

# Euclid Preparation 2

## Analytic Geometry

Vincent Macri

William Lyon Mackenzie C.I. Math Club

© Caroline Liu, Vincent Macri, and Samantha Unger, 2018



# Table of Contents

## 1 Toolkit

## 2 Problems



# Formula 1

## Toolkit

The standard form for a line with slope  $-\frac{A}{B}$  and intercepts  $\left(-\frac{C}{A}, 0\right)$  and  $\left(0, -\frac{C}{B}\right)$

Formula

$$Ax + By + C = 0$$



# Formula 2

## Toolkit

The equation of the line with slope  $m$  through  $(x_0, y_0)$

Formula

$$y - y_0 = m(x - x_0)$$



# Formula 3

## Toolkit

The equation of the line with intercepts at  $(a, 0)$  and  $(0, b)$

Formula

$$\frac{x}{a} + \frac{y}{b} = 1$$



# Formula 4

## Toolkit

The formula for the midpoint  $M$  of  $A(x_1, y_1)$  and  $B(x_2, y_2)$

Formula

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



# Formula 5

## Toolkit

The distance  $D$  between the points  $A(x_1, y_1)$  and  $B(x_2, y_2)$

Formula

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



# Formula 6

## Toolkit

The distance  $D$  between the line  $Ax + By + C = 0$  and the point  $(x_0, y_0)$

### Formula

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$





# Formula 7

## Toolkit

The area of the triangle  $A(x_1, y_1)$ ,  $B(x_2, y_2)$ ,  $C(x_3, y_3)$

Formula

$$A = \frac{1}{2} |x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3|$$



# Formula 8

## Toolkit

The equation of the circle with centre  $(h, k)$  and radius  $r$

Formula

$$(x - h)^2 + (y - k)^2 = r^2$$



# Table of Contents

1 Toolkit

2 Problems



# Analytic geometry problem 1

## Problems

### Problem

If the line  $2x - 3y - 6 = 0$  is reflected in the line  $y = -x$ , find the equation of the image line.



# Analytic geometry problem 1

## Problems

### Problem

If the line  $2x - 3y - 6 = 0$  is reflected in the line  $y = -x$ , find the equation of the image line.

### Solution

First, we find two points on the original line.



# Analytic geometry problem 1

## Problems

### Problem

If the line  $2x - 3y - 6 = 0$  is reflected in the line  $y = -x$ , find the equation of the image line.

### Solution

First, we find two points on the original line. It's easiest to find the intercepts, since we have formulas for those.



# Analytic geometry problem 1

## Problems

### Problem

If the line  $2x - 3y - 6 = 0$  is reflected in the line  $y = -x$ , find the equation of the image line.

### Solution

First, we find two points on the original line. It's easiest to find the intercepts, since we have formulas for those.

The line given has the intercepts  $(3, 0)$  and  $(0, -2)$ .



# Analytic geometry problem 1

## Problems

### Problem

If the line  $2x - 3y - 6 = 0$  is reflected in the line  $y = -x$ , find the equation of the image line.

### Solution

First, we find two points on the original line. It's easiest to find the intercepts, since we have formulas for those.

The line given has the intercepts  $(3, 0)$  and  $(0, -2)$ .

When reflecting in  $y = -x$ , we swap  $x$  and  $y$ , and change the sign.





# Analytic geometry problem 1

## Problems

### Problem

If the line  $2x - 3y - 6 = 0$  is reflected in the line  $y = -x$ , find the equation of the image line.

### Solution

First, we find two points on the original line. It's easiest to find the intercepts, since we have formulas for those.

The line given has the intercepts  $(3, 0)$  and  $(0, -2)$ .

When reflecting in  $y = -x$ , we swap  $x$  and  $y$ , and change the sign.

So, we get:

$$(3, 0) \rightarrow (0, -3)$$

$$(0, -2) \rightarrow (2, 0)$$



# Analytic geometry problem 1 solution continued

## Problems

### Solution

Next, we find an equation for a line which goes through  $(0, -3)$  and  $(2, 0)$ .



# Analytic geometry problem 1 solution continued

## Problems

### Solution

Next, we find an equation for a line which goes through  $(0, -3)$  and  $(2, 0)$ .

$$\frac{x}{a} + \frac{y}{b} = 1$$



# Analytic geometry problem 1 solution continued

## Problems

### Solution

Next, we find an equation for a line which goes through  $(0, -3)$  and  $(2, 0)$ .

$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{2} + \frac{y}{-3} = 1$$



# Analytic geometry problem 1 solution continued

## Problems

### Solution

Next, we find an equation for a line which goes through  $(0, -3)$  and  $(2, 0)$ .

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{2} + \frac{y}{-3} = 1$$

$$3x - 2y - 6 = 0$$



# Analytic geometry problem 1 solution continued

## Problems

### Solution

Next, we find an equation for a line which goes through  $(0, -3)$  and  $(2, 0)$ .

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{2} + \frac{y}{-3} = 1$$

$$3x - 2y - 6 = 0$$

And that is our line.



# Analytic geometry problem 2

## Problems

### Problem

If  $A(3, 5)$  and  $B(11, 11)$  are fixed points, find the point(s)  $P$  on the  $x$ -axis such that the area of the triangle  $ABP$  equals 30.



# Analytic geometry problem 2

## Problems

### Problem

If  $A(3, 5)$  and  $B(11, 11)$  are fixed points, find the point(s)  $P$  on the  $x$ -axis such that the area of the triangle  $ABP$  equals 30.

### Solution

Let  $P = (p, 0)$ .

Then, we use the formula for area:

$$A = \frac{1}{2} |x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3|$$





# Analytic geometry problem 2

## Problems

### Problem

If  $A(3, 5)$  and  $B(11, 11)$  are fixed points, find the point(s)  $P$  on the  $x$ -axis such that the area of the triangle  $ABP$  equals 30.

### Solution

Let  $P = (p, 0)$ .

Then, we use the formula for area:

$$A = \frac{1}{2} |x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3|$$

$$A = \frac{1}{2} |33 + 0 + 5p - 55 - 11p - 0|$$



# Analytic geometry problem 2

## Problems

### Problem

If  $A(3, 5)$  and  $B(11, 11)$  are fixed points, find the point(s)  $P$  on the  $x$ -axis such that the area of the triangle  $ABP$  equals 30.

### Solution

Let  $P = (p, 0)$ .

Then, we use the formula for area:

$$A = \frac{1}{2} |x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3|$$

$$A = \frac{1}{2} |33 + 0 + 5p - 55 - 11p - 0|$$

$$|-22 - 6p| = 60$$



# Analytic geometry problem 2 solution continued

## Problems

Solution

$$|-22 - 6p| = 60$$



# Analytic geometry problem 2 solution continued

## Problems

### Solution

$$|-22 - 6p| = 60$$

$$p = \frac{19}{3}, -\frac{41}{3}$$



# Analytic geometry problem 2 solution continued

## Problems

### Solution

$$|-22 - 6p| = 60$$

$$p = \frac{19}{3}, -\frac{41}{3}$$

So the points are  $\left(\frac{19}{3}, 0\right)$  and  $\left(-\frac{41}{3}, 0\right)$ .



# Analytic geometry problem 3

## Problems

### Problem

Given the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 - 6x + 2 = 0$ , find the length of their common chord.



# Analytic geometry problem 3

## Problems

### Problem

Given the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 - 6x + 2 = 0$ , find the length of their common chord.

### Solution

It is helpful to rewrite the second circle's equation as  $(x - 3)^2 + y^2 = 7$  because this shows us more easily that the second circle is centred at  $(3, 0)$ .



# Analytic geometry problem 3

## Problems

### Problem

Given the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 - 6x + 2 = 0$ , find the length of their common chord.

### Solution

It is helpful to rewrite the second circle's equation as  $(x - 3)^2 + y^2 = 7$  because this shows us more easily that the second circle is centred at  $(3, 0)$ . Since the line joining the centres is horizontal, the common chord is vertical.





# Analytic geometry problem 3

## Problems

### Problem

Given the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 - 6x + 2 = 0$ , find the length of their common chord.

### Solution

It is helpful to rewrite the second circle's equation as  $(x - 3)^2 + y^2 = 7$  because this shows us more easily that the second circle is centred at  $(3, 0)$ . Since the line joining the centres is horizontal, the common chord is vertical. We can solve this easily:

$$\begin{aligned}x^2 + y^2 - 4 &= x^2 + y^2 - 6x + 2 \\x &= 1\end{aligned}$$



# Analytic geometry problem 3 solution continued

## Problems

### Solution

$$x = 1$$

If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at  $(1, \pm\sqrt{3})$ .



# Analytic geometry problem 3 solution continued

## Problems

### Solution

$$x = 1$$

If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at  $(1, \pm\sqrt{3})$ .

So, the length of the entire chord is



# Analytic geometry problem 3 solution continued

## Problems

### Solution

$$x = 1$$

If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at  $(1, \pm\sqrt{3})$ .

So, the length of the entire chord is  $2\sqrt{3}$ .



# Analytic geometry problem 4

## Problems

### Problem

A line has slope  $-2$  and is a distance of 2 units from the origin.  
What is the area of the triangle formed by this line and the axes?



# Analytic geometry problem 4

## Problems

### Problem

A line has slope  $-2$  and is a distance of 2 units from the origin. What is the area of the triangle formed by this line and the axes?

### Solution

Let the  $x$ -intercept be  $k$ .



# Analytic geometry problem 4

## Problems

### Problem

A line has slope  $-2$  and is a distance of 2 units from the origin.  
What is the area of the triangle formed by this line and the axes?

### Solution

Let the  $x$ -intercept be  $k$ .

What does this make the  $y$ -intercept?



# Analytic geometry problem 4

## Problems

### Problem

A line has slope  $-2$  and is a distance of 2 units from the origin.  
What is the area of the triangle formed by this line and the axes?

### Solution

Let the  $x$ -intercept be  $k$ .

What does this make the  $y$ -intercept?  $2k$ .





# Analytic geometry problem 4

## Problems

### Problem

A line has slope  $-2$  and is a distance of 2 units from the origin.  
What is the area of the triangle formed by this line and the axes?

### Solution

Let the  $x$ -intercept be  $k$ .

What does this make the  $y$ -intercept?  $2k$ .

So the equation of the line can be written as:

$$2x + y - 2k = 0$$



# Analytic geometry problem 4 solution continued

## Problems

### Solution

We can use the formula for distance from a point to a line:

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$



# Analytic geometry problem 4 solution continued

## Problems

### Solution

We can use the formula for distance from a point to a line:

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$
$$2 = \frac{|2(0) + 1(0) - 2k|}{\sqrt{2^2 + 1^2}}$$



# Analytic geometry problem 4 solution continued

## Problems

### Solution

We can use the formula for distance from a point to a line:

$$\begin{aligned} D &= \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \\ 2 &= \frac{|2(0) + 1(0) - 2k|}{\sqrt{2^2 + 1^2}} \\ 2 &= \frac{|-2k|}{\sqrt{5}} \end{aligned}$$



# Analytic geometry problem 4 solution continued

## Problems

### Solution

We can use the formula for distance from a point to a line:

$$\begin{aligned} D &= \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \\ 2 &= \frac{|2(0) + 1(0) - 2k|}{\sqrt{2^2 + 1^2}} \\ 2 &= \frac{|-2k|}{\sqrt{5}} \\ k &= \pm\sqrt{5} \end{aligned}$$



# Analytic geometry problem 4 solution continued

## Problems

### Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:



# Analytic geometry problem 4 solution continued

## Problems

### Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

$$A = \frac{1}{2} \times k \times 2k = \quad = \quad =$$



# Analytic geometry problem 4 solution continued

## Problems

### Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

$$A = \frac{1}{2} \times k \times 2k = k^2 = \quad =$$





# Analytic geometry problem 4 solution continued

## Problems

### Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

$$A = \frac{1}{2} \times k \times 2k = k^2 = (\pm\sqrt{5})^2 =$$



# Analytic geometry problem 4 solution continued

## Problems

### Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

$$A = \frac{1}{2} \times k \times 2k = k^2 = (\pm\sqrt{5})^2 = 5$$

