Financial Market Analysis (FMAx)
Module 4

Term Structure of Interest Rates

The Relevance to You

You might be…

- An investment manager
- A debt manager
- An economic analyst/forecaster

Central Banker

Before We Begin

The Term Structure of Interest Rates:

- Provide useful insight about how the market thinks about future interest rate movements.
    - Allows central bankers to gauge market expectation about inflation, growth and risks
- Allows you to price any asset

Simplifying Assumption

To simplify the discussion, we assume the following:

- The face value of all bonds, M, is $100
- Each period is a year
- Coupon payments, *c*, are made annually
- Consider only “clean bond pricing”

time, t 0

Defining the Term Structure of Interest Rates – 1

Question: What is the term structure of interest rate?

- It is the relationship between the yield-to-maturity of zero-coupon bonds and their respective maturity
- It is often called the spot curve
- It can be derived mathematically

Defining the Term Structure of Interest Rates – 2

The bond pricing formula for a t-year zero coupon bond.

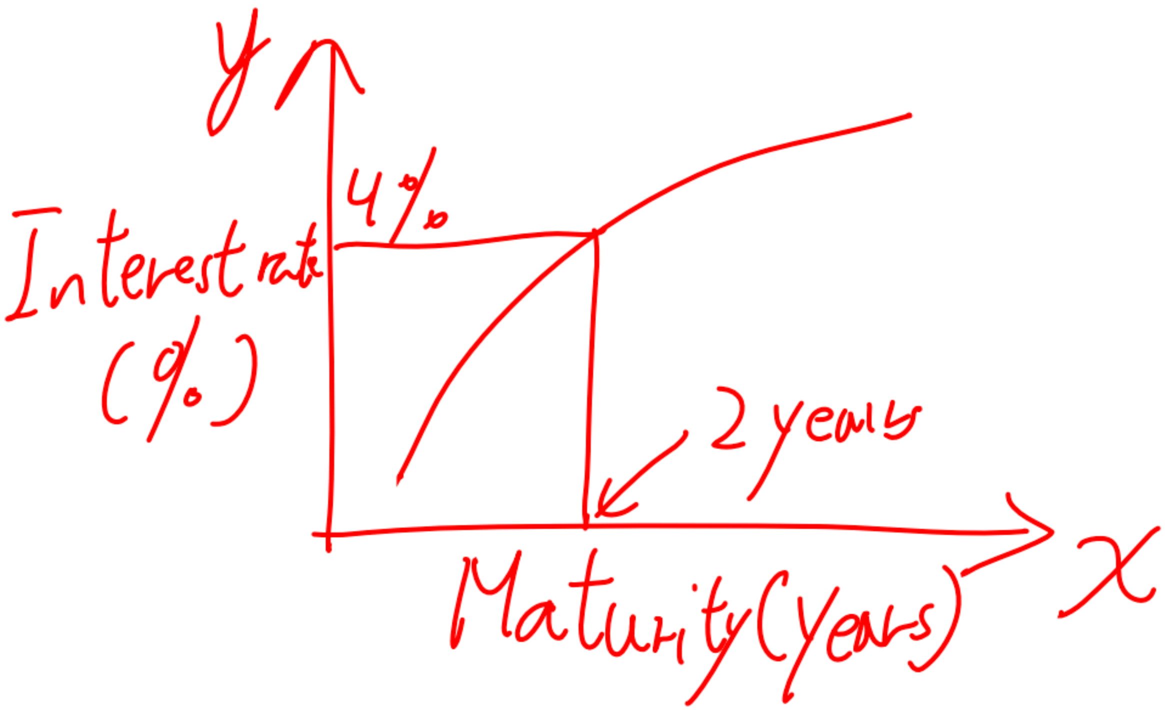
$$P_t = \frac{1}{(1 + y_t)^t} \times M$$

$$y_t = \left(\frac{M}{P_t}\right)^{\frac{1}{t}} - 1$$

$$P_t = DF_t \times M$$

Defining the Term Structure of Interest Rates – 3

A graphical representation of a term structure of interest rates



Defining the Zero Coupon Bond

Why use zero-coupon bond?

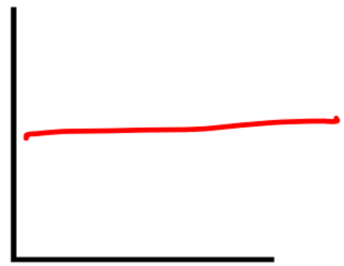
- Consider the bond pricing equation of a coupon bond
  - The YTM of a coupon bond defines "implicitly" in the following equation

$$P_t = \frac{c}{1+\tilde{y}_t} + \frac{c}{\left(1+\tilde{y}_t\right)^2} + \cdots + \frac{c+M}{\left(1+\tilde{y}_t\right)^t}$$

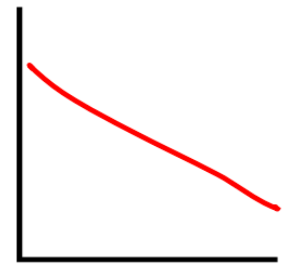
Common Shapes of the Yield Curve

Upward Sloping:

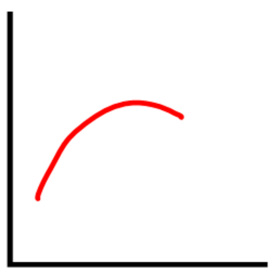
Flat:



Downward Sloping:



Hump-Shaped:



Trough-Shaped:

Common Shapes of the Yield Curve – Remark

On the shapes:

- The shapes showed here are not the only possible shapes of a yield curve
- The shape of a yield curve could be more complicated than those

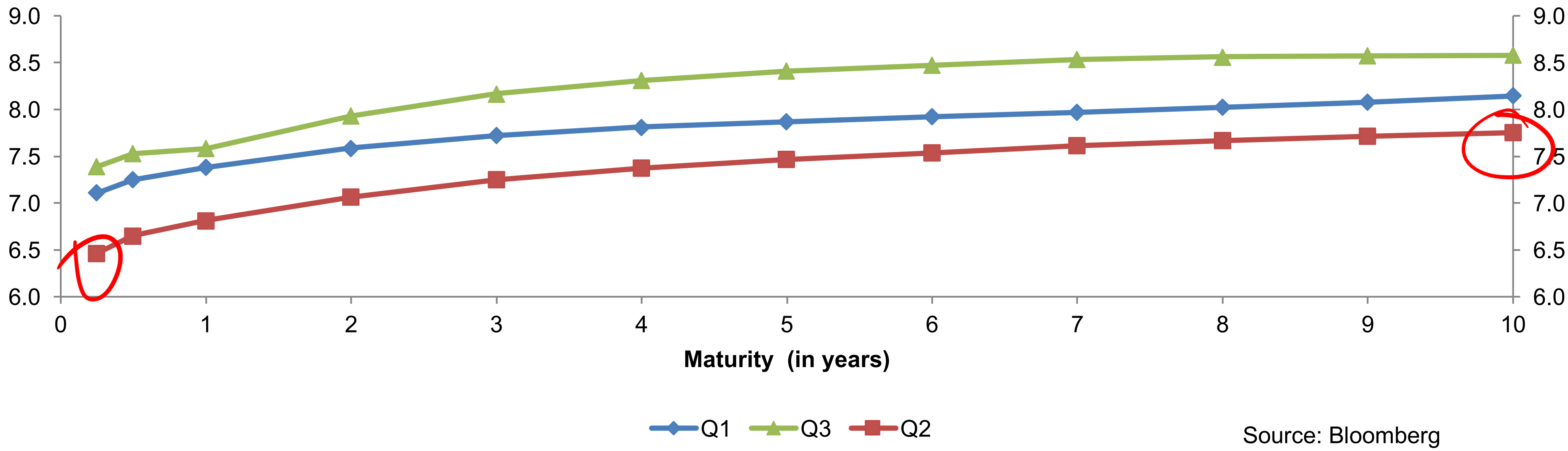
On the interpretations:

- The interpretation given to each shape is not the only “correct” interpretation
- Competing theories to explain the relationship between the shape of the yield curve and interest rates movements.

The Yield Curves for Indonesian Sovereign Bond

Example:

Zero Coupon Yield derived from Indonesian Sovereign Bond on Jan/2nd (blue), Apr/1st (Red) and Jul/1st (Green).

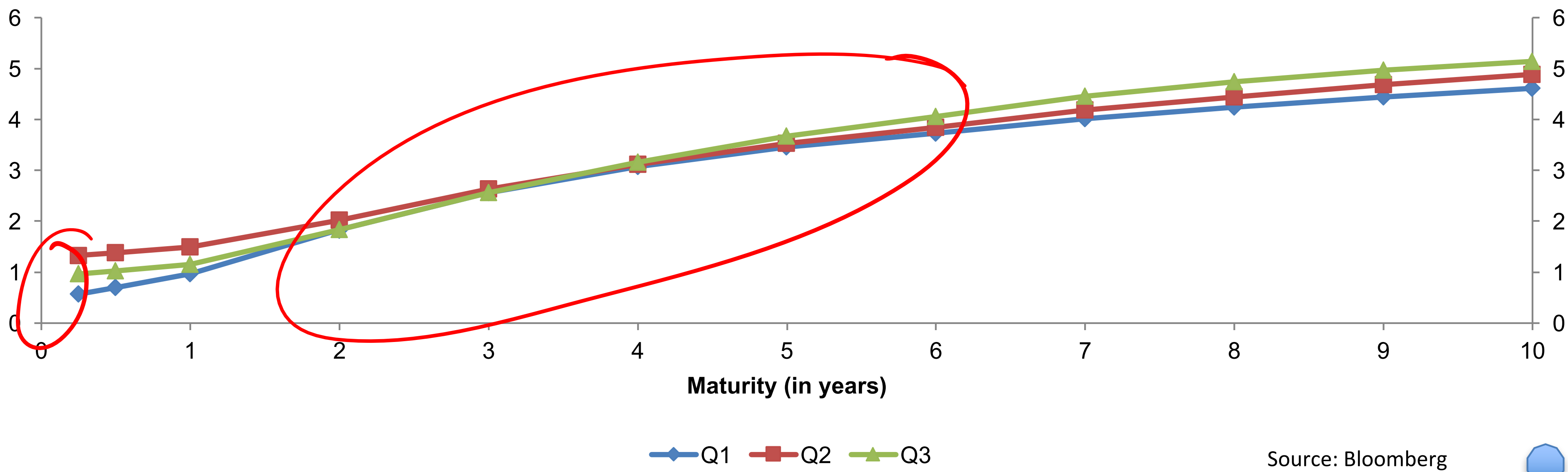


Source: Bloomberg

The Spot Curves for Brazil Sovereign Bond

Example:

Zero Coupon Yield derived from Brazil Sovereign Bond on Jan/2nd (blue), Apr/1st (Red) and Jul/1st (Green). 2015.



Source: Bloomberg

Bond Pricing with the Yield Curve – 1

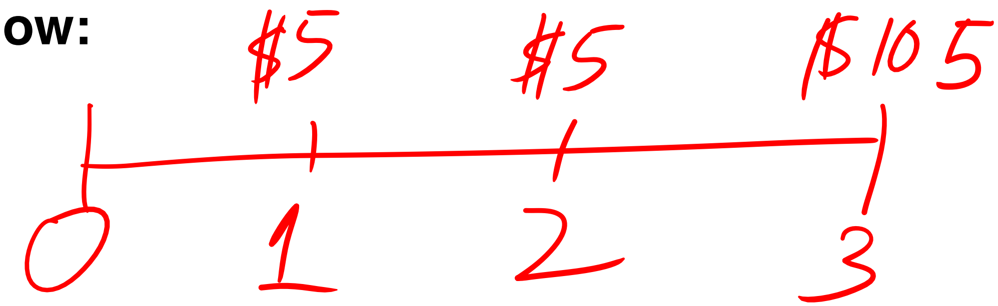
A stream of cash flow:

The Price of Strategy A:

$$P(3)$$

Bond Pricing with the Yield Curve – 2

A stream of cash flow:



The Price of Strategy B:

$$\tilde{P}_0(3) \equiv \frac{5}{100} P_0(1) + \frac{5}{100} P_0(2) + \frac{(100+5)}{100} P_0(3)$$

Defining the Spot Rate – 1

The "No-Arbitrage" Condition:

Given the Law of One Price, the price of alternatives A and B should be identical.

$$P(3) = \tilde{P}_0(3)$$

$$= 5\frac{P_0(1)}{100} + 5\frac{P_0(2)}{100} + (100+5)\frac{P_0(3)}{100}$$

$$= \frac{5}{1+y_1} + \frac{5}{(1+y_2)^2} + \frac{105}{(1+y_3)^3}$$

Defining the Spot Rate – 2

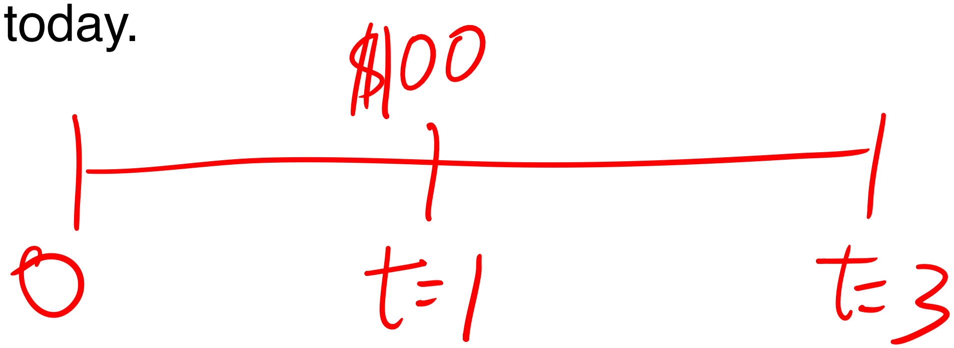
Let us generalize the formula to maturity T:

$$P(T) = \widetilde{P}_0(T) = \sum_{t=1}^{T} \frac{c}{M} P_0(t) + P_0(T)$$

$$= \sum_{t=1}^{T} \frac{c}{\left(1+y_t\right)^t} + \frac{M}{\left(1+y_T\right)^T}$$

Defining the Forward Interest Rate – 1

Forward Loan:
Agreement today to borrow/lend on some future date at an interest rate that is determined on today.



"Forward Rate" (or "forward interest rate"):
Interest rate on a forward loan. ⇒ 4%

The forward rate is **not necessarily** equal to spot short rate that will prevail in the future.

Defining the Forward Interest Rate – 2

Forward rates are tightly related to the spot rates by the no-arbitrage condition.

- Gross Return: $(1+y_2)^2$
- Gross Return: $(1+y_1)(1+f_{1,1})$

Defining the Forward Interest Rate – 3

The no-arbitrage condition ensures that…

$$(1+y_2)^2 = (1+y_1)(1+f_{1,1}) \Longrightarrow f_{1,1} = \frac{(1+y_2)^2}{(1+y_1)} - 1$$

Defining the Forward Interest Rate – 4

We can generalize the idea…

$$(1+y_T)^T = (1+y_{T-i})^{T-i}(1+f_{T-i,i})^i \Rightarrow f_{T-i,i} = \left(\frac{(1+y_T)^T}{(1+y_{T-i})^{T-i}}\right)^{\frac{1}{i}} - 1$$

By definition: $f_{0,1} = y_1$ $\quad y_t = f_{0,t}$

Defining the Forward Interest Rate – 5

It is important to notice and remember that …

- Forward lending/borrowing agreement may not exist or may not be allowed in reality in some countries

- However, forward interest rates can still be calculated as long as the yield curve exists!!

$$f_{T-i,i} = \left(\frac{(1+y_T)^T}{(1+y_{T-i})^{T-i}} \right)^{\frac{1}{i}} - 1$$

The Relationship between Spot and Forward Rates – 1

Forward interest rates are tightly related to spot interest rates.

They are also tightly related to…

- The discount factors (DF)

- The prices of the corresponding zero-coupon bonds (P)

The Relationship between Spot and Forward Rates – 2

Let us recall that…

$$\Rightarrow f_{T-i,i} = \left(\frac{(1+y_T)^T}{(1+y_{T-i})^{T-i}} \right)^{\frac{1}{i}} - 1$$

$$= \left(\frac{DF_{T-i}}{DF_T} \right)^{\frac{1}{i}} - 1$$

$$= \left(\frac{P_{T-i}}{P_T} \frac{M}{M} \right)^{\frac{1}{i}} - 1 = \left(\frac{P_{T-i}}{P_T} \right)^{\frac{1}{i}} - 1$$

$$DF_t = \frac{1}{(1+y_t)^t}$$

$$P_t = DF_t \times M$$

The Relationship between Spot and Forward Rates – 3

Spot rates can also be rewritten as a (geometric) average of forward rates.

$$
\begin{aligned}
(1+y_T)^T &= (1+y_{T-1})^{T-1}(1+f_{T-1,1}) \\
&= (1+y_{T-2})^{T-1}(1+f_{T-2,1})(1+f_{T-1,1}) \\
&= (1+y_1)(1+f_{1,1})\cdots(1+f_{T-2,1})(1+f_{T-1,1}) \\
&= (1+f_{0,1})(1+f_{1,1})\cdots(1+f_{T-2,1})(1+f_{T-1,1})
\end{aligned}
$$

$$
\Rightarrow \quad 1+y_T = \left[(1+f_{0,1})(1+f_{1,1})\cdots(1+f_{T-2,1})(1+f_{T-1,1})\right]^{\frac{1}{T}}
$$

The Relationship between Spot and Forward Rates – 4

$$\Rightarrow \quad 1+y_T=\left[\left(1+f_{0,1}\right)\left(1+f_{1,1}\right)\cdots\left(1+f_{T-2,1}\right)\left(1+f_{T-1,1}\right)\right]^{\frac{1}{T}}$$

$$\Rightarrow \ln\left(1+y_T\right)=\frac{1}{T}\left[\ln\left(1+f_{0,1}\right)+\ln\left(1+f_{1,1}\right)\cdots+\ln\left(1+f_{T-2,1}\right)+\ln\left(1+f_{T-1,1}\right)\right]$$

Since interest rates are generally of a small magnitude…

$$y_T \approx \frac{1}{T}\left[f_{0,1}+f_{1,1}+\cdots f_{T-1,1}\right]$$

The spot interest rate is **approximately** equal to the **arithmetic average of forward interest rate.**

The Relationship between Spot and Forward Rates – 5

This expression has an economic interpretation:

- The spot interest rate at maturity T *year* is the **simple average 1-year borrowing cost** over T years.

- The x-year forward 1-year interest rate $f_{x,1}$ is the **marginal borrowing cost** when the loan is extended by one year.

The Relationship between Spot and Forward Rates – 6

For example:

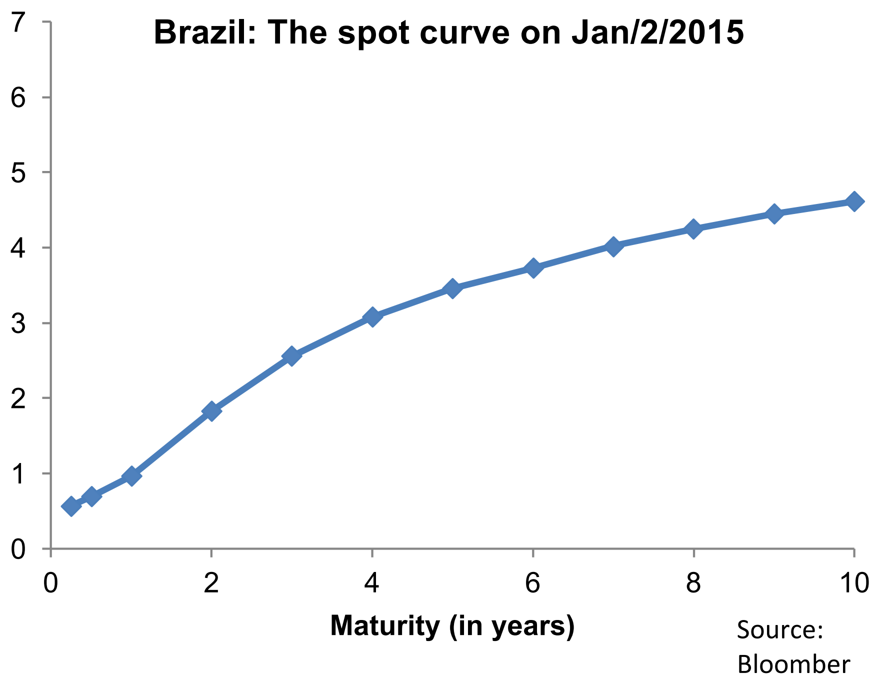
- A 3-year zero coupon bond is trading at YTM of 3%
- A 2-year zero coupon bond is trading at YTM of 2%

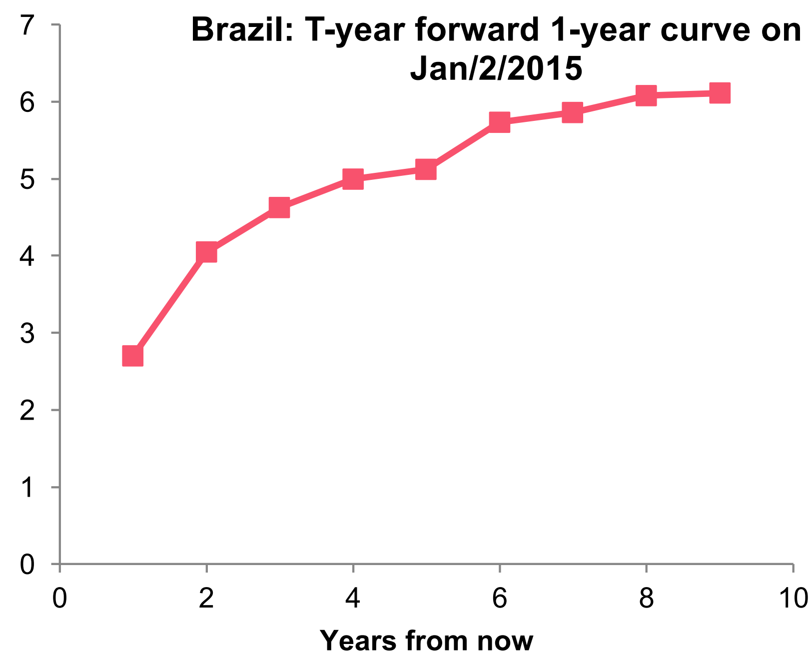
$$f_{2,1} = \left(\frac{(1+0.03)^3}{(1+0.02)^2} \right) - 1 = 5.03\%$$

The interest rate for the additional year is 5.03% (after rounding).

The Relationship between Spot and Forward Rates – 7

Just as we did with spot rates, we can define a forward yield curve.





The Relationship between Spot and Forward Rates – 8

Because the spot interest rate is the average of forward interest rates:

- spot curve is upward-sloping = Forward curve is above the spot curve
- spot curve is downward-sloping = Forward curve is below the spot curve
- spot curve is flat = Forward curve is equal to spot curve.

The Par-Yield – 1

Defining the par-yield:

- Measures the coupon rate (in percentage term) at which a coupon bond would be traded at par.
- The par-yield is often used as a reference for pricing new issues.
- It is NOT the spot interest rate but is closely related to it.
  - Thus, you cannot use it directly for bond pricing or discount future cash-flow

The Par-Yield – 2

Given the term-structure of interest rates, the par-yield of maturity T, c_T^*, is defined implicitly as…

$$M = \sum_{t=1}^{T} \frac{c_T^* M}{(1+y_t)^t} + \frac{M}{(1+y_T)^T}$$

$$c_T^* = \frac{1 - \dfrac{1}{(1+y_T)^T}}{\displaystyle\sum_{t=1}^{T} \frac{1}{(1+y_t)^t}} = \frac{1 - DF_T}{\displaystyle\sum_{t=1}^{T} DF_t}$$

The Par-Yield – 3

Similar to the relationship between the spot curve and the forward curve, the par-yield…

- The par-yield can be viewed as a kind of average of the spot interest rate.
- The par-yield curve is flatter than both the spot curve and the forward curve.

Bootstrapping – 1

Defining bootstrapping:

- “If you fall into a well and no one is around, you use your bootstrap to help yourself climb up from a well”

- Use available data on **coupon bonds** to construct the spot curve.

- The method relies on the assumption that no-arbitrage condition holds.

Bootstrapping – 2

Recall that the bond price equation, under the assumption of no-arbitrage…

$$P(T)=\sum_{t=1}^{T}\frac{c}{(1+y_t)^t}+\frac{M}{(1+y_T)^T}$$

$$\Rightarrow \quad y_T=\left(\frac{c+M}{P(T)-\sum_{t=1}^{T-1}\frac{c}{(1+y_t)^t}}\right)^{\frac{1}{T}}-1$$

Bootstrapping – 3

Suppose you are given the following:

Maturity	Coupon	Price
1	5.25	101.5
2	4.37	99.8

Bootstrapping – 4

The **T-year spot interest rate**: $$y_T = \left(\frac{c+M}{P(T) - \sum_{t=1}^{T-1} \frac{c}{(1+y_t)^t}} \right)^{\frac{1}{T}} - 1$$

The **1-year spot interest rate**:

$$y_1 = \frac{100+5.25}{101.5} - 1 = 3.69\%$$

The **2-year spot interest rate** is then:

$$y_2 = \left(\frac{100+4.27}{99.8 - \frac{4.27}{1+y_1}} \right)^{\frac{1}{2}} - 1 = 4.39\%$$

Bootstrapping – 5

Suppose we have **T** coupon bonds and the maximum maturity among these bond is **T** years.

$$
\left\{
\begin{aligned}
P(T_1) &= CF^1{}_1 DF_1 + CF^1{}_2 DF_2 \cdots + CF^1{}_T DF_T \\
P(T_2) &= CF^2{}_1 DF_1 + CF^2{}_2 DF_2 \cdots + CF^2{}_T DF_T \\
&\;\vdots \\
P(T_T) &= CF^T{}_1 DF_1 + CF^T{}_2 DF_2 \cdots + CF^T{}_T DF_T
\end{aligned}
\right\}
$$

Bootstrapping – 6

Stack the equations in matrix form…

$$\underbrace{\begin{bmatrix} P(T_1) \\ \vdots \\ P(T_T) \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} CF^1{}_1 & \cdots & CF^1{}_T \\ \vdots & \ddots & \vdots \\ CF^T{}_1 & \cdots & CF^T{}_T \end{bmatrix}}_{CF} \underbrace{\begin{bmatrix} DF_1 \\ \vdots \\ DF_T \end{bmatrix}}_{DF}$$

$\sum_{t=1}^{T} CF_t^T DF_t$

Bootstrapping – 7

Pre- (Left-) multiply the inverse of cash-flow matrix to the price vector P…

$$\underbrace{\begin{bmatrix} DF_1 \\ \vdots \\ DF_T \end{bmatrix}}_{DF} = inv \underbrace{\begin{bmatrix} CF^1{}_1 & \cdots & CF^1{}_T \\ \vdots & \ddots & \vdots \\ CF^T{}_1 & \cdots & CF^T{}_T \end{bmatrix}}_{\text{Inverse of CF}, CF^{-1}} \underbrace{\begin{bmatrix} P(T_1) \\ \vdots \\ P(T_T) \end{bmatrix}}_{P}$$

Regression Approach – 1

More generally, suppose we have N coupon bonds and the maximum maturity is T.

$$\underbrace{\begin{bmatrix} P(T_1) \\ \vdots \\ P(T_N) \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} CF^1{}_1 & \cdots & CF^1{}_T \\ \vdots & \ddots & \vdots \\ CF^N{}_1 & \cdots & CF^N{}_T \end{bmatrix}}_{CF} \underbrace{\begin{bmatrix} DF_1 \\ \vdots \\ DF_T \end{bmatrix}}_{DF}$$

Regression Approach – 2

- If $N = T$, then regression approach reduce to the bootstrap approach.
- If $N<T$, then too many discount factors satisfy the system.
- If $N>T$, then unique discount factors with “pricing error”.

Regression Approach – 3

We can estimate the vector DF by the method of **Ordinary-Least-Square** (OLS).

The pricing error: $\varepsilon = P - CF \times DF$

Find the DF such that $\sum_{i=1}^{N} \varepsilon_i^{\,2} = \left(P(T_i) - \sum_{t=1}^{T} CF^{i}{}_{t} DF_t \right)^2$ is minimized

OLS estimated discount factor: $DF = \left(CF' CF \right)^{-1} \left(CF' P \right)$

Parametric Yield Curve Models – 1

The resulting estimated yield curve is usually not smooth.

Interpolation is often used to calculate the discount factor/spot rate for maturity that we do not have.

Parametric Yield Curve Models – 2

Parametric function to model the discount factor or the spot curve (or the forward curve) can give you the desired "smoothness".

- Pros
  - Give the desired "smoothness"
  - Alleviate the difficulty of not enough number of coupon bonds relative to the maximum maturity

- Cons
  - At the potential expense of higher pricing error.
  - May requires non-linear method to estimate.

Parametric Yield Curve Models – 3

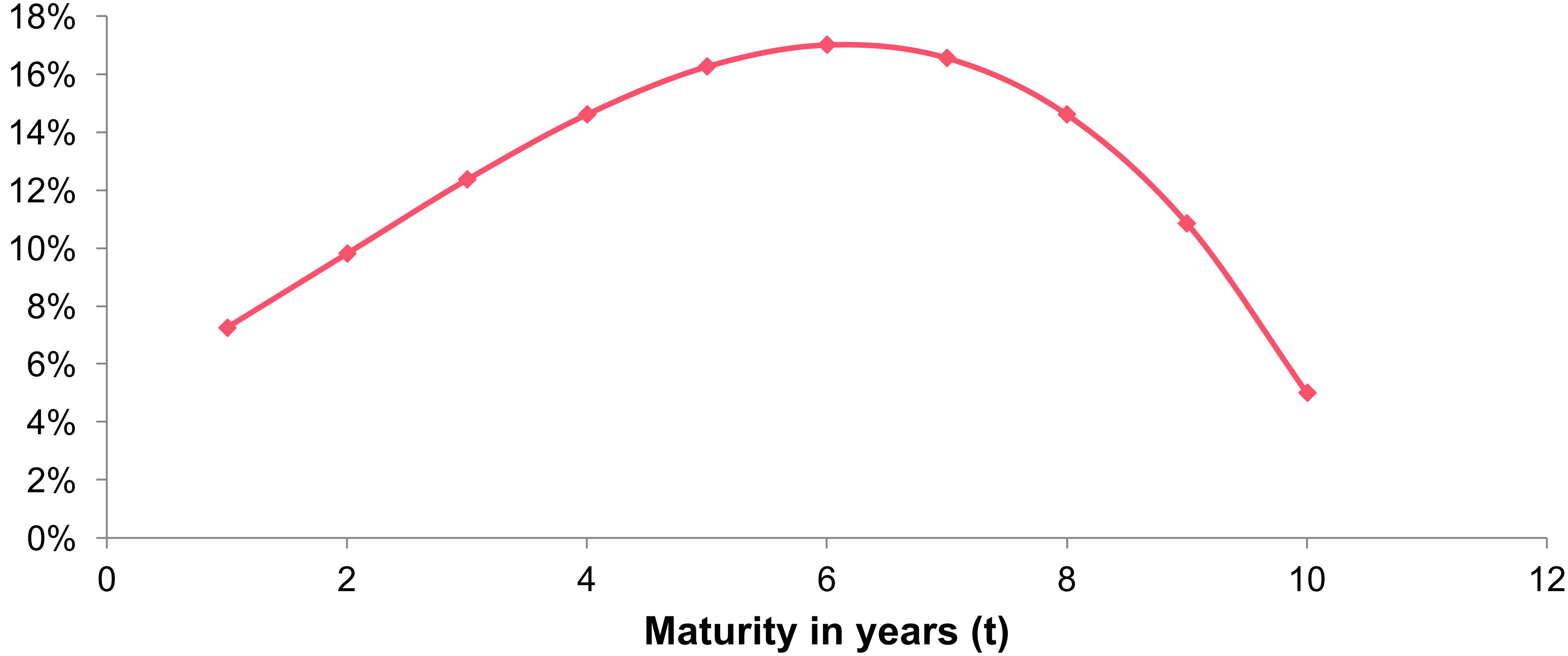
For example… The Polynominal Yield Curve.

The spot curve can be modeled as a polynominal equation with any order…

$$\Rightarrow \quad y_t = \alpha + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$$

Parametric Yield Curve Models – 4

$$y_t = 0.05 + 0.02t + 0.003t^2 - 0.0005t^3$$



Parametric Yield Curve Models – 5

Two other common models:

Nelson-Siegel –

$$y_i = \alpha + \beta_1 \left[1 - \exp\left(-\frac{i}{\tau_1}\right)\right]\frac{\tau_1}{i} + \beta_2 \left\{\left[1 - \exp\left(-\frac{i}{\tau_1}\right)\right]\frac{\tau_1}{i} - \exp\left(-\frac{i}{\tau_1}\right)\right\}$$

Svennson –

$$y_t(i) = \alpha + \beta_1 \left[1 - \exp\left(-\frac{i}{\tau_1}\right)\right]\frac{\tau_1}{i} + \beta_2 \left\{\left[1 - \exp\left(-\frac{i}{\tau_1}\right)\right]\frac{\tau_1}{i} - \exp\left(-\frac{i}{\tau_1}\right)\right\} + \beta_3 \left\{\left[1 - \exp\left(-\frac{i}{\tau_2}\right)\right]\frac{\tau_2}{i} - \exp\left(-\frac{i}{\tau_2}\right)\right\}$$

The Pure Expectation Hypothesis – 1

The strong form of pure expectation hypothesis…

- Forward rates reflect today's expectation of what spot rates will be in the future.

- Investors are ASSUMED to be risk-neutral.

  Investors care about the return only. Risk is not a concern.

The Pure Expectation Hypothesis – 2

Consider the three following strategies:

A. Buy a 2-year zero-coupon bond **on today**.

$$(1+y_2)^2$$

B. Buy a 1-year zero-coupon bond and then roll-over with a one-year forward 1-year zero-coupon bond **on today**.

$$(1+y_1)(1+f_{1,1})$$

C. Buy a 1-year zero-coupon bond **on today** and then roll-over the proceed with another 1-year zero-coupon **one year later**

$$(1+y_1)(1+E[y_{1,1}])$$

The Pure Expectation Hypothesis – 3

Recall that no-arbitrage condition ensures that…

$$(1+y_2)^2 = (1+y_1)(1+f_{1,1})$$

Pure expectation hypothesis: Investors will be indifferent between all three strategies if the **expected** return is the same:

$$(1+y_1)(1+f_{1,1}) = (1+y_1)(1+E[y_{1,1}])$$

$$\Rightarrow f_{1,1} = E[y_{1,1}]$$

The Pure Expectation Hypothesis – 4

Moreover, this implies that…

$$(1+y_2)^2 = (1+y_{1,1})(1+E[y_{1,1}])$$

$$y_2 \approx \frac{y_1 + E[y_{1,1}]}{2}$$

Extending the logic to the T-year case:

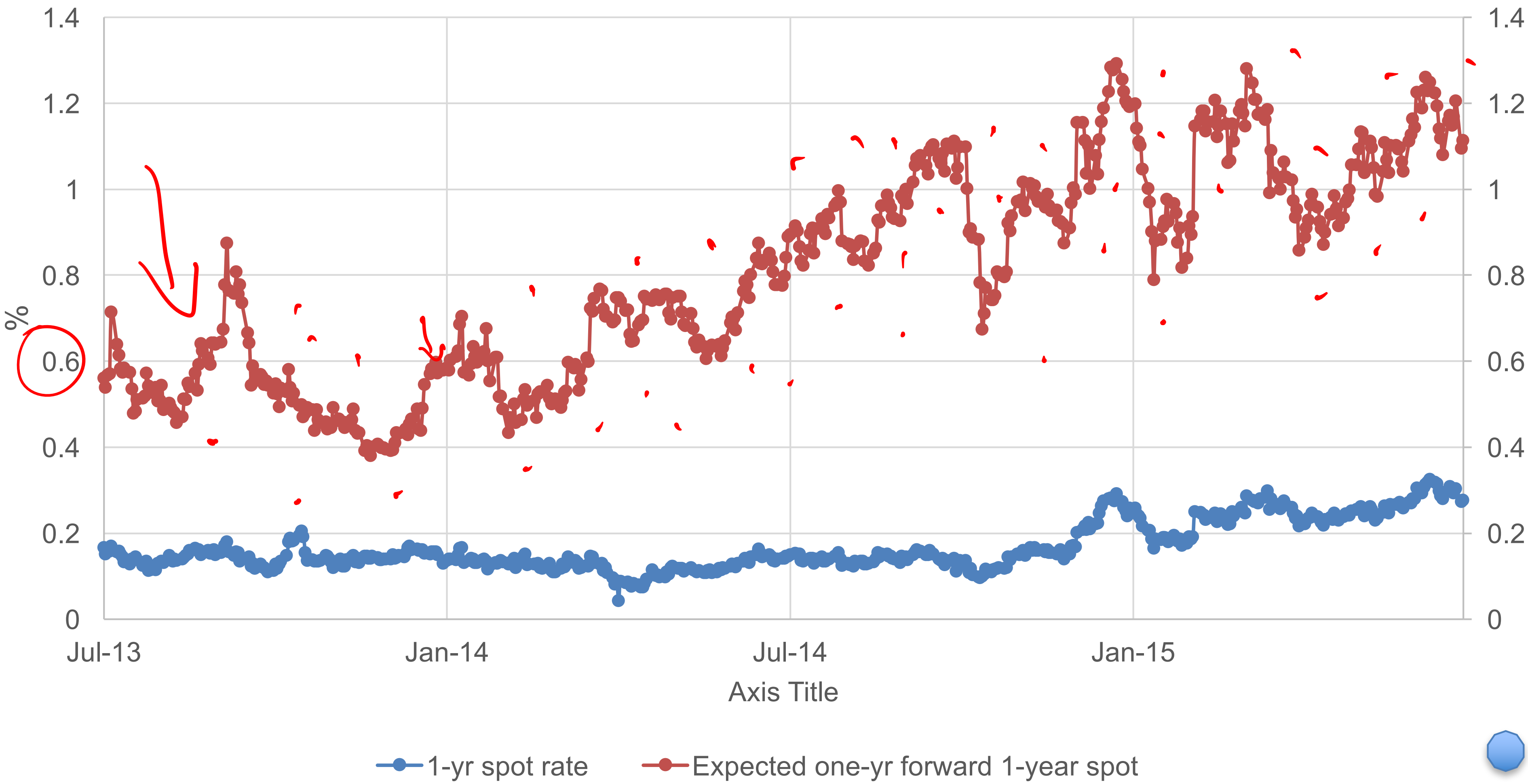
$$\Rightarrow y_T \approx \frac{y_1 + E[y_{1,1}] + \cdots E[y_{T-1,1}]}{T}$$

The Pure Expectation Hypothesis – 5

Forward rates correspond to today's expectations of future spot rates…

- If $y_2 > y_1 \Rightarrow E[y_{1,1}] > y_1$ i.e., market participants are expecting an increase in the spot rate in the future.
- An upward-sloping spot curve suggests that the market is expecting rates to rise.

The Pure Expectation Hypothesis – 6



Term Premium – 1

There must be something else that helps explain….

Suppose that $y_2 = \frac{y_1 + E[y_{1,1}] + tp_{1,1}}{2}$ and ASSUME that $y_1 = E[y_{1,1}]$

Then $y_2 = y_1 + \frac{tp_{1,1}}{2}$

If $tp_{1,1} > 0$, then $y_2 > y_1$ (even though the market does not expect rates to rise.)

Term Premium – 2

Question: What is $tp_{1,1}$?

It is often called the "term premium" --- The premium that investors require to hold a bond with a particular maturity over another maturity
(We usually consider the term premium of longer-bond over short-term bond!)

Alternatively, the no-arbitrage condition suggests that we can define it as

$$tp_{1,1} = f_{1,1} - E[y_{1,1}]$$

(The excess of the forward rate over pure expectation of future interest rate.)

Term Premium – 3

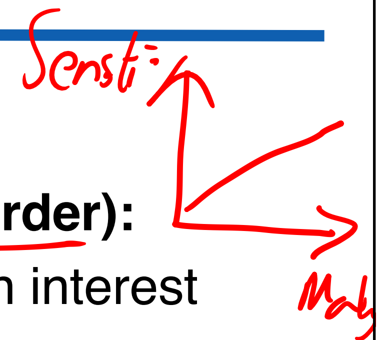
The term-premium of longer-term bond over shorter-term bond is usually (but not always).

- Positive: Investors are assumed to be risk-averse in order to justify the positive term-premium

- Increasing: Longer the maturity, higher the price sensitivity.

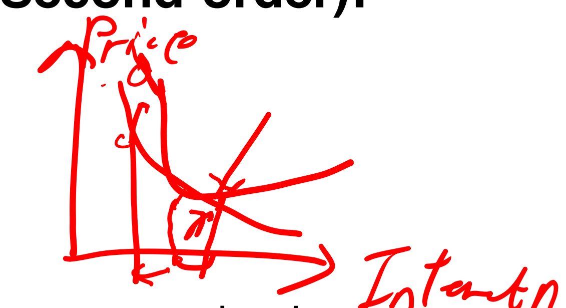
Term Premium – 4

The term-premium can then be decomposed into two components.



- **[Price] risk premium (increasing with the maturity; First-order):**
  Higher the **duration**, more sensitive is the price to a change in interest rates;

- **Convexity premium (decreasing with the maturity Second-order):**
  Higher the **convexity**, for the same amount of interest rate
  - Drop ➔ larger the price increases
  - Increase ➔ smaller the price decreases



The term-premium can be positive or negative depending on which component dominates.

Market Segmentation – 5

Market Segmentation or **Preferred Habitat Hypothesis**

- Bonds of a given maturity are mainly traded by a particular group of investors.

  Longer-term bond ⇔ Pension fund

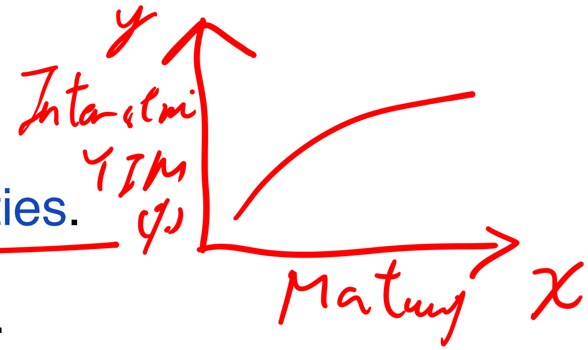
- The supply and demand conditions of a bond with a given maturity are independent to the supply and demand conditions of bonds of other maturity.

- Arbitrage opportunity across maturities is missing.

- This explanation is less popular nowadays.

Module Wrap-Up – 1

The Term Structure of Interest Rates…



- Relates the zero coupon yields to different maturities.
- Zero coupon yields are often not readily available.
  Need to construct the curve using coupon bonds and their prices.
  - Bootstrap and Regression
  - Parametric Models provide smoother yields

Module Wrap-Up – 2

If the Pure Expectation Hypothesis holds…

- The yield curve essentially reflects the market forecast of future interest rate movements.
- The presence of a term-premium, necessitates a theory to model the term-premium.