1 Quaternions

For this section, consider the group $(\mathcal{Q}_8, \times)$, where $\mathcal{Q}_8$ is the set of quaternion elements. That is, $\mathcal{Q}_8 \coloneqq \{-1, 1, -i, i, -j, j, -k, k\}$, and multiplication has the following additional rules:

$$i^2 = j^2 = k^2 = -1 \qquad ij = k$$

1.1 ji

What is ji? Remember, multiplication is not commutative for quaternions, so $ij \neq k$!

1.2 Multiplication table

Draw out the multiplication table for this group:

$\times$	-1	1	$-i$	i	$-j$	j	$-k$	k
-1								
1								
$-i$								
i								
$-j$								
j								
$-k$								
k								

2 Subgroups of $(\mathbb{Z}, +)$

2.1 $2\mathbb{Z}$

We will say that the subset $2\mathbb{Z} \subset \mathbb{Z}$, where $2\mathbb{Z} \coloneqq \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$. That is, $2\mathbb{Z}$ is the set of even integers.

We will call $(2\mathbb{Z}, +)$ a subgroup of $(\mathbb{Z}, +)$ is $(2\mathbb{Z}, +)$ is also a group. Is it a group? Show that it either does or doesn't satisfy all four group axioms.

2.2 $\{-4, -2, 0, 2, 4\}$

Is $(\{-4, -2, 0, 2, 4\}, +)$ a subgroup of $(\mathbb{Z}, +)$? Show that it either does or doesn't satisfy all four group axioms.

3 Order of a group

Similar to the cardinality of a set S, the order of a group $(G, \cdot)$ is defined as the number of elements in G. We write this as $|G|$.

3.1 $(\mathbb{Z}_{12}, +)$

For the group $(\mathbb{Z}_{12}, +)$, what is $|\mathbb{Z}_{12}|$?

4 Equivalent groups

Fill out the follow multiplication tables.

While they are called multiplication tables, we still use the group's operation, which may or may not be multiplication.

4.1 $(\mathbb{Z}_6^*, \times)$

Remember, $\mathbb{Z}_6^* \coloneqq \{1, 5\}$.

$\times$	1	5
1		
5		

4.2 $(\mathbb{Z}_2, +)$

Remember, $\mathbb{Z}_2 \coloneqq \{0, 1\}$.

$+$	0	1
0		
1		

4.3 $(\{a, b\}, \cdot)$

You may have noticed a pattern by now for groups with an order of 2. Without knowing the operation or the elements, fill in the following:

$\cdot$	a	b
a		
b		

Remember that a group must have an identity. Based on how you filled the above tables, is a or b the identity element in this final question?

