

Euclid Preparation 3

Circle Geometry

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■ Tangent chord theorem

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Theorem

Star Trek Theorem

Theorem (“Star Trek” Theorem)

*The central angle **subtended** by any arc is twice any of the inscribed angles on that arc.*

This means that in the diagram, $\angle AOB = 2\angle ACB$.



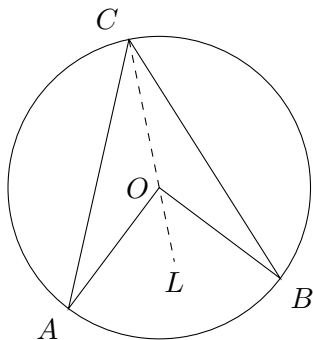
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The central angle *subtended* by any arc is twice any of the inscribed angles on that arc.

This means that in the diagram, $\angle AOB = 2\angle ACB$.



Here, $\angle AOB$ is *subtended* by the *minor arc* from A to B.



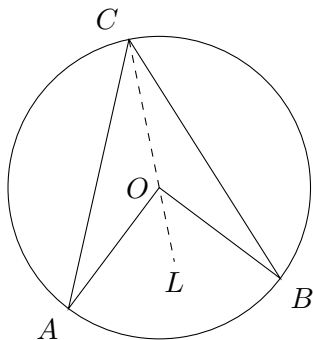
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A **minor arc** is the smaller of the two arcs that can be formed by two points on a circle.



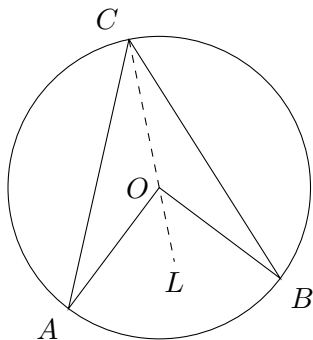
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Also, note that $\triangle OAC$ and $\triangle OBC$ are isosceles.



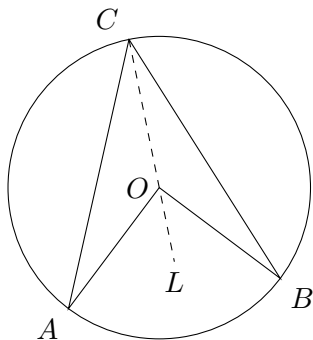
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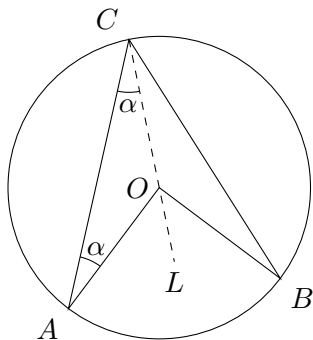
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A **minor arc** is the smaller of the two arcs that can be formed by two points on a circle.

Also, note that $\triangle OAC$ and $\triangle OBC$ are isosceles. This is because OA , OB , and OC are all radii. So, $\angle OAC = \angle OCA$



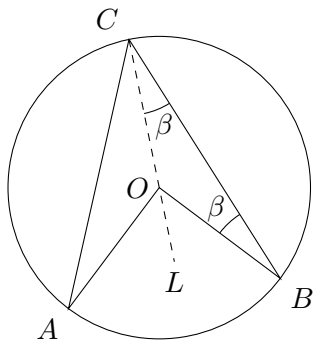
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The central angle *subtended* by any arc is twice any of the inscribed angles on that arc.

This means that in the diagram, $\angle AOB = 2\angle ACB$.



Here, $\angle AOB$ is *subtended* by the *minor arc* from A to B .

A *minor arc* is the smaller of the two arcs that can be formed by two points on a circle.

Also, note that $\triangle OAC$ and $\triangle OBC$ are isosceles. This is because OA , OB , and OC are all radii. So, $\angle OAC = \angle OCA$ and $\angle OCB = \angle OBC$.

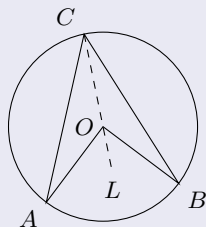


Proof of the Star Trek Theorem

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

We know that $\angle OAC = \angle OCA$.

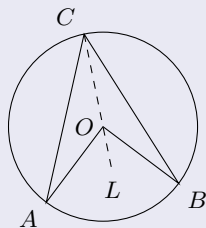


Proof of the Star Trek Theorem

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

We know that $\angle OAC = \angle OCA$. So: $2\angle OCA + \angle AOC = 180^\circ$.



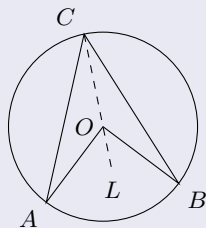
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Proof that $\angle AOB = 2\angle ACB$.

We know that $\angle OAC = \angle OCA$. So: $2\angle OCA + \angle AOC = 180^\circ$.

And we know that $\angle AOC + \angle AOL = 180^\circ$.



Proof of the Star Trek Theorem

Star Trek Theorem

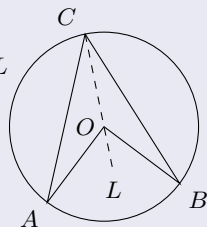
Proof that $\angle AOB = 2\angle ACB$.

We know that $\angle OAC = \angle OCA$. So: $2\angle OCA + \angle AOC = 180^\circ$.

And we know that $\angle AOC + \angle AOL = 180^\circ$.

$$2\angle OCA + \angle AOC = \angle AOC + \angle AOL$$

$$\angle OCA = \frac{1}{2}\angle AOL$$



Proof of the Star Trek Theorem

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

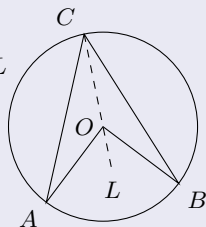
We know that $\angle OAC = \angle OCA$. So: $2\angle OCA + \angle AOC = 180^\circ$.

And we know that $\angle AOC + \angle AOL = 180^\circ$.

$$2\angle OCA + \angle AOC = \angle AOC + \angle AOL$$

$$\angle OCA = \frac{1}{2}\angle AOL$$

And similarly for $\triangle OBC$: $\angle OCB = \frac{1}{2}\angle BOL$.



Proof of the Star Trek Theorem

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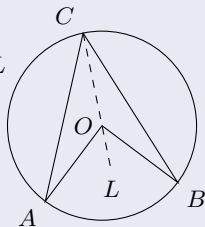
And we know that $\angle AOC + \angle AOL = 180^\circ$.

$$2\angle OCA + \angle AOC = \angle AOC + \angle AOL$$

$$\angle OCA = \frac{1}{2}\angle AOL$$

And similarly for $\triangle OBC$: $\angle OCB = \frac{1}{2}\angle BOL$.

$$\angle ACB = \angle OCA + \angle OCB$$



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Proof that $\angle AOB = 2\angle ACB$.

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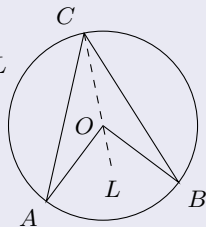
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$$\angle ACB = \angle OCA + \angle OCB$$

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Proof of the Star Trek Theorem

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$$2\angle OCA + \angle AOC = \angle AOC + \angle AOL$$

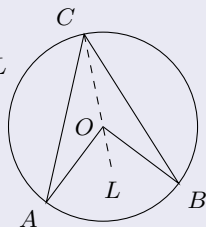
$$\angle OCA = \frac{1}{2}\angle AOL$$

And similarly for $\triangle OBC$: $\angle OCB = \frac{1}{2}\angle BOL$.

$$\angle ACB = \angle OCA + \angle OCB$$

$$\angle ACB = \frac{1}{2}\angle AOL + \frac{1}{2}\angle BOL$$

$$\angle ACB = \frac{1}{2}(\angle AOL + \angle BOL)$$



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We know that $\angle OAC = \angle OCA$. So: $2\angle OCA + \angle AOC = 180^\circ$.

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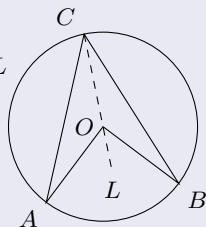
And similarly for $\triangle OBC$: $\angle OCB = \frac{1}{2}\angle BOL$.

$$\angle ACB = \angle OCA + \angle OCB$$

$$\angle ACB = \frac{1}{2}\angle AOL + \frac{1}{2}\angle BOL$$

$$\angle ACB = \frac{1}{2}(\angle AOL + \angle BOL)$$

$$2\angle ACB = \angle AOB$$



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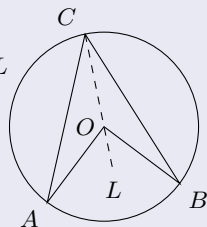
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$$\angle ACB = \frac{1}{2}\angle AOL + \frac{1}{2}\angle BOL$$

$$\angle ACB = \frac{1}{2}(\angle AOL + \angle BOL)$$

$$2\angle ACB = \angle AOB$$



Extending



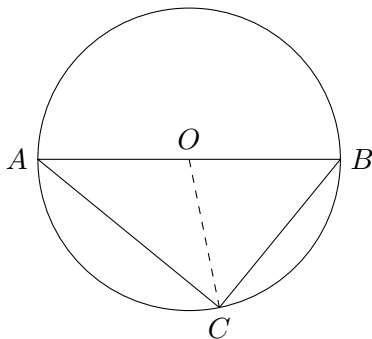
Diameters and right angles

Star Trek Theorem

Example

Show that if the chord AB is a diameter then $\angle ACB = 90^\circ$.

In other words, show that the angle subtended by a diameter is a right angle.



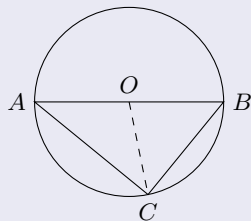
Extension 1 proof

Star Trek Theorem

Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \quad (1)$$



Extension 1 proof

Star Trek Theorem

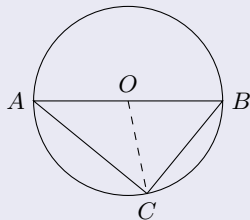
Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \quad (1)$$

Similarly:

$$2\angle BCO + \angle BOC = 180^\circ \quad (2)$$



Extension 1 proof

Star Trek Theorem

Proof that $\angle ACB = 90^\circ$.

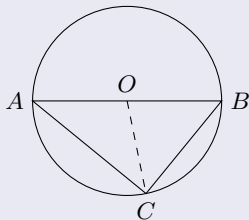
We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \quad (1)$$

Similarly:

$$2\angle BCO + \angle BOC = 180^\circ \quad (2)$$

We also know that $\angle AOC = 180^\circ - \angle BOC$.



Extension 1 proof

Star Trek Theorem

Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

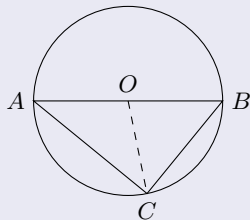
$$2\angle ACO + \angle AOC = 180^\circ \quad (1)$$

Similarly:

$$2\angle BCO + \angle BOC = 180^\circ \quad (2)$$

We also know that $\angle AOC = 180^\circ - \angle BOC$.

We substitute this into (1) to get $2\angle ACO = \angle BOC$.



Extension 1 proof

Star Trek Theorem

Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \quad (1)$$

Similarly:

$$2\angle BCO + \angle BOC = 180^\circ \quad (2)$$

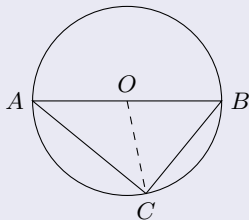
We also know that $\angle AOC = 180^\circ - \angle BOC$.

We substitute this into (1) to get $2\angle ACO = \angle BOC$.

We substitute this into (2) to get:

$$2\angle BCO + 2\angle ACO = 180^\circ$$

$$\angle BCO + \angle ACO = 90^\circ$$



Extension 1 proof

Star Trek Theorem

Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \quad (1)$$

Similarly:

$$2\angle BCO + \angle BOC = 180^\circ \quad (2)$$

We also know that $\angle AOC = 180^\circ - \angle BOC$.

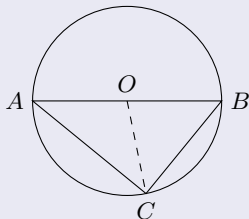
We substitute this into (1) to get $2\angle ACO = \angle BOC$.

We substitute this into (2) to get:

$$2\angle BCO + 2\angle ACO = 180^\circ$$

$$\angle BCO + \angle ACO = 90^\circ$$

Since $\angle BCO + \angle ACO = \angle ACB$, we arrive at:



Extension 1 proof

Star Trek Theorem

Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \quad (1)$$

Similarly:

$$2\angle BCO + \angle BOC = 180^\circ \quad (2)$$

We also know that $\angle AOC = 180^\circ - \angle BOC$.

We substitute this into (1) to get $2\angle ACO = \angle BOC$.

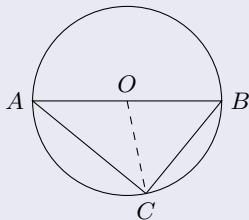
We substitute this into (2) to get:

$$2\angle BCO + 2\angle ACO = 180^\circ$$

$$\angle BCO + \angle ACO = 90^\circ$$

Since $\angle BCO + \angle ACO = \angle ACB$, we arrive at:

$$\angle ACB = 90^\circ$$



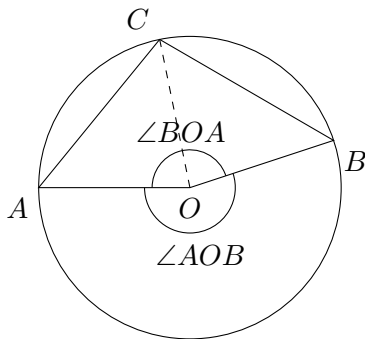
On the major arc

Star Trek Theorem

Example

Show that the Star Trek theorem is still true if $\angle AOB > 180^\circ$.

That is, show that $\angle AOB = 2\angle ACB$ is true in this diagram.



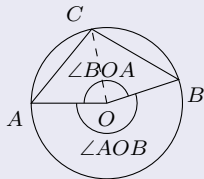
Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$



Extension 2 proof

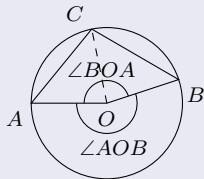
Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$

We add these two equations to get:



Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

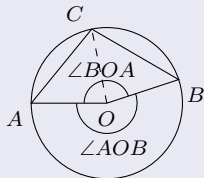
$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$

We add these two equations to get:

$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$

$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$



Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$

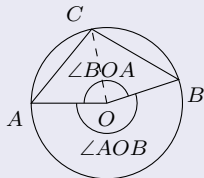
We add these two equations to get:

$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$

$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

We know that $\angle AOC + \angle BOC = \angle BOA$, so:

$$2(\angle ACO + \angle BCO) = 360^\circ - \angle BOA$$



Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$

We add these two equations to get:

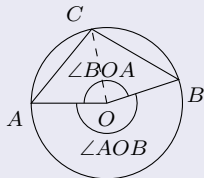
$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$

$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

We know that $\angle AOC + \angle BOC = \angle BOA$, so:

$$2(\angle ACO + \angle BCO) = 360^\circ - \angle BOA$$

We also know that $\angle AOB = 360^\circ - \angle BOA$.



Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$

We add these two equations to get:

$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$

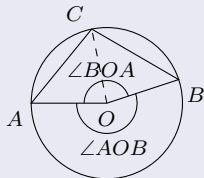
$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

We know that $\angle AOC + \angle BOC = \angle BOA$, so:

$$2(\angle ACO + \angle BCO) = 360^\circ - \angle BOA$$

We also know that $\angle AOB = 360^\circ - \angle BOA$.

And $\angle ACB = \angle ACO + \angle BCO$.



Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$

We add these two equations to get:

$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$

$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

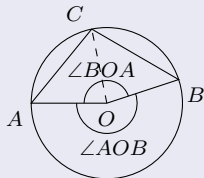
We know that $\angle AOC + \angle BOC = \angle BOA$, so:

$$2(\angle ACO + \angle BCO) = 360^\circ - \angle BOA$$

We also know that $\angle AOB = 360^\circ - \angle BOA$.

And $\angle ACB = \angle ACO + \angle BCO$.

$$\therefore 2\angle ACB = \angle AOB$$



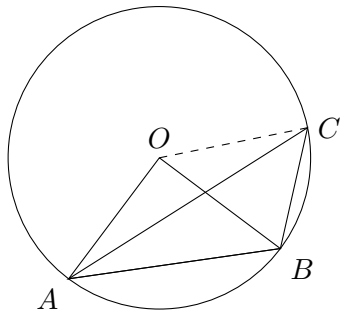
Intersecting

Star Trek Theorem

Example

Show that the Star Trek theorem is still true if the point C is chosen so that AB and OB intersect.

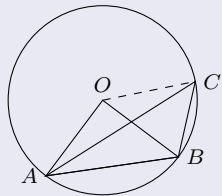
Prove that $\angle AOB = 2\angle ACB$.



Extension 3 proof

Star Trek Theorem

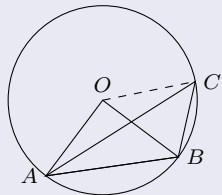
Proof that $\angle AOB = 2\angle ACB$.



Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA$$

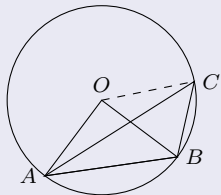
(3)



Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \quad (3)$$

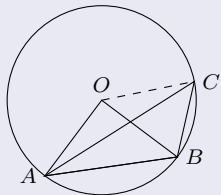
$$\angle COB = 180^\circ - 2\angle OBC \quad (4)$$



Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \quad (3)$$

$$\angle COB = 180^\circ - 2\angle OBC \quad (4)$$

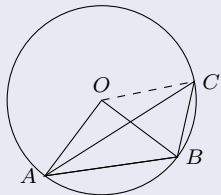
$$\angle AOB = \angle AOC - \angle COB \quad (5)$$



Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \quad (3)$$

$$\angle COB = 180^\circ - 2\angle OBC \quad (4)$$

$$\angle AOB = \angle AOC - \angle COB \quad (5)$$

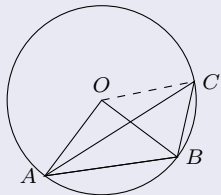
We can substitute (3) and (4) into (5):



Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \quad (3)$$

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We can substitute (3) and (4) into (5):

$$\angle AOB = 180^\circ - 2\angle OCA - (180^\circ - 2\angle OBC)$$

$$\angle AOB = -2\angle OCA + 2\angle OBC$$

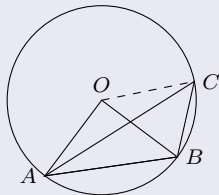
$$\angle AOB = 2(\angle OBC - \angle OCA)$$



Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \quad (3)$$

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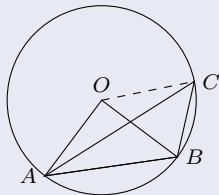
We know that know that $\angle ACB = \angle OCB - \angle OCA$.



Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \quad (3)$$

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We can substitute (3) and (4) into (5):

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$$\angle AOB = -2\angle OCA + 2\angle OBC$$

$$\angle AOB = 2(\angle OBC - \angle OCA)$$

We know that $\angle ACB = \angle OCB - \angle OCA$.

$$\therefore \angle AOB = 2\angle ACB$$



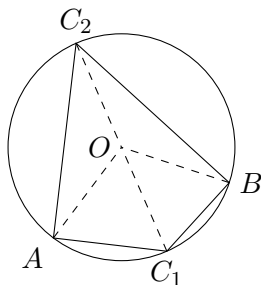
Cyclic quadrilaterals

Star Trek Theorem

Example

If C_1 and C_2 are two points on the circle, one on the minor arc AB and the other on the major arc, prove that $\angle AC_1B + \angle AC_2B = 180^\circ$.

This is equivalent to proving that the opposite angles of a cyclic quadrilateral are supplementary.



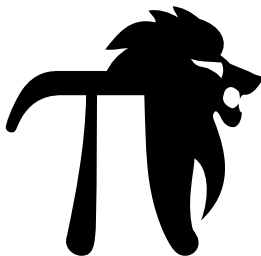
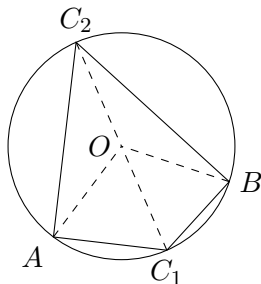
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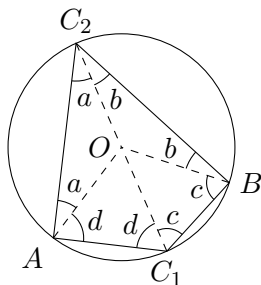
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Extension 4 proof

Star Trek Theorem

Proof that opposite angles of a cycle quadrilateral are supplementary.

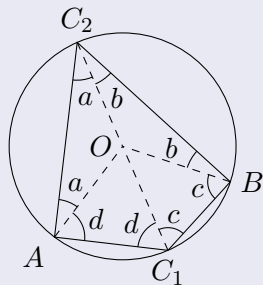


Extension 4 proof

Star Trek Theorem

Proof that opposite angles of a cycle quadrilateral are supplementary.

The sum of the interior angles of a quadrilateral equals

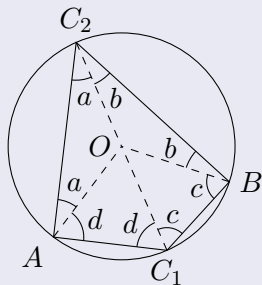


Extension 4 proof

Star Trek Theorem

Proof that opposite angles of a cycle quadrilateral are supplementary.

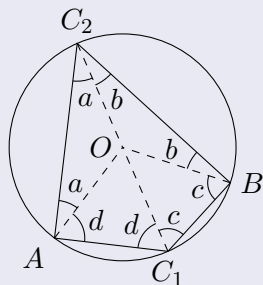
The sum of the interior angles of a quadrilateral equals 360° .



Extension 4 proof

Star Trek Theorem

Proof that opposite angles of a cycle quadrilateral are supplementary.



The sum of the interior angles of a quadrilateral equals 360° .

So:

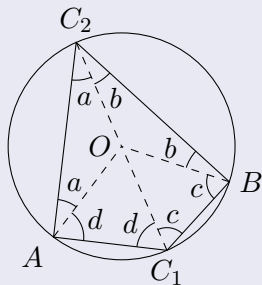
$$a + b + c + d + a + b + c + d = 360^\circ$$



Extension 4 proof

Star Trek Theorem

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The sum of the interior angles of a quadrilateral equals 360° .

So:

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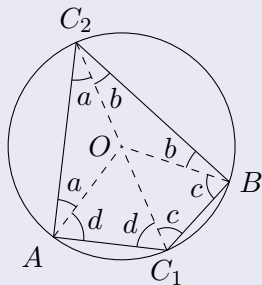
$$2(a + b + c + d) = 360^\circ$$



Extension 4 proof

Star Trek Theorem

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The sum of the interior angles of a quadrilateral equals 360° .

So:

$$a + b + c + d + a + b + c + d = 360^\circ$$

$$2(a + b + c + d) = 360^\circ$$

$$a + b + c + d = 180^\circ$$



Angles subtended by the same arc

Star Trek Theorem

Example

Show that if C_1 and C_2 are two different choices for the position of the point C along the same arc AB then $\angle AC_1B = \angle AC_2B$.

This is equivalent to saying that angles subtended by the same arc are equal.



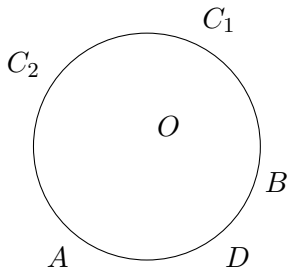
Angles subtended by the same arc

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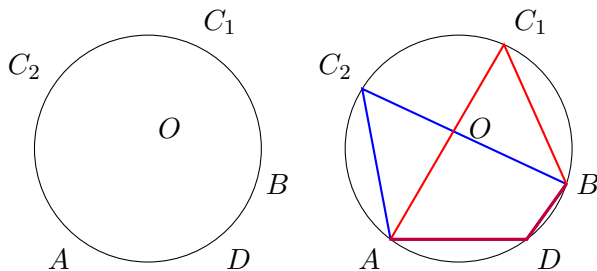
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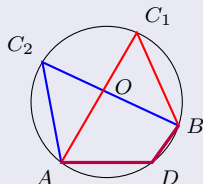


Extension 5 proof

Star Trek Theorem

Proof that $\angle AC_1B = \angle AC_2B$.

Using extension 4, we know that:



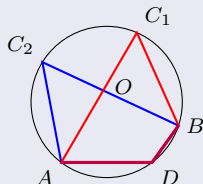
Extension 5 proof

Star Trek Theorem

Proof that $\angle AC_1B = \angle AC_2B$.

Using extension 4, we know that:

$$\angle AC_1B + \angle ADB = 180^\circ$$

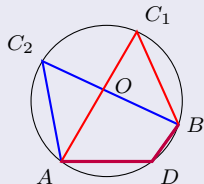


Extension 5 proof

Star Trek Theorem

Proof that $\angle AC_1B = \angle AC_2B$.

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Extension 5 proof

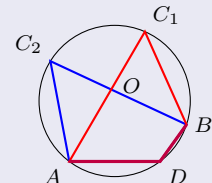
Star Trek Theorem

Proof that $\angle AC_1B = \angle AC_2B$.

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So:

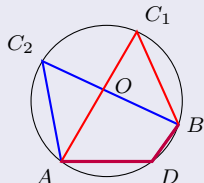


Extension 5 proof

Star Trek Theorem

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$$\angle AC_1B + \angle ADB = 180^\circ$$

$$\angle AC_2B + \angle ADB = 180^\circ$$

So:

$$\angle AC_1B + \angle ADB = \angle AC_2B + \angle ADB$$

$$\angle AC_1B = \angle AC_2B$$



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- Tangent chord theorem
- Proof

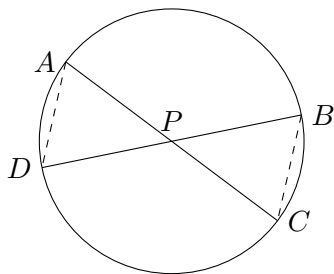


Theorem

Crossed Chord Theorem

Theorem (Crossed Chord Theorem)

If two chords AB and CD of a circle intersect at point P , then $(PA)(PB) = (PC)(PD)$.

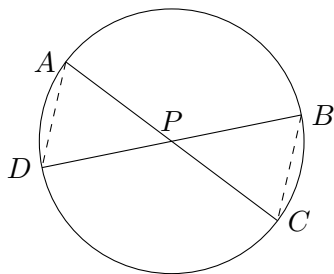


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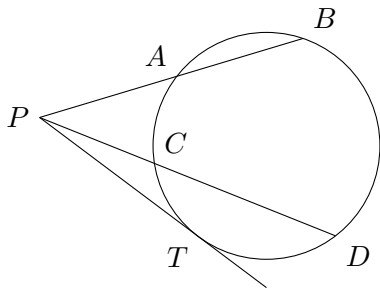


This is proved using similar triangles and the fifth extension we developed for the Star Trek theorem. Try to prove it yourself!



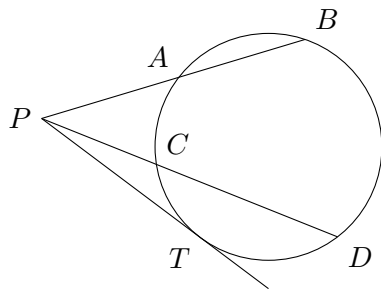
Secant and tangents

Crossed Chord Theorem



Secant and tangents

Crossed Chord Theorem



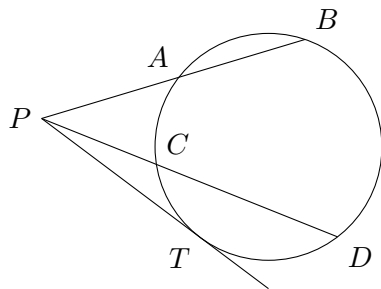
Example

In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.



Secant and tangents

Crossed Chord Theorem



Example

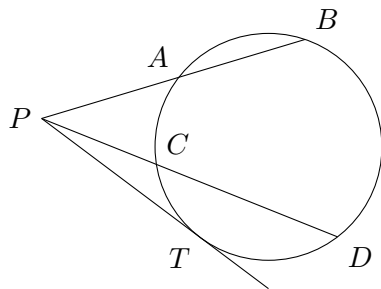
In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.

Prove that
 $(PA)(PB) = (PC)(PD)$.



Secant and tangents

Crossed Chord Theorem



Example

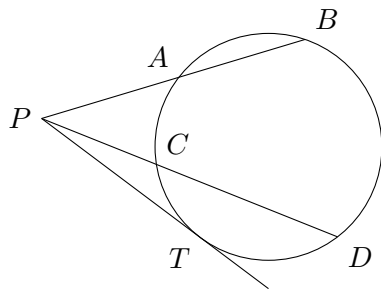
In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.

Prove that $(PA)(PB) = (PC)(PD)$. This proof also uses similar triangles. Try it yourself!



Secant and tangents

Crossed Chord Theorem



Example

In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.

Prove that $(PA)(PB) = (PC)(PD)$. This proof also uses similar triangles. Try it yourself!

Example

If PT is a tangent to the circle, prove that $(PA)(PB) = (PT)^2$.



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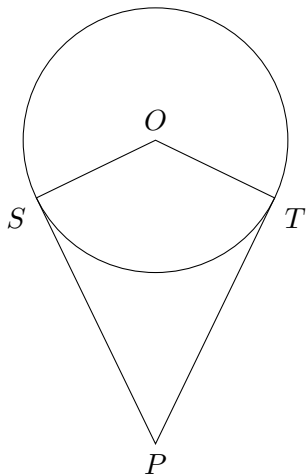
4 Tangent Chord Theorem

- Tangent chord theorem
- Proof



Properties of two tangents

Important Tangent Properties



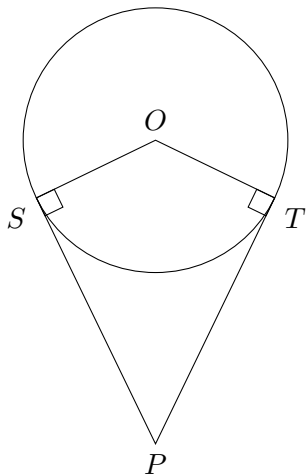
Example

If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:



Properties of two tangents

Important Tangent Properties



Example

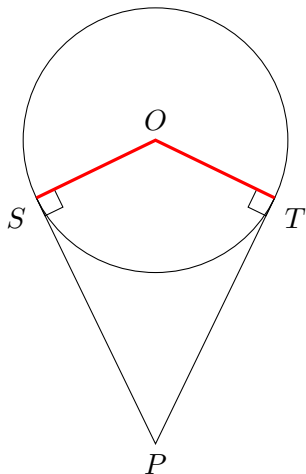
If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:

- 1 A tangent at a point on a circle is perpendicular to the radius drawn to the point.



Properties of two tangents

Important Tangent Properties



Example

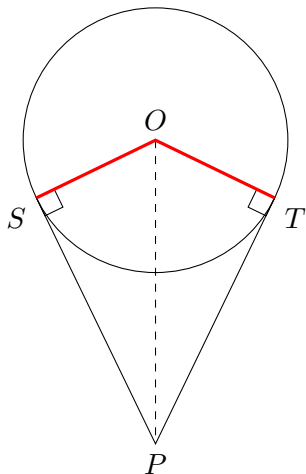
If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:

- 1 A tangent at a point on a circle is perpendicular to the radius drawn to the point.
- 2 $PS = PT$: tangents to a circle from an external point are equal.



Properties of two tangents

Important Tangent Properties



Example

If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:

- 1 A tangent at a point on a circle is perpendicular to the radius drawn to the point.
- 2 $PS = PT$: tangents to a circle from an external point are equal.
- 3 OP bisects the angle between the tangents.



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- Proof

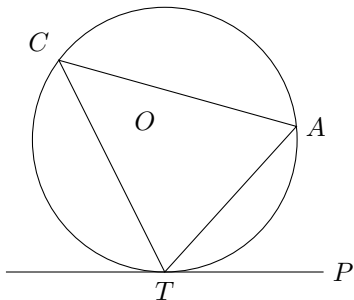


Theorem

Tangent Chord Theorem

Theorem (Tangent chord theorem)

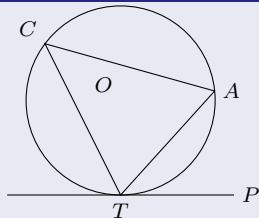
Given that TA is any chord of a circle and PT is a tangent to the circle at T . If C is a point on the circle chosen to be on the side of the chord opposite to the tangent then $\angle TCA = \angle PTA$.



Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



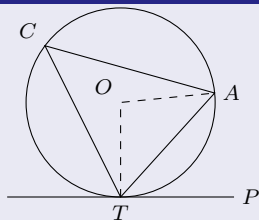
We know that:



Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$

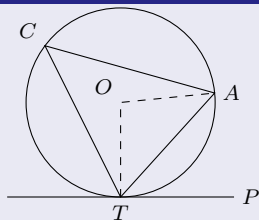
$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$



Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$

$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

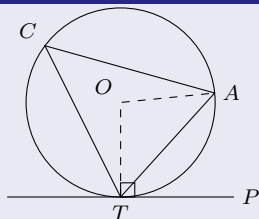
We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.



Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$

$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

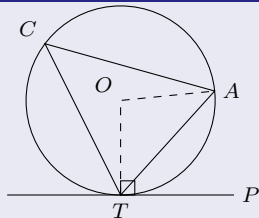
We also know that $\angle PTA = 90^\circ - \angle ATO$.



Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$

$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

We also know that $\angle PTA = 90^\circ - \angle ATO$.

We put this all together:

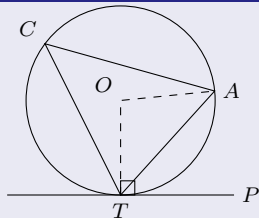
$$\angle PTA = 90^\circ - \frac{1}{2}(180^\circ - 2\angle TCA)$$



Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$

$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

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We also know that $\angle PTA = 90^\circ - \angle ATO$.

We put this all together:

$$\angle PTA = 90^\circ - \frac{1}{2}(180^\circ - 2\angle TCA)$$

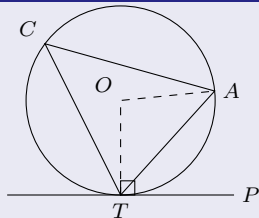
$$\angle PTA = 90^\circ - 90^\circ + \angle TCA$$



Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



We know that:

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$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

We also know that $\angle PTA = 90^\circ - \angle ATO$.

We put this all together:

$$\angle PTA = 90^\circ - \frac{1}{2}(180^\circ - 2\angle TCA)$$

$$\angle PTA = 90^\circ - 90^\circ + \angle TCA$$

$$\angle PTA = \angle TCA$$

