IHS Economics Series
Working Paper 330
September 2017

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Impressum

Author(s):
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Title:
Linear Time Iteration

ISSN: 1605-7996

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May 16, 2017

Abstract

This paper proposes a simple iterative method – time iteration – to solve linear rational expectation models. I prove that this method converges to the solution with the smallest eigenvalues in absolute value, and provide the conditions under which this solution is unique. In particular, if conditions similar to those of Blanchard and Kahn (1980) are met, the procedure converges to the unique stable solution. Apart from its transparency and simplicity of implementation, the method provides a straightforward approach to solving models with less standard features, such as regime switching models. For large-scale problems the method is 10-20 times faster than existing solution methods.

[†]The author would like to thank SeHyoun Ahn, Ben Moll, Wouter J. den Haan, Greg Kaplan, Tamas Papp, and Michael Reiter for helpful comments and suggestions. The usual disclaimer applies. E-mail: pontus.rendahl@gmail.com.

1 Introduction

Solving linear(ized) rational expectation models is a cornerstone in any macroeconomist's toolkit. In a seminal paper, Blanchard and Kahn (1980) provided the foundations for how to solve a certain class of such models, and how to verify the stability and uniqueness of a solution. As their class of models, however, was confined to deal with frameworks absent of *intratemporal* relationships – such as accounting identities, or optimal static choices – a plethora of alternative solution methods has emerged to (successfully) address this seemingly trivial problem.[1] Nevertheless, while these alternative methods are both fast and accurate, they are quite esoteric and lack in transparency.[2] As a consequence, it is common for most researchers to instead rely on pre-canned routines.

This paper takes a step forward by providing a method that improves both the simplicity and transparency of solving linear rational expectation models, without any loss of applicability or efficiency. In particular, I propose an iterative schedule for which the sequence of approximate solutions converges to the solution to the problem with the smallest eigenvalues in absolute value, and provided conditions under which the solution is unique and stable.

The logic underlying the procedure is simple enough to be described in words. Envisage an agent having a certain amount of an asset, facing the choice between how much of this asset to consume and how much to save. An optimal choice would trade off the marginal benefit of saving (future consumption) with its marginal cost (forgone current consumption). The resulting optimal decision is implied by a linear(ized) second-order difference equation.

However, while the marginal cost associated with any choice is clear – since both the current value of the asset is known, and the choice of saving pins down current consumption – the marginal benefit is not readily known. In particular, the future marginal benefit of saving depends on the optimal saving choice in the *future*. Thus, an optimal choice today can only be determined under the condition that the optimal choice in the future is known; thus the problem amounts to finding a fixed point. To solve this problem, this paper proposes to guess for the optimal choice of saving in the future as a linear function of the associated state (which is given by the optimal choice in the present). Given such a guess, the optimal choice in the present is then trivially given by solving a linear equation. However, the current optimal choice provides us with another suggestion regarding future optimal behavior, and the guess is updated accordingly.

This procedure is shown to be convergent, and, as previously mentioned converges to the solution with the smallest eigenvalues in absolute value. After having proved the papers main

[1] Examples include Klein (2000), Uhlig (2001), Christiano (2002), and Sims (2002). King and Watson (2002) show that *any* linearized system can be recast into the original formulation of Blanchard and Kahn (1980).

[2] For instance, Sims (2002) writes "While these more general solution methods are themselves harder to understand, they shift the burden of analysis from the individual economist/model-solver toward the computer, and are therefore useful."

Propositions, I subsequently illustrate how the method works using four examples, including problems in continuous-time, and regime switching models. Lastly, I compare the performance of the algorithm to existing solution methods, and find that the proposed method is 10-20 times faster for large-scale problems. Thus, the method may provide useful in addressing problems such as those considered in Reiter (2009).

2 Method

The model of interest is given by

$$Ax_{t-1} + Bx_t + CE_t x_{t+1} + u_t = 0, \tag{1}$$

where x_t is an $n \times 1$ vector containing endogenous and exogenous variables, u_t is an $n \times 1$ vector of mean-zero disturbances, and A, B and C are conformable matrices.

We are interested in a recursive solution of the type $x_t = Fx_{t-1} + Qu_t$. Inserting this rule into equation (1) gives,

$$Ax_{t-1} + Bx_t + CFx_t + u_t = 0.$$

Thus, the matrix Q is given by

$$Q = -(B + CF)^{-1}, \tag{2}$$

and F must be the solution to the quadratic matrix equation (cf. Uhlig (2001))

$$A + BF + CF^2 = 0. \tag{3}$$

If F is known, finding Q is a trivial operation following equation (2). From hereon, the focus of this paper is therefore on solving equation (3).

A solution to equation (3) is called a solvent.

Definition 1. *A solvent* S_1 *of* (3) *is a dominant solvent if* $\lambda(S_1) = \{\lambda_1, \ldots, \lambda_n\}$ *and* $|\lambda_n| > |\lambda_{n+1}|$ *where the eigenvalues are ordered according to*

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{2n}|.$$

A solvent S_2 *of* (3) *is a minimal solvent if* $\lambda(S_2) = \{\lambda_{n+1}, \ldots, \lambda_{2n}\}$ *and* $|\lambda_n| > |\lambda_{n+1}|$.

Theorem 1. *(Higham and Kim (2000), Theorem 6) Suppose that the Eigenvalues that solves*

$$(A + B\lambda + C\lambda^2)x = 0, \tag{4}$$

satisfy the ordering and conditions of Definition 1, and that corresponding to $\{\lambda_i\}_{i=1}^{n}$ *and* $\{\lambda_i\}_{i=n+1}^{2n}$ *there are two sets of linearly independent eigenvectors* $\{v_1, \ldots, v_n\}$ *and* $\{v_{n+1}, \ldots, v_{2n}\}$*, then there exist a dominant,* S_1*, and a minimal solvent,* S_2*, of* (3)*.*

Theorem 2. *(Gohberg, Lancaster and Rodman (1982), Theorem 4.1) If a dominant and a minimal solvent exist they are unique.*

2.1 Time iteration

The main equation of interest is given by[3]

$$Ax_{t-1} + Bx_t + Cx_{t+1} = 0.$$

Suppose we have a candidate solvent, F_n. Then consider solving the following problem for x_t

$$Ax_{t-1} + Bx_t + CF_n x_t = 0.$$

The solution is trivially given by

$$x_t = -(B + CF_n)^{-1} A x_{t-1}.$$

Thus, we will therefore update our guess, F_{n+1}, as[4]

$$F_{n+1} = -(B + CF_n)^{-1} A.$$

In order to prove that this is a convergent procedure, we will need the following lemma.

Lemma 1. *Let* Z_1 *and* Z_2 *be square matrices such that*

$$\min\{|\lambda| : \lambda \in \lambda(Z_1)\} > \max\{|\lambda| : \lambda \in \lambda(Z_2)\}.$$

[3]Recall that it is sufficient to solve the deterministic part of the problem in order to retrieve F. The matrix Q the follows straightforwardly from equation (2).

[4]Binder and Pesaran (1995), footnote 26, propose a similar iterative procedure, but do not provide any proof of convergence, nor conditions for stability/uniqueness.

Then Z_1 is nonsingular and for any matrix norm

$$\lim_{i\to\infty} \|Z_2^i\| \|Z_1^{-i}\| = 0$$

Proof. Gohberg et al. (1982), Lemma 4.9 □

Proposition 1. *Suppose a dominant, S_1, and a minimal solvent, S_2, exist for equation* (3)*. Then, if the matrix $(B+CF_n)$ is nonsingular for all n, the sequence $\{F_n\}_{n=0}^{\infty}$ defined as*

$$F_{n+1} = (B+CF_n)^{-1}(-A), \quad F_0 \neq S_1, \tag{5}$$

converges to S_2.

Proof. In Appendix A. □

The proof of Proposition 1 follows Higham and Kim (2000) quite closely, with some tweaks to fit the purposes of this paper.

The procedure is guaranteed to converge the minimal solvent, S_2. If all the eigenvalues of S_2 is less than one in absolute value, S_2 is indeed a (stable) solution to the quadratic matrix equation. Thus a stable solution exist. However, that does not necessarily imply that S_2 is the *only* stable solution. Indeed, the dominant solvent S_1 – or some other solvent – may also be stable. To verify that this is not the case, the proposition below provides an iterative procedure to find S_1^{-1}. Thus, if the eigenvalues of S_1^{-1} are smaller than one in absolute value, the eigenvaluea of S_1 is greater than one in absolute value, and S_1 is not a stable solution.

Proposition 2. *Suppose a dominant, S_1, and a minimal solvent, S_2, exist. Then, if the matrix $(B+A\hat{F}_n)$ is nonsingular for all n, the sequence $\{\hat{F}_n\}$, defined by*

$$\hat{F}_{n+1} = (B+A\hat{F}_n)^{-1}(-C), \tag{6}$$

converges to S_1^{-1}.

Proof. In Appendix A. □

Thus, given the (inverse) of a dominant, S_1^{-1}, and minimal, S_2, solvent to equation (3) if

(i) If $\max|\lambda(S_2)| < 1$, and $\max|\lambda(S_1^{-1})| < 1$, S_2 is the unique stable solution to equation (3).

(ii) If $\max|\lambda(S_2)| > 1$ there exist no stable solution.

(iii) If $\max|\lambda(S_1^{-1})| > 1$ there are multiple stable solutions.

2.1.1 Singular solvents

One particularly pertinent issue that arises is to which extent the above method can deal with situations in which solvents contain eigenvalues that are zero. It is straightforward to verify that a dominant solvent cannot have those properties, as Definition 1 would not be satisfied. Unfortunately, this is a quite common feature of dynamic systems that contain static relationships, such as accounting identities or the first order conditions of static optimization problems. Fortunately, there is a straightforward workaround to this issue.

Consider the (modified) quadratic matrix equation

$$\hat{A}S^2 + \hat{B}S + \hat{C} = 0, \tag{7}$$

with

$$\hat{A} = C\mu^2 + B\mu + A, \quad \hat{B} = B + C2\mu, \quad \hat{C} = C,$$

where μ represents a small positive real number multiplied by a conformable identity matrix. It is straightforward to verify that if S satisfies equation (7), then $F = S^{-1} + \mu$ satisfies the original quadratic matrix equation (here repeated for exposition)

$$A + BF + CF^2 = 0. \tag{8}$$

Conversely, any F which satisfies the quadratic matrix equation (8), then $S = (F - \mu)^{-1}$ satisfies equation (7). It is worth pointing out here that S is indeed nonsingular unless μ happens to contain an eigenvalue of F. Ruling this scenario out, S is invertible and has no eigenvalues equal to zero.

Proposition 3. *Suppose for some μ equation* (7) *has dominant, S_1, and a minimal solvent, S_2, such that $|\lambda_i(S_1)| > 1/M$ and $|\lambda_i(S_2)| < 1/M$. If $\mu < M - |\lambda_i(S_1^{-1})|$, for $i = 1, \ldots, n$, then for $F = (S_1^{-1} + \mu)$*

(i) *F solves the quadratic matrix equation* (8).

(ii) *$|\lambda_i(F)| < M$, for $i = 1, \ldots, n$.*

(iii) *F is the unique solvent satisfying* (i) *and* (ii).

Proof. Part (i). Inserting $F = S_1^{-1} + \mu$ into the left-hand side of equation (8) gives

$$\begin{aligned} A + B(S_1^{-1} + \mu) + C(S_1^{-2} + 2S_1\mu + \mu^2) & \\ &= (A + B\mu + C\mu^2) + (B + 2\mu)S_1^{-1} + CS_1^{-2} \\ &= (A + B\mu + C\mu^2)S_1^2 + (B + 2\mu)S_1 + C \\ &= 0. \end{aligned}$$

Part (ii). Since $|\lambda_i(S_1^{-1})| < M$ for each $i = 1, \ldots, n$, we have that

$$\begin{aligned} |\lambda_i(F)| &= |\lambda_i(S_1^{-1}) + \mu| \\ &\leq |\lambda_i(S_1^{-1})| + \mu \\ &< |\lambda_i(S_1^{-1})| + (M - |\lambda_i(S_1^{-1})|) \\ &= M \end{aligned}$$

Part (iii). Suppose there exist an $\hat{F} \neq F$ which solves equation (8) such that $|\lambda_i(\hat{F})| < M$ for $i = 1, \ldots, n$. Then $\hat{S} = (\hat{F} - \mu)^{-1}$ solves equation (7). In addition, for any $\mu > 0$ such that $\mu < M - |\lambda_i(\hat{F})|$ for $i = 1, \ldots, n$ we have

$$\begin{aligned} |\lambda_i(\hat{S}^{-1})| &= |\lambda_i(\hat{F}) - \mu| \\ &\leq |\lambda_i(\hat{F})| + \mu \\ &< |\lambda_i(\hat{F})| + (M - |\lambda_i(\hat{F})|) \\ &= M. \end{aligned}$$

As a consequence $|\lambda_i(\hat{S})| > 1/M$ for $i = 1, \ldots, n$, which contradicts that S_1 is the unique dominant solvent to equation (7). Hence, F is unique. □

In practice, Proposition 3 suggests to pick a (small) value μ; solve equation (7) using Propositions 1 and 2; and then evaluate the interval $\mathscr{M} = (\max|\lambda(S_2)|, \min|\lambda(S_1)|)$, to obtain the admissible values of $1/M \in \mathscr{M}$. In addition, it ought to be noted that, for practical purposes, there is no harm done in setting μ equal to a very small number, as the convergence properties appear to improve with a lower value of μ (see the example in Section 3.2.1 below).

Corollary 1. *If* (7) *has dominant, S_1, and a minimal solvent, S_2, such that $|\lambda_i(S_1)| > 1$ and $|\lambda_i(S_2)| < 1$, with $0 < \mu < 1 - |\lambda_i(S_1^{-1})|$, for $i = 1, \ldots, n$. Then, $F = S_1^{-1} + \mu$ is the unique stable solvent to equation* (8).

While Proposition 3 provides useful guidance on how to find and characterize solvents that are smaller or larger than a certain threshold in *absolute* value, it is far less useful to find and characterize

solvents that are smaller or larger than a certain threshold on the real line. As stability conditions for continuous time problem regularly require that all eigenvalues are negative, Proposition 4 below provides a solution to this issue.

Proposition 4. *Suppose for some $\mu < 0$ equation* (7) *has dominant, S_1, and a minimal solvent, S_2. If $\lambda_i(S_1) \geq -1/\mu$ and $0 \leq \lambda_i(S_2) < -1/\mu$, for $i = 1,\ldots,n$, then for $F = (S_1^{-1} + \mu)$*

(i) *F solves the quadratic matrix equation* (8).

(ii) *$\lambda_i(F) \leq 0$, for $i = 1,\ldots,n$.*

(iii) *F is the unique solvent satisfying* (i) *and* (ii).

Proof. Part (i). See the associated proof for Proposition 3.

Part (ii). Since $\lambda_i(S_1^{-1}) < -\mu$ for each $i = 1,\ldots,n$, we have that

$$\begin{aligned}\lambda_i(F) &= \lambda_i(S_1^{-1}) + \mu \\ &\leq -\mu + \mu \\ &= 0.\end{aligned}$$

Part (iii). Suppose there exist an $\hat{F} \neq F$ with $\lambda_i(\hat{F}) \leq 0$ for all $i = 1,\ldots,n$ which solves equation (8). Then $S = (\hat{F} - \mu)^{-1}$ is a solvent for equation (7), with $\lambda_i(S) \in \{\lambda : \lambda < 0, \text{ or } \lambda \geq -1/\mu\}$, for all $i = 1,\ldots,n$. Suppose that $\lambda_i(S) < 0$ for some $i = 1,\ldots,n$, then $\lambda(S)$, $\lambda(S_1)$, and, $\lambda(S_2)$ all constitute eigenvalues to the quadratic eigenvalue problem in (4), with the total number of eigenvalues exceeding $2n$, which is an impossibility. Thus $\lambda_i(S) \geq -1/\mu$, for all $i = 1,\ldots,n$. However, if $\lambda_i(S) \geq -1/\mu$ for all $i = 1,\ldots,n$ this contradicts that S_1 is unique. Hence, there can not exist an $\hat{F} \neq F$ with $\lambda_i(\hat{F}) \leq 0$ for all $i = 1,\ldots,n$. □

The ideas underlying Proposition 3 and 4 are similar but operate in opposite directions. Proposition 3 requires a perturbation of the system by a sufficiently small amount to ensure that a dominant solvent exists, while at the same time leaving the system sufficiently unperturbed to allow for inference. Conversely, Proposition 4 suggests to perturb the system sufficiently to shift all eigenvalues that were negative in the original system to be the smallest positive roots of the modified system.

3 Examples

This section provides five examples of how to use the proposed method. The first is intended to show the convergence properties of both S_2 and S_1^{-1} in the simplest possible case. The second, to illustrate how to deal with singular solvents using Proposition 3. The third illustrates a model set in

continuous time using Proposition 4. The fourth considers a regime switching model. And the fifth is intended to show the large-scale properties of the method.

Convergence is assumed to be obtained once the absolute maximum error of the quadratic matrix equation is smaller than $1e(-12)$, which is close to double-precision machine epsilon.

3.1 A second-order difference equation.

Consider the following second-order difference equation

$$0.75x_{t-1} - 2x_t + x_{t+1} = 0.$$

The associated quadratic equation is

$$0.75 - 2F + F^2 = 0,$$

with dominant, S_1, and minimal solvent, S_2, equal to 1.5 and 0.5, respectively.

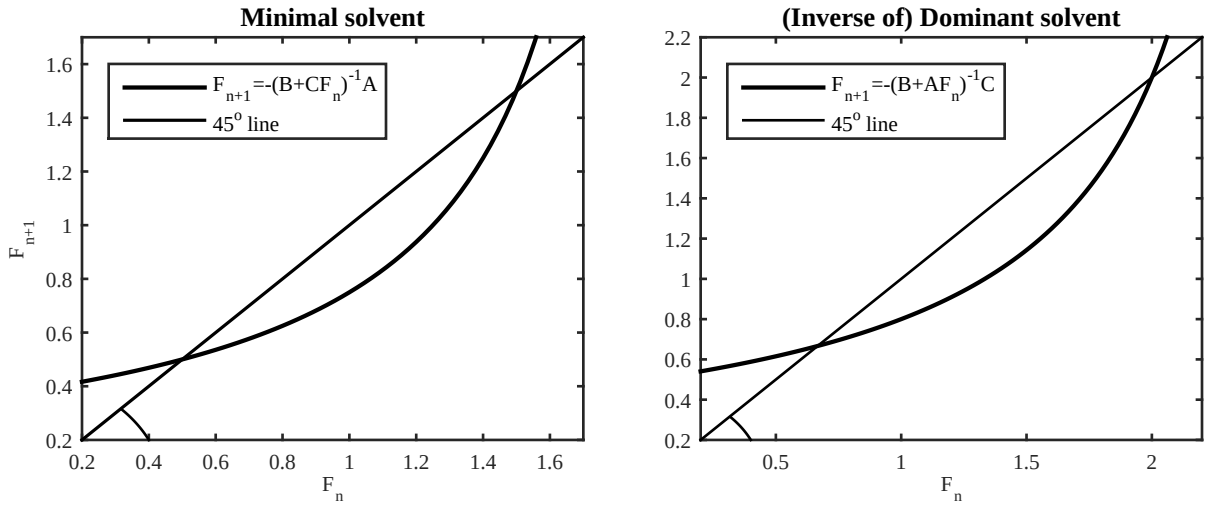

Figure 1: Mapping of F_n to F_{n+1}.

Figure 1 illustrates the mappings of the iterations $F_{n+1} = -0.75/(-2+F_n)$ (left graph) and $F_{n+1} = -1/(-2+0.75F_n)$ (right graph). From the left graph it is clear that for any guess F_0 such that $F_0 < 1.5$, the sequence F_n converges to the minimal solvent 0.5. What is less obvious is that for any $F_n > 1.5$ the resulting sequence also converges to 0.5.[5] The reason is that for any $F_n > 2$ it follows that $F_{n+1} < 0$, which guarantees convergence. Numerically, even at the singularity that arises at $F_n = 2$ is not a problem either, since most numerical programs interpret $1/0$ as infinity and $1/\infty$ as zero. Thus, even for an initial guess of $F_0 = 2$, the procedure converges to the minimal solvent.

[5]Notice that $\lim_{|F|\to\infty} -0.75/(-2+F) = 0$.

The only initial guess for which convergence (towards the minimal solvent) does not occur is if $F_0 = 1.5$; that is, if the initial guess for the minimal solvent is equal to the dominant solvent.

The right graph illustrates the convergence properties of the inverse of the dominant solvent. For any $F_0 < 2$, the sequence F_n converges to the inverse of the dominant solvent, $F = 2/3$. In addition, for any $F_n > 2.67$, $F_{n+1} < 0$ and convergence follows. Again, the only initial guess for which convergence (towards the inverse of the dominant solvent) does not occur is if $F_0 = 2$; that is, if the initial guess for the inverse of the dominant solvent is equal to the inverse of the minimal solvent.

3.2 Singular solvents

Consider a variation of the previous problem

$$\begin{aligned} 0.75y_t - 0.5y_{t+1} &= 0, \\ -2x_t + x_{t-1} - y_t &= 0, \end{aligned}$$

or,

$$AX_{t-1} + BX_t + CX_{t+1} = 0,$$

with $X_t = (y_t, x_t)'$, and

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.75 & 0 \\ -1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} -0.5 & 0 \\ 0 & 0 \end{pmatrix}.$$

There are three solvents to this problem

$$S_1 = \begin{pmatrix} 0 & -2 \\ 0 & 1.5 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1.5 & 0 \\ -0.75 & 0.5 \end{pmatrix},$$

with eigenvalues of $(0, 1.5)$, $(0, 0.5)$, and $(0.5, 1.5)$, respectively. Thus, S_1 and S_3 represent the unstable solutions, and S_2 the (unique) stable solution.

It would be tempting to refer to S_1 as the dominant solvent and S_2 as the minimal solvent, but this would be incorrect; both the dominant and minimal solvent have a zero eigenvalue, and do therefore not satisfy Definition 1. Hence, neither Proposition 1 nor 2 are valid. However, it may still be interesting to conduct the iterations suggested in Proposition 1 and 2 in order to explore what they may deliver.

The resulting "inverse" of the "dominant" solvent, S_1^{-1} and, "minimal" solvent, S_2 are

$$S_1^{-1} = \begin{pmatrix} 0.667 & 0 \\ -0.5 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix},$$

with nonzero eigenvalues of 0.667 and 0.5, respectively. Thus, the nonzero eigenvalue of the inverse of the "dominant" solvent corresponds exactly with the inverse of the nonzero eigenvalue of the unstable solution S_1 above; and the nonzero eigenvalue of the "minimal" solvent corresponds exactly with the nonzero eigenvalue of the stable solution – indeed, the procedure correctly identifies the stable solution S_2 and also suggests that there exist another solution which is unstable. Furthermore, inspecting S_1^{-1} suggests that the unstable solution satisfies

$$\begin{aligned} y_{t-1} &= 0.667 y_t \\ x_{t-1} &= -0.5 y_t. \end{aligned}$$

Thus, "inverting" these policy functions gives

$$\begin{aligned} y_t &= -2 x_{t-1} \\ x_t &= 1.5 x_{t-1}, \end{aligned}$$

which exactly replicates the unstable solution S_1 above.

Is this a coincidence that is particular to this example? My experience tells me that it is not; for a wide range of models and calibrations the procedure delivers a stable solvent, together with a matrix, S_1^{-1}, containing the coefficients of the inverse of the unstable policy function.[6] However, acknowledging that not every reader may be as convinced by "my experience" as I am, we can invoke Proposition 3 to address this issue.

3.2.1 Using Proposition 3

In order to exploit the results in Proposition 3, we need to pick a value for μ. Setting μ, conservatively, equal to 0.1 gives rise to

$$\hat{A} = \begin{pmatrix} 0.07 & 0 \\ -0.1 & 0.8 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0.65 & 0 \\ -1 & -2 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} -0.5 & 0 \\ 0 & 0 \end{pmatrix},$$

[6] Provided, of course, that a stable and unstable solution exist. If there is no stable solution – or if both solvents are stable – the procedure still delivers a solvent with the smallest nonzero eigenvalues in absolute value, and a matrix containing the inverse of a solvent with larger eigenvalues in absolute value, provided that the smallest nonzero eigenvalues of the latter strictly exceeds the largest nonzero eigenvalue of the former. Only if these eigenvalues happen to coincide have I run into convergence issues.

and Proposition 3 delivers the solvents

$$S_1^{-1} = \begin{pmatrix} -0.1 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0.71 & 0 \\ -0.54 & 0 \end{pmatrix},$$

with associated eigenvalues

$$|\lambda(S_1)| = \begin{pmatrix} 10 \\ 2.5 \end{pmatrix}, \quad |\lambda(S_2)| = \begin{pmatrix} 0.71 \\ 0 \end{pmatrix}.$$

Thus S_1 is indeed the dominant solvent and S_2 is the minimal solvent. Since $\mu < 1 - 1/2.5$, all the conditions of Proposition 3 are met, and $F = S_1^{-1} + \mu$ is the unique stable solution.

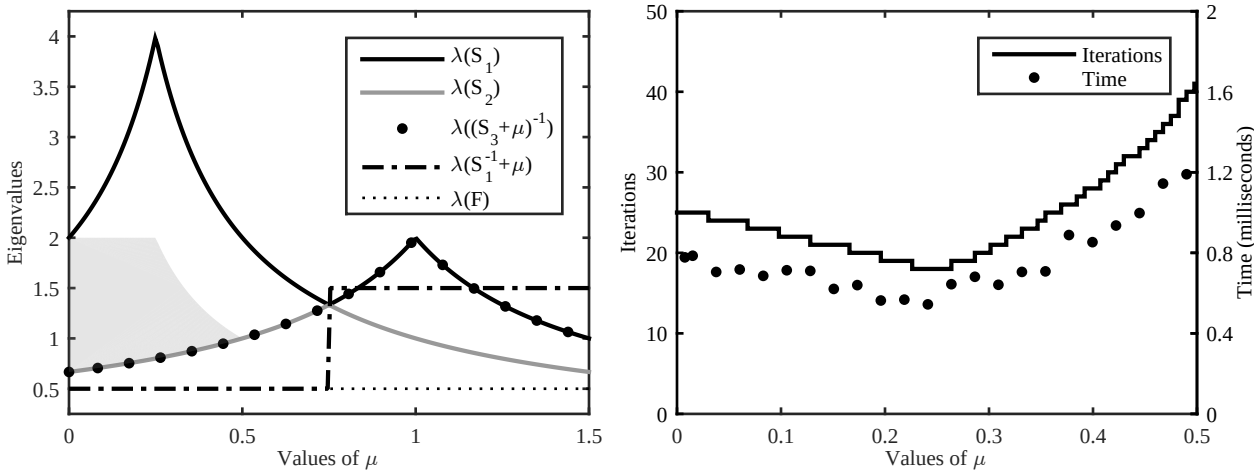

Figure 2: The relationship between μ and eigenvalues.

Notes. (Left panel) The black solid line shows the smallest eigenvalue in absolute value of the resulting dominant solvent, S_1; the grey solid line the largest eigenvalue in absolute value of the resulting minimal solvent, S_2; the black dashed line the maximum eigenvalue of the solvent $F = S_1^{-1} + \mu$; the fine dotted line the largest eigenvalue in absolute value of the (true) unique stable solution; the black dots marks the eigenvalues of the solvent $(S_3 + \mu)^{-1}$. The shaded area illustrates the set of admissible $(\mu, 1/M)$-pairs such that the conditions of Proposition 3 are met.

Can μ be set at an arbitrarily large value? The answer is no. Following Proposition 3, $\mu < M - |\lambda_i(S_1^{-1})|$, for $i = 1, 2$. That is, μ must be sufficiently small. The reason is that μ perturbs the original system in order to ensure invertibility of $\hat{A}$ and S_1. If the system is perturbed to far away from its original formulation, the procedure may pick up a solvent which is not equal to the unique stable solution (if such a solution exist). Figure 2 illustrates the consequences of setting $\mu \in (0, 1.5]$.

The black solid line shows the smallest eigenvalue in absolute value of the resulting dominant solvent, S_1; the grey solid line shows the largest eigenvalue in absolute value of the resulting minimal

solvent, S_2; the black dashed line illustrates the maximum eigenvalue of the solvent $F = S_1^{-1} + \mu$; the finely dotted line the largest eigenvalue in absolute value of the (true) unique stable solution; and the black dots marks the eigenvalues of the solvent $(S_3 + \mu)^{-1}$, with S_3 defined in section 3.2. Lastly, the shaded area illustrates the set of admissible $(\mu, 1/M)$-pairs such that $|\lambda_i(S_1)| > 1/M$, $|\lambda_i(S_2)| < 1/M$, and $\mu < M - |\lambda_i(S_1^{-1})|$, for $i = 1, 2$; i.e. such that the sufficient conditions stated in Proposition 3 are satisfied.

Obviously, for each admissible $(\mu, 1/M)$-pair the solution $S_1^{-1} + \mu$ coincides with the unique stable solution. In addition, for any value of $\mu < 0.75$ the solution still coincides with the unique stable solution. However, for $\mu > 0.75$, the solution $S_1^{-1} + \mu$ instead corresponds to the solvent S_3 in section 3.2. The reason is straightforward: the eigenvalues to the solvents are given by

$$|\lambda(S_1)| = \begin{pmatrix} \frac{1}{\mu} \\ \frac{1}{|0.5-\mu|} \end{pmatrix}, \quad |\lambda(S_2)| = \begin{pmatrix} \frac{1}{\mu} \\ \frac{1}{|1.5-\mu|} \end{pmatrix}, \quad |\lambda(S_3)| = \begin{pmatrix} \frac{1}{|0.5-\mu|} \\ \frac{1}{|1.5-\mu|} \end{pmatrix}.$$

Thus, for any $\mu < 0.75$, $|\lambda_i(S_1)| > |\lambda_i(S_2)|$, but for $\mu > 0.75$, $|\lambda_i(S_3)| > |\lambda_i(S_1)|$. That is, the large value of μ has perturbed the system such that S_3 is picked up as the dominant solvent, and S_1 as the minimal solvent. Thus, this example serves as a cautionary tale of ensuring that the sufficient conditions of Proposition 3 are met.

3.3 Continuous time models

Consider the continuous time problem

$$\begin{aligned} 1.19x_t + y_t - \dot{y}_t &= 0, \\ -1.4x_t - y_t - \dot{x}_t &= 0, \end{aligned}$$

or,

$$AX_t + B\dot{X}_t + C\ddot{X}_t = 0,$$

with $\dot{X}_t = (y_t, \dot{x}_t)'$, and

$$A = \begin{pmatrix} 0 & 1.19 \\ 0 & -1.4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

There are three solvents to this problem

$$S_1 = \begin{pmatrix} 0 & -1.7 \\ 0 & 0.3 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & -0.7 \\ 0 & -0.7 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0.15 & -0.85 \\ -0.15 & -0.55 \end{pmatrix},$$

with eigenvalues of $(0, 0.3)$, $(0, -0.7)$, and $(0.3, -0.7)$, respectively. Thus, S_1 and S_3 represent the unstable solutions, and S_2 the (unique) stable solution. Since the eigenvalues of S_1 are smaller in absolute value than both S_2 and S_3, there is no hope that Propositions 1 and 2 will successfully identify the unique stable solution. Thus, to solve this problem it is useful to invoke Proposition 4. Figure 3 illustrates the results for values of $\mu \in [-1.5, 0)$.

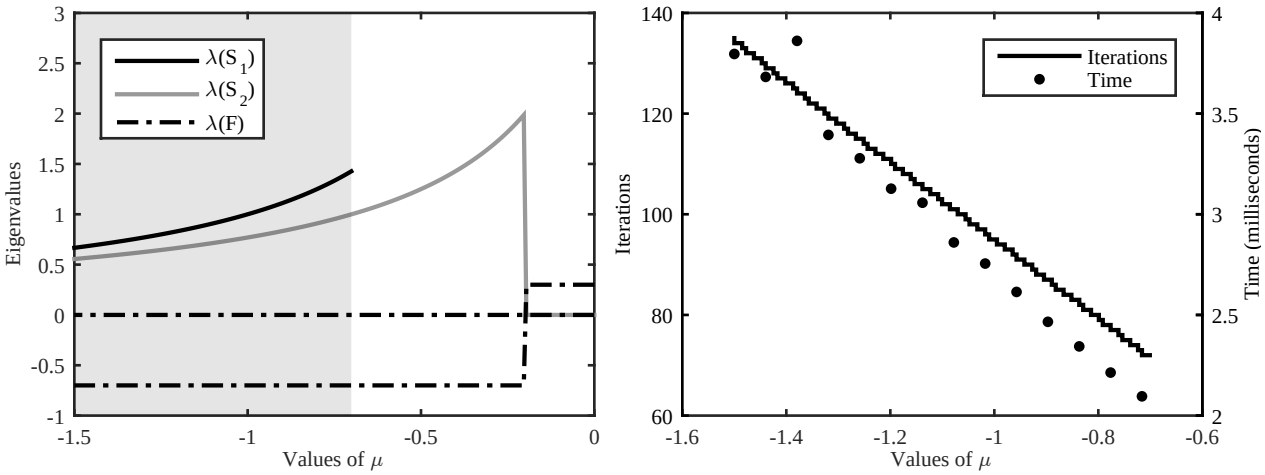

Figure 3: The relationship between μ and eigenvalues.

Notes. (Left panel) The black solid line shows the smallest eigenvalue of the dominant solvent, S_1, such that $\lambda(S_1) \geq -1/\mu$; the grey solid line the largest eigenvalue of the minimal solvent, S_2, such that $0 < \lambda(S_1) < -1/\mu$; the black dashed line the eigenvalues of the solvent $F = S_1^{-1} + \mu$. The shaded area illustrates the set of admissible values of μ such that the conditions of Proposition 4 are met.

The black solid line in the left graph illustrates the smallest eigenvalue of the resulting dominant solvent, S_1, provided that $\lambda_i(S_1) \geq -1/\mu$ for $i = 1, \ldots, n$. Similarly, the grey solid line illustrates the largest eigenvalue of the minimal solvent, S_2. The dashed line shows both eigenvalues of $F = S_1^{-1} + \mu$. For values of $\mu \in [-1.5, -0.7)$ the conditions of Proposition 4 are met, and the procedure indeed identifies the unique stable solution.

Can μ be set too low? No, the lower the value of μ is, the more likely it is that the results align with the conditions of Proposition 4. However, as can be seen from Figure 3, a lower value of μ also implies that the eigenvalues of both S_1 and S_2 are closer together, which slows down the procedure and increases the required number of iterations. Thus, while a low value of μ increases the chance

of Proposition 4 being valid, it comes at the cost of slowing down the procedure. This is a drawback of the method proposed in this paper.

3.4 Regime switching models.

Eggertsson (2011) considers a model which in "normal times" satisfies the system

$$\begin{aligned} y_t &= y_{t+1} - \sigma(i_t - \pi_{t+1}), \\ \pi_t &= \kappa y_t + \beta \pi_{t+1}, \\ i_t &= \phi_\pi \pi_t + \phi_y y_t, \end{aligned}$$

but in "crises times" is instead characterized by

$$\begin{aligned} y_t &= E[y_{t+1}] + \sigma(E[\pi_{t+1}]) + (g_t - E[g_{t+1}]), \\ \pi_t &= \kappa y_t + \kappa\psi(-\sigma^{-1} g_t) + \beta E[\pi_{t+1}], \\ i_t &= 0. \end{aligned}$$

Eggertsson (2011) assumes that in some period, s, the economy (unexpectedly) enters a crisis, and is therefore described by the latter system. With probability $(1-q)$, however, the economy recovers in the subsequent period, and becomes characterised by the first system. If not, the economy remains in the crisis in period $s+1$, and with (conditional) probability $(1-q)$ instead recovers in period $s+2$, and so on. Once the economy has recovered, it will remain in the normal state for perpetuity. Thus, in period s the economy enters a crisis with expected duration $1/(1-q)$. Furthermore, Eggertsson (2011) assumes – in the benchmark exploration – that government spending increases simultaneously with the crisis, with an identical (stochastic) duration. Thus, in period s, g_s increases by some amount, and remains high for the duration of the liquidity trap.

Thus, the above equations can represented as the regime switching system

$$A^c X^c_{t-1} + B^c X^c_t + qC^c X^c_{t+1} + (1-q)CX_{t+1} = 0, \tag{9}$$

$$AX_{t-1} + BX_t + CX_{t+1} = 0, \tag{10}$$

with with $X_t = (y_t, \pi_t, i_t, g_t)'$, and

$$A^c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B^c = \begin{pmatrix} -1 & 0 & 0 & 1 \\ \kappa & -1 & 0 & -\kappa\psi\sigma^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C^c = \begin{pmatrix} 1 & \sigma & 0 & -1 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & -\sigma & 0 \\ \kappa & -1 & 0 & -\kappa\psi\sigma^{-1} \\ \phi_y & \phi_\pi & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \sigma & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

To solve system (9)-(10), we begin by solving (10) using Proposition 3 to find F such that[7]

$$A + BF + CF^2 = 0.$$

Subsequently, define

$$\hat{A}^c = A^c + B^c\mu + qC^c\mu^2 + (1-q)CF\mu, \quad \hat{B}^c = B^c + qC^c 2\mu + (1-q)CF, \quad \hat{C}^c = qC^c,$$

and again invoke Proposition 3 to find the dominant solvent, S_c, of

$$\hat{A}^c S_c^2 + \hat{B}^c S_c + \hat{C}^c = 0,$$

and uncover the solvent of (9) as $F_c = S_c^{-1} + \mu$. Using the calibrated values of the parameters from Eggertsson (2011) reveals that F_c is the unique dominant solvent and given by[8]

$$F_c = \begin{pmatrix} 0 & 0 & 0 & 2.29 \\ 0 & 0 & 0 & 0.16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which suggests a fiscal multiplier of 2.29.

Figure 4 illustrates how the fiscal multiplier relates to the expected duration of the crisis, $1/(1-q)$. As can be seen from the figure, the fiscal multiplier is moderate at short duration, and reaches arbitrarily large values as the duration approaches (approximately) 12 quarters. For values of $q > 0.919$ no solution can be found.

[7] Notice that in this particular example, the stable solution F is simply a 4×4 matrix of zeros.

[8] The values are: $\sigma = 1/1.16$, $\beta = 0.997$, $\psi = 0.3664$, and $\kappa = 0.0086$. Stability is verified slightly differently in regime switching models, but in this particular case the eigenvalues of the "explosive solvent" *multiplied* by q must be greater than one (see Farmer, Waggoner and Zha (2009)).

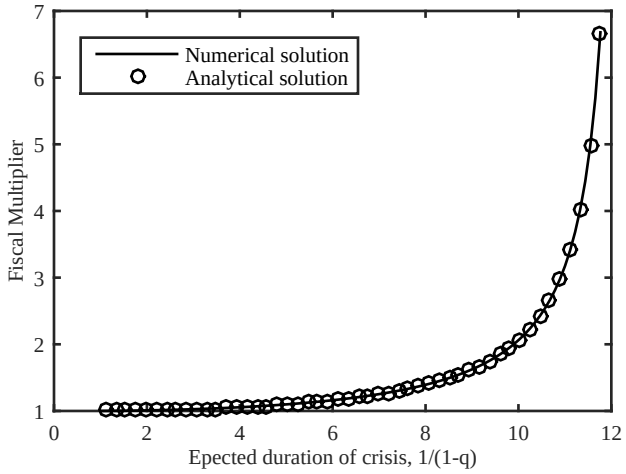

Figure 4: The relationship between q and the fiscal multiplier.

Notes. The multiplier increases exponentially in q and eventually reaches infinity. The analytical solution is obtained by evaluating equation (30) in Eggertsson (2011).

3.5 A large-scale problem

To explore the potential virtues of the proposed method for large-scale problems, I follow Higham and Kim (2000) and Tisseur (2000) and consider the $n \times n$ quadratic matrix equation with

$$A = \begin{pmatrix} 15 & -5 & & & & \\ -5 & 15 & -5 & & & \\ & -5 & & \ddots & & \\ & & \ddots & \ddots & & \\ & & & \ddots & & -5 \\ & & & & -5 & 15 \end{pmatrix}, \quad B = \begin{pmatrix} 20 & -10 & & & & \\ -10 & 30 & -10 & & & \\ & -10 & 30 & -10 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 30 & -10 \\ & & & & -10 & 20 \end{pmatrix},$$

and with C equal to the conformable identity matrix.

I then compare the execution time for using the iterations in both Propositions 1 and 2, and compare them with two popular alternative approaches by Klein (2000) and Sims (2002). The results are summarized in Table 1 below.

As can be seen from the table, the proposed method significantly outperforms both Klein's (2000) and Sims's (2002) methods for $n \geq 100$. In fact, the method is about 20 times faster for very large-scale problems, such as $n = 2,000$. Given that computing time falls from around 20 minutes to 1 minute for such problems, the proposed method appears to dominate existing methods for problems as those in Reiter (2009).

Table 1: Speed comparison of algorithms.

Method	$n =$	10	100	500	1,000	2,000
Klein (2000)		0.7	8.89	13.81	8.45	20.71
Sims (2002)		1.13	10.56	17.23	8.9	21.05
Time *(seconds)*		0.0007	0.0104	0.8423	15.0439	64.4173

Notes. The reported numbers in the two first rows represent the computational speed of the indicated algorithm *relative* to that of time iteration. To make the comparison as fair as possible, I use the authors' own codes. Computational speed is measured by the "timeit" function in Matlab, which evaluates a function several times and reports the median results. The last line reports the time in seconds solving the problem using time iteration.

4 Concluding remarks

This paper has proposed a new approach to solve linear rational expectation models, and provided conditions under which a stable solution can be ascertained to be unique. The main advantages of the proposed method relative to existing methods is its transparency and ease of implementation without relying on pre-canned routines, in addition to significant efficiency advantages for large-scale problems.

A Proofs

A.1 Proof of Proposition 1

The recursion in (5) can be rewritten as

$$A + BF_{n+1} + CF_nF_{n+1} = 0.$$

Conjecture that F_n is given by

$$F_n = (S_1^{-(n-1)}\Omega S_2^n + S_2)(S_1^{-n}\Omega S_2^n + I)^{-1}.$$

Then,

$$\begin{aligned}
F_nF_{n+1} &= (S_1^{-(n-1)}\Omega S_2^n + S_2)(S_1^{-n}\Omega S_2^n + I)^{-1}(S_1^{-n}\Omega S_2^{n+1} + S_2)(S_1^{-(n+1)}\Omega S_2^{n+1} + I)^{-1} \\
&= (S_1^{-(n-1)}\Omega S_2^n + S_2)(S_1^{-n}\Omega S_2^n + I)^{-1}(S_1^{-n}\Omega S_2^n + I)S_2(S_1^{-(n+1)}\Omega S_2^{n+1} + I)^{-1} \\
&= (S_1^{-(n-1)}\Omega S_2^{n+1} + S_2^2)(S_1^{-(n+1)}\Omega S_2^{n+1} + I)^{-1}.
\end{aligned}$$

Inserting this expression into the recursion gives

$$A + B(S_1^{-n}\Omega S_2^{n+1} + S_2)(S_1^{-(n+1)}\Omega S_2^{n+1} + I)^{-1} + C(S_1^{-(n-1)}\Omega S_2^{n+1} + S_2^2)(S_1^{-(n+1)}\Omega S_2^{n+1} + I)^{-1} = X_n,$$

where X_n is an unknown. The left hand side of the above equation can alternatively be rewritten as

$$A(S_1^{-(n+1)}\Omega S_2^{n+1} + I) + B(S_1^{-n}\Omega S_2^{n+1} + S_2) + C(S_1^{-(n-1)}\Omega S_2^{n+1} + S_2^2) = X_n(S_1^{-(n+1)}\Omega S_2^{n+1} + I).$$

Since S_2 is a solvent it follows that

$$\begin{aligned}
A(S_1^{-(n+1)}\Omega S_2^{n+1} + I) + B(S_1^{-n}\Omega S_2^{n+1} + S_2) + C(S_1^{-(n-1)}\Omega S_2^{n+1} + S_2^2) & \\
&= AS_1^{-(n+1)}\Omega S_2^{n+1} + BS_1^{-n}\Omega S_2^{n+1} + CS_1^{-(n-1)}\Omega S_2^{n+1} \\
&= (A + BS_1 + CS_1^2)S_1^{-(n+1)}\Omega S_2^{n+1} \\
&= 0,
\end{aligned}$$

where the second to last equality follows from S_1 being a solvent. Thus $X_n = 0$ for all n. This confirms the conjecture.

By Lemma 1 we have that

$$\lim_{n\to\infty} \|S_1^{-n}\|\|S_2^n\| = 0.$$

Thus,

$$\lim_{n\to\infty} F_n = \lim_{n\to\infty} (S_1^{-(n-1)}\Omega S_2^n + S_2)(S_1^{-n}\Omega S_2^n + I)^{-1} = S_2,$$

Lastly, the initial value F_0 is given by

$$F_0 = (S_1\Omega + S_2)(\Omega + I)^{-1}.$$

Setting $\Omega \neq (S_1+I)^{-1} - S_1^{-1}S_2$ is necessary and sufficient to guarantee that $F_0 \neq S_1$, which completes the proof.

A.2 Proof of Proposition 2

The recursion in (6) can be rewritten as

$$A\hat{F}_n\hat{F}_{n+1} + B\hat{F}_{n+1} + C = 0.$$

Conjecture that $\hat{F}_n$ is given by

$$\hat{F}_n = (S_2^n\Omega S_1^{-n} + I)(S_2^{n+1}\Omega S_1^{-n} + S_1)^{-1}.$$

Then,

$$\begin{aligned}
\hat{F}_n\hat{F}_{n+1} &= (S_2^n\Omega S_1^{-n} + I)(S_2^{n+1}\Omega S_1^{-n} + S_1)^{-1}(S_2^{n+1}\Omega S_1^{-(n+1)} + I)(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1} \\
&= (S_2^n\Omega S_1^{-n} + I)(S_2^{n+1}\Omega S_1^{-n} + S_1)^{-1}(S_2^{n+1}\Omega S_1^{-n} + S_1)S_1^{-1}(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1} \\
&= (S_2^n\Omega S_1^{-n} + I)S_1^{-1}(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1}.
\end{aligned}$$

Inserting this expression into the recursion gives

$$A(S_2^n\Omega S_1^{-n} + I)S_1^{-1}(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1} + B(S_2^{n+1}\Omega S_1^{-(n+1)} + I)(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1} + C = X_n,$$

where X_n is an unknown. The left hand side of the above equation can alternatively be rewritten as

$$A(S_2^n\Omega S_1^{-n} + I)S_1^{-1} + B(S_2^{n+1}\Omega S_1^{-(n+1)} + I) + C(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1) = X_n(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1}.$$

Since S_1 is a solvent it follows that

$$AS_1^{-1} + B + CS_1 = (A + BS_1 + CS_1^2)S_1^{-1} = 0.$$

Thus,

$$AS_2^n\Omega S_1^{-n}S_1^{-1} + BS_2^{n+1}\Omega S_1^{-(n+1)} + CS_2^{n+2}\Omega S_1^{-(n+1)} = X_n(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1}.$$

or

$$\begin{aligned}
(A + BS_2 + CS_2^2)S_2^n\Omega S_1^{-(n+1)} &= X_n(S_2^{n+2}\Omega S_1^{-(n+1)} + S_1)^{-1} \\
&= 0,
\end{aligned}$$

which confirms the conjecture. By Lemma 1 we have that

$$\lim_{n\to\infty}\hat{F}_n = \lim_{n\to\infty}(S_2^n\Omega S_1^{-n} + I)(S_2^{n+1}\Omega S_1^{-n} + S_1)^{-1} = S_1^{-1}.$$

Lastly the initial value of $\hat{F}_0$ is given by

$$\hat{F}_0 = (\Omega + I)(S_2\Omega + S_1)^{-1}.$$

Setting $\Omega = I$, gives $\hat{F}_0 = 0$, which completes the proof.

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