Numerical Methods Bootcamp

Lecture 1
Recursive methods and Value Function Iteration

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2018

Introduction

- This course will focus on solving and simulating dynamic and stochastic models using numerical methods
- In particular we will consider
  1. The concept of recursive problems and value function iteration
  2. Solving nonlinear equations and functional approximation
  3. How to apply those methods to typical macroeconomic models with a representative agent
  4. How to apply those methods to typical macroeconomic models with heterogenous agents
  5. How to develop and solve models in continuous time.

Introduction – Recursive problems

- Recursion refers simply to repetition
- Recursion refers to the idea that one large problem can be broken down into pieces
- And each piece is, in some metric, identical to the other
- So if we can solve one piece, we can solve them all!
- Unfortunately, these pieces are often linked to each other
  - So solving for one piece requires having solved for all other pieces!
  - This is were recursion kicks in

Recursive problems

- Let me illustrate this idea using a familiar problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 0, 1, 2, \ldots . \quad a_0 \text{ is given}$$

- The solution is an infinite sequence $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$

Recursive problems

- Let me illustrate this idea using a familiar problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 0, 1, 2, \ldots . \quad a_0 \text{ is given}$$

- The solution is an infinite sequence $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$
- This infinite sequence if often called an optimal plan or an optimal policy.

Recursive problems

- It can be useful to write this as

$$V(a_0) = \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 0, 1, 2, \ldots. \quad a_0 \text{ is given}$$

- The solution is an optimal plan $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$ such that ...

Recursive problems

- we have

$$V(a_0) = \sum_{t=0}^{\infty} \beta^t u(c_t^*)$$
$$\text{subject to} \quad c_t^* + a_{t+1}^* = y + (1+r)a_t^*,$$
$$t = 0, 1, 2, \ldots . \quad a_0 \text{ is given} \tag{1}$$

Recursive problems

- we have

$$V(a_0) = \sum_{t=0}^{\infty} \beta^t u(c_t^*)$$

$$\text{subject to} \quad c_t^* + a_{t+1}^* = y + (1+r)a_t^*,$$

$$t = 0, 1, 2, \ldots . \quad a_0 \text{ is given} \tag{1}$$

- $V(a_0)$ is called the *value function*.
  - Notice that $V(a_0)$ is indeed a function, since if we vary a_0 the net present value utility will (probably) change as well
- Sometime we say that the sequence $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$ attains $V(a_0)$.

Recursive problems

- Let's spell it out

$$V(a_0) = u(c_0^*) + \beta u(c_1^*) + \beta^2 u(c_2^*) + \beta^3 u(c_3^*) \dots$$

- Why is this any useful? I still don't know what $\{c_t^*, a_{t+1}^*\}$ looks like!

Recursive problems

- Let's spell it out

$$V(a_0) = u(c_0^*) + \beta u(c_1^*) + \beta^2 u(c_2^*) + \beta^3 u(c_3^*) \dots$$

- Why is this any useful? I still don't know what $\{c_t^*, a_{t+1}^*\}$ looks like!
- Well let's see what we can do assuming that we know
  1. $\{c_t^*, a_{t+1}^*\}$ exists
  2. $\{c_t^*, a_{t+1}^*\}$ is feasible. That is,

$$c_t^* + a_{t+1}^* = y + (1+r)a_t^*, \quad \text{for } t = 0, 1, 2, \dots$$

  3. $\{c_t^*, a_{t+1}^*\}$ attains $V(a_0)$

Recursive problems

- Let's spell it out

$$V(a_0) = u(c_0^*) + \beta[\underbrace{u(c_1^*) + \beta u(c_2^*) + \beta^2 u(c_3^*) \ldots]}_{\vec{V}_1}$$

$$= u(c_0^*) + \beta \vec{V}_1$$

- This is pure accounting, but at least this looks pretty tidy!

Recursive problems

- Let's spell it out

$$V(a_0) = u(c_0^*) + \beta[\underbrace{u(c_1^*) + \beta u(c_2^*) + \beta^2 u(c_3^*) \ldots]}_{\vec{V}_1}$$

$$= u(c_0^*) + \beta \vec{V}_1$$

- This is pure accounting, but at least this looks pretty tidy!
- $\vec{V}_1$ is known as a continuation value.
- $\{c_t^*, a_{t+1}^*\}_{t=1}^{\infty}$ is known as a continuation plan
- And if we knew $\vec{V}_1$, it cannot be that hard to find c_0^* can it?
- And then we would at least know what $V(a_0)$ is!

Recursive problems

- Entertain the following idea: $\vec{V}_1$ is equal to $V(a_1^*)$, where

$$V(a_1^*) = \max_{\{c_t, a_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \qquad \text{(SP)}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 1, 2, 3, \ldots . \quad a_1^* \text{ is given}$$

- That is, the continuation plan, $\{c_t^*, a_{t+1}^*\}_{t=1}^{\infty}$, is an optimal plan with regards to “the state resulting from our first decision”, i.e. a_1^*

Bellman's Principle of Optimality

"An optimal policy [plan] has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision".

Bellman's Principle of Optimality

- I would like to phrase it perhaps more simply:
- "An optimal plan is such that given the opportunity of changing your mind (i.e "re-optimizing") in any period t, you can do no better than sticking to your initial plan."
- But the above statement requires one additional qualification: That your possible re-optimization in period t is constrained by some state which resulted from your previous decisions.

Recursive problems

- Can we prove that $\vec{V}_1 = V(a_1^*)$?
- You bet we can!

Recursive problems

- Can we prove that $\vec{V}_1 = V(a_1^*)$?
- You bet we can!
- Could it be the case that $\vec{V}_1 > V(a_1^*)$?
- No this is impossible
- Why? Look again at our optimisation problem, $V(a_1^*)$

Recursive problems

$$V(a_1^*) = \max_{\{c_t, a_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \qquad \text{(SP)}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 1, 2, 3, \ldots . \quad a_1^* \text{ is given}$$

- Obviously, $\{c_t^*, a_{t+1}^*\}_{t=1}^{\infty}$ is at least a feasible plan.
- And $\{c_t^*, a_{t+1}^*\}_{t=1}^{\infty}$ yields, by definition, utility $\vec{V}_1$.
- So since the continuation plan is feasible and yields utility $\vec{V}_1$, $V(a_1^*)$ must at least be as good as $\vec{V}_1$.
- Hence, $\vec{V}_1 > V(a_1^*)$ is impossible, which leave us with the possibility $\vec{V}_1 \leq V(a_1^*)$

Recursive problems

$$V(a_1^*) = \max_{\{c_t, a_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 1, 2, 3, \ldots. \quad a_1^* \text{ is given}$$

- Could $V(a_1^*) > \vec{V}_1$? Let's suppose it can.

Recursive problems

$$V(a_1^*) = \max_{\{c_t, a_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 1, 2, 3, \ldots. \quad a_1^* \text{ is given}$$

- Could $V(a_1^*) > \vec{V}_1$? Let's suppose it can.
- Let $\{c_t', a_{t+1}'\}_{t=1}^{\infty}$ denote the optimal plan that attains $V(a_1^*)$.

Recursive problems

- That is

$$
\begin{aligned}
V(a_1^*) &= u(c_1') + \beta u(c_2') + \beta^2 u(c_3') + \dots \\
\text{and} \quad c_t' + a_{t+1}' &= y + (1+r)a_t'
\end{aligned}
$$

- With $V(a_1^*) > \vec{V}_1$
- Then consider the following plan: $\{c_0^*, c_1', c_2', \dots a_1^*, a_2', a_3', \dots\}$

Recursive problems

- This plan satisfies

$$
\begin{aligned}
c_0^* + a_1^* &= y + (1+r)a_0 \\
c_1' + a_2' &= y + (1+r)a_1^* \\
c_t' + a_{t+1}' &= y + (1+r)a_t', \quad \text{for } t = 2, 3, \ldots
\end{aligned}
$$

- That is, it was feasible in period 0.

Recursive problems

- This plan satisfies

$$
\begin{aligned}
c_0^* + a_1^* &= y + (1+r)a_0 \\
c_1' + a_2' &= y + (1+r)a_1^* \\
c_t' + a_{t+1}' &= y + (1+r)a_t', \quad \text{for } t = 2, 3, \ldots
\end{aligned}
$$

- That is, it was feasible in period 0.
- And yields utility

$$
\begin{aligned}
\hat{V}(a_0) =& u(c_0^*) + \beta \underbrace{[u(c_1') + \beta u(c_2') + \beta^2 u(c_3') + \ldots]}_{V(a_1^*)} \\
=& u(c_0^*) + \beta V(a_1^*) \\
>& u(c_0^*) + \beta \vec{V}_1 = V(a_0)
\end{aligned}
$$

Recursive problems

- But since $\{c_t^*, a_{t+1}^*\}$ was the optimal plan in period 0, there cannot exist another feasible plan such that

$$u(c_0^*) + \beta u(c_1') + \beta^2 u(c_2') + \beta^3 u(c_3') + \ldots > V(a_0)$$

so we have reached a contradiction

Recursive problems

- But since $\{c_t^*, a_{t+1}^*\}$ was the optimal plan in period 0, there cannot exist another feasible plan such that

$$u(c_0^*) + \beta u(c_1') + \beta^2 u(c_2') + \beta^3 u(c_3') + \ldots > V(a_0)$$

so we have reached a contradiction

- What led us to this contradiction?
  - The conjecture that $V(a_1^*) > \vec{V}_1$!
- Hence $V(a_1^*) = \vec{V}_1$!!

Recursive problems

- Ok, so recall that we started with

$$V(a_0) = u(c_0^*) + \beta[\underbrace{u(c_1^*) + \beta u(c_2^*) + \beta^2 u(c_3^*) \dots]}_{\vec{V}_1}$$

$$= u(c_0^*) + \beta \vec{V}_1$$

- And now we can confidently state

$$V(a_0) = u(c_0^*) + \beta V(a_1^*)$$

- With

$$c_0^* + a_1^* = y + (1 + r)a_0$$

Recursive problems

- But now, another question arises.
- Could there exist a pair $(\tilde{c}_0, \tilde{a}_1)$ with

$$\tilde{c}_0 + \tilde{a}_1 = y + (1 + r)a_0$$

such that

$$u(\tilde{c}_0) + \beta V(\tilde{a}_1) > u(c_0^*) + \beta V(a_1^*) = V(a_0)$$

?

- Again, suppose such a pair existed.

Recursive problems

- Then associated with $V(\tilde{a}_1)$ is an optimal plan $\{c''_t, a''_{t+1}\}_{t=1}^{\infty}$.
- Then again consider the plan $\{\tilde{c}_0, c''_1, c''_2, \ldots \tilde{a}_1, a''_2, a''_3, \ldots\}$

Recursive problems

- This plan satisfies

$$
\begin{aligned}
\tilde{c}_0 + \tilde{a}_1 &= y + (1+r)a_0 \\
c_1'' + a_2'' &= y + (1+r)\tilde{a}_1 \\
c_t'' + a_{t+1}'' &= y + (1+r)a_t'', \quad \text{for } t = 2, 3, \ldots
\end{aligned}
$$

- That is, it was feasible in period 0.

Recursive problems

- This plan satisfies

$$
\begin{aligned}
\tilde{c}_0 + \tilde{a}_1 &= y + (1+r)a_0 \\
c_1'' + a_2'' &= y + (1+r)\tilde{a}_1 \\
c_t'' + a_{t+1}'' &= y + (1+r)a_t'', \quad \text{for } t = 2, 3, \ldots
\end{aligned}
$$

- That is, it was feasible in period 0.
- And yields utility

$$
\begin{aligned}
\tilde{V}(a_0) =& u(\tilde{c}_0) + \beta \underbrace{\left[u(c_1'') + \beta^2 u(c_2'') + \beta^3 u(c_3'') + \ldots\right]}_{V(\tilde{a}_1)} \\
=& u(\tilde{c}_0) + \beta V(\tilde{a}_1) \\
>& u(c_0^*) + \beta V(a_1^*) = V(a_0)
\end{aligned}
$$

Recursive problems

- But since $\{c_t^*, a_{t+1}^*\}$ was the optimal plan in period 0, there cannot exist another feasible plan such that

$$u(\tilde{c}_0) + \beta u(c_1'') + \beta^2 u(c_2'') + \beta^3 u(c_3'') + \ldots > V(a_0)$$

so we have reached a contradiction

Recursive problems

- What led us to this contradiction?
  - The conjecture that there exist a pair $(\tilde{c}_0, \tilde{a}_1)$ with

$$\tilde{c}_0 + \tilde{a}_1 = y + (1 + r)a_0$$

such that

$$u(\tilde{c}_0) + \beta V(\tilde{a}_1) > u(c_0^*) + \beta V(a_1^*) = V(a_0)$$

- Hence no such pair can exist.

Recursive problems

- What led us to this contradiction?
  - The conjecture that there exist a pair $(\tilde{c}_0, \tilde{a}_1)$ with

$$\tilde{c}_0 + \tilde{a}_1 = y + (1+r)a_0$$

such that

$$u(\tilde{c}_0) + \beta V(\tilde{a}_1) > u(c_0^*) + \beta V(a_1^*) = V(a_0)$$

- Hence no such pair can exist.
- Thus (c_0^*, a_1^*) must satisfy

$$V(a_0) = \max_{c_0, a_1} \{u(c_0) + \beta V(a_1)\}$$

$$\text{subject to} \quad c_0 + a_1 = y + (1+r)a_0$$

Recursive problems

- So our complicated problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$

$$t = 0, 1, 2, \ldots . \quad a_0 \text{ is given}$$

- Boils down to

$$V(a_0) = \max_{c_0, a_1} \{u(c_0) + \beta V(a_1)\} \tag{FE}$$

$$\text{subject to} \quad c_0 + a_1 = y + (1+r)a_0$$

Recursive problems

$$V(a_0) = \max_{c_0, a_1} \{u(c_0) + \beta V(a_1)\} \tag{FE}$$

$$\text{subject to} \quad c_0 + a_1 = y + (1 + r)a_0$$

- Notice that time subscripts are redundant.
- In particular, in period 1 the problem is the same (but with a_0 replaced by a_1).
- The same goes for period 500 or 5,000.
- Time has no meaning, only that state, a, has

Recursive problems

$$V(a) = \max_{c,a'}\{u(c) + \beta V(a')\} \tag{FE}$$
$$\text{subject to} \quad c + a' = y + (1+r)a$$

- The proof incredibly useful and general (see notes)
- In particular, it is a constructive proof.
  - We go from (SP) and construct (FE)
  - Allows you to check that you got the state variables right!

Recursive problems

- For instance, we have assumed that income, y, is constant.
- Suppose that y follows the law of motion $y' = f(y)$
- The (FE) is then

$$V(a, y) = \max_{c,a'}\{u(c) + \beta V(a', y')\} \tag{FE}$$
$$\text{subject to} \quad c + a' = y + (1 + r)a$$
$$y' = f(y)$$

- (Try to repeat the proof at home for this case)
- The same applies to a law of motion for the interest rate, $r' = h(r)$

Recursive problems

More generally

$$V(a,y,r) = \max_{c,a'}\{u(c) + \beta V(a',y',r')\} \tag{FE}$$

$$\text{subject to} \quad c + a' = y + (1+r)a$$

$$y' = f(y,r), r' = h(y,r)$$

or

$$V(a,z) = \max_{c,a'}\{u(c) + \beta V(a',z')\} \tag{FE}$$

$$\text{subject to} \quad c + a' = y + (1+r)a$$

$$y = f(z), r = h(z), z' = g(z)$$

where z can be an arbitrary vector.

Recursive problems

$$V(a) = \max_{c,a'}\{u(c) + \beta V(a')\} \tag{FE}$$
$$\text{subject to} \quad c + a' = y + (1+r)a$$

We can prove that

- $V(a)$ is montone and concave
- $V(a)$ is differentiable such that

$$V'(a) = u'(c)(1+r)$$

- $\Rightarrow$ first order conditions are necessary and sufficient

Recursive problems

$$V(a) = \max_{c,a'}\{u(c) + \beta V(a')\} \tag{FE}$$
$$\text{subject to} \quad c + a' = y + (1+r)a$$

FOC

$$u'(c) = \beta V'(a')$$
$$\Rightarrow \quad u'(c) = \beta(1+r)u'(c')$$

where c' refers to consumption "tomorrow".

Recursive problems

$$V(a) = \max_{c,a'}\{u(c) + \beta V(a')\} \tag{FE}$$
$$\text{subject to} \quad c + a' = y + (1+r)a$$

- Associated with (FE) are policy functions $c = h(a)$ and $a' = g(a)$ such that

$$V(a) = u(h(a)) + \beta V(g(a))$$
$$\text{subject to} \quad h(a) + g(a) = y + (1+r)a$$

- These are the ones we really wish to lay our hands on, since they can be used to simulate the model.

Recursive problems

- Why do we want the policy functions $c = h(a)$ and $a' = g(a)$?
- Recall the optimal plan $\{c_t^*, a_{t+1}^*\}$. Then given a_0 we can retrieve it as
  1. $c_0^* = h(a_0)$, $a_1^* = g(a_0)$
  2. $c_1^* = h(a_1^*)$, $a_2^* = g(a_1^*)$
  3. $c_2^* = h(a_2^*)$, $a_3^* = g(a_2^*)$
  4. and so on ...

Recursive problems

$$V(a) = \max_{c,a'} \{u(c) + \beta V(a')\} \tag{FE}$$
$$\text{subject to} \quad c + a' = y + (1+r)a$$

- But the policy functions can only be found if we know the (unknown) function $V(a)$.
- And we can only find $V(a)$ if we know $V(a)$ (since $V(a')$ appears on the RHS)
- This is why it is called a functional equation.
    - One equation in one unknown (function!)
- Luckily, there are very well developed methods to solve this functional equation.

Value function iteration

1. Start with a guess (!) for the value function, $V_0(a)$.
2. Update your guess by solving

$$V_1(a) = \max_{c,a'}\{u(c) + \beta V_0(a')\} \tag{2}$$
$$\text{subject to} \quad c + a' = y + (1+r)a$$

3. Keep doing this

$$V_{n+1}(a) = \max_{c,a'}\{u(c) + \beta V_n(a')\} \tag{3}$$
$$\text{subject to} \quad c + a' = y + (1+r)a$$

until

$$||V_{n+1} - V_n|| < \varepsilon$$

Value function iteration

- Since (FE) is a “contraction mapping” (see notes) this is guaranteed to converge.
- There are many ways of implementing this on a computer
- In today’s exercise you will use the “beginner’s algorithm” known as “discretised value function iteration”.

Value function iteration

It goes something like this

- Create a grid for assets, $\mathcal{A} = (a_1, a_2, \dots a_N)$.
- Guess that $V_0(a) = 0$ for all $a \in \mathcal{A}$.
- Iterate on

$$V_{n+1} = \max_{a' \in \mathcal{A}} \{u(a(1+r) + y - a') + \beta V_n(a')\}$$

  until convergence.
- Notice how the choices a' is restricted to only take on values on the grid.
- $\rightarrow$ having a fine grid is important to have a good approximation!

The Deterministic Ramsey Growth Model

The deterministic Ramsey growth model

$$V(k) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + k_{t+1} = f(k_t) + (1-\delta)k_t,$$

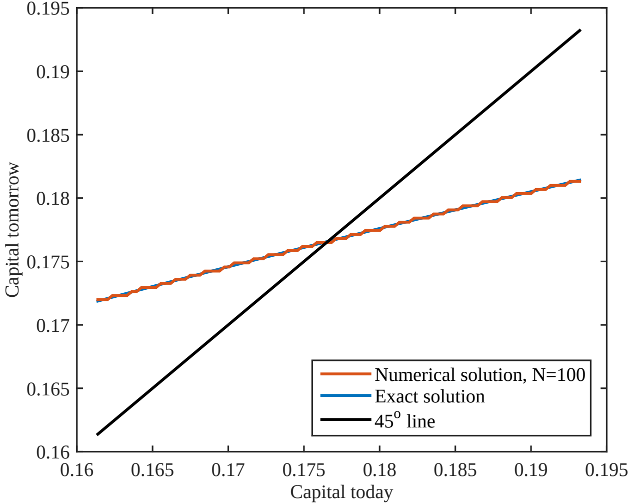
$$t = 0, 1, 2, \ldots. \quad k_0 \text{ is given}$$

Bellman equation

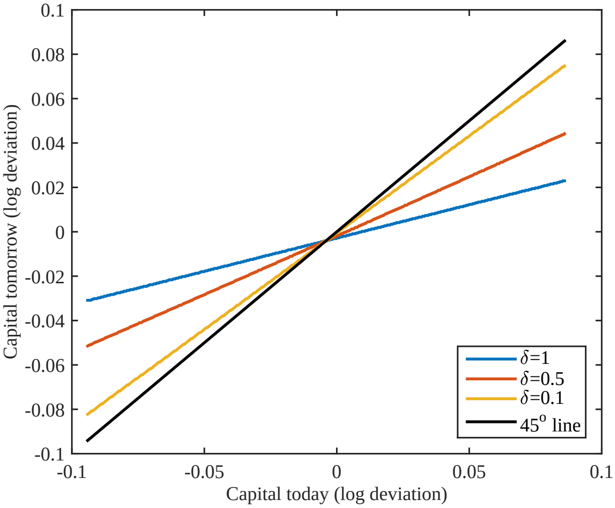
$$V(k) = \max_{c,k'} \{u(c) + \beta V(k')\} \tag{FE}$$

$$\text{subject to} \quad c + k' = f(k) + (1-\delta)k$$

Ramsey growth model, $\delta = 1$



Ramsey growth model, various values for δ



Transition matrices

- Before we proceed to stochastics, it is useful to introduce the notion of transition matrices.
- A transition matrix, T, is a matrix of probabilities that, for each state, tells you the probabilities of ending up at all possible states tomorrow.

$$
T \quad := \quad \begin{array}{c c}
 & \begin{array}{ccc} s_1 & \dots & s_N \end{array} \\
\begin{array}{c} s_1 \\ \vdots \\ s_N \end{array} & \begin{pmatrix} Pr(s_1|s_1) & \dots & Pr(s_N|s_1) \\ \vdots & \ddots & \vdots \\ Pr(s_1, s_N) & \dots & Pr(s_N, s_N) \end{pmatrix}
\end{array}
$$

Transition matrices

- Row n tells you the probabilities of ending up in states $\{s_1, \ldots, s_N\}$ conditional on being in state s_n today.
- All rows must sum to one.
- Element $T_{n,m}$ tells you the probability of ending up in state s_m tomorrow conditional on being in state s_n today.

Transition matrices

Example

- You can be employed, $s_1 = e$, or unemployed, $s_2 = u$
- If you are employed, the probability of remaining employed in the next year is 0.9.
- If you are unemployed, the probability of remaining unemployed in the next year is 0.2.

$$T \quad := \quad \begin{array}{c c} & \begin{array}{cc} e & u \end{array} \\ \begin{array}{c} e \\ u \end{array} & \begin{pmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{pmatrix} \end{array}$$

Transition matrices

- Suppose you are employed today. What is the probability of being employed in 2 years?
- Answer: $T_{1,1}T_{1,1} + T_{1,2}T_{2,1} = 0.89$.
- Suppose you are unemployed today. What is the probability of being employed in 2 years?
- Answer: $T_{2,1}T_{1,1} + T_{2,2}T_{2,1} = 0.88$.

$$T^2 = \begin{array}{c c} & \begin{array}{cc} e & u \end{array} \\ \begin{array}{c} e \\ u \end{array} & \begin{pmatrix} 0.89 & 0.11 \\ 0.88 & 0.12 \end{pmatrix} \end{array}$$

Transition matrices

- Let v_0 be a 2×1 vector indicating your current employment status
  - I.e. $v_0 = (1, 0)'$ if employed, or $v_0 = (0, 1)'$ if unemployed.
- Then the probability of being employed or unemployed in t years if given by the 2×1 vector

$$v_t = (T')^t v_0 \tag{4}$$

$$= T' v_{t-1} \tag{5}$$

Transition matrices

- What is the unconditional probability of being employed or unemployed?
- Under some mild conditions, this is equal to asking: What is v_∞?
- Call this distribution just v. It should then satisfy

$$v = T'v$$

or

$$(I - T')v = 0$$

- That is, v is the eigenvector associated with a unit eigenvalue of matrix T' and normalised to sum to one.

Policy functions and transition matrices

- When I solved the Ramsey growth model I got policy functions $k' = g(k)$.
- In matlab they are given as vectors:

$$\begin{pmatrix} 0.16 \\ 0.18 \\ 0.20 \\ 0.22 \\ 0.24 \end{pmatrix} \rightarrow \begin{pmatrix} 0.18 \\ 0.20 \\ 0.20 \\ 0.20 \\ 0.22 \end{pmatrix}$$

Policy functions and transition matrices

This policy function can be rewritten as transition matrix

$$T = \begin{array}{c} \\ 0.16 \\ 0.18 \\ 0.20 \\ 0.22 \\ 0.24 \end{array} \begin{array}{c} \begin{array}{ccccc} 0.16 & 0.18 & 0.20 & 0.22 & 0.24 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

with $v = (0, 0, 1, 0, 0)'$.

Stochastics

- Introducing stochastic elements to our problem is surprisingly easy
- Conceptually it will look difficult, but once we get our Bellman equation set up, the pieces will hopefully fall into place
- Notationally it's a nightmare! So bare with me!

Stochastics

Recall our (SP)

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

$$\text{subject to} \quad c_t + a_{t+1} = y + (1+r)a_t,$$
$$t = 0, 1, 2, \dots \quad a_0 \text{ is given}$$

- And let's make one simple change: Income is no longer deterministic, but the outcome of a random variable $s_t \in \mathcal{S} = \{0, 1\}$.
- Income in period t is given by $y_t = y \times s_t$ (but we could have written something like $y_t = y(s_t)$).

Stochastics

- Moreover, the probability of *some* s_{t+1} occurring given a *particular* s_t is denoted $p(s_{t+1}, s_t)$.
- Of course, $\sum_{s_{t+1} \in \mathcal{S}} p(s_{t+1}, s_t) = 1$.
- Let $s^t = (s_0, s_1, s_2, \ldots, s_t) \in \mathcal{S}^{t+1}$ denote a history of events. Notice that $s^0 = s_0$.
- And let $P(s^t, s_0)$ denote the probability of such a history occurring.
- By construction $P(s^t, s_0) = p(s_t, s_{t-1}) \times p(s_{t-1}, s_{t-2}) \times \ldots \times p(s_1, s_0)$.
- And $\sum_{s^t \in \mathcal{S}^{t+1}} P(s^t, s_0) = 1$

Stochastics

- Lastly, it is useful to recognise that

$$P(s^t, s_1) = P(s^t, s^1) = \frac{P(s^t, s_0)}{p(s_1, s_0)}$$

- Markov property
- Law of iterated expectations

Stochastics

Our problem is now

$$\max_{\{c_t(s^t), a_{t+1}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{t+1}} \beta^t u(c_t(s^t)) P(s^t, s_0) \qquad \text{(SP)}$$

$$\text{subject to} \quad c_t(s^t) + a_{t+1}(s^t) = y(s^t) + (1+r)a_t(s^t),$$

$$\forall t, \forall s^t \in \mathcal{S}^{t+1} \quad a_0, s_0 \text{ are given}$$

- This looks awful, but let's write things out one period forward and see where it takes us

Stochastics

Spelled out (a little)

$$\max_{\{c_t(s^t),a_{t+1}(s^t)\}_{t=0}^{\infty}} u(c_0(s^0)) + \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{S}^{t+1}} \beta^{t-1} u(c_t(s^t)) P(s^t, s_0)$$

$$\text{subject to} \quad c_t(s^t) + a_{t+1}(s^t) = y(s^t) + (1+r)a_t(s^t),$$

$$\forall t, \forall s^t \in \mathcal{S}^{t+1} \quad a_0, s_0 \text{ are given}$$

Stochastics

Spelled out (a little)

$$\max_{\{c_t(s^t), a_{t+1}(s^t)\}_{t=0}^{\infty}} u(c_0(s^0)) + \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{S}^{t+1}} p(s_1, s_0)\beta^{t-1} u(c_t(s^t)) P(s^t, s_1)$$

$$\text{subject to} \quad c_t(s^t) + a_{t+1}(s^t) = y(s^t) + (1+r)a_t(s^t),$$

$$\forall t, \forall s^t \in \mathcal{S}^{t+1} \quad a_0, s_0 \text{ are given}$$

Stochastics

Spelled out (a little)

$$\max_{\{c_t(s^t), a_{t+1}(s^t)\}_{t=0}^{\infty}} u(c_0(s^0)) + \sum_{s_1 \in \mathcal{S}} p(s_1, s_0) \sum_{t=1}^{\infty} \sum_{s^{t-1} \in \mathcal{S}^t} \beta^{t-1} u(c_t(s^t)) P(s^t, s$$

$$\text{subject to} \quad c_t(s^t) + a_{t+1}(s^t) = y(s^t) + (1+r) a_t(s^t),$$

$$\forall t, \forall s^t \in \mathcal{S}^{t+1} \quad a_0, s_0 \text{ are given}$$

Stochastics

Spelled out (a little)

$$\max_{\{c_t(s^t),a_{t+1}(s^t)\}_{t=0}^{\infty}} u(c_0(s^0)) + \sum_{s_1\in\mathcal{S}} p(s_1,s_0) \sum_{t=1}^{\infty} \sum_{s^{t-1}\in\mathcal{S}^t} \beta^{t-1} u(c_t(s^t)) P(s^t, s$$

$$\text{subject to} \quad c_t(s^t) + a_{t+1}(s^t) = y(s^t) + (1+r)a_t(s^t),$$

$$\forall t, \forall s^t \in \mathcal{S}^{t+1} \quad a_0, s_0 \text{ are given}$$

- Define

$$\vec{V}(s_1) = \sum_{t=1}^{\infty} \sum_{s^{t-1}\in\mathcal{S}^t} \beta^{t-1} u(c_t(s^t)) P(s^t, s_1)$$

Stochastics

Such that at an optimal plan $\{c_t^*(s^t), a_{t+1}^*(s^t)\}_{t=0}^{\infty}$ (which is now a stochastic process, and not a sequence), we have

$$V(a_0, s_0) = u(c_0^*(s^0)) + \beta \sum_{s_1 \in \mathcal{S}} p(s_1, s_0)\vec{V}(s_1)$$

- We can then follow the logic from the deterministic case to show that $\vec{V}(s_1) = V(a_1^*(s^1), s_1)$ for all $s_1 \in \mathcal{S}$, and that the Bellman equation is

$$V(a, s) = \max_{c,a'}\{u(c) + \beta \sum_{s' \in \mathcal{S}} V(a', s')p(s', s)\} \qquad \text{(FE)}$$

$$\text{subject to} \quad c + a' = y(s) + (1 + r)a$$

The stochastic Ramsey Growth Model

$$\max_{\{c_t(s^t),k_{t+1}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{t+1}} \beta^t u(c_t(s^t))P(s^t, s_0) \tag{SP}$$

subject to $\quad c_t(s^t) + k_{t+1}(s^t) = f(s_t, k_t(s^t)) + (1-\delta)k_t(s^t),$

$\forall t, \forall s^t \in \mathcal{S}^{t+1} \quad a_0, s_0$ are given

$$V(k,s) = \max_{c,k'}\{u(c) + \beta \sum_{s' \in \mathcal{S}} V(k', s')p(s', s)\} \tag{FE}$$

subject to $\quad c + k' = f(s,k) + (1-\delta)k$

The stochastic Ramsey Growth Model

Today's assignment

- We will use the production function $y = f(z, k) = zk^{\alpha}$
- And assume that productivity can only take two values, "good" or "bad"; i.e. $z \in \{z_g, z_b\}$.
- Whose law of motion is determined by a transition matrix, T.

The stochastic Ramsey Growth Model

This will give us “two” Bellman equations

$$V(k, z_g) = \max_{c,k'}\{u(c) + \beta[p(z_g, z_g)V(k', z_g) + p(z_b, z_g)V(k', z_b)]\}$$

$$\text{subject to} \quad c + k' = z_g k^\alpha + (1 - \delta)k$$

and

$$V(k, z_b) = \max_{c,k'}\{u(c) + \beta[p(z_g, z_b)V(k', z_g) + p(z_b, z_b)V(k', z_b)]\}$$

$$\text{subject to} \quad c + k' = z_b k^\alpha + (1 - \delta)k$$

The stochastic Ramsey Growth Model

And we will iterate according to

$$V_{n+1}(k, z_g) = \max_{c,k'}\{u(c) + \beta[p(z_g, z_g)V_n(k', z_g) + p(z_b, z_g)V_n(k', z_b$$

$$\text{subject to} \quad c + k' = z_g k^\alpha + (1-\delta)k$$

and

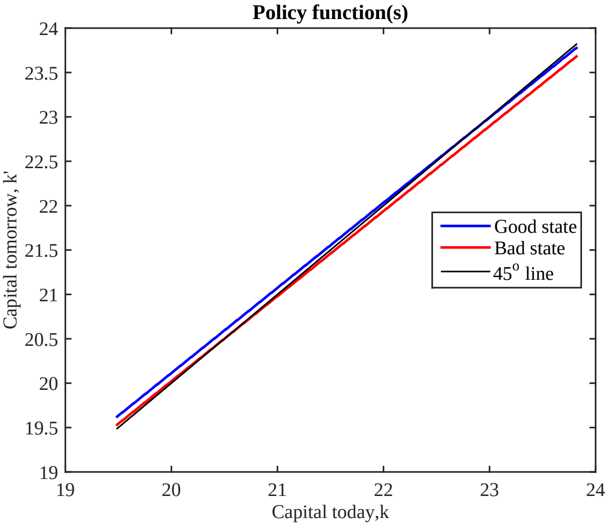
$$V_{n+1}(k, z_b) = \max_{c,k'}\{u(c) + \beta[p(z_g, z_b)V_n(k', z_g) + p(z_b, z_b)V_n(k', z_b$$

$$\text{subject to} \quad c + k' = z_b k^\alpha + (1-\delta)k$$

until convergence.

The stochastic Ramsey Growth Model

From these we will get two policy functions



The stochastic Ramsey Growth Model

Policy functions for $N = 5$

$$\begin{pmatrix} 0.16 \\ 0.17 \\ 0.18 \\ 0.19 \\ 0.20 \end{pmatrix} \underset{\text{if good state}}{\longrightarrow} \begin{pmatrix} 0.18 \\ 0.18 \\ 0.19 \\ 0.19 \\ 0.19 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0.16 \\ 0.17 \\ 0.18 \\ 0.19 \\ 0.20 \end{pmatrix} \underset{\text{if bad state}}{\longrightarrow} \begin{pmatrix} 0.17 \\ 0.17 \\ 0.17 \\ 0.18 \\ 0.18 \end{pmatrix}$$

With some transition matrix for good and bad states

$$T = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

Policy functions and transition matrices

These policy functions can be rewritten as transition matrices

$$T_g = \begin{array}{c} \\ 0.16 \\ 0.17 \\ 0.18 \\ 0.19 \\ 0.20 \end{array} \begin{array}{c} \begin{array}{ccccc} 0.16 & 0.17 & 0.18 & 0.19 & 0.20 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Policy functions and transition matrices

These policy functions can be rewritten as transition matrices

$$
T_b = \begin{array}{c} \\ 0.16 \\ 0.17 \\ 0.18 \\ 0.19 \\ 0.20 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0.16 & 0.17 & 0.18 & 0.19 & 0.20 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\end{array}
$$

Policy functions and transition matrices

The transition matrix for the economy is then

$$T_e = \begin{pmatrix} T_{1,1}\,T_g & T_{1,2}\,T_g \\ T_{2,1}\,T_b & T_{2,2}\,T_b \end{pmatrix}$$

Policy functions and transition matrices

Or

$$T_e = \begin{array}{c} \\ 0.16 \\ 0.17 \\ 0.18 \\ 0.19 \\ 0.20 \\ 0.16 \\ 0.17 \\ 0.18 \\ 0.19 \\ 0.20 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0.16 & 0.17 & 0.18 & 0.19 & 0.20 & 0.16 & 0.17 & 0.18 & 0.19 & 0.20 \end{array} \\ \begin{pmatrix} 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \end{pmatrix} \end{array}$$

with $v = (0, 0.11, 0.12, 0.27, 0, 0, 0.27, 0.12, 0.11, 0)'$.

Plotting the long run distribution