

James Tanton's Exploding Dots

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Table of Contents

1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

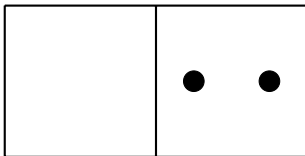
6 Eks

7 Infinitia



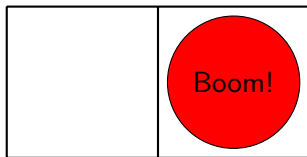
Mechania

The $2 \leftarrow 1$ machine



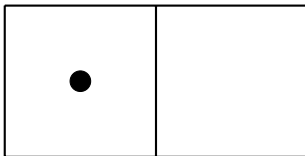
Mechania

The $2 \leftarrow 1$ machine



Mechania

The $2 \leftarrow 1$ machine



Mechanica

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2 \leftarrow 1}$$

$$2 \xrightarrow{2 \leftarrow 1}$$

$$3 \xrightarrow{2 \leftarrow 1}$$

$$4 \xrightarrow{2 \leftarrow 1}$$

$$5 \xrightarrow{2 \leftarrow 1}$$

$$13 \xrightarrow{2 \leftarrow 1}$$



Mechanica

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2 \leftarrow 1} 1$$

$$2 \xrightarrow{2 \leftarrow 1}$$

$$3 \xrightarrow{2 \leftarrow 1}$$

$$4 \xrightarrow{2 \leftarrow 1}$$

$$5 \xrightarrow{2 \leftarrow 1}$$

$$13 \xrightarrow{2 \leftarrow 1}$$



Mechanica

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2 \leftarrow 1} 1$$

$$2 \xrightarrow{2 \leftarrow 1} 10$$

$$3 \xrightarrow{2 \leftarrow 1}$$

$$4 \xrightarrow{2 \leftarrow 1}$$

$$5 \xrightarrow{2 \leftarrow 1}$$

$$13 \xrightarrow{2 \leftarrow 1}$$



Mechanica

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2 \leftarrow 1} 1$$

$$2 \xrightarrow{2 \leftarrow 1} 10$$

$$3 \xrightarrow{2 \leftarrow 1} 11$$

$$4 \xrightarrow{2 \leftarrow 1}$$

$$5 \xrightarrow{2 \leftarrow 1}$$

$$13 \xrightarrow{2 \leftarrow 1}$$



Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2 \leftarrow 1} 1$$

$$2 \xrightarrow{2 \leftarrow 1} 10$$

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Mechanica

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

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Mechania

$2 \leftarrow 1$ examples

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2 \leftarrow 1} 1$$

$$2 \xrightarrow{2 \leftarrow 1} 10$$

$$3 \xrightarrow{2 \leftarrow 1} 11$$

$$4 \xrightarrow{2 \leftarrow 1} 100$$

$$5 \xrightarrow{2 \leftarrow 1} 101$$

$$13 \xrightarrow{2 \leftarrow 1} 1101$$



What if...

What if we had a $3 \leftarrow 1$ machine?



What if...

What if we had a $10 \leftarrow 1$ machine?



Table of Contents

1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia



These machines are another way of handling arithmetic, but they work in any base.

A $b \leftarrow 1$ machine handles numbers in base b .

We'll use a $10 \leftarrow 1$ machine for now.



Table of Contents

1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia



We can count with these machines, but what else can we do?



We can count with these machines, but what else can we do?

What's the first thing you learn to do with numbers after counting?



Addition!



What is $234 + 125$?



Arithmos

Examples

What is $234 + 125$?

$$\begin{array}{r} 234 \\ + 125 \\ \hline 359 \end{array}$$



Arithmos

Examples

What is $234 + 125$?

$$\begin{array}{r} 234 \\ + 125 \\ \hline 359 \end{array}$$

What is $234 + 187$?



Arithmos

Examples

What is $234 + 125$?

$$\begin{array}{r} 2 \quad 3 \quad 4 \\ + \quad 1 \quad 2 \quad 5 \\ \hline 3 \quad 5 \quad 9 \end{array}$$

What is $234 + 187$?

$$\begin{array}{r} 2 \quad 3 \quad 4 \\ + \quad 1 \quad 8 \quad 7 \\ \hline 3 \quad 11 \quad 11 \end{array}$$



Arithmos

Examples

What is $234 + 125$?

$$\begin{array}{r} 2 \ 3 \ 4 \\ + \ 1 \ 2 \ 5 \\ \hline 3 \ 5 \ 9 \end{array}$$

What is $234 + 187$?

$$\begin{array}{r} 2 \ 3 \ 4 \\ + \ 1 \ 8 \ 7 \\ \hline 3 \ 11 \ 11 \end{array}$$

Three hundred eleventy eleven!

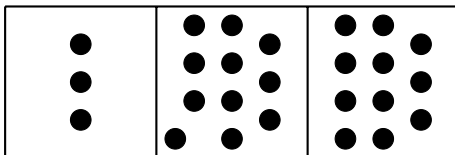
But now society thinks I'm weird. Let's fix that.



Arithmos

Explode the dots!

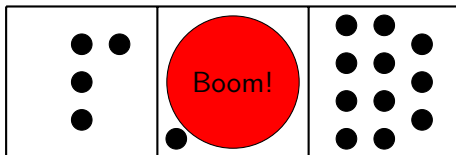
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

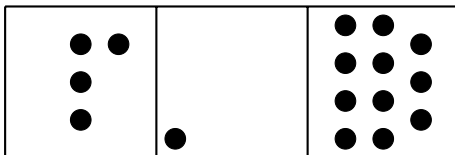
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

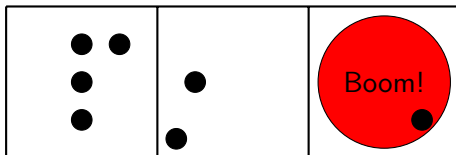
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

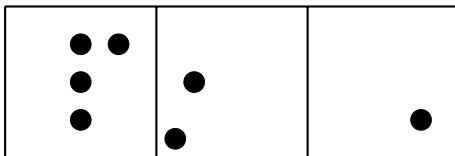
Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



Arithmos

Explode the dots!

Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



421



What do we learn after
addition?



Multiplication!



Arithmos

Examples

What is 2876×3 ?

$$\begin{array}{r} 2 \quad 8 \quad 7 \quad 6 \\ \times 3 \\ \hline 6 \quad 24 \quad 21 \quad 18 \end{array}$$

Let's fix this one together for society.

6	24	21	18
---	----	----	----



Table of Contents

1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia



Antidotia

Subtraction

Theorem 1

Subtraction does not exist.

Theorem 2

What we call subtraction is just the addition of negative numbers.

Or, subtraction is the addition of the opposite.

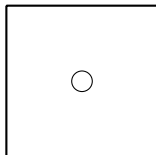


Antidotia

The antidot

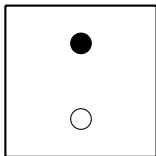
The opposite of a dot is an antidot. I'll call these tods.

This is one tod in one of our machines:



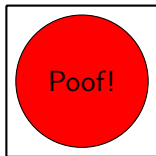
Antidotia

How do todos behave?



Antidotia

How do todos behave?



Antidotia

How do todos behave?



Antidotia

Examples

What is $564 - 123$?



Antidotia

Examples

What is $564 - 123$?

$$\begin{array}{r} 564 \\ - 123 \\ \hline 441 \end{array}$$



Antidotia

Examples

What is $564 - 123$?

$$\begin{array}{r} 564 \\ - 123 \\ \hline 441 \end{array}$$

What is $441 - 254$?



Antidotia

Examples

What is $564 - 123$?

$$\begin{array}{r} 564 \\ - 123 \\ \hline 441 \end{array}$$

What is $441 - 254$?

$$\begin{array}{r} 441 \\ - 254 \\ \hline 2-1-3 \end{array}$$



Antidotia

Examples

What is $564 - 123$?

$$\begin{array}{r} 5 \ 6 \ 4 \\ - \ 1 \ 2 \ 3 \\ \hline 4 \ 4 \ 1 \end{array}$$

What is $441 - 254$?

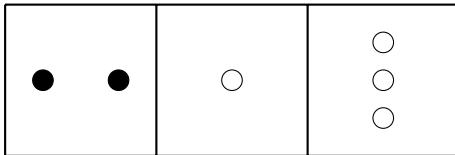
$$\begin{array}{r} 4 \ \ 4 \ 1 \\ - \ 2 \ 5 \ 4 \\ \hline 2 \ -1 \ -3 \end{array}$$

Let's fix this together on the board for society's sake. We'll use the exploding dots method.



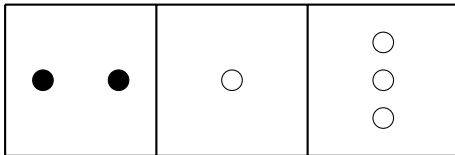
Antidotia

Another method



Antidotia

Another method

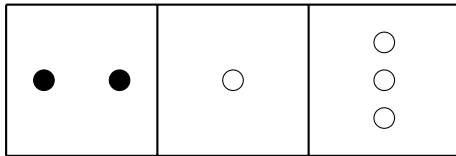


There is another way to fix this which is helpful for doing math mentally.



Antidotia

Another method



There is another way to fix this which is helpful for doing math mentally.

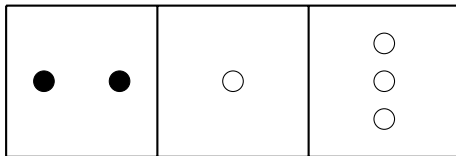
Let's look the place values:

$$200 + -10 + -3$$



Antidotia

Another method



There is another way to fix this which is helpful for doing math mentally.

Let's look the place values:

$$200 + -10 + -3$$

This is very easy to do mentally.

$$200 + -10 = 190 + -3$$

$$190 + -3 = 187$$



Table of Contents

1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia



Obelus

Division

Fun fact before we get started on division, did you know that the \div sign is called an obelus?



Obelus

Long division method

What is $276 \div 12$?

Don't use a calculator!



Obelus

Long division method

What is $276 \div 12$?

Don't use a calculator!

$$\begin{array}{r} 23 \\ 12 \overline{) 276} \\ \underline{240} \\ 36 \\ \underline{36} \\ 0 \end{array}$$

This is stupid and convoluted. Let's use exploding dots instead.

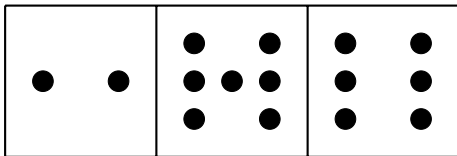


Obelus

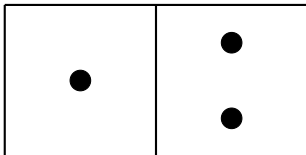
Exploding dots method

$$276 \div 12$$

276:



12:

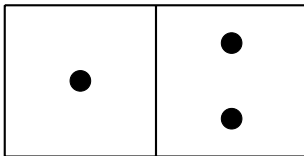


Let's look for groups of 12 in 276.



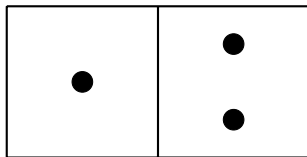
Obelus

Exploding dots solution



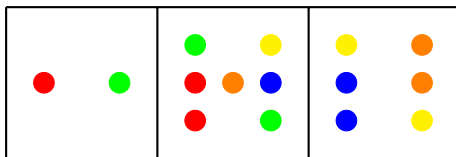
Obelus

Exploding dots solution



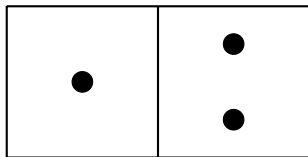
2

3



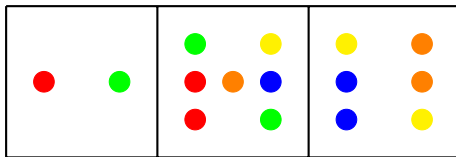
Obelus

Exploding dots solution



2

3



$$273 \div 12 = 23$$



What is $2783 \div 23$?



What is $2783 \div 23$? 121



What is $2783 \div 23$? 121

What is $2785 \div 23$?



What is $2783 \div 23$? 121

What is $2785 \div 23$? 121 R2 or $121 + \frac{2}{23}$



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4 Antidotia

5 Obelus

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7 Infinitia



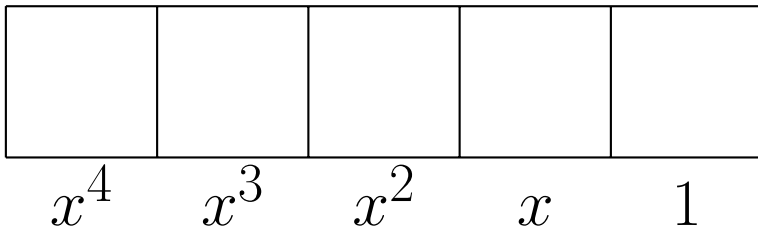
What if...

What if we had a $1 \leftarrow x$ machine? What would it look like?



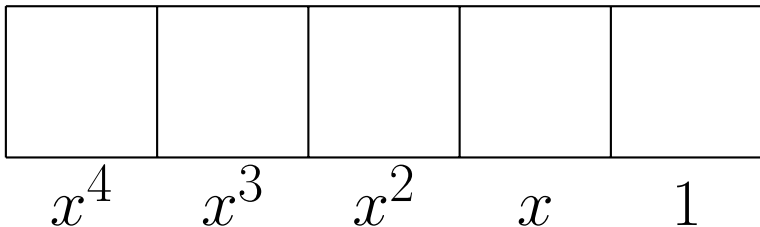
What if...

What if we had a $1 \leftarrow x$ machine? What would it look like?



What if...

What if we had a $1 \leftarrow x$ machine? What would it look like?



With this knowledge, we can now manipulate polynomials just like we do regular numbers.



What is $\frac{2x^2 + 7x + 6}{x + 2}$?



What is $\frac{2x^2 + 7x + 6}{x + 2}$? $2x + 3$



What is $\frac{2x^2 + 7x + 6}{x + 2}$? $2x + 3$

What is $\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$?



What is $\frac{2x^2 + 7x + 6}{x + 2}$? $2x + 3$

What is $\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$? $x^2 + 2x + 1$



What is $\frac{2x^2 + 7x + 6}{x + 2}$? $2x + 3$

What is $\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$? $x^2 + 2x + 1$

What is $\frac{x^3 - 3x + 2}{x + 2}$?



What is $\frac{2x^2 + 7x + 6}{x + 2}$? $2x + 3$

What is $\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$? $x^2 + 2x + 1$

What is $\frac{x^3 - 3x + 2}{x + 2}$? $x^2 - 2x + 1$



Table of Contents

1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia



Infinitia

Now it gets *really* fun

$$\frac{1}{1-x}$$



Using dots and boxes, we get what is known as the **geometric series formula**:



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$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 \dots$$



Using dots and boxes, we get what is known as the **geometric series formula**:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 \dots$$

In calculus, we would call this the **Taylor series** of $\frac{1}{1-x}$.



Using the dots and boxes method, find the Taylor series of:

$$\frac{1}{1 - x - x^2}$$



Congratulations!

We just went from kindergarten arithmetic to advanced calculus in 40 minutes!

