Numerical Methods Bootcamp

Lecture 4
Incomplete markets models

Pontus Rendahl
rpk22@cam.ac.uk

2018

Introduction

- By the end of yesterday's lecture we looked at an income fluctuation problem.
- The long run distribution could also be seen as the stationary cross-section distribution of wealth
- As I mentioned, suppose that, in equilibrium bonds are in zero net supply
- Then we need to find an interest rate such that the integral over the cross-sectional distribution of wealth is zero.

Introduction

- In this lecture we will learn how to do this (the Hugget model).
- Then we will analyse a very similar economy but in which "excess savings" are used as capital in production (the Aiygari model).
- Lastly, we will look at an Aiyagari-style economy with aggregate shocks (the Krusell and Smith model).

A Familiar Problem

- Consider the following two-period problem

$$\max\{u(c_0) + \beta \sum_{s \in \mathcal{S}} u(c_1(s))Pr(s)\}$$

$$\text{s.t.} \quad c_0 + \sum_{s \in \mathcal{S}} q(s)a(s) = \mathcal{S}_0$$

$$c_1(s) = \mathcal{S}_1(s) + a(s)$$

- Where s is *idiosyncratic*: $\sum_s \mathcal{S}_1(s)Pr(s) = \mathcal{S}_1$

A Familiar Problem

- First order conditions

$$u'(c_0)q(s) = \beta u'(c_1(s))Pr(s), \quad \forall s \in \mathcal{S}$$

- Then there exist equilibrium asset prices

$$q(s) = \frac{\beta u'(\mathcal{S}_1)}{u'(\mathcal{S}_0)}Pr(s)$$

- Such that $c_0 = \mathcal{S}_0$ and $c_1 = \mathcal{S}_1$.

Another Familiar Problem

▶ Consider the following two-period problem

$$\max\{u(c_0) + \beta u(c_1)\}$$
$$\text{s.t.} \quad c_0 + qa = \mathcal{S}_0$$
$$c_1 = \mathcal{S}_1 + a$$

Another Familiar Problem

- First order conditions

$$u'(c_0)q = \beta u'(c_1)$$

- Then there exist an equilibrium asset price

$$q = \frac{\beta u'(\mathcal{S}_1)}{u'(\mathcal{S}_0)}$$

- Such that $c_0 = \mathcal{S}_0$ and $c_1 = \mathcal{S}_1$.

What does this mean?

- ▶ The first problem: Loads of agents and loads of outcomes
- ▶ Lots of trade in assets
- ▶ The second problem: One representative agent, one outcome
- ▶ No trade in assets.
- ▶ But same aggregate outcome!

What does this mean?

- A representative agent is not one agent.
- A representative agent does not exclude trade – it just occurs under the hood.
- A representative agent does not preclude coordination failures (e.g. prisoners' dilemma)
- Another word for rep. agent models: Complete markets models.

What does this mean?

- ► But sometimes we wish to depart from representative agent
  - ► Distributions matter/ Aggregation matter
  - ► Precautionary risk/savings from incomplete markets
  - ► Could lead to interesting dynamics
- ► This is what we will analyze today.

The Hugget Model

- The model of Hugget basically takes a standard bond economy as the main starting point.

- In similarity to representative agent models, the continuum of agents are still *ex-ante* identical, but, importantly, they are now *ex-post* heterogenous.

- In particular, the agents are hit by only *partially insurable, idiosyncratic shocks*.

The Hugget Model

- One-period obligation contracts is the only source of insurance (bonds).
- That is, there are many more goods (states), than markets, and sometimes these models are referred to *incomplete markets models*.
- Lastly, there are no *aggregate shocks* to the economy, and therefore no *aggregate risk* (something that is relaxed in Krusell and Smith's (1998) model).

The Hugget Model

The Bellman equation that summarises the households' problem is given by

$$v(b, s) = \max_{c, b'}\{u(c) + \beta \sum_{s' \in \mathcal{S}} v(b', s')p(s', s)\}$$

$$\text{s.t} \quad c + b' = (1 + r)b + w(s)$$

$$b' \geq \underline{b}$$

With associated policy function $b' = g(b, s)$.

The Hugget Model

The first order conditions are given by

$$u'((1 + r)b + w(s) - b') - \mu(b, s)$$
$$= \beta(1 + r) \sum_{s' \in \mathcal{S}} u'((1 + r)b' + w(s') - b'')p(s', s)$$

where μ is the Lagrange multiplier associated with the borrowing constraint.

The Hugget Model

- Solution method: Find $\tilde{g}(b, s)$ as

$$u'((1 + r)b + w(s) - \tilde{g}(b, s))$$
$$= \beta(1 + r) \sum_{s' \in \mathcal{S}} [u'((1 + r)\tilde{g}(b, s) + w(s')$$
$$- g_n(\tilde{g}(b, s), s'))]p(s', s)$$

- Then $g_{n+1}(b, s) = \max\{\tilde{g}(b, s), \underline{b}\}$.

The Hugget Model

- Ok, so suppose that we have found $g(b, s)$ conditional on some r, now what?
- As there are idiosyncratic risk, each individual will be exposed to different shocks in different periods.
- More precisely, an individual is identified through his history of shocks.
- Clearly, as the history of shocks affect an individual's wealth, and individuals have experienced different types of histories, there will be a cross-sectional distribution of wealth-holdings.

The Hugget Model

- ▶ Now, as we have solved the household's problem for a given (constant) interest-rate, we have the policy function $b' = g(b, s)$.

- ▶ Together with (stochastic) law of motion for income, $p(s', s)$, we can easily describe the law of motion for ψ_t as follows,

$$\psi_{t+1}(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b, s)\}} \psi_t(b, s) p(s', s)$$

- ▶ One can show that under quite weak assumptions, ψ_t (and for any ψ_0) converges to a unique stationary distribution ψ such that,

$$\psi(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b, s)\}} \psi(b, s) p(s', s)$$

The Hugget Model

- This stationary distribution is very important to us.
- Given a constant interest-rate, the optimal household decision yields a stationary distribution with a constant excess-demand for bonds.
- Moreover, even if aggregates are constant (aggr. wealth, consumption, endowments etc.) individual specific variables are not: Agents jump frequently around in the distribution, but aggregates never change.
- Lastly, excess demand is given by

$$B(r) = \sum_{s \in \mathcal{S}} \sum_b b \psi(b, s)$$

The Hugget Model

Definition of a competitive equilibrium,

- A competitive equilibrium is an interest-rate r such that
- Given r, the policy function $g(b,s)$ solves the household's optimization problem.
- The stationary distribution satisfies,

$$\psi(b',s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b,s)\}} \psi(b,s)p(s',s)$$

- Markets clear. That is $B(r) = 0$.

The Hugget Model

- We will compute the market clearing interest-rate in a second, but there is also an alternative interpretation of ψ that is important.
- Consider a household at time-zero, having a certain amount of wealth and a certain s.
- Q: What is the probability that the agent will have some arbitrary wealth b and arbitrary s in t-periods?
- A: We can calculate this using the exact same recursive formula as for the cross-sectional distribution!
- But as for any ψ_0, $\psi_t \to \psi$, the probability of (b, s) occurring when $t \to \infty$ must be $\psi(b, s)$.
- So ψ is not only the cross-sectional distribution of wealth, but also the *unconditional* distribution of wealth for an individual agent.

The Hugget Model

Algorithm to find an equilibrium

1. Guess for an interest-rate, r.
2. Solve the consumer's problem.
3. Calculate the cross-sectional distribution $\psi(b, s)$.
4. If excess demand is positive, adjust the interest-rate downwards. If negative upwards.

The Hugget Model

- Let's guess for the interest rate r, and solve the problem
  - This gives policy functions $b' = g(b, s; r)$.
- Convert these into transition matrices using the "lottery" method showed yesterday.
- Find the invariant distribution $\psi(b, s; r)$

Hugget distribution
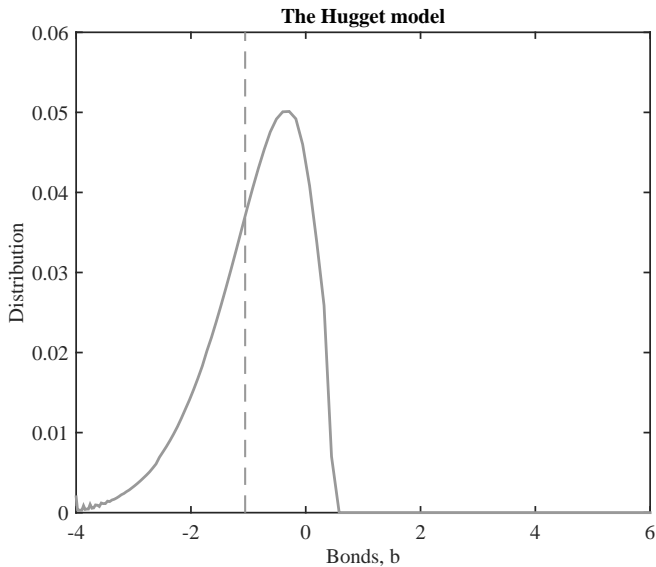
The Hugget model

The Hugget Model

- Clearly there is no market clearing! The interest rate is far too low so there is excess supply of bonds.
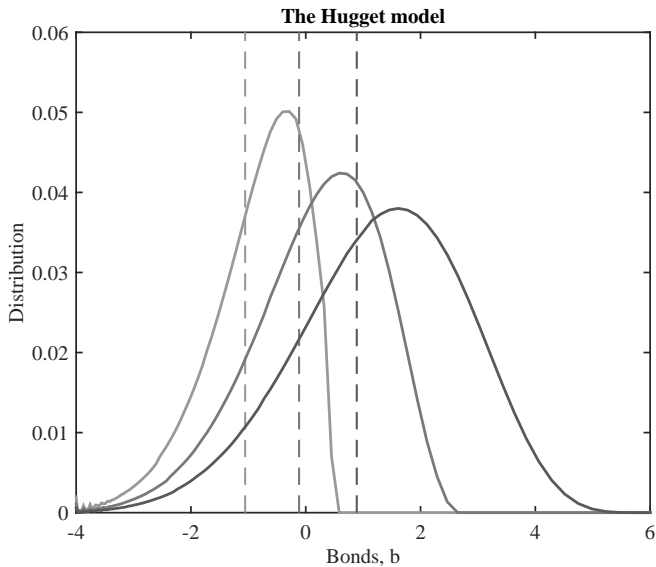- So let's adjust the interest rate, and iterate!

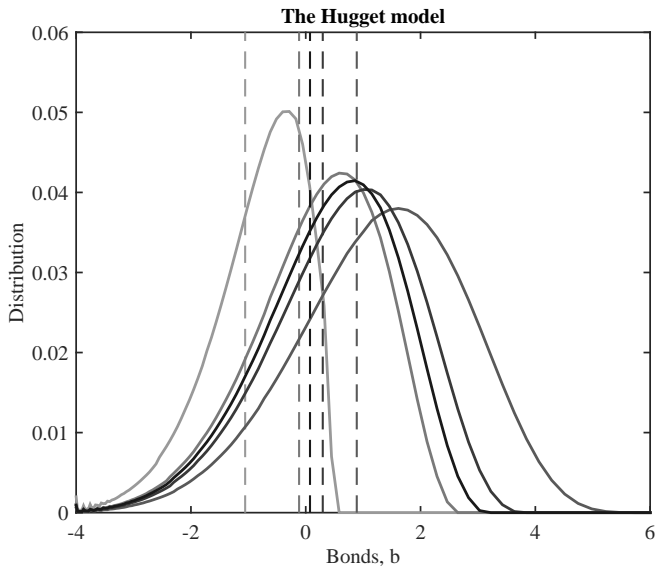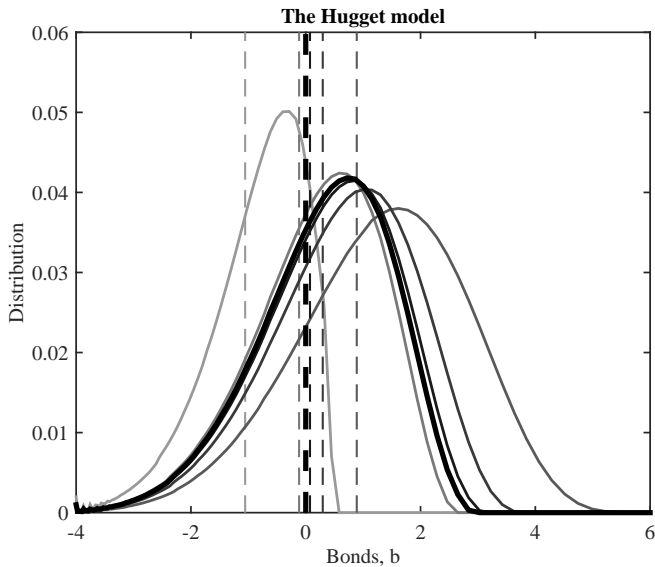# Hugget distribution
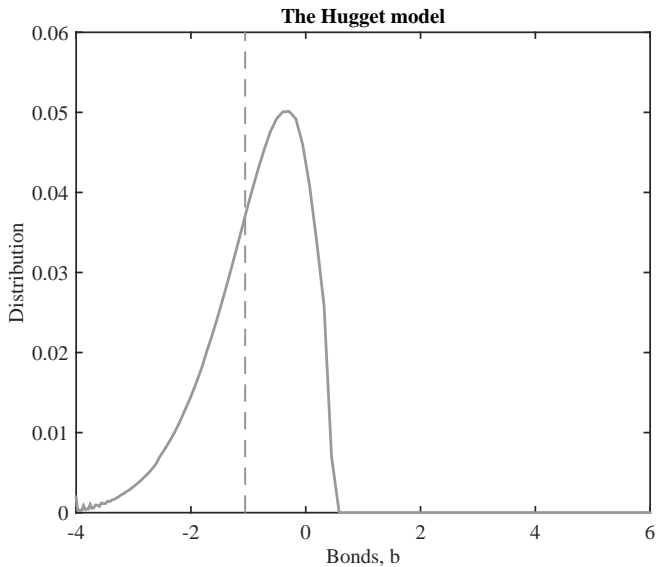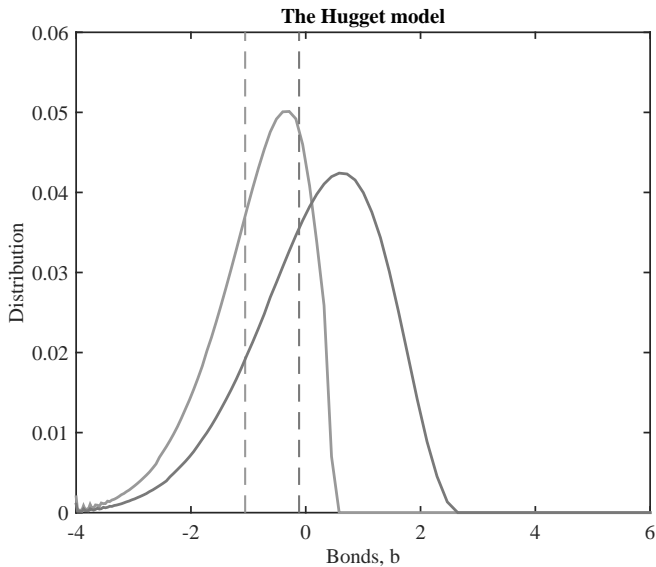
Hugget distribution

The Hugget model

Distribution vs. Bonds, b

Hugget distribution

Hugget distribution

Hugget distribution
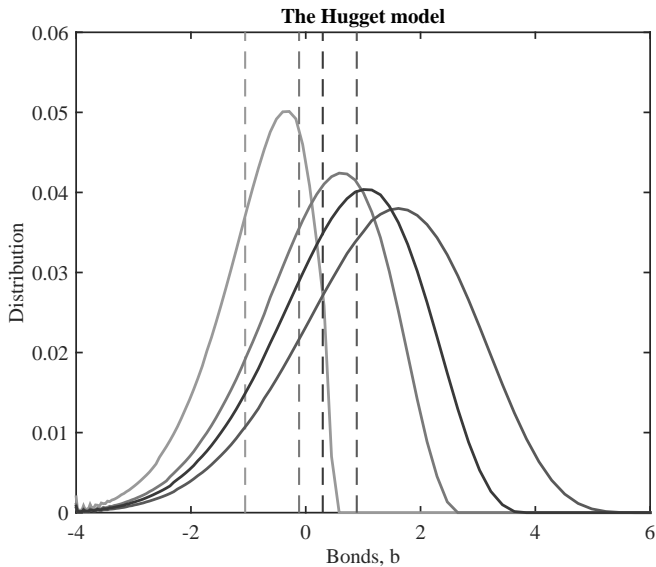
The Hugget model

Hugget distribution

The Hugget model

The Hugget Model

- How did I adjust the interest rate?
- One way is to say simply
    - If $\sum_{s' \in \mathcal{S}} \sum_b b \psi(b, s; r_n) > 0$, set $r_{n+1} = r_n - \varepsilon$
    - Else set $r_{n+1} = r_n + \varepsilon$.

The Hugget Model

- But there is a smarter way of doing things.
- Let $B(r) = \sum_{s \in \int} \sum_b b\psi(b, s)$ denote excess demand.
- B is increasing in r.
- We also know that $B(\frac{1}{\beta} - 1) > 0$ and, hopefully, $B(0) < 0$.
- Luckily, $B(r)$ also happens to be continuous.
- An ideal numerical procedure to find $B(r) = 0$ is *the bisection method*.

The Hugget Model

The bisection method works in the following way.

- Suppose that $f(x)$ is continuous and monotone and $f(\overline{x}) > 0$ but $f(\underline{x}) < 0$.
- Then pick $x = \frac{\overline{x} + \underline{x}}{2}$, and evaluate $f(x)$.
- If $f(x) > 0$, set $\overline{x} = x$ and repeat. Else set $\underline{x} = x$ and repeat.
- Eventually you will find an x such that $f(x) = 0$.
- In our case, $f(x) = B(r)$, and $\overline{x} = \frac{1}{\beta} - 1 - \varepsilon$ and $\underline{x} = 0$.
- This is what I used

The Aiyagari Model

- ▶ The Aiyagari model was developed by Rao Aiyagari and published in QJE 1994.
- ▶ It was simultaneous and independent work of Hugget.
- ▶ The main difference is that in the Aiyagari-world, households both underwrite debt contracts to each other, but also lend out resources to firms which are then used as investments.
- ▶ Of course, this means that there can be positive savings in the economy which determines the capital stock.
- ▶ In addition, wages are not simple endowments, but paid by firms in a competitive market.
- ▶ We will again focus on the stationary distribution.

The Aiyagari Model

Given w and r, the household's problem is given by,

$$v(a, s) = \max_{c, a'}\{u(c) + \beta \sum_{s' \in \mathcal{S}} v(a', s')p(s', s)\}$$

$$\text{s.t} \quad c + a' = (1 + r)a + ws + \mu w(1 - s)$$

$$a' \geq \underline{a}$$

Notice too that v (the stationary distribution of the transition matrix) gives us the employment rate, $(1 - u)$, and the unemployment-rate, u.

The Aiyagari Model

- Firms can hire workers on a labor sport market at wage rate w, and rent capital at the interest rate $\tilde{r}$.
- They operate a constant returns to scale technology $F(k, n)$, so we can work with a representative firm.
- Their problem is therefore static and given by

$$\max_{k,n}\{F(k, n) - nw - k\tilde{r}\}$$

The Aiyagari Model

First order conditions,

$$\tilde{r} = F_k(k, n), \quad w = F_n(k, n)$$

Two things to observe

1. If capital depreciates at rate δ, the (net) interest rate received by the households must be $r = \tilde{r} - \delta$.

2. Let $A(r) = \sum_{s \in \mathcal{S}} \sum_a a\psi(a, s)$. Then market clearing implies that $k = A$ and $n = (1 - u)$.

The definition of an equilibrium is now trivial.

The Aiyagari Model

Definition of a competitive equilibrium.

- A competitive equilibrium is an interest-rate r and a wage-rate w, such that
- Given r and w, $g(a, s)$ solves the households problem.
- Given r and w, k and n solves the firms problem.
- The stationary distribution ψ satisfies

$$\psi(a', s') = \sum_{s \in \mathcal{S}} \sum_{\{a : a' = g(a,s)\}} \psi(a, s) p(s', s) \qquad (1)$$

- Markets clear: $k = \sum_{s \in \mathcal{S}} \sum_a a \psi(a, s)$ and $n = (1 - u)$

The Aiyagari Model

Algorithm,

1. Guess for r.

2. With constant returns to scale, $r = F_k(k, n) - \delta$ uniquely pins down the capital-labor ratio – for instance $r = \alpha \frac{k}{n}^{\alpha-1} - \delta$ – which also pins down $w = F_n(k, n)$ ($w = (1 - \alpha) \frac{k}{n}^{\alpha}$)

3. Solve the households problem.

4. Evaluate $A(r) = \sum_{s \in \mathcal{S}} \sum_a a \psi(a, s)$.

5. Evaluate r' as $r' = F_k(A, 1 - u) - \delta$.

6. If $r' > r$, adjust r upwards. Else downwards. Until $r' = r$.

In practice we will use the bisection method again.
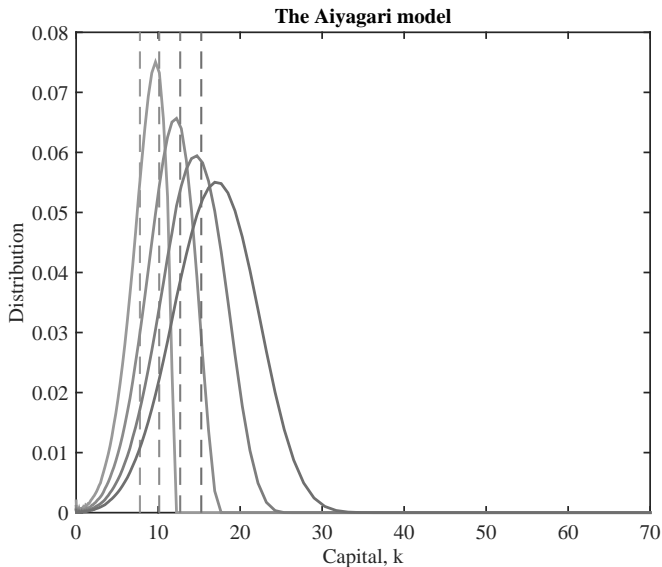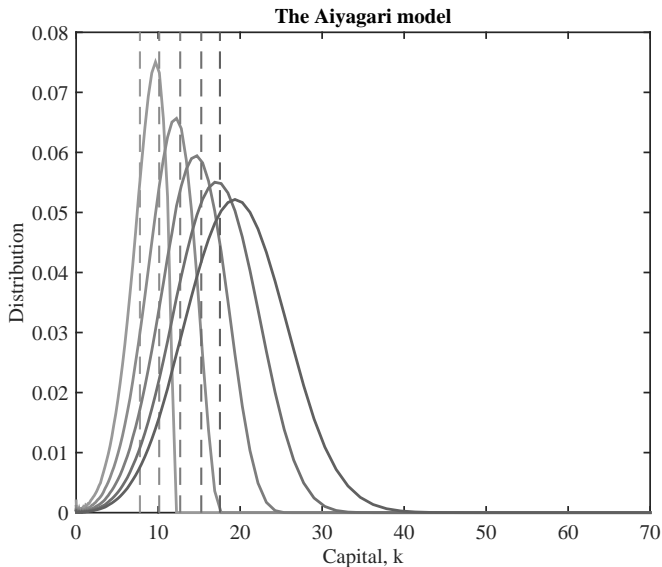
Aiyagari distribution

The Aiyagari model

Aiyagari distribution

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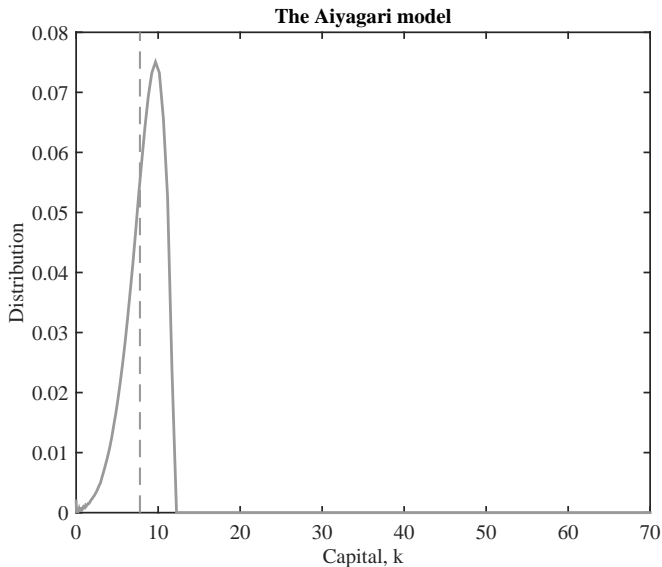
(x-axis: Capital, k; y-axis: Distribution)

Aiyagari distribution

The Aiyagari model

Aiyagari distribution

The Aiyagari model

Aiyagari distribution

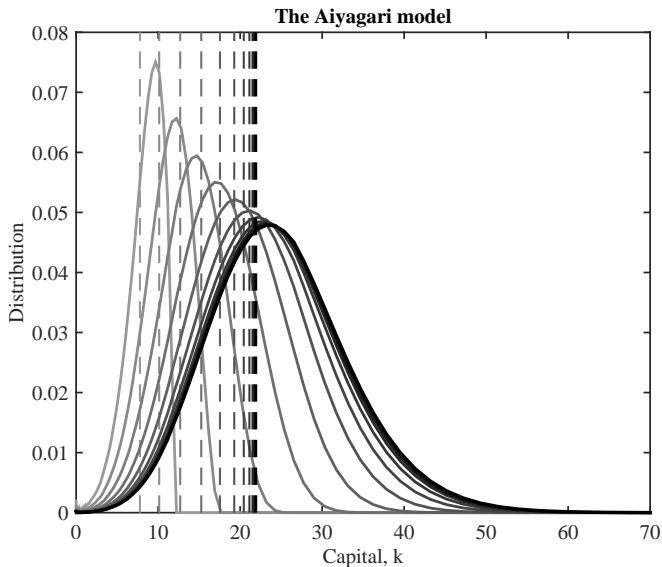The Aiyagari model
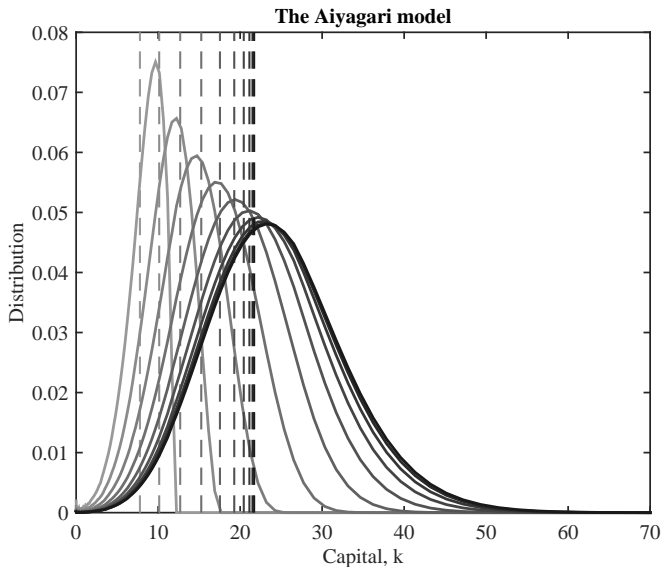
Aiyagari distribution

The Aiyagari model

Calculating the transition

- Beautiful isn't it?
- But the Aiyagari model has one very compromising downside
    - Aggregate variables like output, consumption and investment are all constant!
- So we cannot analyze business cycle properties.

Calculating the transition

- Analyzing the business cycle was the contribution of Krusell and Smith (1998).
- But there are alternative shortcuts that have gained popularity
- The general principle is to consider an Aiyagari economy that is at its steady-state in period t.
- Then we expose the economy to some temporary shock, under the assumption that the economy will fully have reverted back to its steady-state in period $t + T$.
- Then it's not that hard to calculate the transition of the economy between period t and $t + T$.

Calculating the transition

- Suppose the economy is hit by a negative technology shock in period t, such that $y_{t+s} = exp(z_{t+s})F(k_{t+s}, n_{t+s})$, with $z_{t+s+1} = \rho z_{t+s}$, and $z_t < 0$.
- Assume that $z_{t+T} = 0$.
- We know that labor won't change, such that $n_{t+s} = (1 - u) = n$.
- However, the capital stock will!

Calculating the transition

- General principle: Guess how the capital stock will evolve from period $t+1$ to period $t+T$, assuming that it's back to it's steady state value in $t+T+1$. Call this guess $\{k_{t+s}^0\}_{s=1}^{T}$.

- Given this guess, we can calculate wages and interest rates as $w_{t+s}^0 = exp(z_{t+s})F_n(k_t^0, n)$ and $\tilde{r}_{t+1}^0 = exp(z_{t+s})F_k(k_t^0, n)$

Calculating the transition

- Using time iteration again, we can find $\tilde{g}(a, s)$ as

$$u'((1 + r_{t+1}^0)a + w_{t+s}^0 s + \mu w_{t+s}^0(1 - s) - \tilde{g}(a, s))$$
$$= \beta(1 + r_{t+s+1}^0) \sum_{s' \in \mathcal{S}} u'((1 + r_{t+s+1}^0)\tilde{g}(a, s) + w_{t+s+1}^0 s'$$
$$+ \mu w_{t+s+1}^0(1 - s') - g_{t+s+1}^0(\tilde{g}(a, s), s'))p(s', s)$$

- Then $g_{t+s}^0(a, s) = \max\{\tilde{g}(a, s), \underline{a}\}$, and with $g_{t+T+1}(a, s) = g(a, s)$.

Calculating the transition

- Then for each $g_t^0, g_{t+1}^0, \ldots, g_{t+s}^0$ we can find a transition matrix M_{t+s}^0.
- Given that we know $\psi_t(a, s) = \psi(a, s)$, we can update as $\psi_{t+s+1}^0 = \psi_{t+s}^0 \times M_{t+s}^0$, and calculate the implied aggregate asset holdings A_{t+s}^0.
- Then we update our guess for the sequence of capital as $\{k_{t+s}^1\}_{s=1}^T = \eta \{k_{t+s}^0\}_{s=1}^T + (1 - \eta)\{A_{t+s}^0\}_{s=1}^T$, with $\eta \in (0, 1]$.
- And repeat until $\|\{k_{t+s}^{n+1}\}_{s=1}^T - \{k_{t+s}^n\}_{s=1}^T\|$

Krusell & Smith (1998)

- The Aiyagari model has some impressive properties.
  - Each agent faces a tremendous amount of uncertainty.
  - Yet, at the aggregate level there is none (law of large numbers . . .).
  - Implication: Equilibrium prices are constant at the stationary distribution.

Krusell & Smith (1998)

- One problem is, however, that the Aiyagari model gives no predictions on the business cycle frequency.
- Clearly, it is of interest to understand whether representative agent models do a good job compared to more realistic models with idiosyncratic risk.
- This is precisely the question explored by Krusell and Smith (1998).
- Hugely influential paper with 1700 citation on Google Scholar.

Krusell & Smith (1998)

- Simple idea: Take Aiyagari's model and include aggregate shocks to TFP.
- Huge problem: No stationary distribution. No constant prices.
- The main contribution of K&S was to show how to work around this problem in a very accurate way.
- Their idea is referred to as "approximate aggregation".

Krusell & Smith (1998)

- First off, there will be technology shocks

$$z = \{z_g, z_b\} \tag{2}$$

- We set $z_g = 1.02$ and $z_b = 0.98$.
- Second, there will be a transition matrix for these shocks. In particular, we set these to match an expected duration of both good and bad times to 8 quarters.

$$P_{z,z'} \;\; := \;\; \begin{array}{c} \\ z_g \\ z_b \end{array} \begin{pmatrix} z_g & z_b \\ 1 - \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & 1 - \frac{1}{8} \end{pmatrix}$$

Krusell & Smith (1998)

- Naturally, there will be a transition matrix, P, for employment as well.
- Is this transition matrix independent of the aggregate state, z?
- That would be counterintuitive. The probability of finding a job etc. clearly depends on the aggregate state.
- Would you expect that the probability of finding a job at $t + 1$ is independent of the aggregate state in $t + 1$ (and therefore only depends on the state in period t)?
- Maybe, but it brings some salient properties assuming that it differs.

Krusell & Smith (1998)

- Therefore, we will have a transition matrix for employment to unemployment for each respective (z, z')-pair (i.e. 4 different matrices).

- Denote these $P_{e,e'|z,z'}$.

$$
P_{e,e'|z,z'} \quad := \quad
\begin{array}{c}
 \\
e = 1 \\
e = 0
\end{array}
\begin{array}{cc}
e' = 1 & e' = 0 \\
\begin{pmatrix}
\pi_{1,1|z,z'} & \pi_{1,0|z,z'} \\
\pi_{0,1|z,z'} & \pi_{0,0|z,z'}
\end{pmatrix}
\end{array}
$$

- That is we have a total of 8 values of π to calibrate.

Krusell & Smith (1998)

- First off, let's start with $(z, z') = (z_g, z_g)$.
- We want the duration of unemployment in this case to be 1.5 quarters.
- That is $\pi_{0,0|g,g} = 1 - 1/1.5$. Of course $\pi_{0,1|g,g} = 1 - \pi_{0,0|g,g}$.
- Second, we want the unemployment rate in good times to be $u_g = 4\%$.
- Since $(1 - u_g) = (1 - u_g)\pi_{1,1|g,g} + u_g\pi_{0,1|g,g}$, we set $\pi_{1,1|g,g} = ((1 - u_g) - u_g\pi_{0,1|g,g})/(1 - u_g)$.

Krusell & Smith (1998)

- ▸ Second, consider $(z, z') = (z_b, z_b)$.
- ▸ We want the duration of unemployment in this case to be 2.5 quarters.
- ▸ That is $\pi_{0,0|b,b} = 1 - 1/2.5$. Of course $\pi_{0,1|b,b} = 1 - \pi_{0,0|g,g}$.
- ▸ Second, we want the unemployment rate in bad times to be $u_b = 10\%$.
- ▸ Since $(1 - u_b) = (1 - u_b)\pi_{1,1|b,b} + u_b\pi_{0,1|b,b}$, we set $\pi_{1,1|b,b} = ((1 - u_b) - u_b\pi_{0,1|b,b})/(1 - u_b)$.

Krusell & Smith (1998)

good to bad

- Now let's consider the matrix $P_{e,e'|g,b}$.
- First off, we set $\pi_{0,0|g,b} = \boxed{1.25\pi_{0,0|b,b}}$
- Second, as $(1 - u_b) = (1 - u_g)\pi_{1,1|g,b} + u_g\pi_{0,1|g,b}$, we set $\pi_{1,1|g,b} = ((1 - u_g) - u_g\pi_{0,1|g,b})/(1 - u_b)$.

Krusell & Smith (1998)

- Lastly, let's consider the matrix $P_{e,e'|b,g}$.
- First off, we set $\pi_{0,0|b,g} = 0.75\pi_{0,0|g,g}$
- Second, as $(1 - u_g) = (1 - u_b)\pi_{1,1|b,g} + u_b\pi_{0,1|b,g}$, we set $\pi_{1,1|b,g} = ((1 - u_b) - u_b\pi_{0,1|b,g})/(1 - u_g)$.

Krusell & Smith (1998)

- Our final transition matrix is therefore given by,

$$P := \begin{pmatrix} P_{z=g,z'=g} \times P_{e,e'|g,g} & P_{z=g,z'=b} \times P_{e,e'|g,b} \\ P_{z=b,z'=g} \times P_{e,e'|b,g} & P_{z=b,z'=b} \times P_{e,e'|b,b} \end{pmatrix}$$

Krusell & Smith (1998)

▶ Or,

$$P := \begin{pmatrix} 0.8507 & 0.0243 & 0.1159 & 0.0091 \\ 0.5833 & 0.2917 & 0.0313 & 0.0938 \\ 0.1229 & 0.0021 & 0.8361 & 0.0389 \\ 0.0938 & 0.0313 & 0.3500 & 0.5250 \end{pmatrix}$$

Krusell & Smith (1998)

- Let $s_t = (e_t, z_t)$, and $s^t = ((e_0, z_0), (e_1, z_1), \ldots (e_t, z_t))$.
- Then taking price processes $r_t(s^t)$ and $w_t(s^t)$ as given, the household's optimization problem is given by,

$$\max_{\{c_t(s^t), a_{t+1}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s_t \in \mathcal{S}^{t+1}} \beta^t u(c_t(s^t)) f_t(s^t) \tag{3}$$

$$c_t(s^t) + a_{t+1}(s^{t+1}) = (1 + r_t) a_t(s^{t-1}) + w(s_t) \tag{4}$$

$$a_{t+1}(s^t) \geq \underline{a} \tag{5}$$

$$a_0, s_0, \quad \text{given} \tag{6}$$

- Where, of course, f_t (and p), is defined as previously using the transition matrix P.

Krusell & Smith (1998)

- Now, if we assume for a second that agents have access to some variable x_t, such that $r_t = r(z_t, x_t)$ and $w_t = w(z_t, x_t)$.

- Moreover, $x_{t+1} = \phi(z_t, z_{t+1}, x_t)$.

- The the Bellman equation associated with the household problem is given by,

$$v(a, e, z, x) = \max_{c, a'}\{u(c) + \beta \sum_{(z', e') \in \{z_g, z_b\} \times \{0,1\}} v(a', e', z', x')p(e', z'|e, z)\}$$

$$\text{s.t} \quad c + a' = a(1 + r(z, x)) + ew(z, x)$$
$$x' = \phi(z, z', x)$$

- With policy function $a' = g(a, e, z, x)$.

Krusell & Smith (1998)

- Firms can hire workers on a labor spot market at wage rate w_t, and rent capital at the interest rate $\tilde{r}_t$.
- They operate a constant returns to scale technology $z_t F(k_t, n_t)$, so we can work with a representative firm.
- Their problem is therefore static and given by

$$\max_{k_t, n_t}\{z_t F(k_t, n_t) - n_t w_t - k_t \tilde{r}_t\} \tag{7}$$

Krusell & Smith (1998)

First order conditions,

$$\tilde{r}_t = z_t F_k(k_t, n_t), \quad w_t = z_t F_n(k_t, n_t) \tag{8}$$

Two things to observe

1. If capital depreciates at rate δ, the (net) interest rate received by the households must be $r_t = \tilde{r}_t - \delta$.

2. Let $A_t = \sum_{e \in \{0,1\}} \sum_a a \psi_t(a, e)$. Then market clearing implies that $k_t = A_t$ and $n_t = (1 - u_t)$.

The definition of an equilibrium is now straightforward.

Krusell & Smith (1998)

Definition of a recursive competitive equilibrium.

- A competitive equilibrium are prices $\{r_t(s^t), w_t(s^t)\}_{t=0}^{\infty}$ such that

- Given prices, $g(a, e, z, x)$ solves the households problem.

- Given prices, $\{k_t(s^t), n_t(s^t)\}_{t=0}^{\infty}$ solves the firms problem.

- Markets clear: $k_t = \sum_{e \in \{0,1\}} \sum_a a \psi_t(a, e)$ and $n_t = (1 - u_t)$.

- *Rationality*: $r_{t+1} = r(z_{t+1}, \phi(z_{t+1}, z_t, x_t))$ and $w_{t+1} = w(z_{t+1}, \phi(z_{t+1}, z_t, x_t))$.

Krusell & Smith (1998)

- The million dollar question (or one JPE publication question) is however what x is!

- A natural candidate is that $x_t = \psi_t$, that is the entire cross-sectional distribution.

- Notice that if $x_t = \psi_t$, we can easily calculate prices (through A_t – excess demand – together with firm's first order conditions).

- In addition $\psi_{t+1}(a', e') = \sum_{e \in \{0,1\}} \sum_{a: a' = g(a,e,z,x)} \psi_t(a, e) Pr(e'|e, z, z')$.

- So $x' = \phi(z', z, x)$ would hold!

- But as ψ_t is an infinite-dimensional object, we can't solve the Bellman equation!

Krusell & Smith (1998)

- ▸ Krusell & Smith propose that maybe we can approximate ψ_t with some finite collection of moments.
- ▸ In particular, K&S hypothesize that maybe *only the mean matters* (but they also explore the *variance*).
- ▸ So suppose that $x = K$ where K is the aggregate (and average) capital stock in period t.
- ▸ Together with our knowledge of the unemployment-rate, we can easily calculate prices w and r.
- ▸ Moreover, K&S postulate that

$$K' = \alpha_z + \beta_z K \tag{9}$$

Krusell & Smith (1998)

- Hopefully, this will lead to "approximate aggregation" in that

$$K' = \alpha_z + \beta_z K \approx \sum_{e'} \sum_{a'} a' \psi_{t+1}(a', e') \qquad (10)$$

- But how will we find α_z and β_z (four coefficients)?
- This is where their algorithm comes in.
- But first off, let's define an equilibrium with approximate aggregation.

Krusell & Smith (1998)

Definition of a competitive equilibrium with approximate aggregation.

- A competitive equilibrium are prices $\{r_t(s^t), w_t(s^t)\}_{t=0}^{\infty}$ such that
- Given prices, $g(a, e, z, x)$ solves the households problem.
- Given prices, $\{k_t(s^t), n_t(s^t)\}_{t=0}^{\infty}$ solves the firms problem.
- Markets clear: $k_t = \sum_{e \in \{0,1\}} \sum_a a \psi_t(a, e)$ and $n_t = (1 - u_t)$.
- *Approximate Aggregation*:
  $K' = \alpha_z + \beta_z K \approx \sum_{e'} \sum_{a'} a' \psi_{t+1}(a', e')$.

Krusell & Smith (1998)

Krusell & Smith's algorithm

- Guess for α_z and β_z (I go for $\alpha_z = 0$ and $\beta_z = 1$).
- Solve the household's problem to get the policy function $a' = g(a, e, z, K)$.
- Use this policy rule and simulate the savings behavior of a long (T=6000) panel of many individuals (a continuum).
- Regress K_{t+1} on K_t and a constant, conditional on z_t.
- If your coefficient from the regression matches your guessed α_z and β_z you're done!
- Otherwise update α_z and β_z, and repeat.
- Once convergence is obtained, check for accuracy of your laws of motion for aggregate capital.

Krusell & Smith (1998)

- So let's do exactly this, but only for one iteration.
- That is, guess that $\alpha_z = 0$ and $\beta_z = 1$ and simulate a panel.
- Then just plot K_{t+1} vs. K_t.

Krusell and Smith

The Krusell and Smith Model
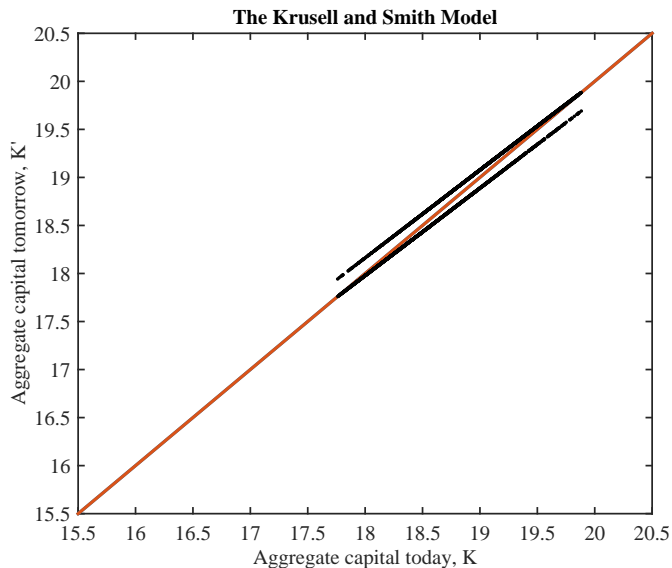
Krusell & Smith (1998)

- A linear guess appears very close to optimal.
- But our coefficients were far off! (see red, 45 degree line)
- Regression reveals that the updated coefficients are: $\alpha_g = 1.69$, $\alpha_b = 1.59$, $\beta_g = 0.92$, $\beta_b = 0.91$.
- This change may be a bit too drastic, so I set the new coefficient as a linear combination between the old and the new estimates (50% weight on the new).
- Iterating on this converges really nicely and yields,

Krusell and Smith
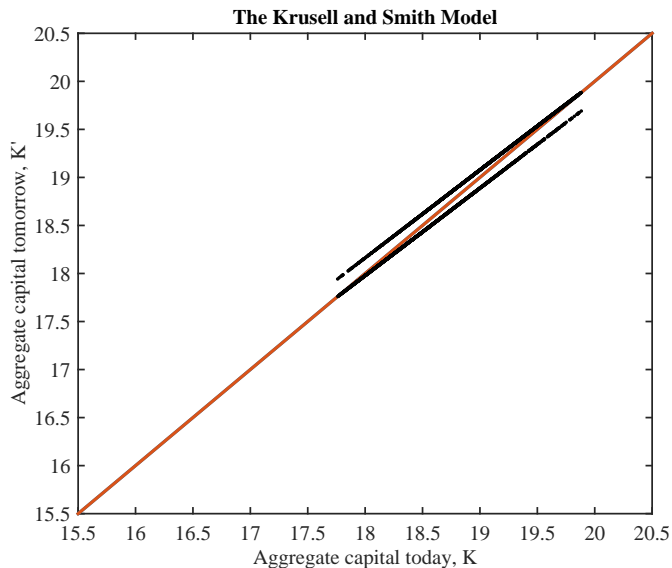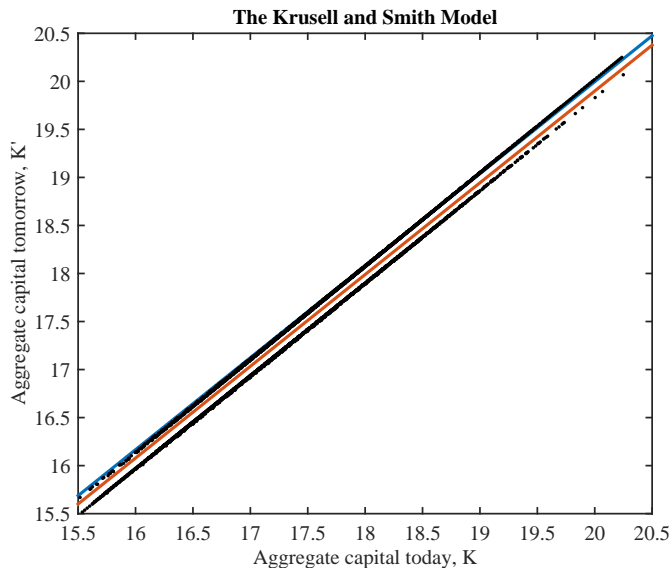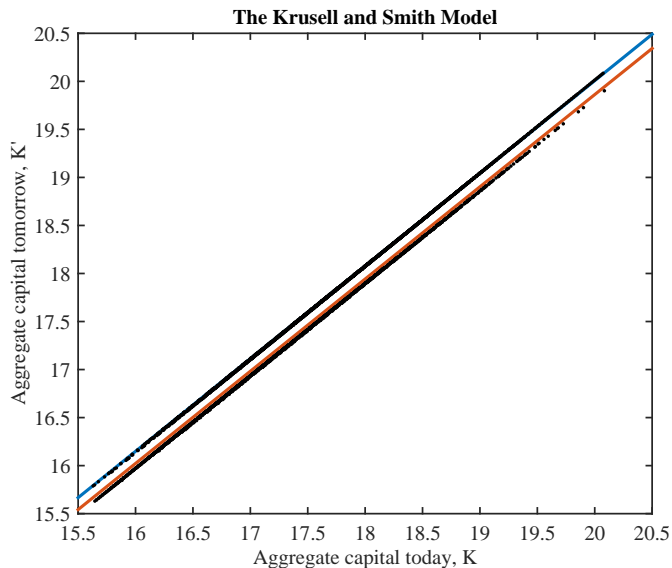
The Krusell and Smith Model

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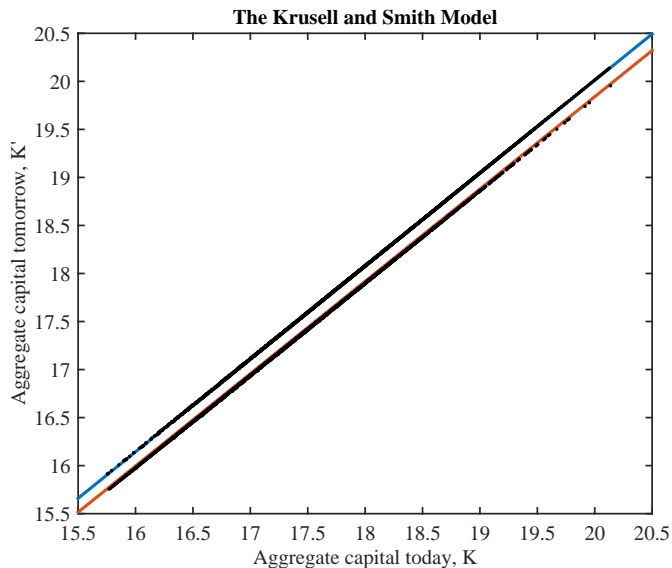
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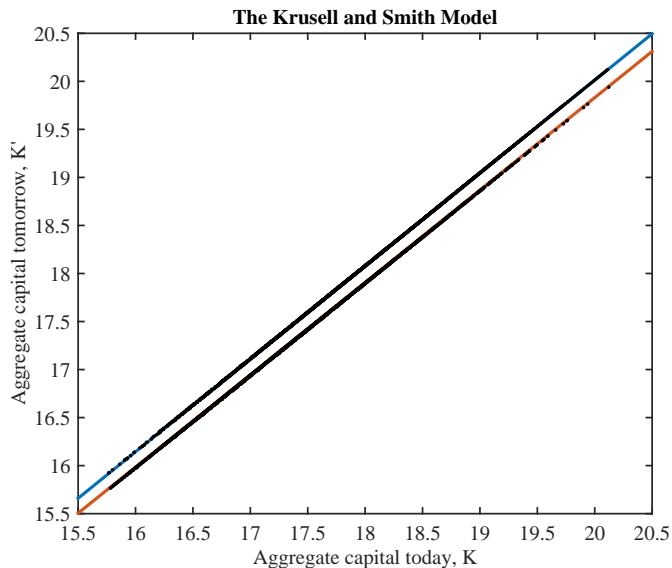
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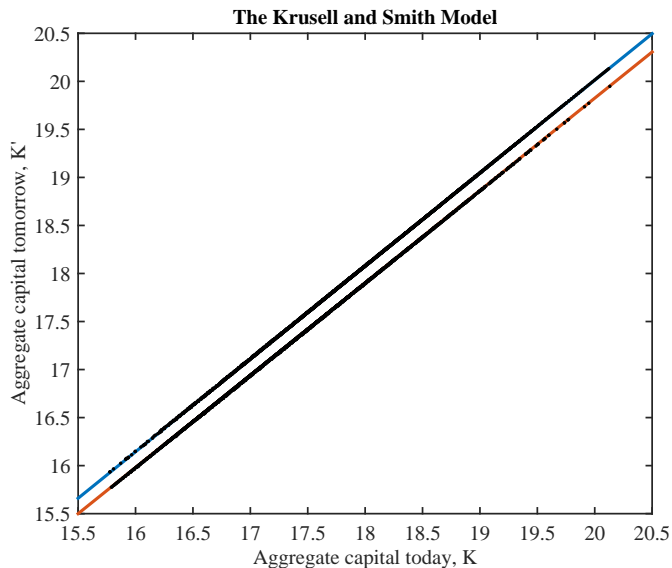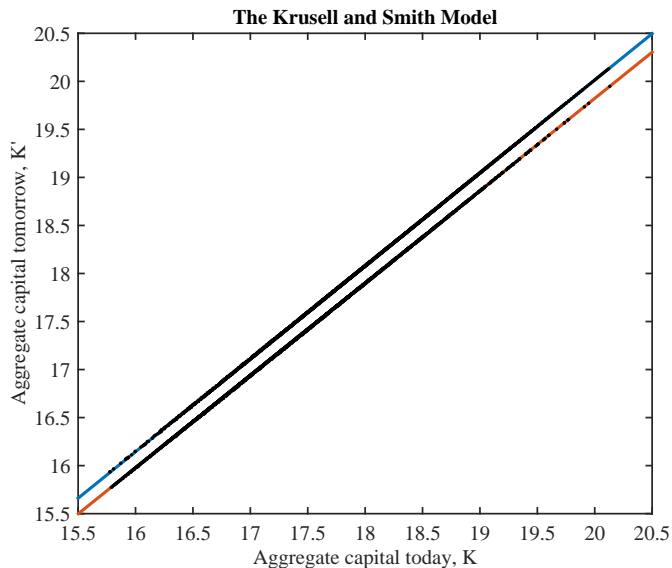
Krusell and Smith
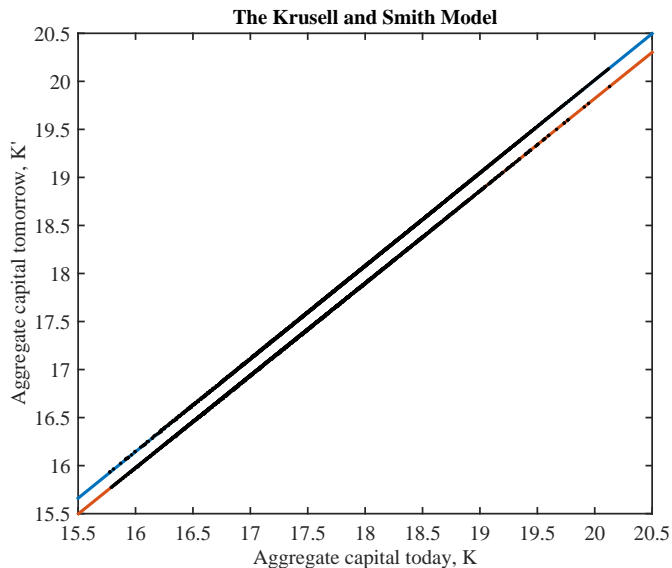
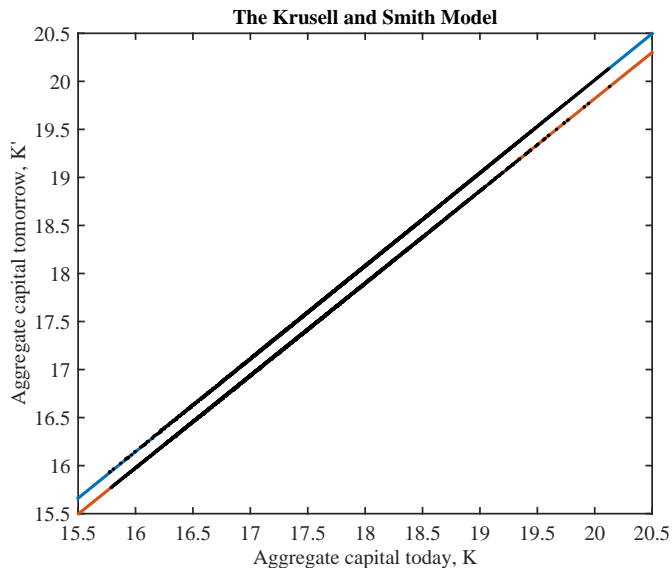The Krusell and Smith Model
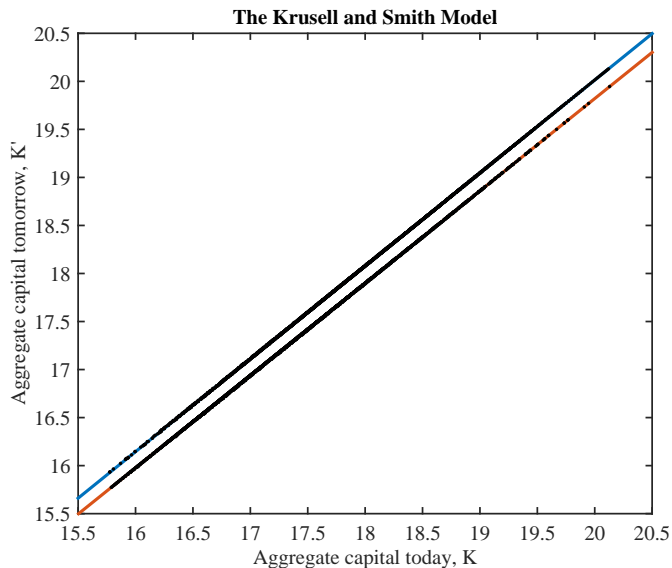
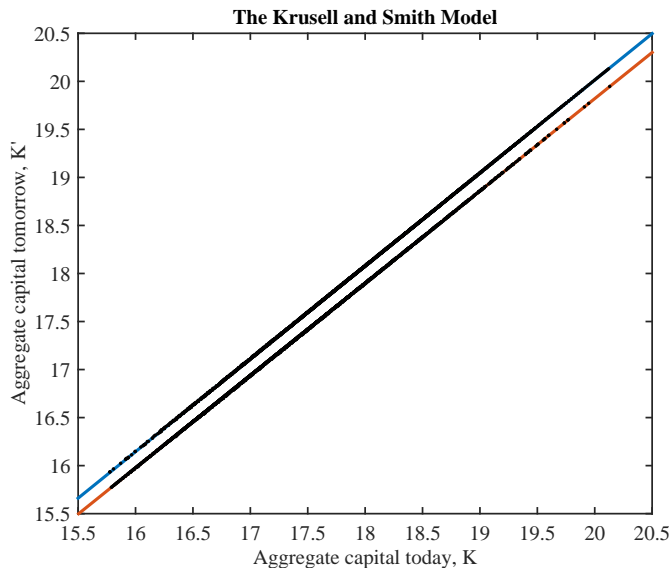# Krusell and Smith

The Krusell and Smith Model

Krusell & Smith (1998)

- The main drawback with K&S' algorithm is that it is slow (8 min on my laptop).
- This is using the absolutely fastest methods available.
- The simulation step uses a lot of RAM and CPU power.
- And we need several simulations before convergence (perhaps a hundred).
- An alternative algorithm to approximate aggregation is *explicit aggregation*.
- It's much faster (around 50×) as it doesn't use simulation.
- However, I also have to admit it's a little less accurate (see den Haan (2010)).
- An optimal procedure may be a combination.