BELLMAN'S PRINCIPLE OF OPTIMALITY

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> *"An optimal [plan] has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal [plan] with regard to the state resulting from the first decision"*
>
> Richard Bellman, 1957.

Consider the optimization problem

$$\begin{aligned} v^*(x_0) &= \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ &\text{subject to} \quad x_{t+1} \in \Gamma(x_t),\ x_0 \text{ given.} \end{aligned} \tag{SP}$$

Any sequence $\{x_{t+1}\}_{t=0}^{\infty}$ which satisfies the constraint in each and every period is called a *plan*. Let $\{x_{t+1}^*\}_{t=0}^{\infty}$ denote the optimal plan. That is, $\{x_{t+1}^*\}_{t=0}^{\infty}$ attains $v^*(x_0)$ in the sense that

$$v^*(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*), \tag{1}$$

with $x_0^* = x_0$, $x_1^* \in \Gamma(x_0^*)$, $x_2^* \in \Gamma(x_1^*)$, and so on.

The objective of this note is to show that the function $v^*(x)$ satisfies the functional equation

$$v^*(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v^*(y)\}. \tag{FE}$$

To achieve this goal, I will first prove that

$$v^*(x_0) = F(x_0, x_1^*) + \beta v^*(x_1^*). \tag{2}$$

Then it will be shown that x_1^* must be the solution to

$$v^*(x_0) = \max_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta v^*(y), \}. \tag{3}$$

Lastly, since this can be shown to hold for $t = 0, 1, 2, \ldots$, time is irrelevant, and (FE) follows.

Step 1. Rewrite (SP) as

$$v^*(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) \tag{4}$$

$$= F(x_0, x_1^*) + \beta \vec{V}_1, \tag{5}$$

with $\vec{V}_1$ defined as

$$\vec{V}_1 = \sum_{t=0}^{\infty} \beta^t F(x_{t+1}^*, x_{t+2}^*). \tag{6}$$

Now consider the optimization problem

$$\begin{aligned} v^*(x_1^*) = & \max_{\{x_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \\ \text{subject to} \quad & x_{t+2} \in \Gamma(x_{t+1}), \end{aligned} \tag{7}$$

and notice that $v^*(\cdot)$ is identical to that in (SP). That is, it is the same function. The goal of this step of the proof is to show that $v^*(x_1^*) = \vec{V}_1$.

To show this, ask yourself first whether $v^*(x_1^*) < \vec{V}_1$ is possible? The answer is no. To see this, notice that $\{x_{t+2}^*\}_{t=0}^{\infty}$ is a *feasible* plan for the problem in (7), since $x_2^* \in \Gamma(x_1^*)$, $x_3^* \in \Gamma(x_2^*)$, and so on. Since $\{x_{t+2}^*\}_{t=0}^{\infty}$ is feasible for (7) and attains $\vec{V}_1$, it must be the case that $v^*(x_1^*) \geq \vec{V}_1$. That is, by having the opportunity of reoptimizing in period one, you can do no worse than sticking to your original choice, as this choice is indeed feasible.

This leaves the possibility that $v^*(x_1^*) > \vec{V}_1$. Suppose indeed that this is true. Then associated with $v^*(x_1^*)$ is a plan $\{x'_{t+2}\}_{t=0}^{\infty}$ which attains $v^*(x_1^*)$. That is,

$$v^*(x_1^*) = \sum_{t=0}^{\infty} \beta^t F(x'_{t+1}, x'_{t+2}), \tag{8}$$

with $x'_1 = x_1^*$, $x'_2 \in \Gamma(x_1^*)$, $x'_3 \in \Gamma(x'_2)$, and so on. Now consider the plan $\{x_0, x_1^*, x'_2, x'_3, x'_4, \ldots\}$. This plan is obviously a feasible choice for (SP), since $x_1^* \in \Gamma(x_0)$ $x'_2 \in \Gamma(x_1^*)$, $x'_3 \in \Gamma(x'_2)$, and attains

$$F(x_0, x_1^*) + \beta v^*(x_1^*) > F(x_0, x_1^*) + \beta \vec{V}_1 > v^*(x_0).$$

However, since $\{x_{t+1}^*\}_{t=0}^{\infty}$ was the optimal plan for (SP), this is a contradiction. Thus, $v^*(x_1^*) = \vec{V}_1$, and

$$v^*(x_0) = F(x_0, x_1^*) + \beta v^*(x_1^*). \tag{9}$$

Step 2. Could there exist a $y \in \Gamma(x_0)$ such that

$$F(x_0, y) + \beta v^*(y) > F(x_0, x_1^*) + \beta v^*(x_1^*) = v^*(x_0). \tag{10}$$

Suppose there is. Then associated with $v^*(y)$ is a plan $\{x''_{t+2}\}_{t=0}^{\infty}$ which attains $v^*(y)$. That is,

$$v^*(y) = \sum_{t=0}^{\infty} \beta^t F(x''_{t+1}, x''_{t+2}), \tag{11}$$

with $x''_1 = y$, $x''_2 \in \Gamma(y)$, $x''_3 \in \Gamma(x''_2)$, and so on. Then consider the plan $\{x_0, y, x''_2, x''_3, x''_4, \ldots\}$. This plan is, again, a feasible choice for (SP), since $y \in \Gamma(x_0)$ $x''_2 \in \Gamma(y)$, $x''_3 \in \Gamma(x''_2)$, and attains

$$F(x_0, y) + \beta v^*(y) > v^*(x_0).$$

Again, since $\{x^*_{t+1}\}_{t=0}^{\infty}$ was the optimal plan for (SP), this is a contradiction. Since this holds for every $y \in \Gamma(x_0)$, x_1^*, must be the solution to

$$v^*(x_0) = \max_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta v^*(y)\}. \tag{12}$$

Finally, since this argument can be repeated for any $t = 0, 1, 2, \ldots$, the time subscripts are unnecessary and we just write

$$v^*(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v^*(y)\}. \tag{13}$$

Lastly, under some (weak) conditions, it can be show that a function, $v(\cdot)$, that satisfies

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}, \tag{14}$$

must equal $v^*(x)$. See Theorem 4.3 in Stokey and Lucas, for which the proof is relatively straightforward.

Exercise. What is the (FE) corresponding to the below optimization problem?

$$\begin{aligned} v^*(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ \text{subject to} \quad & (x_{t+1} + x_{t+2}) \in \Gamma(x_t),\ x_0 \text{ given.} \end{aligned} \tag{15}$$