Tutorial II

EABCN Training School

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Tutorial Objectives

1. A systematic way of thinking about transition dynamics
    - Reduce to system of $n_x \times T$ equations in $n_x \times T$ unknowns
    - Interpretation: certainty equivalence, but keep non-linearity
    - How to solve systems of equatons? Newton updating
2. Jacobians in simple models
    - Easy to obtain using automatic differentiation/finite differences
    - Get solution in one step for linearized model
3. Updating heterogeneous-agent models
    - Get full Jacobian using automatic differentiation/finite differences
    - Update using Jacobian from limiting rep-agent model [Auclert and Rognlie (2018)]
    - Discuss coding implementation
4. The economics of Jacobians [Auclert et al. (2018), Koby and Wolf (2018)]

Outline

1. Transition Dynamics: Getting Started
    - Two Examples
    - Relation to Perturbation
    - Solution Ideas

2. A Representative-Agent Discrete-Time Model

3. Transition Dynamics in Heterogeneous-Agent Models
    - A Heterogeneous-Household Model
    - Fudge Models

4. The Economics of Updating Matrices
    - MPCs and the $\mathcal{M}$-Matrix
    - Price Sensitivity and Firm Behavior

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The Krusell-Smith Model

- The problem of households is

$$\max \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_{it}) dt\right]$$

 such that

$$\begin{aligned} \dot{a}_{it} &= r_t a_{it} + w_t y_{it} - c_{it} \\ a_{it} &\geq 0 \end{aligned}$$

- The firm sector is completely standard
  - Single representative firm with production function

$$Y_t = e^{z_t} K_t^\alpha L_t^{1-\alpha}$$

  - Firms rent capital at rate $r_t + \delta$ and hire labor at wage w_t
  - TFP evolves exogenously

$$dz_t = -\theta_z z_t dt + \sigma_z dW_t^z$$

The Krusell-Smith Model

- Market-clearing dictates that

$$
\begin{aligned}
L_t &= \int_{\mathcal{S}} y(s) d\lambda(s) \equiv 1 \\
K_t &= \int_{\mathcal{S}} a(s) d\lambda(s) \\
Y_t &= C_t + \delta K_t + \dot{K}_t
\end{aligned}
$$

- We study transition dynamics after a completely unexpected TFP shock

The Krusell-Smith Model

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Y_t &= C_t + \delta K_t + \dot{K}_t
\end{aligned}
$$

- We study transition dynamics after a completely unexpected TFP shock
- Suffices to guess interest rate path and clear capital market
    - Given $\{r_t\}_{t=0}^T$ get $\{w_t\}_{t=0}^T$ via

$$r_t + \delta = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha}, \quad w_t = (1-\alpha) e^{z_t} K_t^{\alpha} L_t^{-\alpha}$$

    - Firm side gives $\{K_t^D\}_{t=0}^T$, solve household side to get $\{A_t\}_{t=0}^T = \{K_t^S\}_{t=0}^T$
    - Everything has been reduced to a system of $\frac{T}{dt}$ equations in $\frac{T}{dt}$ unknowns

$$K^D(\{r_t\}_{t=0}^T) = K^S(\{r_t\}_{t=0}^T)$$

The Khan-Thomas Model

- The problem of households is

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\gamma} - 1}{1-\gamma} - \chi \frac{L_t^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}} \right\} \right]$$

  such that

$$A_{t+1} + C_t = (1 + r_t) A_t + w_t L_t + D_t$$

- The problem of firms is

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{1}{\prod_{s=1}^{t-1}(1 + r_s)} d_{it} \right]$$

  such that

$$d_{it} = e^{z_{it}} \left(k_{it}^{\alpha} \ell_{it}^{1-\alpha} \right)^{\nu} - w_t \ell_{it} + (1-\delta) k_{it} - k_{it+1} - \phi(k_{it}, k_{it+1})$$

  giving aggregate output, capital and dividends

$$Y_t = \int_{\mathcal{S}} e^{z(s)} \left(k(s)^{\alpha} \ell(s)^{1-\alpha} \right)^{\nu} d\lambda(s), \quad K_t = \int_{\mathcal{S}} k(s) d\lambda(s), \quad D_t = \int_{\mathcal{S}} d(s) d\lambda(s)$$

The Khan-Thomas Model

- Market-clearing dictates that

$$
\begin{aligned}
L_t &= \int_{\mathcal{S}} \ell(s) d\lambda(s) \\
A_t &= 0 \\
Y_t &= C_t + \underbrace{K_{t+1} - (1-\delta)K_t}_{\equiv I_t}
\end{aligned}
$$

- We study transition dynamics after a completely unexpected TFP shock

The Khan-Thomas Model

- Market-clearing dictates that

$$
\begin{aligned}
L_t &= \int_{\mathcal{S}} \ell(s) d\lambda(s) \\
A_t &= 0 \\
Y_t &= C_t + \underbrace{K_{t+1} - (1-\delta)K_t}_{\equiv I_t}
\end{aligned}
$$

- We study transition dynamics after a completely unexpected TFP shock
- Suffices to guess consumption path and clear output market
  - Given $\{C_t\}_{t=0}^T$, get $\{r_t\}_{t=0}^T$ and $\{w_t\}_{t=0}^T$ via

$$C_t^{-\gamma} = \beta(1+r_{t+1})C_{t+1}^{-\gamma}, \quad \chi = w_t C_t^{-\gamma}$$

  - Solve firm side to get $\{K_t\}_{t=1}^T$, $\{Y_t\}_{t=0}^T$
  - Everything has been reduced to a system of T equations in T unknowns

$$C = Y(\{C_t\}_{t=0}^T) - I(\{C_t\}_{t=0}^T)$$

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MIT Shocks vs. Perturbation

- How do MIT transition paths to TFP shock relate to normal linearization?
  - Both feature certainty-equivalence, MIT shock keeps non-linearity
  - Result: solutions are identical for infinitesimal shocks
  - Intuition: certainty equivalence + Taylor's theorem
- Attractions of MIT shocks
  - Can study size-/sign-/state-dependence of shocks
  - Sometimes reveals interesting economics (later)
  - Lends itself to sufficient-statistics decompositions (later)
- Drawbacks of MIT shocks
  - Not as convenient to study existence/uniqueness
  - Usually slower than perturbation
  - VMA representation vs. state-space representation for estimation (later)

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Solving for Transition Dynamics

- We have successfully reduced the problem:

$$F(\underbrace{X}_{(T\cdot n_x)\times 1}) = \underbrace{0}_{(T\cdot n_x)\times 1}$$

- How are we supposed to find the solution?

1. Standard approach: "trial-and-error"

    - Look at market-clearing period-by-period
    - Adjust prices: down if excess supply, up if excess demand
    - Example: higher (lower) rate if too little (much) savings

2. A smarter approach: Newton updating

    - Why not treat it like a normal root-finding problem?
    - Use Jacobian updating:

$$X_{i+1} = X_i - F_X^{-1} \times F(X_i)$$

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Model

- Environment: neoclassical growth model
  - Conventional representative household
  - Representative firm with DRS and adjustment costs
  - Linearize everything $\rightarrow$ identical to perturbation solution
- Codes/model set up in a way to generalize to heterogeneous-firm model
- Code structure
  1. Map consumption sequence into prices
     - Get SDF immediately
     - From SDF easy to get r, w
  2. Map prices into firm behavior
     - Solve out labor statically
     - Get capital FOC to reduce to system $A \times k = b$
     - From k back out ℓ, y and i
  3. Aggregate firm behavior into net output supply

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A Heterogeneous-Household Model

- Model environment: simple Huggett economy
  - Exogenous labor income for all households, 0 net supply of savings
  - Interest rate is set to clear markets at all time
  - Experiment: transition path after change in value of second income state

- We'll walk through two sets of almost identical codes
  1. Solve for initial and target distribution
  2. Guess initial path for interest rate r_0
  3. (a) Find matrix A with entries

     $$A_{i,j} = \frac{\partial \text{ net asset demand in } i}{r_j}$$

     evaluated at guess r_0, Newton-update *with fixed A* until convergence

     (b) Compute period-by-period excess demand/supply and adjust rates period by period depending on static departure from market-clearing

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The Basic Idea

- Proper Jacobian updating in heterogenous-agent models is prohibitive
  - Would need to differentiate many highly complicated functions, and so many times (once for each updating step)
  - Previous codes used mini-fudge: use first-step updating matrix throughout
  - Take it further: use Jacobian from limiting representative-agent model [Auclert and Rognlie (2018)]
- Loose intuition on why it may work
  - In firm application: Jacobian has entries $\frac{d(Y-I)_i}{dC_j}$
  - Collapses two parts: how prices respond to consumption, and how production responds to prices
  - First block is identical across models, second block may be roughly similar
  - Example: interest rates crowd out investment in rep- and het-agent models
- Ultimately trial-and-error: if it converges you're done!
  - That's a drawback: not as robustly general-purpose as perturbation

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The Intertemporal Keynesian Cross

- Environment: Ultra-Keynesian 1-asset model [Auclert et al. (2018)]
  - Household problem:
$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{it}) \right]$$
    such that
$$\begin{aligned} a_{it+1} &= \frac{1+i_{t-1}}{1+\pi_t} a_{it} + w_t \ell_{it} - c_{it}, \quad a_{it+1} \geq 0 \\ \ell_{it} &= \ell_i(\ell_t) \end{aligned}$$
  - Firm problem: $\max y_t - w_t \ell_t$
  - Rigid prices and wages: $\pi_t = 0$, $w_t = 1$
  - Central bank sets $\{i_t\}$ and so $\{r_t\}$
- Need to only iterate over aggregate output path $\{y_t\}$
  - $\{y_t\}$ gives $\{\ell_t\}$ and $\{\ell_{it}\}$ for all i
  - Given $\{\ell_{it}\}$ and $\{r_t\}$ we find $\{c_{it}\}$ and so $\{c_t\} = \{y_t\}$

The Intertemporal Keynesian Cross

- So we've reduced the equilibrium to a system like this: $Y = C(Y)$
- Let's take a first-order approximation to this system:

$$dY = \frac{\partial C}{\partial Y} dY + \frac{\partial C}{\partial r} dr$$

- We can thus restrict attention to two very interesting objects:

  1. Partial equilibrium impact: $\frac{\partial C}{\partial r} dr$
     - How do households response to a change in interest rates?
     - Micro evidence suggests that these responses may not be big
  2. GE adjustment/Keynesian multiplier: $\frac{\partial C}{\partial Y} dY$
     - More consumption leads to more income, which leads to more consumption
     - Dynamic relationship is summarized by the matrix $\mathcal{M} = \frac{\partial C}{\partial Y}$
     - This is the intertemporal Keynesian cross: dC_i / dY_j

- Empirical discipline: dC/dr and MPCs (dC/dY)

Applications

- Transparent framework to see micro irrelevance results [Werning (2016)]
  - Fraction λ of HtM households, rest is conventional
  - Weaker PE effect: $\frac{\partial C}{\partial r} = (1-\lambda)\frac{\partial C^{\text{RANK}}}{\partial r}$
  - Bigger GE multiplier: $\frac{\partial C}{\partial Y} = \lambda I + (1-\lambda)\frac{\partial C^{\text{RANK}}}{\partial Y}$
  - Putting everything together:

$$dY = \left(I - \lambda I - (1-\lambda)\frac{\partial C^{\text{RANK}}}{\partial Y}\right)^{-1}(1-\lambda)\frac{\partial C^{\text{RANK}}}{\partial r} = dY^{\text{RANK}}$$

- May help understand MP bank transmission puzzle [Koby and Wolf (2018)]
  - Sluggish + incomplete pass-through of rate changes through banking sector
  - Result: does not impair aggregate transmission if transmission to rate-responsive agents is complete

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Model Set-Up

- Recall the basic Khan-Thomas economy:

$$C = Y(C, \varepsilon) - I(C, \varepsilon) \equiv \tilde{Y}(C, \varepsilon)$$

- Again do a first-order expansion:

$$dC = \frac{\partial Y}{\partial \varepsilon} d\varepsilon + \frac{\partial Y}{\partial C} dC$$

  - PE impact features interesting asymmetries
  - GE adjustment: $\mathcal{G} \equiv \frac{\partial Y}{\partial C} = \frac{\partial Y}{\partial p} \circ \frac{\partial p}{\partial C}$

- Now compare with a conventional growth model:

$$dC = dC_{\text{ref}} + \mathcal{G} \times \underbrace{\left(\tilde{Y}_{\varepsilon} \times d\varepsilon - \mathcal{G}^{-1} \times d\tilde{Y}_{\text{ref}}\right)}_{\text{supply - demand at } p_{\text{ref}}} \equiv dC_{\text{ref}} + \mathcal{G} \times \xi_{\text{ref}}$$

  - Different behavior at given prices: ξ_{ref}
  - GE adjustment to changes in prices: $\mathcal{G}$

Irrelevance Results

- Price sensitivity on the firm side
  - Investment in conventional neoclassical models is very price-sensitive

$$\frac{\partial \log(i)}{\partial r} = -\frac{1}{1-\nu}\frac{1}{\delta}\frac{1}{r+\delta} \approx -300\%$$

  - Popular micro frictions (lumpy AC, financial frictions) don't change this
- Fringe of neoclassical firms dominates the equilibrium
  - Setting: neoclassical growth model, no AC, $\nu < 1$, fraction of $\xi < 1$ does not follow optimal investment rule
  - Result: as $\nu \to 1$, the equilibrium becomes independent of ξ
- Important implications for heterogeneous-firm models
  - Explains well-known irrelevance results as coming from price sensitivity
  - Micro identification suggests dampened elasticity: PE matters for GE
  - Suggests that all heterogeneous-form modeling should feature substantial micro frictions to (capital) adjustment

Thanks for your attention – Questions?