1   Inequalities and Extrema

If $f(x) \geq c$, where c is some constant, then the minimum value for $f(x)$ is c.

If $f(x) \leq c$, where c is some constant, then the maximum value for $f(x)$ is c.

This can be extended to any number of variables.

i.e. $f(x)$ can be replaced by any expression involving any number of variables.

Practice

What is the minimum value of $(x - y)^2 + 5$, where $x, y \in \mathbb{R}$?

2   Quadratics

In $f(x) = ax^2 + bx + c$:

If $a < 0$, $f(x)$ must have a local maximum.

If $a > 0$, $f(x)$ must have a local minimum.

The local extremum (the vertex) can be found using the formula:

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a} \right)$$

Practice

If $\frac{3}{x} = \frac{2}{y}$, what is the minimum value of $4x + 2xy + 3y + 6y^2 + 6x^2$?

3 Jensen's Inequality

An interval of a function is **convex** if the line segment connecting any 2 points in the interval lies above or on the function.

An interval of a function is **concave** if the line segment connecting any 2 points in the interval lies below or on the function.

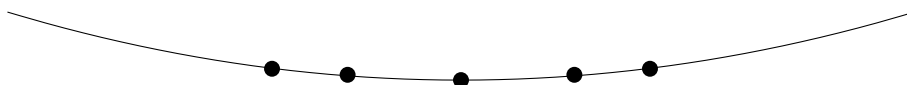Jensen's Inequality states that

For a convex function:

$$\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right)$$

For a concave function:

$$\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right)$$

Practice

A symmetric catenary bridge forms a "U" shape. If I some equally-spaced rocks on the bridge, will the average height be higher or lower than the height of the rock in the middle?

4 Cauchy-Bunyakovsky-Schwarz Inequality

For some 2 sequences of real numbers a_n and b_n,

$$(a_1^2 + a_2^2 + ... + a_n^2) \cdot (b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1 \cdot b_1 + a_2 \cdot b_2 + ... + a_n \cdot b_n)^2$$

5 Arithmetic Mean - Geometric Mean Inequality

For a sequence a_n of **non-negative** numbers:

$$(a_1 + a_2 + ... + a_n) \cdot \frac{1}{n} \geq (a_1 \times a_2 \times ... \times a_n)^{\frac{1}{n}}$$

Practice

If I have n positive numbers whose product is 1, what is the minimum possible sum of the numbers?

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