GMM, Indirect Inference and Bootstrap

Method of Moments

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- Multiple linear regression model

$$\begin{aligned} y &= X\beta + u \\ u &\sim N\left(0, \sigma^2 I\right) \end{aligned}$$

- OLS estimator

$$\hat{\beta} = \left(X'X\right)^{-1} X'y$$

- Covariance matrix

$$Cov\left(\hat{\beta}\right) = \sigma^2 \left(X'X\right)^{-1}$$

- Gauss-Markov theorem

- Notation of Davidson and MacKinnon (2004),

$$
\begin{aligned}
y_t &= x_t(\beta) + u_t \\
u_t &\sim IID(0, \sigma^2)
\end{aligned}
$$

- $x_t(\beta)$ is a nonlinear function of the parameter vector β
- Example:

$$y_t = \beta_1 + \beta_2 x_{t1} + \frac{1}{\beta_2} x_{t2} + u_t$$

- Minimize the sum of squared residuals

$$\sum_{t=1}^{T} \left(y_t - x_t\left(\beta\right)\right)^2$$

with respect to β

- Usually, the minimization must be done numerically

- Raw moment of order p

$$\mu_p = E(X^p)$$

- Empirical raw moment of order p

$$\hat{\mu}_p = \frac{1}{n}\sum_{i=1}^{n} X_i^p$$

for a simple random sample $X_1, \ldots, X_n$

- Write r theoretical moments as functions of r unknown parameters

$$\begin{aligned} \mu_1 &= g_1(\theta_1, \ldots, \theta_r) \\ &\vdots \\ \mu_r &= g_r(\theta_1, \ldots, \theta_r) \end{aligned}$$

- Of course, central moments may be used as well

- Invert the system of equations:
  Write the r unknown parameters
  as functions of the r theoretical moments

$$\begin{aligned}\theta_1 &= h_1(\mu_1, \dots, \mu_r)\\ &\vdots\\ \theta_r &= h_r(\mu_1, \dots, \mu_r)\end{aligned}$$

Method of moments

Basic idea: Step 3

- Replace all theoretical moments by empirical moments

$$\begin{aligned}\hat{\theta}_1 &= h_1(\hat{\mu}_1, \ldots, \hat{\mu}_r) \\ &\vdots \\ \hat{\theta}_r &= h_r(\hat{\mu}_1, \ldots, \hat{\mu}_r)\end{aligned}$$

- The estimators $\hat{\theta}_1, \ldots, \hat{\theta}_r$ are **moment estimators**

- Moment estimators are consistent since

$$
\begin{aligned}
plim\hat{\theta}_1 &= plim\left(h_1(\hat{\mu}_1, \hat{\mu}_2, \ldots)\right)\\
&= h_1(plim\hat{\mu}_1, plim\hat{\mu}_2, \ldots)\\
&= h_1(\mu_1, \mu_2, \ldots)\\
&= \theta_1
\end{aligned}
$$

- In general, moment estimators are not unbiased and not efficient
- Since the empirical moments are asymptotically normal (why?), moment estimators are also asymptotically normal
  $\longrightarrow$ delta method [P]

Method of moments

Example

- Let $X \sim Exp(\lambda)$ with unknown parameter λ and let $X_1, \ldots, X_n$ be a random sample
- Step 1: We know that $E(X) = \mu_1 = 1/\lambda$
- Step 2 (inversion): $\lambda = 1/\mu_1$
- Step 3: The estimator is

$$\hat{\lambda} = \frac{1}{\hat{\mu}_1} = \frac{1}{\frac{1}{n}\sum_i X_i} = \frac{1}{\bar{X}_n}$$

- Is $\hat{\lambda}$ unbiased?
- Alternative: $Var(X) = 1/\lambda^2$, then $\hat{\lambda} = 1/\sqrt{S^2}$