Game Theory

Caroline Liu Vincent Macri Samantha Unger

Table of Contents

What is game theory?
Introduction

"Game theory is the study of rigging games in your favour."

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— Vincent Macri

Table of Contents

Pick Up Sticks!

The most accurately named game of all time.

Pick up sticks is a 2 player game that is often used as a game theory question on math contests. The game involves:

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Rules and intro

Pick Up Sticks!

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Variations include the winner being the one who takes the last stick, or the loser being the one who takes the last stick.

The game can be structured in such ways that certain players can win every time. Lets look at some of these scenarios!

Let's go over some scenarios.

The scenarios: your worksheet
Pick Up Sticks!

Let's go over some scenarios.

Scenario 1 Whoever picks the last stick **loses**. Which player can guarantee a win?

There are 24 sticks in total, and the players can take up to 4 sticks.

The scenarios: your worksheet

Pick Up Sticks!

Let's go over some scenarios.

Scenario 1 Whoever picks the last stick **loses**. Which player can guarantee a win?

There are 24 sticks in total, and the players can take up to 4 sticks.

Scenario 2 Whoever picks the last stick **wins**. Which player can guarantee a win?

There are 35 sticks, and the players can take up to 4 sticks on their turns.

Try it out yourself!

Answers: The Strats

Pick Up Sticks!

Let's take those scenarios up together.

Answers: The Strats

Pick Up Sticks!

Let's take those scenarios up together.

Scenario 1 Player 1 always wins. Why?

Let's take those scenarios up together.

- Scenario 1 Player 1 always wins. Why?
- Scenario 2 Player 2 always wins. Why?

Table of Contents

Nim

Vincent's favourite impartial combinatorial game.

What is an impartial combinatorial game?
Nim

- 2 players
- Players alternate turns
- Both players can see everything (this rules out most card games)
- No chance (this rules out most board games)
- Players can make the same moves – the game is **impartial** (this rules out games such as checkers and chess)

Nim is similar to **pick up sticks**, but with more piles.

What is Nim?

Nim

Nim is similar to **pick up sticks**, but with more piles.

- In Nim, there are n piles, with k_i stones in each pile.
- A player can take **as many stones as they want** (as long as they take at least one) from only **one pile** on their turn.
- We will be looking at the version where the person who picks up the last stone wins, but Nim can also be played where the person who picks up last loses.

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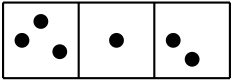
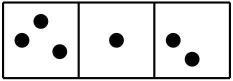
Here is an example Nim game:



Here: $n = 3$, $k_1 = 3$, $k_2 = 1$, and $k_3 = 2$.

Some Nim notation

Instead of drawing out all the boxes with circles in them, we use this notation, because it's shorter and mathematicians are lazy:



becomes

$$3 \oplus 1 \oplus 2$$

Let's play on the board!

Nim

$$4 \oplus 5 \oplus 8$$

Let's play on the board!

Nim

$$4 \oplus 5 \oplus 8$$

$$5 \oplus 7$$

$$4 \oplus 5 \oplus 8$$

$$5 \oplus 7$$

$$2 \oplus 2$$

$$4 \oplus 5 \oplus 8$$

$$5 \oplus 7$$

$$2 \oplus 2$$

Now, let's look at how to solve Nim.

Usually in game theory, we use the following notation:

- N The next player wins
- P The previous player wins

We write who wins a give game as an ordered pair.
For example:

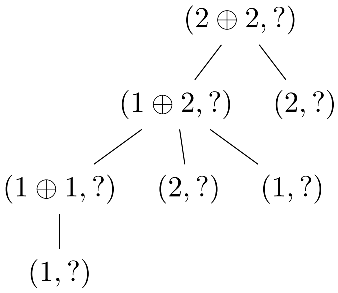
$$(1, N)$$

$$(1 \oplus 1, P)$$

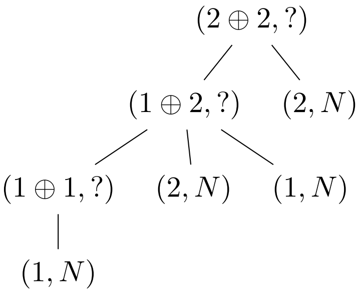
Game trees
Nim

$$(2 \oplus 2, ?)$$

Game trees
Nim

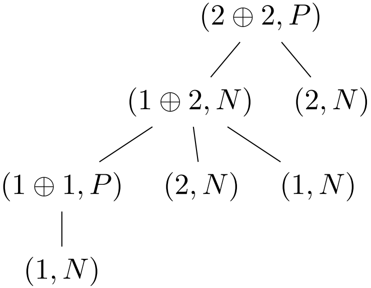


Game trees
Nim



Game trees

Nim



What are the rules for game trees?
Nim

- If the only option is to leave your opponent with an N, then it is a P.
- If you can leave your opponent with a P, then it is an N.

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Game trees are useful, but they take a long time to draw out.
For example, consider the game tree for:

$$53 \oplus 242 \oplus 21 \oplus 5 \oplus 13 \oplus 241$$

It will take a **very** long time to draw this out.
For some games, including Nim, there is a better way.

First, some quick maths review of base systems and binary.
In base 10 (decimal) the number 1452 is really just short for:

$$1452_{10} = 1 \times 10^3 + 4 \times 10^2 + 5 \times 10^1 + 2 \times 10^0$$

In base 2 (binary), we only have the digits 0 and 1. So:

$$1001_2 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 9_{10}$$

Take the Nim game $2 \oplus 4 \oplus 7 \oplus 9 \oplus 15$.
Convert the numbers to binary:

$$2_{10} = 10_2$$

$$4_{10} = 100_2$$

$$7_{10} = 111_2$$

$$9_{10} = 1001_2$$

$$15_{10} = 1111_2$$

$$
\begin{array}{ccccccccc}
2_{10} & \oplus & 4_{10} & \oplus & 3_{10} & \oplus & 9_{10} & \oplus & 15_{10} \\
10_2 & \oplus & 100_2 & \oplus & 111_2 & \oplus & 1001_2 & \oplus & 1111_2
\end{array}
$$

$$
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
\hline
2 & 3 & 3 & 3
\end{array}
$$

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If any column sum is odd, it is a next player win. If no column sums are odd, it is a previous player win.

Sum the binary digits

Nim

$$
\begin{array}{ccccccccc}
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If any column sum is odd, it is a next player win. If no column sums are odd, it is a previous player win.

How do we apply this?

Let's go through the following game together on the board, keeping in mind that we want to leave the other player with no odd column sums.

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$$2 \oplus 5 \oplus 4$$

Proof of general Nim solution

The proof is left as an exercise to the reader, but I'll give two hints:

- It is **not** a coincidence that the $\oplus$ symbol used in Nim is the same symbol used for XOR in boolean algebra.
- You can prove it by induction.

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- You can prove it by induction.

If there is time, we can take it up at the end.

Table of Contents

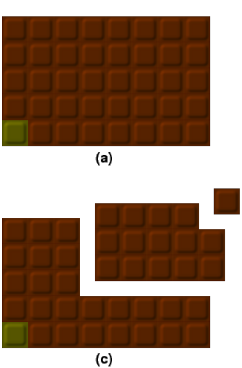
The third game

Chomp!

Math has never been this exciting (or delicious) 🍫

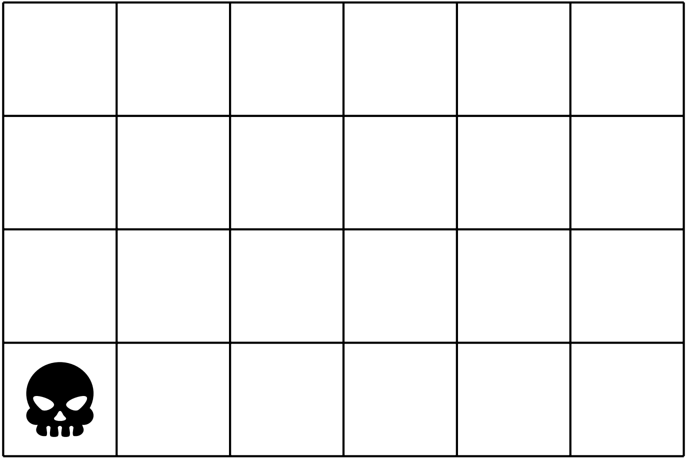
What's the game?
Chomp!

Chomp is played on a rectangular grid, such as squares of a candy bar. The lower left square is considered poison. Players take turns picking a square. With each choice, all squares above and to the right of the picked square are no longer available – they are eaten. The person forced to take the poison square loses.



Let's play! (4×6 board)

Chomp!



Is there a winning strategy? (Or, why this game so cool.)

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The real question is. . .

What is the winning strategy?

Chomp!

- Does anyone know one right away?

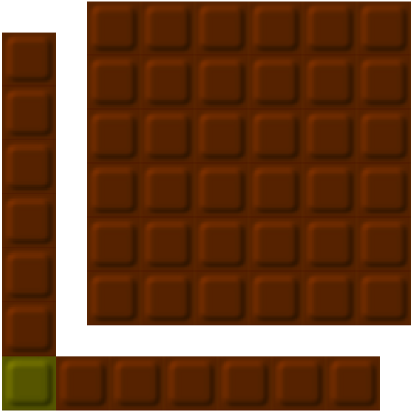
What is the winning strategy?

Chomp!

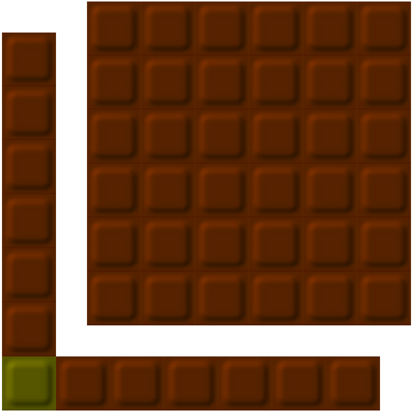
- Does anyone know one right away?
- No!

- Does anyone know one right away?
- No!
- Let's analyze some cases!

- What's the strategy here?



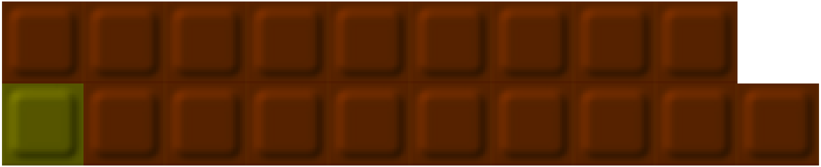
- What's the strategy here?
- Make an "L", and then take symmetrical moves!



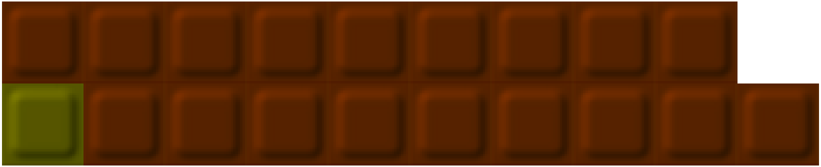
$2 \times n$ grid

Chomp!

- What's the strategy here?



- What's the strategy here?
- Make sure that player 2 encounters a rectangle... with a square missing!



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Chomp!

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Chomp!

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Chomp!

- Just because there is always a winning strategy for player 1 doesn't mean that we know what it is!
- We know the strategy for $n \times n$ and $2 \times n$, as well as particular small grids… but not all…

What is the winning strategy?

Chomp!

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  - In 2002, Steven Byrnes (a high school senior!!) solved the $3 \times n$ case and won over $\$100\,000$

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  - In 2002, Steven Byrnes (a high school senior!!) solved the $3 \times n$ case and won over $\$100\,000$
  - Computers can calculate winning moves for grids of reasonable size

Some cool extensions

Chomp!

- 3D or nD chomp
- Infinite/ordinal chomp:
  - Here is how player "Too" can win on a $2 \times \omega$ board