1 Inequalities and Extrema

If $f(x) \geq c$, where c is some constant, then the minimum value for $f(x)$ is c.

If $f(x) \leq c$, where c is some constant, then the maximum value for $f(x)$ is c.

This can be extended to any number of variables.

i.e. $f(x)$ can be replaced by any expression involving any number of variables.

Practice

What is the minimum value of $(x-y)^2+5$, where $x, y \in \mathbb{R}$?

2 Quadratics

In $f(x) = ax^2 + bx + c$:

If $a < 0$, $f(x)$ must have a local maximum.

If $a > 0$, $f(x)$ must have a local minimum.

The local extremum (the vertex) can be found using the formula:

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$$

Practice

If $\frac{3}{x} = \frac{2}{y}$, what is the minimum value of $4x + 2xy + 3y + 6y^2 + 6x^2$?

3 Jensen's Inequality

An interval of a function is **convex** if the line segment connecting any 2 points in the interval lies above or on the function.

An interval of a function is **concave** if the line segment connecting any 2 points in the interval lies below or on the function.

Jensen's Inequality states that

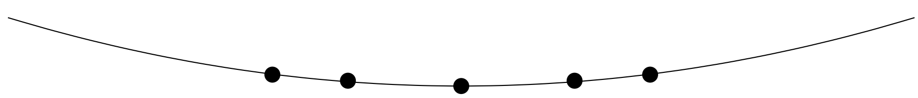
For a convex function:

$$\frac{f(x_1)+f(x_2)+...+f(x_n)}{n} \geq f\left(\frac{x_1+x_2+...+x_n}{n}\right)$$

For a concave function:

$$\frac{f(x_1)+f(x_2)+...+f(x_n)}{n} \leq f\left(\frac{x_1+x_2+...+x_n}{n}\right)$$

Practice



A symmetric catenary bridge forms a U shape. If I some equally-spaced rocks on the bridge, will the average height be higher or lower than the height of the rock in the middle?

4 Cauchy-Bunyakovsky-Schwarz Inequality

For some 2 sequences of real numbers a_n and b_n,

$$(a_1^2+a_2^2+...+a_n^2)\cdot(b_1^2+b_2^2+...+b_n^2) \geq (a_1\cdot b_1+a_2\cdot b_2+...+a_n\cdot b_n)^2$$

5 Arithmetic Mean - Geometric Mean Inequality

For a sequence a_n of **non-negative** numbers:

$$(a_1+a_2+...+a_n)\cdot\frac{1}{n} \geq (a_1\times a_2\times ...\times a_n)^{\frac{1}{n}}$$

Practice

If I have n positive numbers whose product is 1, what is the minimum possible sum of the numbers?

