Euclid Preparation 1

Logarithms, Exponents, Functions, and Equations

Vincent Macri

William Lyon Mackenzie C.I. Math Club

Workshop Overview

Part I:

1. Exponents
   - Overview
   - Exponent problems
2. Logarithms
   - Overview
   - Logarithm problems

Part II:

3. Parabolas
   - Overview
   - Parabola problems
4. Polynomials
   - Overview
   - Polynomial problems

Part I

Logarithms and Exponents

Table of Contents

When $a, b, x, y \in \mathbb{R}$ and $n \in \mathbb{R} \mid n \neq 0$:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^0 = 1 \text{ if } a \neq 0$$

$$a^{-x} = \frac{1}{a^x} \text{ if } a \neq 0$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x \text{ if } b \neq 0$$

0^0 is not defined.

$$\frac{a^x}{a^y} = a^{x-y} \text{ if } a \neq 0$$

$$(a^x)^y = a^{xy}$$

$$a^x \cdot b^x = (ab)^x$$

$$a^x a^y = a^{x+y}$$

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

Exponents problem 1

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

Solution

$$9(1 + 7^2)7^k = 5^m(5^3 + 1)$$

Exponents problem 1

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

Solution

$$9(1 + 7^2)7^k = 5^m(5^3 + 1)$$

$$(450)7^k = 5^m(126)$$

Exponents problem 1

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

Solution

$$9(1 + 7^2)7^k = 5^m(5^3 + 1)$$

$$(450)7^k = 5^m(126)$$

$$(2 \cdot 3^2 \cdot 5^2)7^k = 5^m(2 \cdot 3^2 \cdot 7)$$

Exponents problem 1

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

Solution

$$\begin{aligned} 9(1 + 7^2)7^k &= 5^m(5^3 + 1) \\ (450)7^k &= 5^m(126) \\ (2 \cdot 3^2 \cdot 5^2)7^k &= 5^m(2 \cdot 3^2 \cdot 7) \\ 5^2 \cdot 7^k &= 5^m \cdot 7 \end{aligned}$$

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

Solution

$$\begin{aligned} 9(1+7^2)7^k &= 5^m(5^3+1) \\ (450)7^k &= 5^m(126) \\ (2\cdot 3^2\cdot 5^2)7^k &= 5^m(2\cdot 3^2\cdot 7) \\ 5^2\cdot 7^k &= 5^m\cdot 7 \end{aligned}$$

Since the integer factorization of numbers is always unique and both m and k are integers:

Exponents problem 1

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

Solution

$$\begin{aligned} 9(1+7^2)7^k &= 5^m(5^3+1) \\ (450)7^k &= 5^m(126) \\ (2 \cdot 3^2 \cdot 5^2)7^k &= 5^m(2 \cdot 3^2 \cdot 7) \\ 5^2 \cdot 7^k &= 5^m \cdot 7 \end{aligned}$$

Since the integer factorization of numbers is always unique and both m and k are integers: $m = 2$ and $k = 1$.

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$m = \pm\sqrt{5}$$

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$\begin{aligned} m &= \pm\sqrt{5} \\ n &= (\pm\sqrt{5})^5 \end{aligned}$$

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$m = \pm\sqrt{5}$$
$$n = (\pm\sqrt{5})^5$$

$$mn = \qquad = \qquad = \qquad = \quad =$$

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$\begin{aligned} m &= \pm\sqrt{5} \\ n &= (\pm\sqrt{5})^5 \\ mn &= \left(\pm\sqrt{5}\right)^6 = \quad = \quad = \quad = \end{aligned}$$

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$\begin{aligned} m &= \pm\sqrt{5} \\ n &= (\pm\sqrt{5})^5 \\ mn &= \left(\pm\sqrt{5}\right)^6 = \left(\sqrt{5}\right)^6 = \qquad\qquad = \quad = \end{aligned}$$

Exponents problem 2

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$m = \pm\sqrt{5}$$
$$n = (\pm\sqrt{5})^5$$
$$mn = \left(\pm\sqrt{5}\right)^6 = \left(\sqrt{5}\right)^6 = \left(\sqrt{5}\cdot\sqrt{5}\right)^3 = \quad =$$

Exponents problem 2

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$m = \pm\sqrt{5}$$

$$n = (\pm\sqrt{5})^5$$

$$mn = \left(\pm\sqrt{5}\right)^6 = \left(\sqrt{5}\right)^6 = \left(\sqrt{5}\cdot\sqrt{5}\right)^3 = 5^3 =$$

Exponents problem 2

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$\begin{aligned} m &= \pm\sqrt{5} \\ n &= (\pm\sqrt{5})^5 \\ mn &= \left(\pm\sqrt{5}\right)^6 = \left(\sqrt{5}\right)^6 = \left(\sqrt{5}\cdot\sqrt{5}\right)^3 = 5^3 = 125 \end{aligned}$$

Table of Contents

When $a, x, y \in \mathbb{R} \mid a, x, y \neq 0$:

$$\log_a(xy) = log_a x + log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^y) = y \log_a x$$
$$\log_a(a^x) = a^{\log_a x} = x$$

$$\log_a 1 = 0$$
$$\log_a x = \frac{1}{\log_x a}$$
$$\frac{\log_a x}{\log_a y} = \log_y x$$

Formulas
Logarithms

When $a, x, y \in \mathbb{R} \mid a, x, y \neq 0$:

$$\log_a(xy) = log_a x + log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^y) = y \log_a x$$
$$\log_a(a^x) = a^{\log_a x} = x$$

$$\log_a 1 = 0$$
$$\log_a x = \frac{1}{\log_x a}$$
$$\frac{\log_a x}{\log_a y} = \log_y x$$

Also, $\log_b c$ has the restrictions:

$$b \in \mathbb{R} \mid b > 0 \text{ and } b \neq 1$$
$$c \in \mathbb{R} \mid c > 0$$

When $a, x, y \in \mathbb{R} \mid a, x, y \neq 0$:

$$\log_a(xy) = log_a x + log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^y) = y \log_a x$$
$$\log_a(a^x) = a^{\log_a x} = x$$

$$\log_a 1 = 0$$
$$\log_a x = \frac{1}{\log_x a}$$
$$\frac{\log_a x}{\log_a y} = \log_y x$$

Also, $\log_b c$ has the restrictions:

$$b \in \mathbb{R} \mid b > 0 \text{ and } b \neq 1$$
$$c \in \mathbb{R} \mid c > 0$$

Finally, if $f(x) = a^x$ then $f^{-1} = \log_a(x)$. That is, the exponential and logarithmic functions are each other's inverses. More formally:

$$y = a^x \iff x = \log_a y$$

Logarithms problem 1

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y)=\log_5(2x)+\log_5(2y)$.

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

Solution

First, we state our **restrictions**:

Logarithms problem 1
Logarithms

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

Solution

First, we state our **restrictions**: $x > 0$, $y > 0$, and $x > 3y$.

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

Solution

First, we state our **restrictions**: $x > 0$, $y > 0$, and $x > 3y$.

$$2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$$
$$\log_5(x-3y)^2 = \log_5(4xy)$$

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

Solution

First, we state our **restrictions**: $x > 0$, $y > 0$, and $x > 3y$.

$$\begin{aligned} 2\log_5(x-3y) &= \log_5(2x) + \log_5(2y) \\ \log_5(x-3y)^2 &= \log_5(4xy) \end{aligned}$$

We know that the logarithmic function is an **injective function**.

Logarithms problem 1

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

Solution

First, we state our **restrictions**: $x > 0$, $y > 0$, and $x > 3y$.

$$\begin{aligned} 2\log_5(x-3y) &= \log_5(2x) + \log_5(2y) \\ \log_5(x-3y)^2 &= \log_5(4xy) \end{aligned}$$

We know that the logarithmic function is an **injective function**.

An **injective function** is one where $f(a) = b$ is only true for one value of a. More formally:

$f \colon A \to B$ is injective if $\forall a, b \in A, f(a) = f(b) \implies a = b$

Solution

Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

Solution

Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

$$\begin{aligned} \log_5(x-3y)^2 &= \log_5(4xy) \\ (x-3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \end{aligned}$$

Solution

Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

$$\begin{aligned} \log_5 (x-3y)^2 &= \log_5(4xy) \\ (x-3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \end{aligned}$$

$$\begin{aligned} x^2 - 10xy + 9y^2 &= 0 \\ x^2 - xy - 9xy + 9y^2 &= 0 \\ (x-y)(x-9y) &= 0 \end{aligned}$$

Solution

Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

$$\begin{aligned} \log_5(x-3y)^2 &= \log_5(4xy) \\ (x-3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \end{aligned}$$

$$\begin{aligned} x^2 - 10xy + 9y^2 &= 0 \\ x^2 - xy - 9xy + 9y^2 &= 0 \\ (x-y)(x-9y) &= 0 \end{aligned}$$

From here we have two cases:

Solution

Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

$$\begin{aligned}
\log_5(x-3y)^2 &= \log_5(4xy) \\
(x-3y)^2 &= 4xy \\
x^2 - 6xy + 9y^2 &= 4xy
\end{aligned}$$

$$\begin{aligned}
x^2 - 10xy + 9y^2 &= 0 \\
x^2 - xy - 9xy + 9y^2 &= 0 \\
(x-y)(x-9y) &= 0
\end{aligned}$$

From here we have two cases:

$$\begin{aligned}
x - y &= 0 \\
x &= y
\end{aligned}$$

But this violates our restriction $x > 3y$, so the answer must be found via the next case.

Logarithms problem 1 solution continued

Solution

Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

$$\begin{aligned} \log_5(x-3y)^2 &= \log_5(4xy) \\ (x-3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \end{aligned}$$

$$\begin{aligned} x^2 - 10xy + 9y^2 &= 0 \\ x^2 - xy - 9xy + 9y^2 &= 0 \\ (x-y)(x-9y) &= 0 \end{aligned}$$

From here we have two cases:

$$\begin{aligned} x - y &= 0 \\ x &= y \end{aligned}$$

But this violates our restriction $x > 3y$, so the answer must be found via the next case.

$$\begin{aligned} x - 9y &= 0 \\ x &= 9y \\ \therefore \frac{x}{y} &= 9 \end{aligned}$$

Problem

Determine the points of intersection of the curves $y = \log_{10}(x - 2)$ and $y = 1 - \log_{10}(x + 1)$.

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**:

Logarithms problem 2
Logarithms

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Next, we simply equate the two curves:

$$\log_{10}(x-2) = 1 - \log_{10}(x+1)$$

Logarithms problem 2
Logarithms

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Next, we simply equate the two curves:

$$\log_{10}(x-2) = 1 - \log_{10}(x+1)$$
$$\log_{10}(x-2) + \log_{10}(x+1) = 1$$

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Next, we simply equate the two curves:

$$\begin{aligned} \log_{10}(x-2) &= 1 - \log_{10}(x+1) \\ \log_{10}(x-2) + \log_{10}(x+1) &= 1 \\ \log_{10}\big((x-2)(x+1)\big) &= 1 \end{aligned}$$

Logarithms problem 2 solution continued

Solution

$$\log_{10}\left((x-2)(x+1)\right) = 1$$

Solution

$$\log_{10}\big((x-2)(x+1)\big) = 1$$
$$(x-2)(x+1) = 10$$

Solution

$$
\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10
\end{aligned}
$$

Solution

$$
\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10 \\
x^2 - x - 12 &= 0
\end{aligned}
$$

Logarithms problem 2 solution continued

Solution

$$\log_{10}\left((x-2)(x+1)\right) = 1$$
$$(x-2)(x+1) = 10$$
$$x^2 - x - 2 = 10$$
$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$

Solution

$$\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10 \\
x^2 - x - 12 &= 0 \\
(x-4)(x+3) &= 0
\end{aligned}$$

So we get $x = 4, -3$.

Solution

$$
\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10 \\
x^2 - x - 12 &= 0 \\
(x-4)(x+3) &= 0
\end{aligned}
$$

So we get $x = 4, -3$. We have the restriction $x > 2$, so we are left with $x = 4$.

Solution

$$
\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10 \\
x^2 - x - 12 &= 0 \\
(x-4)(x+3) &= 0
\end{aligned}
$$

So we get $x = 4, -3$. We have the restriction $x > 2$, so we are left with $x = 4$. This leaves us with the point of intersection $(4, \log_{10} 2)$.

Logarithms problem 3

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**:

Logarithms problem 3

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Logarithms problem 3

Logarithms

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\log_2(9 - 2^x) = 3 - x$$

Logarithms problem 3

Logarithms

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\begin{aligned} \log_2(9 - 2^x) &= 3 - x \\ 9 - 2^x &= 2^{3-x} \end{aligned}$$

Logarithms problem 3

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\begin{aligned} \log_2(9 - 2^x) &= 3 - x \\ 9 - 2^x &= 2^{3-x} \\ 9 - 2^x &= \frac{2^3}{2^x} \end{aligned}$$

Logarithms problem 3

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\begin{aligned}
\log_2(9 - 2^x) &= 3 - x \\
9 - 2^x &= 2^{3-x} \\
9 - 2^x &= \frac{2^3}{2^x} \\
9 - 2^x &= \frac{8}{2^x}
\end{aligned}$$

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

Logarithms problem 2 solution continued

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

$$9 - y = \frac{8}{y}$$

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

$$\begin{aligned} 9 - y &= \frac{8}{y} \\ -y^2 + 9y - 8 &= 0 \end{aligned}$$

Logarithms problem 2 solution continued

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

$$\begin{aligned} 9 - y &= \frac{8}{y} \\ -y^2 + 9y - 8 &= 0 \\ y &= 1, 8 \end{aligned}$$

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

$$\begin{aligned} 9 - y &= \frac{8}{y} \\ -y^2 + 9y - 8 &= 0 \\ y &= 1, 8 \end{aligned}$$

We substitute these solutions back into $y = 2^x$ to find that $x = 0$ or $x = 3$, both of which satisfy our restrictions.

Part II

Functions and Equations

Table of Contents

For the quadratic function $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R} \mid a \neq 0$, there are two zeroes (roots) given by the quadratic formula:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where the **discriminant** (Δ) is the value inside the square root:

$$\Delta = b^2 - 4ac$$

These roots are:

Real and distinct if $\Delta > 0$

Real and equal if $\Delta = 0$

Non-real and distinct if $\Delta < 0$

The sum of the roots of a parabola is $r_1 + r_2 = -\frac{b}{a}$.
The product of the roots of a parabola is $r_1 r_2 = \frac{c}{a}$.
We can rearrange $y = ax^2 + bx + c$ as:

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Since the vertex is halfway between the roots, the vertex is at:

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

Problem

If $x^2 - x - 2 = 0$, determine all possible vales of $1 - \frac{1}{x} - \frac{6}{x^2}$.

Parabola problem 1
Parabolas

Problem

If $x^2 - x - 2 = 0$, determine all possible vales of $1 - \frac{1}{x} - \frac{6}{x^2}$.

Solution

We can apply the quadratic formula to find the values of x.

Problem

If $x^2 - x - 2 = 0$, determine all possible vales of $1 - \frac{1}{x} - \frac{6}{x^2}$.

Solution

We can apply the quadratic formula to find the values of x.

We use $a = 1$, $b = -1$, and $c = -2$:

Problem

If $x^2 - x - 2 = 0$, determine all possible vales of $1 - \frac{1}{x} - \frac{6}{x^2}$.

Solution

We can apply the quadratic formula to find the values of x.

We use $a = 1$, $b = -1$, and $c = -2$:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_1, x_2 = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$x_1, x_2 = 2, -1$$

Parabola problem 1 solution continued

Parabolas

Solution

We were asked to find the possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.
We simply plug in our two values of x.

Solution

We were asked to find the possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.
We simply plug in our two values of x.

$x = 2$:

$$1 - \frac{1}{2} - \frac{6}{2^2} = -1$$

Solution

We were asked to find the possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.
We simply plug in our two values of x.

$x = 2$:

$$1 - \frac{1}{2} - \frac{6}{2^2} = -1$$

$x = -1$:

$$1 - \frac{1}{-1} - \frac{6}{(-1)^2} = -4$$

Parabola problem 1 solution continued

Parabolas

Solution

We were asked to find the possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.
We simply plug in our two values of x.

$x = 2$:

$$1 - \frac{1}{2} - \frac{6}{2^2} = -1$$

$x = -1$:

$$1 - \frac{1}{-1} - \frac{6}{(-1)^2} = -4$$

Therefore the possible values are -1 and -4.

Parabola problem 2

Parabolas

Problem

If the graph of the parabola $y = x^2$ is translated to a position such that its x intercepts are $-d$ and e and its y intercept is $-f$, (where $d, e, f > 0$), show that $de = f$.

Parabola problem 2

Problem

If the graph of the parabola $y = x^2$ is translated to a position such that its x intercepts are $-d$ and e and its y intercept is $-f$, (where $d, e, f > 0$), show that $de = f$.

Solution

We know the formula for a parabola given the roots:
$y = a(x - r_1)(x - r_2)$.

Problem

If the graph of the parabola $y = x^2$ is translated to a position such that its x intercepts are $-d$ and e and its y intercept is $-f$, (where $d, e, f > 0$), show that $de = f$.

Solution

We know the formula for a parabola given the roots: $y = a(x - r_1)(x - r_2)$. We can plug in $r_1 = -d$ and $r_2 = e$:

$$y = a(x + d)(x - e)$$

Problem

If the graph of the parabola $y = x^2$ is translated to a position such that its x intercepts are $-d$ and e and its y intercept is $-f$, (where $d, e, f > 0$), show that $de = f$.

Solution

We know the formula for a parabola given the roots: $y = a(x - r_1)(x - r_2)$. We can plug in $r_1 = -d$ and $r_2 = e$:

$$y = a(x + d)(x - e)$$

Since we only performed a translation, and no stretch or compression, we know that $a = 1$.

Parabola problem 2

Parabolas

Problem

If the graph of the parabola $y = x^2$ is translated to a position such that its x intercepts are $-d$ and e and its y intercept is $-f$, (where $d, e, f > 0$), show that $de = f$.

Solution

We know the formula for a parabola given the roots: $y = a(x - r_1)(x - r_2)$. We can plug in $r_1 = -d$ and $r_2 = e$:

$$y = a(x + d)(x - e)$$

Since we only performed a translation, and no stretch or compression, we know that $a = 1$.

So: $y = (x + d)(x - e)$.

Parabola problem 2 solution continued

Parabolas

Solution

$$y = (x+d)(x-e)$$

Solution

$$y = (x + d)(x - e)$$

And since the y-intercept occurs at $x = 0$, we plug in $x = 0$ and $y = -f$:

$$y = (x + d)(x - e)$$

Solution

$$y = (x + d)(x - e)$$

And since the y-intercept occurs at $x = 0$, we plug in $x = 0$ and $y = -f$:

$$\begin{aligned} y &= (x + d)(x - e) \\ -f &= (0 + d)(0 - e) \end{aligned}$$

Solution

$$y = (x + d)(x - e)$$

And since the y-intercept occurs at $x = 0$, we plug in $x = 0$ and $y = -f$:

$$\begin{aligned} y &= (x + d)(x - e) \\ -f &= (0 + d)(0 - e) \\ -f &= (d)(-e) \end{aligned}$$

Solution

$$y = (x + d)(x - e)$$

And since the y-intercept occurs at $x = 0$, we plug in $x = 0$ and $y = -f$:

$$\begin{aligned} y &= (x + d)(x - e) \\ -f &= (0 + d)(0 - e) \\ -f &= (d)(-e) \\ -f &= -de \end{aligned}$$

Solution

$$y = (x+d)(x-e)$$

And since the y-intercept occurs at $x = 0$, we plug in $x = 0$ and $y = -f$:

$$\begin{aligned} y &= (x+d)(x-e) \\ -f &= (0+d)(0-e) \\ -f &= (d)(-e) \\ -f &= -de \\ f &= de \end{aligned}$$

Q.E.D.

Parabola problem 3

Parabolas

Problem

Find all values of x such that $x + \frac{36}{x} \geq 13$.

Problem

Find all values of x such that $x + \frac{36}{x} \geq 13$.

Solution

First, we state our restriction:

Problem

Find all values of x such that $x + \frac{36}{x} \geq 13$.

Solution

First, we state our restriction: $x \neq 0$.

Problem

Find all values of x such that $x + \frac{36}{x} \geq 13$.

Solution

First, we state our restriction: $x \neq 0$.

We can also see that this inequality will certainly be false for any negative values of x. This means that $x > 0$.

Problem

Find all values of x such that $x + \frac{36}{x} \geq 13$.

Solution

First, we state our restriction: $x \neq 0$.

We can also see that this inequality will certainly be false for any negative values of x. This means that $x > 0$.

Since we know the sign of x, we can algebraically rearrange the inequality.

Solution

Next, we rearrange the inequality:

$$x + \frac{36}{x} \geq 13$$

$$x^2 + 36 \geq 13x$$

$$x^2 - 13x + 36 \geq 0$$

Solution

Next, we rearrange the inequality:

$$\begin{aligned} x + \frac{36}{x} &\geq 13 \\ x^2 + 36 &\geq 13x \\ x^2 - 13x + 36 &\geq 0 \end{aligned}$$

We can factor it as $(x-4)(x-9) \geq 0$.

Solution

Next, we rearrange the inequality:

$$x + \frac{36}{x} \geq 13$$
$$x^2 + 36 \geq 13x$$
$$x^2 - 13x + 36 \geq 0$$

We can factor it as $(x-4)(x-9) \geq 0$. After taking into account our restrictions, we arrive at:

$$0 < x \leq 4 \cup x \geq 9$$

Solution

Next, we rearrange the inequality:

$$x + \frac{36}{x} \geq 13$$
$$x^2 + 36 \geq 13x$$
$$x^2 - 13x + 36 \geq 0$$

We can factor it as $(x-4)(x-9) \geq 0$. After taking into account our restrictions, we arrive at:

$$0 < x \leq 4 \cup x \geq 9$$

More formally, we can state that: $x \in (0,4] \cup [9,\infty)$.

Table of Contents

Theorem (Remainder Theorem and Factor Theorem)

The remainder theorem states that when a polynomial $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, *of degree* n, *is divided by* $(x - k)$ *the remainder is* $p(k)$.
The factor theorem then follows: $p(k) = 0$ *if and only if* $(x - k)$ *is a factor of* $p(x)$.
A polynomial equation of degree n *has at most* n **real** *roots. It will always have* n **complex** *roots.*

Theorem (Rational Root Theorem)

The rational root theorem states that all rational roots $\frac{a}{b}$ *have the property that* a *and* b *are factors of the last and first coefficient,* a_n *and* a_0 *respectively.*

Problem

If a polynomial leaves a remainder of 5 when divided by $x - 3$ and a remainder of -7 when divided by $x + a$, what is the remainder when the polynomial is divided by $x^2 - 2x - 3$?

Problem

If a polynomial leaves a remainder of 5 when divided by $x - 3$ and a remainder of -7 when divided by $x + a$, what is the remainder when the polynomial is divided by $x^2 - 2x - 3$?

Solution

Let's examine how we we divide polynomials:

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

Where $p(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

Problem

If a polynomial leaves a remainder of 5 when divided by $x - 3$ and a remainder of -7 when divided by $x + a$, what is the remainder when the polynomial is divided by $x^2 - 2x - 3$?

Solution

Let's examine how we we divide polynomials:

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

Where $p(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

This can be rearranged as: $p(x) = d(x)q(x) + r(x)$.

Polynomial problem 1 solution continued

Polynomials

Solution

We know that (generally) when dividing a polynomial by another polynomial of degree n, the remainder will have a degree of $n-1$ (fun exercise: why did I say generally?).

Polynomial problem 1 solution continued

Solution

We know that (generally) when dividing a polynomial by another polynomial of degree n, the remainder will have a degree of $n-1$ (fun exercise: why did I say generally?). So, dividing our polynomial by x^2-2x-3 should leave a linear remainder.

Solution

We know that (generally) when dividing a polynomial by another polynomial of degree n, the remainder will have a degree of $n-1$ (fun exercise: why did I say generally?). So, dividing our polynomial by x^2-2x-3 should leave a linear remainder.

We will call the polynomial we are dividing by $p(x)$. Then:

$$p(x) = (x^2-2x-3)q(x) + ax + b$$

where $q(x)$ is the quotient, and $ax+b$ is the remainder.

Polynomial problem 1 solution continued

Solution

$$p(x) = (x^2 - 2x - 3)q(x) + ax + b$$

We can factor $x^2 - 2x - 3$ as $(x-3)(x+1)$.

$$p(x) = (x-3)(x+1)q(x) + ax + b$$

Solution

$$p(x) = (x^2 - 2x - 3)q(x) + ax + b$$

We can factor $x^2 - 2x - 3$ as $(x - 3)(x + 1)$.

$$p(x) = (x - 3)(x + 1)q(x) + ax + b$$

The remainder for when we divide by these was given in the problem statement. From the remainder theorem, we know that:

$$p(3) = 5 = ax + b = 3a + b$$

$$p(-1) = -7 = ax + b = -a + b$$

Polynomial problem 1 solution continued

Polynomials

Solution

$$\begin{array}{rrl} & 5 & = 3a + b \\ - & -7 & = -a + b \\ \hline & 12 & = 4a \\ & a & = 3 \end{array}$$

Polynomial problem 1 solution continued

Polynomials

Solution

$$
\begin{array}{rrl}
 & 5 & = 3a + b \\
- & -7 & = -a + b \\
\hline
 & 12 & = 4a \\
 & a & = 3
\end{array}
$$

$$
\begin{aligned}
-7 &= -a + b \\
-7 &= -3 + b \\
b &= -4
\end{aligned}
$$

Polynomial problem 1 solution continued

Solution

$$\begin{array}{rrl} & 5 & = 3a + b \\ - & -7 & = -a + b \\ \hline & 12 & = 4a \\ & a & = 3 \end{array}$$

$$-7 = -a + b$$
$$-7 = -3 + b$$
$$b = -4$$

Therefore, the remainder is $3x - 4$.