Dynamic Stochastic General Equilibrium Models

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Households

- In the economy there is a continuum of households indexed by $\tau \, \epsilon \, (0, \; 1)$.
- each household maximizes an intertemporal utility function over an endless horizon:

$$E_0 \sum_{t=0}^{\infty} \beta^t U_t^{\tau}$$

where β is a discount factor and U_t^{τ} is the contemporaneous utility function depending on:

- consumption C_t^{τ},
- labour l_t^{τ}
- real money holding M_t^{τ} / P_t^{τ}

The utility function is separable in its three arguments:

$$U_t^\tau = \varepsilon_t^B \left[\frac{1}{1-\sigma_c} (C_t^\tau - H_t)^{1-\sigma_c} - \frac{\varepsilon_t^L}{1+\sigma_l} (l_t^\tau)^{1+\sigma_l} + \frac{\varepsilon_t^M}{1-\sigma_m} \left(\frac{M_t^\tau}{P_t} \right)^{1-\sigma_m} \right] \tag{1}$$

where:

- σ_c is the measure of relative risk aversion:
- $1/\sigma_l$ is the elasticity of labour with respect to the real wage;
- $1/\sigma_m$ is the elasticity of money holding with respect to the real interest rate.

The utility function (1) includes three shocks:

- ε_t^B is a general preference shock affecting intertemporal substitution of households;
- ε_t^L is a labour supply shock;
- ε_t^M is a money demand shock.

Shocks follow autoregressive processes of the first order with n.i.d. error terms:

$$\begin{aligned} \varepsilon_t^B &= \rho_B \varepsilon_{t-1}^B + \eta_t^B, \quad \varepsilon_t^L = \rho_L \varepsilon_{t-1}^L + \eta_t^L, \quad \varepsilon_t^M = \rho_M \varepsilon_{t-1}^M + \eta_t^M \\ \text{with } \eta^i &\sim N(0,1) \end{aligned} \tag{2}$$

The utility function (1) also includes a habit H_t that is proportional to the aggregate past consumption:

$$H_t = h \cdot C_{t-1} \tag{3}$$

this allows the optimal behavior of households to respond to shocks with an hump-shaped and lagged path of the optimal consumption, as stylized facts show.

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- $\left(r_t^k z_t K_{t-1}^\tau - \Psi(z_t) K_{t-1}^\tau\right)$ the return on the real capital stock minus the cost associated with variations in the degree of capital utilization; where z_t is the utilization rate; the cost of capital utilization is zero when capital utilization is one $(\Psi(1) = 0)$ as it would in the steady-state.

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- (Div_t^τ) the dividends derived from the imperfect competitive intermediate firms;
- $$Y_t^\tau = \frac{W_t^\tau}{P_t} l_t^\tau + A_t^\tau + (r_t^k z_t - \Psi(z_t)) K_{t-1}^\tau + Div_t^\tau \tag{4}$$

- Households maximize their objective function subject to an intertemporal budget constraint:

$$\underbrace{\overbrace{C_t^{\tau}}^{\text{Consumption}} + \overbrace{I_t^{\tau}}^{\text{Investment}} + \overbrace{\frac{M_t^{\tau}}{P_t}}^{\substack{\text{real}\\\text{money}}} + b_t \overbrace{\frac{B_t^{\tau}}{P_t}}^{\text{Bonds}}}_{\text{Expenses}} = \underbrace{Y_t^{\tau}}_{\text{Income}} + \underbrace{\overbrace{\frac{M_{t-1}^{\tau}}{P_t}}^{\substack{\text{real money}\\\text{in (-1)}}} + \overbrace{\frac{B_{t-1}^{\tau}}{P_t}}^{\substack{\text{Bonds}\\\text{in (-1)}}}}_{\text{Wealth}} \tag{5}$$

- Households hold their financial wealth in the form of cash balances M_t and bonds B_t.
- Bonds are one-period securities with price b_t. The gross return of bonds is:

$$R_t = 1 + i_t = \frac{1}{b_t} \tag{6}$$

- Current income and financial wealth can be used for consumption and investment in physical capital.

The maximization of the objective function (1) subject to the budget constraint (5) with respect to consumption and holdings of bonds, may be performed through the Lagrangen based technique:

$$
\begin{aligned}
L \quad = \quad & E_0 \sum_{t=0}^{\infty} \beta^t \varepsilon_t^B \left(\begin{array}{c} \frac{1}{1-\sigma_c} (C_t^\tau - H_t)^{1-\sigma_c} + \\ -\frac{\varepsilon_t^L}{1+\sigma_l} (l_t^\tau)^{1+\sigma_l} + \frac{\varepsilon_t^M}{1-\sigma_m} \left(\frac{M_t^\tau}{P_t} \right)^{1-\sigma_m} \end{array} \right) + \\
& -\beta^t \lambda_t \left(C_t^\tau + I_t^\tau + b_t \frac{B_t^\tau}{P_t} - \frac{B_{t-1}^\tau}{P_t} + \frac{M_t^\tau}{P_t} - \frac{M_{t-1}^\tau}{P_t} - Y_t^\tau \right)
\end{aligned} \tag{7}
$$

- The derivatives with respect to C_t^τ, B_t^τ and M_t^τ are:

$$\frac{\partial L}{\partial C_t^\tau} = E_t\left[\beta^t\left(\varepsilon_t^B\left(C_t^\tau - H_t\right)^{-\sigma_c} - \lambda_t\right)\right] = 0$$

$$\Leftrightarrow \lambda_t \ = \varepsilon_t^B(C_t^\tau - H_t)^{-\sigma_c} = \frac{\partial U_t^\tau}{\partial C_t^\tau} = U_t^c \tag{8}$$

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$$\begin{aligned} \frac{\partial \$}{\partial B_t^\tau} &= -E_t\left[\beta^t\lambda_t\frac{b_t}{P_t}\right] - E_t\left[\beta^{t+1}\lambda_{t+1}\frac{-1}{P_{t+1}}\right] = 0 \qquad (9) \\ \Leftrightarrow & \quad E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{b_t}\frac{P_t}{P_{t+1}}\right] = 1 \end{aligned}$$

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$$\frac{\partial L}{\partial M_t^\tau} = E_t\left[\beta^t\left(\frac{\varepsilon_t^B\varepsilon_t^M}{P_t}\left(\frac{M_t^\tau}{P_t}\right)^{-\sigma_m} - \frac{\lambda_t}{P_t}\right)\right] + E_t\left[\frac{\beta^{t+1}\lambda_{t+1}}{P_{t+1}}\right] = 0$$

$$\Leftrightarrow \quad \varepsilon_t^B\varepsilon_t^M\left(\frac{M_t^\tau}{P_t}\right)^{-\sigma_m}\frac{1}{\lambda_t} + E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{P_t}{P_{t+1}}\right] = 1 \qquad (10)$$

- Eq. (8) with (9) and (10) yield the intertemporal optimization condition.
- one exploits the fact that households are homogenous in their consumption-saving decisions, in other words, the marginal utility of consumption is for all τ identical.
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$$\begin{aligned} E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{b_t}\frac{P_t}{P_{t+1}}\right] &= E_t\left[\beta\frac{U_{t+1}^c}{U_t^c}(1+i_t)\frac{P_t}{P_{t+1}}\right] \\ &= E_t\left[\beta\frac{\varepsilon_{t+1}^B(C_{t+1}-h\cdot C_t)^{-\sigma_c}}{\varepsilon_t^B(C_t-h\cdot C_{t-1})^{-\sigma_c}}R_t\frac{P_t}{P_{t+1}}\right] = 1 \end{aligned} \tag{11}$$

- The Euler equation (11) shows a Trade-Off between present and future consumption.

Rearranging (11) yields:

$$\beta(1+i_t)E_t\left[\frac{U^c_{t+1}}{P_{t+1}}\right]=\frac{U^c_t}{P_t}$$

Substituting eq. (8) and (9) in (10) yields the intratemporal optimization condition:

$$\varepsilon_t^M \cdot \left(\frac{M_t^\tau}{P_t}\right)^{-\sigma_m} = (C_t^\tau - H_t)^{-\sigma_c} \cdot (1-b_t) = (C_t^\tau - H_t)^{-\sigma_c} \cdot \frac{i_t}{1+i_t} \quad (12)$$

- (12) is the money demand or the newkeynesian LM curve;
- the nominal interest rate is the instrument of monetary policy;
- cash holdings are additively separable in the utility function, cash holding will not enter in any of the other structural equations. Eq. (12) becomes recursive to the rest of the system of equations