned} |-22 - 6p| &= 60 \\ p &= \frac{19}{3}, -\frac{41}{3} \end{aligned}$$

Analytic geometry problem 2 solution continued

Problems

Solution

$$|-22 - 6p| = 60$$
$$p = \frac{19}{3}, -\frac{41}{3}$$

So the points are $\left(\frac{19}{3}, 0\right)$ and $\left(-\frac{41}{3}, 0\right)$.

Problem

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Solution

It is helpful to rewrite the second circle's equation as $(x-3)^2 + y^2 = 7$ because this shows us more easily that the second circle is centred at $(3, 0)$. Since the line joining the centres is horizontal, the common chord is vertical. We can solve this easily:

$$x^2 + y^2 - 4 = x^2 + y^2 - 6x + 2$$
$$x = 1$$

Solution

$$x = 1$$

If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at $(1, \pm\sqrt{3})$.

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So, the length of the entire chord is

Solution

$$x = 1$$

If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at $(1, \pm\sqrt{3})$.

So, the length of the entire chord is $2\sqrt{3}$.

Analytic geometry problem 4

Problem

A line has slope -2 and is a distance of 2 units from the origin. What is the area of the triangle formed by this line and the axes?

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What does this make the y-intercept? $2k$.

Problem

A line has slope -2 and is a distance of 2 units from the origin. What is the area of the triangle formed by this line and the axes?

Solution

Let the x-intercept be k.

What does this make the y-intercept? $2k$.

So the equation of the line can be written as:

$$2x + y - 2k = 0$$

Analytic geometry problem 4 solution continued

Solution

We can use the formula for distance from a point to a line:

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

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$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$2 = \frac{\left|2(0) + 1(0) - 2k\right|}{\sqrt{2^2 + 1^2}}$$

Solution

We can use the formula for distance from a point to a line:

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$2 = \frac{\left|2(0) + 1(0) - 2k\right|}{\sqrt{2^2 + 1^2}}$$

$$2 = \frac{|-2k|}{\sqrt{5}}$$

Solution

We can use the formula for distance from a point to a line:

$$
\begin{aligned}
D &= \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \\
2 &= \frac{\left|2(0) + 1(0) - 2k\right|}{\sqrt{2^2 + 1^2}} \\
2 &= \frac{|-2k|}{\sqrt{5}} \\
k &= \pm\sqrt{5}
\end{aligned}
$$

Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

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$$A = \frac{1}{2} \times k \times 2k = k^2 = \qquad\qquad =$$

Solution

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Next, we find the area:

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Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

$$A = \frac{1}{2} \times k \times 2k = k^2 = (\pm\sqrt{5})^2 = 5$$