Euclid Preparation 3

Circle Geometry

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William Lyon Mackenzie C.I. Math Club

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Theorem ("Star Trek" Theorem)

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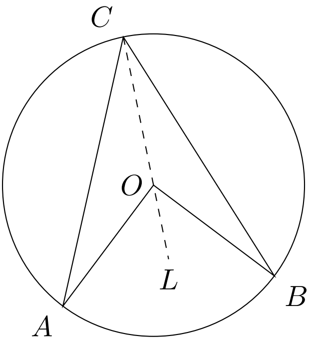

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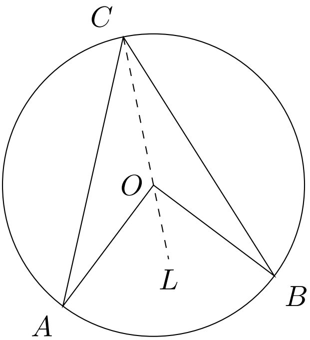

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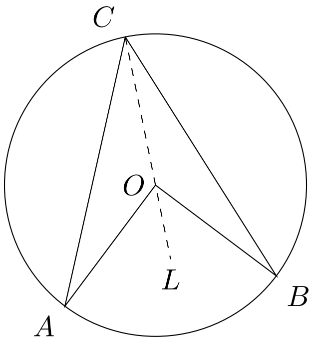

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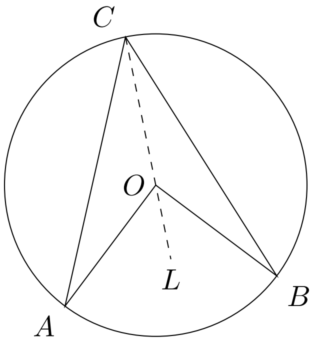

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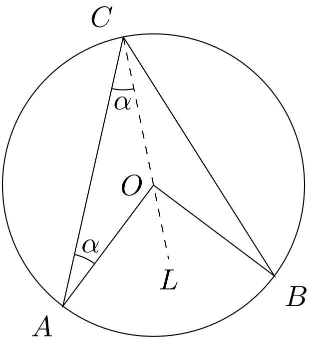

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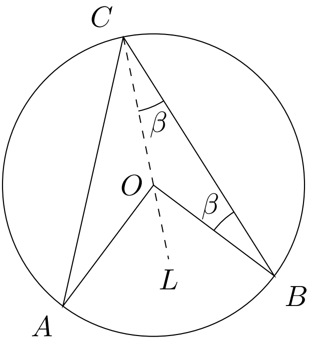

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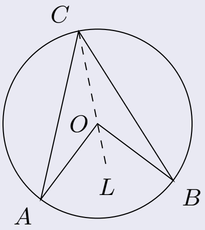
Also, note that $\triangle OAC$ and $\triangle OBC$ are isosceles. This is because OA, OB, and OC are all radii. So, $\angle OAC = \angle OCA$ and $\angle OCB = \angle OBC$.

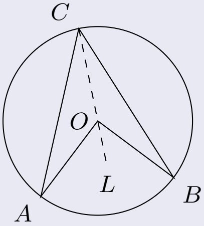
Proof that $\angle AOB = 2\angle ACB$.

We know that $\angle OAC = \angle OCA$.



Proof that $\angle AOB = 2\angle ACB$.

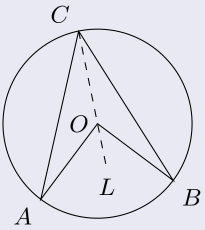
We know that $\angle OAC = \angle OCA$. So: $2\angle OCA + \angle AOC = 180°$.



Proof that $\angle AOB = 2\angle ACB$.

We know that $\angle OAC = \angle OCA$. So: $2\angle OCA + \angle AOC = 180°$.

And we know that $\angle AOC + \angle AOL = 180°$.



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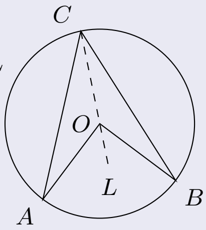

$$2\angle OCA + \angle AOC = \angle AOC + \angle AOL$$

$$\angle OCA = \frac{1}{2}\angle AOL$$

Proof that $\angle AOB = 2\angle ACB$.

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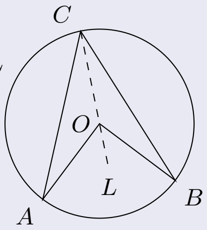
$$2\angle OCA + \angle AOC = \angle AOC + \angle AOL$$
$$\angle OCA = \frac{1}{2}\angle AOL$$

And similarly for $\triangle OBC$: $\angle OCB = \frac{1}{2}\angle BOL$.



Proof of the Star Trek Theorem

Star Trek Theorem

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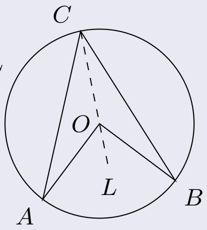

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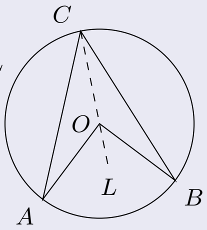

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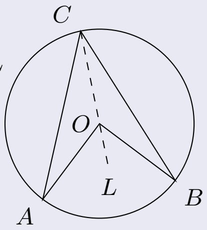

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$$\angle ACB = \angle OCA + \angle OCB$$
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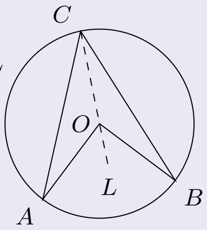

And similarly for $\triangle OBC$: $\angle OCB = \frac{1}{2}\angle BOL$.

$$\angle ACB = \angle OCA + \angle OCB$$
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$$2\angle ABC = \angle AOB$$

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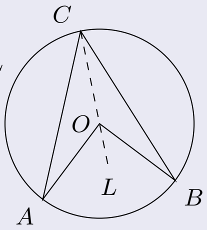

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$$2\angle ABC = \angle AOB$$

$\square$

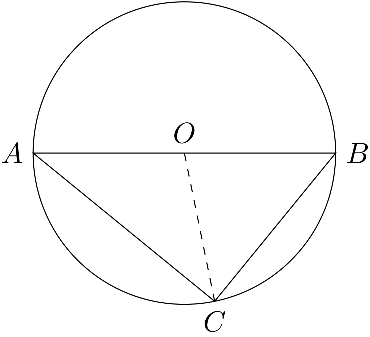
Extending

Diameters and right angles

Star Trek Theorem

Example

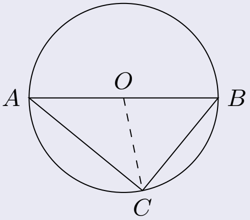
Show that if the chord AB is a diameter then $\angle ACB = 90°$.
In other words, show that the angle subtended by a diameter is a right angle.



Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \qquad (1)$$



Extension 1 proof
Star Trek Theorem

Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

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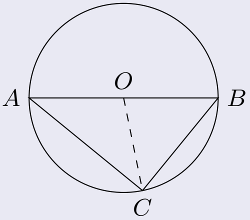
Similarly:

$$2\angle BCO + \angle BOC = 180^\circ \qquad (2)$$



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Similarly:

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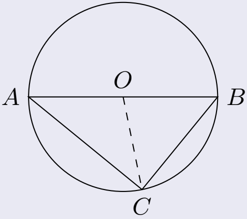
We also know that $\angle AOC = 180^\circ - \angle BOC$.



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Similarly:

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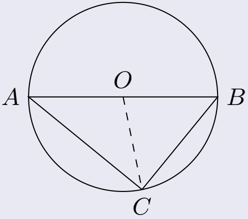
We also know that $\angle AOC = 180^\circ - \angle BOC$.
We substitute this into (1) to get $2\angle ACO = \angle BOC$.



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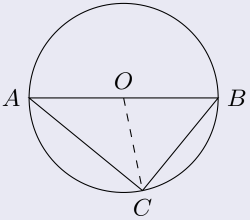

We also know that $\angle AOC = 180^\circ - \angle BOC$.
We substitute this into (1) to get $2\angle ACO = \angle BOC$.
We substitute this into (2) to get:

$$2\angle BCO + 2\angle ACO = 180^\circ$$
$$\angle BCO + \angle ACO = 90^\circ$$

Proof that $\angle ACB = 90^\circ$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180^\circ \qquad (1)$$

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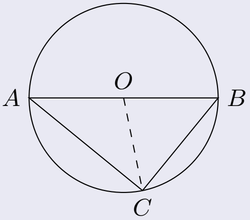


We also know that $\angle AOC = 180^\circ - \angle BOC$.
We substitute this into (1) to get $2\angle ACO = \angle BOC$.
We substitute this into (2) to get:

$$2\angle BCO + 2\angle ACO = 180^\circ$$
$$\angle BCO + \angle ACO = 90^\circ$$

Since $\angle BCO + \angle ACO = \angle ACB$, we arrive at:

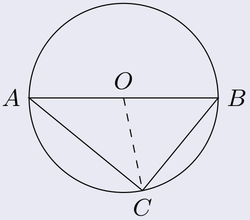
Extension 1 proof

Star Trek Theorem

Proof that $\angle ACB = 90°$.

We know that $\angle ACO = \angle CAO$. So:

$$2\angle ACO + \angle AOC = 180° \qquad (1)$$



Similarly:

$$2\angle BCO + \angle BOC = 180° \qquad (2)$$

We also know that $\angle AOC = 180° - \angle BOC$.
We substitute this into (1) to get $2\angle ACO = \angle BOC$.
We substitute this into (2) to get:

$$2\angle BCO + 2\angle ACO = 180°$$
$$\angle BCO + \angle ACO = 90°$$

Since $\angle BCO + \angle ACO = \angle ACB$, we arrive at:

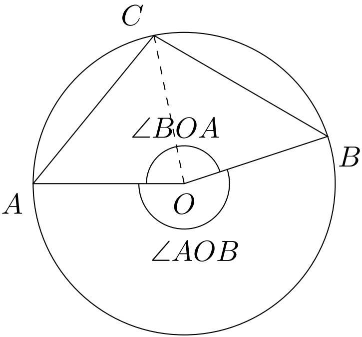
$$\angle ACB = 90°$$

□

Example

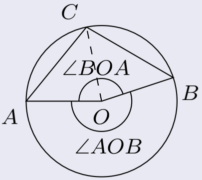
Show that the Star Trek theorem is still true if $\angle AOB > 180°$.

That is, show that $\angle AOB = 2\angle ACB$ is true in this diagram.



Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$
$$2\angle BCO + \angle BOC = 180^\circ$$

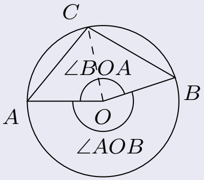


Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

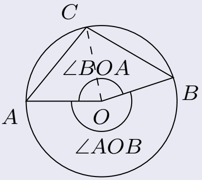
$$2\angle ACO + \angle AOC = 180^\circ$$
$$2\angle BCO + \angle BOC = 180^\circ$$



We add these two equations to get:

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$
$$2\angle BCO + \angle BOC = 180^\circ$$



We add these two equations to get:

$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$
$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

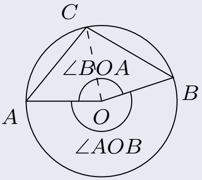
Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$

$$2\angle BCO + \angle BOC = 180^\circ$$



We add these two equations to get:

$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$

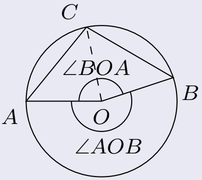
$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

We know that $\angle AOC + \angle BOC = \angle BOA$, so:

$$2(\angle ACO + \angle BCO) = 360^\circ - \angle BOA$$

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$
$$2\angle BCO + \angle BOC = 180^\circ$$



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$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

We know that $\angle AOC + \angle BOC = \angle BOA$, so:

$$2(\angle ACO + \angle BCO) = 360^\circ - \angle BOA$$

We also know that $\angle AOB = 360^\circ - \angle BOA$.

Extension 2 proof

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Proof that $\angle AOB = 2\angle ACB$.

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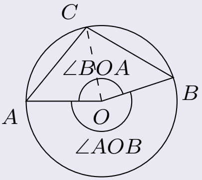
$$2\angle BCO + \angle BOC = 180^\circ$$



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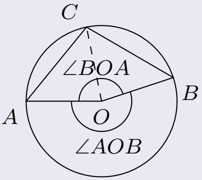
We also know that $\angle AOB = 360^\circ - \angle BOA$.
And $\angle ACB = \angle ACO + \angle BOC$.

Extension 2 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

$$2\angle ACO + \angle AOC = 180^\circ$$
$$2\angle BCO + \angle BOC = 180^\circ$$



We add these two equations to get:

$$2(\angle ACO + \angle BCO) + \angle AOC + \angle BOC = 360^\circ$$
$$2(\angle ACO + \angle BCO) = 360^\circ - (\angle AOC + \angle BOC)$$

We know that $\angle AOC + \angle BOC = \angle BOA$, so:

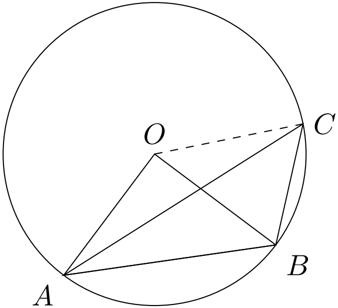
$$2(\angle ACO + \angle BCO) = 360^\circ - \angle BOA$$

We also know that $\angle AOB = 360^\circ - \angle BOA$.
And $\angle ACB = \angle ACO + \angle BOC$.

$$\therefore 2\angle ACB = \angle AOB \qquad \square$$

Example

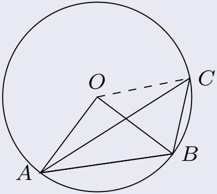
Show that the Star Trek theorem is still true if the point C is chosen so that AB and OB intersect.
Prove that $\angle AOB = 2\angle ACB$.



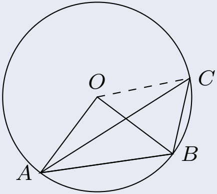
Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



Proof that $\angle AOB = 2\angle ACB$.

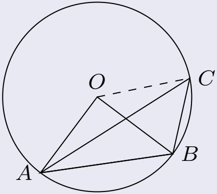


$$\angle AOC = 180^\circ - 2\angle OCA \tag{3}$$

Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

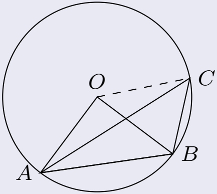


$$\angle AOC = 180^\circ - 2\angle OCA \tag{3}$$

$$\angle COB = 180^\circ - 2\angle OBC \tag{4}$$

Extension 3 proof
Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.

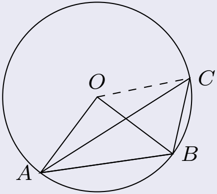


$$\angle AOC = 180^\circ - 2\angle OCA \tag{3}$$

$$\angle COB = 180^\circ - 2\angle OBC \tag{4}$$

$$\angle AOB = \angle AOC - \angle COB \tag{5}$$

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \tag{3}$$

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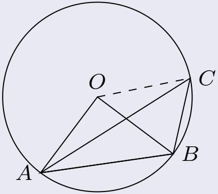
$$\angle AOB = \angle AOC - \angle COB \tag{5}$$

We can substitute (3) and (4) into (5):

Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



$$\angle AOC = 180^\circ - 2\angle OCA \quad (3)$$

$$\angle COB = 180^\circ - 2\angle OBC \quad (4)$$

$$\angle AOB = \angle AOC - \angle COB \quad (5)$$

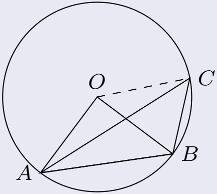
We can substitute (3) and (4) into (5):

$$\angle AOB = 180^\circ - 2\angle OCA - (180^\circ - 2\angle OBC)$$

$$\angle AOB = -2\angle OCA + 2\angle OBC$$

$$\angle AOB = 2(\angle OBC - \angle OCA)$$

Proof that $\angle AOB = 2\angle ACB$.



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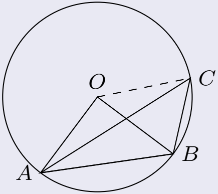
Extension 3 proof

Star Trek Theorem

Proof that $\angle AOB = 2\angle ACB$.



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We can substitute (3) and (4) into (5):

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$$\angle AOB = 2(\angle OBC - \angle OCA)$$

We know that know that $\angle ACB = \angle OCB - \angle OCA$.

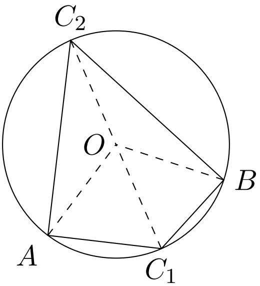
$$\therefore \angle AOB = 2\angle ACB$$

□

Example

If C_1 and C_2 are two points on the circle, one on the minor arc AB and the other on the major arc, prove that
$\angle AC_1B + \angle AC_2B = 180°$.

This is equivalent to proving that the opposite angles of a cyclic quadrilateral are supplementary.



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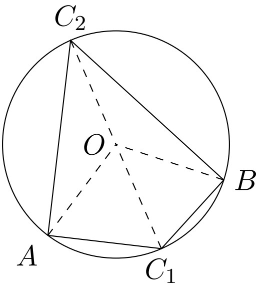

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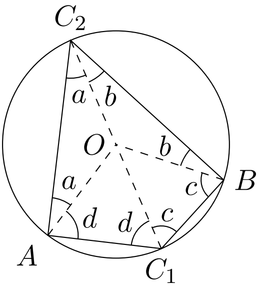

Proof that opposite angles of a cycle quadrilateral are supplementary.

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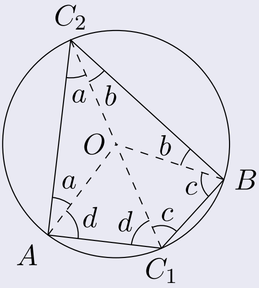
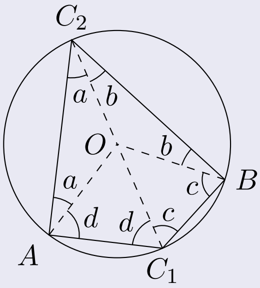


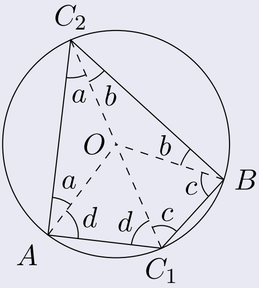
The sum of the interior angles of a quadrilateral equals

Proof that opposite angles of a cycle quadrilateral are supplementary.



The sum of the interior angles of a quadrilateral equals $360°$.

Proof that opposite angles of a cycle quadrilateral are supplementary.

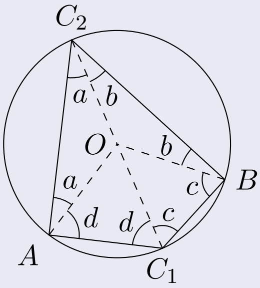


The sum of the interior angles of a quadrilateral equals 360°.

So:

$$a + b + c + d + a + b + c + d = 360^\circ$$

Proof that opposite angles of a cycle quadrilateral are supplementary.

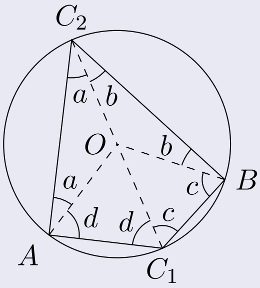


The sum of the interior angles of a quadrilateral equals 360°.

So:

$$\begin{aligned} a+b+c+d+a+b+c+d &= 360^\circ \\ 2(a+b+c+d) &= 360^\circ \end{aligned}$$

Proof that opposite angles of a cycle quadrilateral are supplementary.



The sum of the interior
angles of a quadrilateral equals $360°$.

So:

$$a+b+c+d+a+b+c+d = 360°$$
$$2(a+b+c+d) = 360°$$
$$a+b+c+d = 180°$$

$\square$

Angles subtended by the same arc

Star Trek Theorem

Example

Show that if C_1 and C_2 are two different choices for the position of the point C along the same arc AB then $\angle AC_1B = \angle AC_2B$.

This is equivalent to saying that angles subtended by the same arc are equal.

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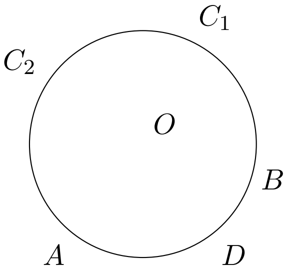

Angles subtended by the same arc

Star Trek Theorem

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Show that if C_1 and C_2 are two different choices for the position of the point C along the same arc AB then $\angle AC_1B = \angle AC_2B$.

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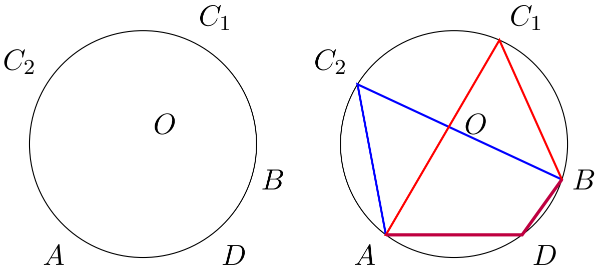
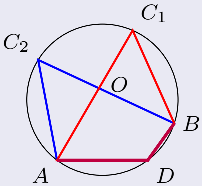
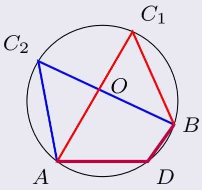


Proof that $\angle AC_1B = \angle AC_2B$.

Using extension 4, we know that:



Proof that $\angle AC_1B = \angle AC_2B$.



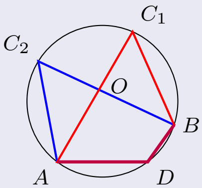
Using extension 4, we know that:

$$\angle AC_1B + \angle ADB = 180^\circ$$

Extension 5 proof

Star Trek Theorem

Proof that $\angle AC_1B = \angle AC_2B$.



Using extension 4, we know that:

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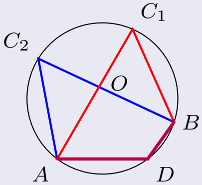
Extension 5 proof

Star Trek Theorem

Proof that $\angle AC_1B = \angle AC_2B$.



Using extension 4, we know that:

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$$\angle AC_2B + \angle ADB = 180^\circ$$

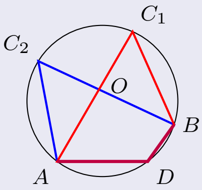
So:

Extension 5 proof

Star Trek Theorem

Proof that $\angle AC_1B = \angle AC_2B$.

Using extension 4, we know that:



$$\angle AC_1B + \angle ADB = 180^\circ$$

$$\angle AC_2B + \angle ADB = 180^\circ$$

So:

$$\angle AC_1B + \angle ADB = \angle AC_2B + \angle ADB$$
$$\angle AC_1B = \angle AC_2B \qquad \square$$

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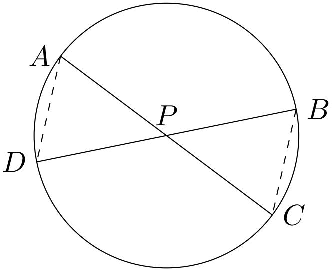
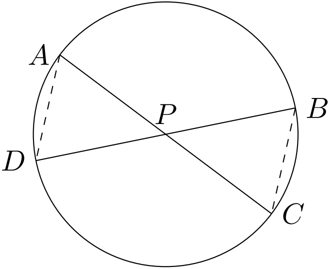
Theorem (Crossed Chord Theorem)

If two chords AB and CD of a circle intersect at point P, then $(PA)(PB) = (PC)(PD)$.



Theorem (Crossed Chord Theorem)

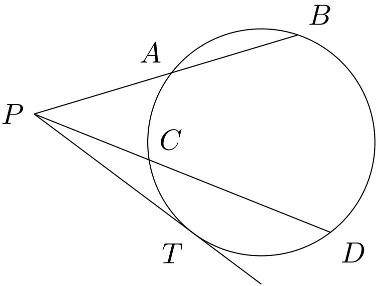
If two chords AB and CD of a circle intersect at point P, then $(PA)(PB) = (PC)(PD)$.

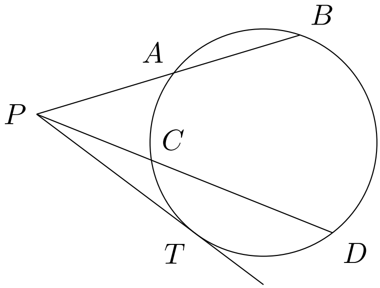


This is proved using similar triangles and the fifth extension we developed for the Star Trek theorem. Try to prove it yourself!

Secant and tangents

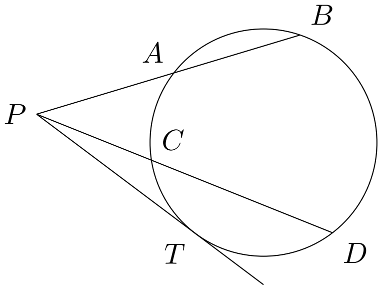
Crossed Chord Theorem





Example

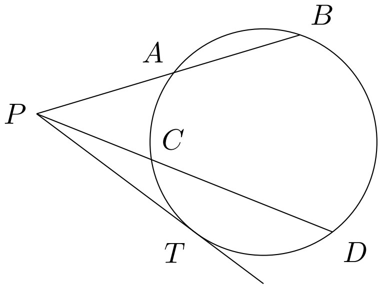
In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.



Example

In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.

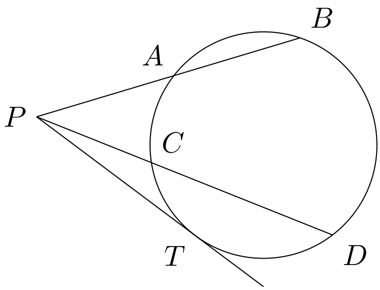
Prove that $(PA)(PB) = (PC)(PD)$.



Example

In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.

Prove that $(PA)(PB) = (PC)(PD)$. This proof also uses similar triangles. Try it yourself!



Example

In the diagram PAB and PCD are two secants of the same circle and they intersect at a point P outside the circle.

Prove that $(PA)(PB) = (PC)(PD)$. This proof also uses similar triangles. Try it yourself!

Example

If PT is a tangent to the circle, prove that $(PA)(PB) = (PT)^2$.

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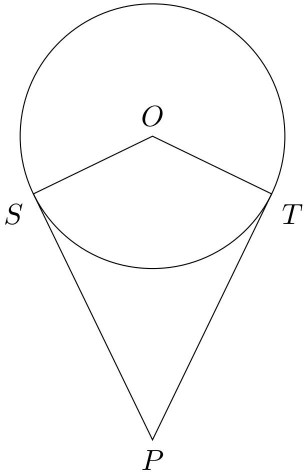
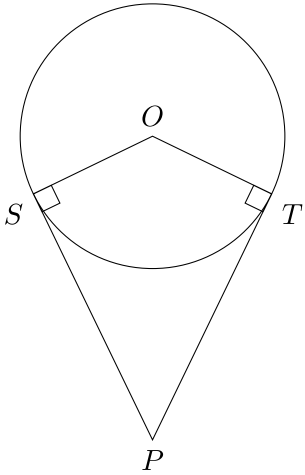


Example

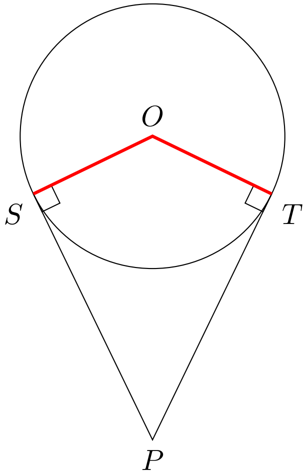
If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:



Example

If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:

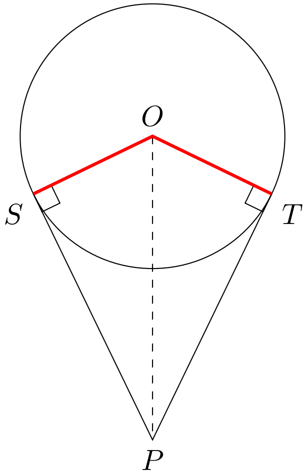
1. A tangent at a point on a circle is perpendicular to the radius draw to the point.



Example

If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:

1. A tangent at a point on a circle is perpendicular to the radius draw to the point.
2. $PS = PT$: tangents to a circle from an external point are equal.



Example

If P is a point outside of a circle and PT and PS are two tangents to the circle, then the following are true:

1. A tangent at a point on a circle is perpendicular to the radius draw to the point.
2. $PS = PT$: tangents to a circle from an external point are equal.
3. OP bisects the angle between the tangents.

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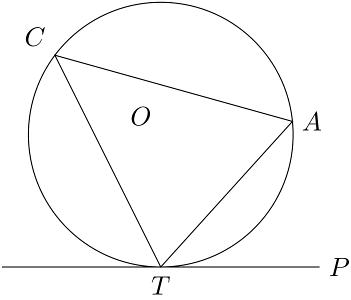
Theorem

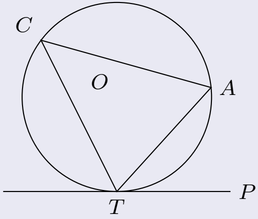
Tangent Chord Theorem

Theorem (Tangent chord theorem)

Given that TA is any chord of a circle and PT is a tangent to the circle at T. If C is a point on the circle chosen to be on the side of the chord opposite to the tangent then $\angle TCA = \angle PTA$.

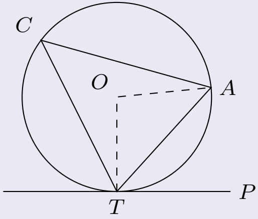


Proof that $\angle TCA = \angle PTA$.



We know that:

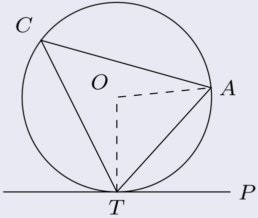
Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$
$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

Proof that $\angle TCA = \angle PTA$.

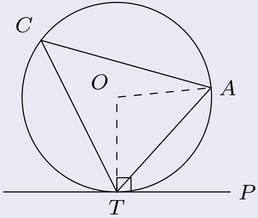


We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$
$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

Proof that $\angle TCA = \angle PTA$.



We know that:

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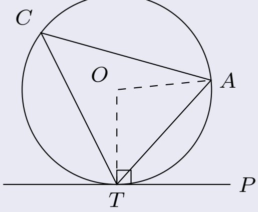
$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

We also know that $\angle PTA = 90^\circ - \angle ATO$.

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$
$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

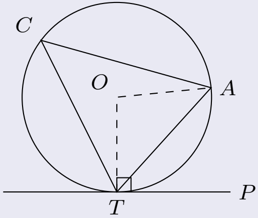
We also know that $\angle PTA = 90^\circ - \angle ATO$.

We put this all together:

$$\angle PTA = 90^\circ - \frac{1}{2}(180^\circ - 2\angle TCA)$$

$\square$

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$
$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

We also know that $\angle PTA = 90^\circ - \angle ATO$.

We put this all together:

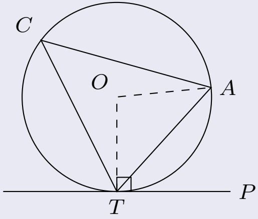
$$\angle PTA = 90^\circ - \frac{1}{2}(180^\circ - 2\angle TCA)$$
$$\angle PTA = 90^\circ - 90^\circ + \angle TCA$$

□

Proof of tangent chord theorem

Tangent Chord Theorem

Proof that $\angle TCA = \angle PTA$.



We know that:

$$2\angle ATO = 180^\circ - \angle AOT$$
$$\angle ATO = \frac{1}{2}(180^\circ - \angle AOT)$$

We know from the Star Trek theorem that $\angle AOT = 2\angle TCA$.

We also know that $\angle PTA = 90^\circ - \angle ATO$.

We put this all together:

$$\angle PTA = 90^\circ - \frac{1}{2}(180^\circ - 2\angle TCA)$$
$$\angle PTA = 90^\circ - 90^\circ + \angle TCA$$
$$\angle PTA = \angle TCA$$

□