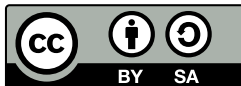


Game Theory

Caroline Liu Vincent Macri Samantha Unger



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What is game theory?

Introduction

“Game theory is the study of rigging games in your favour.”



What is game theory?

Introduction

“Game theory is the study of rigging games in your favour.”

— Vincent Macri



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Pick Up Sticks!

The most accurately named game of all time.



Rules and intro

Pick Up Sticks!

Pick up sticks is a 2 player game that is often used as a game theory question on math contests. The game involves:



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Rules and intro

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Variations include the winner being the one who takes the last stick, or the loser being the one who takes the last stick.

The game can be structured in such ways that certain players can win every time. Lets look at some of these scenarios!



The scenarios: your worksheet

Pick Up Sticks!

Let's go over some scenarios.



The scenarios: your worksheet

Pick Up Sticks!

Let's go over some scenarios.

Scenario 1 Whoever picks the last stick **loses**. Which player can guarantee a win?

There are 24 sticks in total, and the players can take up to 4 sticks.



The scenarios: your worksheet

Pick Up Sticks!

Let's go over some scenarios.

Scenario 1 Whoever picks the last stick **loses**. Which player can guarantee a win?

There are 24 sticks in total, and the players can take up to 4 sticks.

Scenario 2 Whoever picks the last stick **wins**. Which player can guarantee a win?

There are 35 sticks, and the players can take up to 4 sticks on their turns.



Try it out yourself!
Pick Up Sticks!

Try it out yourself!



Answers: The Strats

Pick Up Sticks!

Let's take those scenarios up together.



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Scenario 1 Player 1 always wins. Why?



Answers: The Strats

Pick Up Sticks!

Let's take those scenarios up together.

Scenario 1 Player 1 always wins. Why?

Scenario 2 Player 2 always wins. Why?



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Nim

Vincent's favourite impartial combinatorial game.



What is an impartial combinatorial game?

Nim

- 2 players
- Players alternate turns
- Both players can see everything (this rules out most card games)
- No chance (this rules out most board games)
- Players can make the same moves – the game is **impartial** (this rules out games such as checkers and chess)



What is Nim?

Nim

Nim is similar to **pick up sticks**, but with more piles.



What is Nim?

Nim

Nim is similar to **pick up sticks**, but with more piles.

- In Nim, there are n piles, with k_i stones in each pile.
- A player can take **as many stones as they want** (as long as they take at least one) from only **one pile** on their turn.
- We will be looking at the version where the person who picks up the last stone wins, but Nim can also be played where the person who picks up last loses.



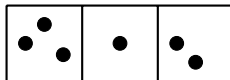
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Here is an example Nim game:



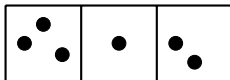
Here: $n = 3$, $k_1 = 3$, $k_2 = 1$, and $k_3 = 2$.



Some Nim notation

Nim

Instead of drawing out all the boxes with circles in them, we use this notation, because it's shorter and mathematicians are lazy:



becomes

$$3 \oplus 1 \oplus 2$$



Let's play on the board!

Nim

$$4 \oplus 5 \oplus 8$$



Let's play on the board!

Nim

$$4 \oplus 5 \oplus 8$$

$$5 \oplus 7$$



Let's play on the board!

Nim

$$4 \oplus 5 \oplus 8$$

$$5 \oplus 7$$

$$2 \oplus 2$$



Let's play on the board!

Nim

$$4 \oplus 5 \oplus 8$$

$$5 \oplus 7$$

$$2 \oplus 2$$

Now, let's look at how to solve Nim.



First, game theory some notation

Nim

Usually in game theory, we use the following notation:

N The next player wins

P The previous player wins

We write who wins a give game as an ordered pair.

For example:

$$(1, N)$$

$$(1 \oplus 1, P)$$



Game trees

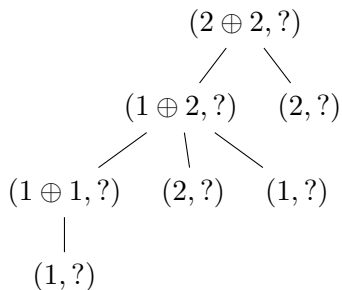
Nim

$$(2 \oplus 2, ?)$$



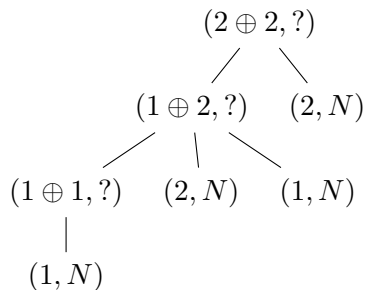
Game trees

Nim



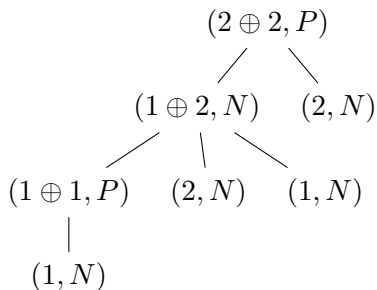
Game trees

Nim



Game trees

Nim



What are the rules for game trees?

Nim

- If the only option is to leave your opponent with an N , then it is a P .
- If you can leave your opponent with a P , then it is an N .



What are the rules for game trees?

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- If the only option is to leave your opponent with an N , then it is a P .
- If you can leave your opponent with a P , then it is an N .

Game trees are useful, but they take a long time to draw out.
For example, consider the game tree for:

$$53 \oplus 242 \oplus 21 \oplus 5 \oplus 13 \oplus 241$$

It will take a **very** long time to draw this out.
For some games, including Nim, there is a better way.



First, some quick maths review of base systems and binary.
In base 10 (decimal) the number 1452 is really just short for:

$$1452_{10} = 1 \times 10^3 + 4 \times 10^2 + 5 \times 10^1 + 2 \times 10^0$$

In base 2 (binary), we only have the digits 0 and 1. So:

$$1001_2 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 9_{10}$$



Binary and Nim

Nim

Take the Nim game $2 \oplus 4 \oplus 7 \oplus 9 \oplus 15$.

Convert the numbers to binary:

$$2_{10} = 10_2$$

$$4_{10} = 100_2$$

$$7_{10} = 111_2$$

$$9_{10} = 1001_2$$

$$15_{10} = 1111_2$$



Sum the binary digits

Nim

$$\begin{array}{ccccccccc} 2_{10} & \oplus & 4_{10} & \oplus & 3_{10} & \oplus & 9_{10} & \oplus & 15_{10} \\ 10_2 & \oplus & 100_2 & \oplus & 111_2 & \oplus & 1001_2 & \oplus & 1111_2 \end{array}$$

$$\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 2 & 3 & 3 & 3 \end{array}$$



Sum the binary digits

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If any column sum is odd, it is a next player win. If no column sums are odd, it is a previous player win.



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How do we apply this?



Applying the general Nim solution

Nim

Let's go through the following game together on the board, keeping in mind that we want to leave the other player with no odd column sums.



Applying the general Nim solution

Nim

Let's go through the following game together on the board, keeping in mind that we want to leave the other player with no odd column sums.

$$2 \oplus 5 \oplus 4$$



Proof of general Nim solution

Nim

The proof is left as an exercise to the reader, but I'll give two hints:

- It is **not** a coincidence that the \oplus symbol used in Nim is the same symbol used for XOR in boolean algebra.
- You can prove it by induction.



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The proof is left as an exercise to the reader, but I'll give two hints:

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- You can prove it by induction.

If there is time, we can take it up at the end.



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Chomp!

Math has never been this exciting (or delicious)



What's the game?

Chomp!

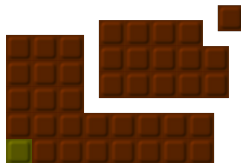
Chomp is played on a rectangular grid, such as squares of a candy bar. The lower left square is considered **poison**. Players take turns picking a square. With each choice, all squares above and to the right of the picked square are no longer available – they are eaten. The person forced to take the **poison** square loses.



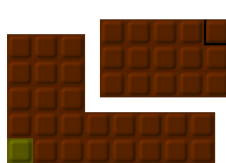
(a)



(b)



(c)

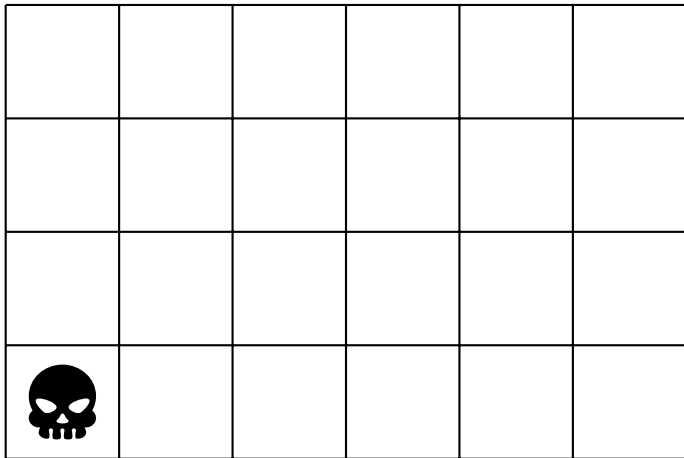


(d)



Let's play! (4×6 board)

Chomp!



Is there a winning strategy? (Or, why this game so cool.)

Chomp!

Yes!



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The real question is...



What is the winning strategy?

Chomp!

- Does anyone know one right away?



What is the winning strategy?

Chomp!

- Does anyone know one right away?
- No!



What is the winning strategy?

Chomp!

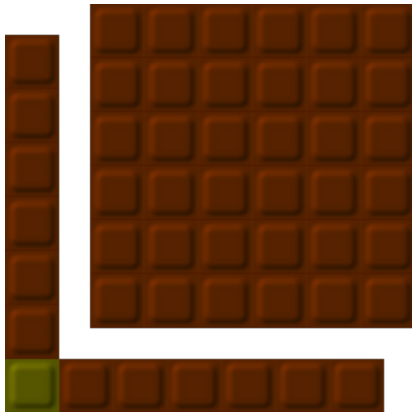
- Does anyone know one right away?
- No!
- Let's analyze some cases!



$n \times n$ grid

Chomp!

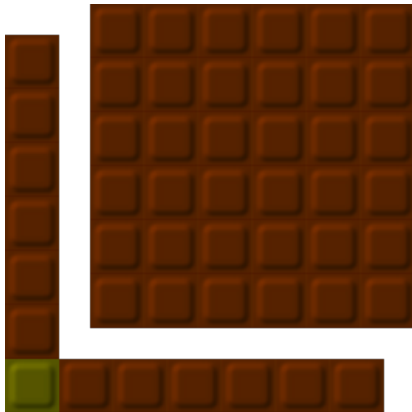
- What's the strategy here?



$n \times n$ grid

Chomp!

- What's the strategy here?
- Make an "L", and then take symmetrical moves!



$2 \times n$ grid

Chomp!

- What's the strategy here?



$2 \times n$ grid

Chomp!

- What's the strategy here?
- Make sure that player 2 encounters a rectangle. . . with a square missing!



What is the winning strategy?

Chomp!



What is the winning strategy?

Chomp!

- Just because there is always a winning strategy for player 1 doesn't mean that we know what it is!



What is the winning strategy?

Chomp!

- Just because there is always a winning strategy for player 1 doesn't mean that we know what it is!
- We know the strategy for $n \times n$ and $2 \times n$, as well as particular small grids... but not all...



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- We know the strategy for $n \times n$ and $2 \times n$, as well as particular small grids... but not all...
 - In 2002, Steven Byrnes (a high school senior!!) solved the $3 \times n$ case and won over \$100 000
 - Computers can calculate winning moves for grids of reasonable size



Some cool extensions

Chomp!

- 3D or n D chomp
- Infinite/ordinal chomp:
 - Here is how player “Too” can win on a $2 \times \omega$ board

