GMM, Indirect Inference and Bootstrap

Stochastic convergence and limit theorems

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- Convergence of real sequences: Let $a_1, a_2, \ldots$ be a sequence of real numbers
- The sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to the limit a if for every (arbitrarily small) $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that

$$|a_n - a| < \varepsilon$$

  for all $n \geq N(\varepsilon)$
- Notation:

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \to a$$

Important questions:

- How can we transfer the idea of convergence to sequences of random variables?
- How can we visualize a sequence of random variables?
- What does convergence of sequences of random variables mean?
- Which sequences of random variables do we typically encounter in econometrics?

- Let $X_1, X_2, \ldots$ be random variables
$$X_i : \Omega \to \mathbb{R}.$$
Then $X_1, X_2, \ldots$ is called a **sequence of random variables**
- $X_1, X_2, \ldots$ are (countably infinite) multivariate random variables
- Formally, it is a sequence of functions (not real numbers)

- A sequence $X_1, X_2, \ldots$ of random variables converges **almost surely (fast sicher)** to a random variable X, if

$$P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1$$

- Notation

$$X_n \stackrel{a.s.}{\to} X$$

- This definition of convergence is not very important in econometrics

- A sequence $X_1, X_2, \ldots$ of random variables converges **in probability (nach Wahrscheinlichkeit)** to a random variable X, if

$$\lim_{n\to\infty} P\left(|X_n - X| < \varepsilon\right) = 1$$

- Notation

$$\begin{aligned} X_n &\overset{p}{\to} X \\ plim\, X_n &= X \end{aligned}$$

- This definition of convergence is very important in econometrics

- Special case: convergence in probability to a constant
- A sequence $X_1, X_2, \ldots$ of random variables converges **in probability** to a constant $a \in \mathbb{R}$, if

$$\lim_{n\to\infty} P\left(|X_n - a| < \varepsilon\right) = 1$$

- Notation

$$\begin{aligned} X_n &\stackrel{p}{\to} a \\ plim\, X_n &= a \end{aligned}$$

- This special case is very often encountered in econometrics

- A sequence $X_1, X_2, \dots$ of random variables with distribution functions $F_1, F_2, \dots$ converges **in distribution (weakly; in law; nach Verteilung)** to a random variable X with distribution function F, if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all $x \in \mathbb{R}$ where $F(x)$ is continuous
- Notation

$$X_n \overset{d}{\to} X$$

- Let $plim\ X_n = a$ and $plim\ Y_n = b$, then

$$\begin{aligned} plim\ (X_n \pm Y_n) &= a \pm b \\ plim\ (X_n Y_n) &= ab \\ plim\ \left(\frac{X_n}{Y_n}\right) &= \frac{a}{b}, \quad \text{if } b \neq 0 \end{aligned}$$

- If a function g is continuous in a, then

$$plim\ g\left(X_n\right) = g\left(a\right)$$

- If $Y_n \overset{d}{\to} Z$ and h is a continuous function, then
$$h(Y_n) \overset{d}{\to} h(Z)$$
- Cramér's theorem: If $X_n \overset{p}{\to} a$ and $Y_n \overset{d}{\to} Z$, then
$$X_n + Y_n \overset{d}{\to} a + Z$$
$$X_n Y_n \overset{d}{\to} aZ$$