

Stress Testing of Migration Matrix and Loss Given Default

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Outline

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- 2 Stressing of Migration Matrix
- 3 Stress Testing LGD

Section

Symbols Used

Symbols Used

X_i	firm's log asset value
ρ	correlation between asset returns and systematic risk factor
Z	systematic risk factor
z_t	value of systematic risk factor Z at time period t
ϵ	idiosyncratic risk factor
DR_i^t	default rate of rating group i observed during time period t
DR_i^{TTC}	average (aka through-the-cycle) observed default rate of rating group i
PD_i	probability of default of rating group i
PD_{TTC}	through-the-cycle probability of default
PD_{stress}	stressed probability of default
$PM_{i,j}$	probability of migrating from rating group i to rating group j
EL	expected loss
$loss_{stress}$	stressed loss rate
$TH_{i,j}$	threshold of rating group i separating rating group $j - 1$ and rating group j
n_i^t	counterparties within rating group i during time period t
$k_{i,n}^t$	defaults / migrations within rating group i during time period t
M	number of macroeconomic variables included in regression model
T	number of time periods used to determine z_t and to estimate thresholds
$LM_{i,j}$	maximum likelihood measure used to estimate threshold $TH_{i,j}$
q	quantile of stressed probability of default
k	LGD index risk

Section

Stressing of Migration Matrix

Introduction

- Consensus has been reached as most papers are based on Merton and Vasicek model.
- The proposed solution nicely fits into Basel framework.
- KBC credit guidelines introduced the same approach, which was subsequently applied in CSOB and K&H.
- The main difference between current and proposed approach is that RMO suggests to stress the whole migration matrix rather than only vector of probability of defaults.
- RMO approach is based on "Stress-Testing Probability of Default and Migration Rate with respect to Basel II Requirements" by Peter Miu & Bogie Ozdemir.

The Model

- We assume that firm's log asset value is normally distributed (cornerstone of Merton / Vasicek model).

$$X = \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon \quad (1)$$

- $Z \sim N[0, 1]$ represents systematic risk factor.
- $\epsilon \sim N[0, 1]$ represents idiosyncratic risk factor.
- ρ represents correlation between systematic risk factor and asset returns. Value of ρ is set up in line with Basel methodology.

The Model

- A firm defaults if value of its assets falls under a certain threshold TH_i .

$$PD_i = P[X_i < TH_i] = P[\sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i < TH_i] = P\left[\epsilon_i < \frac{TH_i - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right] = \Phi\left[\frac{TH_i - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right]. \quad (2)$$

- Similar approach could be used for migration probabilities.

$$PM_{i,j} = P[TH_{i,j+1} \leq X_i < TH_{i,j}] = P[TH_{i,j+1} \leq \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i < TH_{i,j}] = \Phi\left[\frac{TH_{i,j} - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right] - \Phi\left[\frac{TH_{i,j+1} - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right]. \quad (3)$$

The Model

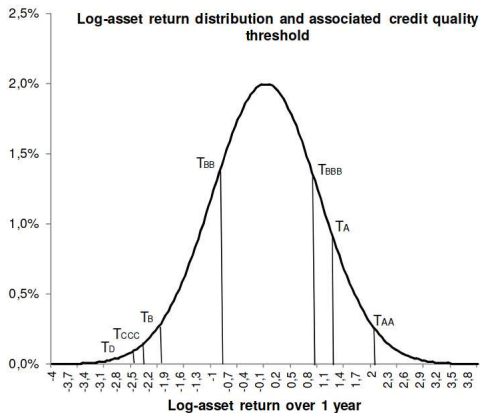


Figure: Illustration of threshold concept

Systematic Risk

- Systematic risk factor Z could be thought of as an economic cycle indicator.
 - expansion year $\Rightarrow z_t > 0$
 - average year $\Rightarrow z_t \approx 0$
 - recession year $\Rightarrow z_t < 0$
- If we knew value z_t of systematic risk factor Z , we could have calculated conditioned (aka stressed or point-in-time) probability of default.

$$PD_i = \Phi \left[\frac{TH_i - \sqrt{\rho} z_t}{\sqrt{1 - \rho}} \mid Z = z_t \right]. \quad (4)$$

Systematic Risk

- However, we will go the other way around. Using observed default rates as proxy for conditioned probability of default and exploiting $TH_i = \Phi^{-1}(DR_i^{TTC})$, we determine z_t .

$$DR_i^t = \Phi \left[\frac{\Phi^{-1}[DR_i^{TTC}] - \sqrt{\rho}z_t}{\sqrt{1-\rho}} \mid Z = z_t \right]. \quad (5)$$

$$z_t = \frac{\Phi^{-1}[DR_i^{TTC}] - \Phi^{-1}[DR_i^t]\sqrt{1-\rho}}{\sqrt{\rho}}. \quad (6)$$

- Please note that in theory default rate of any rating group could be used. However, since high ratings are less impacted by economic cycle, one should use default rates aggregated over lower ratings. Lower ratings are more sensitive to changes in economic cycle and therefore better in "extracting" information on level of systematic risk Z .

Macroeconomic Variables

- Once we determine historical values z_t of systematic risk factor Z , we can link them to macroeconomic variables via a regression model. Macroeconomic variables could be both current and lagged.
- Macroeconomic variables should be ideally normally distributed with zero mean and variance equal to one. Therefore one should try various transformations (log-odd, log ratio or log transformation) and standardize the result.
- Given the number of possible candidates, it is challenging to come up with an appropriate regression model.
- An example of a regression model is provided below.

$$z_t = \alpha + \beta_1 GDP_{t-1} + \beta_2 IR_t^{5Y} + \beta_3 z_{t-1} + \varepsilon_t \quad (7)$$

- When selecting macroeconomic variables into the model not only their statistical significance but also their importance for stress scenario formulation should be considered.

Threshold Calibration

- Conditioned (aka stressed or point-in-time) migration matrix is defined through

$$PM_{i,j} = \Phi \left[\frac{TH_{i,j} - \sqrt{\rho}Z}{\sqrt{1-\rho}} \right] - \Phi \left[\frac{TH_{i,j+1} - \sqrt{\rho}Z}{\sqrt{1-\rho}} \right]. \quad (8)$$

- Remember we know z_t and ρ . Therefore we have to estimate threshold $TH_{i,j}$.
- Please note that the thresholds are not function of time, i.e. they do not change with economic cycle.
- We estimate the thresholds using maximum likelihood so that for given z_t and ρ they imply historical point-in-time migration matrices which match observed migrations / defaults as close as possible.

Threshold Calibration

- Consider a migration matrix with N rating groups and let us focus on a general rating group i .
- In theory we are to estimate $N + 1$ thresholds. However, the first threshold $TH_{i,1}$ and the last threshold $TH_{i,N+1}$ are not estimated as they are by definition equal to ∞ and $-\infty$ respectively. That leaves us with $N - 1$ thresholds to estimate.
- Let us focus at time period t . Assume that the rating group consists of n_i^t counterparties and that we observe $k_{i,j}^t$ migrations / defaults. Then (for a given threshold) probability of this event is

$$p_{i,N}^t = \binom{n_i^t}{k_{i,j}^t} (PM_{i,j}^t)^{k_{i,j}^t} (1 - PM_{i,j}^t)^{n_i^t - k_{i,j}^t}. \quad (9)$$

- To find an optimal threshold $TH_{i,j}$ we have to maximize probabilities across all time periods, i.e. we maximize $LM_{i,j} = \sum_{t=1}^T \log(p_{i,j}^t)$ by changing value of threshold $TH_{i,j}$.
- Optimal value of threshold $TH_{i,j}$ could be easily found using a grid search method.
- Optimization process starts with default state and continues towards the best rating.
- In this way we estimate all thresholds one by one for all rating groups.

Link Between PIT and TTC

- PIT migration matrix could be thought of as a migration matrix conditioned on state of an economic cycle.
- TTC migration matrix could be thought of as a migration matrix over the whole economic cycle.
- Please note that state of economic cycle is indicated by value z_t of systematic risk factor Z .
- Migration matrix is fully determined by z_t , ρ and thresholds.
- Since $z_{TTC} \approx 0$, the estimated thresholds define not only PIT but also TTC migration matrix.
- Therefore the above procedure enables us to estimate consistent PIT and TTC migration matrices.

Stress Testing

- 1 Split historical data in several non-overlapping time periods $t = 1, 2, \dots, T$. The periods typically represent quarters, half-years or years.
- 2 Determine number of observed migrations and defaults per rating group for each time period $t = 1, 2, \dots, T$.
- 3 Using data from the previous step, determine aggregated default rates for non-investment rating groups for each time period $t = 1, 2, \dots, T$.
- 4 Fix value of correlation ρ in line with Basel methodology.
- 5 Using default rates from step (3) determine values z_t of systematic risk factor Z for each time period $t = 1, 2, \dots, T$.
- 6 Construct an appropriate regression model that establishes a link between systematic risk factor Z and macroeconomic variables.
- 7 Using observed defaults and migrations during time periods $t = 1, 2, \dots, T$ calibrate thresholds, which define historical point-in-time migration matrices and consistent through-the-cycle migration matrix (which could be for example determined via $z_{TTC} = 0$).
- 8 Take stressed macroeconomic variables and determine corresponding value z_{stress} of systematic risk factor Z using regression model introduced in step (6).
- 9 Using z_{stress} and thresholds calibrated in step (7) construct point-in-time migration matrix that corresponds to stressed macroeconomic variables.

Section

Stress Testing LGD

Introduction

- Consensus has not been reached as several papers are introducing different approaches to the problem.
- RMO approach follows "Loss given default as a function of the default rate" by Jon Frye as we found his approach as the most pragmatic and since it naturally fits into concept of stressing migration matrix.

The Model

- The main idea is that LGD is increasing with increasing probability of default, i.e. LGD and probability of default are comonotonic.
- This means that LGD quantiles map directly to quantiles of probability of default, e.g. 20% LGD quantile corresponds to 20% quantile of probability of default.
- Therefore, if we knew quantile of a stressed probability of default, we could use the quantile to determine corresponding stressed LGD.
- Let us assume that through-the-cycle probability of default is known. Using Vasicek model, quantile of a stressed probability of default is

$$q = \Phi \left[\frac{\sqrt{1-\rho}\Phi^{-1}[PD_{stress}] - \Phi^{-1}[PD_{TTC}]}{\sqrt{\rho}} \right]. \quad (10)$$

- Suppose that stressed loss rate also obeys Vasicek distribution. Using the above derived quantile q and expected loss EL , the stressed loss rate is defined by

$$loss_{stress} = \Phi \left[\Phi^{-1}[PD_{stress}] - \frac{\Phi^{-1}[PD_{TTC}] - \Phi^{-1}[EL]}{\sqrt{1-\rho}} \right]. \quad (11)$$

The Model

- Stressed LGD could be easily calculated as a ratio of stressed loss rate and stressed probability of default

$$LGD_{stress} = \frac{loss_{stress}}{PD_{stress}} = \frac{\Phi [\Phi^{-1}[PD_{stress}] - k]}{PD_{stress}} \quad (12)$$

where

$$k = \frac{\Phi^{-1}[PD_{TTC}] - \Phi^{-1}[EL]}{\sqrt{1 - \rho}} \quad (13)$$

is called LGD risk index and which fully determines LGD function.

The Model

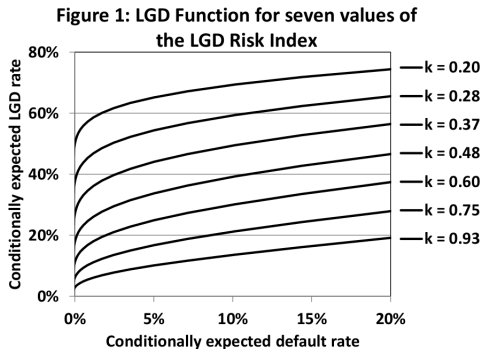


Figure: LGD function for different levels of k

The Model

- Advantages

- LGD function is very easy to calibrate.
- The model fits into concept of migrating matrix stressing.

- Disadvantages

- Loss rates is assumed to follow Vasicek distribution. However, no sound economic explanation is provided.
- The model is tested on US corporate bonds. There is no guarantee it will work on retail portfolios as well.
- Loss rate function is fully determined by expected loss. This means one can only change level but not shape of LGD curve; for illustration see figure on the previous slide.

The Model

- Comonotonicity of LGD and probability of default could be easily used to introduce other probability distributions.
- We replaced Vasicek distribution with beta distribution. The distribution could be fitted to LGDs historically observed through out economic cycle or calibrated on expert opinion.
- Given quantile implied by stressed probability of default, we use the quantile to sample from beta distribution to get a consistently stressed LGD.
- Beta distribution could be set up in a way that it matches LGD function based on Vasicek distribution. Besides it is more flexible in terms of possible shapes.

The Model

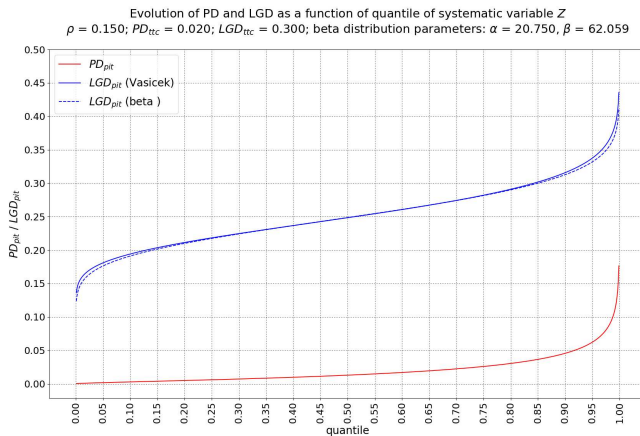


Figure: LGD function based on Vasicek and beta distribution

Mortgage Portfolio

- Concept introduced in "Stress Testing the Credit Risk of Mortgage Loans: The Relationship between Portfolio-LGD and the Loan-to-Value Distribution" by Christian Greve and Lutz Hahnenstein could be applied to mortgage (and other collateralized) portfolios.
- The main idea is that LGD is fully determined by actual value of loan collateral. Therefore LGD of a mortgage loan is driven by its loan-to-value and real-estate shocks.
- To apply the concept on portfolio level, it is assumed that portfolio's loan-to-value composition could be described via beta distribution and that LGD for a given loan-to-value X and recovery rate rr is $1 - \frac{rr}{\max[X, rr]}$.
- It can be proved that under these assumptions average portfolio LGD is

$$E[LGD] = 1 - F^{\alpha, \beta}(rr) - rr \frac{\alpha + \beta - 1}{\alpha - 1} (1 - F^{\alpha-1, \beta}(rr)), \quad (14)$$

where $F^{\alpha, \beta}$ is cumulative distribution of beta distribution with parameters α and β .

Mortgage Portfolio

- We expressed mean portfolio LGD as a function of the mean recovery rate rr and of the two parameters α and β that capture the portfolio's loan-to-value distribution.
- Please note portfolio LGD is stressed through stressing average recovery rate rr rather than through stressing current loan-to-value.
- As illustrated on next slide, two portfolios with the same average loan-to-value could have quite different average LGD; check portfolios' LGD for a real estate shock of -30%. General rule is that portfolios with more widespread distribution of loan-to-values tend to have larger average LGD.

Mortgage Portfolio

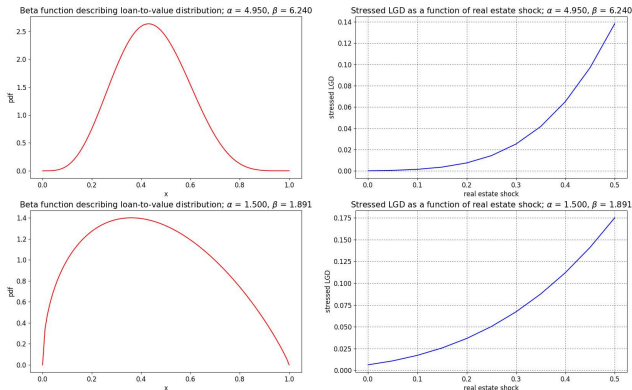


Figure: Loan-to-value based LGD as a function of a real estate price shock

Mortgage Portfolio

- Advantages

- The model is intuitive and easy to understand.
- Beta function describing portfolio's composition of loan-to-value could be easily calibrated on real data; no expert input is needed.

- Disadvantages

- Only secured recoveries are considered.
- LGD is stressed only via real estate shocks. Other potentially important macroeconomic variables are ignored.