Numerical Methods Bootcamp

Tuesday Assignment
Miscellaneous stuff

In this assignment you are asked to do three things. First, you will simulate draws from a weird probability distribution and illustrate the Central Limit Theorem.[1] Second you are asked to solve a nonlinear equation on a grid containing several values of a parameter, numerically approximate the solution for parameter values not on the grid, and graphically illustrate the results. Lastly, you will solve a fixed point problem. As usual you are provided with some incomplete scripts to get you started.

The Central Limit Theorem in action. Consider the probability density function (pdf)

$$f(x) = a(1-x), \text{ if } x \geq \bar{x}$$
$$f(x) = 0, \text{ otherwise.}$$

for $x \in [0,1]$. Figure 1 illustrates what this pdf looks like. In Figure 1 (and the subsequent figures) $\bar{x}$ is set to 0.7 and $a = 1/(1/2+1/2\bar{x}^2-\bar{x})$. Can you verify that this indeed is a pdf? Derive the cdf, $F(z) = Pr(x \leq z)$, and its inverse. Figure 2 illustrates what these should look like. The inverse of a cdf is incredibly useful for the purpose of simulating peculiar looking distributions.

Calculate the mean, μ, and the variance, σ^2, of the distribution using the `integral` function in Matlab.

Set the random seed to `1979` and make $N = 100$ draws of a random uniform variable on $[0,1]$. Calculate S_n as

$$S_n = \sqrt{N}\left(\frac{\sum_i F^{-1}(e_i)}{N} - \mu\right).$$

Repeat 10,000 times and plot a histogram. Figure 3 shows the results you should get, accompanied by a fitted normal distribution. According to the CLT, $\lim_{N\to\infty} S_n$ should be normally distributed with mean μ and variance σ^2. Isn't the CLT pretty damn amazing after all? What happens if you set $N = 2$?

[1]This may seem slightly divorced from the course content. But it is a useful exercise to illustrate how numerical computations can provide some intuition on theoretical results. Seeing is believing.

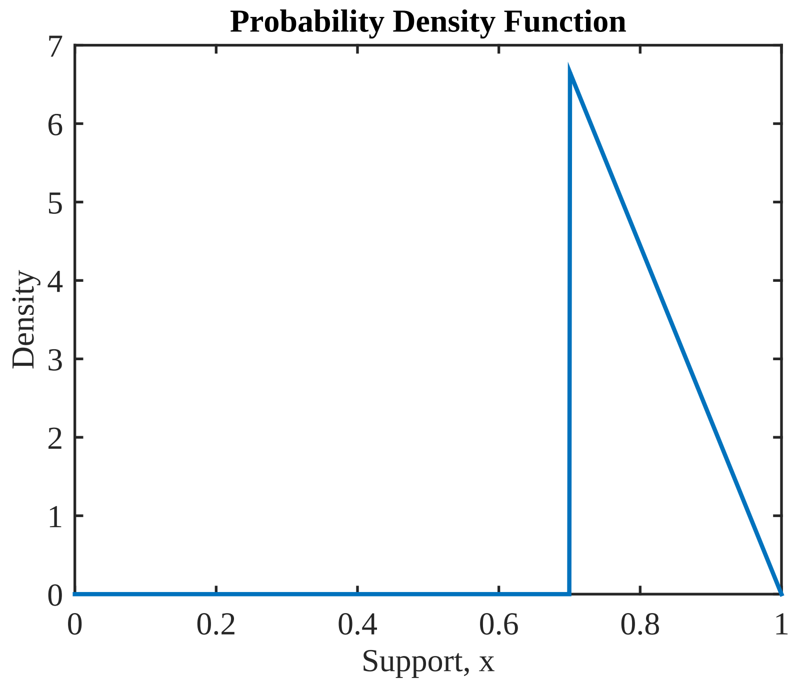

Figure 1. Probability Density Function (pdf).

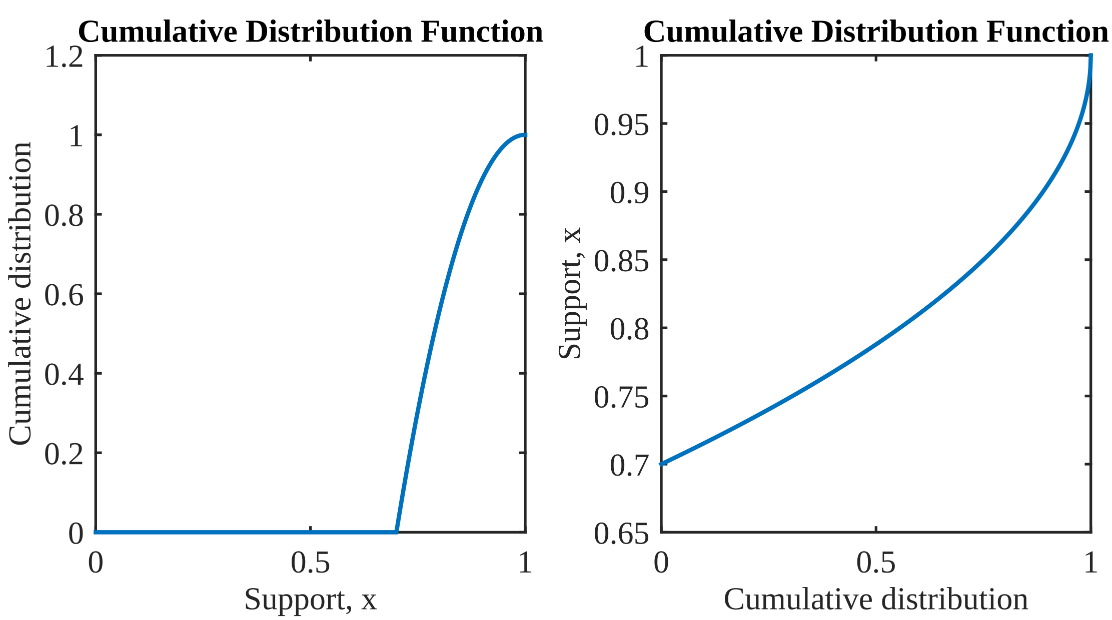

Figure 2. Cumulative Density Functions (cdf).

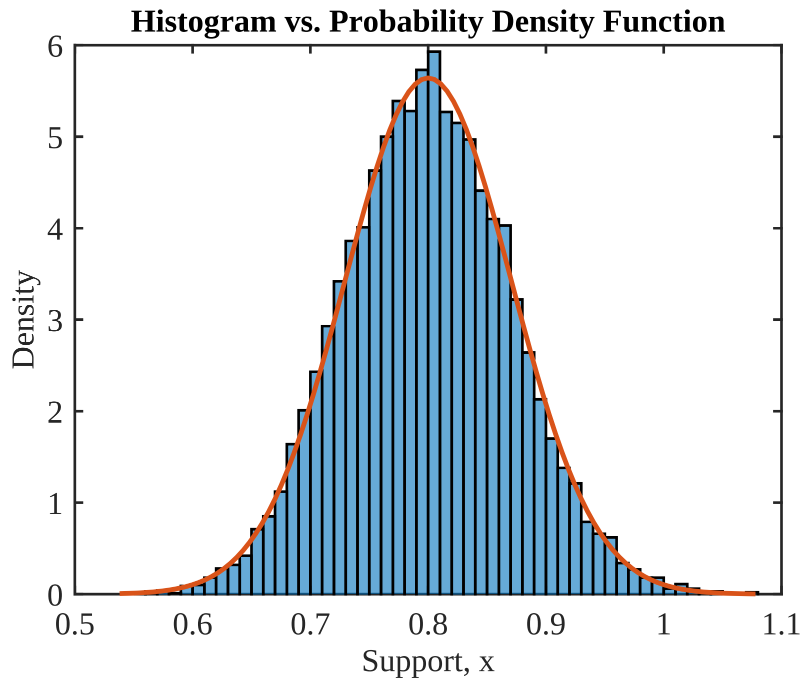

FIGURE 3. Histogram with fitted normal.

A nonlinear equation. The nonlinear equation you will solve is given by

$$f(y) = (x^\eta - y)^{-1} - \delta\eta y^{\eta-1}(y^\eta - z)^{-1},$$

where x and z are treated as parameters. z is set equal to $(\eta\delta)^{1/(1-\eta)}$.

For each value of x on a grid solve for y such that $f(y) = 0$ using Newton's method. Plot your results in the (x, y) plane. Compare your results with those in Figure 4 which illustrates the mapping $y = g(x)$. That is the function $y = g(x)$ such that,

$$(x^\eta - g(x))^{-1} - \delta\eta g(x)^{\eta-1}(g(x)^\eta - z)^{-1} = 0 \quad \forall x$$

What is the relationship between z and the intersection of your solution with the 45^o line?

A Functional Equation. For this exercise we will think of z as being an endogenous variable which relates to y in exactly the same way as y relates to x. That is if $y = g(x)$ then $z = g(y)$, or, equivalently, that $z = g(g(x))$. As a consequence, the function $g(x)$ should satisfy the *functional equation*

$$(x^\eta - g(x))^{-1} - \delta\eta g(x)^{\eta-1}(g(x)^\eta - g(g(x)))^{-1} = 0 \quad \forall x \tag{1}$$

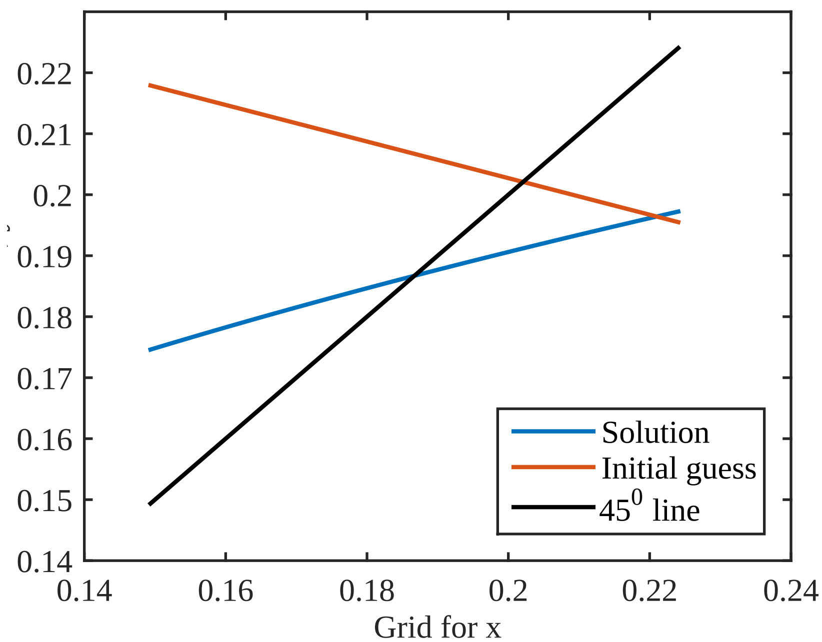

FIGURE 4. The solution to a nonlinear equation.

Notice that the only difference between this equation and the previous equation is that z is now not fixed, but a function of y. And it takes the same for as y as a function of x. Hence, z is replaced by $g(g(x))$.

There are several ways of solving this functional equation, but I would like you to use what is known as *time iteration* – a concept that will be further clarified tomorrow.

Time iteration in this setting works as follows. Guess for a solution to equation (1) and call this guess $g_0(\cdot)$. For instance, a stupid, but perfectly operational guess, would be $g_0(x) = x$.[2] Now, use your knowledge of solving nonlinear equations on a grid for x to solve

$$(x^\eta - y)^{-1} - \delta\eta y^{\eta-1}(y^\eta - g_0(y))^{-1} = 0 \quad \forall x$$

Thus, for all x we have a solution for y. Combining these solutions we can use an approximation method to update our guess $y = g_1(x)$. Thus, in general, for any guess $g_n(x)$, solve

$$(x^\eta - y)^{-1} - \delta\eta y^{\eta-1}(y^\eta - g_n(y))^{-1} = 0$$

for a grid of x. Combine to create $y = g_{n+1}(x)$, and repeat until

$$(x^\eta - g_n(x))^{-1} - \delta\eta g_n(x)^{\eta-1}(g_n(x)^\eta - g_n(g_n(x)))^{-1} \approx 0, \quad \forall x.$$

[2]This is not a general statement. It is a good idea to choose an initial guess with a bit of care.

If you have done things correctly, your results should correspond to Figure 5

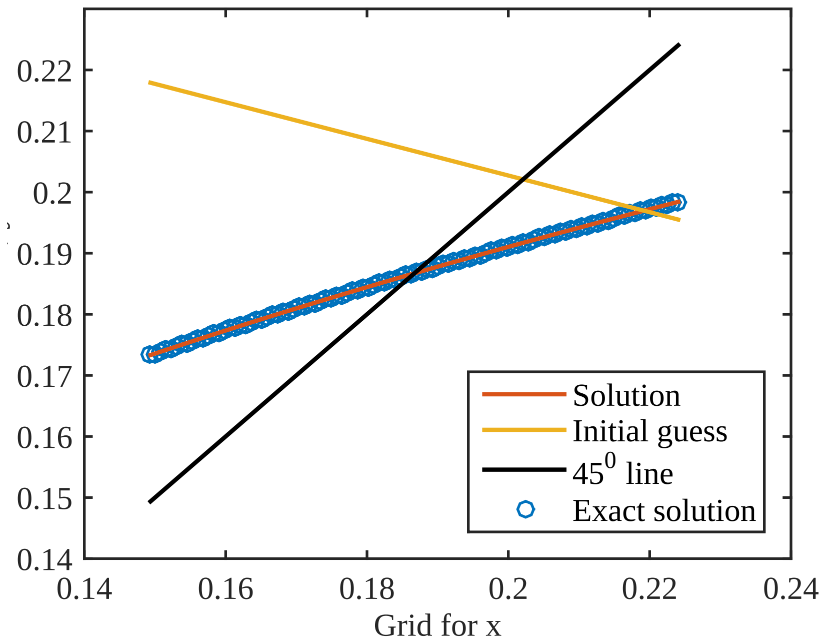

FIGURE 5. The solution to a functional equation.