Lecture X

Perturbation Methods

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Local Approximations

- DSGE models are characterized by a set of nonlinear equations, some of which are intertemporal Euler equations (stochastic difference equations), others are intratemporal FOCs, others feasibility/budget constraints.
- We want to transform the nonlinear system into a system of linear equations and then use well known techniques from linear RE
- To do this transformation, we use local approximations: the idea is then to obtain a good approximation of the function in a neighborhood of a benchmark point
- Benchmark point needs to have two features: 1) easy to compute, 2) visited often by the dynamic system. Usually, the SS.
- These methods are useful to solve economies with distortions, but with small shocks and no occasionally binding constraints.

Taylor principle

- The starting point of this theory is Taylor's theorem. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a C^{∞} function, then for any x^* in its domain we have

$$\begin{aligned} f(x) &\simeq f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2} f''(x^*)(x - x^*)^2 + ... \\ &+ \frac{1}{n!} f^{(n)}(x^*)(x - x^*)^n + \mathcal{O}_{n+1} \end{aligned}$$

  so the error is asymptotically smaller than any of its terms.

- Taylor's theorem says the RHS polynomial approximation is good near x^*. But how near is near?

- More in general, write the power series expansion of f around x^* :

$$f(x) = \sum_{k=0}^{\infty} \alpha_k (x - x^*)^k \qquad \text{where} \quad \alpha_k = \frac{1}{k!} f^{(k)}(x^*)$$

Taylor principle

- Theorem: Consider the power series $\sum_{k=0}^{\infty} \alpha_k \left(x - x^*\right)^k$. If it converges for x^r, it converges for all $x < x^r$.

- Then, we can define the radius of convergence of the power series as the quantity $r > 0$ s.t.:

$$r = \sup \left\{ |x - x^*| : |\sum_{k=0}^{\infty} \alpha_k \left(x - x^*\right)^k | < \infty \right\}$$

  r provides maximal radius around x^* for which series converges.

- Definition. A function $f : \Omega \rightarrow C$ is said to be analytic on the open set Ω if, for every $x^* \in \Omega$, there exists a sequence $\{\alpha_k\}$ and a radius $r > 0$ such that:

$$f(x) = \sum_{k=0}^{\infty} \alpha_k \left(x - x^*\right)^k \text{ for } |x - x^*| < r$$

Taylor principle

- f analytic $\leftrightarrow$ locally given by a convergent power series
- If f is analytic at x^*, then f has continuous derivatives of all orders at x^* (not too surprising...)
- A point where f is not analytic is called a singularity of f. That's where the local approximation goes berserk...
- Example: tangent function with power series representation:

$$\tan(x) = \frac{\sin x}{\cos x} = \frac{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2k!}}{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

but the function has a singularity at $x^0 = \pi/2$ for which $\cos(x) = 0$.

Taylor principle

- Then, we can state an important theorem for local approximations.

- Theorem: Let f be analytic at x^*. If f or any derivative of f exhibits a singularity at x^0, then the radius of convergence of the Taylor series expansion of f in the neighborhood of x^* is bounded above by $|x^0 - x^*|$.

- This theorem gives us a guideline to use Taylor series expansions, as it actually tells us that the series at x^* cannot deliver any accurate, and therefore reliable, approximation for f at any point farther away from x^* than any singular point of f.

Taylor principle

- Example I: $\log(1-x)$ with singularity at 1.

x	$\log(1-x)$	Taylor$_{100}$
0.9	-2.3026	-2.3026
0.99	-4.6052	-4.3894
0.999	-9.2103	-5.1774

- Example II: production function $y = k^{\alpha}$ with $\alpha \in (0,1)$ and steady-state $k^* = 0.5$.
- The function has a singularity at $x^0 = 0$ (first derivative), and hence the Taylor expansion has a radius bounded above by 0.5.
- Bottom line: when you linearize, think about the radius of convergence (calibration-specific)

Linearization of FOCs

- Consider the standard deterministic growth model:

$$\begin{aligned} k_{t+1} - k_t^{\alpha} + c_t - (1-\delta)\,k_t &= 0 \\ \frac{1}{c_t} - \beta\frac{1}{c_{t+1}}\left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right) &= 0 \end{aligned}$$

- Let $\hat{k}_t \equiv k_t - k^*$ and $\hat{c} \equiv c_t - c^*$ be deviations from SS
- The feasibility constraint is an equation $g\left(k_{t+1}, k_t, c_t\right) = 0$:

$$\begin{aligned} g\left(k_{t+1}, k_t, c_t\right) &= g^* + g_1^*\hat{k}_{t+1} + g_2^*\hat{k}_t + g_3^*\hat{c}_t \\ &= \hat{k}_{t+1} - \left[\alpha\left(k^*\right)^{\alpha-1} + (1-\delta)\right]\hat{k}_t + \hat{c} \end{aligned}$$

since $g^* = 0$ (SS equilibrium condition)

Linearization of Euler equation

$$\frac{1}{c_t} - \beta \frac{1}{c_{t+1}} \left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta \right) = 0$$

- The FOC is an equation $g\left(c_t, c_{t+1}, k_{t+1}\right) = 0$:

$$0 = g^* + g_1^* \hat{c}_t + g_2^* \hat{c}_{t+1} + g_3^* \hat{k}_{t+1}$$

$$0 = -\frac{1}{\left(c^*\right)^2} \hat{c}_t + \beta \frac{\left[\alpha \left(k^*\right)^{\alpha-1} + 1 - \delta\right]}{\left(c^*\right)^2} \hat{c}_{t+1} - \beta \frac{\alpha \left(\alpha - 1\right) \left(k^*\right)^{\alpha-2}}{c^*} \hat{k}_{t+1}$$

- Simplifying

$$0 = -\hat{c}_t + \beta \left[\alpha \left(k^*\right)^{\alpha-1} + 1 - \delta\right] \hat{c}_{t+1} - \beta\alpha \left(\alpha - 1\right) \left(\frac{c^*}{k^*}\right) \left(k^*\right)^{\alpha-1} \hat{k}_{t+1}$$

Log-linearization of FOCs

- Another common practice is to take a log-linear approximation to the equilibrium.
- It delivers a natural interpretation of the coefficients in front of the variables: these can be interpreted as elasticities.
- Define log deviations from SS:

$$\hat{x} = \log(x) - \log(x^*)$$

and linearize around $\log(x^*)$

- First order Taylor expansion around $\log(x^*)$

$$g(x) \simeq g(x^*) + g_x(\exp(\log x^*))\exp(\log x^*)\,\hat{x} = g(x^*) + g_x^* x^* \hat{x}$$

Log-linearization of feasibility constraint

$$g\left(k_{t+1}, k_t, c_t\right) = k_{t+1} - k_t^{\alpha} + c_t - (1-\delta)\, k_t = 0$$

- Log-linearization:

$$\begin{aligned} g\left(k_{t+1}, k_t, c_t\right) &= g^* + g_1 k^* \hat{k}_{t+1} + g_2 k^* \hat{k}_t + g_3 c^* \hat{c}_t \\ &= k^* \hat{k}_{t+1} - \alpha \left(k^*\right)^{\alpha} \hat{k}_t + c^* \hat{c}_t - (1-\delta)\, k^* \hat{k}_t \end{aligned}$$

Alternative approach

There is a simple way to do it w/o computing derivatives based on three building blocks:

1. Log approximation:

$$ax = ax^* \exp(\log x - \log x^*) = ax^* \exp(\hat{x}) \simeq ax^* (1 + \hat{x})$$

2. Steady-state relation $\rightarrow$ all additive constants, like ax^* above, drop out because each log-linearized equation is a SS relationship:

$$g(k^*, k^*, c^*) = 0$$

3. Second order terms negligible and set to zero:

$$\hat{x}\hat{y} = 0$$

Log-linearization of feasibility constraint

$$\begin{aligned}
k_{t+1} - k_t^{\alpha} + c_t - (1-\delta)\, k_t &= 0 \\
k^* \exp\left(\hat{k}_{t+1}\right) - \left[(k^*) \exp\left(\hat{k}_t\right)\right]^{\alpha} + c^* \exp\left(\hat{c}_t\right) - (1-\delta)\, k^* \exp\left(\hat{k}_t\right) &= 0 \\
k^* \exp\left(\hat{k}_{t+1}\right) - \left[(k^*)^{\alpha} \exp\left(\alpha \hat{k}_t\right)\right] + c^* \exp\left(\hat{c}_t\right) - (1-\delta)\, k^* \exp\left(\hat{k}_t\right) &= 0 \\
k^* \left(1 + \hat{k}_{t+1}\right) - \left[(k^*)^{\alpha} \left(1 + \alpha \hat{k}_t\right)\right] + c^* \left(1 + \hat{c}_t\right) - (1-\delta)\, k^* \left(1 + \hat{k}_t\right) &= 0 \\
k^* \hat{k}_{t+1} - \alpha\, (k^*)^{\alpha}\, \hat{k}_t + c^* \hat{c} - (1-\delta)\, k^* \hat{k} &= 0
\end{aligned}$$

where the last line uses the SS restriction:

$$k^* - (k^*)^{\alpha} + c^* - (1-\delta)\, k^* = 0$$

Log-linearization of Euler equation

$$
\begin{aligned}
0 &= \frac{1}{c_t} - \beta \frac{1}{c_{t+1}} \left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta \right) \\
0 &= \frac{1}{c^* \exp\left(\hat{c}_t\right)} - \beta \frac{1}{c^* \exp\left(\hat{c}_{t+1}\right)} \left(\alpha \left(k^*\right)^{\alpha-1} \exp\left(\hat{k}_{t+1}\right)^{\alpha-1} + 1 - \delta \right) \\
c^* \exp\left(\hat{c}_{t+1}\right) &= \beta c^* \exp\left(\hat{c}_t\right) \left(\alpha \left(k^*\right)^{\alpha-1} \exp\left(\left(\alpha-1\right)\hat{k}_{t+1}\right) + 1 - \delta \right) \\
\exp\left(\hat{c}_{t+1}\right) &= \beta \exp\left(\hat{c}_t\right) \left[\alpha \left(k^*\right)^{\alpha-1} \exp\left(\left(\alpha-1\right)\hat{k}_{t+1}\right) + 1 - \delta \right] \\
&= \beta \exp\left(\hat{c}_t\right) \left[\alpha \left(k^*\right)^{\alpha-1} \left(1 + \left(\alpha-1\right)\hat{k}_{t+1}\right) + 1 - \delta \right] \\
&= \exp\left(\hat{c}_t\right) \left[\beta\alpha \left(k^*\right)^{\alpha-1} + \beta\left(\alpha-1\right)\hat{k}_{t+1}\alpha\left(k^*\right)^{\alpha-1} + \beta\left(1-\delta\right) \right] \\
&= \exp\left(\hat{c}_t\right) \left[1 + \beta\left(\alpha-1\right)\hat{k}_{t+1}\alpha\left(k^*\right)^{\alpha-1} \right]
\end{aligned}
$$

where the last line uses the SS version of the EE. Taking logs:

$$
\hat{c}_{t+1} = \hat{c}_t + \beta\left(\alpha-1\right)\hat{k}_{t+1}\alpha\left(k^*\right)^{\alpha-1}
$$

The method of undetermined coefficients

- This is Uhlig's toolkit method
- Other classic references for solving linear RE models:
  - Blanchard-Kahn (ECMA, 1980)
  - Christiano (Comp. Economics, 2002)
  - King-Watson (Comp. Economics, 2002)
  - Klein (JEDC, 2000)
  - Sims (Comp. Econ, 2001)

Strategy

1. Find the nonlinear dynamic system that characterizes the model economy (same number of equations and unknowns).

2. Stationarize the economy, if needed. Compute the steady state.

3. (Log) linearize the nonlinear system around the SS. The system is linear in controls and (exogenous and endogenous) state variables. Some eqns are difference eqns.

4. Express the linear system in some matrix representation. This representation is what differs by solution method.

5. Use a matrix decomposition method to derive:

   (a) Decision rules, i.e., linear functions from states to controls.

   (b) Laws of motion for the endogenous state variables, i.e., linear functions from the states at t to the states at $t+1$.

Stochastic growth model with leisure and 2 shocks

- Household problem:

$$\max_{\{c_t,k_{t+1},h_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t - \varphi \frac{h_t^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}}$$

$$s.t.$$

$$c_t + k_{t+1} = (1-\tau) w_t h_t + (1 + r_t - \delta) k_t$$

- Firm problem:

$$\max_{h_t,k_t} e^{z_t} k_t^{\alpha} h_t^{1-\alpha} - w_t h_t - r_t k_t$$

- Government:

$$\bar{g} e^{g_t} = \tau w_t h_t, \quad \text{with} \quad g_{t+1} = \rho_g g_t + \eta_{t+1}$$

- Laws of motion for endogenous and exogenous states:

$$\begin{aligned} k_{t+1} &= (1-\delta) k_t + i_t \\ z_{t+1} &= \rho_z z_t + \varepsilon_{t+1} \end{aligned}$$

Step 1: Equilibrium conditions

$$
\begin{aligned}
\frac{1}{c_t} &= \beta \mathbb{E}_t \left[\frac{1 + r_{t+1} - \delta}{c_{t+1}} \right] \\
\frac{w_t}{c_t}(1-\tau) &= \varphi h_t^{\frac{1}{\gamma}} \\
w_t &= (1-\alpha) e^{z_t} k_t^{\alpha} h_t^{-\alpha} \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} h_t^{1-\alpha} \\
\bar{g} e^{g_t} &= \tau w_t h_t \\
c_t + k_{t+1} &= (1-\tau) w_t h_t + (1 + r_t - \delta) k_t \\
k_{t+1} &= (1-\delta) k_t + i_t \\
z_{t+1} &= \rho z_t + \varepsilon_{t+1} \\
g_{t+1} &= \rho_g g_t + \eta_{t+1}
\end{aligned}
$$

We can exclude the law of motion for k from the system because, once we know k we know i, and i does not enter anywhere else. And we can substitute the tax revenues out from the household BC.

Step 1: Reduced system of equilibrium conditions

$$
\begin{aligned}
\frac{1}{c_t} &= \beta \mathbb{E}_t \left[\frac{1 + r_{t+1} - \delta}{c_{t+1}} \right] \\
\frac{w_t}{c_t} (1 - \tau) &= \varphi h_t^{\frac{1}{\gamma}} \\
w_t &= (1 - \alpha) e^{z_t} k_t^{\alpha} h_t^{-\alpha} \\
r_t &= \alpha e^{z_t} k_t^{\alpha - 1} h_t^{1-\alpha} \\
c_t + k_{t+1} + \bar{g} e^{g_t} &= (1 - \tau) w_t h_t + (1 + r_t - \delta) k_t \\
z_{t+1} &= \rho z_t + \varepsilon_{t+1} \\
g_{t+1} &= \rho_g g_t + \eta_{t+1}
\end{aligned}
$$

- At any t, (k_t, z_t, g_t) are pre-determined
- 3 choice variables (c_t, k_{t+1}, h_t) and 2 prices (w_t, r_t) to be determined (by the first 5 equations) plus 2 equations for law of motions of the exogenous states

Step 2: SS system

- Set $z_t = 0, g_t = 0$. Evaluate at SS the first 5 equations:

$$
\begin{aligned}
\bar{r} &= \frac{1}{\beta} - 1 + \delta \\
\frac{\bar{k}}{\bar{h}} &= \left(\frac{1/\beta - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha - 1}} \\
\bar{w} &= (1 - \alpha) \left(\frac{\bar{k}}{\bar{h}} \right)^{\alpha} \\
\bar{c} + \bar{g} &= (1 - \tau) \, \bar{w}\bar{h} + \bar{r}\bar{k} \\
\frac{\bar{w}}{\bar{c}} (1 - \tau) &= \varphi \bar{h}^{\frac{1}{\gamma}}
\end{aligned}
$$

- Solve for: $(\bar{k}, \bar{h}, \bar{c}, \bar{w}, \bar{r})$

Step 3: Log-linearize around SS

- Euler equation:

$$
\begin{aligned}
\frac{1}{c_t} &= \beta \mathbb{E}_t \left[\frac{1 + r_{t+1} - \delta}{c_{t+1}} \right] \\
\frac{1}{\bar{c} e^{\hat{c}_t}} &= \beta \mathbb{E}_t \left[\frac{1 + \bar{r} e^{\hat{r}_{t+1}} - \delta}{\bar{c} e^{\hat{c}_{t+1}}} \right] \\
\frac{1}{e^{\hat{c}_t}} &= \beta \mathbb{E}_t \left[\frac{1 + \bar{r} \left(1 + \hat{r}_{t+1}\right) - \delta}{e^{\hat{c}_{t+1}}} \right] \\
\frac{1}{e^{\hat{c}_t}} &= \beta \mathbb{E}_t \left[\frac{1 + \bar{r} + \bar{r}\hat{r}_{t+1} - \delta}{e^{\hat{c}_{t+1}}} \right] \\
e^{-\hat{c}_t} &= \mathbb{E}_t \left[\left(1 + \beta \bar{r} \hat{r}_{t+1}\right) e^{-\hat{c}_{t+1}} \right] \\
1 - \hat{c}_t &= \mathbb{E}_t \left[\left(1 + \beta \bar{r} \hat{r}_{t+1}\right) \left(1 - \hat{c}_{t+1}\right) \right] \\
0 &= \mathbb{E}_t \left[\beta \bar{r} \hat{r}_{t+1} - \hat{c}_{t+1} + \hat{c}_t \right]
\end{aligned}
$$

Step 3: Log-linearize

- Intratemporal FOC:

$$
\begin{aligned}
\frac{w_t}{c_t}(1-\tau) &= \varphi h_t^{\frac{1}{\gamma}} \\
\frac{e^{\hat{w}_t}}{e^{\hat{c}_t}} &= e^{\hat{h}_t \frac{1}{\gamma}} \\
\hat{h}_t &= \gamma(\hat{w}_t - \hat{c}_t)
\end{aligned}
$$

- Firm's FOC for labor:

$$
\begin{aligned}
w_t &= (1-\alpha) e^{z_t} k_t^{\alpha} h_t^{-\alpha} \\
\hat{w}_t &= z_t + \alpha\left(\hat{k}_t - \hat{h}_t\right)
\end{aligned}
$$

- Firm's FOC for capital:

$$
\hat{r}_t = z_t + (1-\alpha)\left(\hat{h}_t - \hat{k}_t\right)
$$

Step 3: Log-linearize

- Budget constraint:

$$c_t + \bar{g}e^{g_t} + k_{t+1} = (1-\tau)\, w_t h_t + (1 + r_t - \delta)\, k_t$$

$$\bar{c}e^{\hat{c}_t} + \bar{g}e^{g_t} + \bar{k}e^{\hat{k}_{t+1}} = (1-\tau)\, \bar{w}\bar{h}e^{\hat{w}_t + \hat{h}_t} + \left(\bar{r}e^{\hat{r}_t} + 1 - \delta\right) \bar{k}e^{\hat{k}_t}$$

$$\begin{aligned}\bar{c}\,(1+\hat{c}_t) + \bar{g}\,(1+g_t) + \bar{k}\left(1+\hat{k}_{t+1}\right) &= (1-\tau)\, \bar{w}\bar{h}\,(1+\hat{w}_t)\left(1+\hat{h}_t\right) \\ &\quad + \bar{r}\,(1+\hat{r}_t)\,\bar{k}\left(1+\hat{k}_t\right) + (1-\delta)\,\bar{k}\left(1+\hat{k}_t\right)\end{aligned}$$

$$\bar{c}\hat{c}_t + \bar{g}g_t + \bar{k}\hat{k}_{t+1} = (1-\tau)\, \bar{w}\bar{h}\left(\hat{w}_t + \hat{h}_t\right) + \bar{r}\bar{k}\left(\hat{r}_t + \hat{k}_t\right) + (1-\delta)\,\bar{k}\hat{k}_t$$

Step 3: Log-linearize

- Final log-linear system:

$$
\begin{aligned}
0 &= \mathbb{E}_t \left[\beta \bar{r} \hat{r}_{t+1} - \hat{c}_{t+1} + \hat{c}_t\right] \\
\hat{h}_t &= \gamma \left(\hat{w}_t - \hat{c}_t\right) \\
\hat{r}_t &= z_t + (1-\alpha)\left(\hat{h}_t - \hat{k}_t\right) \\
\hat{w}_t &= z_t + \alpha \left(\hat{k}_t - \hat{h}_t\right) \\
\bar{c}\hat{c}_t + \bar{g} g_t + \bar{k}\hat{k}_{t+1} &= (1-\tau)\,\bar{w}\bar{h}\left(\hat{w}_t + \hat{h}_t\right) + \bar{r}\bar{k}\left(\hat{r}_t + \hat{k}_t\right) + (1-\delta)\,\bar{k}\hat{k}_t
\end{aligned}
$$

- Substitute away factor prices (last eqn. uses CRS: $\bar{y} = \bar{r}\bar{k} + \bar{w}\bar{h}$)

$$
\begin{aligned}
0 &= \mathbb{E}_t \left\{\beta \bar{r}\left[z_{t+1} + (1-\alpha)\left(\hat{h}_{t+1} - \hat{k}_{t+1}\right)\right] - \hat{c}_{t+1} + \hat{c}_t\right\} \\
\hat{h}_t &= \gamma \left[z_t + \alpha\left(\hat{k}_t - \hat{h}_t\right) - \hat{c}_t\right] \\
\bar{c}\hat{c}_t + \bar{g} g_t + \bar{k}\hat{k}_{t+1} &= (1-\tau)\,\bar{y}\left[z_t + \alpha\hat{k}_t + (1-\alpha)\,\hat{h}_t\right] + (1-\delta)\,\bar{k}\hat{k}_t
\end{aligned}
$$

Step 3: Log-linearize

- Now substitute away consumption in the EE using the intratemporal FOC:

$$\mathbb{E}_t \left\{ \beta \bar{r} \left[z_{t+1} + (1-\alpha)\left(\hat{h}_{t+1} - \hat{k}_{t+1}\right)\right] - z_{t+1} - \alpha\left(\hat{k}_{t+1} - \hat{h}_{t+1}\right) + \frac{1}{\gamma}\hat{h}_{t+1} \right\} =$$

$$z_t + \alpha\left(\hat{k}_t - \hat{h}_t\right) - \frac{1}{\gamma}\hat{h}_t$$

$$\bar{c}\left[z_t + \alpha\left(\hat{k}_t - \hat{h}_t\right) - \frac{1}{\gamma}\hat{h}_t\right] + \bar{g}g_t + \bar{k}\hat{k}_{t+1} = \bar{y}\left[z_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t\right] + (1-\delta)\bar{k}\hat{k}_t$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}$$

$$g_{t+1} = \rho_g g_t + \eta_{t+1}$$

- We cannot reduce the dimensionality of the system further.

- Note: the variance of the shock does not affect any of the first two equilibrium equations. Certainty equivalence holds.

Step 4: Solve the system

- To ease the notation, express the system in recursive form and collect terms:

$$
\begin{aligned}
0 &= \mathbb{E}\left[\hat{k}' + \phi_1 \hat{k} + \phi_2 \hat{h}' + \phi_3 \hat{h} + \phi_4 z' + \phi_5 z\right] \\
0 &= \hat{k}' + \phi_8 \hat{k} + \phi_9 \hat{h} + \phi_{10} z + \phi_{11} g \\
z' &= \rho_z z + \varepsilon' \\
g' &= \rho_g g + \eta'
\end{aligned}
$$

- We are looking for a linear solution to the system:

$$
\begin{aligned}
\hat{k}' &= \lambda_1 \hat{k} + \lambda_2 z + \lambda_3 g \\
\hat{h} &= \lambda_4 \hat{k} + \lambda_5 z + \lambda_6 g
\end{aligned}
$$

- In other words, we are looking to express the λ's as a function of the ϕ's (and ρ's) only

Step 4: Solve the system

- Substituting the decision rules into the first two equations:

$$
\begin{aligned}
0 &= \left(\lambda_1 + \phi_1 + \phi_2\lambda_4\lambda_1 + \phi_3\lambda_4\right)\hat{k} + \\
&\quad \left(\lambda_2 + \phi_2\lambda_4\lambda_2 + \phi_2\lambda_5\rho_z + \phi_3\lambda_5 + \phi_4\rho + \phi_5\right)z + \\
&\quad \left(\lambda_3 + \phi_2\lambda_4\lambda_3 + \phi_2\lambda_6\rho_g + \phi_3\lambda_6\right)g \\
0 &= \left(\lambda_1 + \phi_8 + \phi_9\lambda_4\right)\hat{k} + \left(\lambda_2 + \phi_9\lambda_5 + \phi_{10}\right)z + \left(\lambda_3 + \phi_9\lambda_6 + \phi_{11}\right)g
\end{aligned}
$$

- Since both equations need to be satisfied for all $\left(\hat{k}, z, g\right)$:

$$
\begin{aligned}
\lambda_1 + \phi_1 + \phi_2\lambda_4\lambda_1 + \phi_3\lambda_4 &= 0 \\
\lambda_2 + \phi_2\lambda_4\lambda_2 + \phi_2\lambda_5\rho_z + \phi_3\lambda_5 + \phi_4\rho + \phi_5 &= 0 \\
\lambda_3 + \phi_2\lambda_4\lambda_3 + \phi_2\lambda_6\rho_g + \phi_3\lambda_6 &= 0 \\
\lambda_1 + \phi_8 + \phi_9\lambda_4 &= 0 \\
\lambda_2 + \phi_9\lambda_5 + \phi_{10} &= 0 \\
\lambda_3 + \phi_9\lambda_6 + \phi_{11} &= 0
\end{aligned}
$$

- Linear system of 6 equations into 6 unknowns (λ's)

General case

- Let x be the $m = 1$ endogenous states (k), y the $n = 1$ control (h), z as the $k = 2$ exogenous states (z, g).
- Write the system as:

$$\begin{aligned} 0 &= Ax' + Bx + Cy + Dz \\ 0 &= \mathbb{E}[Fx'' + Gx' + Hx + Jy' + Ky + Lz' + Mz] \\ z' &= Nz + \varepsilon' \end{aligned}$$

- Note that in our model $F = 0$ and you can substitute out z' and collapse the other terms with M so $L = 0$.
- Note also that the C is a square matrix, in our case $(1, 1)$ so it can be inverted. When it is not square, you have to amend a bit the strategy.

General case

- The decision rules can be expressed as:

$$\begin{aligned} x' &= Px + Qz \\ y &= Rx + Sz \end{aligned}$$

where P is $(m \times m)$, Q is $(m \times k)$, R is $(n \times m)$, and S is $(n \times k)$

- Substitute the decision rules into the system:

$$\begin{aligned} 0 &= (AP + B + CR)\, x + (AQ + CS + D)z \\ 0 &= (GP + H + JRP + KR)x + (GQ + JRQ + JSN + KS + M)z \end{aligned}$$

- Since these equations must hold for any x and z:

$$\begin{aligned} 0 &= AP + B + CR && (1) \\ 0 &= AQ + CS + D && (2) \\ 0 &= GP + H + JRP + KR && (3) \\ 0 &= GQ + JRQ + JSN + KS + M && (4) \end{aligned}$$

General case

- Solving equation (1) for R, we obtain:

$$R = -C^{-1}(AP + B)$$

  ... and here is where it's useful that C is a square matrix (invertible)

- Substituting this expression for R into equation (3):

$$-JC^{-1}AP^2 - \left(JC^{-1}B + KC^{-1}A - G\right)P - \left(KC^{-1}B - H\right) = 0$$

- This is a quadratic matrix equation in P. Number of ways to solve this problem.

- Uhlig suggests using the generalized eigenvalue problem (also called the QZ decomposition or generalized Schur decomposition).

General case

- Now, solve for S as a function of Q with equation (2):

$$S = -C^{-1}\left(AQ + D\right)$$

- To solve for Q substitute expression above into equation (4):

$$\begin{aligned} GQ + JRQ - JC^{-1}\left(AQ + D\right)N - KC^{-1}\left(AQ + D\right) + M &= 0 \\ \left(G + JR - KC^{-1}A\right)Q - JC^{-1}AQN - JC^{-1}DN - KC^{-1}D + M &= 0 \end{aligned}$$

- Q is the only unknown, but not trivial as Q is sandwiched in some terms. Use $vec\left(AXB\right) = \left(B^T \otimes A\right) vec\left(X\right)$, where $\otimes$ is the Kronecker product, to pull out the sandwiched X matrix.

- Apply this vectorization to equation (4) to obtain Q:

$$0 = I \otimes \left(G + JR - KC^{-1}A\right) vec(Q) - N^T \otimes \left(JC^{-1}A\right) vec(Q) - vec(JC^{-1}DN - KC^{-1}D + M)$$

$$vec(Q) = \left[I \otimes \left(G + JR - KC^{-1}A\right) - N^T \otimes \left(JC^{-1}A\right)\right]^{-1} vec(JC^{-1}DN - KC^{-1}D + M)$$

Perturbation methods: a baby example

- Stochastic growth model with inelastic leisure and z shock only

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{1}{c_{t+1}}\beta\left(\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}+1-\delta\right)-\frac{1}{c_t}\right] &= 0\\
k_{t+1}-e^{z_t}k_t^{\alpha}+c_t-(1-\delta)\,k_t &= 0\\
z_{t+1}-\rho_z z_t-\sigma\varepsilon_{t+1} &= 0
\end{aligned}
$$

- Rewrite as:

$$
\mathbb{E}_t\left[f\left(y_{t+1},y_t,x_{t+1},x_t\right)\right]=\mathbb{E}_t\left[\begin{array}{c} y_{t+1}^{-1}\beta\left(\alpha e^{x_{2,t+1}}x_{1,t+1}^{\alpha-1}+1-\delta\right)-y_t^{-1}\\ x_{1,t+1}-e^{x_{2,t}}x_{1t}^{\alpha}+y_t-(1-\delta)\,x_{1t}=0\\ x_{2,t+1}-\rho_z x_{2t}\end{array}\right]
$$

where x_t is vector of endogenous and exogenous states of size $m=2$ and y_t vector of controls of size $n=1$: $n+m$ eqns in f.

- Note more compact notation, x denotes both endo and exo states

Main idea of perturbation methods

- Transform the problem rewriting it in terms of a perturbation parameter

- Approximate the solution around a particular choice of the perturbation parameter

- Perturbation parameter: SD of the shock σ

- Particular choice for local approximation: $\bar{\sigma} = 0$

- The proposed solution of this model is of the form:

$$\begin{aligned} y_t &= g\left(x_t, \sigma\right) \\ x_{t+1} &= h\left(x_t, \sigma\right) + \eta\sigma\varepsilon_{t+1} \end{aligned}$$

where $\eta = (0 \quad 1)'$ and σ is the SD of the ε shock

Linearization as a first-order perturbation

- We wish to find a first-order approximation of the functions g and h around $\sigma = 0$, i.e. around the non-stochastic steady state $x_t = \bar{x}$.

- We define the non-stochastic steady state as vectors $(\bar{x}, \bar{y})$ s.t.:

$$f\left(\bar{y}, \bar{y}, \bar{x}, \bar{x}\right) = 0$$

  It is clear that $\bar{y} = g(\bar{x}, 0)$ and $\bar{x} = h(\bar{x}, 0)$.

- Substituting the solution into the system, we use the recursive formulation to define:

$$F\left(x, \sigma\right) = \mathbb{E}\left[f\left(g\left(h\left(x, \sigma\right) + \eta\sigma\varepsilon', \sigma\right), g\left(x, \sigma\right), h\left(x, \sigma\right) + \eta\sigma\varepsilon', x\right)\right] = 0$$

- Since this must hold for any x and σ, it is also true that the derivatives of any order of F wrt to x and σ must also be zero.

Linearization as a first-order perturbation

- Recall that we are looking for first-order approximations of g and h at the point $(x,\sigma) = (\bar{x},0)$ of the form:

$$\begin{aligned} g(x,\sigma) &= \bar{y} + \bar{g}_x (x-\bar{x}) + \bar{g}_\sigma \sigma \\ h(x,\sigma) &= \bar{x} + \bar{h}_x (x-\bar{x}) + \bar{h}_\sigma \sigma \end{aligned}$$

- where we used the notation:

$$\bar{g}_x = g_x(\bar{x},0), \quad \text{etc}$$

- The four unknown coefficients $\{\bar{g}_x, \bar{g}_\sigma, \bar{h}_x, \bar{h}_\sigma\}$ of the first-order approximation of g and h are found by imposing:

$$\begin{aligned} \bar{F}_x &= F_x(\bar{x},0) = 0 \\ \bar{F}_\sigma &= F_\sigma(\bar{x},0) = 0 \end{aligned}$$

Tensor notation

- Tensors are multidimensional arrays and in physics there is a compact notation to deal with them that we can use:

$$\left[\bar{f}_y\right]^i_\alpha = \frac{\partial f^i}{\partial y^\alpha}|_{x=\bar{x},\sigma=0}$$

is the (i,α) element of the derivative of f wrt y, which is a $(n+m\times n)$ matrix. Thus $i=1,...n+m$ and $\alpha=1,...,n$.

- We can also define:

$$\begin{aligned}
\left[\bar{f}_y\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_j &= \sum_{\alpha=1}^{n} \frac{\partial f^i}{\partial y^\alpha}\frac{\partial g^\alpha}{\partial x^j}|_{x=\bar{x},\sigma=0} \\
\left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_\beta \left[\bar{h}_x\right]^\beta_j &= \sum_{\alpha=1}^{n}\sum_{\beta=1}^{m} \frac{\partial f^i}{\partial y^\alpha}\frac{\partial g^\alpha}{\partial x^\beta}\frac{\partial h^\beta}{\partial x^j}|_{x=\bar{x},\sigma=0}
\end{aligned}$$

Linearization as a first-order perturbation

- Imposing $\bar{F}_x = 0$:

$$\begin{aligned}\left[\bar{F}_x\right]^i_j &= \mathbb{E}\left\{\left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_\beta \left[\bar{h}_x\right]^\beta_j + \left[\bar{f}_y\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_j + \left[\bar{f}_{x'}\right]^i_\beta \left[\bar{h}_x\right]^\beta_j + \left[\bar{f}_x\right]^i_j\right\} = 0 \\ &= \left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_\beta \left[\bar{h}_x\right]^\beta_j + \left[\bar{f}_y\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_j + \left[\bar{f}_{x'}\right]^i_\beta \left[\bar{h}_x\right]^\beta_j + \left[\bar{f}_x\right]^i_j = 0\end{aligned}$$

  a system with $(n+m) \times m = 6$ equations and $(n+m)\, m = 6$ unknowns: $nm\ [\bar{g}_x]$ and $m^2\ \left[\bar{h}_x\right]$

- Similarly, imposing $\bar{F}_\sigma = 0$:

$$\begin{aligned}\left[\bar{F}_\sigma\right]^i &= \mathbb{E}\left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_\beta \left[\bar{h}_\sigma\right]^\beta + \left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_\beta [\eta]^\beta_\phi \left[\varepsilon'\right]^\phi + \left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_\sigma\right]^\alpha \\ &\quad + \left[\bar{f}_y\right]^i_\alpha \left[\bar{g}_\sigma\right]^\alpha + \left[\bar{f}_{x'}\right]^i_\beta \left[\bar{h}_\sigma\right]^\beta + \left[\bar{f}_{x'}\right]^i_\beta [\eta]^\beta_\phi \left[\varepsilon'\right]^\phi = 0 \\ &= \left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_\beta \left[\bar{h}_\sigma\right]^\beta + \left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_\sigma\right]^\alpha + \left[\bar{f}_y\right]^i_\alpha \left[\bar{g}_\sigma\right]^\alpha + \left[\bar{f}_{x'}\right]^i_\beta \left[\bar{h}_\sigma\right]^\beta = 0\end{aligned}$$

  a system of $(m+n) = 3$ eqns and unknowns: $m\ \left[\bar{h}_\sigma\right]$ and $n\ [\bar{g}_\sigma]$.

- This system is linearly homogeneous in $\left(\bar{h}_\sigma, \bar{g}_\sigma\right) \rightarrow \bar{h}_\sigma = \bar{g}_\sigma = 0$

Second-order perturbations

- We look for functions g and h at point $(x,\sigma) = (\bar{x}, 0)$ of the form:

$$g(x,\sigma) = \bar{g} + \bar{g}_x (x-\bar{x}) + \bar{g}_\sigma \sigma + \frac{1}{2}(x-\bar{x})' \bar{g}_{xx} (x-\bar{x}) + \frac{1}{2}(x-\bar{x})' \bar{g}_{x\sigma}\sigma + \frac{1}{2}\bar{g}_{\sigma\sigma}\sigma^2$$

and similarly for $h(x,\sigma)$.

- Analogously to FO perturb., to find $\bar{g}_{xx}$ and $\bar{h}_{xx}$ we exploit $\bar{F}_{xx} = 0$.

- A useful result is the following:

$$\left[\bar{F}_{\sigma x}\right]^i_j = \mathbb{E}\left\{\left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_x\right]^\alpha_\beta \left[\bar{h}_{\sigma x}\right]^\beta_j + \left[\bar{f}_{y'}\right]^i_\alpha \left[\bar{g}_{\sigma x}\right]^\alpha_\gamma \left[\bar{h}_x\right]^\gamma_j + \left[\bar{f}_y\right]^i_\alpha \left[\bar{g}_{\sigma x}\right]^\alpha_j + \left[\bar{f}_{x'}\right]^i_\beta \left[\bar{h}_{\sigma x}\right]^\beta_j\right\}$$

which is a system that is linear and homogeneous in unknowns $\bar{h}_{\sigma x}$ and $\bar{g}_{\sigma x}$ and therefore:

$$\bar{h}_{\sigma x} = \bar{g}_{\sigma x} = 0$$

Second-order perturbations

- This is an important results because, coupled with $\bar{h}_\sigma = \bar{g}_\sigma = 0$, it means that the only terms that affect the decision rules are the terms $1/2 g_{\sigma\sigma}\sigma^2$ and $1/2 h_{\sigma\sigma}\sigma^2$ that shifts up and down the *constants* in the decision rules.

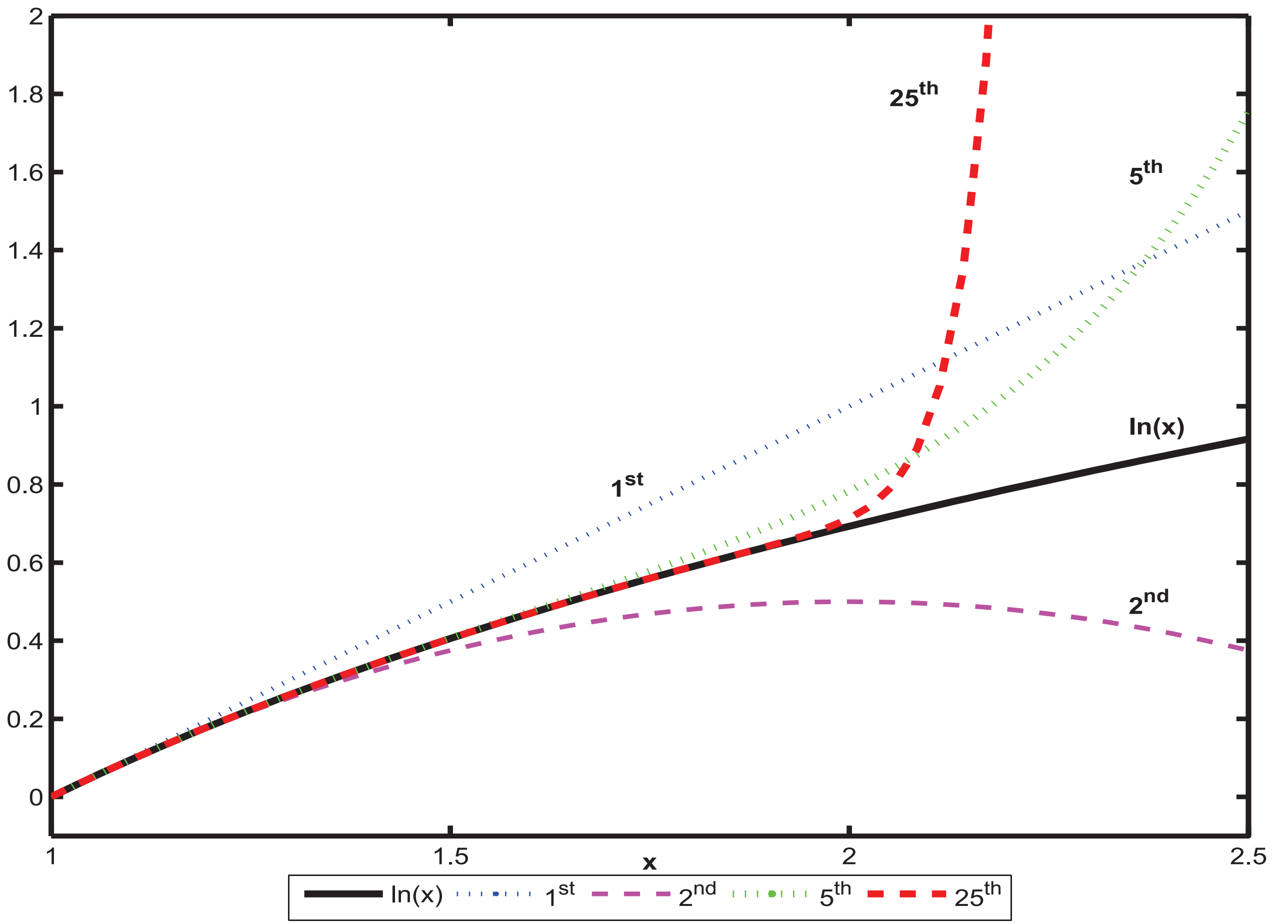
- To summarize:

  1. The coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks. In other words, these coefficients must be the same in the stochastic and the deterministic versions of the model.

  2. Thus, up to second order, the presence of uncertainty affects only the constant term of the decision rules.

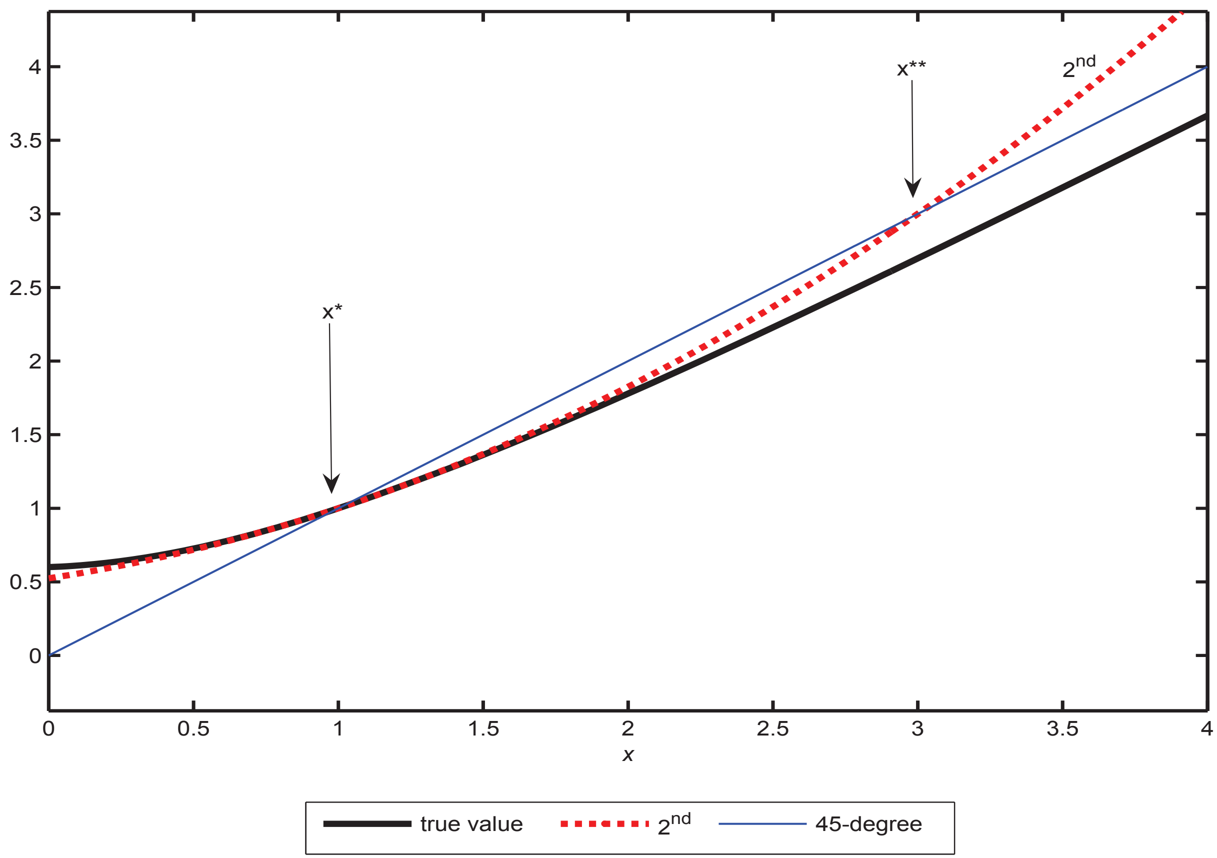
Issues with higher order approximations

- Far from the approximation point they can be worse than linear, and non-monotonic in the states



Issues with higher order approximations

- Unstable dynamics of the approximated system



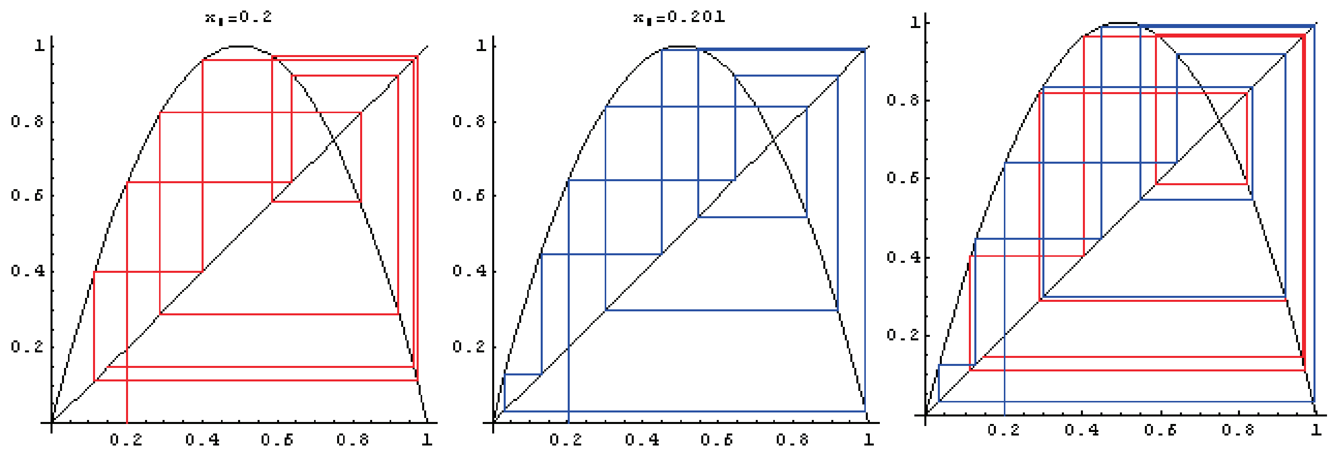
Textbook example of unstable dynamics

- Consider a second order approximation of a univariate decision rule $x_{t+1} = g(x_t)$ around $\bar{x}$ (where $\hat{x}_t = x_t - \bar{x}$):

$$
\begin{aligned}
\hat{x}_{t+1} &= \alpha_0 \hat{x}_t + \alpha_1 \hat{x}_t^2 \\
&= \alpha_0 \hat{x}_t \left(1 - \alpha_2 \hat{x}_t\right)
\end{aligned}
$$

  where $\alpha_2 = -\alpha_1/\alpha_0$

- Logistic map: even if $\alpha_0 < 1$, it can have chaotic dynamics:



Pruning (Kim, Kim, Schaumburg, and Sims, 2008)

$$\hat{x}_{t+1}^{(n)} = g^{(n)}\left(\hat{x}_t, \sigma\right)$$

where (n) denotes n-th order approxim. When we simulate the model:

$$\begin{aligned}
\hat{x}_{t+1}^{(1)} &= g^{(1)}\left(\hat{x}_t^{(1)}, \sigma\right) + \eta\sigma\varepsilon_t = \bar{g}_x \hat{x}_t^{(1)} + \eta\sigma\varepsilon_t \\
\hat{x}_{t+1}^{(2)} &= g^{(2)}\left(\hat{x}_t^{(2)}, \sigma\right) = \bar{g}_x \hat{x}_t^{(2)} + \frac{1}{2}\hat{x}_t^{(2)} \bar{g}_{xx} \hat{x}_t^{(2)} + \frac{1}{2}\bar{g}_{\sigma\sigma}\sigma^2 + \eta\sigma\varepsilon_t
\end{aligned}$$

- Under pruning:

$$\hat{x}_{t+1}^{P} = g^{P}\left(\hat{x}_t^{(2)}, \sigma, \hat{x}_t^{(1)}\right) = \bar{g}_x \hat{x}_t^{(2)} + \frac{1}{2}\hat{x}_t^{(1)} \bar{g}_{xx} \hat{x}_t^{(1)} + \frac{1}{2}\bar{g}_{\sigma\sigma}\sigma^2 + \eta\sigma\varepsilon_t$$

  i.e., we use the dynamics from the first order approximation in the second order terms. It is easy to see that it is stationary.

- It adds new state variable / changes coefficient $\bar{g}_{xx}$ of decision rule

Dynare

- It is a collection of routines which solves and simulates non-linear models with forward looking variables.

- It can be used in either MATLAB or Octave (an open source and free Matlab clone). It is freely downloadable from the Dynare website

  `http://www.dynare.org/`

- Dynare can solve forward looking non-linear models under perfect foresight, when the model is deterministic, and rational expectations, when the model is stochastic.

- In the first case, the solution method preserves all the non-linearities. In the second case, Dynare computes a (first or second order) local approximation around the steady state

Equilibrium conditions for the stochastic growth model

$$
\begin{aligned}
\frac{1}{c_t} &= \beta \mathbb{E}_t \left[\frac{1 + r_{t+1} - \delta}{c_{t+1}} \right] \\
\frac{w_t}{c_t} &= \varphi h_t^{\frac{1}{\gamma}} \\
w_t &= (1-\alpha) e^{z_t} k_t^{\alpha} h_t^{-\alpha} \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} h_t^{1-\alpha} \\
c_t + k_{t+1} &= w_t h_t + (1 + r_t) k_t \\
k_{t+1} &= (1-\delta) k_t + i_t \\
y_t &= c_t + i_t \\
z_t &= \rho z_{t-1} + \varepsilon_t
\end{aligned}
$$

8 “endogenous” variables $(c_t, h_t, k_{t+1}, w_t, r_t, i_t, y_t, z_{t+1})$ and 8 equations. One exogenous variable ε_t

Dynare timing convention

- Predetermined variables at date t must have time index $t-1$
- Rewrite system with this convention:

$$
\begin{aligned}
\frac{1}{c_t} &= \beta \mathbb{E}_t \left[\frac{1 + r_{t+1} - \delta}{c_{t+1}} \right] \\
\frac{w_t}{c_t} &= \varphi h_t^{\frac{1}{\gamma}} \\
w_t &= (1-\alpha) e^{z_t} k_{t-1}^{\alpha} h_t^{-\alpha} \\
r_t &= \alpha e^{z_t} k_{t-1}^{\alpha-1} h_t^{1-\alpha} \\
c_t + k_t &= w_t h_t + (1 + r_t) k_{t-1} \\
k_t &= (1-\delta) k_{t-1} + i_t \\
y_t &= c_t + i_t \\
z_t &= \rho z_{t-1} + \varepsilon_t
\end{aligned}
$$

Dynare program blocks

- Write a file with extension `.mod` and run it in Matlab by typing `dynare filename`. The file must contain 6 blocks:

1. Labeling block: declare endogenous variables, exogenous variables, and parameters.
2. Parameter values block: assigns values to the parameters.
3. Model block: between `model` and `end` write your n equil. conditions
4. Initialization block: Dynare solves for the SS.
5. Shock variance block: declare the parameters that is the variance of the shock
6. Solution block: solves model and computes IRFs

Blocks 1-2

- Labeling block (endo vars, exo vars, parameters)

```
var c h y i w r k z;

varexo eps;

parameters gamma beta alpha delta psi rho sigma_eps;
```

- Parameter values block

```
gamma =      0.5;
beta =       0.99;
alpha =      0.333;
delta =      0.025;
psi =        1.5;
rho =        0.95;
sigma_eps =  0.01;
```

Model Block

- Dynare's default: linear approximation. If we want log-linear approximations, need to specify a variable as $\exp(x)$, so that x is the $\log$ of the variable of interest and $\exp(x)$ is the level
- Dynare knows that when there is a (+1) there is an expectation

```
model;
```

$$\mathtt{1}/\mathtt{exp}(\mathtt{c}) = \mathtt{beta} * (\mathtt{1}/\mathtt{exp}(\mathtt{c}(+\mathtt{1})) * (\mathtt{1} + \mathtt{exp}(\mathtt{r}(+\mathtt{1})) - \mathtt{delta}));$$

$$\mathtt{exp}(\mathtt{w})/\mathtt{exp}(\mathtt{c}) = \mathtt{psi} * \mathtt{exp}(\mathtt{h})\hat{}(\mathtt{1}/\mathtt{gamma});$$

$$\exp(\mathtt{w}) = (\mathtt{1} - \mathtt{alpha}) * \exp(\mathtt{z}) * \exp(\mathtt{k}(-\mathtt{1}))\hat{}\mathtt{alpha} * \exp(\mathtt{h})\hat{}(-\mathtt{alpha});$$

$$\exp(\mathtt{r}) = \mathtt{alpha} * \exp(\mathtt{z}) * \exp(\mathtt{k}(-\mathtt{1}))\hat{}(\mathtt{alpha} - \mathtt{1}) * \exp(\mathtt{h})\hat{}(\mathtt{1} - \mathtt{alpha});$$

$$\exp(\mathtt{c}) + \exp(\mathtt{k}) = \exp(\mathtt{w}) * \exp(\mathtt{h}) + (\mathtt{1} + \exp(\mathtt{r})) * \exp(\mathtt{k}(-\mathtt{1}));$$

$$\exp(\mathtt{k}) = (\mathtt{1} - \mathtt{delta}) * \exp(\mathtt{k}(-\mathtt{1})) + \exp(\mathtt{i});$$

$$\exp(\mathtt{y}) = \exp(\mathtt{c}) + \exp(\mathtt{i});$$

$$\mathtt{z} = \mathtt{rho} * \mathtt{z} + \mathtt{eps};$$

```
end;
```

Initialization block for steady state

- Dynare will solve for the steady state numerically, if you provide an initial guess
- When giving initial values, remember that Dynare is interpreting all the variables as logs

```
initval;
k = log(9);
c = log(0.7);
h = log(0.3);
w = log(2);
r = log(0.02);
y = log(1);
i = log(0.3);
z = 0;
end;
```

Initialization block for steady state

- Shock variance block

```
shocks;
vareps= sigma_eps^2d;
end;
```

- Computation of steady state

```
steady;
```

- Solution block

```
stoch_simul(hpfilter=1600,order=1,irf=40);
```

`order` is the order of approximation of the model around the SS

Output of Dynare

1. steady state values of endogenous variables
2. a model summary, which counts variables by type
3. covariance matrix of shocks (which in this example is a scalar)
4. the policy and transition functions (in state space notation)

$$c_t = \bar{c} + a_{c,k}(k_{t-1} - \bar{k}) + a_{c,z_{-1}}(z_{t-1} - \bar{z}) + a_{c,z}\varepsilon_t$$

   that can be rewritten, in more familiar notation as:

$$c_t = \bar{c} + a_{c,k}(k_{t-1} - \bar{k}) + a_{c,z}(z_t - \bar{z}) \quad \text{where} \quad a_{c,z} = \frac{a_{c,z_{-1}}}{\rho}$$

5. first and second moments, correl. matrix, autocov. up to order 5
6. IRFs

Occbin

- Methodology (and suite of Matlab/Dynare codes) developed by Guerrieri and Iacoviello (JME, 2015) to apply first-order perturbation approach in a piecewise fashion in order to handle occasionally binding constraints

- Key insight: occasionally binding constraints can be handled as different regimes of the same model. Under one regime, the occasionally binding constraint is slack. Under the other regime, the same constraint is binding. The piecewise linear solution method involves linking the first-order approximation of the model around the same point under each regime.

- Solution is nonlinear, i.e., decision rules parameters depend on the values of the state variables.

Occbin algorithm

- Two regimes. Reference regime (R1) that includes the SS, and alternative regime (R2). Assume the alternative regime is the one where the constraint binds and reference where it is slack
- In the reference regime R1, the equilibrium dynamics of the endogenous state X_t linearized around SS can be expressed as:

$$A\mathbb{E}_t X_{t+1} + BX_t + CX_{t-1} + Fz_t = 0$$

- In the alternative regime R2 (when constraint binds), the equilibrium dynamics of the endogenous state X_t linearized around the SS can be expressed as:

$$A^c\mathbb{E}_t X_{t+1} + B^c X_t + C^c X_{t-1} + D^c + F^c z_t = 0$$

- The additional vector D^c is there because the linearization is taken around a point where R1 applies.

Occbin algorithm

- Define the decision rules under the reference regime

$$X_t = PX_{t-1} + Qz_t$$

  and under the alternative regime

$$X_t = P_t^c X_{t-1} + Q_t^c z_t + R_t^c$$

  note that these function are time-dependent, i.e., depend on the values of the states.

- For a given point in the state space (X_{t-1}, z_t) , how do we solve for the decision rule?

Occbin algorithm

1. Guess the number of periods $T \geq 0$ for which the alternative regime applies before returning to R1.

2. Use dynamics under R2 combined with the R1 decision rule at T:

$$A^c \mathbb{E}_T \left[P X_{T-1} + Q z_T \right] + B^c X_{T-1} + C^c X_{T-2} + D^c + F^c z_{T-1} = 0$$

3. Assume agents expect no shocks from t onward (violation of RE)

$$A^c P X_{T-1} + B^c X_{T-1} + C^c X_{T-2} + D^c = 0$$
$$X_{T-1} = -\left(A^c P + B^c\right)^{-1} C^c X_{T-2} - \left(A^c P + B^c\right)^{-1} D^c$$

thus $P^c_{T-1} = -\left(A^c P + B^c\right)^{-1} C^c, \quad R^c_{T-1} = -\left(A^c P + B^c\right)^{-1} D^c$

Occbin algorithm

4. Using $X_{T-1} = P^c_{T-1} X_{T-2} + R^c_{T-1}$ and the dynamics under R2, obtain $X_{T-2} = P^c_{T-2} X_{T-3} + R^c_{T-2}$ and iterate back until X_{t-1} where

$$X_t = P^c_t X_{t-1} + R^c_t + \left[-\left(A^c P + B^c \right)^{-1} F^c \right] z_t$$

... the shock reappears in the decision rule only at t

5. From stochastic paths for X_t compute expected length of R2 to verify the current guess T. If the guess is verified, stop. Otherwise, update the guess and return to step 1.

Occbin algorithm

- To get the entire decision rule, you need to repeat this for very pair (X, z) in the state space
- To calculate an IRF, you need to perform this loop only once, for given initial conditions (X_0, z_1)
- To compute a stochastic simulation of length N, you need to repeat this loop N times for every pair (X_t, z_t) in the state space reached by the simulation
- ...but according to the authors, it is very fast

Occbin

- Main advantage: like any linear solution method, delivers a nonlinear solution easily even for a problem with many state variables

- Main limit: like any linear solution method, lack of precautionary behavior in decision rules linked to the possibility that a constraint may bind in the future
    - It discards all information regarding the realization of future shocks
    - E.g., in SS decision rule ignores that constraint may be binding under some realization of the shock