Numerical Methods Bootcamp

Lecture 4
Incomplete markets models

Pontus Rendahl
rpk22@cam.ac.uk

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Introduction

- By the end of yesterday’s lecture we looked at an income fluctuation problem.
- The long run distribution could also be seen as the stationary cross-section distribution of wealth
- As I mentioned, suppose that, in equilibrium bonds are in zero net supply
- Then we need to find an interest rate such that the integral over the cross-sectional distribution of wealth is zero.

Introduction

- In this lecture we will learn how to do this (the Hugget model).
- Then we will analyse a very similar economy but in which “excess savings” are used as capital in production (the Aiygari model).
- Lastly, we will look at an Aiyagari-style economy with aggregate shocks (the Krusell and Smith model).

A Familiar Problem

- Consider the following two-period problem

$$\max\{u(c_0) + \beta \sum_{s \in \mathcal{S}} u(c_1(s))Pr(s)\}$$

$$\text{s.t.} \quad c_0 + \sum_{s \in \mathcal{S}} q(s)a(s) = \mathcal{S}_0$$

$$c_1(s) = \mathcal{S}_1(s) + a(s)$$

- Where s is *idiosyncratic*: $\sum_s \mathcal{S}_1(s)Pr(s) = \mathcal{S}_1$

A Familiar Problem

- First order conditions

$$u'(c_0)q(s) = \beta u'(c_1(s))Pr(s), \quad \forall s \in \mathcal{S}$$

- Then there exist equilibrium asset prices

$$q(s) = \frac{\beta u'(\mathcal{S}_1)}{u'(\mathcal{S}_0)} Pr(s)$$

- Such that $c_0 = \mathcal{S}_0$ and $c_1 = \mathcal{S}_1$.

Another Familiar Problem

- Consider the following two-period problem

$$\begin{aligned} &\max\{u(c_0) + \beta u(c_1)\} \\ \text{s.t.} \quad & c_0 + qa = \mathcal{S}_0 \\ & c_1 = \mathcal{S}_1 + a \end{aligned}$$

Another Familiar Problem

- First order conditions

$$u'(c_0)q = \beta u'(c_1)$$

- Then there exist an equilibrium asset price

$$q = \frac{\beta u'(\mathcal{S}_1)}{u'(\mathcal{S}_0)}$$

- Such that $c_0 = \mathcal{S}_0$ and $c_1 = \mathcal{S}_1$.

What does this mean?

- The first problem: Loads of agents and loads of outcomes
- Lots of trade in assets
- The second problem: One representative agent, one outcome
- No trade in assets.
- But same aggregate outcome!

What does this mean?

- A representative agent is not one agent.
- A representative agent does not exclude trade – it just occurs under the hood.
- A representative agent does not preclude coordination failures (e.g. prisoners' dilemma)
- Another word for rep. agent models: Complete markets models.

What does this mean?

- But sometimes we wish to depart from representative agent
  - Distributions matter/ Aggregation matter
  - Precautionary risk/savings from incomplete markets
  - Could lead to interesting dynamics
- This is what we will analyze today.

The Hugget Model

- The model of Hugget basically takes a standard bond economy as the main starting point.
- In similarity to representative agent models, the continuum of agents are still *ex-ante* identical, but, importantly, they are now *ex-post* heterogenous.
- In particular, the agents are hit by only *partially insurable*, *idiosyncratic shocks*.

The Hugget Model

- One-period obligation contracts is the only source of insurance (bonds).
- That is, there are many more goods (states), than markets, and sometimes these models are referred to *incomplete markets models*.
- Lastly, there are no *aggregate shocks* to the economy, and therefore no *aggregate risk* (something that is relaxed in Krusell and Smith's (1998) model).

The Hugget Model

The Bellman equation that summarises the households' problem is given by

$$v(b, s) = \max_{c, b'} \{u(c) + \beta \sum_{s' \in \mathcal{S}} v(b', s') p(s', s)\}$$

$$\text{s.t} \quad c + b' = (1 + r)b + w(s)$$

$$b' \geq \underline{b}$$

With associated policy function $b' = g(b, s)$.

The Hugget Model

The first order conditions are given by

$$u'((1+r)b + w(s) - b') - \mu(b,s)$$
$$= \beta(1+r)\sum_{s' \in \mathcal{S}} u'((1+r)b' + w(s') - b'')p(s',s)$$

where μ is the Lagrange multiplier associated with the borrowing constraint.

The Hugget Model

- Solution method: Find $\tilde{g}(b, s)$ as

$$
\begin{aligned}
&u'((1+r)b + w(s) - \tilde{g}(b,s)) \\
&\qquad = \beta(1+r)\sum_{s' \in \mathcal{S}} [u'((1+r)\tilde{g}(b,s) + w(s') \\
&\qquad\qquad - g_n(\tilde{g}(b,s), s'))] p(s', s)
\end{aligned}
$$

- Then $g_{n+1}(b, s) = \max\{\tilde{g}(b,s), \underline{b}\}$.

The Hugget Model

- Ok, so suppose that we have found $g(b, s)$ conditional on some r, now what?
- As there are idiosyncratic risk, each individual will be exposed to different shocks in different periods.
- More precisely, an individual is identified through his history of shocks.
- Clearly, as the history of shocks affect an individual's wealth, and individuals have experienced different types of histories, there will be a cross-sectional distribution of wealth-holdings.

The Hugget Model

- Now, as we have solved the household's problem for a given (constant) interest-rate, we have the policy function $b' = g(b, s)$.
- Together with (stochastic) law of motion for income, $p(s', s)$, we can easily describe the law of motion for ψ_t as follows,

$$\psi_{t+1}(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b,s)\}} \psi_t(b, s) p(s', s)$$

- One can show that under quite weak assumptions, ψ_t (and for any ψ_0) converges to a unique stationary distribution ψ such that,

$$\psi(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b,s)\}} \psi(b, s) p(s', s)$$

The Hugget Model

- This stationary distribution is very important to us.
- Given a constant interest-rate, the optimal household decision yields a stationary distribution with a constant excess-demand for bonds.
- Moreover, even if aggregates are constant (aggr. wealth, consumption, endowments etc.) individual specific variables are not: Agents jump frequently around in the distribution, but aggregates never change.
- Lastly, excess demand is given by

$$B(r) = \sum_{s \in \mathcal{S}} \sum_{b} b\psi(b, s)$$

The Hugget Model

Definition of a competitive equilibrium,

- A competitive equilibrium is an interest-rate r such that
- Given r, the policy function $g(b, s)$ solves the household's optimization problem.
- The stationary distribution satisfies,

$$\psi(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b,s)\}} \psi(b, s) p(s', s)$$

- Markets clear. That is $B(r) = 0$.

The Hugget Model

- We will compute the market clearing interest-rate in a second, but there is also an alternative interpretation of ψ that is important.
- Consider a household at time-zero, having a certain amount of wealth and a certain s.
- Q: What is the probability that the agent will have some arbitrary wealth b and arbitrary s in t-periods?
- A: We can calculate this using the exact same recursive formula as for the cross-sectional distribution!
- But as for any ψ_0, $\psi_t \to \psi$, the probability of (b, s) occurring when $t \to \infty$ must be $\psi(b, s)$.
- So ψ is not only the cross-sectional distribution of wealth, but also the *unconditional* distribution of wealth for an individual agent.

The Hugget Model

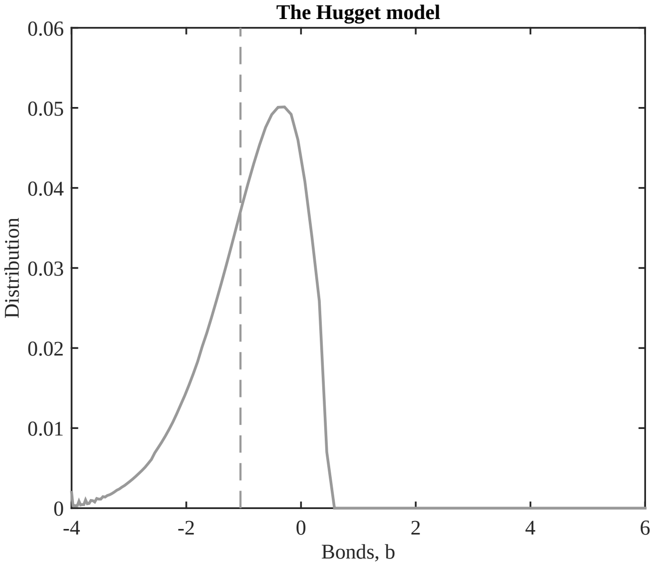
Algorithm to find an equilibrium

1. Guess for an interest-rate, r.
2. Solve the consumer's problem.
3. Calculate the cross-sectional distribution $\psi(b, s)$.
4. If excess demand is positive, adjust the interest-rate downwards. If negative upwards.

The Hugget Model

- Let's guess for the interest rate r, and solve the problem
  - This gives policy functions $b' = g(b, s; r)$.
- Convert these into transition matrices using the "lottery" method showed yesterday.
- Find the invariant distribution $\psi(b, s; r)$

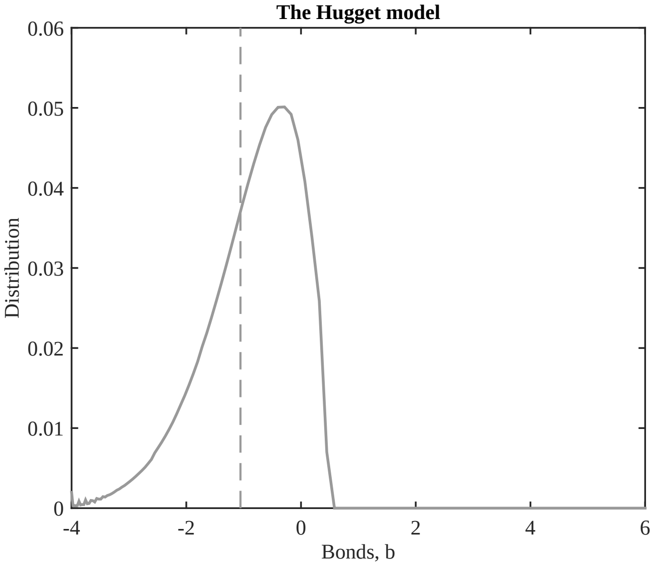
Hugget distribution



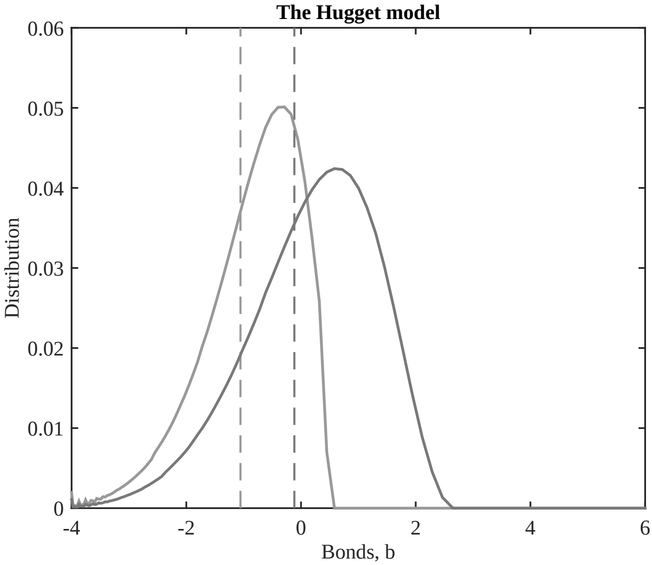
The Hugget Model

- Clearly there is no market clearing! The interest rate is far too low so there is excess supply of bonds.
- So let's adjust the interest rate, and iterate!

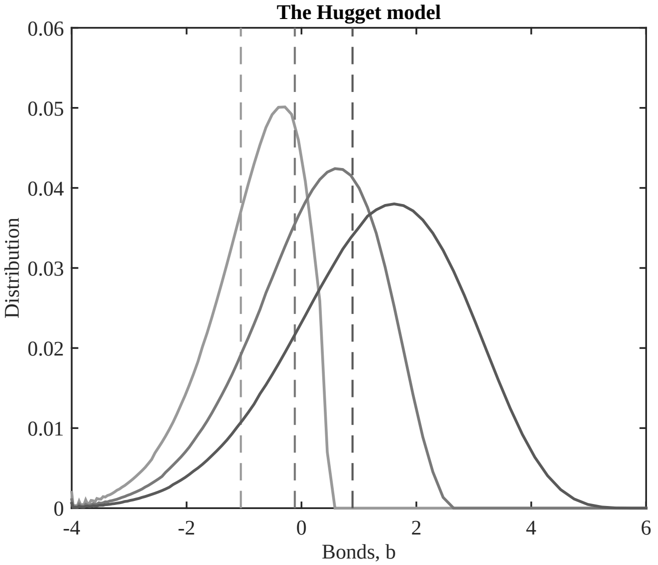
Hugget distribution



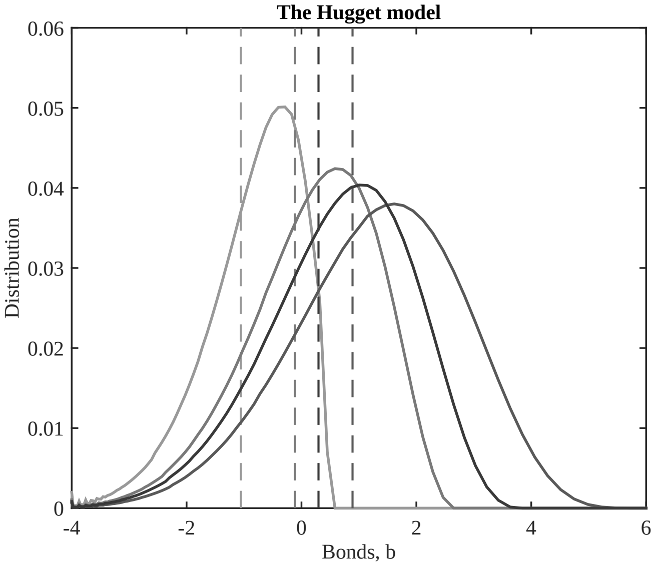
Hugget distribution



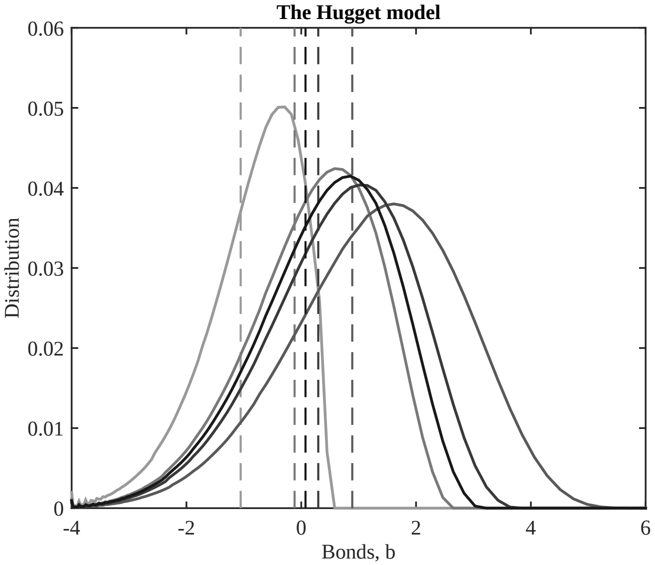
Hugget distribution



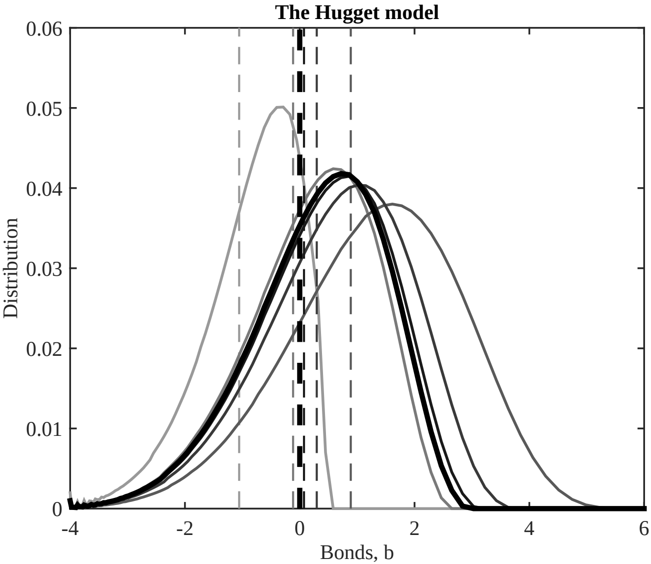
Hugget distribution



Hugget distribution



Hugget distribution



The Hugget Model

- How did I adjust the interest rate?
- One way is to say simply
  - If $\sum_{s' \in \mathcal{S}} \sum_b b\psi(b, s; r_n) > 0$, set $r_{n+1} = r_n - \varepsilon$
  - Else set $r_{n+1} = r_n + \varepsilon$.

The Hugget Model

- But there is a smarter way of doing things.
- Let $B(r) = \sum_{s \in \int} \sum_b b\psi(b, s)$ denote excess demand.
- B is increasing in r.
- We also know that $B(\frac{1}{\beta} - 1) > 0$ and, hopefully, $B(0) < 0$.
- Luckily, $B(r)$ also happens to be continuous.
- An ideal numerical procedure to find $B(r) = 0$ is *the bisection method.*

The Hugget Model

The bisection method works in the following way.

- Suppose that $f(x)$ is continuous and monotone and $f(\overline{x}) > 0$ but $f(\underline{x}) < 0$.
- Then pick $x = \frac{\overline{x}+\underline{x}}{2}$, and evaluate $f(x)$.
- If $f(x) > 0$, set $\overline{x} = x$ and repeat. Else set $\underline{x} = x$ and repeat.
- Eventually you will find an x such that $f(x) = 0$.
- In our case, $f(x) = B(r)$, and $\overline{x} = \frac{1}{\beta} - 1 - \varepsilon$ and $\underline{x} = 0$.
- This is what I used

The Aiyagari Model

- The Aiyagari model was developed by Rao Aiyagari and published in QJE 1994.
- It was simultaneous and independent work of Hugget.
- The main difference is that in the Aiyagari-world, households both underwrite debt contracts to each other, but also lend out resources to firms which are then used as investments.
- Of course, this means that there can be positive savings in the economy which determines the capital stock.
- In addition, wages are not simple endowments, but paid by firms in a competitive market.
- We will again focus on the stationary distribution.

The Aiyagari Model

Given w and r, the household's problem is given by,

$$v(a,s) = \max_{c,a'}\{u(c) + \beta \sum_{s' \in \mathcal{S}} v(a',s')p(s',s)\}$$

$$\text{s.t} \quad c + a' = (1+r)a + ws + \mu w(1-s)$$

$$a' \geq \underline{a}$$

Notice too that v (the stationary distribution of the transition matrix) gives us the employment rate, $(1-u)$, and the unemployment-rate, u.

The Aiyagari Model

- Firms can hire workers on a labor sport market at wage rate w, and rent capital at the interest rate $\tilde{r}$.
- They operate a constant returns to scale technology $F(k, n)$, so we can work with a representative firm.
- Their problem is therefore static and given by

$$\max_{k,n}\{F(k, n) - nw - k\tilde{r}\}$$

The Aiyagari Model

First order conditions,

$$\tilde{r} = F_k(k, n), \quad w = F_n(k, n)$$

Two things to observe

1. If capital depreciates at rate δ, the (net) interest rate received by the households must be $r = \tilde{r} - \delta$.
2. Let $A(r) = \sum_{s \in \mathcal{S}} \sum_a a\psi(a, s)$. Then market clearing implies that $k = A$ and $n = (1 - u)$.

The definition of an equilibrium is now trivial.

The Aiyagari Model

Definition of a competitive equilibrium.

- A competitive equilibrium is an interest-rate r and a wage-rate w, such that
- Given r and w, $g(a, s)$ solves the households problem.
- Given r and w, k and n solves the firms problem.
- The stationary distribution ψ satisfies

$$\psi(a', s') = \sum_{s \in \mathcal{S}} \sum_{\{a : a' = g(a,s)\}} \psi(a, s) p(s', s) \tag{1}$$

- Markets clear: $k = \sum_{s \in \mathcal{S}} \sum_{a} a\psi(a, s)$ and $n = (1 - u)$

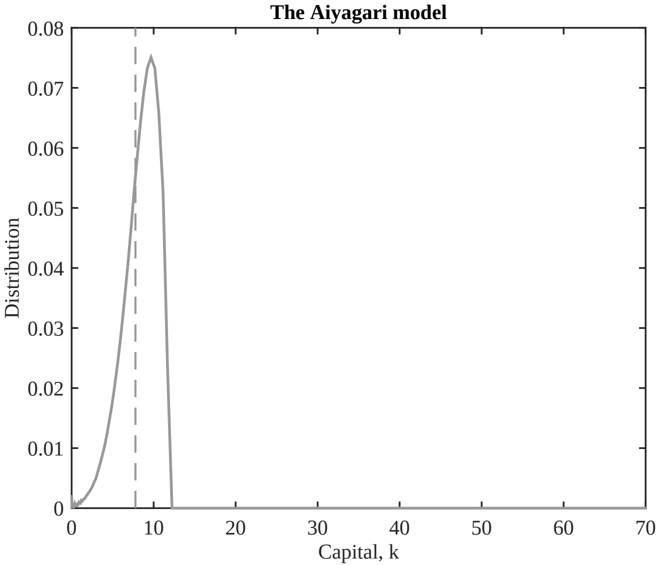
The Aiyagari Model

Algorithm,

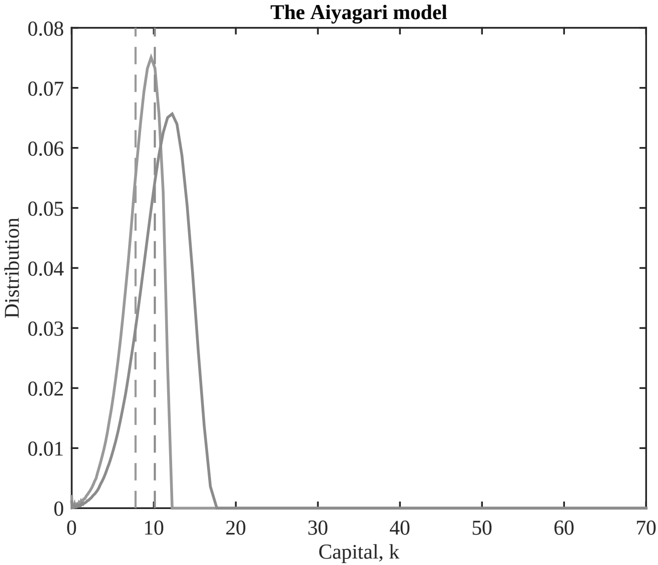
1. Guess for r.
2. With constant returns to scale, $r = F_k(k,n) - \delta$ uniquely pins down the capital-labor ratio – for instance $r = \alpha \frac{k}{n}^{\alpha-1} - \delta$ – which also pins down $w = F_n(k,n)$ $(w = (1-\alpha)\frac{k}{n}^{\alpha})$
3. Solve the households problem.
4. Evaluate $A(r) = \sum_{s \in \mathcal{S}} \sum_a a\psi(a,s)$.
5. Evaluate r' as $r' = F_k(A, 1-u) - \delta$.
6. If $r' > r$, adjust r upwards. Else downwards. Until $r' = r$.

In practice we will use the bisection method again.

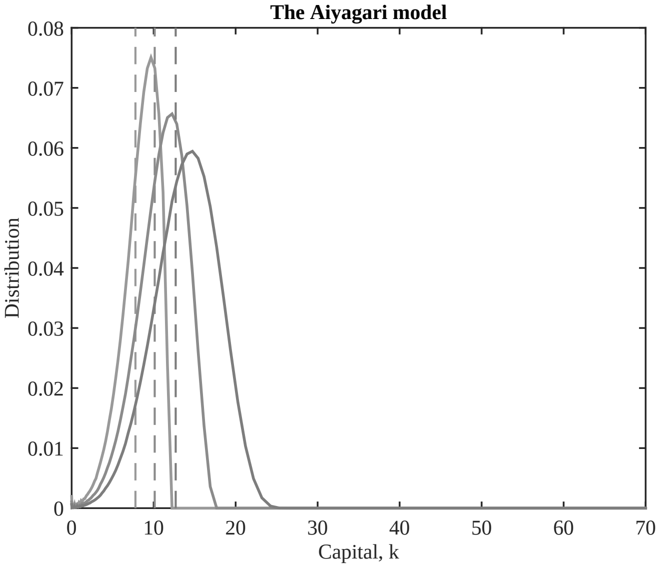
Aiyagari distribution



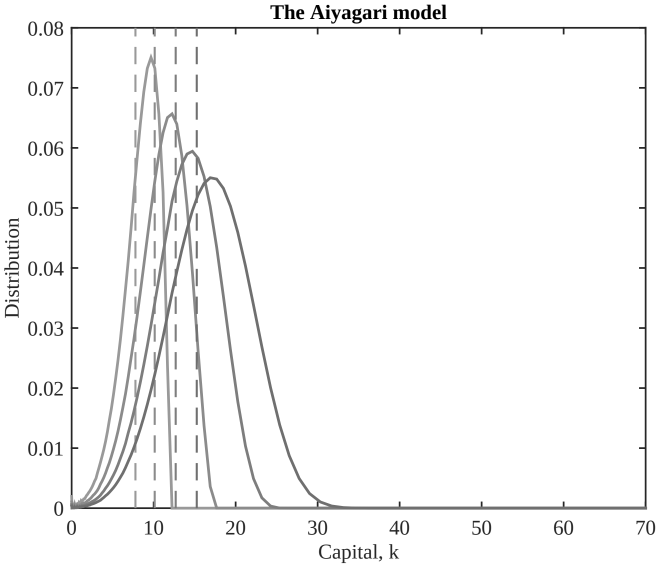
Aiyagari distribution



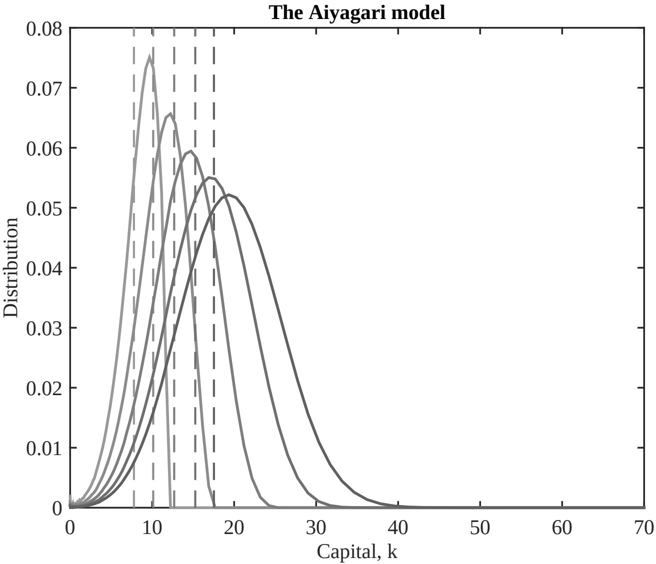
Aiyagari distribution



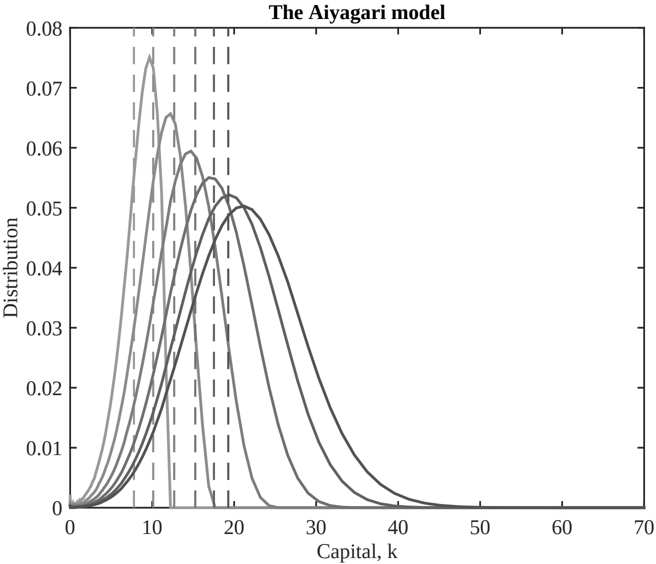
Aiyagari distribution



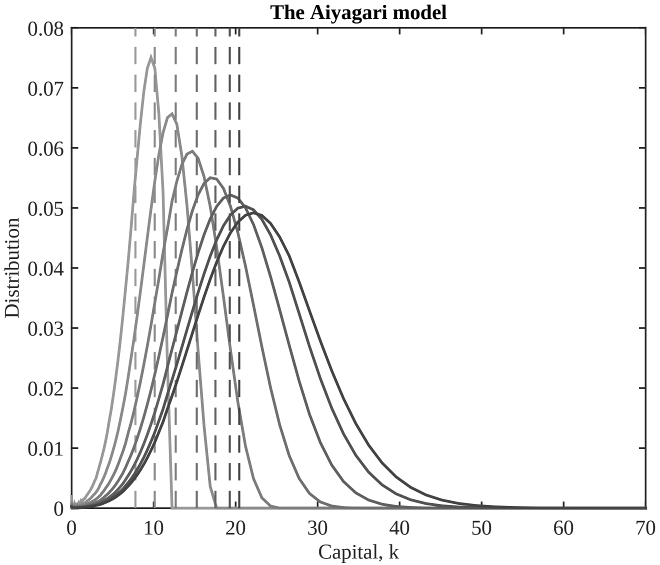
Aiyagari distribution



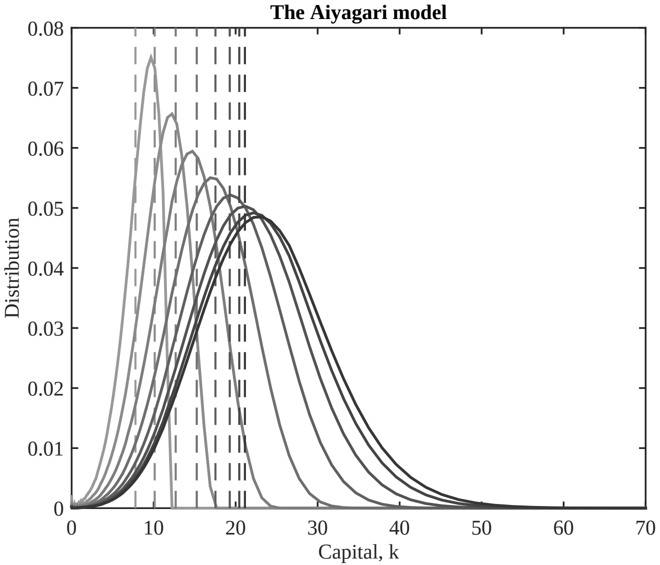
Aiyagari distribution



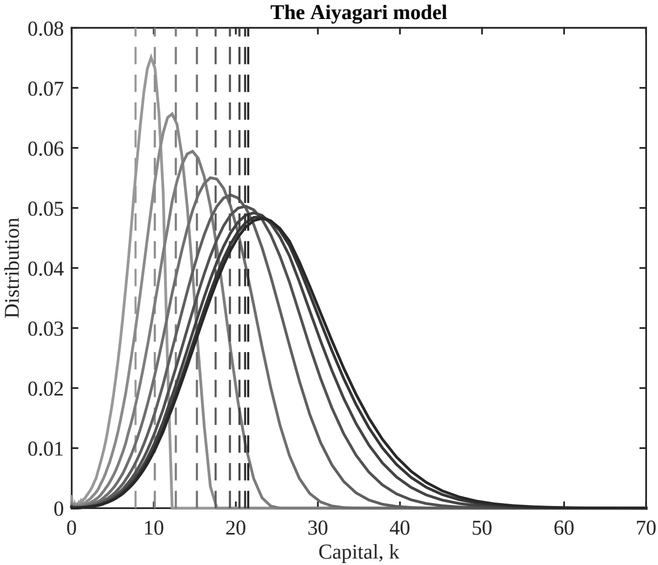
Aiyagari distribution



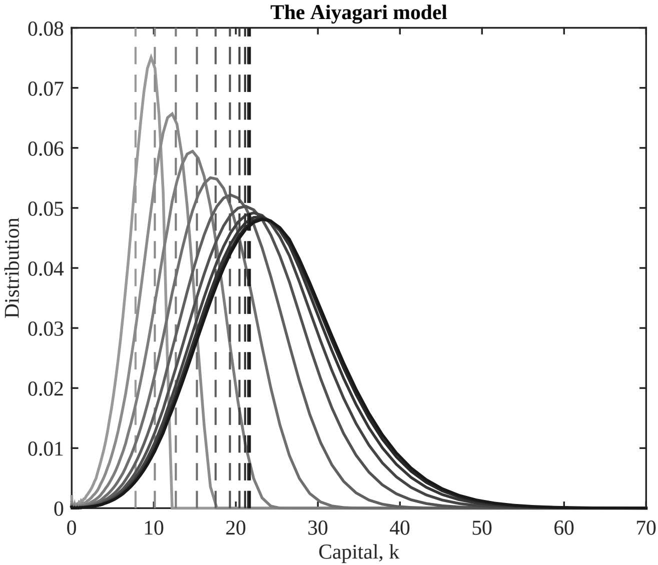
Aiyagari distribution



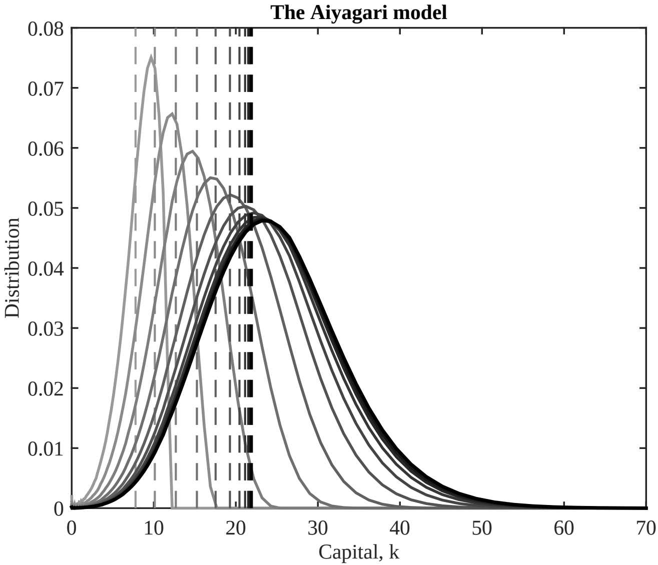
Aiyagari distribution



Aiyagari distribution



Aiyagari distribution



Calculating the transition

- Beautiful isn't it?
- But the Aiyagari model has one very compromising downside
  - Aggregate variables like output, consumption and investment are all constant!
- So we cannot analyze business cycle properties.

Calculating the transition

- Analyzing the business cycle was the contribution of Krusell and Smith (1998).
- But there are alternative shortcuts that have gained popularity
- The general principle is to consider an Aiyagari economy that is at its steady-state in period t.
- Then we expose the economy to some temporary shock, under the assumption that the economy will fully have reverted back to its steady-state in period $t + T$.
- Then it's not that hard to calculate the transition of the economy between period t and $t + T$.

Calculating the transition

- Suppose the economy is hit by a negative technology shock in period t, such that $y_{t+s} = exp(z_{t+s})F(k_{t+s}, n_{t+s})$, with $z_{t+s+1} = \rho z_{t+s}$, and $z_t < 0$.
- Assume that $z_{t+T} = 0$.
- We know that labor won't change, such that $n_{t+s} = (1 - u) = n$.
- However, the capital stock will!

Calculating the transition

- General principle: Guess how the capital stock will evolve from period $t+1$ to period $t+T$, assuming that it's back to it's steady state value in $t+T+1$. Call this guess $\{k^0_{t+s}\}^T_{s=1}$.
- Given this guess, we can calculate wages and interest rates as $w^0_{t+s} = exp(z_{t+s})F_n(k^0_t, n)$ and $\tilde{r}^0_{t+1} = exp(z_{t+s})F_k(k^0_t, n)$

Calculating the transition

- Using time iteration again, we can find $\tilde{g}(a, s)$ as

$$
\begin{aligned}
& u'((1 + r^0_{t+1})a + w^0_{t+s}s + \mu w^0_{t+s}(1 - s) - \tilde{g}(a, s)) \\
& = \beta(1 + r^0_{t+s+1}) \sum_{s' \in \mathcal{S}} u'((1 + r^0_{t+s+1})\tilde{g}(a, s) + w^0_{t+s+1}s' \\
& \qquad + \mu w^0_{t+s+1}(1 - s') - g^0_{t+s+1}(\tilde{g}(a, s), s'))p(s', s)
\end{aligned}
$$

- Then $g^0_{t+s}(a, s) = \max\{\tilde{g}(a, s), \underline{a}\}$, and with $g_{t+T+1}(a, s) = g(a, s)$.

Calculating the transition

- Then for each $g_t^0, g_{t+1}^0, \ldots, g_{t+s}^0$ we can find a transition matrix M_{t+s}^0.
- Given that we know $\psi_t(a, s) = \psi(a, s)$, we can update as $\psi_{t+s+1}^0 = \psi_{t+s}^0 \times M_{t+s}^0$, and calculate the implied aggregate asset holdings A_{t+s}^0.
- Then we update our guess for the sequence of capital as $\{k_{t+s}^1\}_{s=1}^T = \eta\{k_{t+s}^0\}_{s=1}^T + (1-\eta)\{A_{t+s}^0\}_{s=1}^T$, with $\eta \in (0, 1]$.
- And repeat until $\|\{k_{t+s}^{n+1}\}_{s=1}^T - \{k_{t+s}^n\}_{s=1}^T\|$

Krusell & Smith (1998)

- The Aiyagari model has some impressive properties.
  - Each agent faces a tremendous amount of uncertainty.
  - Yet, at the aggregate level there is none (law of large numbers ...).
  - Implication: Equilibrium prices are constant at the stationary distribution.

Krusell & Smith (1998)

- One problem is, however, that the Aiyagari model gives no predictions on the business cycle frequency.
- Clearly, it is of interest to understand whether representative agent models do a good job compared to more realistic models with idiosyncratic risk.
- This is precisely the question explored by Krusell and Smith (1998).
- Hugely influential paper with 1700 citation on Google Scholar.

Krusell & Smith (1998)

- Simple idea: Take Aiyagari's model and include aggregate shocks to TFP.
- Huge problem: No stationary distribution. No constant prices.
- The main contribution of K&S was to show how to work around this problem in a very accurate way.
- Their idea is referred to as “approximate aggregation”.

Krusell & Smith (1998)

- First off, there will be technology shocks

$$z = \{z_g, z_b\} \tag{2}$$

- We set $z_g = 1.02$ and $z_b = 0.98$.
- Second, there will be a transition matrix for these shocks. In particular, we set these to match an expected duration of both good and bad times to 8 quarters.

$$P_{z,z'} \quad := \quad \begin{array}{cc} & \begin{array}{cc} z_g & \quad z_b \end{array} \\ \begin{array}{c} z_g \\ z_b \end{array} & \begin{pmatrix} 1-\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & 1-\frac{1}{8} \end{pmatrix} \end{array}$$

Krusell & Smith (1998)

- Naturally, there will be a transition matrix, P, for employment as well.
- Is this transition matrix independent of the aggregate state, z?
- That would be counterintuitive. The probability of finding a job etc. clearly depends on the aggregate state.
- Would you expect that the probability of finding a job at $t+1$ is independent of the aggregate state in $t+1$ (and therefore only depends on the state in period t)?
- Maybe, but it brings some salient properties assuming that it differs.

Krusell & Smith (1998)

- Therefore, we will have a transition matrix for employment to unemployment for each respective (z, z')-pair (i.e. 4 different matrices).
- Denote these $P_{e,e'|z,z'}$.

$$P_{e,e'|z,z'} \quad := \quad \begin{array}{c c} & \begin{array}{cc} e'=1 & e'=0 \end{array} \\ \begin{array}{c} e=1 \\ e=0 \end{array} & \begin{pmatrix} \pi_{1,1|z,z'} & \pi_{1,0|z,z'} \\ \pi_{0,1|z,z'} & \pi_{0,0|z,z'} \end{pmatrix} \end{array}$$

- That is we have a total of 8 values of π to calibrate.

Krusell & Smith (1998)

- First off, let's start with $(z, z') = (z_g, z_g)$.
- We want the duration of unemployment in this case to be 1.5 quarters.
- That is $\pi_{0,0|g,g} = 1 - 1/1.5$. Of course $\pi_{0,1|g,g} = 1 - \pi_{0,0|g,g}$.
- Second, we want the unemployment rate in good times to be $u_g = 4\%$.
- Since $(1 - u_g) = (1 - u_g)\pi_{1,1|g,g} + u_g\pi_{0,1|g,g}$, we set $\pi_{1,1|g,g} = ((1 - u_g) - u_g\pi_{0,1|g,g})/(1 - u_g)$.

Krusell & Smith (1998)

- Second, consider $(z, z') = (z_b, z_b)$.
- We want the duration of unemployment in this case to be 2.5 quarters.
- That is $\pi_{0,0|b,b} = 1 - 1/2.5$. Of course $\pi_{0,1|b,b} = 1 - \pi_{0,0|g,g}$.
- Second, we want the unemployment rate in bad times to be $u_b = 10\%$.
- Since $(1 - u_b) = (1 - u_b)\pi_{1,1|b,b} + u_b\pi_{0,1|b,b}$, we set $\pi_{1,1|b,b} = ((1 - u_b) - u_b\pi_{0,1|b,b})/(1 - u_b)$.

Krusell & Smith (1998)

- Now let's consider the matrix $P_{e,e'|g,b}$.
- First off, we set $\pi_{0,0|g,b} = 1.25\pi_{0,0|b,b}$
- Second, as $(1 - u_b) = (1 - u_g)\pi_{1,1|g,b} + u_g\pi_{0,1|g,b}$, we set $\pi_{1,1|g,b} = ((1 - u_g) - u_g\pi_{0,1|g,b})/(1 - u_b)$.

Krusell & Smith (1998)

- Lastly, let's consider the matrix $P_{e,e'|b,g}$.
- First off, we set $\pi_{0,0|b,g} = 0.75\pi_{0,0|g,g}$
- Second, as $(1 - u_g) = (1 - u_b)\pi_{1,1|b,g} + u_b\pi_{0,1|b,g}$, we set $\pi_{1,1|b,g} = ((1 - u_b) - u_b\pi_{0,1|b,g})/(1 - u_g)$.

Krusell & Smith (1998)

- Our final transition matrix is therefore given by,

$$P \quad := \quad \begin{pmatrix} P_{z=g,z'=g} \times P_{e,e'|g,g} & P_{z=g,z'=b} \times P_{e,e'|g,b} \\ P_{z=b,z'=g} \times P_{e,e'|b,g} & P_{z=b,z'=b} \times P_{e,e'|b,b} \end{pmatrix}$$

Krusell & Smith (1998)

- Or,

$$P \quad := \quad \begin{pmatrix} 0.8507 & 0.0243 & 0.1159 & 0.0091 \\ 0.5833 & 0.2917 & 0.0313 & 0.0938 \\ 0.1229 & 0.0021 & 0.8361 & 0.0389 \\ 0.0938 & 0.0313 & 0.3500 & 0.5250 \end{pmatrix}$$

Krusell & Smith (1998)

- Let $s_t = (e_t, z_t)$, and $s^t = ((e_0, z_0), (e_1, z_1), \dots (e_t, z_t))$.
- Then taking price processes $r_t(s^t)$ and $w_t(s^t)$ as given, the household's optimization problem is given by,

$$\max_{\{c_t(s^t), a_{t+1}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s_t \in \mathcal{S}^{t+1}} \beta^t u(c_t(s^t)) f_t(s^t) \tag{3}$$

$$c_t(s^t) + a_{t+1}(s^{t+1}) = (1 + r_t) a_t(s^{t-1}) + w(s_t) \tag{4}$$

$$a_{t+1}(s^t) \geq \underline{a} \tag{5}$$

$$a_0, s_0, \quad \text{given} \tag{6}$$

- Where, of course, f_t (and p), is defined as previously using the transition matrix P.

Krusell & Smith (1998)

- Now, if we assume for a second that agents have access to some variable x_t, such that $r_t = r(z_t, x_t)$ and $w_t = w(z_t, x_t)$.
- Moreover, $x_{t+1} = \phi(z_t, z_{t+1}, x_t)$.
- The the Bellman equation associated with the household problem is given by,

$$v(a, e, z, x) = \max_{c,a'} \{u(c) + \beta \sum_{(z',e') \in \{z_g, z_b\} \times \{0,1\}} v(a', e', z', x') p(e', z' | e, z)\}$$

$$\begin{aligned} \text{s.t} \quad c + a' &= a(1 + r(z, x)) + ew(z, x) \\ x' &= \phi(z, z', x) \end{aligned}$$

- With policy function $a' = g(a, e, z, x)$.

Krusell & Smith (1998)

- Firms can hire workers on a labor spot market at wage rate w_t, and rent capital at the interest rate $\tilde{r}_t$.
- They operate a constant returns to scale technology $z_t F(k_t, n_t)$, so we can work with a representative firm.
- Their problem is therefore static and given by

$$\max_{k_t, n_t} \{z_t F(k_t, n_t) - n_t w_t - k_t \tilde{r}_t\} \tag{7}$$

Krusell & Smith (1998)

First order conditions,

$$\tilde{r}_t = z_t F_k(k_t, n_t), \quad w_t = z_t F_n(k_t, n_t) \tag{8}$$

Two things to observe

1. If capital depreciates at rate δ, the (net) interest rate received by the households must be $r_t = \tilde{r}_t - \delta$.
2. Let $A_t = \sum_{e \in \{0,1\}} \sum_a a\psi_t(a, e)$. Then market clearing implies that $k_t = A_t$ and $n_t = (1 - u_t)$.

The definition of an equilibrium is now straightforward.

Krusell & Smith (1998)

Definition of a recursive competitive equilibrium.

- A competitive equilibrium are prices $\{r_t(s^t), w_t(s^t)\}_{t=0}^{\infty}$ such that
- Given prices, $g(a, e, z, x)$ solves the households problem.
- Given prices, $\{k_t(s^t), n_t(s^t)\}_{t=0}^{\infty}$ solves the firms problem.
- Markets clear: $k_t = \sum_{e \in \{0,1\}} \sum_a a\psi_t(a, e)$ and $n_t = (1 - u_t)$.
- *Rationality*: $r_{t+1} = r(z_{t+1}, \phi(z_{t+1}, z_t, x_t))$ and $w_{t+1} = w(z_{t+1}, \phi(z_{t+1}, z_t, x_t))$.

Krusell & Smith (1998)

- The million dollar question (or one JPE publication question) is however what x is!
- A natural candidate is that $x_t = \psi_t$, that is the entire cross-sectional distribution.
- Notice that if $x_t = \psi_t$, we can easily calculate prices (through A_t – excess demand – together with firm's first order conditions).
- In addition $\psi_{t+1}(a', e') = \sum_{e \in \{0,1\}} \sum_{a: a' = g(a,e,z,x)} \psi_t(a, e) Pr(e'|e, z, z')$.
- So $x' = \phi(z', z, x)$ would hold!
- But as ψ_t is an infinite-dimensional object, we can't solve the Bellman equation!

Krusell & Smith (1998)

- Krusell & Smith propose that maybe we can approximate ψ_t with some finite collection of moments.
- In particular, K&S hypothesize that maybe *only the mean matters* (but they also explore the *variance*).
- So suppose that $x = K$ where K is the aggregate (and average) capital stock in period t.
- Together with our knowledge of the unemployment-rate, we can easily calculate prices w and r.
- Moreover, K&S postulate that

$$K' = \alpha_z + \beta_z K \tag{9}$$

Krusell & Smith (1998)

- Hopefully, this will lead to “approximate aggregation” in that

$$K' = \alpha_z + \beta_z K \approx \sum_{e'} \sum_{a'} a' \psi_{t+1}(a', e') \qquad (10)$$

- But how will we find α_z and β_z (four coefficients)?
- This is where their algorithm comes in.
- But first off, let’s define an equilibrium with approximate aggregation.

Krusell & Smith (1998)

Definition of a competitive equilibrium with approximate aggregation.

- A competitive equilibrium are prices $\{r_t(s^t), w_t(s^t)\}_{t=0}^{\infty}$ such that
- Given prices, $g(a, e, z, x)$ solves the households problem.
- Given prices, $\{k_t(s^t), n_t(s^t)\}_{t=0}^{\infty}$ solves the firms problem.
- Markets clear: $k_t = \sum_{e \in \{0,1\}} \sum_a a\psi_t(a, e)$ and $n_t = (1 - u_t)$.
- *Approximate Aggregation*: $K' = \alpha_z + \beta_z K \approx \sum_{e'} \sum_{a'} a'\psi_{t+1}(a', e')$.

Krusell & Smith (1998)

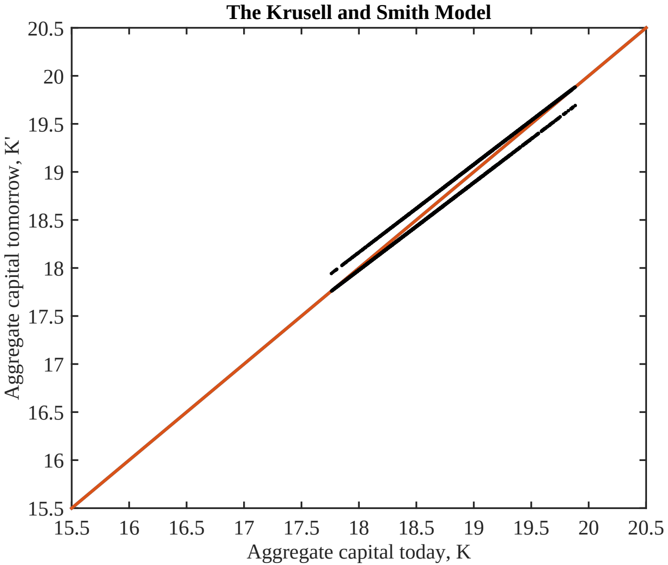
Krusell & Smith's algorithm

- Guess for α_z and β_z (I go for $\alpha_z = 0$ and $\beta_z = 1$).
- Solve the household's problem to get the policy function $a' = g(a, e, z, K)$.
- Use this policy rule and simulate the savings behavior of a long (T=6000) panel of many individuals (a continuum).
- Regress K_{t+1} on K_t and a constant, conditional on z_t.
- If your coefficient from the regression matches your guessed α_z and β_z you're done!
- Otherwise update α_z and β_z, and repeat.
- Once convergence is obtained, check for accuracy of your laws of motion for aggregate capital.

Krusell & Smith (1998)

- So let's do exactly this, but only for one iteration.
- That is, guess that $\alpha_z = 0$ and $\beta_z = 1$ and simulate a panel.
- Then just plot K_{t+1} vs. K_t.

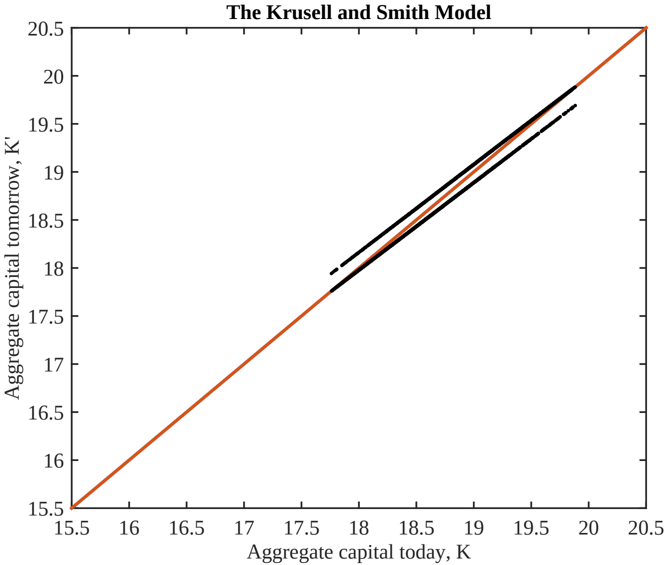
Krusell and Smith



Krusell & Smith (1998)

- A linear guess appears very close to optimal.
- But our coefficients were far off! (see red, 45 degree line)
- Regression reveals that the updated coefficients are: $\alpha_g = 1.69$, $\alpha_b = 1.59$, $\beta_g = 0.92$, $\beta_b = 0.91$.
- This change may be a bit too drastic, so I set the new coefficient as a linear combination between the old and the new estimates (50% weight on the new).
- Iterating on this converges really nicely and yields,

Krusell and Smith



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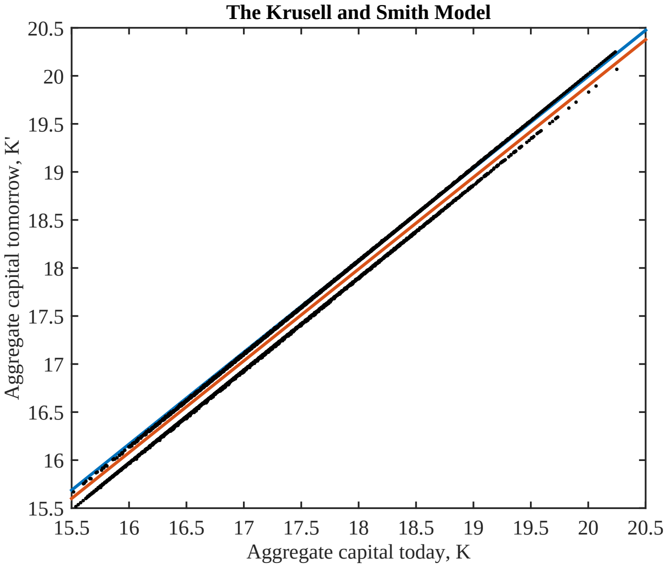

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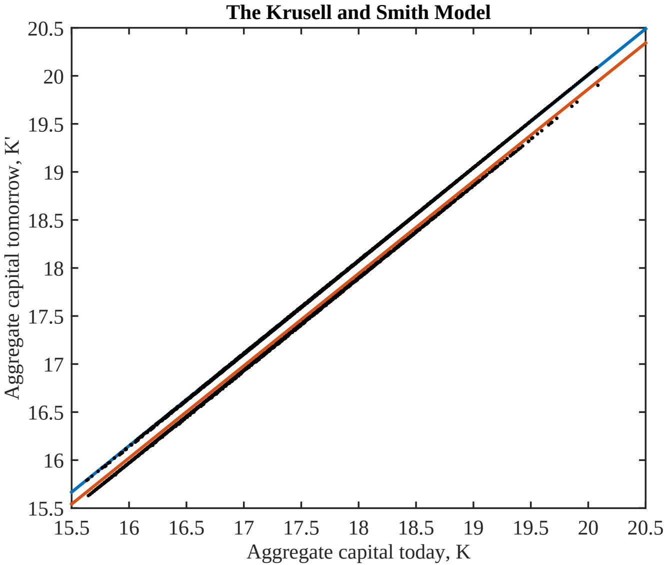

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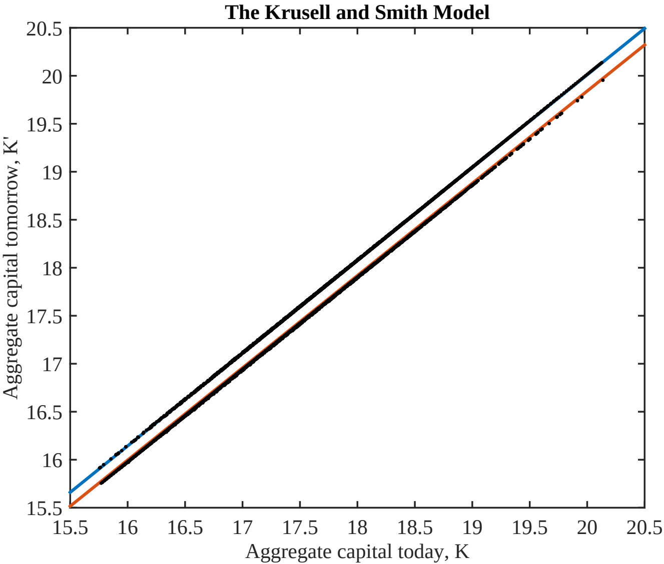

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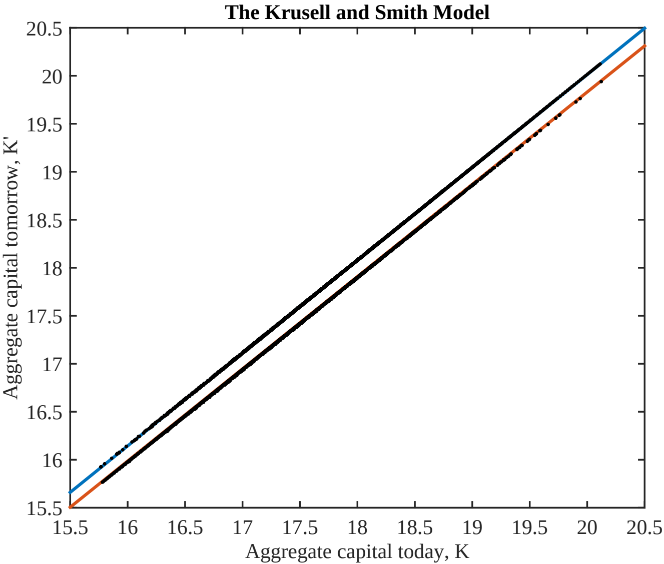

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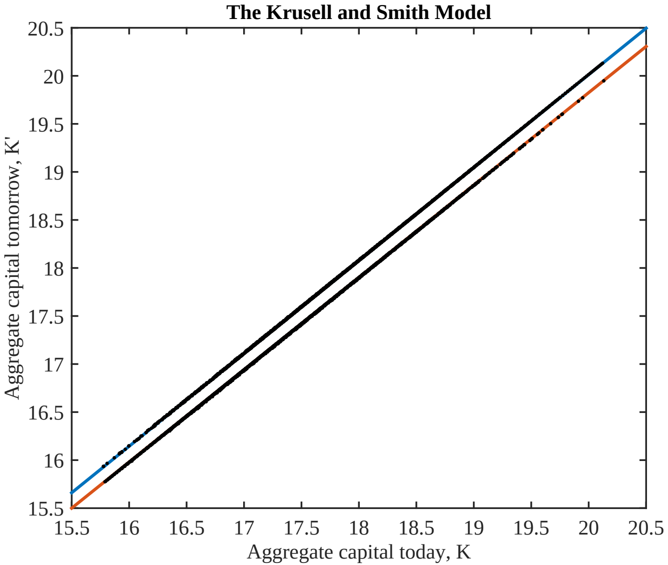

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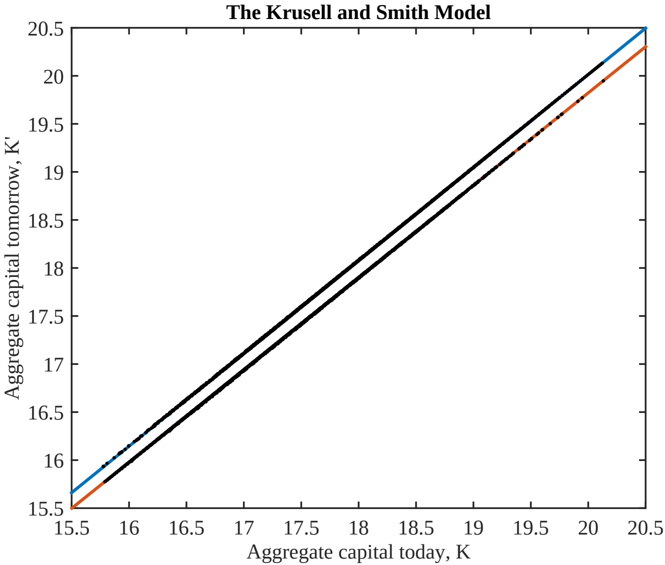

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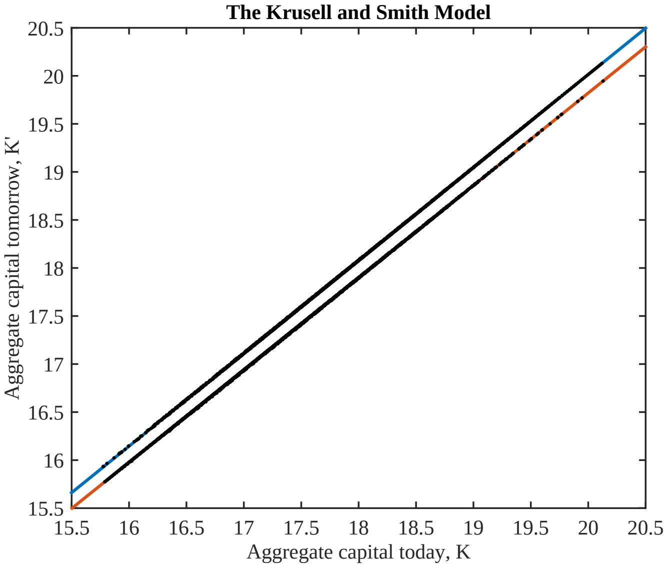

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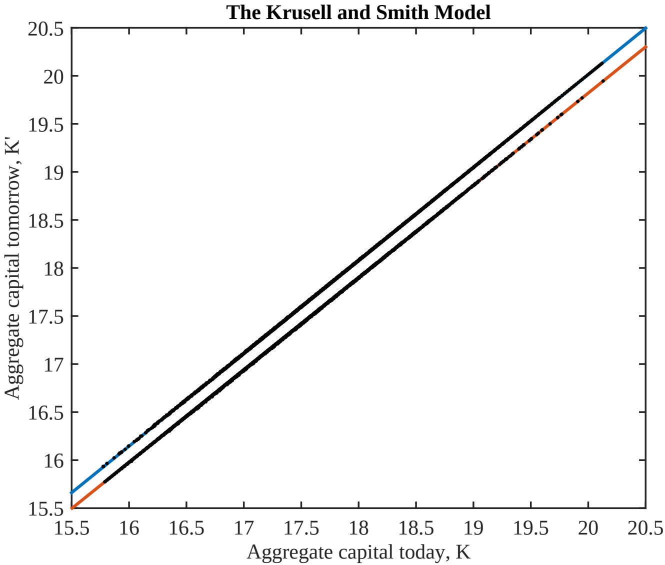

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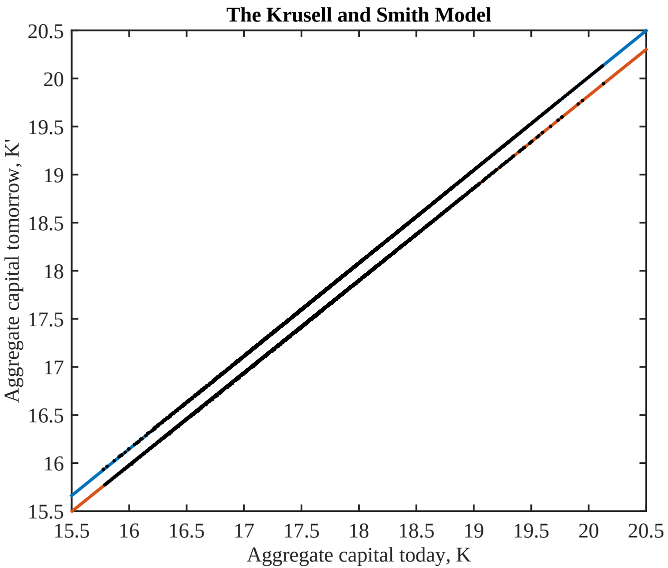

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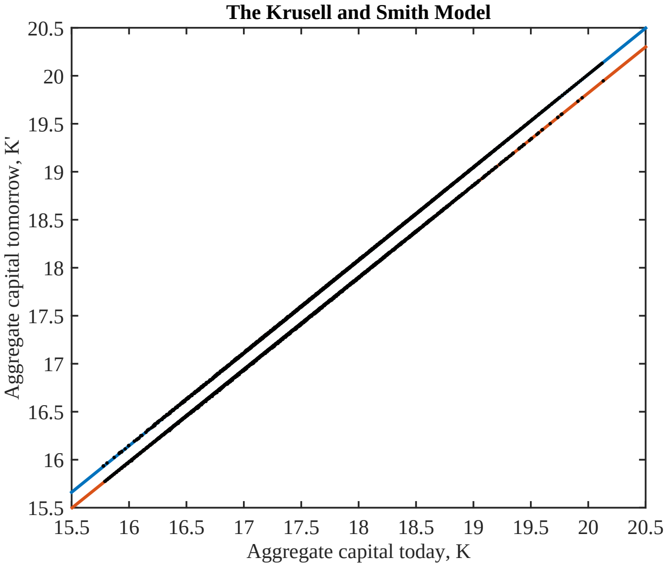

Krusell & Smith (1998)

- The main drawback with K&S' algorithm is that it is slow (8 min on my laptop).
- This is using the absolutely fastest methods available.
- The simulation step uses a lot of RAM and CPU power.
- And we need several simulations before convergence (perhaps a hundred).
- An alternative algorithm to approximate aggregation is *explicit aggregation*.
- It's much faster (around 50$\times$) as it doesn't use simulation.
- However, I also have to admit it's a little less accurate (see den Haan (2010)).
- An optimal procedure may be a combination.