Euclid Preparation 2

Analytic Geometry

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Toolkit

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Formula

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Formula 3

Toolkit

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Toolkit

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Formula 7
Toolkit

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Formula

$$A = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|$$

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Formula

$$(x-h)^2+(y-k)^2=r^2$$

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Problem

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So, we get:

$$(3, 0) \to (0, -3)$$

$$(0, -2) \to (2, 0)$$

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$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{2} + \frac{y}{-3} = 1$$
$$3x - 2y - 6 = 0$$

And that is our line.

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Then, we use the formula for area:

$$A=\frac{1}{2}|x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_3y_2-x_1y_3|$$

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$$|-22-6p|=60$$

Solution

$$|-22 - 6p| = 60$$

Analytic geometry problem 2 solution continued

Solution

$$
\begin{aligned}
|-22 - 6p| &= 60 \\
p &= \frac{19}{3}, -\frac{41}{3}
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So the points are $\left(\frac{19}{3}, 0\right)$ and $\left(-\frac{41}{3}, 0\right)$.

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$$\begin{aligned} x^2 + y^2 - 4 &= x^2 + y^2 - 6x + 2 \\ x &= 1 \end{aligned}$$

Solution

$$x = 1$$

If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at $(1, \pm\sqrt{3})$.

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If we substitute this back into the equation for either of our circle, we find that the chord intersects the circles at $(1, \pm\sqrt{3})$.

So, the length of the entire chord is $2\sqrt{3}$.

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Solution

Let the x-intercept be k.

What does this make the y-intercept? $2k$.

So the equation of the line can be written as:

$$2x + y - 2k = 0$$

Solution

We can use the formula for distance from a point to a line:

$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

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$$D = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$2 = \frac{\left|2(0) + 1(0) - 2k\right|}{\sqrt{2^2 + 1^2}}$$

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$$k = \pm\sqrt{5}$$

Analytic geometry problem 4 solution continued

Solution

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Solution

$$k = \pm\sqrt{5}$$

Next, we find the area:

$$A = \frac{1}{2} \times k \times 2k = k^2 = (\pm\sqrt{5})^2 = 5$$