Stress-Testing Probability of Default and Migration Rate with respect to Basel II Requirements

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Abstract

Basel II implementation requires the estimations of probability of default (PD) and migration rate under hypothetical or historically observed stress scenarios. Typically, financial institutions first forecast selected macroeconomic variables under these stress scenarios and then estimate the corresponding stressed PD and migration rates. These stressed parameters are in turn used in estimating the capital requirement and marked-to-market losses within the capital adequacy assessment framework. In this paper, we demonstrate a practical methodology to estimate both stressed PD and migration rates in a consistent fashion while conditioning on the selected explanatory variables. The estimation methodology allows for the robust use of external data, which is essential in the absence of long enough internal data. The proposed methodology is consistent with the economic model underlying Basel II Pillar 1 risk-weight function, and it is compatible with the financial institutions' existing internal risk rating systems making the implementation and operations cost effective. It also allows for the stress testing at the overall portfolio level as well as that for specific industries. An example of its implementation on the oil and gas sector is provided at the end of the paper. Although the paper is presented within the context of Basel II implementation, the proposed approach can be used as a sound risk management practice to identify, examine and quantify the effect of stress events.

Keywords: Basel II, Stress Testing, Correlation, Probability of Default and Transition, Migration Rate

I. Introduction

Stress testing plays an important role in risk and capital management for financial institutions (FIs). FIs need to assess how their portfolios behave under stress conditions, historically observed or otherwise plausible. What would be the changes in the expected probability of default (PD), loss-given-default (LGD) and migration rates in their credit portfolios if the economy experiences a moderate or severe downturn? Also, can one estimate the default and migration rates that might be realized over the next year based on currently observed macroeconomic leading indicators? It is a timely issue given the uncertainty surrounding the credit markets subsequent to the melt-down of the US sub-prime mortgage market. A tool to assist FIs in generating these "what if" scenarios would be invaluable. Stress testing is also a required component of the implementation of Basel II. Within its Pillar 1's credit risk framework, the parameters governing PD, LGD, exposure-at-default (EAD) and migration rates need to be estimated under stress conditions. Paragraphs 434-437 of BCBS (2004) outline the principles of regulatory stress testing. Specifically, "Stress testing must involve identifying possible events or future changes in economic conditions that could have unfavourable effects on a bank's credit exposures and assessment of the bank's ability to withstand such changes." In achieving this objective, the use of information on external (rating agencies') ratings in supplementing FI's own default and migration experiences is suggested. Under Pillar II of Basel II, the stress parameters (PD, LGD, EAD and migration rates) are translated into marked-to-market (MTM) losses and capital requirements, which serve as important inputs when FIs examine the adequacy of their capital level. Figure 1 depicts the role of stress testing in the two pillars of Basel II.

INSERT FIGURE 1 ABOUT HERE

The implementation of Pillar II requires the design of a capital assessment process where alternative inputs like economic capital and stressed economic capital, potential MTM losses under stress events, earnings retention and dividend policy, planned growth and risk taking policy, pro-cyclicality management policy are incorporated within the context of FIs' risk appetite and capital management policy. The building blocks in estimating the stressed regulatory and economic capital, and MTM losses under the stress events for credit risk are stressed PDs, LGDs, EADs and migration rates.

A typical stress testing framework is presented in Figure 2, in which selected historical and hypothetical stress scenarios are first expressed in terms of selected observable variables (e.g. macroeconomic variables). These "stressed" macroeconomic variables are then used to determine the stressed PDs, LGDs, EADs, and migration rates, which are in turn used to generate stressed capital requirements and MTM losses corresponding to the stress scenario selected. In this framework, we need a model to compute the conditional PDs, LGDs, EADs, and migration rates contingent on the stressed levels of the macroeconomic variables. The inputs to this model are the current (unconditional) PDs, LGDs and EADs assigned to the risk ratings and their long-run migration rates.

INSERT FIGURE 2 ABOUT HERE

With the objective of satisfying the above requirements, we consider in this study a stress testing model of PDs and migration rates which is based on the factor model of firm's asset value. It is the structural model underlying Pillar 1's risk-weight formulation. It is also the model used by Vasicek (1987) in deriving the loss distribution of an infinitely granular credit portfolio. The characteristics and properties of this model are derived by Gordy (2003). The proposed model allows for the stressing of both PDs and migration rates within a consistent framework, thus allowing us to characterize both the default rates and portfolio compositions under stress events which are defined by observable macroeconomic variables.[1] Essentially, we adopt a specific version of the Bernoulli regression models of default rates considered by Frey and McNeil (2003) in formulating our stress testing models. It can also be considered as the multifactor version of the conditionally independent credit risk models introduced by Schönbucher (2000), in which defaults become (cross-sectionally) independent when conditioning on the realizations of (a small number of) systematic factors. Rösch and Scheule (2005) adopt a similar multifactor approach in modeling and estimating the systematic variations of both defaults and recovery rates.[2] McNeil and Wendin (2007) document the importance of incorporating an unobservable while serially-correlated random component in the systematic factor and demonstrate how it can be calibrated by using the Bayesian techniques.

Different from the previous research, we extend the framework by modeling both the dynamics of default rates and migration rates of all risk ratings of an internal risk rating system in a consistent fashion for the purpose of stress testing under Basel II requirement.[3] We can therefore model the increase in not only default rates but also portfolio downgrades during a specific stress event. Modeling of default rates allows us to assess the marked-to-market losses immediately after the stress event. Modeling of downgrade rates on the other hand is required to quantify the effect of portfolio deteriorations following the stress event. Capturing the increases in both default rates and downgrade rates are therefore required to fully quantify the effect of the stress event.

In this study, we also outline an estimation methodology that can cater for the use of external (rating agencies') default rate data in supplementing the potential lack of internal default rate data; while at the same time able to account for the differences in the two sets of data. The ability to utilize external default rate data robustly is essential in practice given the fact that data internal to banks rarely cover more than a single credit cycle and thus the resultant model is unlikely to be representative of the different forms of

[1] A number of studies (e.g. Helwege and Kleiman (1996), Keenan et al (1999), Chava et al (2006), and Carling et al (2007)) have documented the importance of macroeconomic conditions in explaining the observed corporate default rates.

[2] Rather than adopting the structural credit risk models, other researchers (e.g. Duffie et al (2007)) study the relation between conditional probability of default and macroeconomic variables by considering the intensity-based credit risk models.

[3] Our model can be considered as an extension of the one-parameter representation of transition matrics considered by Belkin et al (1998). In their study, the single systematic factor is latent and there is no attempt to establish the time-series relation with any observable explanatory variables.

historically observed stress events.[4] In estimating the model using external data while implementing it internally, we need to fully account for the differences in the characteristics between the internal and external credit portfolio. For example, the levels of long-run probabilities of defaults, the degree of asset correlations among the borrowers, and the risk rating philosophies may be quite different for the internal and external portfolios.[5,6] The proposed estimation methodology allows for these portfolio-specific parameters to be adjusted independently based on banks' internal default experiences, whereas the relation between the systematic default risk and the explanatory variables being established through the use of external default rate data. Other desirable implementation characteristics of the proposed methodology are:

- It is consistent with the structural credit risk model underlying Basel II Pillar 1 formulation of risk-weight function;
- It is compatible and thus can be implemented as "add-on" to banks' currently adopted internal risk rating systems and portfolio value-at-risk models;
- Besides allowing for the stress testing of PD and migration rates, it can be readily extended to include the stress testing of LGD and EAD within the same framework;
- It caters for the developments of both *broad-base* (covering the overall portfolio of a bank) and *sector-specific* (only covering a specific business or geographical sector) stress models, thus allowing banks to address different business questions in its stress testing endeavors.

Unlike commonly-used credit risk models (e.g. *CreditMetrics* and Moody's *Credit Transition Model*) in which stress testing can be conducted by changing the values of the pre-specified explanatory variables which have been calibrated against external default experiences (of typically US-issuers), the proposed methodology allows for customization to suit the specific characteristics of the banks' internal portfolios and the circumstances of the specific economic environment in which the banks conduct their businesses. For example, if a bank is stressing its entire US credit portfolio, it would most likely consider those macroeconomic variables that explain the general health of the economy (e.g. GDP, unemployment rate, short-term interest rates etc). On the other hand, if the bank wants to develop a model specifically for a sub-segment of its US portfolio, say only consisting of issuers in the oil and gas sector, crude oil prices and international demand and supply of oil products are likely to be better explanatory variables than the more general economic indicators. Another advantage of adopting a customized-approach is the ability to calibrate the model against historical default rates obtained from portfolios of which the characteristics resemble those of the internal portfolio (as data permit). For example, a model calibrated against the default experiences of all the corporations in Canada is unlikely to be suitable in conducting stress testing on corporations in the Province of Alberta, given that its economy is more sensitive to any changes in the commodity prices and the demand of natural resources

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[4] On the other hand, external default rate data (e.g. those compiled by Standard & Poor's) dated back to the early 1980s covering multiple credit cycles.

[5] For example, asset correlations are typically higher for large corporate borrowers than for mid-market borrowers.

[6] For example, internal risk ratings are more likely to be *point-in-time* than external risk ratings, resulting in a higher transition mobility of the borrowers.

than the rest of Canada. Default rate data specific to the Province should be used instead in establishing the macroeconomic drivers of the systematic credit risks. Finally, customization allows users to select those explanatory variables which are considered to be essential in describing the (hypothetical) stress scenarios which are of interest to the senior management of the banks.

The rest of the paper is organized as follows. We introduce the proposed model in the next section. In Section III, we outline a three-step process in calibrating the model and computing the stressed probabilities of defaults and transitions. In Section IV, we demonstrate the approach by developing a stress testing model for the oil and gas sector. Finally, we conclude the paper with Section V.

II. Proposed Model

We adopt the structural credit risk model, where borrowers are uniform in terms of their credit risks within a certain risk rating m (m = 1, 2, …, N for a risk rating system of N different risk ratings) of the portfolio. Their asset values p_t at time t is driven by both the systematic PD risk P_t and the borrower-specific PD risk $e_{PD,t}$. For example, for borrower i, asset value at time t is given by

$$p_t^i = R \times P_t + \sqrt{1-R^2} \times e_{PD,t}^i \tag{1}$$

Equation (1) depicts the single-factor model considered by Vasicek (1987) in deriving the loss distribution of a credit portfolio. The idiosyncratic risk factor $e_{PD,t}^i$ is modeled as a standard normal distribution, where $e_{PD,t}^i$ is independent of $e_{PD,t}^j$ for $i \neq j$. Borrower in risk rating m defaults when asset value p_t becomes less than some constant default point (DP_m). The coefficient R is uniform across borrowers and measures the sensitivity of individual asset values to the systematic PD risk. Under this setting, variance and pair-wise correlation of asset values are given by:[7]

$$Var\left(p_t^i\right) = 1-(1-\sigma_P^2)R^2 \tag{2a}$$

$$Correl\left(p_t^i, p_t^j\right) = \frac{R^2\sigma_P^2}{1-(1-\sigma_P^2)R^2} \quad \text{for } i \neq j \tag{2b}$$

where σ_P is the standard deviation of P_t.

This is the framework underlying the credit risk model implicit in Basel II Pillar 1 risk-weight function, in which P_t is assumed to follow a standard normal distribution and the credit portfolio is made up of infinite number of (uniform) borrowers.[8] Miu and Ozdemir

[7] Variance and pair-wise asset correlation are equal to unity and R^2 respectively if P_t has unit standard deviation.

[8] For the reconciliation between the Basel II Pillar 1 credit capital formula and Vasicek (1987) model, see for example Gordy (2003) and Miu and Ozdemir (2008).

(2008) show that Basel II long-run probability of default (LRPD) requirement is equivalent to the *unconditional probability* of p_t being smaller than DP_m . That is,

$$\Pr\left[p_t^i < DP_m\right]. \tag{3}$$

LRPD of risk rating m is therefore governed by DP_m, R and the unconditional distribution of P_t via equation (1).[9] It can also be shown that, conditioning on realizing P_t, the probability of observing a default rate $\theta_{m,t}$ at time t of a portfolio made up of $n_{m,t}$ number of borrowers in risk rating m is given by:

$$\Omega\left(\theta_{m,t}; P_t, R, DP_m, n_{m,t}\right) = \begin{pmatrix} n_{m,t} \\ n_{m,t} \cdot \theta_{m,t} \end{pmatrix} \times \left(\Phi\left(z(P_t, R, DP_m\right)\right)^{n_{m,t} \cdot \theta_{m,t}} \times \left(1 - \Phi\left(\left(z(P_t, R, DP_m\right)\right)\right)^{n_{m,t}\left(1-\theta_{m,t}\right)} \tag{4}$$

where $\Phi(\bullet)$ is the cumulative standard normal distribution function, and

$$z\left(P_t, R, DP_m\right) = \frac{1}{\sqrt{1-R^2}}\left(DP_m - R \cdot P_t\right). \tag{5}$$

When $n_{m,t}$ approaches infinity, the conditional distribution of $\theta_{m,t}$ *collapses* into a deterministic function of P_t.

$$\theta_{m,t} = \Phi\left[\frac{1}{\sqrt{1-R^2}}\left(DP_m - R \cdot P_t\right)\right] \tag{6}$$

Analogous to the (unconditional) LRPD and the distribution of default rates conditional on P_t, we can also consider the (unconditional) long-run probability of transition (LRPT) and the conditional distribution of portfolio migration rates across different risk ratings by defining a set of migration thresholds $TH_{m,q}$ for each risk rating m (where m = 1, 2, …, N and q = 1, 2, …, N-1), for a rating system with N different risk ratings. LRPT from rating m_1 to m_2 is therefore the unconditional probability of p_t lying among the threshold levels.

$$LRPT_{m_1,m_2} = \begin{cases} \Pr\left[p_t^i \geq TH_{m_1,m_2}\right] & \text{for } m_2 = 1 \\ \Pr\left[TH_{m_1,m_2-1} > p_t^i \geq TH_{m_1,m_2}\right] & \text{for } m_2 = 2 \text{ to } N-1 \\ \Pr\left[TH_{m_1,m_2-1} > p_t^i \geq DP_{m_1}\right] & \text{for } m_2 = N \end{cases} \tag{7}$$

The LRPT from risk rating m_1 to a risk rating **equal to or higher than** m_2 (i.e. including the default state) is therefore equal to:

$$LRPD_{m_1} + \sum_{s=m_2}^{N} LRPT_{m_1,s} = \begin{cases} \Pr\left[p_t^i < TH_{m_1,m_2-1}\right] & \text{for } m_2 \neq 1 \\ 1 & \text{for } m_2 = 1 \end{cases} \tag{8}$$

[9] Financial institutions, which adopt the internal rating based approach of Basel II, are required to estimate LRPD for each of their internal risk ratings.

Let $\varphi_{m_1,m_2,t}$ be the migration rate from risk rating m_1 to a risk rating equal to or higher than m_2 at time t. Suppose $\varphi_{m_1,m_2,t}$ is observed for a portfolio of $n_{m_1,t}$ borrowers in risk rating m_1. It can be shown that, conditioning on realizing P_t, the probability of observing $\varphi_{m_1,m_2,t}$ is equal to:

$$\Omega^*\left(\varphi_{m_1,m_2,t}; P_t, R, TH_{m_1,m_2-1}, n_{m_1,t}\right) = \binom{n_{m_1,t}}{n_{m_1,t}\cdot\varphi_{m_1,m_2,t}} \times \left(\Phi\left(z^*(\bullet)\right)\right)^{n_{m_1,t}\cdot\varphi_{m_1,m_2,t}} \times \left(1-\Phi\left(\left(z^*(\bullet)\right)\right)\right)^{n_{m,t}(1-\varphi_{m_1,m_2,t})} \quad (9)$$

where

$$z^*(\bullet) = \begin{cases} \dfrac{1}{\sqrt{1-R^2}}\left(TH_{m_1,m_2-1} - R\cdot P_t\right) & \text{for } m_2 \neq 1 \\ \infty & \text{for } m_2 = 1 \end{cases}. \quad (10)$$

When $n_{m_1,t}$ approaches infinity, the conditional distribution of $\varphi_{m_1,m_2,t}$ again *collapses* into deterministic functions of P_t.

$$\varphi_{m_1,m_2,t} = \begin{cases} \Phi\left[\dfrac{1}{\sqrt{1-R^2}}\left(TH_{m_1,m_2-1} - R\cdot P_t\right)\right] & \text{for } m_2 \neq 1 \\ 1 & \text{for } m_2 = 1 \end{cases} \quad (11)$$

To allow for the stress testing of probability of default and transition among risk ratings, we explain the systematic PD risk P_t with J explanatory variables, $X_t^1, X_t^2, ..., X_t^J$.[10,11]

$$P_t = f\left(X_t^1, X_t^2, ..., X_t^J\right) + e_t \quad (12)$$

These explanatory variables could be macroeconomic variables, leading economic indicators and/or industry-specific variables, which can explain the systematic PD risk of the credit portfolio.[12] In practice, banks would like to develop both *broad-base* and *sector-specific* stress testing models. Broad-base models which cover the overall portfolio help banks to quantify the impact of the stress events on the performance of the aggregate credit portfolio. Sector-specific models are, on the other hand, industry specific, such as for the sectors of oil and gas, real estates, or financial services. Through the uses of these sector-specific models, banks can achieve a more detail understanding of the impact of the stress events on certain industries which are particularly important to them. The sensitivities of the credit risks to the explanatory variables are likely to be different for different sectors. For example, the price of crude oil could be an important explanatory variable of credit risks for those companies in the oil and gas sector, whereas it is unrelated to the credit risks of those in real estates.

The first term of Equation (12) (i.e. $f(\bullet)$) can be interpreted as the *explainable* component of P_t, whereas the second term e_t is the residual representing the

[10] Besides the contemporaneous values of the explanatory variables, we can also include the lagged values of these variables as possible explanatory variables.
[11] Rösch and Scheule (2005) consider a similar model where DP (rather than P_t) is time varying and is a function of the explanatory variables.
[12] A list of criteria in selecting explanatory variables can be found in the Appendix.

unexplainable component.[13] The residual e_t is assumed to be independent of $X_t^1, X_t^2, ..., X_t^J$ and normally distributed with zero mean and standard deviation equals to σ_e. However, e_t is unlikely to be serially independent. Historically observed default rates are *persistent*. That is, an above (below) average default rate is more likely to be followed by another above (below) average one in the subsequent year. In the proposed model, part of this serial correlation in PD risk can be explained by the serial correlation inherent in the explanatory variables. Nevertheless, residual serial correlation effect might still remain after the explanatory variables are fully accounted for. Using default rates of US corporate obligors observed from 1981 to 2000, McNeil and Wendin (2007) document a statistically significant positive serial correlation. We cater for this residual effect by modeling e_t itself as a time-series process. We follow McNeil and Wendin (2007) and model it as an AR(1) process as depicted in Equation (13).

$$e_t = \rho \cdot e_{t-1} + u_t \tag{13}$$

The residuals e_t therefore has a variance-covariance matrix of

$$\sigma_e^2 \Lambda_e = \sigma_e^2 \begin{bmatrix} 1 & \rho & \rho^2 & . & \rho^{T-1} \\ \rho & 1 & \rho & . & \rho^{T-2} \\ \rho^2 & \rho & 1 & . & \rho^{T-3} \\ . & . & . & . & . \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & . & 1 \end{bmatrix} \tag{14}$$

where ρ is the serial correlation and T is the number of time periods under consideration.

Once we specify the functional form of $f(\bullet)$ and all the governing parameters (i.e. R, DP_m, $TH_{m,q}$, σ_e, ρ, and the parametric values of $f(\bullet)$) estimated, the proposed model allows us to calculate the conditional (i.e. stressed) probability of default and transition under different stress scenarios expressed in terms of the selected explanatory variables (e.g. $X_t^1 = x_t^1, X_t^2 = x_t^2, ..., X_t^J = x_t^J$).[14] It can be shown that the PD of risk rating m conditional on the realization of the stress scenario can be estimated by evaluating the conditional probability:

$$\Pr\left[p_t^i < DP_m \middle| X_t^1 = x_t^1, X_t^2 = x_t^2, ..., X_t^J = x_t^J\right] = \Phi\left[\frac{DP_m - R \times f\left(x_t^1, x_t^2, ..., x_t^J\right)}{\sqrt{1 - R^2 + \sigma_e^2 R^2}}\right]; \tag{15}$$

[13] In practice, we may adopt different functional forms for $f(\bullet)$, with the simplest being the linear and quadratic versions.

[14] Please refer to the next section on estimation of parametric values of the proposed model.

whereas the stressed probability of transition (PT) from risk rating m_1 to m_2 is given by:

for $m_2 = 1$, $\Pr\left[p_t^i \geq TH_{m_1,m_2} \middle| X_t^1 = x_t^1, X_t^2 = x_t^2, ..., X_t^J = x_t^J\right]$

$$=1-\Phi\left[\frac{TH_{m_1,m_2} - R \times f\left(x_t^1, x_t^2, ..., x_t^J\right)}{\sqrt{1-R^2+\sigma_e^2 R^2}}\right] \tag{16a}$$

for $m_2 = 2$ to N-1, $\Pr\left[TH_{m_1,m_2-1} > p_t^i \geq TH_{m_1,m_2} \middle| X_t^1 = x_t^1, X_t^2 = x_t^2, ..., X_t^J = x_t^J\right]$

$$=\Phi\left[\frac{TH_{m_1,m_2-1} - R \times f\left(x_t^1, x_t^2, ..., x_t^J\right)}{\sqrt{1-R^2+\sigma_e^2 R^2}}\right] - \Phi\left[\frac{TH_{m_1,m_2} - R \times f\left(x_t^1, x_t^2, ..., x_t^J\right)}{\sqrt{1-R^2+\sigma_e^2 R^2}}\right] \tag{16b}$$

for $m_2 = N$, $\Pr\left[TH_{m_1,m_2-1} > p_t^i \geq DP_{m_1} \middle| X_t^1 = x_t^1, X_t^2 = x_t^2, ..., X_t^J = x_t^J\right]$

$$=\Phi\left[\frac{TH_{m_1,m_2-1} - R \times f\left(x_t^1, x_t^2, ..., x_t^J\right)}{\sqrt{1-R^2+\sigma_e^2 R^2}}\right] - \Phi\left[\frac{DP_{m_1} - R \times f\left(x_t^1, x_t^2, ..., x_t^J\right)}{\sqrt{1-R^2+\sigma_e^2 R^2}}\right] \tag{16c}$$

III. Estimation Methodology

We can simultaneously estimate all the parameters of the proposed model, namely R, DP_m (for m = 1 to N), $TH_{m,q}$ (for m = 1 to N and q = 1 to N-1), σ_e, ρ, and the parametric values of specified function $f(\bullet)$, by maximum likelihood estimation if the following are observed for a particular credit portfolio:

- Default rates $\theta_{m,t}$ of risk ratings m = 1, 2, …, N at time t = 1, 2, …, T;
- Transition matrices (each defined by $N \times N$ elements of $\varphi_{m_1,m_2,t}$) of the risk ratings at time 1, 2, …, T; and
- Panel data of explanatory variables $X_t^1, X_t^2, ..., X_t^J$ at time t = 1, 2, …, T.

Specifically, we can estimate the parameters by maximizing the sum of the logarithmic of the conditional probabilities of Equations (4) and (9), after integrating out the residual e_t.

$$\log L = \sum_{m=1}^{N} \log\left[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \Omega\left(\theta_{m,t};\bullet\right)\phi\left(e_1,...,e_T;0,\sigma_e^2\Lambda_e\right)de_1...de_T\right]$$
$$+\sum_{m_1=1}^{N}\sum_{m_2=1}^{N} \log\left[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \Omega^*\left(\varphi_{m_1,m_2,t};\bullet\right)\phi\left(e_1,...,e_T;0,\sigma_e^2\Lambda_e\right)de_1...de_T\right] \tag{17}$$

where $\phi\left(e_1,...,e_T;0,\sigma_e^2\Lambda_e\right)$ is the density function of the T-dimension multivariate normal distribution with zero mean and covariance matrix equals to $\sigma_e^2\Lambda_e$.

There are however a couple of practical issues which might forbid us to adopt this approach which is supposed to be the most efficient. First of all, rarely we have the luxury of having historical default rates on each internal risk ratings of the banks which span more than a single credit cycle. With such a short data series, it is difficult if at all possible to establish a statistically robust relation between internal default rates and the selected explanatory variables, which may be used to model stress conditions. It is more practical to adopt a three-step approach. In the first step, the relation between systematic PD risk and the explanatory variables (i.e. the parametric values of function $f(\bullet)$, σ_e, and ρ) are established by estimating the proposed model with external default rate data (e.g. compiled by Standard & Poor's or Moody's) which easily dated back to early 1980. In the second step, pair-wise correlation R, DP_m (for m = 1 to N), $TH_{m,q}$ of the banks' internal risk ratings are then estimated by assuming the systematic PD risk P_t implicit in the external default rate data is identical to that of the banks' internal portfolios.[15] The estimations are conducted by using the banks' limited internal default rate data and transition matrices. This ensures the resulting stressed PD and PT will be consistent with the default experiences of the banks' internal portfolios. In the third step, we combine the estimated results of Step 1 (i.e. parametric values of function $f(\bullet)$, σ_e, and ρ) and Step 2 (i.e. R, DP_m, $TH_{m,q}$) in evaluating the stressed PD and PT conditional on selected stress scenarios.

The second issue is about computational efficiency. Even though we might have a rich data set of long history, in practice we still might not want to estimate the full-scale model by maximizing the log-likelihood of Equation (17). Optimizing Equation (17) involves numerical integrations and closed-form expressions of the derivatives of the likelihood function are not available. The inefficiency of the related computation efforts might not be acceptable given the fact that a large number of specifications of the functional form of $f(\bullet)$ needed to be evaluated and their performances compared in order to identify the model that can best explain the variation of systematic PD risks. One way to improve the computational efficiency is to consider the limiting conditions where the number of borrowers $n_{m,t}$ approaches infinity. Under this approximation, realized default rates and migration rates become deterministic functions of P_t (and thus e_t) via Equations (6) and (11). Computational efficiency can thus be significantly enhanced.

In the rest of this section, we describe in details the three-step approach adopted in estimating the proposed model for stress testing internal risk ratings.

Step 1: Estimation of systematic PD risk (i.e. parametric values of function $f(\bullet)$, σ_e, and ρ) by using external default rate data

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[15] The validity of this assumption can be enhanced by ensuring the compilation of the external default rate data is conducted in a way such that they are consistent with the characteristics of the internal portfolio. For example, in extracting historical data from S&P's CreditPro[TM] (which is one of the sources of external default rate data), it allows user to specify a number of characteristics (e.g. industrial compositions) of the portfolio over which default rate data are compiled.

Suppose we observed the time-series of default rates $\theta_{x,t}$ of an external portfolio and J explanatory variables $X_t^1, X_t^2, ..., X_t^J$ at time t = 1, 2, …, T_x.[16,17] If we consider the limiting condition in which the number of borrowers $n_{x,t}$ approach infinity, we can express e_t as a deterministic function of the observables $\theta_{x,t}$ and $X_t^1, X_t^2, ..., X_t^J$ by combining Equations (6) and (12).[18,19]

$$e_t = \frac{DP_x - R_x f\left(X_t^1, X_t^2, ..., X_t^J\right) - \sqrt{1-R_x^2}\,\Phi^{-1}\left(\theta_{x,t}\right)}{R_x} \tag{18}$$

The likelihood of observing the time-series of $\theta_{x,t}$ and $X_t^1, X_t^2, ..., X_t^J$ are therefore proportional to the density of the following multivariate normal distribution.

$$L_x = \frac{\exp\left[-\frac{1}{2}\left(\frac{DP_x - R_x \mathcal{F}(\bullet) - \sqrt{1-R_x^2}\,\Phi^{-1}\left(\Theta_x\right)}{R_x}\right)'\left(\sigma_e^2 \Lambda_e\right)^{-1}\left(\frac{DP_x - R_x \mathcal{F}(\bullet) - \sqrt{1-R_x^2}\,\Phi^{-1}\left(\Theta_x\right)}{R_x}\right)\right]}{\left(2\pi\right)^{T/2}\left|\sigma_e^2 \Lambda_e\right|^{1/2}} \tag{19}$$

where $\Theta_x = \left[\theta_{x,1} \quad \theta_{x,2} \quad ... \quad \theta_{x,T_x}\right]'$ and

$$\mathcal{F}(\bullet) = \left[f\left(X_1^1, ..., X_1^J\right) \quad f\left(X_2^1, ..., X_2^J\right) \quad ... \quad f\left(X_{T_x}^1, ..., X_{T_x}^J\right)\right]'.$$

We can therefore obtain the MLE estimators of $DP_x, \sigma_e, R_x, \rho$, and the parameters governing $f(\bullet)$ by maximizing log(L_x).[20] As discussed in the previous section, LRPD of

[16] The length of the time-series of external default rate data T_x is most likely to be much longer than that of the default rate data of any of the banks' internal risk ratings.

[17] Both *non-overlapping* (e.g. annually-observed 12-month default rates) and *overlapping* (e.g. quarterly-observed 12-month default rates) time-series may be used in the analysis. The use of the latter would result in a higher estimate of the serial correlation coefficient ρ of Equation (13).

[18] Subscript x denotes parameters specific to the external portfolio which might not be relevant to banks' internal risk ratings.

[19] We suggest the use of the aggregated default rates of all speculative-grade issuers in Step 1. Speculative-grade default rates are believed to be more sensitive to any changes in the selected explanatory variables than those of investment-grade, and thus able to more readily capture the time-series relation with these variables. Default rates of any single letter grade might not be appropriate given the fact that the number of issuers in any single letter grade in any particular year could be limited and thus violating the limiting condition of infinite granularity being invoked here. The simulation results of Miu and Ozdemir (2008) suggest that the limiting condition becomes inappropriate when the number of borrowers is less than 100. Arguably, rather than using default rates, we could also use observed migration rates of an external portfolio to retrieve the underlying systematic risk in Step 1. However, observed migration rates are philosophy specific. By focusing on only the default rates of speculative grade borrowers, we do not need to concern ourselves with the difference in risk rating philosophy because, unlike re-rating, default is an objective event.

[20] Alternatively, if prior information is available with regard to the values of some of these parameters (e.g. R_x), we may assume they take on these known parametric values in conducting the estimation of the remaining parameters.

the external portfolio is defined by the level of DP_x. This LRPD however might not be relevant for any of the banks' internal risk ratings. The appropriate LRPDs of the banks' internal risk ratings will be estimated in Step 2 using the default rates of these internal risk ratings. As *by-products* of the estimation process, we can also retrieve the systematic PD risk $\hat{\mathcal{P}}$ (by reverting Equation (6)) and its *explainable* ($\hat{\mathcal{F}}(\bullet)$) and *unexplainable* ($\hat{\mathcal{E}}$) components.

$$\hat{\mathcal{P}} = \left(\frac{DP_x - \sqrt{1-R_x^2}\,\Phi^{-1}(\Theta_x)}{R_x} \right) \tag{20}$$

$$\hat{\mathcal{F}}(\bullet) = \left[\hat{f}\left(X_1^1,...,X_1^J\right) \quad \hat{f}\left(X_2^1,...,X_2^J\right) \quad ... \quad \hat{f}\left(X_{T_x}^1,...,X_{T_x}^J\right) \right]' \tag{21}$$

$$\hat{\mathcal{E}} = \left(\frac{DP_x - R_x\hat{\mathcal{F}}(\bullet) - \sqrt{1-R_x^2}\,\Phi^{-1}(\Theta_x)}{R_x} \right) \tag{22}$$

In this step, we need to evaluate and compare the performances of competing specifications and parameterizations of $f(\bullet)$ and the inclusion or exclusion of each potential explanatory variable. We first examine all potential explanatory variables in a univariate setting. Essentially, we calibrate the proposed model repeatedly, every time using only a single but different explanatory variable in the specification of $f(\bullet)$ in Equation (12). The objective is to arrive at a *short-list* of statistically significant variables. Besides judging by their statistical power, we also look for variables which are related to the observed default rates in an intuitive fashion. For example, we expect GDP growth rate to be negatively related to the change in default rates. Specifically, during a credit downturn, GDP growth rate decreases while default rates increase.

We then consider specifications of $f(\bullet)$ which involve multiple explanatory variables from the short-list of variables identified above.[21] Besides, we also consider an *intercept model* in which no variables are used (i.e. the *explainable* component of P_t becomes a constant). McFadden's pseudo adjusted R-square is used to gauge the in-sample goodness-of-fit. We also assess the out-of-sample performances of the models by computing the root mean square errors of the calibrated models over hold-out samples. Detail discussions on these fitness tests can be found in the subsequent section when we consider an example of implementing the proposed methodology on a particular credit portfolio.

<u>Step 2:</u> Estimation of pair-wise correlation R, DP_m (for m = 1 to N), $TH_{m,q}$ of the banks' internal risk ratings

Using historical default rates $\theta_{m,t}$ and transition matrices (each defined by $N \times N$ elements of $\varphi_{m_1,m_2,t}$) of the banks' internal risk ratings m = 1, 2, …, N at time t = 1, 2, …, T, we can estimate R, DP_m (for m = 1 to N), $TH_{m,q}$ (for m = 1 to N and q = 1 to N-1) by maximum

[21] In this process, to avoid multicollinearity, we make sure highly-correlated variables are not included in the same model.

likelihood estimation (MLE). But unlike the full-scale version of the MLE as depicted in Equation (17), we assume the systematic PD risk $\hat{\mathcal{P}}$ (and thus also its *explainable* ($\hat{\mathcal{F}}(\bullet)$) and *unexplainable* ($\hat{\mathcal{E}}$) components) retrieved in Step 1 using the external default rate data are the true process. Specifically, we estimate the parameters by maximizing the sum of the logarithmic of the conditional probabilities of Equations (4) and (9), after substituting in $\hat{\mathcal{P}}$. In estimating DP_m for each risk rating m = 1, 2, …, N, we therefore maximize:

$$\log L = \sum_{t=1}^{T} \log\left[\Omega\left(\theta_{m,t}; \hat{P}_t, R, DP_m, n_{m,t}\right)\right], \tag{23}$$

where $\hat{\mathcal{P}} = \begin{bmatrix}\hat{P}_1 & \hat{P}_2 & \ldots & \hat{P}_T\end{bmatrix}'$. In estimating $TH_{m1,m2-1}$ for each transition from risk rating m_1 = 1, 2, …, N to risk rating m_2 = 1, 2, …, N, we maximize:

$$\log L^* = \sum_{t=1}^{T} \log\left[\Omega^*\left(\varphi_{m_1,m_2,t}; \hat{P}_t, R, TH_{m_1,m_2-1}, n_{m_1,t}\right)\right]. \tag{24}$$

Unlike in Step 1, we do not invoke the infinite granularity assumption in conducting the MLE. Unlike external portfolio, the number of borrowers $n_{m,t}$ in each internal risk rating at any particular time period is unlikely to be large enough to satisfy this assumption. Once DP_m and $TH_{m1,m2-1}$ are estimated, we can then compute LRPD and LRPT by numerically evaluating Equations (3) and (7) through the simulations of p_t^i. Individual asset values p_t^i are constructed by using Equations (1) and (12) through the bootstrapping of the explanatory variables $X_t^1, X_t^2, ..., X_t^J$ and the evaluation of $f(\bullet)$ with its estimated parametric values.

Step 3: Evaluation of stressed PD and PT under selected stress scenarios based on the estimations conducted in Step 1 and 2.

Using the estimated results of Step 1 (i.e. parametric values of function $f(\bullet)$, σ_e, and ρ) and Step 2 (i.e. R, DP_m, $TH_{m,q}$), we can compute conditional PD and PT of each internal risk rating using Equations (15), (16a), (16b), and (16c) under different stress scenarios articulated in terms of selected explanatory variables (i.e. $X_t^1 = x_t^1, X_t^2 = x_t^2, ..., X_t^J = x_t^J$).

Table 1 presents an illustration of the stressed effects on a hypothetical internal risk rating system (with five levels of risk ratings). Upon the occurrence of a stress event, not only expected default probabilities but also expected downgrade (upgrade) probabilities increase (decrease) relative to their unconditional (i.e. long-run) counterparts.

INSERT TABLE 1 ABOUT HERE

If we look at Risk Rating 3 (in Panel A) for instance, LRPD is 8%, expected downgrade probability to Risk Rating 4 is 20%, whereas expected upgrade probability to Risk Rating 2 is 5% over a one-year period. These are unconditional probabilities. During a stress period (in Panel B), for the same Risk Rating 3, conditional PD increases to 14%.

Besides, expected conditional downgrade probability to Risk Rating 4 increases to 30%, while expected conditional upgrade probability to Risk Rating 2 decreases to 1%. Increases in PDs represent downgrades of borrowers immediately following the stress event (i.e. at the beginning of the risk horizon) assuming the bank adopts a constant risk rating-to-PD mapping.[22] Increases (decreases) in downgrade (upgrade) transition probabilities, on the other hand, represent the expected deterioration of the portfolio within a year after the stress event.[23] In order to quantify the total effect of the stress event, both effects needs to be captured in devising a comprehensive stress testing model.[24]

IV. Example

In this section, we demonstrate the application of the proposed stress testing models on a portfolio of the oil and gas sector in the US.

Data:

Speculative-grade default rate data (Step 1 analysis): We extract annual default rates of speculative-grade US oil and gas obligors from Standard & Poor's CreditPro over the 26-year period from 1981 to 2006.

Explanatory variables (Step 1 analysis): They are annual observations (from 1981 to 2006) of both broad-base and sector-specific variables (collected from various sources) which can be estimated by the bank's economics department under identified high-level stress scenarios. The variables are normalized with the respectively sample means and standard deviations.

- Broad-base variables
  - US real GDP
  - US industrial production
  - US unemployment rate
  - US corporate profit
  - Slope of yield curve of US Treasuries (yield on 10-year minus yield on 3-month Treasuries)
  - Short term interest rate in US (yield on 3-month Treasuries)
- Sector-specific variables
  - Spot crude oil price (WTI)
  - IEA global oil supply

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[22] Under Basel II's Advanced Internal Rating Based Approach, borrowers are first assigned to internal ratings (based on their ordinal ranking). Then, each of the internal ratings has a PD (more precisely LRPD) assigned to it. Many banks keep the mapping between their internal ratings and the corresponding PDs constant. Under this approach when the default risk of a borrower increases, the borrower is downgraded to a lower risk rating with a higher PD assigned to it.

[23] Assuming the risk horizon is one year and thus the transition matrices presented in Table 1 are one-year transition matrices.

[24] For example, if we want to estimate stressed economic capital through the simulations of a loss distribution, we need to take into account both the immediate downgrade events at the beginning of the risk horizon and further deterioration of credit quality within the risk horizon.

- IEA global oil demand
- Total OECD stocks of oil
- Government controlled days of forward demand of oil
- US annual inflation rate
- Scotiabank All Commodity Price index

Migration and default rates of internal risk ratings (Step 2 analysis): We collected historical migration and default rates for 12 internal risk ratings (AAA/AA+, AA/AA-/A+, A/A-, BBB+, BBB, BBB-, BB+, BB, BB-, B+, B/B-, and CCC) over the 19-year period from 1988 to 2006.

Step 1 (Estimation of systematic PD risk):

We first consider the explanatory variables one at a time in explaining the systematic PD risk P_t implicit in the speculative-grade default rates of the oil and gas sector. We consider both the linear and quadratic relations in the specification of function $f(\bullet)$. That is,

$$P_t = \beta_1 \cdot X_t^j + e_t \tag{25a}$$

and

$$P_t = \beta_1 \cdot X_t^j + \beta_2 \cdot \left(X_t^j\right)^2 + e_t \tag{25b}$$

where e_t is assumed to follow the AR(1) process of Equation (13). For each explanatory variable, we consider the use of the contemporaneous and lag-one period values respectively in explaining P_t. We conduct the estimations by maximizing the likelihood function of Equation (19).[25] Table 2 reports the t-statistics of the estimated coefficients of $f(\bullet)$.

INSERT TABLE 2 ABOUT HERE

From Table 2, the following 13 (short-listed) variables are found to be statistically significant. These variables are examined in the subsequent multiple-variable analysis.

1. Annual change in real GDP (Contemporaneous)
2. Quarterly change in real GDP (Contemporaneous)
3. Quarterly change in industrial production (Contemporaneous)
4. Annual change in corporate profit (Contemporaneous)
5. Annual change in spot crude oil price (WTI) (Lag 12-month)
6. Annual change in global oil supply (IEA) (Lag 12-month)
7. Quarterly change in global oil supply (IEA) (Lag 12-month)
8. Annual change in global oil demand (IEA) (Lag 12-month)
9. Annual change in total OECD stocks (Contemporaneous)
10. Quarterly change in total OECD stocks (Contemporaneous)
11. US annual inflation rate in % (Lag 12-month)
12. Annual change in Scotiabank All Commodity Price index (Lag 12-month)
13. Quarterly change in Scotiabank All Commodity Price index (Lag 12-month)

[25] In this example, rather than simultaneously estimating R_x^2, we assume it is equal to 0.20.

Before we proceed any further, let us examine the characteristic of the credit cycle specific to the oil and gas sector. In Figure 3, we plot the speculative-grade (SG) default rates of the oil and gas sector along with those of all SG companies in US. The credit cycle of the oil and gas sector is found to be out-of-phase (if not completely counter-cyclical) when compared with the market-wide credit cycle. For example, during the two recessions of 1990-91 and 2000-01 when overall SG default rate is at its all time high, the default rate of the oil and gas sector attains a relatively low level. On the other hand, oil and gas sector's default rate peaks in 1986-87 and 1998-99, while the overall default rate is at a more normal (if not low) level. This counter-cyclical phenomenon can also be observed by comparing the oil and gas sector's default rate with the annual change in US real GDP (again in Figure 3). During the periods of negative or zero GDP growth (i.e. 1981-82, 1990-91 and 2000-01), oil and gas sector's default rate is zero or at a relatively low level. This explains the negative signs of the statistically significant *t*-statistics in the contemporaneous model of GDP, industrial production and corporate profit as reported in Table 2. The financial health of the oil and gas sector seems to be negatively related to contemporaneous changes in these variables.[26]

INSERT FIGURE 3 ABOUT HERE

The 13 variables identified above to be of statistically significant can be categorized into four groups.

- Group A ("Macroeconomics variables"): variables 1, 2, 3, 4, and 11
- Group B ("Price variables"): variables 5, 12, and 13
- Group C ("Demand/supply variables"): variables 6, 7, and 8
- Group D ("Inventory variables"): variables 9 and 10

The within-group pair-wise correlations (not reported here) are found to be high. Thus, no more than one member in each group will enter into the subsequent multiple-variable models simultaneously. Moreover, as observed in Table 2, for those variables of which the linear version is statistically significant, none of the corresponding square term is statistically significant. We therefore do not consider any square terms in the multiple-variable analysis. Given these criteria, we exhausted all possible combinations and consider a total of 154 different models. We also consider an "intercept" model in which no variables are used. Equation (26) is a generic representation of the multiple-variable model considered here. Again, we conduct the estimation by maximizing the likelihood function of Equation (19).

$$P_t = \beta_1 \cdot X_t^1 + \beta_2 \cdot X_t^2 + ... + \beta_J \cdot X_t^J + e_t \qquad (26)$$

To conserve space, we only report the results of the top twelve models (out of a total of 154 models considered) in terms of their performances as defined below. The estimated coefficients together with their corresponding *t*-statistics are reported in Table 3. The

[26] In the proposed model, higher the value of P_t, lower the chance of realizing an asset value which is below the default point, and thus better the financial health of the company. Any explanatory variable, which is negatively related to P_t, is therefore also negatively related the financial health of the company.

financial health (systematic default risk) of the oil and gas companies are found to be negatively (positively) related to the change in real GDP; while positively (negatively) related to the change in spot crude oil price, total OECD stocks and the Scotiabank All Commodity Price Index. These relations are found to be statistically significant and robust to different specifications of the model. As expected, most of the estimated serial correlations of the unexplainable component of the systematic PD risk are positive and many of them are also found to be statistically significant.

INSERT TABLE 3 ABOUT HERE

We compare the in-sample goodness-of-fit of the models by ranking them according to their McFadden's adjusted pseudo R-squares (also reported in Table 3), which can be expressed as:

$$1-\frac{\ln L_{model}-J}{\ln L_{intercept}} \tag{27}$$

where ln L_{model} and ln $L_{\text{intercept}}$ are the log-likelihood of the specific model we want to appraise and that of the intercept model respectively, and J is the number of explanatory variables in the model. Besides comparing the in-sample goodness-of-fit, we also conduct out-of-sample tests by exhausting all the different cases of holding out each time one data point of the full sample. We estimate each of the 154 models using the remaining data points and evaluate the root mean square errors (RMSE) of the hold out sample (i.e. computing the mean of the square of the residual u_t of Equation (13) at the single hold out sample point). The median of the RMSE (across all the different hold-out cases) for each of the models are reported in Table 3. We then rank the models from the smallest median RMSE to the largest. To obtain a model with both high in-sample goodness-of-fit and out-of-sample performance, we add the two rankings and compute an overall ranking for each model. Based on this overall ranking, Model 1 (in Table 3) is the best performer among the 154 models considered. Model 1 uses three explanatory variables: quarterly change in real GDP at t, annual change in global oil demand at t-1, and annual change in Scotiabank All Commodity Price Index at t-1. In Figure 4, we plot the *observed* (total) P_t against the *explainable component* of P_t (i.e. $f(\bullet)$) of Model 1. A substantial amount of the variation of systematic default risk of the oil and gas sector can be explained with these three explanatory variables, especially in explaining the downturns of the sector in 1991-1993, 1998-1999 and 2002 respectively.

INSERT FIGURE 4 ABOUT HERE

Step 2 (Estimations of migration thresholds and default points of internal risk ratings):

Using the historical migration and default rates of 12 internal risk ratings (AAA/AA+, AA/AA-/A+, A/A-, BBB+, BBB, BBB-, BB+, BB, BB-, B+, B/B-, and CCC) over the 19-year period from 1988 to 2006, we estimate the default points DP_m (for m = 1 to 12) and migration thresholds $TH_{m,q}$ (for m = 1 to 12 and q = 1 to 11) by maximizing the log-likelihood functions of Equations (23) and (24) respectively.[27] We conduct the MLE by assuming the systematic PD risk $\hat{\mathcal{P}}$ retrieved in Step 1 over the same sample period using the external default rate data are in fact the true process. The estimated results are tabulated in Table 4. The estimations of risk ratings "AAA/AA+" and "CCC" are ignored since there are less than or equal to seven years in which these two risk ratings have more than or equal to five (starting number of) obligors. In Table 4, those entries denoted by "Inf" indicate the values of the corresponding migration thresholds are at infinity. That is, we never observe any corresponding upgrade events throughout the sample period.

INSERT TABLE 4 ABOUT HERE

Given the estimated default points and migration thresholds as reported in Table 4, we then compute LRPD and LRPT by numerically evaluating Equations (3) and (7) through the simulations of p_t^i. Asset values p_t^i are constructed by using Equations (1) and (12) through the bootstrapping of the explanatory variables and the evaluation of Model 1 (the *best* model) at its estimated parametric values obtained in Step 1 above. The results are tabulated in Table 5.

INSERT TABLE 5 ABOUT HERE

Step 3 (Evaluation of stressed PD and PT under selected stress scenario):

Using the selected stress model (i.e. Model 1) of the oil and gas sector estimated in Step 1 and the default points and migration rates obtained in Step 2 for the internal risk ratings, we compute conditional PD and PT for each internal risk rating using Equations (15), (16a), (16b), and (16c) under a hypothetical stress scenario expressed in terms of the three explanatory variables of Model 1. Specifically, the hypothetical scenario is defined as:

- A quarterly change in real GDP at time *t* which is one standard deviation above average; together with
- An annual change in global oil demand at time *t*-1 which is one standard deviation below average; and
- An annual change in Scotiabank All Commodity Price Index at time *t*-1 which is one standard deviation below average.

The conditional PD and PT under this stress event are reported in Table 6.

INSERT TABLE 6 ABOUT HERE

[27] In this example, rather than simultaneously estimating R^2, we assume it is equal to 0.20.

V. Conclusions

In this paper, we outline a methodology in generating both PDs and migration rates conditional on macroeconomic explanatory variables corresponding to selected stress scenarios. The proposed methodology is consistent with the economic model underlying Basel II Pillar 1 risk-weight function, while allowing the use of external data for the development of the models. This is possible as the methodology adjusts for the differences between internal and external data, namely the levels of long-run probabilities of default and asset correlations. The proposed methodology allows full customization. We can select the most relevant explanatory variables for the portfolio under consideration and use the external default data capturing the regional specificities. Lastly, it is compatible with the financial institutions' existing internal risk rating system making the implementation and operations cost effective.

Appendix: Selection of Explanatory Variables

The following is a list of criteria in selecting explanatory variables in practice:

- **Explanatory power:** Naturally, we would like to use variables with high explanatory power in the modeling of the systematic credit risk.
- **Forecastability:** We need to make sure the variables selected can be forecasted by the bank (typically its Economic Department) under the specified stress events in a consistent fashion.
- **Stress testing vs. forecasting models:** If we would like to use these models also for forecasting of PD using the current values of the explanatory variables (rather than during the stressed scenario), we need to consider the use of various *leading indicators* as explanatory variables (e.g. the national composite leading indicators).
- **Model coverage:** For broad-base models, we need to use more generic macroeconomics factors (e.g. interest rate and GDP); whereas, for sector-specific models, we should also include industry-specific factors. Figure 5 provides a few examples.
- **Data availability:** Data availability is essential for modelling and forecasting. We need to be careful that even though historical time series of some of the variables are available, the calculation methodology of the variables has been changed creating consistency problems.
- **Correlations among variables:** We should not include highly correlated variables in the same model to prevent multicollinearity**.**
- **Representation and coverage:** We should try to include variables explaining the credit environment (e.g. ratio of downgrade to total rating actions, S&P's outlook distribution etc.) as well as general economic and financial indicators.

INSERT FIGURE 5 ABOUT HERE

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Table 1: An Illustrative Comparison of Unconditional and Conditional Transition Matrices

Panel A: Unconditional (Long-Run) Rating Transition Matrix

		Expected Transition and Default Probabilities by Year-End					
		Risk Rating 1	**Risk Rating 2**	**Risk Rating 3**	**Risk Rating 4**	**Risk Rating 5**	**Unconditional PD**
Current Ratings	**Risk Rating 1**	80%	10%	5%	3%	1%	1%
	Risk Rating 2	3%	65%	15%	10%	3%	4%
	Risk Rating 3	1%	**5%**	60%	**20%**	6%	**8%**
	Risk Rating 4	0%	2%	3%	50%	30%	15%
	Risk Rating 5	0%	0%	10%	20%	45%	25%

Panel B: Conditional (Stressed) Rating Transition Matrix

		Expected Transition and Default Probabilities by Year-End					
		Risk Rating 1	**Risk Rating 2**	**Risk Rating 3**	**Risk Rating 4**	**Risk Rating 5**	**Conditional PD**
Current Ratings	**Risk Rating 1**	62%	15%	10%	6%	4%	3%
	Risk Rating 2	0%	45%	25%	15%	7%	8%
	Risk Rating 3	0%	**1%**	35%	**30%**	20%	**14%**
	Risk Rating 4	0%	0%	0%	30%	40%	30%
	Risk Rating 5	0%	0%	5%	10%	35%	50%

Table 2: t-statistics of estimated coefficients in single-variable specification of $f(\bullet)$

	Lag-one period			**Contemporaneous**		
	Linear	**Quadratic**		**Linear**	**Quadratic**	
	β_1	β_1	β_2	β_1	β_1	β_2
Annual change in real GDP	0.548	0.353	-2.028	-2.197	-2.195	-0.029
Quarterly change in real GDP	0.101	0.457	1.014	-2.809	-2.966	-0.882
Annual change in industrial production	1.248	0.701	-1.238	-1.719	-1.321	0.752
Quarterly change in industrial production	0.988	1.003	0.188	-2.520	-2.347	1.583
Unemployment rate	-1.863	-1.616	0.170	-0.076	-1.123	1.780
Annual change in unemployment rate	-1.121	-1.032	-1.430	1.820	1.729	0.337
Quarterly change in unemployment rate	-0.206	-0.149	-0.435	1.849	1.856	0.308
Annual change in corporate profit	1.532	1.871	-1.238	-2.396	-2.349	0.161
Quarterly change in corporate profit	0.726	0.983	-0.910	-1.329	-1.878	1.399
Slope of yield curve (10 yr - 3-m)	-0.992	-0.845	-0.136	-1.673	-1.276	1.130
Annual change in slope of yield curve	-0.016	-0.980	1.304	-0.248	-1.223	1.534
Quarterly change in slope of yield curve	0.292	0.631	0.868	-0.727	-1.617	2.187
Short term interest rate (3-m)	1.168	0.636	0.416	1.216	1.298	-0.325
Annual change in short term interest rate (3-m)	1.572	1.865	1.059	-0.260	0.243	1.422
Quarterly change in short term interest rate (3-m)	0.231	-0.311	1.468	-0.296	0.125	0.407
Annual change in spot crude oil price (WTI)	2.409	2.346	0.098	-0.517	-0.322	-1.827
Quarterly change in spot crude oil price (WTI)	1.893	1.909	0.392	-1.885	-1.822	0.632
Annual change in global oil supply (IEA)	2.298	2.523	-1.004	0.400	0.394	0.450
Quarterly change in global oil supply (IEA)	2.644	2.173	-1.260	1.754	1.579	-1.167
Annual change in global oil demand (IEA)	2.074	1.483	-0.979	-1.363	-1.105	0.201
Quarterly change in global oil demand (IEA)	1.760	2.440	1.552	-1.824	-0.492	1.767
Annual change in total OECD stocks	-0.218	-0.446	-0.394	2.560	3.051	1.721
Quarterly change in total OECD stocks	0.472	0.493	0.131	2.377	2.388	0.357
Annual change in gov't controlled days of fwd demand	0.890	1.609	1.358	0.848	0.730	0.238
Quarterly change in gov't controlled days of fwd demand	0.405	0.683	1.693	-0.375	-0.208	1.005
US annual inflation rate in %	2.470	2.870	-1.395	0.496	-0.116	1.077
US quarterly inflation rate in %	1.879	2.293	-1.210	-1.687	-2.044	1.577
Annual change in Scotiabank All Commodity Price Index	2.801	2.880	0.727	-0.957	-0.540	0.844
Quarterly change in Scotiabank All Commodity Price Index	2.707	2.841	0.825	-0.449	0.171	1.669

Table 3: Estimated coefficients, variances of residuals, serial correlations, and log-likelihoods of top twelve multiple-variable models in terms of overall performance ranking (out of 154 models). The corresponding *t*-statistics are presented in italics. Performances are compared based on both in-sample goodness-of-fit (McFadden's pseudo adjusted R-square) and out-of-sample median RMSE.

Model	Intercept	1	2	3	4	5	6	7	8	9	10	11	12
$\Phi(DP)$	0.032	0.032 *6.474*	0.032 *8.025*	0.031 *5.753*	0.032 *6.107*	0.032 *6.234*	0.031 *7.000*	0.030 *4.808*	0.032 *8.470*	0.032 *11.980*	0.031 *5.458*	0.032 *9.289*	0.031 *7.289*
Annual change in real GDP (at t)													-0.254 *-1.954*
Quarterly change in real GDP (at t)		-0.460 *-4.641*	-0.382 *-3.338*	-0.395 *-3.606*	-0.414 *-3.901*	-0.497 *-4.612*		-0.344 *-3.191*	-0.355 *-3.101*	-0.284 *-2.969*	-0.471 *-3.890*		
Annual change in spot crude oil price (WTI) (at t-1)							0.218 *1.966*						0.338 *3.061*
Annual change in global oil supply (IEA) (at t-1)				0.218 *2.440*									
Quarterly change in global oil supply (IEA) (at t-1)							0.224 *2.167*	0.219 *2.570*	0.072 *0.643*	0.108 *1.036*			
Annual change in global oil demand (IEA) (at t-1)		0.236 *3.056*									0.186 *1.856*		0.429 *3.920*
Annual change in total OECD stocks (at t)				0.314 *3.565*	0.248 *2.700*		0.371 *3.517*	0.387 *4.712*				0.274 *2.186*	
Quarterly change in total OECD stocks (at t)						0.224 *2.215*					0.271 *2.446*		
Annual change in Scotiabank All Commodity Price Index (at t-1)		0.385 *4.212*	0.406 *3.667*		0.296 *2.837*	0.322 *2.998*			0.391 *3.487*				
Quarterly change in Scotiabank All Commodity Price Index (at t-1)										0.466 *4.638*		0.367 *2.942*	
Variance of residuals (e_t)	0.502	0.187	0.220	0.225	0.205	0.214	0.227	0.255	0.213	0.187	0.270	0.238	0.221
Serial correlation (ρ)	0.080	0.443 *2.120*	0.150 *0.645*	0.465 *2.258*	0.455 *2.198*	0.417 *1.970*	0.272 *1.205*	0.560 *2.936*	0.109 *0.465*	-0.185 *-0.802*	0.429 *2.037*	-0.046 *-0.197*	0.246 *1.079*
Log L	-19.443	-8.186	-11.462	-9.700	-8.922	-9.696	-11.286	-9.673	-11.267	-9.833	-11.798	-12.393	-11.156
McFadden's pseudo adjusted R-sq (Model ranking)		0.425 (1)	0.308 (11)	0.347 (8)	0.387 (2)	0.347 (7)	0.265 (22)	0.348 (6)	0.266 (21)	0.340 (9)	0.239 (33)	0.260 (24)	0.272 (17)
Out-of-sample median RMSE (Model ranking)		0.306 (10)	0.280 (6)	0.315 (11)	0.354 (24)	0.339 (19)	0.269 (4)	0.349 (23)	0.321 (12)	0.376 (29)	0.277 (5)	0.329 (15)	0.396 (41)
Overall Rank		1	2	3	4	5	6	7	8	9	10	11	12

Table 4: Estimations of default points and migration thresholds of internal risk ratings. The estimations of risk ratings “AAA/AA+” and “CCC” are ignored since there are less than or equal to seven years of valid migration and default data for these two ratings. “Inf” denotes the fact that the migration thresholds approach infinity (i.e. we never observe any corresponding upgrade events throughout the sample period).

	Migration Thresholds												**Default Point**
	AAA/AA+	AA/AA-/A+	A/A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B/B-	CCC	
AAA/AA+		N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
AA/AA-/A+	2.340		-1.160	-1.960	-2.400	-2.400	-2.930	-2.930	-2.930	-2.930	-2.930	-2.930	-2.930
A/A-	2.480	1.600		-1.220	-1.740	-2.260	-2.260	-2.420	-2.420	-2.420	-2.420	-2.420	-2.860
BBB+	2.300	1.990	1.170		-1.090	-1.720	-2.190	-2.190	-2.360	-2.360	-2.360	-2.360	-2.800
BBB	Inf	Inf	1.700	1.020		-1.070	-1.640	-2.000	-2.180	-2.730	-2.730	-2.730	-2.730
BBB-	Inf	Inf	Inf	1.480	0.710		-1.030	-1.560	-1.900	-1.900	-1.900	-1.900	-2.670
BB+	1.470	1.470	1.470	1.370	1.370	0.940		-0.980	-1.430	-2.550	-2.550	-2.550	-2.550
BB	Inf	Inf	2.060	1.780	1.480	1.410	1.030		-0.890	-1.440	-1.800	-2.430	-2.430
BB-	Inf	Inf	Inf	Inf	Inf	1.840	1.590	1.030		-0.930	-1.480	-2.310	-2.310
B+	Inf	Inf	Inf	Inf	Inf	Inf	1.750	1.430	0.750		-1.070	-1.650	-2.060
B/B-	Inf	Inf	Inf	Inf	Inf	Inf	1.600	1.480	1.380	0.800		-1.030	-1.780
CCC	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A		N/A

Table 5: Estimations of long-run probabilities of default and long-run probabilities of transition of internal risk ratings. The estimations of risk ratings “AAA/AA+” and “CCC” are ignored since there are less than or equal to seven years of valid migration and default data for these two ratings.

	Long-Run Probability of Transition												**Long-Run Prob. of Default**
	AAA/AA+	AA/AA-/A+	A/A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B/B-	CCC	
AAA/AA+	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
AA/AA-/A+	0.74%	87.87%	9.24%	1.49%	0.00%	0.53%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.12%
A/A-	0.49%	4.33%	84.91%	6.68%	2.62%	0.00%	0.36%	0.00%	0.00%	0.00%	0.00%	0.46%	0.15%
BBB+	0.83%	1.09%	9.38%	75.76%	9.15%	2.61%	0.00%	0.45%	0.00%	0.00%	0.00%	0.53%	0.19%
BBB	0.00%	0.00%	3.86%	10.61%	72.16%	8.86%	2.60%	0.71%	0.96%	0.00%	0.00%	0.00%	0.23%
BBB-	0.00%	0.00%	0.00%	6.20%	16.84%	62.66%	8.94%	2.90%	0.00%	0.00%	0.00%	2.17%	0.29%
BB+	6.31%	0.00%	0.00%	1.39%	0.00%	8.75%	67.92%	8.70%	6.51%	0.00%	0.00%	0.00%	0.41%
BB	0.00%	0.00%	1.60%	1.58%	2.99%	0.93%	7.07%	68.03%	10.97%	3.67%	2.56%	0.00%	0.59%
BB-	0.00%	0.00%	0.00%	0.00%	0.00%	2.76%	2.13%	9.49%	68.91%	10.49%	5.38%	0.00%	0.84%
B+	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	3.46%	3.46%	15.07%	64.67%	8.91%	2.77%	1.66%
B/B-	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	4.87%	1.28%	1.38%	12.77%	65.40%	11.04%	3.27%
CCC	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A

Table 6: Stressed (conditional) probabilities of default and transition of internal risk ratings under a hypothetical stress scenario.

	Stressed Probability of Transition												**Stressed Prob. of Default**
	AAA/AA+	AA/AA-/A+	A/A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B/B-	CCC	
AAA/AA+	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
AA/AA-/A+	0.07%	76.00%	18.41%	3.66%	0.00%	1.48%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.37%
A/A-	0.04%	0.89%	77.20%	13.07%	6.11%	0.00%	0.93%	0.00%	0.00%	0.00%	0.00%	1.28%	0.47%
BBB+	0.09%	0.18%	2.80%	70.64%	17.16%	5.94%	0.00%	1.13%	0.00%	0.00%	0.00%	1.49%	0.57%
BBB	0.00%	0.00%	0.71%	3.75%	68.60%	16.30%	5.64%	1.74%	2.55%	0.00%	0.00%	0.00%	0.71%
BBB-	0.00%	0.00%	0.00%	1.38%	7.46%	62.83%	16.01%	6.08%	0.00%	0.00%	0.00%	5.39%	0.85%
BB+	1.48%	0.00%	0.00%	0.46%	0.00%	3.63%	64.42%	14.63%	14.18%	0.00%	0.00%	0.00%	1.21%
BB	0.00%	0.00%	0.25%	0.37%	0.88%	0.32%	2.88%	62.07%	18.25%	7.38%	5.91%	0.00%	1.68%
BB-	0.00%	0.00%	0.00%	0.00%	0.00%	0.56%	0.62%	3.75%	63.83%	17.45%	11.50%	0.00%	2.30%
B+	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.89%	1.25%	7.66%	65.05%	15.39%	5.65%	4.12%
B/B-	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	1.63%	0.59%	0.63%	7.06%	64.63%	18.22%	7.24%
CCC	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A

Figure 1. Stress Testing in Pillar I and Pillar II of Basel II

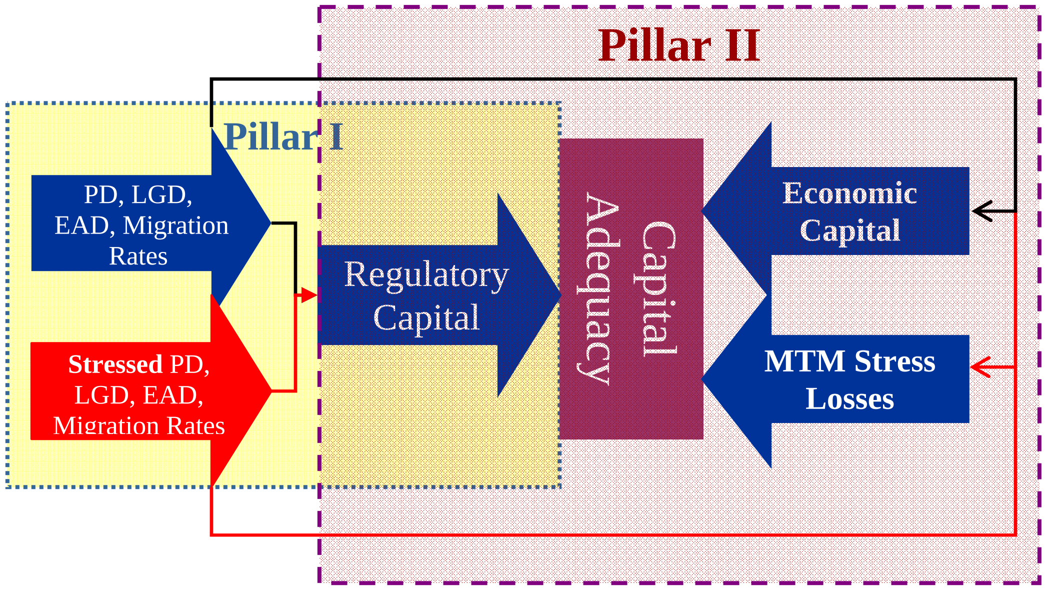

Figure 2. A Typical Stress Testing Framework

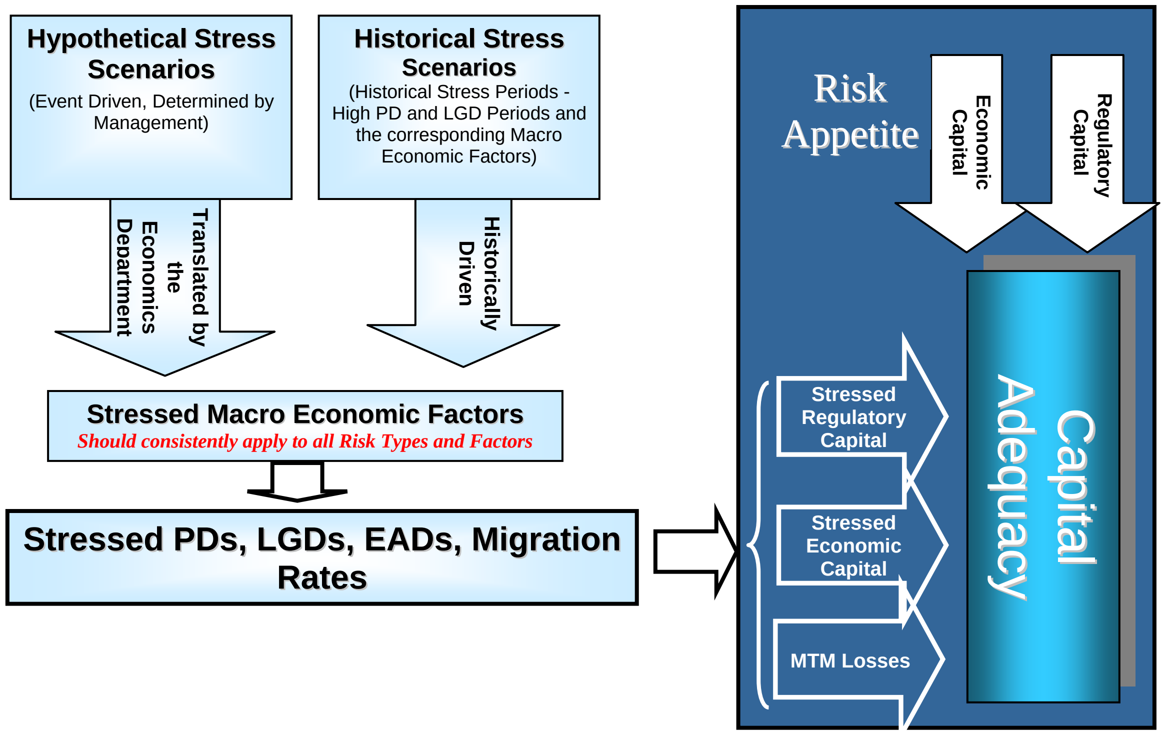

Figure 3: Characteristics of the credit cycle of oil and gas sector

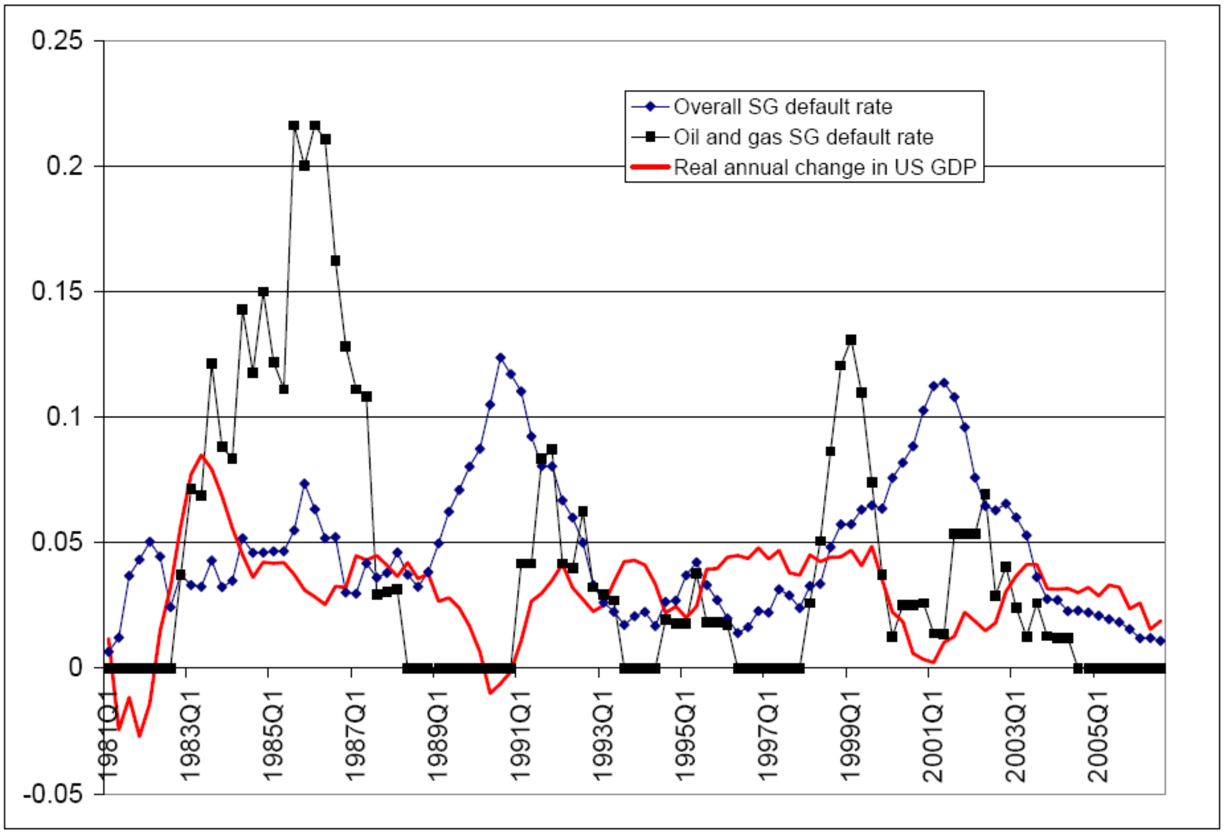

Figure 4: Observed (Total) P_t vs. Explainable Component of P_t of Model 1

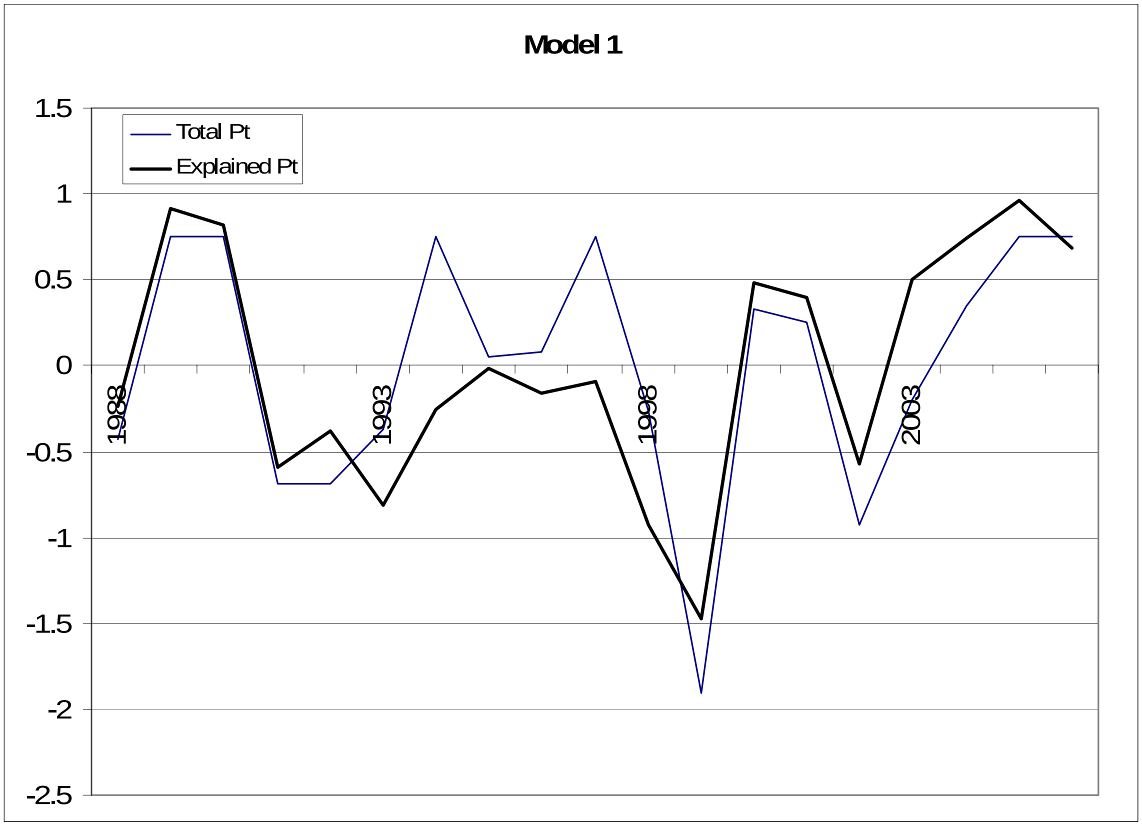

Figure 5: Examples of Explanatory Variables

Canada General Model
(a *broad-base* model)

- Real GDP
- Industrial production
- Unemployment
- Corporate profit
- Slope of yield curve
- High-yield spread
- Equity indices
- Composite index of leading indicators for Canada
- Consumer credit
- Delinquency rate
- Short-term interest rate

Real Estate Model
(a *sector-specific* model)

General health of the economy:

- GDP
- Unemployment
- Interest rate
- Retail sales
- Composite index of leading indicators
- Consumer sentiment index
- Durable goods
- S&P 500, TSE Composite Index

Specific to the sector:

- S&P/TSX Capped Real Estate Index
- New building permits
- New housing starts
- Residential real estate (U.S.)
- House price appreciation
- Housing affordability index
- Commercial real estate (U.S.)
    - Vacancy rates
    - Rents per square foot