Dynamic Stochastic General Equilibrium Models

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Note on Dynamic Programming

One considers a representative agent whose objective is to find a control (or decision) sequence $\{u_t\}_{t=0}^{\infty}$ such that

$$E_0\left[\sum_{t=0}^{\infty}\beta^t U(x_t, u_t)\right] \tag{1}$$

is maximized, subject to

$$x_{t+1} = F(x_t, u_t, z_t) \tag{2}$$

where

- x_t is a vector of m state variables at period t; u_t is a vector of n control variables;
- z_t is a vector of s exogenous variables whose dynamics does not depend on x_t and u_t;
- $\beta \in (0, 1)$ denotes the discount factor.
- The uncertainty of the model only comes from the exogenous z_t .
- Assume that z_t follow an AR(1) process:

$$z_{t+1} = a + bz_t + \epsilon_{t+1} \tag{3}$$

- where ϵ is independently and identically distributed (i.i.d.).
- The initial condition (x_0, z_0) in this formulation is assumed to be given.

- The problem to solve a dynamic decision problem is to seek a time-invariant policy function G mapping from the state and exogenous (x, z) into the control u.

$$u_t = G(x_t, z_t) \tag{4}$$

- With such a policy function (or control equation), the sequences of state $\{x_t\}_{t=0}^{\infty}$ and control $\{u_t\}_{t=0}^{\infty}$ can be generated by iterating the control equation as well as the state eq. (2), given the initial condition (x_0, z_0) and the exogenous sequence $\{z_t\}_{t=0}^{\infty}$ generated by eq. (3).

The Standard Recursive Method

To find the policy function G by the recursive method, one first defines a value function V:

$$V(x_0, z_0) = \max_{\{u_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_t, u_t) \right] \tag{5}$$

- Expression (5) could be transformed to unveil its recursive structure.

- For this purpose, one first rewrites eq. (5) as follows:

$$V(x_0, z_0) = \max_{\{u_t\}_{t=0}^{\infty}} \left\{ U(x_0, u_0) + \beta E_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_{t+1}, u_{t+1}) \right] \right\} \qquad (6)$$

- It is easy to find that the second term in eq. (6) can be expressed as being β times the value V as defined in eq. (5) with the initial condition (x_1, z_1).
  Therefore, we could rewrite eq. (5) as

$$V(x_0, z_0) = \max_{\{u_t\}_{t=0}^{\infty}} \{U(x_0, u_0) + \beta E_0 [V(x_1, z_1)]\} \qquad (7)$$

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- In every period t, the planner faces the same decision problem.
- Choosing the control variable u_t that maximizes the current return plus the discounted value of the optimum plan from period $t + 1$ onwards.
- Since the problem repeats itself every period the time subscripts become irrelevant.

- One thus can write eq. (7) as

$$V(x,z) = \max_u \left\{ U(x,u) + \beta E \left[V(\tilde{x}, \tilde{z}) \right] \right\} \tag{8}$$

  where the tilde $(\sim)$ over x and z denotes the corresponding next period values.
  They are subject to eq. (2) and (3).
- Eq. (8) is said to be the Bellman equation, named after Richard Bellman (1957).
- If one knows the function V, we then can solve u via the Bellman equation but in generally, this is not the case.

- The typical method in this case is to construct a sequence of value functions by iterating the following equation:

$$V_{j+1}(x, z) = \max_{u} \left\{ U(x, u) + \beta E \left[V_j(\tilde{x}, \tilde{z}) \right] \right\} \tag{9}$$

- In terms of an algorithm, the method can be described as follows:

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- Step 3. If $V_{j+1} = V_j$, stop. Otherwise, update V_j and go to step 1.

- Under some regularity conditions regarding the function U and F, the convergence of this algorithm is warranted.
- However, the difficulty of this algorithm is that in each Step 2, we need to find the optimum u that maximize the right side of eq. (9).
- This task makes it difficult to write a closed form algorithm for iterating the Bellman equation.

The Euler Equation

Alternatively, one starts from the Bellman equation (8).

- The first-order condition for maximizing the right side of the equation takes the form:

$$\frac{\partial U(x,u)}{\partial u} + \beta E\left[\frac{\partial F(x,u,z)}{\partial u}\frac{\partial V(\tilde{x},\tilde{z})}{\partial \tilde{x}}\right] = 0 \tag{10}$$

- The objective here is to find $\partial V/\partial x$. Assume V is differentiable and thus from eq. (8) and exploiting eq. (10) it satisfies

$$\frac{\partial V(x,z)}{\partial x} = \frac{\partial U(x,G(x,z))}{\partial x} + \beta E\left[\frac{\partial F(x,G(x,z),z)}{\partial x}\frac{\partial V(\tilde{x},\tilde{z})}{\partial F}\right] \tag{11}$$

- This equation is often called the Benveniste-Scheinkman formula.

- Assume $\partial F/\partial x = 0$. The above formula becomes

$$\frac{\partial V(x,z)}{\partial x} = \frac{\partial U(x, G(x,z))}{\partial x} \tag{12}$$

- Substituting this formula into eq (10) gives rise to the Euler equation:

$$\frac{\partial U(x,u)}{\partial u} + \beta E\left[\frac{\partial F(x,u,z)}{\partial u}\frac{\partial U(\tilde{x},\tilde{z})}{\partial \tilde{x}}\right] = 0 \tag{13}$$

- In economic analysis, one often encounters models, after some transformation, in which x does not appear in the transition law so that $\partial F/\partial x = 0$ is satisfied.
- However, there are still models in which such transformation is not feasible.

Deriving the First-Order Condition from the Lagrangian

- Suppose for the dynamic optimization problem as represented by eq.(1) and (2), we can define the Lagrangian L:

$$L = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(x_t, u_t) - \beta^{t+1} \lambda'_{t+1} \left[x_{t+1} - F(x_t, u_t, z_t) \right] \right\} \quad (14)$$

- where λ_t, the Lagrangian multiplier, is a $m \times 1$ vector.

- Setting the partial derivatives of L to zero with respect to λ_t, x_t and u_t will yield eq. (2) as well as

$$\frac{\partial U(x_t, u_t)}{\partial x_t} + \beta E_t \left[\frac{\partial F(x_{t+1}, u_{t+1}, z_{t+1})}{\partial x_{t+1}} \lambda_{t+1} \right] = \lambda_t$$

$$\frac{\partial U(x_t, u_t)}{\partial u_t} + E_t \lambda_{t+1} \frac{\partial F(x_t, u_t, z_{t)}}{\partial u_t} = 0 \tag{15}$$

- In comparison with the Euler equation, we find that there is an unobservable variable λ_t appearing in the system.
- Yet, using eq. (14) and (15), one does not have to transform the model into the setting that $\partial F / \partial x = 0$.
- This is an important advantage over the Euler equation.

The Log-linear Approximation Method

- Solving nonlinear dynamic optimization model with log-linear approximation has been proposed in particular by King et al. (1988) and Campbell (1994) in the context of Real Business Cycle models.
- In principle, this approximation method can be applied to the first-order condition either in terms of the Euler equation or derived from the Langrangean.

- Formally, let X_t be the variables, $\bar{X}$ the corresponding steady state. Then,

$$x_t \equiv lnX_t - ln\bar{X}$$

is regarded to be the log-deviation of X_t. In particular, $100x_t$ is the percentage of X_t that it deviates from $\bar{X}$.

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- Step 1. Find the necessary equations characterizing the equilibrium law of motion of the system. These necessary equations should include the state equation (2), the exogenous equation (3) and the first-order condition derived either as Euler equation (13) or from the Lagrangian (14) and (15).

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- Step 3. Log-linearize the necessary equations characterizing the equilibrium law of motion of the system.

- The following building block for such log-linearization can be used:

$$X_t \approx \bar{X} e^{x_t} \tag{16}$$

$$e^{x_t + a y_t} \approx 1 + x_t + a y_t \tag{17}$$

$$x_t y_t \approx 0 \tag{18}$$

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- Step 4. Solve the log-linearized system for the decision rule (which is also in log-linear form) with the method of undetermined coefficients.

- Ramsey (1928) posed a problem of optimal resource allocation, which is now often used as a prototype model of dynamic optimization.
- Let C_t denote consumption, Y_t output and K_t the capital stock.
- Assume that the output is produced by capital stock and it is either consumed or invested, that is, added to the capital stock. Formally,

$$Y_t = A_t K_t^{\alpha} \tag{19}$$

$$Y_t = C_t + K_{t+1} - K_t \tag{20}$$

- where $\alpha \in (0, 1)$ and A_t is the technology which may follow an AR(1) process:

$$A_{t+1} = a_0 + a_1 A_t + \epsilon_{t+1} \tag{21}$$

- Assume ϵ_t to be $i.i.d.$ Equation (19) and (20) indicates that we could write the transition law of capital stock as

$$K_{t+1} = A_t K_t^{\alpha} - C_t \tag{22}$$

- Note that one has assumed here that the depreciation rate of capital stock is equal to 1.
- This is a simplified assumption by which the exact solution is computable.

- The representative agent is assumed to find the control sequence $\{C_t\}_{t=0}^{\infty}$ such that

$$maxE_0 \left[\sum_{t=0}^{\infty} \beta^t \ln C_t \right] \tag{23}$$

given the initial condition (K_0, A_0).

- The first task is to transform the model into a setting so that the state variable K_t does not appear in $F(\cdot)$.
- This can be done by assuming K_{t+1} (instead of C_t) as model's decision variable.
- To achieve a notational consistency in the time subscript, one may denote the decision variable as Z_t. Therefore the model can be rewritten as

$$maxE_0 \left[\sum_{t=0}^{\infty} \beta^t \ln \left(A_t K_t^{\alpha} - K_{t+1} \right) \right] \tag{24}$$

subject to
$K_{t+1} = Z_t$

- Note that here one has used (22) to express C_t in the utility function.
- Also note that in this formulation the state variable in period t is still K_t.
- Therefore $\partial F/\partial x = 0$ and $\partial F/\partial u = 1$. The Bellman equation in this case can be written as

$$V(K_t, A_t) = \max_{K_{t+1}} \{ln(A_t K_t^{\alpha} - K_{t+1}) + \beta E_t[V(K_{t+1}, A_{t+1})]\} \quad (25)$$

- The necessary condition for maximizing the right hand side of the Bellman equation (25) is given by

$$\frac{-1}{A_t K_t^{\alpha} - K_{t+1}} + \beta E_t \left[\frac{\partial V(K_{t+1}, A_{t+1})}{\partial K_{t+1}} \right] = 0 \quad (26)$$

- Meanwhile applying the Benveniste-Scheinkman formula,

$$\frac{\partial V(K_t, A_t)}{\partial K_t} = \frac{\alpha A_t K_t^{\alpha-1}}{A_t K_t^{\alpha} - K_{t+1}} \tag{27}$$

- Substituting eq. (27) into (26) allows us to obtain the Euler equation:

$$\frac{-1}{A_t K_t^{\alpha} - K_{t+1}} + \beta E_t \left[\frac{\alpha A_{t+1} K_{t+1}^{\alpha-1}}{A_{t+1} K_{t+1}^{\alpha} - K_{t+2}} \right] = 0$$

- which can further be written as

$$-\frac{1}{C_t} + \beta E_t \left[\frac{\alpha A_{t+1} K_{t+1}^{\alpha-1}}{A_{t+1} K_{t+1}^{\alpha} - K_{t+2}} \right] = 0 \tag{28}$$

- This Euler equations (28) along with (22) and (21) determine the transition sequences of $\{\mathrm{K}_{t+1}\}_{t=1}^{\infty}$, $\{\mathrm{A}_{t+1}\}_{t=1}^{\infty}$ and $\{\mathrm{C}_t\}_{t=0}^{\infty}$ given the initial condition K_0 and A_0.

- The First-Order Condition Derived from the Lagrangian
- Next, derive the first-order condition from the Lagrangian. Define the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \ln C_t - \sum_{t=0}^{\infty} E_t \left[\beta^{t+1} \lambda'_{t+1} (K_{t+1} - A_t, K_t^{\alpha} + C_t) \right]$$

- Setting to zero the derivatives of L with respect to λ_t, C_t and K_t, one obtains (22) as well as

$$1/C_t - \beta E_t \lambda_{t+1} = 0 \tag{29}$$

$$\beta E_t \lambda_{t+1} \alpha A_t K_t^{\alpha-1} = \lambda_t \tag{30}$$

- These are the first-order conditions derived from the Lagrangian.
- Substituting out for λ one obtains the Euler equation (28).

- The exact solution for this model which could be derived from the standard methods described below can be written as

$$K_{t+1} = \alpha\beta A_t K_t^{\alpha} \tag{31}$$

- This further implies from (22) that

$$C_t = (1-\alpha\beta)A_t K_t^{\alpha} \tag{32}$$

- Given the solution paths for C_t and K_{t+1}, one is then able to derive the steady state.
- One steady state is on the boundary, that is $\bar{K} = 0$ and $\bar{C} = 0$.

- To obtain a more meaningful interior steady state, we take logarithm for both sides of (31) and evaluate the equation at its certainty equivalence form:

$$logK_{t+1} = log(\alpha\beta A_t) + \alpha logK_t \tag{33}$$

- At the steady state, $K_{t+1} = K_t = \bar{K}$. Solving (33) for $log\bar{K}$, we obtain $log\bar{K} = \frac{log(\alpha\beta A)}{1-\alpha}$.

Therefore,

$$\bar{K} = (\alpha\beta\bar{A})^{1/(1-\alpha)} \tag{34}$$

Given $\bar{K}$, $\bar{C}$ is resolved from (22):

$$\bar{C} = \bar{A}\bar{K}^{\alpha} - \bar{K} \tag{35}$$