FRM II - Notes and Formulas

Michal Mackanic

2016

Contents

Chapter 1

Notes

1.1 Book I

1.1.1 Statistics

- The GEV is not particularly useful for VaR estimation since VaR does not consider the distribution of maximum, but it is useful for stress testing.
- GPD is defined through a scale parameter $\beta \geq 0$ and tail shape parameter ξ, which can be any real number.
- Fat tails generate a positive GPD shape parameter, which indicates that VaR estimates are probably too small.
- POT approach requires a choice of a threshold, which may introduce additional uncertainty.
- POT approach serves as a basis for an expanded model of risk estimation.
  - Under POT method, in the case of fat tails, not all moments are defined.
  - POT is often estimated with a GPD.
- The chi-square distribution is used to test how closely the selected distribution fits the actual data.

1.1.2 VaR and ES

- The major improvement of the non-parametric approach over the traditional historical simulation approach is that VaR can be calculated for a continuum of points in the data set.
- Assuming normal distribution, both VaR and ES satisfy all properties of coherent risk measure. Assuming non-normal distribution, only ES satisfies the requirements.
- VaR is the most widely used risk measure for both capital allocation and absolute risk calculation purposes. However, ES is increasingly being used for capital allocation purposes.

- When backtesting VaR Error Type I is "rejecting correct model" and Error Type II is "accepting incorrect model". Regulators are more affraid of Error Type II and of Error Type I.
- Economic capital models are quite similar to VaR models despite the longer time horizons, higher confidence levels, and greater lack of data.

1.1.3 Interest Rate Models

- Basis point volatility of CIR model increases at rate $\sigma\sqrt{r}$, while log-normal basis-point volatility increases at rate σr. Thus CIR basis-point volatility is less than log-normal basis point volatility.

1.1.4 Fixed Income Valuation and Options

- Position-based risk measures evaluate the manager's holdings on a current basis and can thus detect style drift more quickly. Return-based risk measures evalute risk using historical returns and are therefore not convenient for style drift detection.
- LIBOR is commonly used as a risk-free rate for non-collateralised transactions.
- Local valuation methods measure portfolio risk by valuing assets at one point in time, then making adjustments to relevant risk factors that are expected to cause changes in the overall portfolio value. The delta-normal valuation is an example of a local method.
- The assumption that the relationship between an option's price and the ratio of the underlying to strike price applies in subsequent periods is known as sticky-delta rule. The assumption that an option implied volatility is the same over a short period of time is called the sticky-strike rule.
- When put feature expires on a bond (so that it becomes option-free), the curve depicting the price and yield relationship of the bond becomes less convex.

1.2 Book II

1.2.1 Credit Risk Models

- Under Merton model, value of a firm at maturity of the ZCB is defined as $max(A_T - D, 0)$.
- Under Merton model the option on equity is a call-on-call compound option and cannot be valued using the Black-Scholes-Merton option pricing model (Geske model is used instead).
- KMV model is most sensitive to asset return volatility and asset return correlation estimates. It employs equity price volatility as a proxy for asset price volatility.

- In CreditRisk+ model, the bankruptcy / recovery process are exogenous. It only focuses on default events and ignores prices, spreads, and mitigations. It measures the credit risk of a portfolio using a set of common risk factors for each obligor. The model assumes that the defaults are uncorrelated across obligors.
- CreditMetrics calculates the change in portfolio value due to credit migration of the underlying bonds.
- Shimko, Tejima and van Deverter (1993) model states
  - interest rate volatility increase $\rightarrow$ lower value of debt
  - higher correlation between firm values and changes in interest rates $\rightarrow$ lower value of debt
  - higher value of speed term $\rightarrow$ lower value of debt
- Neymann-Pearson rule minimises type I error (lending to a risky company) with type II error (not lending to a non-risky company) remaining constant.

1.2.2 Securitized Instruments and Credit Derivatives

- A downgrade by a bond-rating agency is not used as a trigger for payments on a default swap. The reason is that downgrades are often rare and using them might harm independence of a rating agency.
- Invocation of cross-default clause is a credit event.
- CDS will provide a hedge with regard to operations risk, provided that the operational problem is serious enough.
- A digital CDS will pay off a pre-determined fixed amount in the event of a default. Digital CDS is often used against highly illiquid reference assets that would be difficult to price.
- CLN issuer has no counterparty risk.
- Originator / manager of a securitization would most likely bear the first loss in the capital structure in order to help align their interest with investors and minimise conflicts of interest in order to make the security less risky.
- A collateral loan obligation (CLO) is a specialised form of CDO that only invests in bank loans.
- Government securities are the least likely used as collateral for CDO, since CDOs are constructed to give access to certain risk / return opportunities. Bonds, loans and mortgages are used individually or collectively to collateralized CDOs.
- All early prepayments within a lock-out period are reinvested into new loans. After lock-out period the prepayments are used to amortise the highest tranches.

- When securitizing pool of credits, at least one tranche has an investment grade (even if the underlying credits have rating of B or CCC).
- In case of CDO, SPV is typically AAA rated.
- When calculating default sensitivities of equity, mezzanine and senior tranches, various hypothetical default probabilities (not a CDS curve!) are shocked. Default sensitivities are the largest at values that create losses close to the attachment points.
- The two trust structure allows the assets conveyed to be sufficiently distanced from the originator. Thus the assets are considered a true sale and the trust is bankruptcy remote.
- Assets have to be transferred from the loan originator to the SPV through a "true sale" for a securitization to have transferred real ownership of the assets.
- Ring fencing is undertaken to provide a higher credit rating to a subsidiary than is available to the parent. Derivative products companies or unregulated subsidiaries of investment banks are examples of such structures. Aim is to create a structure that is bankruptcy remove from its parent.
- The behaviour of default sensitivities for tranches are most similar to high gammas for at-the-money options. Default sensitivities are always positive and are the largest when the resulting loss is close to the attachment point.
- Mortgage-back securities exhibit negative convexity - their NPV declines more when interest rates rise then it increases when interest rates fall. Standard bond / loan exhibits positive convexity.

1.2.3 Counterparty Risk, Collateral Management and Central Counterparty

- Banker's paradox is associated with a break clause (also called a liquidity put or early termination option).
- A swap paying the higher interest rate has a greater exposure than swap paying lower interest rate due to the fact that it has a significantly higher gain on the notional value at the maturity of both swaps. In addition, over the long term, the interest reate drift dominates the implied volatility measure. This causes the PFE for the swap receiving the higher interest rate to remain relatively flat.
- ISDA master agreements lower transaction costs because of their operation efficiency.
- Collateralization should be viewed as proposition to, but not a replacement for, ongoing due diligence review of credit quality and exposures. Collateralization may improve asset recovery in the event of default, give rise to additional risks, and can be viewed as a "double-edge" sword.

- Under the limited two-way payment covenant clause, the non-defaulting counterparty is not required to pay if the final net amount is favourable to the defaulting party.

- Legal risk, documentation risk and liquidity risk are residual risks that may arise when credit risk mitigation techniques are applied. Therefore, inability to secure collateral would be a residual risk.

- Contract is (a) bilateral with counterparty risk or (b) unilateral with lending risk.

- With an unfunded bilateral contract, both parties take on risk. As such, a trade on the OTC market would be a bilateral contract and carry counterparty risk. Bank loans and exchange traded options are unilateral contracts as only one party takes on risk.

- Traditionally, when quantifying credit exposure, there is an assumption of zero recovery value.

- *running spread CVA = EPE(%) * credit spread (%)*

- Assumptions to approximate CVA as a running spread include
  - EPE and PD are constant,
  - EE or PD are symmetric

- CVA falls to zero in the case of default.

- In case of counterparties with which bank has other credit exposure, only facility LGD (= LGD for the specific OTC derivative) needs to be calculated (PD is facility independent).

- Only some (not all) central counterparties (CCP) are too important to fail (TIFF).

- General clearing members (GCMs), individual clearing members (ICMs) and non-clearing members (NCMs) can all trade through a central counterparty.

- A bank run is the most common example of systemic risk.

- Multilateral netting requires knowledge of the positions of all members. However, institutions may wish to keep the information secret. As a result, disclosure is one of the disadvantages of multilateral netting.

- Credit exposure is defined both for pricing and risk management wheres VaR is defined only for risk management.

- A credit spread based on a single liquid observation can be used as an alternative to a credit spread curve based on multiple issues and maturities. The credit spread is estimated from a single observation and is added to the reference curve at a constant rate of percentage.

- Upward-sloping spread curves generate cumulative default distributions that are flatter in the short-term but steeper afterwards. Downward-slopping spread curves are steeper in the short-term as the near probability of default is higher but then moderates to flatter curve afterwards.
- Specific risk exposure refers to securities that undergo adverse price movements as a result of idiosyncratic factors related to individual issuers. Thus, government debt issues and futures contracts based on diversified indices (e.g. S&P 500) are not exposed to specific risk.
- Correlation risk applies to market risk, credit risk, systemic risk and concentration risk as well.

1.2.4 Mortgages

- Sub-prime market is characterised by high debt service ratios of 50% or higher.
- For sub-prime hybrid loans are typically of 2/28 or 3/27 schemes where only the first 2Y or 3Y are fixed. The bank reduces its exposure to a floating rate inflow (asset) by entering into pay-floating / receive fixed swap.
- Score cards
  - behaviour score - assess existing customer credit usage and historical delinquencies
  - revenue score - evaluate existing customers on potential profitability
  - response score - assign a probability to whether a customer is likely to respond to an offer
  - attrition score - probability of reduction or elimination of outstanding debt by existing customer
- A FICO score of 660 or above is considered by most credit scoring firms as a prime credit.
- A charge-off is declaration by creditor (usually a credit card company) that an amount of debt is unlikely to be collected.
- Deliquency ratio is calculated as $\frac{\textit{credit cards receivables over 90D}}{\textit{total credit card pool}}$.

1.3 Book III

1.3.1 Basel and Capital Requirements

- capital requirements - CRE > OPR > ALM
- Since Basel II parameters are calibrated on G-7 banks, they tend to underestimate the risk at emerging market banks.
- Within the banking industry, systemic risk is more of an issue, thus the stability of the overall financial system is more of a focus within Basel II / III. In contrast, Solvency II focuses more on the individual policyholders.

- The following six risk types are all related to underwritting risk within the insurance sector - life and health, non-life, market, counterparty default, operational, and intangibles.
- Tier 1 capital (also called core capital) is composed of common equity, non-cumulative perpetual preferred stocks, and minority equity interest in consolidated subsidiaries, less goodwill and other deductions.
- Tier 2 capital (also called supplementary capital) is composed of hybrid instruments that are structured to be more or less permanent. These include perpetual shares, subordinated debt of initial maturity of not less than 5Y and quality 99Y debt. Tier 2 capital also includes unrealised gains on long term investments.
- Under Basel II, Tier 2 capital must be lower or equal to Tier 1 capital. This implies that *total capital = Tier 1 + min(Tier 1, Tier2) - deductions.*
- For Basel III purposes, the leverage ratio is Tier 1 capital / total exposure.
- For operational risk under Basel II there can be within-cell dependencies which include both frequency and severity dependencies.
- When modelling LDA, extrapolation of observed losses, in order to develop a more complete severity distribution of losses, could result in overestimation of the needed capital charge due to "including" severe hypothetical losses.
- Under Basel II securitized positions rated below BB- / unrated are subject to deductions. A deduction implies that capital equivalents to the securitization tranche must be hold. Deductions are excluded only for estimating capital change for market risk.
- Revision to Basel II market risk framework requires banks to establish and uphold procedures for computing adjustments to the current value of liquid securities. The adjustment is made regardless of which of the positions is market to market, market to model or obtained through third-party valuation. When determining accuracy and sensibility of adjustment, the following factors are taken into account
  - market concentration,
  - average volatility of bid-ask spread,
  - average trading volume and
  - age of the position
- Under Basel II market risk include
  - equities
  - interest rates
  - foreign exchange
  - commodities (single factor model is sufficient for institutions with limited or aggregated positions)

  - options

- Under IRB approach to the credit risk first loss piece (equity tranche) must be fully deducted from regulatory capital.
- To compute regulatory capital under IRB approach for securitization exposures under the Basel II framework, the following can be used
  - rating based approach - must be used if external rating is available
  - internal assessment approach
  - supervisory formula
- Both Internal Assessment Approach (IAA) and Supervisory Formula (SF) approach could be used for unrated securitized assets. IAA is only used in limited situations, with specific permission from the regulatory authority. IAA is based on methodology used by external rating agencies.
- Under Pillar II, regulator must assess impact of interest rate risk on bank's capital position through a +200bps shock.
- The purpose for addressing market discipline in Basel II is to benefit market participants, banks, investors and regulators of timely and relevant banking operations and activities while simultaneously recognising the sensitivity of proprietary internal operations.
- Under IRB approach, calculation of weights should be independent of specific portfolio. This so-called portfolio invariance referes to the fact that risk weights do not explicitly incorporate correlations among assets within the portfolio.
- Basel II risk weight function for IRB is based on ASRF model, under which the system-wide risks that affect all obligors are modelled with only one systematic risk factor. The major reason for using ASRF is that the model should be portfolio invariant, so that the capital required for any given loan depends only on the risk of that loan and does not depend on the portfolio it is added to.
- With respect to including insurance, an analyst is allowed to reduce the severity of losses that exceed a given deductible but she cannot adjust the frequency.
- Capital requirements under IRB approach increase concavely with PD and linearly with LGD. With increasing PD, capital requirements are increasing at a decreasing rate due to decreasing correlations.
- Under advanced IRB, banks are allowed to provide their own estimates of PD, LGD and EAD, but must use correlation coefficient formula provided by the supervisor.
- When adopting IRB method, a voluntarily return to standard approach is not permitted.
- When estimating credit portfolio losses, LGD is not a critical parameter to estimate. PD and default correlations are of much higher importance.

- Under Basel II, banks cannot use external rating for some counterparites and internal ratings for other. The reason is to avoid "cherry picking".
- HQLA assets received through rehypothecation can be included in the stock of HQLA if they cannot be withdrawn during a stress period by owner.
- In contrast to credit risk IRB approach, banks using AMA may have to set aside capital for both expected and unexpected operational loss.

	Moody's	S&P	Fitch	Meaning
Investment Grade	Aaa	AAA	AAA	Prime
	Aa1	AA+	AA+	High Grade
	Aa2	AA	AA	
	Aa3	AA-	AA-	
	A1	A+	A+	Upper Medium Grade
	A2	A	A	
	A3	A-	A-	
	Baa1	BBB+	BBB+	Lower Medium Grade
	Baa2	BBB	BBB	
	Baa3	BBB-	BBB-	
Junk	Ba1	BB+	BB+	Non Investment Grade Speculative
	Ba2	BB	BB	
	Ba3	BB-	BB-	
	B1	B+	B+	Highly Speculative
	B2	B	B	
	B3	B-	B-	
	Caa1	CCC+	CCC+	Substantial Risks
	Caa2	CCC	CCC	Extremely Speculative
	Caa3	CCC-	CCC-	In Default w/ Little Prospect for Recovery
	Ca	CC	CC+	
		C	CC	In Default
			CC-	
	D	D	DDD	

Figure 1.1: Credit ratings

1.3.2 Model Risk

- model risks
  - incorrect implementation - model is correctly specified and calibrated but incorrectly implemented
  - incorrect calibration - inaccurate / infrequent calibration
  - incorrect model application - e.g. low number of Monte-Carlo simulations
- Models should be judged on their predictive ability rather than on their assumptions.
- At institutional level, the goal of model risk management is to reduce the likelihood that recorded values and future observed market values will differ.

- Basis risk arises when a position or its hedge is mapped to the same set of risk factors, which can be done when it is difficult to distinguish between two closely related positions.

- In early 2012, JP Morgan informed its regulator Office of the Controller (OCC) that the new VaR model cut bank's VaR by 50%. The OCC failed to inquire about the dramatic drop in risk, the efficiency of the new model, or reasons for the change. The OCC never concluded that the new VaR model failed to properly reflect risks.

1.3.3 Liquidity Risk

- Liquidity risk is the degree to which a trader cannot trade a position without excess cost, excess risk, or inconvenience.

- Trade processing costs are typically not a substantial component of liquidity risk.

- For endogenous markets, if a trader attempts to liquidate (buy) a large position, the trader should expect the bid (ask) price to fall (increase) and bid-ask spread to widen.

- ABCD does not have an active secondary market and has significant liquidity risk. ABCP pools assets from multiple issuers and continually purchases new assets and offers new issues.

1.4 Book IV

1.4.1 Portfolio Management

- Screening technique strives for risk control by including a sufficient number of stocks that meet the screening parameters and by weighting them to avoid concentration in any particular stock.

- Linear programming builds on stratification technique. It does not necessarily select the portfolio with the lowest level of active risk. Rather, linear programming attempts to improve stratification by introducing many more dimensions of risk control and ensuring that the portfolio approximates the benchmark for all these dimensions.

- Quadratic programming allows for risk control through parameter estimations but generally requires many more inputs estimated from market data than other method requires because it entails estimating volatilities and pair-wise correlations between all assets in a portfolio. Quadratic programming is a powerful process, but given the large number of inputs it introduces the potential for noise and poor calibration.

1.4.2 Hedge Funds

- Hedge fund assets grew to 1.4 trillion USD by 2007.

- During the 2001 - 2010 time period, all three hedge fund databases substantionally outperformed equities, accompanied by less than half the standard deviation of equities.

- For investment funds historical performance review is not part of an operational due diligence questionnaire. It is reviewed separately.

- For investment funds

  – business model risk can be assessed by considering revenues and expenses, sufficiency of working capital, existence of budgets, computation of break-even points, ability to increase investment asset base, existence of key person insurance and existence of a succession plan,

  – fraud risk can be assessed by considering the existence of related-party transactions, liquidity, litigation, unreasonably high (stated) investment returns, personal trading by the manager of the same or similar securities as those held by the fund and shorting transactions.

- Typical long / short hedge strategy would target net long position.

- Merger arbitrage hedge funds are exposed to the event risk of a collapsed merger deal. Thus, the fund is exposed to large downside risk that is similar to credit risk.

- Managed futures hedge funds may help mitigate risk associated with a market wide funding crisis due to their convex performace profile.

- After a hedge fund is closed, the historical data of the fund is seldom included in data for future analysis. The reasons are (a) investors desire the analysis of funds that are investable and (b) funds that are shut down do not wish to disclose data for legal reasons. This effect is called survivorship bias.

- A higher level or risk aversion and lower transaction costs lead to lower dispersion risk.

- When testing Jensen α we use t-statistics in form of $t = \frac{\alpha}{\sigma_\alpha}$ or $t = \frac{\alpha}{\sigma/\sqrt{N}}$.

- When calculating a hurdle rate to be compared with RAROC, weights of equity are based on market rather than on book value.

1.4.3 Pension Risk

- From the plan sponsor's perspective nominal pension obligations are similar to a short position in a long-term bond. Therefore nominal pension obligations are similar to a short position in a bond.

- In case of a pension fund a mismatch between asset and liability value is called a funding risk.

1.4.4 Other

- The same individual or team responsible for identifying and clasifying the firm's information assets should be also responsible for assessing those assets and managing and reporting the risk.
- Stress scenarios do not typically incorporate the possibility that traders will re-hedge positions during times of market shocks.

Chapter 2

Examples

2.1 Example 1

2.1.1 Question

Based on the following information, what are the risk-neutral and real-world default probabilities?

- Market price of bond is 88.
- Liquidity premium is 3%.
- Credit risk premium is 4%.
- Risk-free rate is 2.5%.
- Expected inflation is 1.5%.
- Recovery rate is 0%.

2.1.2 Answer

The risk-neutral default probability is approximately 12% because the market price is 88% of par. Risk-neutral probability = real-world probability + credit risk premium + liquidity premium. Therefore 12% = real-world probability + 4% + 3% which implies real-world probability = 12% - 7% = 5%.

2.2 Example 2

2.2.1 Question

If a credit position has a correlation with the market factor of 0.4, what is the approximate realised market value at the 99% confidence level?

2.2.2 Answer

The default loss level has a default probability of 1%, which corresponds to a value of -2.33 on the standard normal distribution. Also, the credit position's correlation to the market is β, which equals 0.4. Using this information, realised market value is calculated as follows.

$$p(m) = \Phi\left(\frac{k - \beta\bar{m}}{\sqrt{1-\beta^2}}\right)$$
$$-2.33 = \Phi\left(\frac{-2.33 - 0.4\bar{m}}{\sqrt{1-0.4^2}}\right)$$
$$\bar{m} = -0.486$$

2.3 Example 3

2.3.1 Question

Using the Merton model to value the firm's debt and equity, which of the following scenarios in not possible? Assume the other three are true.

- Equity = 0 USD; Debt = 20 USD
- Equity = 10 USD; Debt = 35 USD
- Equity = 10 USD; Debt = 20 USD
- Equity = 0 USD; Debt = 35 USD

2.3.2 Answer

Combination of equity = 10 USD and debt = 20 USD is not possible, since equity cannot have a positive value until debt has reached its face value of 35 USD.

2.4 Example 4

2.4.1 Question

The XYZ Retirement Fund has 400 MUSD in assets and 370 MUSD in liabilities. Assume that the expected return on the surplus scaled by assets is 6% and the expected growth in liabilities is 5%. The volatility of asset growth is 10% and the volatility of the liability growth is 7%. Compute the volatility of the surplus growth assuming the correlation between assets and liabilities is 0.4.

2.4.2 Answer

$$\sigma^2_{surpl} = w^2_{ass}\sigma^2_{ass} + w^2_{liab}\sigma^2_{liab} - 2w_{ass}w_{liab}\sigma_{ass}\sigma_{liab}\rho$$
$$\sigma^2_{surpl} = 400^2 \cdot 0.010^2 + 370^2 \cdot 0.07^2 - 2 \cdot 400 \cdot 370 \cdot 0.10 \cdot 0.07 \cdot 0.40$$
$$\sigma^2_{surpl} = 1442.01$$
$$\sigma_{surpl} = 37.97$$

2.5 Example 5

2.5.1 Question

The ABC Retirement Fund has 300 MUSD in assets and 290 MUSD in liabilities. Assume that the expected return on the surplus scaled by assets is 6%. This means that the surplus is expected to grow by 18 MUSD over the first year. The volatility of the surplus 10%. Using a z-score of 1.65, compute VaR and the associated deficit that would occur with the loss associated with the VaR.

2.5.2 Answer

Surplus in 1Y is $300 - 290 + 18 = 28$ MEUR. 1Y VaR based on the z-score is $1.65 \cdot 300 \cdot 0.10 = 49.5$ MUSD. Therefore, the associated deficit is $28 - 49.5 = 21.5$ MUSD.

2.6 Example 6

2.6.1 Question

A share of Avedon, Inc. currently has a bid price of 59.50 USD and ask price of 60.00 USD. The sample standard deviation of bid-ask spread is 0.0003. Given this information, determine the expected transactions costs and 99% spread risk factor for transaction involving Avedon.

2.6.2 Answer

Midprice is defined as $P = (60.00 + 59.50)/2 = 59.75$ USD and bid-ask spread as $s = (60.00 - 59.50)/59.75 = 0.00837$. 99% spread risk factor is $f = \frac{1}{2}(0.00837 + 2.33 \cdot 0.0003) = 0.0045345$. Expected transaction costs are defined as $59.75 \cdot 0.0045345 = 2.71$ USD.

2.7 Example 6

2.7.1 Question

An analyst estimates that the hazard rate of a company is 0.16 per year. What is the probability of survival in the first year followed by a default in the second year?

2.7.2 Answer

$$p = \frac{\textit{probability of default in 2Y - probability of default in 1Y}}{\textit{probability of survival in 1Y}} = \frac{(1 - e^{-2\lambda}) - (1 - e^{-\lambda})}{e^{-\lambda}} = 0.14786$$

2.8 Example 7

2.8.1 Question

Suppose that you want to estimate the implied default probability for a BB-rated discount corporate bond.

- The T-bond (a risk-free bond) yields 8% per year.
- The one-year BB-rated discount bond yields 14% per year.
- The two-year BB-rated discount bond yields 21% per year.

If the recovery on a BB-rated bond is expected to be 0%, and the marginal default probability in year one is 7%, what is the estimate of the risk-neutral probability that the BB-rated discount bond defaults within the next 2Y?

2.8.2 Answer

$$(1 + 0.08)^2 = PD_{2Y} \cdot (1 + 0.21^2)$$
$$PD_{2Y} = 0.2033$$

2.9 Example 8

2.9.1 Question

Let us assume yearly hazard rate of 0.2. What is a default probability in the second year and conditional default probability after one year?

2.9.2 Answer

The default probability in the second year is $(1-e^{-0.2\cdot 2})-(1-e^{-0.2\cdot 1}) = 0.149$. Thus, the conditional probability of default after one year is $\frac{0.149}{e^{-0.2\cdot 1}} = 0.182$. Is to be noted that the conditional probability is memoryless and constant.

Chapter 3

Formulas

3.1 Book I

3.1.1 VaR and Extreme Value Theory

- arithmetic return $r_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}$
- geometric return $r_t = ln\left(\frac{P_t + D_t}{P_{t-1}}\right)$
- normal VaR $= -\mu + \sigma z_\alpha$
- log-normal VaR $= 1 - e^{\mu - \sigma z_\alpha}$
- spectral risk measure $M(\Phi) = \int_0^1 \Phi(p) q(p) dp$
- standard deviation of a risk measure $\sigma = \frac{\sqrt{p(1-p)/n}}{f(q)}$ where $p = 1 - \Phi(q + h/2)$ and $f(q) = \Phi(q + h/2) - \Phi(q - h/2)$
- weight in age-weighted historical simulation $w_i = \lambda^{i-1};\ w_1 = \frac{\lambda^{i-1}(1-\lambda)}{1-\lambda^n}$
- re-scaling of return under volatility-weighted historical simulation $r^*_{t,i} = \frac{\sigma_{T,i}}{\sigma_{t,i}} r_{t,i}$
- probability distribution over threshold $F_u(x) = P[X - u \leq u | X > u] = \frac{F(x+u) - F(u)}{1 - F(u)}$
- $\mu_P = w_x \mu_x + w_y \mu_y$
- $\sigma_P^2 = w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2 w_x w_y cov(x, y)$
- $VaR = \sqrt{\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \sigma_{x^2} & \rho_{xy} \\ \rho_{xy} & \sigma_{y^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$
- $VaR_{tot}^2 = VaR_{bnd}^2 + VaR_{eq}^2 + 2\rho VaR_{bnd} VaR_{eq}$
- joint default probability $P(AB) = \rho_{AB} \sqrt{PD_A(1 - PD_A) \cdot PD_B(1 - PD_B)} + PD_A \cdot PD_B$

3.1.2 Covariance and Correlation

- $cov(x, y) = \frac{\sum(x_i-\mu_x)(y_i-\mu_y)}{n-1}$
- $\rho_{xy} = \frac{cov(x,y)}{\sigma_x\sigma_y}$
- realised correlation from correlation swap $\rho_{realized} = \frac{2}{n^2-n}\sum_{i>j}\rho_{ij}$
- payoff for the investor buying the correlation swap $N \cdot (\rho_{realized} - \rho_{fixed})$
- mean-reversion model
  - $\Delta S_t = a(\mu - S_{t-t})\Delta t + \sigma\epsilon\sqrt{\Delta t}$
  - assuming $\Delta t = 1$ we get $E[\Delta S_t] = a(\mu - S_{t-t})$
  - applying regression on historical data we get $E[\Delta S_t] = \alpha - aS_{t-1}$
  - one-period lag autocorrelation equals $1 - a$
- Pearson correlation $\rho_{xy} = \frac{cov(x,y)}{\sigma_x\sigma_y}$
- Spearman's rank correlation $\rho_S = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$
- Kendall's rank correlation $\tau = \frac{n_c-n_d}{n(n-1)/2}$
- copula
  - $C : [0, 1]^n \to [0, 1]$
  - $C[G_1(u_1), ..., G_n(u_n)] = F_n[F_1^{-1}(G_1(u_1)), ..., F_n^{-1}(G_n(u_n)); \rho_F]$

3.1.3 Empirical Approaches to Risk Metrics and Hedges

- hedging nominal yields with real yields
  - $\Delta y_t^N = \alpha + \beta\Delta y_t^R + \epsilon_t$
  - $F^R = F^N \frac{DV01^N}{DV01^R}\beta$
- hedging 20Y swap rate with 10Y and 30Y swap rates
  - $\Delta y_t^{20} = \alpha + \beta^{10}\Delta y_t^{10} + \beta^{30}\Delta y_t^{30} + \epsilon_t$
  - $F^{10} = F^{20}\frac{DV01^{20}}{DV01^{10}}\beta^{10}$
  - $F^{30} = F^{20}\frac{DV01^{20}}{DV01^{30}}\beta^{30}$
- change-on-change regression $\Delta y_t = \alpha + \beta\Delta x_t + \Delta\epsilon_t$
- level-on-level regression $y_t = \alpha + \beta x_t + \epsilon_t$
- to solve problem with serial correlation one can use $\epsilon_t = \rho\epsilon_{t-1} + v_t$

3.1.4 Interest Rate Models

- Jenson's inequality $E[B(y)] > B(E[y])$
- risk premium to compensate interest rate uncertainty $(1+r_{1Y})(1+r_{2Y}+s)B_{t=0}^{averse} = (1+r_{1Y})(1+r_{2Y})B_{t=0}^{neutral}$
- model 1 $dr = \sigma dw = \sigma\sqrt{\Delta t}\epsilon$
- model 2 $dr = \lambda dt + \sigma dw$
- Ho-Lee model $dr = \lambda(t)dt + \sigma dw$
- Vasicek model
  - $dr = k(\theta - r)dt + \sigma dw$
  - drift term $\lambda \approx k(\theta - r_l)$
  - $E[r_T] = \theta - (\theta - r_0)e^{-kt}$
  - half-life T is defined through $e^{-kT} = \frac{1}{2}$
- model 3
  - $dr = \lambda(t)dt + \sigma(t)dw$
  - $dr = \lambda(t)dt + \sigma e^{-\alpha t}dw$
- CIR model $dr = k(\theta - r)dt + \sigma\sqrt{r}dw$
- log-normal model (model 4) $dr = ardt + \sigma rdw$
- log-normal model with drift $d[ln(r)] = a(t) + \sigma dw$
- Black-Karasinski model $d[ln(r)] = k(t)[ln(\theta(t)) - ln(r)]dt + \sigma(t)dw$

3.1.5 OIS Discounting

- LIBOR discounting is based on $\frac{swp_{2Y}}{1+r_{1Y}^{swp}} + \frac{swp_{2Y}}{(1+r_{2Y}^{swp})^2} = 1.00$. We know all but r_{2Y}^{swp}, which we calculate based on the formula.
- OIS discounting is based on $\frac{swp_{2Y}}{1+r_{1Y}^{ois}} + \frac{swp_{2Y}}{(1+r_{2Y}^{ois})^2} = \frac{fwd_{1Y}}{1+r_{1Y}^{ois}} + \frac{fwd_{2Y}}{(1+r_{2Y}^{ois})^2}$. We know all but fwd_{2Y}, which we calculate based on the formula.

3.1.6 Volatility Smiles

- put-call parity $c + Xe^{-rT} = S_0 + p$
- put-call parity holds both for Black-Scholes model and real market, therefore $p_{mkt} - p_{bsm} = c_{mkt} - c_{mbs}$

3.2 Book II

3.2.1 Credit Risk Models

- Black-Scholes-Merton option formula
  - $c = S_0 N(d_1) - Xe^{-rT} N(d_2)$
  - $p = Xe^{-rT} N(-d_2) - S_0 N(-d_1)$
  - $d_1 = \frac{ln(S_0/X)+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$
  - $d_2 = \frac{ln(S_0/X)+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$
- Merton model
  - payment to shareholders $= max(A_T - D, 0)$
  - payment to debtholders $= A_T - max(A_T - D, 0) = D - max(D - A_T, 0)$
  - probability of default $= N(-d_2)$; using r we get risk-neutral probability of default vs. using $E[ROA]$ we get real-world probability of default
  - LGD is absolute terms $= DN(-d_2) - A_0 e^{rT} N(-d_1)$
  - value of subordinated debt with nominal value of U is defined as $call(D) - call(D + U)$ where D represents senior debt
- KMV model
  - default threshold = short-term liabilities + 0.5 * long-term liabilities
  - distance to threshold $= \frac{\textit{expected asset value - default threshold}}{\sigma_{\textit{expected asset value}}}$
- credit factor model
  - asset return is defined as $a_i^T = ln(A_i^T / A_i^t)$
  - asset return can be modelled as $a_i^T = \beta_i m + \sqrt{1-\beta_i^2}\epsilon_i$
  - default occurs when $A_i^T < D_i$, which corresponds to $a_i^T < ln(D_i/A_i^t) = ln(1 - E_i^t/A_i^t) \approx -E_i^t/A_i^t$
  - default threshold is defined as $k_i = -E_i^t/A_i^t$ and is based on rating
  - default occurs when $a_i \leq k_i$
  - correlation between assets i and j is $\beta_i\beta_j$
  - conditional cumulative default probability is $p(m) = \Phi\left(\frac{k_i - \beta_i \bar{m}}{\sqrt{1-\beta_i^2}}\right)$
- vulnerable option
  - $c^* = max(min(V, S - X), 0)$
  - $c^* = (1 - PD) \cdot c + PD \cdot RR \cdot c$

3.2.2 Credit Risk and Probability of Default

- $EL = PD \cdot LGD \cdot EAD$
- investors would prefer a risky bond if $(1-PD)(1+r+z)+PD \cdot RR > 1+r$
- probability of default based on Bernoulli trial $PD_T = 1-(1-\pi)^T$
- exponential distribution $f(x) = \frac{1}{\beta}e^{-x/\beta}, x \geq 0; E[x] = \beta = 1/\lambda, D[x] = \beta^2 = 1/\lambda^2$
- Poisson distribution $P[x=k] = \frac{\lambda^k e^{-\lambda}}{k!}; \quad E[x] = \lambda, \quad D[x] = \lambda^2$
- cumulative default time distribution $P[t^* < t] = F(t) = 1 - e^{-\lambda t}$
- survival distribution $P[t^* \geq t] = 1 - F(t) = e^{-\lambda t}$
- marginal default $f(t) = \lambda e^{-\lambda t}$
- instantaneous conditional default probability λdt
- risk-neutral hazard rate $\lambda \approx \frac{z}{1-RR}$
- time-varying hazard rate $\lambda(t)$ implies cumulative default time distribution in form of $F(t) = 1 - e^{-\int_0^t} \lambda(s)ds$
- default correlation $\rho_{12} = \frac{\pi_{12}-\pi_1\pi_2}{\sqrt{\pi_1(1-\pi_1)}\sqrt{\pi_2(1-\pi_2)}}$

3.2.3 Credit Exposure

- netting factor $= \frac{\sqrt{n+n(n-1)\rho}}{n}$
- EE during remargin period $= 0.40 \cdot \sigma_E \cdot \sqrt{T_M}$
- PFE during remargin period $= z_\alpha \cdot \sigma_E \cdot \sqrt{T_M}$

3.2.4 Credit Value Adjustment

- $CVA \approx LGD \cdot \sum_i DF_i \cdot EE_i \cdot PD_i$
- CVA as running spread $= s_{CDS} \cdot EPE$
- CVA as IRS spread $=$ *CVA / dollar duration*

3.2.5 Other

- required number of defaults to breach a tranche $= n\left(\frac{X\%}{1-RR}\right)$, where n is number of underlying credits and $X\%$ is an attachment point

3.3 Book III

3.3.1 RAROC

- $RAROC = \frac{\textit{expected revenues - costs -expected losses - taxes + return on economic capital} \pm \textit{transfers}}{\textit{economic capital}}$
- $h_{AT} = \frac{CE \cdot r_{CE} + PE \cdot r_{PE}}{CE + PE}$
- $\textit{adjusted RAROC} = RAROC - \beta_E (r_M - r_F)$

3.3.2 Repo Market

- penalty rate $= \max(3\% - \textit{federal funds rate}, 0)$
- special spread $\in < 0, \textit{penalty rate} >$

3.3.3 Liquity

- constant spread approach
  - $LC = \frac{1}{2} \cdot V \cdot spread$
  - $spread = \frac{\textit{ask price - bid price}}{(askprice + bidprice)/2}$
  - $LVaR = VaR + LC$
- exogenous spread approach
  - $LVaR = VaR + \frac{1}{2}(\mu + \sigma z_\alpha)$
- endogenous spread approach
  - $E = \frac{\Delta P / P}{\Delta N / N}$
  - $LVaR = VaR \cdot \left(1 - \frac{\Delta P}{P}\right) = VaR \cdot \left(1 - E \frac{\Delta N}{N}\right)$
- $\frac{LVAR}{VaR}|_{combined} = \frac{LVaR}{VaR}|_{exogenous} \cdot \frac{LVaR}{VaR}|_{endogenous}$
- CrashMetric $\Pi = \delta \Delta S + \frac{\gamma}{2}(\Delta S)^2$; Π is minimised (= maximum loss) at $\Pi = \frac{\delta^2}{2\gamma}$ for $\Delta S = -\frac{\delta}{\gamma}$

3.3.4 Liquidity

- $L = \frac{A}{E} = 1 + \frac{D}{E}$
- $r_E = L \cdot r_A - (L - 1) r_D$

3.3.5 Basel

- credit equivalent for OTC derivatives $= \max(V, 0) + a \cdot L$
- worst case probability of default $WCDR_i = N\left[\frac{N^{-1}(PD_i) + \sqrt{\rho} N^{-1}(0.999)}{\sqrt{1-\rho}}\right]$
- $\rho = 0.12 \cdot (1 + e^{-50 \cdot PD})$

3.4 Book IV

3.4.1 Portfolio Construction

- alpha scaling α = *volatility * information coefficient * score*
- *risk aversion* $= \frac{\textit{information ratio}}{\textit{2 * active risk}}$
- *marginal value added* = α - *2 * risk aversion * active risk * marginal active risk*
- no-trade range *0* < *marginal value added* < *transaction costs*

3.4.2 Portfolio Risk Management

- $\sigma_P^2 = \sigma\sqrt{\frac{1}{N} + \left(1 - \frac{1}{N}\right)\rho}$
- $MVaR_i = \frac{cov(r_i,r_p)}{\sigma_P} z_\alpha = \frac{VaR_P}{NPV_P}\beta_i,\ \beta_i = \frac{cov(r_i,r_P)}{\sigma_P^2}$
- $VaR = \sum CVaR_i = VaR \sum w_i\beta_i$
- portfolio risk will be at a global minimum when $MVaR_i = MVaR_j$
- optimal portfolio is defined by $\frac{r_i - r_F}{MVaR_i} = \frac{r_j - r_F}{MVaR_j}$
- optimal portfolio for eliptical distributions is defined by $\frac{r_i - r_F}{\beta_i} = \frac{r_j - r_F}{\beta_j}$
- $r_{surplus} = \frac{\Delta Assets}{Assets} - \frac{\Delta Liabilities}{Liabilities}\frac{Liability}{Assets} = r_{asset} - r_{liability}\frac{Liabilities}{Assets}$
- weight of portfolio managed by manager i $= \frac{IR_i \cdot portfolio\text{'s tracking error}}{IR_P \cdot manager\text{'s tracking error}}$
- Sharpe ratio $S = \frac{r - r_F}{\sigma}$
- Treynor ratio $T = \frac{r - r_F}{\beta}$
- Jensen's alpha $r - r_F = \alpha + \beta(r_M - r_F)$
- information ratio $IR = \frac{r_A - r_B}{\sigma_{A-B}} = \frac{\alpha_A}{\sigma(\epsilon_A)}$
- alpha t-statics $t = \frac{\alpha - 0}{\sigma/\sqrt{(N)}}$
- superior market timing skills $r_P - r_F = \alpha + \beta(r_M - r_F) + M(r_M - r_F)D + \epsilon$
- style analysis $r_P = \beta_1 r_1 + \beta_2 r_2 + ... + \beta_b r_n + \epsilon$

3.4.3 Liquidity

- liquidity duration $LD = \frac{Q}{0.10 \cdot V}$
- Amihund measure of bond liquidity $= \frac{\text{daily absolute price change in bps}}{\text{trading volume}}$
- Amihund measure of equity liquidity $= \frac{\text{daily absolute stock return}}{\text{trading volume}}$
- market efficiency coefficient $= \frac{\text{variance of long-term stock returns}}{\text{variance of short-term stock returns}}$