Structural Macroeconometrics

Chapter 2. Approximating and Solving DSGE Models

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Empirical investigations involving DSGE models invariably require the completion of two preparatory stages. One stage involves preparation of the model to be analyzed; this is the focus of the current chapter. The other involves preparation of the data; this is the focus of Chapter 3.

Regarding the model-preparation stage, DSGE models typically include three components: a characterization of the environment in which decision makers reside; a set of decision rules that dictate their behavior; and a characterization of the uncertainty they face in making decisions. Collectively, these components take the form of a non-linear system of expectational difference equations. Such systems are not directly amenable to empirical analysis, but can be converted into empirically implementable systems through the completion of the general two-step process outlined in this chapter.

The first step involves the construction of a linear approximation of the model. Just as non-linear equations may be approximated linearly via the use of Taylor-series expansions, so too may non-linear systems of expectational difference equations. The second step involves the solution of the resulting linear approximation of the system. The solution is written in terms of variables expressed as deviations from steady state values, and is directly amenable to empirical implementation.

While this chapter is intended to be self-contained, far more detail is provided in the literature cited below. Here, the goal is to impart an intuitive understanding of the model-preparation stage, and to provide guidance regarding its implementation. In addition, we note that there are alternatives to the particular approaches to model approximation and solution presented in this chapter. A leading alternative to model approximation is provided by perturbation methods; for a textbook discussion see Judd (1998). And a leading alter-

native to the approaches to model solution presented here is based on the use of projection methods. This alternative solution technique is discussed in this text in Chapter 10; for additional textbook discussions, see Judd (1998), Adda and Cooper (2003) and Sargent and Ljungqvist (2004).

1 Linearization

1.1 Taylor Series Approximation

Consider the following n-equation system of non-linear difference equations:

$$\Psi(x_{t+1}, x_t) = 0, \tag{1}$$

where the x's and 0 are $n \times 1$ vectors. The parameters of the system are contained in the vector μ. DSGE models are typically represented in terms of such a system, augmented to include sources of stochastic behavior. We abstract from the stochastic component of the model in the linearization stage, since models are typically designed to incorporate stochastic behavior directly into the linearized system (a modest example is provided in Section 2.2; detailed examples are provided in Chapter 5). Also, while expectational terms are typically included among the variables in x (e.g., variables of the form $E_t(x_{t+j})$, where E_t is the conditional expectations operator), these are not singled out at this point, as they receive no special treatment in the linearization stage.

Before proceeding, note that while (1) is written as a first-order system, higher-order specifications may be written as first-order systems by augmenting x_t to include variables

observed at different points in time. For example, the p^{th}-order equation

$$\omega_{t+1} = \rho_1\omega_t + \rho_2\omega_{t-1} + ... + \rho_p\omega_{t-p+1}$$

may be written in first-order form as

$$\begin{bmatrix} \omega_{t+1} \\ \omega_t \\ \vdots \\ \omega_{t-p+2} \end{bmatrix} - \begin{bmatrix} \rho_1 & \rho_2 & \cdots & \cdots & \rho_p \\ 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \omega_t \\ \omega_{t-1} \\ \vdots \\ \omega_{t-p+1} \end{bmatrix} = 0,$$

or more compactly, as

$$x_{t+1} - \Pi x_t = 0, \qquad x_{t+1} = [\omega_{t+1}, \omega_t, ..., \omega_{t-p+2}]'.$$

Thus (1) is sufficiently general to characterize a system of arbitrary order.

The goal of the linearization step is to convert (1) into a linear system, which can then be solved using any of the procedures outlined below. Anticipating the notation that follows in Section 2.2, the form for the system we seek is given by

$$Ax_{t+1} = Bx_t. \tag{2}$$

Denoting the steady state of the system as $\Psi(\overline{x}) = 0$, where $\overline{x}$ is understood to be a function of μ, linearization is accomplished via a first-order Taylor series approximation of (1) around

its steady state, given by [1]

$$0 \approx \Psi(\overline{x}) + \frac{\partial \Psi}{\partial x_t}(\overline{x}) \times (x_t - \overline{x}) + \frac{\partial \Psi}{\partial x_{t+1}}(\overline{x}) \times (x_{t+1} - \overline{x}), \tag{3}$$

where $(x_t - \overline{x})$ is $n \times 1$, and the $n \times n$ matrix $\frac{\partial \Psi}{\partial x_t}(\overline{x})$ denotes the Jacobian of $\Psi(x_{t+1}, x_t)$ with respect to x_t evaluated at $\overline{x}$. That is, the $(i,j)^{th}$ element of $\frac{\partial \Psi}{\partial x_t}(\overline{x})$ is the derivative of the i^{th} equation in (1) with respect to the j^{th} element of x_t. Defining $A = \frac{\partial \Psi}{\partial x_{t+1}}(\overline{x})$ and $B = -\frac{\partial \Psi}{\partial x_t}(\overline{x})$ yields (2), where variables are expressed as deviations from steady state values.

1.2 Logarithmic Approximations

It is often useful to work with log-linear approximations of (1), due to their ease of interpretation. For illustration, we begin with a simple example in which the system is 1×1, and can be written as

$$x_{t+1} = f(x_t).$$

Taking logs and using the identity $x_t = e^{\log x_t}$, the system becomes

$$\log x_{t+1} = \log \left[f(e^{\log x_t}) \right].$$

Then approximating,

$$\log x_{t+1} \approx \log \left[f(\overline{x}) \right] + \frac{f'(\overline{x})}{f(\overline{x})} \left(\log(x_t) - \log(\overline{x}) \right),$$

[1] It is also possible to work with higher-order approximations; e.g., see Schmitt-Grohé and Uribe (2002).

or since $\log[f(\overline{x})] = \log \overline{x}$,

$$\log\left(\frac{x_{t+1}}{\overline{x}}\right) \approx \frac{f'(\overline{x})}{f(\overline{x})}\left(\log(\frac{x_t}{\overline{x}})\right).$$

Note that $\frac{f'()}{f()}$ is the elasticity of x_{t+1} with respect to x_t. Moreover, writing x_t as $\overline{x}+\varepsilon_t$, where ε_t denotes a small departure from steady state,

$$\log\left(\frac{x_t}{\overline{x}}\right) = \log\left(1+\frac{\varepsilon_t}{\overline{x}}\right) \approx \frac{\varepsilon_t}{\overline{x}},$$

and thus $\log\left(\frac{x_t}{\overline{x}}\right)$ is seen as expressing x_t in terms of its percentage deviation from steady state.

Returning to the $n \times 1$ case, re-write (1) as

$$\Psi_1(x_{t+1}, x_t) = \Psi_2(x_{t+1}, x_t), \tag{4}$$

since it is not possible to take logs of both sides of (1). Again using the identity $x_t = \log e^{x_t}$, taking logs of (4) and rearranging yields

$$\log \Psi_1(\log e^{x_{t+1}}, \log e^{x_t}) - \log \Psi_2(\log e^{x_{t+1}}, \log e^{x_t}) = 0. \tag{5}$$

The first-order Taylor series approximation of this converted system yields the log-linear approximation we seek. The approximation for the first term is

$$\log \Psi_1(x_{t+1}, x_t) \approx \log[\Psi_1(\overline{x})] + \frac{\partial \log[\Psi_1]}{\partial \log(x_t)}(\overline{x}) \times \left[\log(\frac{x_t}{\overline{x}})\right] + \frac{\partial \log[\Psi_1]}{\partial \log(x_{t+1})}(\overline{x}) \times \left[\log(\frac{x_{t+1}}{\overline{x}})\right], \tag{6}$$

where $\frac{\partial \log[\Psi_1]}{\partial \log(x_t)}(\overline{x})$ and $\frac{\partial \log[\Psi_1]}{\partial \log(x_{t+1})}(\overline{x})$ are $n \times n$ Jacobian matrices, and $\left[\log(\frac{x_t}{\overline{x}})\right]$ and $\left[\log(\frac{x_{t+1}}{\overline{x}})\right]$ are $n \times 1$ vectors. The approximation of the second term in (5) is analogous. Then combining the two approximations and rearranging yields (2), which takes the specific form

$$\left[\frac{\partial \log[\Psi_1]}{\partial \log(x_{t+1})}(\overline{x}) - \frac{\partial \log[\Psi_2]}{\partial \log(x_{t+1})}(\overline{x})\right] \times \left[\log(\frac{x_{t+1}}{\overline{x}})\right] = -\left[\frac{\partial \log[\Psi_1]}{\partial \log(x_t)}(\overline{x}) - \frac{\partial \log[\Psi_2]}{\partial \log(x_t)}(\overline{x})\right] \times \left[\log(\frac{x_t}{\overline{x}})\right]. \tag{7}$$

The elements of A and B are now elasticities, and the variables of the system are expressed in terms of percentage deviations from steady state.

In Part II of the text we will discuss several empirical applications that involve the need to approximate (1) or (5) repeatedly for alternative values of μ. In such cases, it is useful to automate the linearization stage via the use of a numerical gradient calculation procedure. We introduce this briefly here in the context of approximating (1); the approximation of (5) is analogous.

Gradient procedures are designed to construct the Jacobian matrices in (3) without analytical expressions for the required derivatives. Derivatives are instead calculated numerically, given the provision of three components by the user. The first two components are a specification of μ and a corresponding specification of $\overline{x}$. The third component is a procedure designed to return the $n \times 1$ vector of values z generated by (1) for two cases. In the first case x_{t+1} is treated as variable and x_t is fixed at $\overline{x}$; in the second case x_t is treated as variable and x_{t+1} is fixed at $\overline{x}$. The gradient procedure delivers the Jacobian $\frac{\partial \Psi}{\partial x_{t+1}}(\overline{x}) = A$ in the first case and $\frac{\partial \Psi}{\partial x_t}(\overline{x}) = -B$ in the second case. Examples follow.

1.3 Examples

Consider the simple resource constraint

$$y_t = c_t + i_t,$$

indicating that output (y_t) can be either consumed (c_t) or invested (i_t). This equation is already linear. In the notation of (1) the equation appears as

$$y_t - c_t - i_t = 0;$$

and in terms of (3), with $x_t = [y_t \quad c_t \quad i_t]'$ and the equation representing the i^{th} of the system, the i^{th} row of $\frac{\partial \Psi}{\partial x_t}(\overline{x}) = [1 \quad -1 \quad -1]$. In the notation of (5), the equation appears as

$$\log y_t - \log\left[\exp(\log c_t) - \exp(\log i_t)\right] = 0,$$

and in terms of (7), the i^{th} row of the right-hand-side matrix is

$$\left[\frac{\partial \log[\Psi_1]}{\partial \log(x_t)}(\overline{x}) - \frac{\partial \log[\Psi_2]}{\partial \log(x_t)}(\overline{x})\right] = \left[\frac{1}{\overline{y}} \quad \frac{-\overline{c}}{\overline{c}+\overline{i}} \quad \frac{-\overline{i}}{\overline{c}+\overline{i}}\right]. \tag{8}$$

Finally, to use a gradient procedure to accomplish log-linear approximation, the i^{th} return of the system-evaluation procedure would be

$$z_i = \log y_t - \log\left[\exp(\log c_t) - \exp(\log i_t)\right].$$

As an additional example consider the Cobb-Douglas production function

$$y_t = a_t k_t^{\alpha} n_t^{1-\alpha}, \quad \alpha \in (0,1),$$

where output is produced by use of capital (k_t) and labor (n_t) and is subject to a technology or productivity shock (a_t). Linear approximation of this equation is left as an exercise. To accomplish log-linear approximation, taking logs of the equation and rearranging maps into the notation of (5) as

$$\log y_t - \log a_t - \alpha \log k_t - (1-\alpha) \log n_t = 0.$$

With $x_t = \left[\log \frac{y_t}{\overline{y}} \;\; \log \frac{a_t}{\overline{a}} \;\; \log \frac{k_t}{\overline{k}} \;\; \log \frac{n_t}{\overline{n}}\right]'$, the i^{th} row of the right-hand-side matrix in (7) is

$$\left[\frac{\partial \log\left[\Psi_1\right]}{\partial \log(x_t)}(\overline{x}) - \frac{\partial \log\left[\Psi_2\right]}{\partial \log(x_t)}(\overline{x})\right] = \left[1 \;\; -1 \;\; -\alpha \;\; -(1-\alpha)\right]. \tag{9}$$

And to use a gradient procedure to accomplish log-linear approximation, the i^{th} return of the system-evaluation procedure would be

$$z_i = \log y_t - \log a_t - \alpha \log k_t - (1-\alpha) \log n_t.$$

2 Solution Methods

Having approximated the model as in (2), we next seek a solution of the form

$$x_{t+1} = Fx_t + Gv_{t+1}. \tag{10}$$

This solution represents the time series behavior of $\{x_t\}$ as a function of $\{v_t\}$, where v_t is a vector of exogenous innovations, or as frequently referenced, structural shocks.

Here we present four popular approaches to the derivation of (10) from (2). Each approach involves an alternative way of expressing (2), and employs specialized notation. Before describing these approaches, we introduce an explicit example of (2), which we will map into the notation employed under each approach to aid with the exposition.

The example is a linearized stochastic version of Ramsey's (1928) optimal growth model:

$$\widetilde{y}_{t+1} - \widetilde{a}_{t+1} - \alpha\widetilde{k}_{t+1} = 0 \tag{11}$$

$$\widetilde{y}_{t+1} - \gamma_c\widetilde{c}_{t+1} - \gamma_i\widetilde{i}_{t+1} = 0 \tag{12}$$

$$\theta_{1c}E_t(\widetilde{c}_{t+1}) + \theta_a E_t(\widetilde{a}_{t+1}) + \theta_k E_t(\widetilde{k}_{t+1}) + \theta_{2c}\widetilde{c}_t = 0 \tag{13}$$

$$\widetilde{k}_{t+1} - \delta_k\widetilde{k}_t - \delta_i\widetilde{i}_t = 0 \tag{14}$$

$$\widetilde{a}_{t+1} - \rho\widetilde{a}_t = \varepsilon_{t+1}. \tag{15}$$

The variables $\left\{\widetilde{y}_t, \widetilde{c}_t, \widetilde{i}_t, \widetilde{k}_t, \widetilde{a}_t\right\}$ represent output, consumption, investment, physical capital, and a productivity shock, all expressed as logged deviations from steady state values. The

variable ε_t is a serially uncorrelated stochastic process. The vector

$$\mu = [\alpha\, \gamma_c\, \gamma_i\, \theta_{1c}\, \theta_a\, \theta_k\, \theta_{2c}\, \delta_k\, \delta_i, \rho]'$$

contains the 'deep' parameters of the model.

Two modifications enable a mapping of the model into a specification resembling (2). First, the expectations operator $E_t(.)$ is dropped from (13), introducing an expectational error into the modified equation; let this error be denoted as η_{ct+1}. Next, the innovation term ε_{t+1} in (15) must be accommodated. The resulting expression is

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & -\alpha & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -\gamma_c & -\gamma_i & 0 & 0 \\ 0 & \theta_{1c} & 0 & \theta_k & \theta_z \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \widetilde{y}_{t+1} \\ \widetilde{c}_{t+1} \\ \widetilde{i}_{t+1} \\ \widetilde{k}_{t+1} \\ \widetilde{z}_{t+1} \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\theta_{2c} & 0 & 0 & 0 \\ 0 & 0 & \delta_i & \delta_k & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \widetilde{y}_t \\ \widetilde{c}_t \\ \widetilde{i}_t \\ \widetilde{k}_t \\ \widetilde{z}_t \end{bmatrix}}_{x_t}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \varepsilon_{t+1} \end{bmatrix}}_{\upsilon_{t+1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \eta_{ct+1} \\ 0 \end{bmatrix}}_{\eta_{t+1}}. \quad (16)$$

2.1 Blanchard and Kahn's Method

The first solution method we present was developed by Blanchard and Kahn (1980), and is applied to models written as

$$\begin{bmatrix} x_{1t+1} \\ E_t(x_{2t+1}) \end{bmatrix} = \widetilde{A} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + Ef_t, \tag{17}$$

where the model variables have been divided into an $n_1 \times 1$ vector of endogenous predetermined variables x_{1t} (defined as variables for which $E_t x_{1t+1} = x_{1t+1}$), and an $n_2 \times 1$ vector of endogenous non-predetermined variables x_{2t}. The $k \times 1$ vector f_t contains exogenous forcing variables.

Following the approach of King and Watson (2002), a preliminary step is taken before casting a given model into the form (17). The step is referred to as a system reduction: it involves writing the model in terms of a subset of variables that are uniquely determined. In terms of the example, note that observations on $\widetilde{a}_t$ and $\widetilde{k}_t$ are sufficient for determining $\widetilde{y}_t$ using (11), and that given $\widetilde{y}_t$, the observation of either $\widetilde{c}_t$ or $\widetilde{i}_t$ is sufficient for determining both variables using (12). Thus we proceed in working directly with $\left\{\widetilde{c}_t, \widetilde{k}_t, \widetilde{a}_t\right\}$ using (13)−(15), and recover $\left\{\widetilde{y}_t, \widetilde{i}_t\right\}$ as functions of $\left\{\widetilde{c}_t, \widetilde{k}_t, \widetilde{a}_t\right\}$ using (11) and (12). Among $\left\{\widetilde{c}_t, \widetilde{k}_t, \widetilde{a}_t\right\}$, $\widetilde{k}_t$ is predetermined (given $\widetilde{k}_t$ and $\widetilde{i}_t$, $\widetilde{k}_{t+1}$ is determined as in (14)); $\widetilde{c}_t$ is endogenous but not predetermined (as indicated in (13), its time-$(t+1)$ realization is associated with an expectations error); and $\widetilde{a}_t$ is an exogenous forcing variable. Thus in the notation of (17),

we seek a specification of the model in the form

$$\begin{bmatrix} \widetilde{k}_{t+1} \\ E_t(\widetilde{c}_{t+1}) \end{bmatrix} = \widetilde{A} \begin{bmatrix} \widetilde{k}_t \\ \widetilde{c}_t \end{bmatrix} + E\widetilde{a}_t. \tag{18}$$

To obtain this expression, let $\xi_t = \begin{bmatrix} \widetilde{y}_t & \widetilde{i}_t \end{bmatrix}'$, $\zeta_t = \begin{bmatrix} \widetilde{k}_t & \widetilde{c}_t \end{bmatrix}'$, and note that $E_t(\widetilde{a}_{t+1}) = \rho\widetilde{a}_t$. In terms of these variables, the model may be written as

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & -\gamma_i \end{bmatrix}}_{\Psi_0} \xi_t = \underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & \gamma_c \end{bmatrix}}_{\Psi_1} \zeta_t + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\Psi_2} \widetilde{a}_t \tag{19}$$

$$\underbrace{\begin{bmatrix} \theta_k & \theta_{1c} \\ 1 & 0 \end{bmatrix}}_{\Psi_3} E_t(\zeta_{t+1}) = \underbrace{\begin{bmatrix} 0 & -\theta_{2c} \\ \delta_k & 0 \end{bmatrix}}_{\Psi_4} \zeta_t + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \delta_i \end{bmatrix}}_{\Psi_5} \xi_t + \underbrace{\begin{bmatrix} -\theta_a\rho \\ 0 \end{bmatrix}}_{\Psi_6} \widetilde{a}_t. \tag{20}$$

Next, substituting (19) into (20), which requires inversion of Ψ_0, we obtain

$$\Psi_3 E_t(\zeta_{t+1}) = \left[\Psi_4 + \Psi_5\Psi_0^{-1}\Psi_1\right]\zeta_t + \left[\Psi_6 + \Psi_5\Psi_0^{-1}\Psi_2\right]\widetilde{a}_t. \tag{21}$$

Finally, premultiplying (21) by Ψ_3^{-1} yields a specification in the form of (18); Blanchard and Kahn's solution method may now be implemented. Hereafter, we describe its implementation in terms of the notation employed in (17).

The method begins with a Jordan decomposition of $\widetilde{A}$, yielding

$$\widetilde{A} = \Lambda^{-1} J \Lambda, \tag{22}$$

where the diagonal elements of J, consisting of the eigenvalues of $\widetilde{A}$, are ordered in increasing absolute value in moving from left to right.[2] Thus J may be written as

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tag{23}$$

where the eigenvalues in J_1 lie on or within the unit circle, and those in J_2 lie outside of the unit circle. J_2 is said to be unstable or explosive, since J_2^n diverges as n increases. The matrices Λ and E are partitioned conformably as

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \tag{24}$$

where Λ_{11} is conformable with J_1, etc. If the number of explosive eigenvalues is equal to the number of non-predetermined variables, the system is said to be saddle-path stable and a unique solution to the model exists. If the number of explosive eigenvalues exceeds the number of non-predetermined variables no solution exists (and the system is said to be a source); and in the opposite case an infinity of solutions exist (and the system is said to be a sink).

Proceeding under the case of saddle-path stability, substitution for $\widetilde{A}$ in (18) yields

$$\begin{bmatrix} x_{1t+1} \\ E_t(x_{2t+1}) \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} f_t. \tag{25}$$

[2]Eigenvalues of a matrix Θ are obtained from the solution of equations of the form $\Theta e = \lambda e$, where e is an eigenvector and λ the associated eigenvalue. The GAUSS command `eigv` performs this decomposition.

Next, the system is pre-multiplied by Λ, yielding

$$\begin{bmatrix} \acute{x}_{1t+1} \\ E_t(\acute{x}_{2t+1}) \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \acute{x}_{1t} \\ \acute{x}_{2t} \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} f_t, \tag{26}$$

where

$$\begin{bmatrix} \acute{x}_{1t} \\ \acute{x}_{2t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{28}$$

This transformation effectively 'de-couples' the system, so that the non-predetermined variables depend only upon the unstable eigenvalues of $\widetilde{A}$ contained in J_2, as expressed in the lower part of (26).

Having de-coupled the system, we derive a solution for the non-predetermined variables by performing a forward iteration on the lower portion of (26). Using f_{2t} to denote the portion of f_t conformable with D_2, this is accomplished as follows. First, re-express the lower portion of (26) as

$$\acute{x}_{2t} = J_2^{-1} E_t(\acute{x}_{2t+1}) - J_2^{-1} D_2 f_{2t}. \tag{29}$$

This implies an expression for x_{2t+1} of the form

$$\acute{x}_{2t+1} = J_2^{-1} E_{t+1}(\acute{x}_{2t+2}) - J_2^{-1} D_2 f_{2t+1}, \tag{30}$$

which can be substituted into (29) to obtain

$$\acute{x}_{2t} = J_2^{-2} E_t(\acute{x}_{2t+1}) - J_2^{-2} D_2 E_t(f_{2t+1}) - J_2^{-1} D_2 f_{2t}. \tag{31}$$

In writing (31) we have exploited the Law of Iterated Expectations, which holds that $E_t\left[E_{t+1}(x_t)\right] = E_t(x_t)$ for any x_t (e.g., see Ljungqvist and Sargent, 2004). Since J_2 contains explosive eigenvalues, J_2^{-n} disappears as n approaches infinity, thus continuation of the iteration process yields

$$\acute{x}_{2t} = -\sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t(f_{2t+i}). \tag{32}$$

Mapping this back into an expression for x_{2t} using (27), we obtain

$$x_{2t} = -\Lambda_{22}^{-1}\Lambda_{21} x_{1t} - \Lambda_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t(f_{2t+i}). \tag{33}$$

In the case of the example model presented above, $E_t(f_{2t+i}) = \rho^i \widetilde{a}_t$, and thus (33) becomes

$$x_{2t} = -\Lambda_{22}^{-1}\Lambda_{21} x_{1t} - \Lambda_{22}^{-1} J_2^{-1} \left(I - \rho J_2^{-1} D_2\right)^{-1} \widetilde{a}_t. \tag{34}$$

Finally, to solve the non-explosive portion of the system begin by expanding the upper portion of (25):

$$x_{1t+1} = \widetilde{A}_{11} x_{1t} + \widetilde{A}_{22} x_{2t} + E_1 f_t, \tag{35}$$

where $\widetilde{A}_{11}$ and $\widetilde{A}_{22}$ are partitions of $\Lambda^{-1} J \Lambda$ conformable with x_{1t} and x_{2t}. Then substituting for x_{2t} using (33) yields a solution for x_{1t} of the form given by (10).

We conclude this subsection by highlighting two requirements of this solution method.

First, a model-specific system reduction is employed to obtain an expression of the model that consists of a subset of its variables. The variables in the subset are distinguished as being either predetermined or non-predetermined. Second, invertibility of the lead matrices Ψ_0 and Ψ_3 is required in order to obtain a specification of the model amenable for solution.

Exercise 1 *Write computer code for mapping the example model expressed in (11)-(15) into the form of the representation given in (17).*

2.2 Sims's Method

Sims (2001) proposes a solution method applied to models expressed as

$$Ax_{t+1} = Bx_t + E + C\upsilon_{t+1} + D\eta_{t+1}, \tag{36}$$

where E is a matrix of constants.[3] Relative to the notation we have employed above, E is unnecessary because the variables in x_t are expressed in terms of deviations from steady state values. Like Blanchard and Kahn's (1980) method, Sims' method involves a de-coupling of the system into explosive and non-explosive portions. However, rather than expressing variables in terms of expected values, expectations operators have been dropped, giving rise to the expectations errors contained in η_{t+1}. Also, while Blanchard and Kahn's method entails isolation of the forcing variables from x_{t+1}, these are included in x_{t+1} under Sims' method; thus the appearance in the system of the vector of shocks to these variables υ_{t+1}. Third, Sims' method does not require an initial system-reduction step. Finally, it does not

[3]The programs available on Sims' website perform all of the steps of this procedure. The web address is: http://www.princeton.edu/~sims/. The programs are written in Matlab; analogous code written in GAUSS is currently under construction.

entail a distinction between predetermined and non-predetermined variables.

Note from (16) that the example model has already been cast in the form of (36), thus we proceed directly to a characterization of the solution method. The first step employs a 'QZ factorization' to decompose A and B into unitary upper triangular matrices:

$$A = Q'\Lambda Z' \tag{37}$$

$$B = Q'\Omega Z', \tag{38}$$

where (Q, Z) are unitary, and (Λ, Ω) are upper triangular.[4] Next, (Q, Z, Λ, Ω) are ordered such that, in absolute value, the generalized eigenvalues of A and B are organized in Λ and Ω in increasing order moving from left to right, just as in Blanchard and Kahn's Jordan decomposition procedure.[5] Having obtained the factorization, the original system is then pre-multiplied by Q, yielding the the transformed system expressed in terms of $z_{t+1} = Z'x_{t+1}$:

$$\Lambda z_t = \Omega z_{t-1} + QE + QC\upsilon_t + QD\eta_t, \tag{39}$$

where we have lagged the system by one period in order to match the notation (and code) of Sims.

Next, as with Blanchard and Kahn's (1980), method, (39) is partitioned into explosive

[4] A unitary matrix Θ satisfies $\Theta'\Theta = \Theta\Theta' = I$. If Q and/or Z contain complex values, the transpositions reflect complex conjugation, that is, each complex entry is replaced by its conjugate and then transposed.

[5] Generalized eigenvalues of Θ are obtained as the solution to $\Theta e = \lambda \Xi e$, where Ξ is a symmetric matrix. Sims' website also provides a program that orders the eigenvalues appropriately.

and non-explosive blocks:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [E + C\upsilon_t + D\eta_t]. \tag{40}$$

The explosive block (the lower equations) is solved as follows. Letting $w_t = Q(E+C\upsilon_t+D\eta_t)$ (partitioned conformably as w_{1t} and w_{2t}), the lower block of (40) is given by

$$\Lambda_{22} z_{2t} = \Omega_{22} z_{2t-1} + w_{2t}. \tag{41}$$

Leading (41) by one period and solving for z_{2t} yields

$$z_{2t} = M z_{2t+1} - \Omega_{22}^{-1} w_{2t+1}, \tag{42}$$

where $M = \Omega_{22}^{-1}\Lambda_{22}$. Then recursive substitution for z_{2t+1}, z_{2t+2}, ... yields

$$z_{2t} = -\sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} w_{2t+1+i}, \tag{43}$$

since $\lim_{t\to\infty} M^t z_{2t} = 0$. Recalling that w_t is defined as $w_t = Q(E + C\upsilon_t + D\eta_t)$, note that (43) expresses z_{2t} as a function of future values of structural and expectational errors. But z_{2t} is known at time t, and $E_t(\eta_{t+s}) = E_t(\upsilon_{t+s}) = 0$ for $s > 0$, thus (43) may be written as

$$z_{2t} = -\sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} Q_2 E_2, \tag{44}$$

where Q_2E_2 are the lower portions of QE conformable with z_2.[6] Postmultiplying (44) by $\Omega_{22}^{-1}Q_2E_2$ and noting that $-\sum_{i=0}^{\infty}M^i = -(I-M)^{-1}$, the solution of z_{2t} is obtained as

$$z_{2t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 E. \tag{45}$$

Having solved for z_{2t}, the final step is to solve for z_{1t} in (40). Note that the solution of z_{1t} requires a solution for the expectations errors that appear in (40). As Sims notes, when a unique solution for the model exists, it will be the case that a systematic relationship exists between the expectations errors associated with z_{1t} and z_{2t}; exploiting this relationship yields a straightforward means of solving for z_{1t}. The necessary and sufficient condition for uniqueness is given by the existence of a $k \times (n-k)$ matrix Φ that satisfies

$$Q_1 D = \Phi Q_2 D, \tag{46}$$

which represents the systematic relationship between the expectations errors associated with z_{1t} and z_{2t} noted above. Given uniqueness, and thus the ability to calculate Φ as in (46), the solution of z_{1t} proceeds with the pre-multiplication of (39) by $[I \quad -\Phi]$, which yields

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + [Q_1 - \Phi Q_2]\,[E + C\upsilon_t + D\eta_t]\,. \tag{47}$$

Then due to (46), the loading factor for the expectational errors in (47) is zero, and thus the

[6] Sims also considers the case in which the structural innovations υ_t are serially correlated, which leads to a generalization of (44).

system may be written in the form

$$x_t = \Theta_E + \Theta_0 x_{t-1} + \Theta_1 \upsilon_t, \tag{48}$$

where

$$H = Z \begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12} - \Phi\Lambda_{22}) \\ 0 & I \end{bmatrix} \tag{49}$$

$$\Theta_E = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1} Q_2 \end{bmatrix} E \tag{50}$$

$$\Theta_0 = Z\Lambda_{11}^{-1}[\Omega_{11}(\Omega_{12} - \Phi\Omega_{22})]Z' \tag{51}$$

$$\Theta_1 = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ 0 \end{bmatrix} D. \tag{52}$$

Exercise 2 *Using the code cited for this method, compute the solution for (11)-(15) for given values of* μ.

2.3 Klein's Method

Klein (2000) proposes a solution method that is a hybrid of those of Blanchard and Kahn (1980) and Sims (2001).[7] The method is applied to systems written as

$$\widetilde{A}E_t(x_{t+1}) = \widetilde{B}x_t + Ef_t, \tag{53}$$

[7]GAUSS and Matlab code that implement this solution method are available at http://www.ssc.uwo.ca/economics/faculty/klein/.

where the vector f_t (of length n_z) has a zero-mean vector autoregressive (VAR) specification with autocorrelation matrix Φ; additionally $\widetilde{A}$ may be singular.[8]

Like Blanchard and Kahn, Klein distinguishes between the predetermined and non-predetermined variables of the model. The former are contained in x_{1t+1}, the latter in x_{2t+1}: $E_t(x_{t+1}) = \begin{bmatrix} x_{1t+1} & E_t(x_{2t+1}) \end{bmatrix}'$. The solution approach once again involves de-coupling the system in to non-explosive and explosive components, and solving the two components in turn.

Returning to the example model expressed in (11)-(15), the form of the model amenable to the implementation of Klein's method is given by (21), repeated here for convenience:

$$\Psi_3 E_t(\zeta_{t+1}) = \left[\Psi_4 + \Psi_5 \Psi_0^{-1} \Psi_1\right] \zeta_t + \left[\Psi_6 + \Psi_5 \Psi_0^{-1} \Psi_2\right] \widetilde{a}_t. \tag{54}$$

The main advantage of Klein's approach relative to Blanchard and Kahn's is that Ψ_3 may be singular. To proceed with the description of Klein's approach, we revert to the notation employed in (53).

Klein's approach overcomes the potential non-invertibility of $\widetilde{A}$ by implementing a complex generalized Schur decomposition to decompose $\widetilde{A}$ and $\widetilde{B}$. This is in place of the QZ decomposition employed by Sims. In short, the Schur decomposition is a generalization of the QZ decomposition that allows for complex eigenvalues associated with $\widetilde{A}$ and $\widetilde{B}$. Given the decomposition of $\widetilde{A}$ and $\widetilde{B}$, Klein's method closely follows that of Blanchard and Kahn.

[8] See Chapter 4 for a description of VAR models.

The Schur decompositions of $\widetilde{A}$ and $\widetilde{B}$ are given by

$$Q\widetilde{A}Z = S \tag{55}$$

$$Q\widetilde{B}Z = T, \tag{56}$$

where (Q, Z) are unitary and (S, T) are upper triangular matrices with diagonal elements containing the generalized eigenvalues of $\widetilde{A}$ and $\widetilde{B}$. Once again the eigenvalues are ordered in increasing value in moving from left to right. Partitioning Z as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \tag{57}$$

Z_{11} is $n_1 \times n_1$ and corresponds to the non-explosive eigenvalues of the system. Given saddle-path stability, this conforms with x_1, which contains the predetermined variables of the model.

Having obtained this decomposition, the next step in solving the system is to triangularize (53) as was done in working with the QZ decomposition. Begin by defining

$$z_t = Z^H x_t, \tag{58}$$

where Z^H refers to a Hermitian transpose.[9] This transformed vector is divided into $n_1 \times 1$ stable (s_t) and $n_2 \times 1$ unstable (u_t) components. Then since $\widetilde{A} = Q'SZ^H$ and $\widetilde{B} = Q'SZ^H$,

[9] Given a matrix Θ, if the lower triangular portion of Θ is the complex conjugate transpose of the upper triangle portion of Θ, then Θ is denoted as Hermitian.

(53) may be written as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} E f_t; \tag{59}$$

once again, the linear portion of (59) contains the unstable components of the system. Solving this component via forward iteration, we obtain[10]

$$u_t = M f_t \tag{60}$$

$$vec(M) = \left[(\Phi^T \otimes S_{22}) - I_{n_z} \otimes T_{22}\right]^{-1} vec(Q_2 E). \tag{61}$$

This solution for the unstable component is then used to solve the stable component, yielding

$$s_{t+1} = S_{11}^{-1} T_{11} s_t + S_{11}^{-1} \{T_{12} M - S_{12} M \Phi + Q_1 E\} f_t - Z_{11}^{-1} Z_{12} M \upsilon_{t+1}, \tag{62}$$

where υ_{t+1} is a serially uncorrelated stochastic process representing the innovations in the VAR specification for f_{t+1}. In the context of our example model, f_t corresponds to $\widetilde{a}_t$, the

[10]The appearance of the *vec* operator accommodates the VAR specification for f_t. In the context of the example model, Φ^T is replaced by the scalar ρ^T, and (61) becomes $M = \left[\rho^T S_{22} - T_{22}\right]^{-1} Q_2 E$.

innovation to which is ε_t. In terms of the original variables the solution is expressed as

$$x_{2t} = Z_{21}Z_{11}^{-1}x_{1t} + Nf_t \tag{63}$$

$$x_{1t+1} = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}x_{2t} + Lf_t \tag{64}$$

$$N = (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})M \tag{65}$$

$$L = -Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{12}M + Z_{11}S_{11}^{-1}[T_{12}M - S_{12}M\Phi + Q_1E] + Z_{12}M\Phi. \tag{66}$$

This solution can be cast into the form of (10) as

$$x_{1t+1} = \left[Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{21}Z_{11}^{-1}\right]x_{1t} + \left[Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}N + L\right]f_t. \tag{67}$$

Exercise 3 *Apply Klein's code to the example model presented in (11)-(15).*

2.4 An Undetermined Coefficients Approach

Uhlig (1999) proposes a solution method based on the method of undetermined coefficients.[11] The method is applied to systems written as

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Lf_{t+1} + Mf_t] \tag{68}$$

$$f_{t+1} = Nf_t + v_{t+1}, \quad E_t(v_{t+1}) = 0. \tag{69}$$

[11] Matlab code available for implementing this solution method is available at:
http://www.wiwi.hu-berlin.de/wpol/html/toolkit.htm.
GAUSS code is currently under construction.

With respect to the example model in (11)-(14) let $x_t = [y_t\, c_t\, i_t\, k_t]'$. Then lagging the first two equations, which are subject neither to structural shocks nor expectations errors, the matrices in (68) and (69) are given by

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \theta_{1c} & 0 & \theta_k \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 1 & -\gamma_c & -\gamma_i & 0 \\ 0 & \theta_{2c} & 0 & 0 \\ 0 & 0 & -\delta_i & -\delta_k \end{bmatrix}, \tag{70}$$

$H = 0$, $L = [0\, 0\, \theta_a\, 0]'$, $M = [-1\, 0\, 0\, 0]'$, and $N = \rho$.

Solutions to (68)-(69) take the form

$$x_t = Px_{t-1} + Qf_t. \tag{71}$$

In deriving (71), we will confront the problem of solving matrix quadratic equations of the form

$$\Psi P^2 - \Gamma P - \Theta = 0 \tag{72}$$

for the $m \times m$ matrix P. Thus we first describe the solution of such equations.

To begin, define

$$\underset{2m\times 2m}{\Xi} = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_{m\times m} \end{bmatrix}, \quad \underset{2m\times 2m}{\Delta} = \begin{bmatrix} \Psi & 0_{m\times m} \\ 0_{m\times m} & I_m \end{bmatrix}. \tag{73}$$

Given these matrices, let s and λ denote the generalized eigenvector and eigenvalue of Ξ

with respect to Δ, and note that $s' = [\lambda x', x']$ for some $x \in \Re^m$. Then the solution to the matrix quadratic is given by

$$P = \Omega \Lambda \Omega^{-1}, \quad \Omega = [x_1, ..., x_m], \quad \Lambda = diag(\lambda_1, ..., \lambda_m), \tag{74}$$

so long as the m eigenvalues contained in Λ and $(x_1, ..., x_m)$ are linearly independent. The solution is stable if the generalized eigenvalues are all less than one in absolute value.

Returning to the solution of the system in (68)-(69), the first step towards obtaining (71) is to combine these three equations into a single equation. This is accomplished in two steps. First, write x_t in (68) in terms of its relationship with x_{t-1} given by (71), and do the same for x_{t+1}, where the relationship is given by

$$x_{t+1} = P^2 x_{t-1} + PQ f_t + Q f_{t+1}. \tag{75}$$

Next, write f_{t+1} in terms with its relationship with f_t given by (69). Taking expectations of the resulting equation yields

$$0 = [FP^2 + GP + H] x_{t-1} + [(FP + G)Q + M + (FQ + L)N] f_t. \tag{76}$$

Note that in order for (76) to hold, the coefficients on x_{t-1} and f_t must be zero. The first restriction implies that P must satisfy the matrix quadratic equation

$$0 = FP^2 + GP + H, \tag{77}$$

the solution of which is obtained as indicated in (73) and (74). The second restriction requires the derivation of Q which satisfies

$$(FP + G)Q + M + (FQ + L)N = 0. \tag{78}$$

The required Q can be shown to be given by

$$Q = V^{-1}\left[-vec(LN + M)\right], \tag{79}$$

where V is defined as

$$V = N' \otimes F + I_k \otimes (FP + G). \tag{80}$$

The solutions for P and Q will be unique so long as the matrix P has stable eigenvalues.

As noted by Christiano (2002), this solution method is particularly convenient for working with models involving endogenous variables that have differing associated information sets. Such models can be cast in the form of (68)-(69), with the expectations operator $\mathring{E}_t$ replacing E_t. In terms of calculating the expectation of an $n \times 1$ vector X_t, $\mathring{E}_t$ is defined as

$$\mathring{E}_t(X_t) = \begin{bmatrix} E(X_{1t}|\Xi_{1t}) \\ \vdots \\ E(X_{nt}|\Xi_{nt}) \end{bmatrix}, \tag{81}$$

where Ξ_{it} represents the information set available for formulating expectations over the i^{th} element of X_t. Thus systems involving this form of heterogeneity may be accommodated using an expansion of the system (68)-(69) specified for a representative agent. The solution

of the expanded system proceeds as indicated above; for details and extensions, see Christiano (2002).

Exercise 4 *Apply Uhlig's code to the example model presented in (11)-(15).*

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