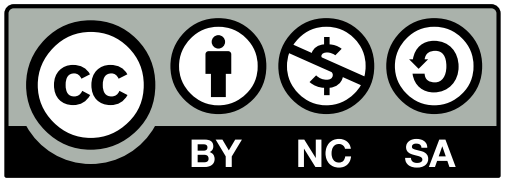
# James Tanton's Exploding Dots

Vincent Macri






# Table of Contents


1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

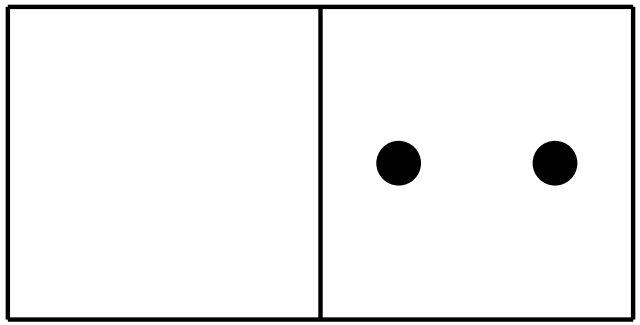
5 Obelus

6 Eks

7 Infinitia

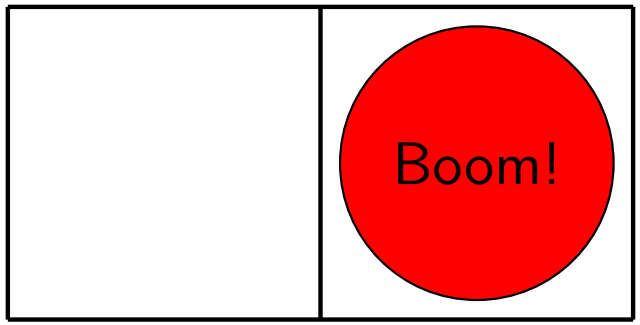

# The $2 \leftarrow 1$ machine

Mechania



# The $2 \leftarrow 1$ machine

## Mechania

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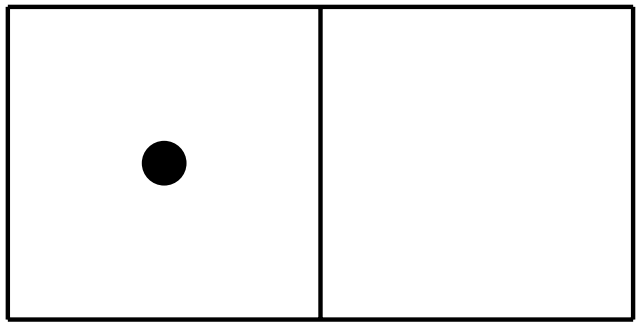
Mechania



# $2 \leftarrow 1$ examples

## Mechania

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1}$$
$$2 \xrightarrow{2\leftarrow 1}$$
$$3 \xrightarrow{2\leftarrow 1}$$
$$4 \xrightarrow{2\leftarrow 1}$$
$$5 \xrightarrow{2\leftarrow 1}$$
$$13 \xrightarrow{2\leftarrow 1}$$

# $2 \leftarrow 1$ examples

## Mechania

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$

$$2 \xrightarrow{2\leftarrow 1}$$

$$3 \xrightarrow{2\leftarrow 1}$$

$$4 \xrightarrow{2\leftarrow 1}$$

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$$13 \xrightarrow{2\leftarrow 1}$$

# $2 \leftarrow 1$ examples

## Mechania

What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$
$$2 \xrightarrow{2\leftarrow 1} 10$$
$$3 \xrightarrow{2\leftarrow 1}$$
$$4 \xrightarrow{2\leftarrow 1}$$
$$5 \xrightarrow{2\leftarrow 1}$$
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# $2 \leftarrow 1$ examples

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What are the following in a $2 \leftarrow 1$ machine?

$$1 \xrightarrow{2\leftarrow 1} 1$$
$$2 \xrightarrow{2\leftarrow 1} 10$$
$$3 \xrightarrow{2\leftarrow 1} 11$$
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## Mechania

What are the following in a $2 \leftarrow 1$ machine?

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$$4 \xrightarrow{2\leftarrow 1} 100$$
$$5 \xrightarrow{2\leftarrow 1} 101$$
$$13 \xrightarrow{2\leftarrow 1} 1101$$

# What if. . .

What if we had a $3 \leftarrow 1$ machine?

# What if. . .

What if we had a $10 \leftarrow 1$ machine?

# Table of Contents


1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia


# What are these machines?
## Insighto

These machines are another way of handling arithmetic, but they work in any base.
A $b \leftarrow 1$ machine handles numbers in base $b$.
We'll use a $10 \leftarrow 1$ machine for now.

# Table of Contents


1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia


# What next?
## Arithmos

We can count with these machines, but what else can we do?

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## Arithmos

We can count with these machines, but what else can we do?
What's the first thing you learn to do with numbers after counting?

# Addition!

# Examples
## Arithmos

What is $234 + 125$?

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$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

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What is $234 + 125$?

$$\begin{array}{cccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

What is $234 + 187$?

# Examples
## Arithmos

What is $234 + 125$?

$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

What is $234 + 187$?

$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 8 & 7 \\ \hline & 3 & 11 & 11 \end{array}$$

# Examples
## Arithmos

What is $234 + 125$?

$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 2 & 5 \\ \hline & 3 & 5 & 9 \end{array}$$

What is $234 + 187$?

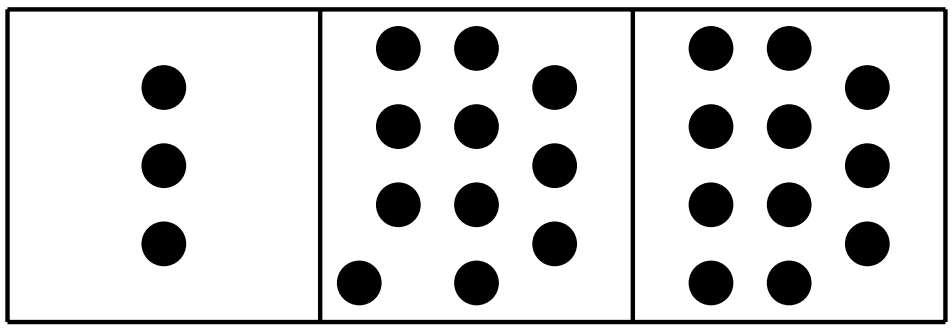
$$\begin{array}{rccc} & 2 & 3 & 4 \\ + & 1 & 8 & 7 \\ \hline & 3 & 11 & 11 \end{array}$$

Three hundred eleventy eleven!

But now society thinks I'm weird. Let's fix that.

# Explode the dots!
## Arithmos

Three hundred eleventy eleven in a $10 \leftarrow 1$ machine.



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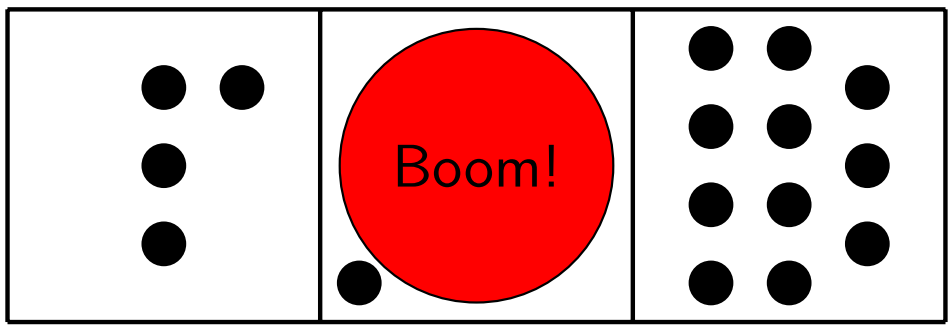


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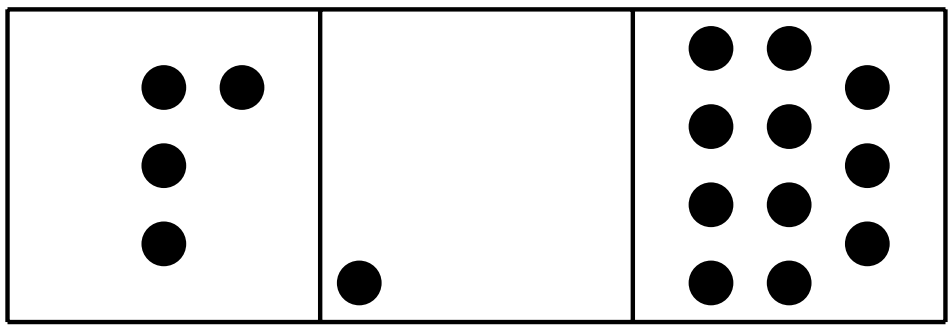

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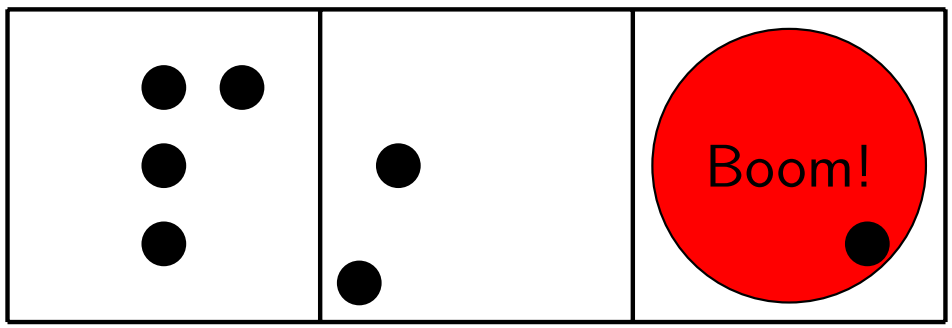


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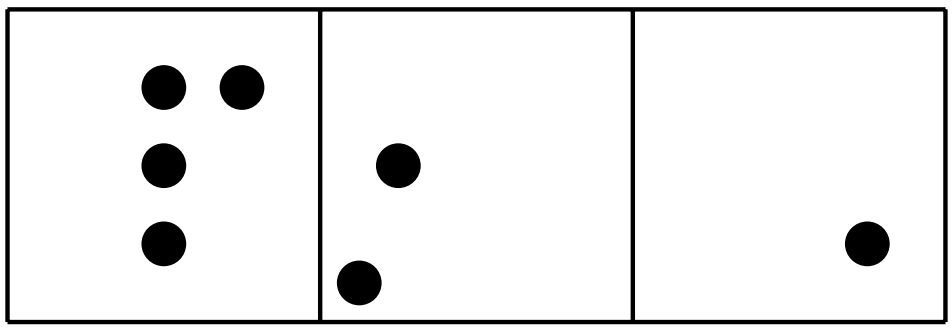


$$421$$

# What do we learn after addition?

# Multiplication!

# Examples
## Arithmos

What is $2876 \times 3$?

$$\begin{array}{cccc} 2 & 8 & 7 & 6 \\ \times & & & 3 \\ \hline 6 & 24 & 21 & 18 \end{array}$$

Let's fix this one together for society.

| | | | |
|---|---|---|---|
| 6 | 24 | 21 | 18 |

# Table of Contents


1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

7 Infinitia


# Subtraction
## Antidotia

**Theorem 1**

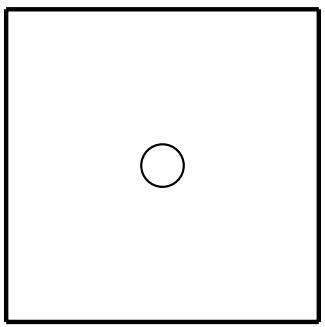
Subtraction does not exist.

**Theorem 2**

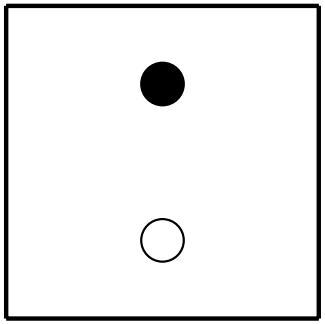
What we call subtraction is just the addition of negative numbers.
Or, subtraction is the addition of the opposite.

# The antidot
## Antidotia

The opposite of a dot is an antidot. I'll call these tods.
This is one tod in one of our machines:



# How do tods behave?
## Antidotia



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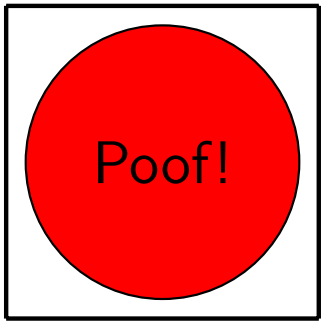


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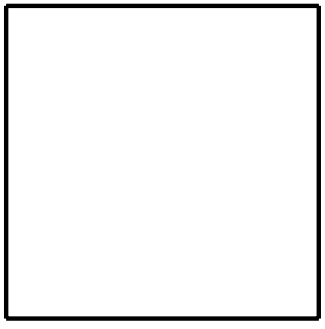

# Examples
## Antidotia

What is $564 - 123$?

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$$\begin{array}{rccc} & 5 & 6 & 4 \\ - & 1 & 2 & 3 \\ \hline & 4 & 4 & 1 \end{array}$$

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What is $564 - 123$?

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What is $441 - 254$?

# Examples
## Antidotia

What is $564 - 123$?

$$\begin{array}{rrrr} & 5 & 6 & 4 \\ - & 1 & 2 & 3 \\ \hline & 4 & 4 & 1 \end{array}$$

What is $441 - 254$?

$$\begin{array}{rrrr} & 4 & 4 & 1 \\ - & 2 & 5 & 4 \\ \hline & 2 & -1 & -3 \end{array}$$

# Examples
## Antidotia

What is $564 - 123$?

$$\begin{array}{rccc} & 5 & 6 & 4 \\ - & 1 & 2 & 3 \\ \hline & 4 & 4 & 1 \end{array}$$

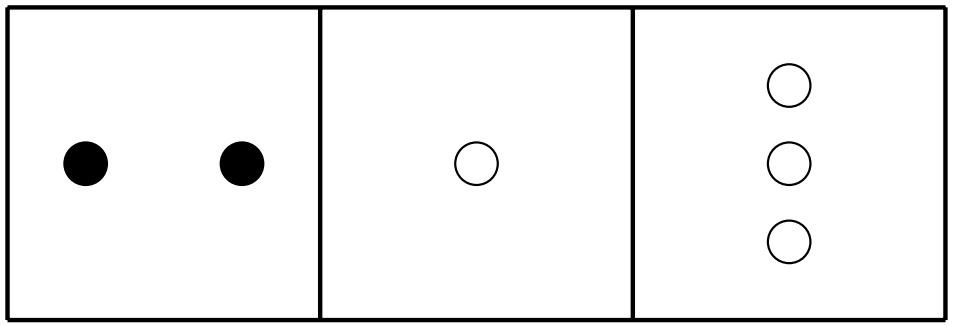
What is $441 - 254$?

$$\begin{array}{rccc} & 4 & 4 & 1 \\ - & 2 & 5 & 4 \\ \hline & 2 & -1 & -3 \end{array}$$

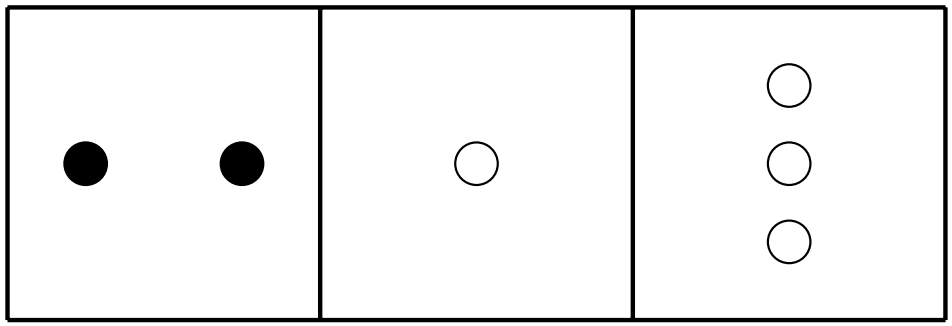
Let's fix this together on the board for society's sake. We'll use the exploding dots method.

# Another method

## Antidotia

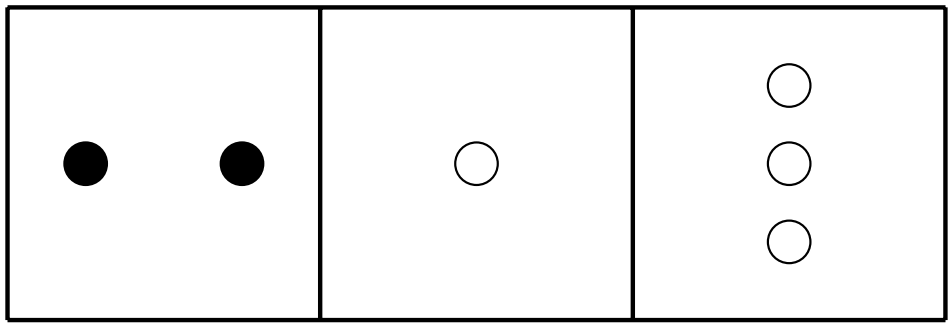


# Another method
## Antidotia



There is another way to fix this which is helpful for doing math mentally.

# Another method
## Antidotia



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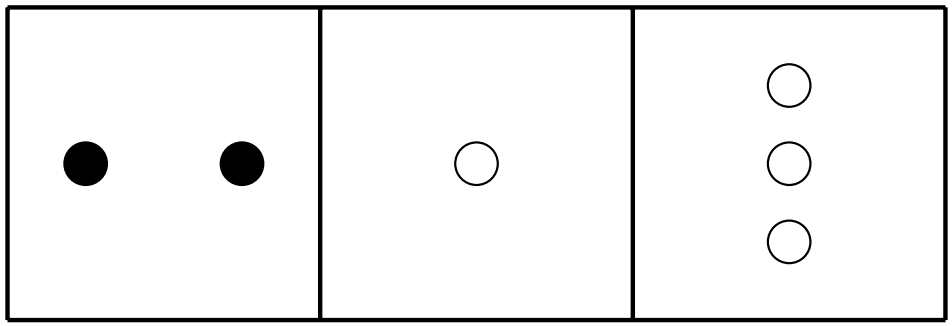
Let's look the place values:

$$200 + -10 + -3$$

# Another method
## Antidotia



There is another way to fix this which is helpful for doing math mentally.
Let's look the place values:

$$200 + -10 + -3$$

This is very easy to do mentally.

$$200 + -10 = 190 + -3$$

$$190 + -3 = 187$$

# Table of Contents


1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

5 Obelus

6 Eks

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# Division
## Obelus

Fun fact before we get started on division, did you know that the $\div$ sign is called an obelus?

# Long division method

## Obelus

What is $276 \div 12$?

Don't use a calculator!

# Long division method

## Obelus

What is $276 \div 12$?

Don't use a calculator!

$$
\begin{array}{r}
23 \\
12\overline{)\,276} \\
\underline{240} \\
36 \\
\underline{36} \\
0
\end{array}
$$

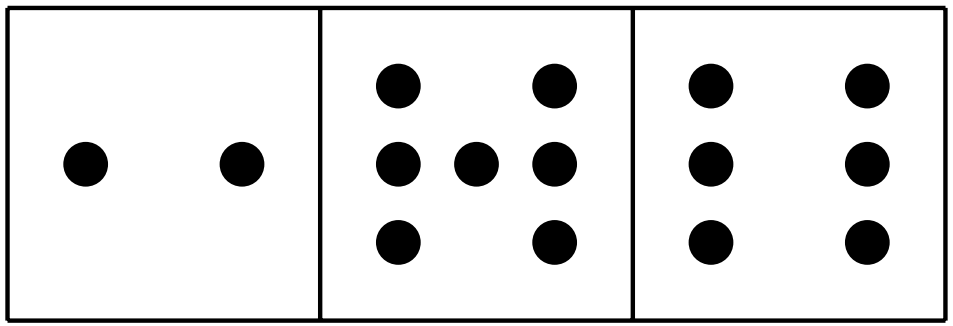
This is stupid and convoluted. Let's use exploding dots instead.

# Exploding dots method

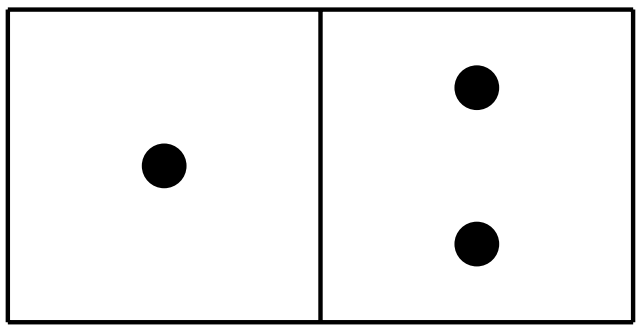
## Obelus

$$276 \div 12$$

276:



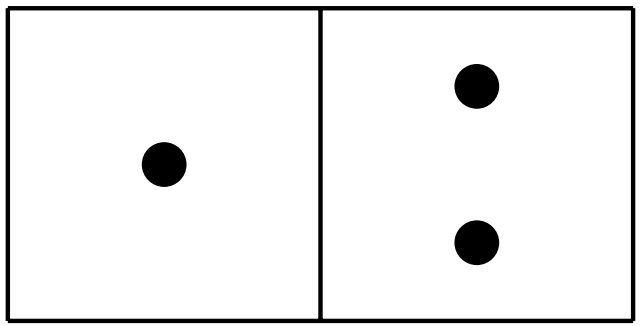
12:



Let's look for groups of 12 in 276.

# Exploding dots solution

## Obelus



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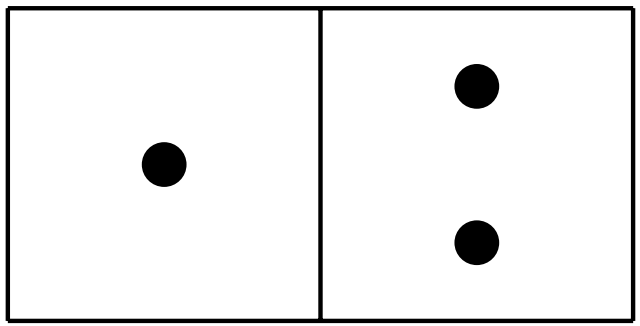
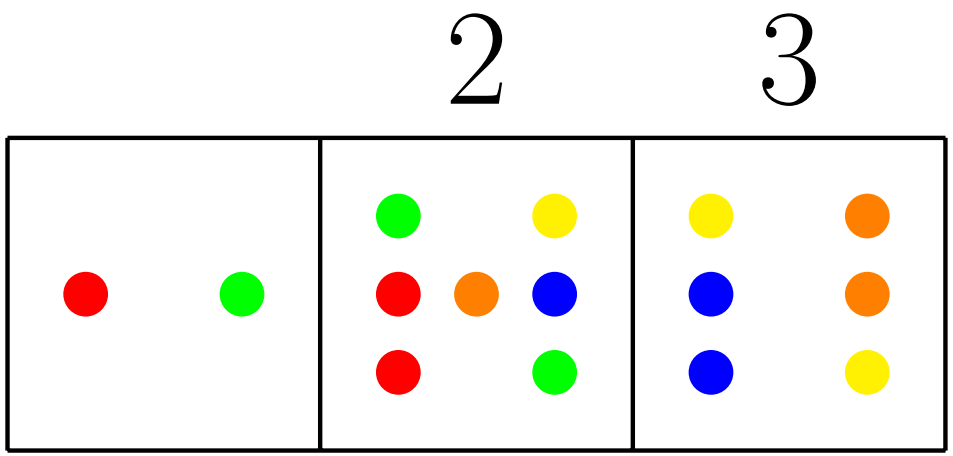
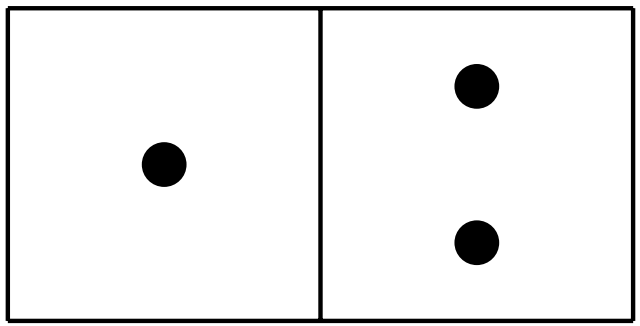
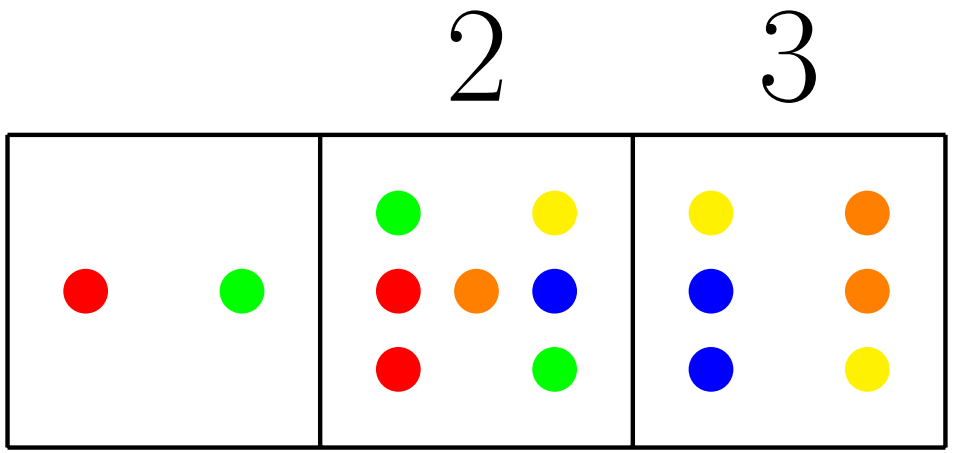





# Exploding dots solution

Obelus






$$273 \div 12 = 23$$

# Examples together
## Obelus

What is $2783 \div 23$?

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What is $2783 \div 23$? $121$

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What is $2783 \div 23$? $121$

What is $2785 \div 23$?

# Examples together

## Obelus

What is $2783 \div 23$? $121$

What is $2785 \div 23$? $121$ R2 or $121 + \frac{2}{23}$

# Table of Contents


1 Mechania

2 Insighto

3 Arithmos

4 Antidotia

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6 Eks

7 Infinitia


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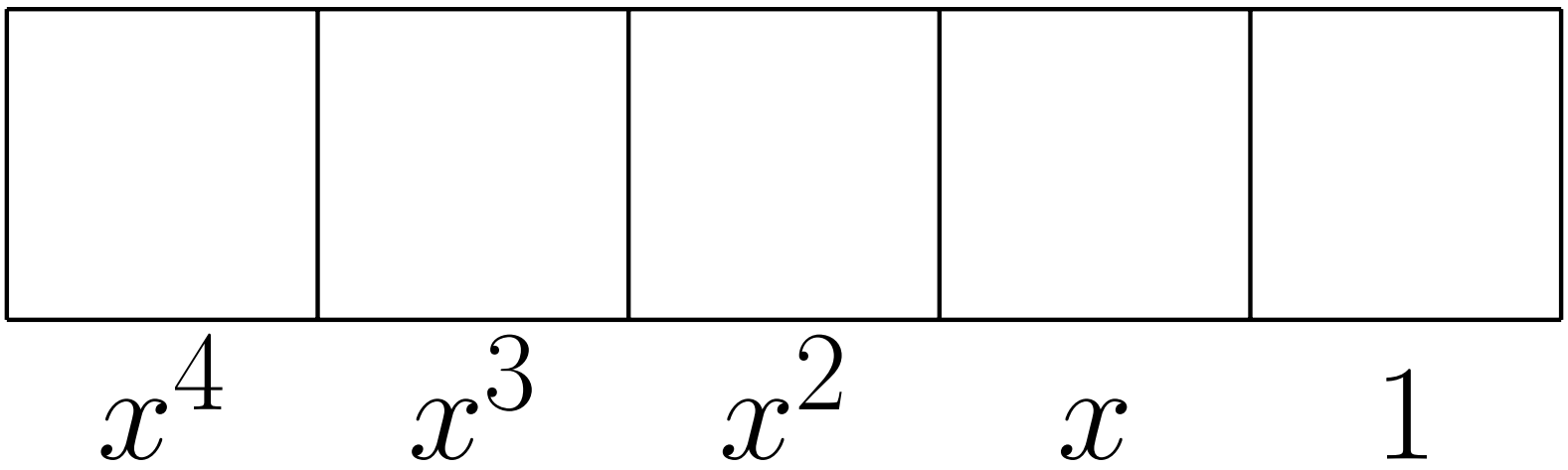


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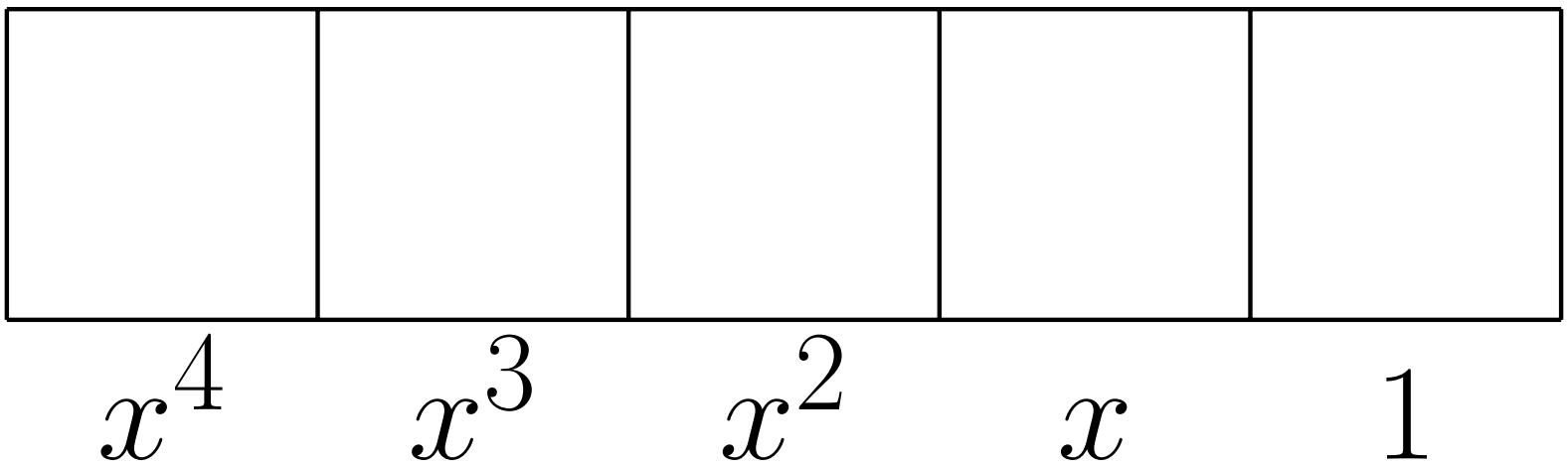


With this knowledge, we can now manipulate polynomials just like we do regular numbers.

# Polynomial division

## Eks

What is $\frac{2x^2 + 7x + 6}{x + 2}$?

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What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$?

# Polynomial division

## Eks

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$? $x^2+2x+1$

# Polynomial division

## Eks

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$? $x^2+2x+1$

What is $\dfrac{x^3-3x+2}{x+2}$?

# Polynomial division

## Eks

What is $\dfrac{2x^2+7x+6}{x+2}$? $2x+3$

What is $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$? $x^2+2x+1$

What is $\dfrac{x^3-3x+2}{x+2}$? $x^2-2x+1$

# Table of Contents


1 Mechania

2 Insighto

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# Now it gets **really** fun

## Infinitia

$$\frac{1}{1-x}$$

# Geometric series formula

## Infinitia

Using dots and boxes, we get what is knows at the geometric series formula:

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## Infinitia

Using dots and boxes, we get what is knows at the geometric series formula:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 \ldots$$

# Geometric series formula

## Infinitia

Using dots and boxes, we get what is knows at the geometric series formula:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 \ldots$$

In calculus, we would call this the Taylor series of $\frac{1}{1-x}$.

# This is really cool

## Infinitia

Using the dots and boxes method, find the Taylor series of:

$$\frac{1}{1 - x - x^2}$$

# Congratulations!

We just went from kindergarten arithmetic to advanced calculus in 40 minutes!