Euclid Preparation 1

Logarithms, Exponents, Functions, and Equations

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Part I:

Part I

Logarithms and Exponents

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When $a, b, x, y \in \mathbb{R}$ and $n \in \mathbb{R} \mid n \neq 0$:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^0 = 1 \text{ if } a \neq 0$$

$$a^{-x} = \frac{1}{a^x} \text{ if } a \neq 0$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x \text{ if } b \neq 0$$

0^0 is not defined.

$$\frac{a^x}{a^y} = a^{x-y} \text{ if } a \neq 0$$

$$(a^x)^y = a^{xy}$$

$$a^x \cdot b^x = (ab)^x$$

$$a^x a^y = a^{x+y}$$

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + k^{k+2}) = 5^{m+3} + 5^m$$

Exponents problem 1

Exponents

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$$9(7^k + k^{k+2}) = 5^{m+3} + 5^m$$

Solution

$$9(1 + 7^2)7^k = 5^m(5^3 + 1)$$

Exponents problem 1

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$$9(1 + 7^2)7^k = 5^m(5^3 + 1)$$
$$(450)7^k = 5^m(126)$$

Exponents problem 1

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Solution

$$9(1+7^2)7^k = 5^m(5^3+1)$$
$$(450)7^k = 5^m(126)$$
$$(2 \cdot 3^2 \cdot 5^2)7^k = 5^m(2 \cdot 3^2 \cdot 7)$$

Problem

If m and k are integers, find all solutions to the equation:

$$9(7^k + k^{k+2}) = 5^{m+3} + 5^m$$

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$$5^2 \cdot 7^k = 5^m \cdot 7$$

Exponents problem 1

Exponents

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If m and k are integers, find all solutions to the equation:

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(2\cdot 3^2\cdot 5^2)7^k &= 5^m(2\cdot 3^2\cdot 7) \\
5^2\cdot 7^k &= 5^m\cdot 7
\end{aligned}$$

Since the integer factorization of numbers is always unique and both m and k are integers:

Exponents problem 1

Problem

If m and k are integers, find all solutions to the equation:

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Solution

$$\begin{aligned} 9(1+7^2)7^k &= 5^m(5^3+1) \\ (450)7^k &= 5^m(126) \\ (2\cdot 3^2 \cdot 5^2)7^k &= 5^m(2\cdot 3^2\cdot 7) \\ 5^2\cdot 7^k &= 5^m\cdot 7 \end{aligned}$$

Since the integer factorization of numbers is always unique and both m and k are integers: $m = 2$ and $k = 1$.

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Exponents problem 2

Problem

The graph of $y = m^x$ passed through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

We know that $m^2 = 5$ and $n = m^5$. The solution is trivial from here:

$$m = \pm\sqrt{5}$$

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$$
\begin{aligned}
m &= \pm\sqrt{5} \\
n &= (\pm\sqrt{5})^5 \\
mn &= \left(\pm\sqrt{5}\right)^6 = \qquad\qquad = \qquad\qquad\qquad = \quad =
\end{aligned}
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Formulas
Logarithms

When $a, x, y \in \mathbb{R} \mid a, x, y \neq 0$:

$$\log_a(xy) = log_a x + log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^y) = y \log_a x$$
$$\log_a(a^x) = a^{\log_a x} = x$$

$$\log_a 1 = 0$$
$$\log_a x = \frac{1}{\log_x a}$$
$$\frac{\log_a x}{\log_a y} = \log_y x$$

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Also, $\log_b c$ has the restrictions:

$$b \in \mathbb{R} \mid b > 0 \text{ and } b \neq 1$$

$$c \in \mathbb{R} \mid c > 0$$

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$$b \in \mathbb{R} \mid b > 0 \text{ and } b \neq 1$$

$$c \in \mathbb{R} \mid c > 0$$

Finally, if $f(x) = a^x$ then $f^{-1} = \log_a(x)$. That is, the exponential and logarithmic functions are each other's inverses. More formally:

$$y = a^x \iff x = \log_a y$$

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

Solution

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Problem

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Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

Solution

First, we state our **restrictions**: $x > 0$, $y > 0$, and $x > 3y$.

$$2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$$
$$\log_5(x-3y)^2 = \log_5(4xy)$$

Problem

Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.

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First, we state our **restrictions**: $x > 0$, $y > 0$, and $x > 3y$.

$$\begin{aligned} 2\log_5(x-3y) &= \log_5(2x) + \log_5(2y) \\ \log_5(x-3y)^2 &= \log_5(4xy) \end{aligned}$$

We know that the logarithmic function is an **injective function**.

Logarithms problem 1

Logarithms

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We know that the logarithmic function is an **injective function**.

An **injective function** is one where $f(a) = b$ is only true for one value of a. More formally:

$$f\colon A \to B \text{ is injective if } \forall a, b \in A, f(a) = f(b) \implies a = b$$

Logarithms problem 1 solution continued

Solution

Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

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Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

$$\begin{aligned} \log_5(x-3y)^2 &= \log_5(4xy) \\ (x-3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \end{aligned}$$

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\log_5(x-3y)^2 &= \log_5(4xy) \\
(x-3y)^2 &= 4xy \\
x^2 - 6xy + 9y^2 &= 4xy
\end{aligned}$$

$$\begin{aligned}
x^2 - 10xy + 9y^2 &= 0 \\
x^2 - xy - 9xy + 9y^2 &= 0 \\
(x-y)(x-9y) &= 0
\end{aligned}$$

Logarithms problem 1 solution continued

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Since $\log_a x$ is injective: $\log_a b = \log_a c \iff b = c$.

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From here we have two cases:

Logarithms problem 1 solution continued

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From here we have two cases:

$$\begin{aligned} x - y &= 0 \\ x &= y \end{aligned}$$

But this violates our restriction $x > 3y$, so the answer must be found via the next case.

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But this violates our restriction $x > 3y$, so the answer must be found via the next case.

$$\begin{aligned}x - 9y &= 0 \\ x &= 9y \\ \therefore \frac{x}{y} &= 9\end{aligned}$$

Problem

Determine the points of intersection of the curves $y = \log_{10}(x - 2)$ and $y = 1 - \log_{10}(x + 1)$.

Solution

First, we state our **restrictions**:

Logarithms problem 2

Problem

Determine the points of intersection of the curves $y = \log_{10}(x - 2)$ and $y = 1 - \log_{10}(x + 1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Next, we simply equate the two curves:

$$\log_{10}(x-2) = 1 - \log_{10}(x+1)$$

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Next, we simply equate the two curves:

$$\begin{aligned} \log_{10}(x-2) &= 1 - \log_{10}(x+1) \\ \log_{10}(x-2) + \log_{10}(x+1) &= 1 \end{aligned}$$

Problem

Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

First, we state our **restrictions**: $x > 2$.

Next, we simply equate the two curves:

$$\begin{aligned} \log_{10}(x-2) &= 1 - \log_{10}(x+1) \\ \log_{10}(x-2) + \log_{10}(x+1) &= 1 \\ \log_{10}\big((x-2)(x+1)\big) &= 1 \end{aligned}$$

Solution

$$\log_{10}\big((x-2)(x+1)\big) = 1$$

Logarithms problem 2 solution continued

Solution

$$
\begin{aligned}
\log_{10}\left((x-2)(x+1)\right) &= 1 \\
(x-2)(x+1) &= 10
\end{aligned}
$$

Solution

$$
\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10
\end{aligned}
$$

Logarithms problem 2 solution continued

Solution

$$\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1\\
(x-2)(x+1) &= 10\\
x^2 - x - 2 &= 10\\
x^2 - x - 12 &= 0
\end{aligned}$$

Solution

$$\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1\\
(x-2)(x+1) &= 10\\
x^2-x-2 &= 10\\
x^2-x-12 &= 0\\
(x-4)(x+3) &= 0
\end{aligned}$$

Solution

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\log_{10}\left((x-2)(x+1)\right) &= 1 \\
(x-2)(x+1) &= 10 \\
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So we get $x = 4, -3$.

Solution

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\begin{aligned}
\log_{10}\big((x-2)(x+1)\big) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10 \\
x^2 - x - 12 &= 0 \\
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So we get $x = 4, -3$. We have the restriction $x > 2$, so we are left with $x = 4$.

Logarithms problem 2 solution continued

Solution

$$
\begin{aligned}
\log_{10}\left((x-2)(x+1)\right) &= 1 \\
(x-2)(x+1) &= 10 \\
x^2 - x - 2 &= 10 \\
x^2 - x - 12 &= 0 \\
(x-4)(x+3) &= 0
\end{aligned}
$$

So we get $x = 4, -3$. We have the restriction $x > 2$, so we are left with $x = 4$. This leaves us with the point of intersection $(4, \log_{10} 2)$.

Logarithms problem 3

Logarithms

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**:

Logarithms problem 3
Logarithms

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Logarithms problem 3
Logarithms

Problem
Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution
First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\log_2(9 - 2^x) = 3 - x$$

Logarithms problem 3
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Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\begin{aligned} \log_2(9 - 2^x) &= 3 - x \\ 9 - 2^x &= 2^{3-x} \end{aligned}$$

Logarithms problem 3

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\begin{aligned} \log_2(9 - 2^x) &= 3 - x \\ 9 - 2^x &= 2^{3-x} \\ 9 - 2^x &= \frac{2^3}{2^x} \end{aligned}$$

Problem

Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

First, we state our **restrictions**: $9 > 2^x$.

Next, we do some algebra:

$$\begin{aligned}
\log_2(9 - 2^x) &= 3 - x \\
9 - 2^x &= 2^{3-x} \\
9 - 2^x &= \frac{2^3}{2^x} \\
9 - 2^x &= \frac{8}{2^x}
\end{aligned}$$

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

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Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

$$\begin{aligned} 9 - y &= \frac{8}{y} \\ -y^2 + 9y - 8 &= 0 \end{aligned}$$

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

$$\begin{aligned} 9 - y &= \frac{8}{y} \\ -y^2 + 9y - 8 &= 0 \\ y &= 1, 8 \end{aligned}$$

Solution

$$9 - 2^x = \frac{8}{2^x}$$

Let $y = 2^x$.

$$\begin{aligned} 9 - y &= \frac{8}{y} \\ -y^2 + 9y - 8 &= 0 \\ y &= 1, 8 \end{aligned}$$

We substitute these solutions back into $y = 2^x$ to find that $x = 0$ or $x = 3$, both of which satisfy our restrictions.