

Euclid Preparation 4

Proofs

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It follows a set of logically true statements, which allow one to arrive from statement A to statement B.

They can rely on axioms, definitions, and earlier proved COMMON KNOWLEDGE theorems.

Proof by example is a sub-category of direct proofs, and is not covered in this lesson. It consists of finding a single case to a supposition in the form of “There exists N such that ...”.



Definitions

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An integer N is even if it can be expressed as $N = 2M$ where M is also an integer.



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An integer N is odd if it can be expressed as $N = 2M + 1$ where M is also an integer.



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An integer N is odd if it can be expressed as $N = 2M + 1$ where M is also an integer. Sometimes, $2M - 1$ is used.



Odd Number Squared

Direct Proof

Example

Given an odd integer N , prove that N^2 is also an odd integer.



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$$\begin{aligned} N^2 &= (2M - 1)^2 \\ &= 4M^2 - 2M + 1 \end{aligned}$$



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If N is odd, it can be expressed as $2M - 1$.

$$\begin{aligned} N^2 &= (2M - 1)^2 \\ &= 4M^2 - 2M + 1 \\ &= 2(2M^2 - M) + 1 \end{aligned}$$



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Given an odd integer N , prove that N^2 is also an odd integer.

If N is odd, it can be expressed as $2M - 1$.

$$\begin{aligned} N^2 &= (2M - 1)^2 \\ &= 4M^2 - 2M + 1 \\ &= 2(2M^2 - M) + 1 \end{aligned}$$

Therefore N^2 is also odd.

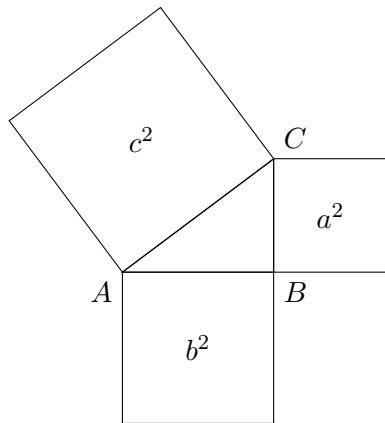


Pythagorean Theorem

Direct Proof

Example

Given a right angled triangle, the sum of the squares of the two legs is equal to the square of the hypotenuse.

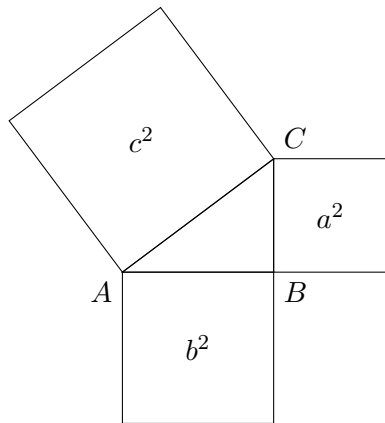


Pythagorean Theorem

Direct Proof

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In Canada, this theorem is provided to students without proof, despite the fact that over 1000 different ones exist.



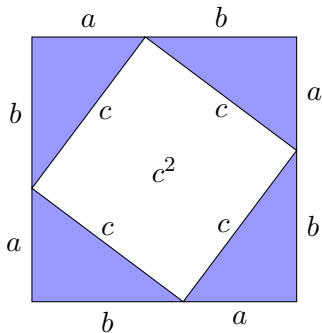
Pythagorean Theorem

Direct Proof



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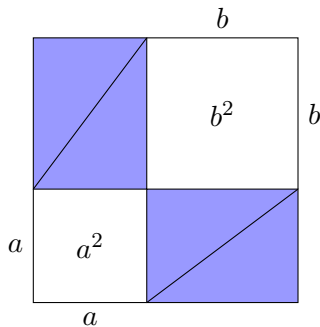
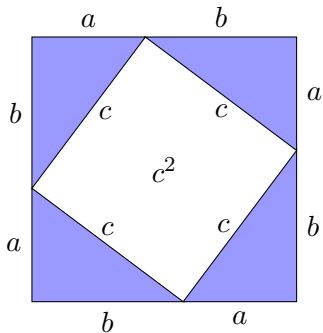


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It makes an assumption, and then follows through a set of logical steps that show that if the assumption is true, the assumption must be false. This results in a contradiction, meaning the assumption is false.



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Proof by Contradiction

A proof by contradiction is another common method of proving a statement.

It makes an assumption, and then follows through a set of logical steps that show that if the assumption is true, the assumption must be false. This results in a contradiction, meaning the assumption is false.

A similar style of proof is called “Proof by Contraposition”, which proves a statement “If A then B ” by proving that “If not A , then not B ”.



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Suppose that N is not an odd integer. Thus, it can be expressed as $2M$.

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Suppose that N is not an odd integer. Thus, it can be expressed as $2M$.

$$\begin{aligned} N^2 &= (2M)^2 \\ &= 4M^2 \\ &= 2(2M^2) \end{aligned}$$

N^2 is not odd. Therefore, N is *not* not an odd integer, and must be odd.



Irrationality of $\sqrt{2}$

Proof by Contradiction

Legend goes that Pythagoras drowned the student that proved the existence of irrational numbers.



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Example

Prove that $\sqrt{2}$ is irrational.

Let's assume that $\sqrt{2}$ is rational.



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Example

Prove that $\sqrt{2}$ is irrational.

Let's assume that $\sqrt{2}$ is rational. Then, it can be expressed as $\frac{a}{b}$ where a and b are integers and have no common factors. If they have common factors, divide both by the factor until the original condition is met. This has been shown to be possible for any two integers in an earlier lesson.



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This means that a is even, and can be expressed as $2c$.



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As before, the above statement implies that b is even.



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$$\sqrt{2}b = a$$

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This means that a is even, and can be expressed as $2c$.

$$2b^2 = (2c)^2$$

$$b^2 = 2c^2$$

As before, the above statement implies that b is even. However, if both a and b are even, then they share a common factor 2. This contradicts the original assumption. Therefore, $\sqrt{2}$ is irrational.



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It splits the question into all the possible cases, and then proves that the statement is true for all possible cases.



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Proof by Exhaustion

This proof is a case by case proof, but must be used with caution.

It splits the question into all the possible cases, and then proves that the statement is true for all possible cases.

It can look very disorganized, especially if the cases are singular numbers instead of sets, but it is still a proof.



Odd Number Squared

Proof by Exhaustion

Example

Given an integer N , prove that N^2 must be even, or one greater than even.*



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Proof by Exhaustion

Example

Given an integer N , prove that N^2 must be even, or one greater than even.*

Separate the problem into two cases: N is odd, or N is even. Then, you will find that one case will yield an even number (namely N is even) and the other case will yield a number one greater than even (namely N is odd).



Actual Example

Proof by Exhaustion

Example

Given a perfect cube N^3 , prove that it is always divisible by 9, or 1 greater or 1 less than that.



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Example

Given a perfect cube N^3 , prove that it is always divisible by 9, or 1 greater or 1 less than that.

For this problem, we will make 3 cases: N is divisible by 3, N is one less than a number divisible by 3, and N is one greater than a number divisible by 3.



Case 1

Proof by Exhaustion

Case 1 (N is divisible by 3, and can be expressed as $3M$):



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Proof by Exhaustion

Case 1 (N is divisible by 3, and can be expressed as $3M$):

$$\begin{aligned} N^3 &= (3M)^3 \\ &= 27M^3 \\ &= 9(3M^3) \end{aligned}$$



Case 2

Proof by Exhaustion

Case 2 (N is one less than a number divisible by 3, and can be expressed as $3M - 1$):



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Proof by Exhaustion

Case 2 (N is one less than a number divisible by 3, and can be expressed as $3M - 1$):

$$\begin{aligned} N^3 &= (3M - 1)^3 \\ &= 27M^3 - 27M^2 + 9M - 1 \\ &= 9(3M^3 - 3M^2 + M) - 1 \end{aligned}$$



Case 3

Proof by Exhaustion

Case 3 (N is one greater than a number divisible by 3, and can be expressed as $3M + 1$):



Case 3

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Case 3 (N is one greater than a number divisible by 3, and can be expressed as $3M + 1$):

$$\begin{aligned} N^3 &= (3M + 1)^3 \\ &= 27M^3 + 27M^2 + 9M + 1 \\ &= 9(3M^3 + 3M^2 + M) + 1 \end{aligned}$$



Conclusion

Proof by Exhaustion

The three cases are *exhaustive* (cover all possibilities), and the result fits the conjecture. Therefore, the statement is true.



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It relies on showing that within a given set of parameters something must be possible, without pinpointing exactly what it is. This is why it is called nonconstructive. It also often relies on exhaustive properties of these parameters.



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This proof is quite rare, and is somewhat similar to an exhaustive proof

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It is also sometimes called “Proof by Ternary Exclusion”, when there are two cases, and no third one is possible.



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Nonconstructive Proof

Example

Given an integer N , prove that N^2 must be even, or one greater than even.*

This problem can be solved non-constructively. Simply noting that every number is either even, or one greater than even (odd), proves that this must be true. This is a nonconstructive proof because you proved the statement without finding the exact answer. A better example follows.



Irrational to the Irrational

Nonconstructive Proof

Example

Prove that it is possible to have an irrational number to the power of another irrational number result in a rational number.



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Prove that it is possible to have an irrational number to the power of another irrational number result in a rational number.

Recall that $\sqrt{2}$ is irrational. Let us consider two cases: $\sqrt{2}^{\sqrt{2}}$ and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.



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Recall that $\sqrt{2}$ is irrational. Let us consider two cases: $\sqrt{2}^{\sqrt{2}}$ and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

If the first case is rational, then we have found an example to match the statement.



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If the first case is rational, then we have found an example to match the statement.

If it is not rational, it must be irrational. This is the exhaustive part.



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If the first case is rational, then we have found an example to match the statement.

If it is not rational, it must be irrational. This is the exhaustive part. However, simplifying the second case allows one to see that the result is 2.

One of the two cases must be an irrational to the power of an irrational resulting in a rational. However, the specific one is unknown. This is the nonconstructive part.



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It relies on proving a case, then assuming that the statement is true, and proving that the statement is always true for an input one greater or one less. This results in a recursive proof of all other values.



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Proof by Induction

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Despite its name, the proof is still deductive, not inductive.



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First, check that it is true for $N = 1$.



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Given an odd integer N , prove that N^2 is also an odd integer.

First, check that it is true for $N = 1$.

$$\begin{aligned} N^2 &= 1^2 \\ &= 1 \end{aligned}$$



Odd Number Squared

Proof by Induction

Next, assume that it is true for all N . Now prove that it is true for any $N + 2$. It cannot be for $N + 1$, because an odd number plus 1 is even, while the proof asks for odd. Recall that if N is odd, it can be expressed as $2M + 1$.

$$\begin{aligned}(N + 2)^2 &= (2M + 3)^2 \\ &= 2M^2 + 12M + 9 \\ &= 2(M^2 + 6M + 4) + 1\end{aligned}$$



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$$\begin{aligned}(N + 2)^2 &= (2M + 3)^2 \\ &= 2M^2 + 12M + 9 \\ &= 2(M^2 + 6M + 4) + 1\end{aligned}$$

Therefore, any $N + 2$ is also odd.



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$$\begin{aligned}(N + 2)^2 &= (2M + 3)^2 \\ &= 2M^2 + 12M + 9 \\ &= 2(M^2 + 6M + 4) + 1\end{aligned}$$

Therefore, any $N + 2$ is also odd. This works because we showed that the statement is true for $N = 1$. From the above, we know that it is also true for $N + 2$, in this case 3. However, this implies that it is also true for 5, which in turn implies that it is true for 7, and so on until all odd integers are covered.



Arithmetic Sequence

Proof by Induction

Example

Prove that $p(N) = 0 + 1 + 2 + 3 + \dots + N - 1 + N = \frac{N(N+1)}{2}$.

First, check that it is true for $N = 0$.



Arithmetic Sequence

Proof by Induction

Example

Prove that $p(N) = 0 + 1 + 2 + 3 + \dots + N - 1 + N = \frac{N(N+1)}{2}$.

First, check that it is true for $N = 0$.

$$\begin{aligned} 0 &= \frac{0(0+1)}{2} \\ &= 0 \end{aligned}$$



Arithmetic Sequence

Proof by Induction

Example

Prove that $p(N) = 0 + 1 + 2 + 3 + \dots + N - 1 + N = \frac{N(N+1)}{2}$.

First, check that it is true for $N = 0$.

$$\begin{aligned} 0 &= \frac{0(0+1)}{2} \\ &= 0 \end{aligned}$$

Assume it is true from $p(N)$. Now prove that it is true for $p(N+1)$.



Inductive Portion

Proof by Induction

What is being proven!

$$0 + 1 + 2 + \dots + N + (N + 1) = \frac{(N+1)((N+1)+1)}{2}$$

$$0 + 1 + 2 + \dots + N + (N + 1) = \frac{N(N + 1)}{2} + (N + 1)$$



Inductive Portion

Proof by Induction

What is being proven!

$$0 + 1 + 2 + \dots + N + (N + 1) = \frac{(N+1)((N+1)+1)}{2}$$

$$\begin{aligned} 0 + 1 + 2 + \dots + N + (N + 1) &= \frac{N(N + 1)}{2} + (N + 1) \\ &= \frac{N(N + 1) + 2(N + 1)}{2} \end{aligned}$$



Inductive Portion

Proof by Induction

What is being proven!

$$0 + 1 + 2 + \dots + N + (N + 1) = \frac{(N+1)((N+1)+1)}{2}$$

$$\begin{aligned} 0 + 1 + 2 + \dots + N + (N + 1) &= \frac{N(N + 1)}{2} + (N + 1) \\ &= \frac{N(N + 1) + 2(N + 1)}{2} \\ &= \frac{(N + 2)(N + 1)}{2} \end{aligned}$$



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What is being proven!

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What is being proven!

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$$\begin{aligned} 0 + 1 + 2 + \dots + N + (N + 1) &= \frac{N(N + 1)}{2} + (N + 1) \\ &= \frac{N(N + 1) + 2(N + 1)}{2} \\ &= \frac{(N + 2)(N + 1)}{2} \\ &= \frac{(N + 1)((N + 1) + 1)}{2} \end{aligned}$$

Which is what needed to be proven. *Q.E.D.*



Addition and Subtraction when Inducing

Proof by Induction

Example

Prove that $4^N - 1$ is always divisible by 3 if N is a positive integer.

First, check that the statement is true for $N = 1$.

$$\begin{aligned}4^N - 1 &= 4^1 - 1 \\ &= 3\end{aligned}$$

Assume it is true for N , now prove for $N + 1$.



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$$\begin{aligned} 4^N - 1 &= 4^1 - 1 \\ &= 3 \end{aligned}$$

Assume it is true for N , now prove for $N + 1$. This time, we will take the difference between the statement for N and $N + 1$. Because the statement is assumed to be true for N , if their difference is divisible by 3, then it must also be divisible by 3 for $N + 1$.



Addition and Subtraction when Inducing

Proof by Induction

$$4^{N+1} - 1 - (4^N - 1) = 4^{N+1} - 4^N$$



Addition and Subtraction when Inducing

Proof by Induction

$$\begin{aligned}4^{N+1} - 1 - (4^N - 1) &= 4^{N+1} - 4^N \\ &= 4^N(4^1 - 1)\end{aligned}$$



Addition and Subtraction when Inducing

Proof by Induction

$$\begin{aligned}4^{N+1} - 1 - (4^N - 1) &= 4^{N+1} - 4^N \\&= 4^N(4^1 - 1) \\&= 4^N(3)\end{aligned}$$



Addition and Subtraction when Inducing

Proof by Induction

$$\begin{aligned}4^{N+1} - 1 - (4^N - 1) &= 4^{N+1} - 4^N \\&= 4^N(4^1 - 1) \\&= 4^N(3)\end{aligned}$$

Thus, the statement is true.

