Fractional Programming in Cooperative Games

Introduction to my PhD work

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Supervised by

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Dr. W. Kern

This is a test frame

Example block

This is an example.

Alerted block.

some alerted text.

Normal block

This is a block.

An allocation problem

Two types of players

- A: Each $i \in A$ possesses an item of size a_i
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- A game is called *ϵ-balanced* if ϵ-core $\neq \emptyset$.

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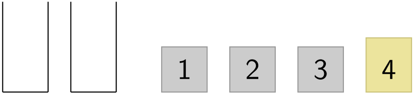

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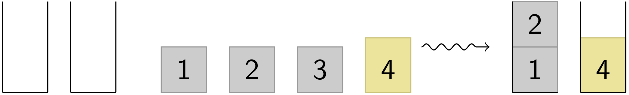

$v \approx \frac{3}{2}$

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- $\frac{1}{4}$-core $\neq \emptyset$. (Kern and Qiu 2013)

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- Integrality gap: $\frac{ILP}{LP}$.

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ϵ-core(N)$\neq \emptyset \Leftrightarrow \epsilon \geq 1 - \frac{v}{v'}$.

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Trivially, $\frac{1}{2}$-core(N) $\neq \emptyset$ (for all N).

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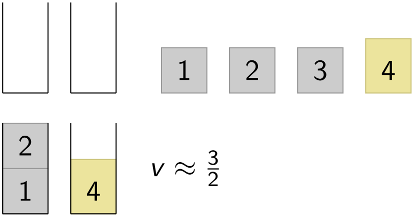

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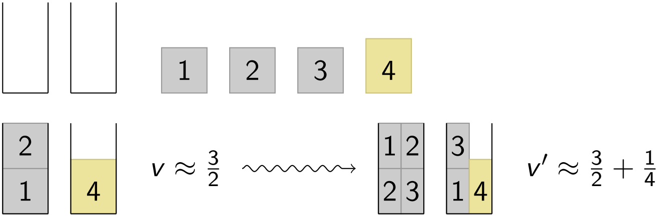

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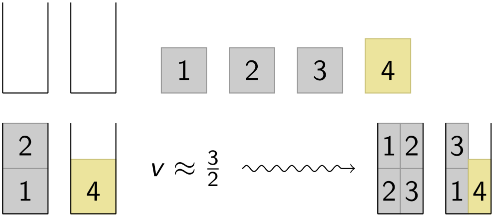

$v \approx \frac{3}{2}$ $\rightsquigarrow$ $v' \approx \frac{3}{2} + \frac{1}{4}$

$$1 - \frac{v}{v'} \approx \frac{1}{7}$$

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- Kern and Qiu (2012) proved the following results:
  1. $1/2$-core $\neq \emptyset$.
  2. $5/12$-core $\neq \emptyset$ if $a_i > 1/3$.
  3. ϵ-core $\neq \emptyset$ if $k \geq O(\epsilon \bar{b})^{-5}$, where $\bar{b}$ is the average bin capacity.

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