Tutorial I

EABCN Training School

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Tutorial Objectives

1. Towards micro heterogeneity: the income fluctuation problem
   - Problem set-up and link to lectures
   - Details on the Kolmogorov forward equation
2. Tying up some loose ends on theory
   - Deriving HJB/KFE as continuous-time limits
   - MPCs and the Feynman-Kac formula
3. Going through codes in detail
   - Implementing HJB + KFE
   - Comparison with discrete-time best-practice
4. Harder problems, extensions
   - Models with non-convexities
   - 2-asset models
   - Sparsity with many income states

Outline

Outline

The Problem

- The income fluctuation/consumption insurance problem
  - Consider a single household subject to uninsurable income risk
  - Self-insure through risk-free borrowing/saving (with borrowing constraint)
  - Significance: this is the canonical "heterogeneous-household" problem
- The problem of the household is

$$\max \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

such that

$$\begin{aligned} \dot{a}_t &= r a_t + y_t - c_t \\ a_t &\geq \underline{a} \end{aligned}$$

and given an exogenous law of motion for y

- We let y follow a two-state Poisson process: $y \in \{y_1, y_2\}$
  - Alternative (in lecture): diffusion process

Optimal Behavior: The HJB

- Optimal household behavior is characterized by the HJB:

$$\rho v_j(a) = \max_c u(c) + \frac{\partial v_j(a)}{\partial a}(ra + y_j - c) + \lambda_j(v_{-j}(a) - v_j(a))$$

- Everywhere on the interior of the state space we have the FOC

$$u'(c) = \frac{\partial v_j(a)}{\partial a}$$

- On the boundary we have the state boundary condition

$$v_j'(\underline{a}) = u'(y_j + r\underline{a})$$

  - Where does this come from? ensure positive drift at $\underline{a}$
  - Formal justification: constrained viscosity solution

Solving the HJB

Solution is exactly analogous to what you saw in lecture:

1. Discretize using finite difference approximation with upwinding
2. Find optimal consumption/savings from FOC to get

$$\rho v_{i,j} = u(c_{i,j}) + (v_{i,j})'[y_j + ra_i - c_{i,j}] + \lambda_j[v_{i,-j} - v_{i,j}]$$

3. Update from v^n to v^{n+1} using explicit or implicit iterative scheme, e.g.

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} + \rho v_{i,j}^{n+1} = u(c_{i,j}^n) + v_{i-1,j}^{n+1} x_{i,j} + v_{i,j}^{n+1} y_{i,j} + v_{i+1,j}^{n+1} z_{i,j} + v_{i,-j}^{n+1} \lambda_j$$

   where

$$x_{i,j} = -\frac{(s_{i,j,B}^n)^-}{\Delta a}, \quad y_{i,j} = -\frac{(s_{i,j,F}^n)^+}{\Delta a} + \frac{(s_{i,j,B}^n)^-}{\Delta a} - \lambda_j, \quad z_{i,j} = -\frac{(s_{i,j,F}^n)^+}{\Delta a}$$

4. Repeat until convergence

Solving the HJB

In more compact notation the implicit system is

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u^n + A^n v^{n+1}$$

where

$$A^n = \begin{pmatrix}
y_{1,1} & z_{1,1} & 0 & \dots & 0 & \lambda_1 & 0 & 0 & \dots & 0 \\
x_{2,1} & y_{2,1} & z_{2,1} & 0 & \dots & 0 & \lambda_1 & 0 & 0 & \dots \\
0 & x_{3,1} & y_{3,1} & z_{3,1} & 0 & \dots & 0 & \lambda_1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & x_{I,1} & y_{I,1} & 0 & 0 & 0 & 0 & \lambda_1 \\
\lambda_2 & 0 & 0 & 0 & 0 & y_{1,2} & z_{1,2} & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 & x_{2,2} & y_{2,2} & z_{2,2} & 0 & 0 \\
0 & 0 & \lambda_2 & 0 & 0 & 0 & x_{2,3} & y_{3,2} & z_{3,2} & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \dots & \dots & 0 & \lambda_2 & 0 & \dots & 0 & x_{I,2} & y_{I,2}
\end{pmatrix}$$

Household Distributions: The KFE

- What happens in an economy populated by many such households?
  - Key object is the distribution of households over the state space: $g_t(a, y)$
  - Evolution of distribution is given by the KFE:

$$\partial_t g_t(a, y_j) = -\frac{d}{da}[s(a, y_j) g_t(a, y_j)] - \lambda_j g_t(a, y_j) + \lambda_{-j} g_t(a, y_{-j})$$

- We get the numerical solution almost for free
  - Discretize

$$0 = -[s_{i,j} g_{i,j}]' - \lambda_j g_{i,j} + \lambda_{-j} g_{i,-j}, \quad 1 = \sum_{i=1}^{I} g_{i,1} \Delta a + \sum_{i=1}^{I} g_{i,2} \Delta a$$

  - Now use

$$-\frac{g_{i,j}(s^n_{i,j,F})^+ - g_{i-1,j}(s^n_{i-1,j,F})^+}{\Delta a} - \frac{g_{i+1,j}(s^n_{i+1,j,B})^- - g_{i,j}(s^n_{i-1,j,B})^-}{\Delta a} - g_{i,j}\lambda_i + g_{i,-j}\lambda_{-j} = 0$$

  - Then easy to see that this is just

$$A^T g = 0$$

  - Interpretation: generator/adjoint generator

Outline

Outline

From Bellman Equation to HJB

- For time periods of length Δ we have the standard Bellman equation

$$v(k_t) = \max_{c_t} \Delta u(c_t) + e^{-\rho\Delta} v(k_{t+\Delta})$$

  such that

$$k_{t+\Delta} = \Delta[F(k_t) - \delta k_t - c_t] + k_t$$

- Limiting behavior of constraint is immediate: $\dot{k}_t = F(k_t) - \delta k_t - c_t$

- For HJB itself use that $e^{-\rho\Delta} \approx 1 - \rho\Delta$:

$$v(k_t) = \max_{c_t} \Delta u(c_t) + (1 - \rho\Delta) v(k_{t+\Delta})$$

- Re-arrange and divide by Δ:

$$\rho v(k_t) = \max_{c_t} u(c_t) + (1 - \Delta\rho) \frac{v(k_{t+\Delta}) - v(k_t)}{k_{t+\Delta} - k_t} \frac{k_{t+\Delta} - k_t}{\Delta}$$

- Taking limits completes the argument

The KFE

- In the income fluctuation problem wealth evolves as $a_{t+\Delta} = a_t + \Delta s_i(a_t)$
- Thus the fraction of people with wealth below a evolves as

$$\begin{aligned} G_i(a, t+\Delta) \equiv \Pr(a_{t+\Delta} \leq a, z_{t+\Delta} = z_i) \quad &= \quad (1-\Delta\lambda_i)\Pr(a_t \leq a - \Delta s_i(a), z_t = z_i) \\ &\quad\quad + \Delta\lambda_j \Pr(a_t \leq a - \Delta s_j(a), z_t = z_j) \end{aligned}$$

- Subtracting $G_i(a,t)$ from both sides, dividing by Δ:

$$\begin{aligned} \frac{G_i(a, t+\Delta) - G_i(a,t)}{\Delta} \quad &= \quad \frac{G_i(a - \Delta s_i(a), t) - G_i(a,t)}{\Delta} \\ &\quad\quad -\lambda_i G_i(a - \Delta s_i(a), t) + \lambda_j G_j(a - \Delta s_j(a), t) \end{aligned}$$

- Taking the limit:

$$\partial_t G_i(a,t) = -s_i(a)\partial_a G_i(a,t) - \lambda_i G_i(a,t) + \lambda_j G_j(a,t)$$

- Differentiate w.r.t. a:

$$\partial_t g_i(a,t) = -\partial_a s_i(a) g_i(a,t) - \lambda_i g_i(a,t) + \lambda_j g_j(a,t)$$

Outline

The Feynman-Kac Formula

- Consider a diffusion

$$dx_t = \mu(x_t)dt + \sigma(x_t)dW_t$$

- Then the solution to the PDE

$$0 = f(x) + \mu(x)\frac{\partial u}{\partial x}(x,t) + \frac{1}{2}\sigma^2(x)\frac{\partial u^2}{\partial x^2}(x,t) + \frac{\partial u}{\partial t}(x,t)$$

  with terminal condition $u(x,\tau) = 0$ satisfies

$$u(x,0) = \mathbb{E}_0\left[\int_0^\tau f(x_t)dt \mid x_0 = x\right]$$

  - This is slightly less general than the full Feynman-Kac formula (which allows for discounting in the integral)
  - General formula features the infinitesimal generator of a stochastic process (a partial differential operator encoding everything you need)

Why this is useful

- How should we define (measurable) MPCs in continuous time?

$$
\begin{aligned}
C_\tau(a,y) &= \mathbb{E}_0\left[\int_0^\tau c(a_t,y_t)dt \mid a_0 = a, y_0 = y\right] \\
MPC_\tau(a,y) &= \frac{\partial C_\tau}{\partial a}(a,y)
\end{aligned}
$$

  - First expression is consumption flow over interval (e.g. quarter)
  - MPC then captures change in consumption over following quarter for change in income today
  - Get more natural discrete MPCs as

$$
MPC_\tau^x(a,y) \equiv \frac{C_\tau(a+x,y) - C_\tau(a,y)}{x}
$$

- This is exactly what the Feynman-Kac formula gives us
  - $s = (a,y)$ follows a diffusion process
  - Thus remains to solve one PDE and take one derivative
  - Used for example in Kaplan, Moll and Violante (2018)

Outline

A Quick Overview

- Consider a generic PDE: $F(x, v, Dv) = 0, \quad x \in \Omega$
- Classical solution concept: find function $v(x)$ such that
  1. v is continuous and differentiable over Ω
  2. Above equation holds for all $x \in \Omega$
- Viscosity solutions
  - Define a weak type of solution: don't require v to be differentiable everywhere, just satisfy PDE in a "weak" sense
  - Basic idea: whenever $v'(x)$ doesn't exist (kink) replace it with derivative of smooth function ϕ touching v
- Main takeaways
  - This is exactly the right solution concept
  - "Constrained" viscosity solutions give us boundary conditions
  - Maximization problems only allow convex kinds

Finite Difference Schemes

- Write the HJB as

$$0 = G(k, v(k), v'(k), v''(k))$$

  and consider the FD scheme

$$0 = S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1})$$

- For example for a growth model

$$G(k, v(k), v'(k), v''(k)) = \rho v(k) - \max_c \{u(c) + v'(k)(F(k) - \delta k - c)\}$$

  and

$$\begin{aligned} S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) &= \rho v_i - u(c_i) - \frac{v_{i+1} - v_i}{\Delta k}(F(k_i) - \delta k_i - c_i)^+ \\ &\quad - \frac{v_i - v_{i-1}}{\Delta k}(F(k_i) - \delta k_i - c_i)^- \end{aligned}$$

Main Result

Three conditions:

1. Monotonicity. S is non-increasing in both v_{i-1} and v_{i+1}
2. Consistency. For every smooth function v with bounded derivatives:

$$S(\Delta k, k_i, v(k_i); v(k_{i-1}), v(k_{i+1})) \to G(v(k), v'(k), v''(k))$$

   as $\Delta k \to 0$ and $k_i \to k$
3. Stability. For every $\Delta k > 0$, there is a solution v_i, $i = 1, \ldots, I$, uniformly bounded independent of Δk.

Theorem

If the FD scheme satisfies 1. - 3., then as $\Delta k \to 0$ its solution v_i converges uniformly to the unique viscosity solution of G.

Discussing Monotonicity

- Consistency and stability are usually easy; monotonicity is the problem
- Interpretation: value today is increasing in continuation value

$$\begin{aligned}\tilde{S}(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) &= u(c_i) + \frac{v_{i+1} - v_i}{\Delta k}(F(k_i) + \delta k_i - c_i)^+ \\ &\quad + \frac{v_i - v_{i-1}}{\Delta k}(F(k_i) - \delta k_i - c_i)^-\end{aligned}$$

  and $\tilde{S}$ is increasing in v_{i-1}, v_{i+1}

- Easy to see that upwind scheme does exactly that in growth model:

$$\begin{aligned}S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) &= \rho v_i - u(c_i) - \frac{v_{i+1} - v_i}{\Delta k}(F(k_i) - \delta k_i - c_i)^+ \\ &\quad - \frac{v_i - v_{i-1}}{\Delta k}(F(k_i) - \delta k_i - c_i)^-\end{aligned}$$

Outline

Outline

A Simple Model

- We consider a simple 1-asset consumption-savings problem
- Recall the basic HJB

$$\rho v(a,y) = \max u(c) + \frac{\partial v}{\partial a}(ra + y - c) + \sum_{y'} \lambda(y,y')\left[v(a,y') - v(a,y)\right]$$

such that

$$a \geq \underline{a}$$

- The FOC is

$$u'(c) = \frac{\partial v}{\partial a}$$

and the state boundary condition is

$$u'(y + r\underline{a}) \leq v_a(\underline{a}, z) \quad \forall z$$

- Let's go through the codes line by line . . .

Outline

Discrete-Time Solution Methods

- At least two robust general-purpose solution methods
  - Value function iteration
    - → Guess value function on grid, optimize on state grid
    - → Iterate function entries until convergence
  - Projection methods
    - → Parameterize value function with basis functions, optimize exactly
    - → Iterate basis function coefficients until convergence
- Why are these methods slower than continuous time?
  - Need to optimize to get decision rules (instead of using FOCs)
  - Do not get distribution update step for free
  - Distribution on grid tends to be less accurate (spikes)
- How problematic are these concerns?

Speeding Things Up

1. Decision rules: exploiting FOCs
    - Discrete time steps mean that constraints always matter
    - Endogenous gridpoint method offers workaround for special models
        - $\rightarrow$ Leading example: classical one-asset model
    - My position: doesn't always work, but competitive when it does
2. Distribution updating: fine-grid histograms with interpolation
    - DON'T use Monte Carlo simulation (slow, inaccurate)
    - Optimize once given value function to get decision rules on fine grid
    - Interpolate to incorporate off-grid information
    - My position: given decision rules, DT is not (much) worse

Numerical Implementation

- We consider the same 1-asset consumption-savings problem:

$$V(a, y) = \max u(c) + \beta \mathbb{E}\left[V(a', y')\right]$$

such that

$$\begin{aligned} a' + c &= (1+r)a + y \\ a' &\geq \underline{a} \end{aligned}$$

- The FOC is

$$u'(c) = \mu + \beta(1+r)\mathbb{E}\left[u'(c')\right]$$

or more carefully

$$u'(c(a, y)) = \mu(a, y) + \beta(1+r)\mathbb{E}\left[u'(c(a', y'))\right]$$

- Codes on github; speed competitive with continuous time

Outline

Outline

Motivation

- Conventional 1-asset models are a bit limited
  - 1 liquid asset: households can always tap wealth to smooth consumption
  - High average MPCs require many households to hold no wealth
- Solution: distinguish between liquid and illiquid wealth
  - Liquid assets b pay low returns, but are always accessible
  - Illiquid assets a are costly to tap into, but pay high returns
  - Main benefit: rationalizes high MPCs for positive-wealth households
- For today: discuss two modeling approaches
  1. Proper two-asset model [Kaplan, Moll & Violante (2018)]
  2. Fudge two-asset model with $b = b(a)$ [Kaplan, Moll & Wolf (2018)]

A Two-Asset Model

- Households now insure against income risk using two assets:

$$
\begin{aligned}
\dot{b}_t &= r^b b_t + y_t - c_t - d_t - \chi(d_t, a_t) \\
\dot{a}_t &= r^a a_t + d_t
\end{aligned}
$$

- The problem of the household then is

$$
\max \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right]
$$

such that

$$
\begin{aligned}
\dot{b}_t &= r^b b_t + y_t - c_t - d_t - \chi(d_t, a_t), \quad b_t \geq \underline{b} \\
\dot{a}_t &= r^a a_t + d_t, \quad a_t \geq \underline{a}
\end{aligned}
$$

and given an exogenous law of motion for y

Solving the Two-Asset Model

- The HJB equation is

$$\begin{aligned}\rho v(a,b,y_j) &= \max_{c,d} u(c) + \partial_b v(a,b,y_j)(y_j + r^b b - d - \chi(d,a) - c) \\ &\quad + \partial_a(d + r^a a) + \lambda_j(v(a,b,y_{-j})v(a,b,y_j))\end{aligned}$$

- This gives the two FOCs

$$\begin{aligned}u'(c) &= \partial_b v(a,b,y_j) \\ \partial_a v(a,b,y_j) &= \partial_b v(a,b,y_j)(1 + \chi_d(d,a))\end{aligned}$$

- Numerical implementation
  - Need to work harder to get a monotone FD scheme
  - Details in appendix of KMV (2018), or talk to me

2-Asset Fudge: Discrete Time

- Let's start with discrete time because it's more transparent
- The Bellman equation is

$$V(a, y) = \max u(c) + \beta \mathbb{E}\left[V(a', y')\right]$$

such that

$$\begin{aligned} a' + c &= (1+r)a + y \\ c &\leq b(a) + ra + y \end{aligned}$$

- Re-arranging the second constraint:

$$a' \geq a - b(a)$$

- The FOC is

$$u'(c) = \mu + \beta(1+r)\mathbb{E}\left[u'(c') - \frac{1}{1+r}\mu'(1 - b_a(a'))\right]$$

- Problem for EGP: may not be monotone

2-Asset Fudge: Continuous-Time Limit

- Let's write the constraints for time periods of length Δ:

$$\begin{aligned} \Delta c + a' &= a + \Delta(ra + y) \\ \Delta c &\leq b(a) + \Delta(ra + y) \end{aligned}$$

- The limit for the first one is as usual:

$$\dot{a} = ra + y - c$$

- For the second one get

$$\begin{cases} b(a) \geq 0 & \text{if } b(a) > 0 \\ c \leq ra + y & \text{if } b(a) = 0 \end{cases}$$

- Relative to the usual problem we just change the state boundary:

$$u'(y + ra) \leq v_a(a, y) \quad \forall y, a \text{ s.t. } b(a) = 0$$

Outline

The Skiba Model

- Growth model with non-convex production function

$$v(k) = \max \int_0^\infty e^{-\rho t} u(c_t) dt$$

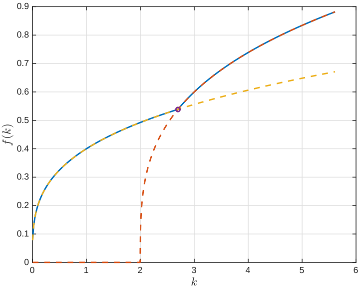
such that

$$\dot{k}_t = f(k_t) - \delta k_t - c_t, \quad k_0 = k$$

and where

$$f(k) = \max \{ f_L(k), f_H(k) \}$$

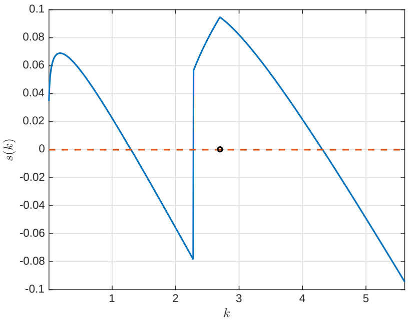
$$f_L(k) = A_L k^\alpha, \quad f_H(k) = A_H((k-\kappa)^+)^\alpha$$



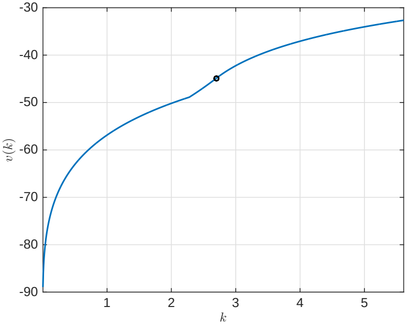
The Skiba Model

- Why FOC-based solution methods struggle in discrete time
  - FOC is
$$u'(f(k) - \delta k - k') = \beta v'(k')$$
  - Problem: FOC is no longer sufficient (for some k's the FOC has multiple solutions as a result of kink in production and so value function)
- But non-convexity causes no problems for our algorithm
  - Can simply change production function in benchmark codes
  - Viscosity solution + FD designed to handle non-convexities and non-differentiabilities with upwinding scheme
- These insights generalize to richer environments [Achdou et al. (2017)]
  - Interesting application: housing/mortgages

Economics of the Skiba Model



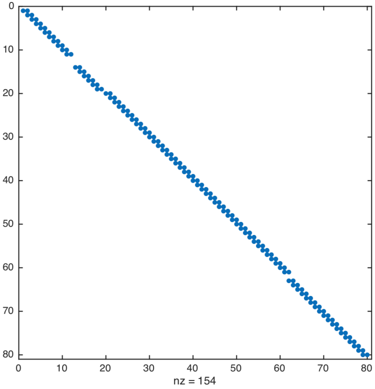
(a) Saving Policy Function



(b) Value Function

- Skiba point is the left of the kink in $f(k)$
  - Intuition: start pushing towards the high technology not just at point where it's used, but earlier
- Two steady states ("poverty trap")

Numerical Solution



- Steady states: no outflow
  - Negative savings with forward approximation, positive savings with backward approximation
- Skiba point: jump in flows

Outline

Losing Sparsity

- Recall the basic (vectorized) updating equation:

$$\frac{1}{\Delta}(V^{n+1} - V^n) + \rho V^{n+1} = u^n + (\tilde{A}^n + \Lambda)V^{n+1}$$

  where $\tilde{A}^n$ contains only the savings transition

  - Uses implicit method for speed
  - But problem: with many income states A^n is not sparse anymore

- Solution: implicit for assets, explicit for income risk

  - Mixture scheme gives

$$\frac{1}{\Delta}(V^{n+1} - V^n) + \rho V^{n+1} = u^n + \tilde{A}^n V^{n+1} + \Lambda V^n$$

    where $\tilde{A}^n$ is tri-diagonal

  - Can thus even parallelize for each income state k:

$$\frac{1}{\Delta}(V_k^{n+1} - V_k^n) + \rho V_k^{n+1} = u_k^n + \tilde{A}_k^n V^{n+1} + \sum_{k' \neq l} \lambda_{k,k'}(V_k'^n - V_k^n)$$

Thanks for your attention – Questions?

Appendix: Viscosity Solutions

Let's specialize to a simple HJB:

$$\rho v(x) = \max_c u(c, x) + v'(x) f(c, x)$$

A viscosity solution is a continuous function v such that:

1. (Subsolution) If ϕ is smooth and $v - \phi$ has a local maximum at x^* (v may have a concave kink), then

$$\rho v(x^*) \leq \max_c u(c, x^*) + \phi'(x^*) f(c, x^*)$$

2. (Supersolution) If ϕ is smooth and $v - \phi$ has a local minimum at x^* (v may have a convex kink), then

$$\rho v(x^*) \geq \max_c u(c, x^*) + \phi'(x^*) f(c, x^*)$$

Appendix: Viscosity Solutions

- Basic idea: sidestep non-differentiability and use monotonicity
- Consider discrete-time Bellman equation:

$$v(x_t) = \max_c \Delta u(c, x_t) + (1 - \rho\Delta) v(x_{t+\Delta})$$

  such that

$$x_{t+\Delta} = \Delta f(x_t, c) + x_t$$

- Now let ϕ be s.t. $\phi(x^*) = v(x^*)$ and $v - \phi$ has a local minimum at x^*
- Then at $x_t = x^*$ we have

$$v(x_t) \geq \max_c \Delta u(c, x_t) + (1 - \rho\Delta) \phi(x_{t+\Delta})$$

  since $v(x) > \phi(x)$ in a neighborhood of x^* and by monotonicity
- Subtract $(1 - \rho\Delta)\phi(x_t)$, take limits, and the supersolution condition follows (subsolution condition is symmetric)

Appendix: Viscosity Solutions

1. Only allow convex kinks

   - Maximization problems only admit convex (downward) kinks
   - Proof strategy: convex Hamiltonian, then contradiction with concave kinks

2. Boundary conditions

   - Easy to motivate when differentiable:

   $$\rho v(x) = H(x, \lambda) = \max u(c, x) + \lambda f(c, x) = \max u(c, x) + v'(x) f(c, x)$$

   and then combine envelope theorem and complementary slackness:

   $$\begin{aligned} H_\lambda(x, \lambda) &= f(c^*(x), x) \\ H_\lambda(x_{\min}.v'(x_{\min})) &\geq 0 \end{aligned}$$

   - Constrained viscosity solution: for maximization subsolution at $x = x_{\min}$, supersolution at $x = x_{\max}$

3. Unique solution: proof using comparison theorems

Appendix: Financial Frictions

- Introduce two kinds of non-convexities to entrepreneurship model
  - Financial frictions
  - Two entrepreneur production technologies
- Entrepreneur problem
  - Preferences: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$
  - Labor income: wz^θ
  - Entrepreneurial technologies

$$
\begin{aligned}
y_u &= F_u(z,k,\ell) = zB_u k^\alpha \ell^\beta \\
y_p &= F_p(z,k,\ell) = zB_p((k-f_k)^+)^\alpha((\ell-f_\ell)^+)^\beta
\end{aligned}
$$

  - Financial constraint: $k \leq \lambda a$
- Rest of the economy: representative firm with

$$
Y_c = F_c(A, K_c, L_c) = AB_c K_c^\eta L_c^{1-\eta}
$$

Appendix: Financial Frictions

- Optimal use of technology:

$$\Pi_j(a, z, A; w, r) = \max_{\ell, k \leq \lambda a} F_j(z, A, k, \ell) - (r + \delta)k - w\ell, \quad j = p, u$$

- Occupation choice:

$$M(a, z, A; w, r) = \max\{wz^\theta, \Pi_u(a, z, A; w, r), \Pi_p(a, z, A; w, r)\}$$

- Consumption-savings problem:

$$\max_{\{c_t\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

such that

$$\begin{aligned} da_t &= [M(a_t, z_t, A_t; w_t, r_t) + r_t a_t - c_t] dt \\ dz_t &= \mu(z_t) dt + \sigma(z_t) dW_t \\ a_t &\geq 0 \end{aligned}$$

Appendix: Financial Frictions

- Optimal entrepreneur behavior

$$\begin{aligned}
\rho v_t(a,z) &= \max_c u(c) + \partial_a v_t(a,z)[M(a,z;w_t,r_t) + r_t a - c] \\
&\quad + \partial_z v(a,z)\mu(z) + \frac{1}{2}\partial_{zz} v_t(a,zt)\sigma^2(z) + \dot{v}_t(a,z) \\
\partial_t g_t(a,z) &= -\partial_a[s_t(a,z)g_t(a,z)] - \partial_z[\mu(z)g_t(a,z)] + \frac{1}{2}\partial_{zz}[\sigma^2(z)g_t(a,z)] \\
s_t(a,z) &= M(a,z;w_t,r_t) + r_t a - c_t(a,z)
\end{aligned}$$

- Public firms

$$r_t = \partial_K F_c(A, K_{ct}, L_{ct}) - \delta, \quad w_t = \partial_L F_c(A, K_{ct}, L_{ct})$$

- Capital market-clearing

$$\text{savings} = K_c + K_u + K_p$$

- Labor market-clearing

$$\text{supply of workers} = L_c + L_u + L_p$$