Inequalities and Extrema

Extreme Maths

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Problem

The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

(A) $\frac{1+\sqrt{5}}{2}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** 4

AMC 12B 2016 Problem 14
Source: Art of Problem Solving

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Extrema (Minima and Maxima) are closely related with inequalities.

Minimum Case

If $f(x) \geq c$, where c is some constant, what is the minimum value of $f(x)$?

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Minimum Case

If $f(x) \geq c$, where c is some constant, what is the minimum value of $f(x)$?

Solution

The minimum value of $f(x)$ is c.
Proof:
Suppose the minimum value of $f(x)$ is less than c. This contradicts the inequality.
Suppose the minimum value of $f(x)$ is more than c. This cannot be true because c is less than this value.

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Maximum Case

If $f(x) \leq c$, where c is some constant, what is the maximum value of $f(x)$?

Extrema (Minima and Maxima) are closely related with inequalities.

Maximum Case

If $f(x) \leq c$, where c is some constant, what is the maximum value of $f(x)$?

Soltion

The maximum value of $f(x)$ is c.

Proof:

Suppose the maximum value of $f(x)$ is more than c. This contradicts the inequality.

Suppose the maximum value of $f(x)$ is less than c. This cannot be true because c is more than this value.

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A quadratic function or expression has exactly one global minimum or maximum.
In $ax^2 + bx + c$:
If $a < 0$, then there exists a global maximum.
If $a > 0$, then there exists a global minimum.
This global minimum/maximum will always be found at $(-\frac{b}{2a}, c - \frac{b^2}{4a})$.
This can be proven by completing the square.

Example Problem

Suppose that x and y are real numbers with $3x + 4y = 10$. Determine the minimum possible value of $x^2 + 16y^2$.

Euclid 2014 6B. Source: CEMC

Solution

$3x + 4y = 10$
$4y = 10 - 3x$
$16y^2 = 100 - 60x + 9x^2$

So in the other equation,

$$\begin{aligned} & x^2 + 16y^2 \\ &= x^2 + 100 - 60x + 9x^2 \\ &= 10x^2 - 60x + 100 \end{aligned}$$

Using the formula $c - \frac{b^2}{4a}$, the minimum value is 10.

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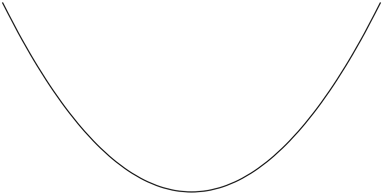
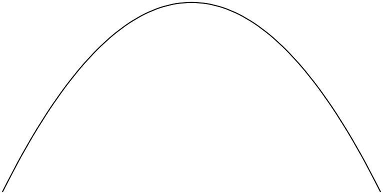
An interval of a function is **convex** if the line segment connecting any 2 points in the interval lies above or on the function.



An interval of a function is **concave** if the line segment connecting any 2 points in the interval lies below or on the function.



Convex Case

$$\frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)$$

Less formally, choosing some points on a convex curve, the average of the y-coordinates is greater than or equal to the y-coordinate of average of x-coordinates.

Concave Case

$$\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right)$$

Less formally, choosing some points on a concave curve, the average of the y-coordinates is less than or equal to the y-coordinate of average of x-coordinates.

Example

Jensen's Inequality

Problem

Prove that for all $n \in \mathbb{N}$,

$$\sqrt{1^2+1}+\sqrt{2^2+1}+...+\sqrt{n^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

Solution

Let $f(x) = \sqrt{x^2 + 1}$.
This also equals $|x| \cdot \sqrt{1 + \frac{1}{x^2}}$, which behaves like $|x|$ but has a minimum value of 1 instead of 0.

Solution

Using Jensen's Inequality, we get

$$\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

$$\frac{f(1) + f(2) + \ldots + f(n)}{n} \leq f\left(\frac{1 + 2 + \ldots + n}{n}\right)$$

$$f(1) + f(2) + \ldots + f(n) \leq f\left(\frac{1 + 2 + \ldots + n}{n}\right) \cdot n$$

Solution (continued)

$$f(1)+f(2)+\ldots+f(n)\leq f\left(\frac{1+2+\ldots+n}{n}\right)\cdot n$$

$$f(1)+f(2)+\ldots+f(n)\leq f\left(\frac{\frac{n\cdot(n+1)}{2}}{n}\right)\cdot n$$

$$f(1)+f(2)+\ldots+f(n)\leq f\left(\frac{n+1}{2}\right)\cdot n$$

$$f(1)+f(2)+\ldots+f(n)\leq \left(\sqrt{\frac{n+1}{2}}+1\right)\cdot n$$

$$\sqrt{1^2+1}+\sqrt{2^2+1}+\ldots+\sqrt{n^2+1}\leq \frac{1}{2}\sqrt{n^2+2n+5}$$

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Statement

For some 2 sequences of real numbers a_n and b_n,

$$(a_1^2+a_2^2+...+a_n^2)\cdot(b_1^2+b_2^2+...+b_n^2) \geq (a_1\cdot b_1+a_2\cdot b_2+...+a_n\cdot b_n)^2$$

Less formally, the sum of squares in a_n multiplied by the sum of squares in b_n is greater or equal to the square of the sum of the one-to-one products of a_n and b_n.

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Simple Case

$$(a^2+b^2)\cdot(c^2+d^2) \geq (ac+bd)^2$$

Problem

Suppose a, b are positive real numbers such that $a + b = 1$. Find the minimum value of $\frac{1}{a} + \frac{1}{b}$.

Solution

We cleverly use the Cauchy-Bunyakovsky-Schwarz Inequality:

$$\left((\sqrt{a})^2 + (\sqrt{b})^2\right) \cdot \left(\frac{1}{(\sqrt{a})^2} + \frac{1}{(\sqrt{b})^2}\right) \geq \left(\left((\sqrt{a}) \cdot \frac{1}{\sqrt{a}}\right) + \left((\sqrt{b}) \cdot \frac{1}{\sqrt{b}}\right)\right)^2$$

$$(a+b) \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \geq (1+1)^2$$

$$1 \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$$

$$\frac{1}{a} + \frac{1}{b} \geq 4$$

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The Arithmetic Mean - Geometric Mean inequality is as follows:

Statement

For a sequence a_n of **non-negative** numbers:

$$(a_1 + a_2 + ... + a_n) \cdot \frac{1}{n} \geq (a_1 \times a_2 \times ... \times a_n)^{\frac{1}{n}}$$

Problem

A jelly shop sells 2 sets of cuboid jellies for the same price.
The red jellies come in packs of 3 cubes, with side length a, b, and c, respectively.
The green jellies come in packs of 3 identical cuboids, each with dimesions $a \times b \times c$.
Which one should you buy?

Solution

The red jellies have a total volume of $a^3 + b^3 + c^3$.
The green jellies have a total volume of $3abc$.
By AM-GM,

$$(a^3 + b^3 + c^3) \cdot \frac{1}{3} \geq \left(a^3 b^3 c^3\right)^{\frac{1}{3}}$$

$$(a^3 + b^3 + c^3) \cdot \frac{1}{3} \geq abc$$

$$a^3 + b^3 + c^3 \geq 3abc$$

$\therefore$ You should always take the red jellies.

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AMC 12B 2016 Problem 14
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Solution

Recall that for the common ratio r and first number a,

$$S = a \cdot \frac{r^n - 1}{r - 1}$$

As n approaches ∞, only $r < 1$ will allow this to converge. As n approaches infinity, r^n becomes infinitely small, so

$$S_\infty = a \cdot \frac{0 - 1}{r - 1}$$

$$S_\infty = \frac{a}{1 - r}$$

Additionally, the second term of a geometric series is always ar, so $ar = 1$.

Solution

Given the formulas $S_\infty = \frac{a}{1-r}$ and $ar = 1$, we can rearrange the terms to get $S_\infty = \frac{1}{r(1-r)}$.

Solution

Creatively using AM-GM:

$$\left(\frac{1}{r}+\frac{1}{1-r}\right)\cdot\frac{1}{2}\geq\left(\frac{1}{r}\times\frac{1}{1-r}\right)^{\frac{1}{2}}$$

$$\frac{1}{r}+\frac{1}{1-r}\geq 2\cdot\sqrt{\frac{1}{r}\times\frac{1}{1-r}}$$

$$\frac{1}{r(1-r)}\geq 2\cdot\sqrt{\frac{1}{r(1-r)}}$$

$$S_\infty\geq 2\cdot\sqrt{S_\infty}$$

$$(S_\infty)^2\geq 4\cdot S_\infty$$

$$(S_\infty)^2-4S_\infty\geq 0$$

$$S_\infty\leq 0\cup S_\infty\geq 4$$

However, S_∞ must be positive, so its minimum possible value is 4.

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Sample Problem and solution taken from *Art of Problem Solving*.
Euclid problem from *Waterloo Centre for Education in Mathematics and Computing*.
Jensen, Cauchy-Bunyakovsky-Schwarz, AM-GM Inequalities: Statements, sample problems, solutions, from *Brilliant*.