Ex 7

$\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\} = \{(0,0);(1,1);(2,1);(3,2);(4,2);(5,3)\}$

modèle $f(x) = ax+b$.

critère : minimisation de $\mathcal{E}(a,b) = \sum_{i=1}^{6}(c_i - f(x_i))^2 = \sum_{i=1}^{6}(c_i - ax_i - b)^2$

calcul : nécessaire et suffisant pr $\vec{\nabla}\mathcal{E}(a,b) = \vec{0}$.

$\boxed{h \cap \mathcal{F} \to a^t A^{n-1}}$

$\frac{\partial f}{\partial a}(x,y)$

$\vec{\nabla}\mathcal{E}(a,b) = \vec{0} \iff \begin{pmatrix} \frac{\partial \mathcal{E}}{\partial a}(a,b) \\ \frac{\partial \mathcal{E}}{\partial b}(a,b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{\frac{df}{dx}(x)}$

$\iff \begin{cases} -\cancel{2}\sum_{i=1}^{6} x_i(c_i - ax_i - b) = 0 \\ -\cancel{2}\sum_{i=1}^{6}(c_i - ax_i - b) = 0 \end{cases}$

$\iff \begin{cases} \sum(ax_i^2 + bx_i) = \sum x_i c_i \\ \sum(ax_i + b) = \sum c_i \end{cases}$

$\iff \begin{pmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & \sum 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum x_i c_i \\ \sum c_i \end{pmatrix}$

$\iff \underbrace{\begin{pmatrix} 106 & 20 \\ 20 & 5 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 192 \\ 34 \end{pmatrix}}_{B}$

$\sum x_i^2$

$\frac{1\times2^2 + 4^2 + 6^2 + 8^2}{5} = \frac{2^2}{49}$

$\sum x_i c_i$

$4 \underbrace{1+6+20}_{27} + \underbrace{6+105}_{165}$

$X = A^{-1}B$, $|A| = 106\times5 - 20^2 = 530 - 400 = 130 \neq 0 \Rightarrow A$ inversible.

$\mathrm{com}\,A = \begin{pmatrix} 5 & -20 \\ -20 & 106 \end{pmatrix}$, $A^{-1} = \frac{1}{|A|}\,{}^t\mathrm{com}\,A$.

$A^{-1} = \frac{1}{130}\begin{pmatrix} 5 & -20 \\ -20 & 106 \end{pmatrix}$

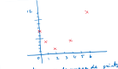
$\frac{1}{130}\begin{pmatrix} 5 & -20 \\ -20 & 106 \end{pmatrix}\begin{pmatrix} 192 \\ 34 \end{pmatrix}$

$\leftarrow a \simeq 12,615$

$\leftarrow b \simeq -57,261$

$\boxed{f(x) = ax + b}$

Ex 8



2 mesures par le compte pris

modèle $f(x) = ax^2 + bx + c$.

$\{(x_1,c_1),(x_2,c_2)\} = \{(1,0),(2,0),(3,1),(4,1),(5,2),(6,2)\}$.

critère de minimisation choisi

$\mathcal{E}$ minimiser $\mathcal{E}(a,b,c) = \sum_{i=1}^{6}(c_i - f(x_i))^2$

$= \sum_{i=1}^{6}(c_i - ax_i^2 - bx_i - c)^2$

$\vec{\nabla}\mathcal{E}(a,b,c)$ nécessaire et suffisant pr $\vec{\nabla}\mathcal{E}(a,b,c) = 0$.

$\vec{\nabla}\mathcal{E}(a,b,c) = \begin{pmatrix} \frac{\partial\mathcal{E}}{\partial a} \\ \frac{\partial\mathcal{E}}{\partial b} \\ \frac{\partial\mathcal{E}}{\partial c} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff \begin{cases} -\cancel{2}\sum x_i^2(c_i - ax_i^2 - bx_i - c) = 0 \\ -\cancel{2}\sum x_i(c_i - ax_i^2 - bx_i - c) = 0 \\ -\cancel{2}\sum(c_i - ax_i^2 - bx_i - c) = 0 \end{cases}$

$\iff \begin{cases} \sum(ax_i^4 + bx_i^3 + cx_i^2) = \sum x_i^2 c_i \\ \sum(ax_i^3 + bx_i^2 + cx_i) = \sum x_i c_i \\ \sum(ax_i^2 + bx_i + c) = \sum c_i \end{cases}$

$\iff \begin{pmatrix} \sum x_i^4 & \sum x_i^3 & \sum x_i^2 \\ \sum x_i^3 & \sum x_i^2 & \sum x_i \\ \sum x_i^2 & \sum x_i & \sum 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum x_i^2 c_i \\ \sum x_i c_i \\ \sum c_i \end{pmatrix}$

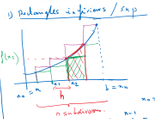
$\iff \underbrace{\begin{pmatrix} 1569 & 289 & 57 \\ 289 & 57 & 13 \\ 57 & 13 & 5 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 508 \\ 96 \\ 29 \end{pmatrix}}_{B}$

$\boxed{X = A^{-1}B}$ $\Rightarrow$ on déduit les paramètres a, b, c.

$\boxed{f(x) \simeq 0,8x^2 - 4,02x + 7,037}$

Éléments de cours

1) Rectangles inférieurs / sup



n subdivisions

$n = a + i.h$.

$h = \frac{b-a}{n}$, $[a,b] = \bigcup_{i=0}^{n-1}[x_i, x_{i+1}] = [x_0,x_1]\cup\ldots\cup[x_{n-1},x_n]$

$x_i = a + ih$.

$I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\underbrace{\int_{x_i}^{x_{i+1}} f(x)\,dx}_{\simeq h.f(x_i)}$

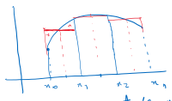
$\simeq h\sum_{i=0}^{n-1} f(x_i)$.

pour les rectangles supérieurs

$I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\underbrace{\int_{x_i}^{x_{i+1}} f(x)\,dx}_{\simeq h.f(x_{i+1})}$

$I \simeq h\sum_{i=0}^{n-1} f(x_{i+1})$.

2) Rectangles points-milieux.



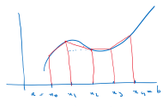
+ précis, mais rajout de $n-1$ points milieux.

$I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,dx$

$\simeq \sum_{i=0}^{n-1} h.\,f\left(\frac{x_i + x_{i+1}}{2}\right)$

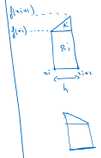
$= h\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)$

3) Méthodes des trapèzes.



$I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\underbrace{\int_{x_i}^{x_{i+1}} f(x)\,dx}_{\simeq T_i}$.

$\simeq T_i$: aire du trapèze i de base h



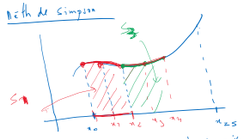
$T_i = R_i + A_i$

$= h f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{2}h$

$= h\,\frac{f(x_i) + f(x_{i+1})}{2}$

$I \simeq \frac{h}{2}\sum_{i=0}^{n-1}\left(f(x_i) + f(x_{i+1})\right)$

4) Méth de Simpson



n = pair, $x_0, \ldots, x_{2s}$: nbr impair de points. $\boxed{n = 2s}$

$[a,b] = \bigcup_{\substack{i=1 \\ i \text{ impair}}}^{n-1}[x_{i-1}, x_{i+1}] = [x_0,x_2]\cup[x_2,x_4]\cup\ldots[x_{n-2},x_n]$.

Sur chaque $[x_{i-1}, x_{i+1}]$, i impair

on calcule $P_i(x)$ de polynôme interpolant $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$

par la méth de Lagrange.

↳ On déduit $I = \int_a^b f(x)\,dx$

$= \sum_{\substack{i=1 \\ i \text{ impair}}}^{n-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx$

$\simeq \sum_{\substack{i=1 \\ i \text{ impair}}}^{n-1}\underbrace{\int_{x_{i-1}}^{x_{i+1}} P_i(x)\,dx}_{S_i = \frac{h}{3}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right]} = \sum_{\substack{i=1 \\ i \text{ impair}}}^{2s-1}\frac{h}{3}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right]$

$n = 2s$

$= \frac{h}{3}\sum_{k=0}^{s-1}\left(f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\right)$ ←

Ex 9 $I = \int_a^b f(x)\,dx$

1) Rectangles inf

$h = \frac{\pi}{8}$.

$I = \sum_{i=0}^{3}\int_{x_i}^{x_{i+1}} f(x)\,dx$

$\simeq h\sum_{i=0}^{3} f(x_i) = \frac{\pi}{8}(0 + 0,382683 + 0,707107 + 0,923880)$

$\simeq 0,79$

2) Rect sup

$I = \sum_{i=0}^{3} h\int_{x_i}^{x_{i+1}} f(x)\,dx \simeq h\sum_{i=0}^{3} f(x_{i+1})$

$\simeq \frac{\pi}{8}(0,382683 + 0,707107 + 0,923880 + 1)$

$\simeq 1,18$.

3) Rect milieu (sans rajout)

$h = \frac{\pi}{4}$

$I \simeq h\left(f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right)\right)$

$\simeq \frac{\pi}{4}(0,382683 + 0,923880) \simeq 1,026$.

4) Trapèze

$I = \int_0^{\frac{\pi}{2}} f(x)\,dx$

$= \sum_{i=0}^{3}\underbrace{\int_{x_i}^{x_{i+1}} f(x)\,dx}_{\simeq h\frac{f(x_i) + f(x_{i+1})}{2}}$

$\simeq \frac{\pi}{8}\times\frac{1}{2}\sum_{i=0}^{3} f(x_i) + f(x_{i+1})$

$\simeq \frac{\pi}{16}\left[f(x_0) + f(x_4) + 2\left(f(x_1) + f(x_2) + f(x_3)\right)\right]$

$\simeq \frac{\pi}{16}\left[1 + 2(0,382683 + 0,707107 + 0,923880)\right]$

$\simeq 0,98$.

5) Simpson

$x_0 \longrightarrow x_4$: nb impair, OK.

$4 = 2s \iff \boxed{s = 2}$, $h = \frac{\pi}{8}$.

$I \simeq \frac{h}{3}\sum_{i=0}^{s-1}\left(f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right)$

$\simeq \frac{\pi}{24}\left[f(x_0) + f(x_4) + 4\sum_{i=0}^{s-1} f(x_{2i+1}) + 2\sum_{i=1}^{s-1} f(x_{2i})\right]$

$\simeq \frac{\pi}{24}\left[0 + 1 + 4(0,382683 + 0,923880) + 2\times0,707107\right]$

$\simeq 1,00013$

6) Calcul direct

$I = \int_0^{\pi/2}\sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = 1$.

10

$v(t) = \int_0^t y(t)\,dt$, $y'(t) = v'(t)$.

↳ Approximation de $v(80) = \int_0^{80} y(t)\,dt$.

↳ $\boxed{0,1}$.