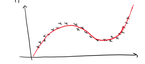
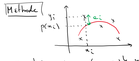
Les moindres carrés

Entrée : nuage de points dans le plan.
Sortie : un polynôme de degré adapté approximant au mieux le nuage.



Exemple de variations, avec une certaine erreur, dans le degré du polynôme.

Méthode



$N = \{(x_1,y_1), \dots, (x_n,y_n)\}$ nuage de points

Modèle $p(x) = a_0 + a_1 x + \dots + a_q x^q$ polynôme de degré q.

on note $e_i = y_i - p(x_i)$ l'erreur entre y_i et $p(x_i)$

Moindres carrés $Q(a_0,\dots,a_q) = \sum_{i=0}^{n} e_i^2$

somme des carrés des erreurs.

$$= \sum_{i=0}^{n} (y_i - p(x_i))^2$$
$$= \sum_{i=0}^{n} (y_i - a_0 - a_1 x_i - \dots - a_q x_i^q)^2$$

On cherche $a_0,\dots,a_q$ tels que $Q(a_0,\dots,a_q)$ soit minimum.

or son extremum est atteint quand $\vec{\nabla} Q = \vec{0}$

comme Q n'a pas de maximum (on peut toujours construire un polynôme ayant des erreurs d'approche + grandes qu'un nombre fixé $q \leq q$), alors les valeurs de $a_0,\dots,a_q$ donnant $\vec{\nabla} Q = \vec{0}$ nous permettra d'avoir $Q(a_0,\dots,a_q)$ minimum et donc un modèle $p(x)$ optimal en fct des données.

Calcul

On cherche $a_0,\dots,a_q$ tels que $\vec{\nabla} Q = \vec{0}$

$$\Leftrightarrow \begin{pmatrix} \frac{\partial Q}{\partial a_0} \\ \frac{\partial Q}{\partial a_1} \\ \vdots \\ \frac{\partial Q}{\partial a_q} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$Q(a_0,\dots,a_q) = \sum_{i=0}^{n} \overbrace{(y_i - a_0 - a_1 x_i - \dots - a_q x_i^q)^2}^{(u_i)^2}$$

$$\vec{\nabla} Q = \vec{0} \Leftrightarrow \begin{pmatrix} \frac{\partial Q}{\partial a_0} \\ \frac{\partial Q}{\partial a_1} \\ \vdots \\ \frac{\partial Q}{\partial a_q} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Rq $(u^2)' = 2 u' u^{2-1}$

$$\Leftrightarrow \begin{pmatrix} \cancel{2} \sum_{i=0}^{n} (-1)(y_i - a_0 - a_1 x_i - \dots - a_q x_i^q) \\ \cancel{2} \sum_{i=0}^{n} (-x_i)(y_i - a_0 - a_1 x_i - \dots - a_q x_i^q) \\ \vdots \\ \cancel{2} \sum_{i=0}^{n} (-x_i^q)(y_i - a_0 - a_1 x_i - \dots - a_q x_i^q) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -\sum y_i + \sum(a_0 + a_1 x_i + \dots + a_q x_i^q) \\ -\sum x_i y_i + \sum(x_i a_0 + a_1 x_i^2 + \dots + a_q x_i^{q+1}) \\ \vdots \\ -\sum x_i^q y_i + \sum(x_i^q a_0 + a_1 x_i^{q+1} + \dots + a_q x_i^{2q}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

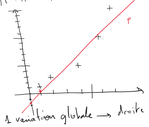
$$\Leftrightarrow \begin{pmatrix} a_0 \sum 1 + a_1 \sum x_i + \dots + a_q \sum x_i^q \\ a_0 \sum x_i + a_1 \sum x_i^2 + \dots + a_q \sum x_i^{q+1} \\ \vdots \\ a_0 \sum x_i^q + a_1 \sum x_i^{q+1} + \dots + a_q \sum x_i^{2q} \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \vdots \\ \sum x_i^q y_i \end{pmatrix}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} \sum 1 & \sum x_i & \dots & \sum x_i^q \\ \sum x_i & \sum x_i^2 & \dots & \sum x_i^{q+1} \\ \vdots & & & \\ \sum x_i^q & \sum x_i^{q+1} & \dots & \sum x_i^{2q} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_q \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \vdots \\ \sum x_i^q y_i \end{pmatrix}}_{Y}$$

$\Leftrightarrow X = A^{-1} Y \rightarrow$ on déduit $\boxed{a_0, \dots, a_q}$

Ex 7

$N = \{(x_i, y_i),\ i = 0 \to 4\}$
$= \{(1,1), (2,3), (4,5), (6,10), (7,11)\}$



1 variation globale $\rightarrow$ droite affine d'approximation.

Modèle : $p(x) = a_0 + a_1 x$

Critère : moindres carrés

$$Q(a_0, a_1) = \sum_{i=0}^{4} (y_i - p(x_i))^2 = \sum_{i=0}^{4} (y_i - a_0 - a_1 x_i)^2$$

Conclusion a_0, a_1 t.q $Q(a_0, a_1)$ soit minimum

càd qd $\vec{\nabla} Q = \vec{0}$.

On a $\begin{pmatrix} \frac{\partial Q}{\partial a_0} \\ \frac{\partial Q}{\partial a_1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cancel{2} \sum_{i=0}^{4} (-1)(y_i - a_0 - a_1 x_i) \\ \cancel{2} \sum_{i=0}^{4} (-x_i)(y_i - a_0 - a_1 x_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} -\sum y_i + \sum(a_0 + a_1 x_i) \\ -\sum x_i y_i + \sum(a_0 x_i + a_1 x_i^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \sum a_0 + \sum a_1 x_i \\ \sum a_0 x_i + \sum a_1 x_i^2 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a_0 \sum 1 + a_1 \sum x_i \\ a_0 \sum x_i + a_1 \sum x_i^2 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} \sum 1 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}}_{Y}$$

On pose $A = \begin{pmatrix} 5 & 20 \\ 20 & 106 \end{pmatrix}$, $Y = \begin{pmatrix} 34 \\ 192 \end{pmatrix}$

On a $AX = Y \Leftrightarrow X = A^{-1} Y$

Calcul de A^{-1} :

$$A^{-1} = \frac{1}{|A|}\, {}^t\mathrm{com}\, A.$$

$$|A| = \begin{vmatrix} 5 & 20 \\ 20 & 106 \end{vmatrix} = 5 \times 106 - 20^2 = 530 - 400 = 130.$$

$$\mathrm{com}\, A = \begin{pmatrix} 106 & -20 \\ -20 & 5 \end{pmatrix} = {}^t\mathrm{com}\, A$$

$$A^{-1} = \frac{1}{130} \begin{pmatrix} 106 & -20 \\ -20 & 5 \end{pmatrix}.$$

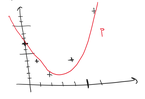
Conl :

$$\frac{1}{130} \begin{pmatrix} 106 & -20 \\ -20 & 5 \end{pmatrix} \begin{pmatrix} 34 \\ 192 \end{pmatrix} = \frac{1}{130} \begin{pmatrix} -236 \\ 280 \end{pmatrix} \simeq \begin{pmatrix} -1{,}81 \\ 2{,}15 \end{pmatrix}$$

donc $\boxed{p(x) \simeq -1{,}81 + 2{,}15\, x}$

Ex 8

$N = \{(x_i, y_i),\ i = 0 \to 4\}$
$= \{(0,7), (1,4), (2,2), (3,4), (4,12)\}$.



Modèle $p(x) = a_0 + a_1 x + a_2 x^2$

↳ 2 variations $\Rightarrow$ degré 2.

Moindres carrés $Q(a_0, a_1, a_2) = \sum_{i=0}^{4} (y_i - p(x_i))^2$
$= \sum_{i=0}^{4} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$

On cherche a_0, a_1, a_2 t.q $\vec{\nabla} Q = \vec{0}$

On a $\begin{pmatrix} 2 \sum (y_i - a_0 - a_1 x_i - a_2 x_i^2) \\ 2 \sum (-x_i)(y_i - a_0 - a_1 x_i - a_2 x_i^2) \\ 2 \sum (-x_i^2)(y_i - a_0 - a_1 x_i - a_2 x_i^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \underbrace{\begin{pmatrix} \sum 1 & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}}_{Y}$$

On a $A = \begin{pmatrix} 5 & 13 & 57 \\ 13 & 57 & 289 \\ 57 & 289 & 1569 \end{pmatrix}$, $Y = \begin{pmatrix} 29 \\ 96 \\ 598 \end{pmatrix}$

On déduit $X = A^{-1} Y = \begin{pmatrix} 7{,}09 \\ -4{,}05 \\ 0{,}81 \end{pmatrix}$

$\Rightarrow p(x) = 7{,}09 - 4{,}05\, x + 0{,}81\, x^2$