# GMM, Indirect Inference and Bootstrap

## Multivariate random variables and multivariate normal distribution

Willi Mutschler

TU Dortmund

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- Let
$$X_i \colon \Omega \to \mathbb{R}, i = 1, \dots, n,$$
be random variables. The vector $X = (X_1, \dots, X_n)'$ is called **random vector** or $n$-dimensional random variable
- Multivariate random variables are a natural generalization of univariate random variables
- For $n = 2$ we often write $(X, Y)$ instead of $(X_1, X_2)$
- In the following we mostly refer to the bivariate case

- The function
$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$
for $(x,y) \in \mathbb{R}^2$ is called joint **cumulative distribution function** (or cdf, or distribution function) of $(X,Y)$
- $F_{X,Y}$ is monotonic increasing in $x$ and $y$ with limits

$$\begin{aligned}
\lim_{x\to-\infty} F_{X,Y}(x,y) &= \lim_{y\to-\infty} F_{X,Y}(x,y) = 0 \\
\lim_{x\to\infty, y\to\infty} F_{X,Y}(x,y) &= 1
\end{aligned}$$

- $(X, Y)$ are called **jointly discrete** if there is a finite (or countably infinite) number of points $x_i$ and $y_j$ such that

$$P(X = x_i, Y = y_j) > 0$$

  and $\sum_i \sum_j P(X = x_i, Y = y_j) = 1$
- Joint distribution function $F_{X,Y}$

$$F_{X,Y}(x, y) = \sum_{i|x_i \leq x} \sum_{j|y_j \leq y} P(X = x_i, Y = y_j)$$

- $(X, Y)$ are called **jointly continuous** if there is a non-negative function $f_{X,Y}$ such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v) dv du$$

- $f_{X,Y}$ is called the **joint density** (or pdf) of $(X, Y)$
- The density is

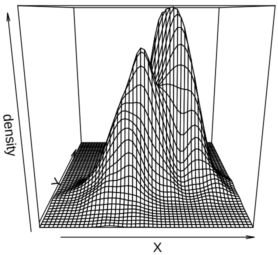
$$f_{X,Y}(x,y) = \frac{\partial^2 \; F_{X,Y}(x,y)}{\partial x \partial y}$$

- The volume under the density is a probability
- Therefore
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x,y\right) dxdy = 1$$
- The probability that $(X, Y)$ is inside the rectangle $[a, b] \times [a', b']$ is

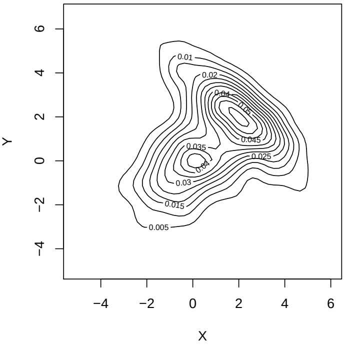
$$\begin{aligned} & P\left(a < X \leq b, a' < Y \leq b'\right) \\ = \; & \int_a^b \int_{a'}^{b'} f_{X,Y}(u,v) dvdu \end{aligned}$$

# Multivariate random variables
## Density

**3D plot**




**Contour plot**



- Let $(X, Y)$ be a random vector, then

$$\begin{aligned} F_X(x) &= F_{X,Y}(x,\infty) = \lim_{y\to\infty} F_{X,Y}(x,y) \\ F_Y(y) &= F_{X,Y}(\infty,y) = \lim_{x\to\infty} F_{X,Y}(x,y) \end{aligned}$$

  are called the **marginal distributions** of $X$ and $Y$

- The marginal densities of $X$ and $Y$ are

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f(x,y)dy \\ f_Y(y) &= \int_{-\infty}^{+\infty} f(x,y)dx \end{aligned}$$

- Reminder: Let $A$ and $B$ be two events (with $P(B) > 0$), then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Let $(X, Y)$ be jointly continuous; the **conditional density** of $X$ given $Y = y$ is

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

  and vice versa for $Y$ given $X = x$

- Conditional expectation, conditional cdf, and conditional variance of $X$ given $Y=y$

$$
\begin{aligned}
E\left(X|Y=y\right) &= \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx \\
P\left(X \leq x|Y=y\right) &= \int_{-\infty}^{x} f_{X|Y=y}(x)dx \\
Var\left(X|Y=y\right) &= \int_{-\infty}^{\infty} \left(x - E(X|Y=y)\right)^2 f_{X|Y=y}(x)dx
\end{aligned}
$$

- Let $(X, Y)$ be a random vector; $X$ and $Y$ are called (stochastically) **independent** if
$$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$$
for all $(x, y) \in \mathbb{R}^2$.
- Equivalently: $X$ and $Y$ are independent if
$$\begin{aligned} F_{X|Y=y}(x) &= F_X(x) \\ F_{Y|X=x}(y) &= F_Y(y) \end{aligned}$$
for all $x, y \in \mathbb{R}$

- Jointly continuous $X$ and $Y$ are independent if for all $(x, y) \in \mathbb{R}^2$

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$$

  or

$$\begin{aligned} f_{X|Y=y}(x) &= f_X(x) \\ f_{Y|X=x}(y) &= f_Y(y) \end{aligned}$$

- If $X$ and $Y$ are independent and $g$ and $h$ two (measurable) functions, then $g(X)$ and $h(Y)$ are also independent

- Generalization to $n$ random variables: The elements of the random vector are independent if for all $(x_1, \dots, x_n) \in \mathbb{R}^n$

$$F_{X_1,\dots,X_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

or

$$f_{X_1,\dots,X_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

- **Covariance** of $X$ and $Y$ (often denoted as $\sigma_{XY}$)

$$
\begin{aligned}
Cov(X,Y) &= E([X-E(X)][Y-E(Y)]) \\
&= E(XY)-E(X)E(Y)
\end{aligned}
$$

- **Correlation** of $X$ and $Y$ (often denoted as $\rho_{XY}$)

$$
Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}
$$

- **Expectation vector** of $X = (X_1, \ldots, X_n)'$

$$E(X) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix}$$

- **Covariance matrix** of $X = (X_1, \ldots, X_n)'$

$$\begin{aligned} Cov(X) &= E\left[(X - E(X))(X - E(X))'\right] \\ &= E(XX') - E(X)E(X)' \end{aligned}$$

Properties of covariance matrices

- Symmetry: $Cov(X) = Cov(X)'$
- $Cov(X)$ is positive semidefinite, i.e. for all real vectors $a \neq 0$

$$a' Cov(X) a \geq 0$$

- All diagonal elements are non-negative
- All eigenvalues of $Cov(X)$ are non-negative
- All sub-determinants are non-negative

- Let $X = (X_1, \ldots, X_n)'$ be a random vector with

$$
\begin{aligned}
E(X) &= \mu_X \\
Cov(X) &= \Sigma_X
\end{aligned}
$$

- Let

$$Y = AX + b$$

where $A$ is a real matrix and $b$ a real vector

- What are $E(Y)$ and $Cov(Y)$?

- The expectation vector is

$$\begin{aligned} E(Y) &= AE(X) + b \\ &= A\mu_X + b \end{aligned}$$

- The covariance matrix is

$$\begin{aligned} Cov(Y) &= ACov(X)A' \\ &= A\Sigma_X A' \end{aligned}$$

- Special case: If $A$ is a row vector, then $A\Sigma_X A'$ is the variance of the univariate random variable $Y$

# Multivariate normal distribution

## Outline

- Univariate standard normal distribution $N(0,1)$
- Univariate normal distribution $N(\mu,\sigma^2)$
- Relation between $N(0,1)$ and $N(\mu,\sigma^2)$
- Generalization to the $K$-dimensional case

- Let $U$ be a random variable with density function

$$\varphi\left(u\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^{2}\right),$$

then $U$ is called **standard normally (Gaussian) distributed**

- The distribution function $\Phi\left(u\right)=\int_{-\infty}^{u}\varphi\left(t\right)dt$ is tabulated and implemented in R etc.
- Moments: $E(U)=0$ and $Var(U)=1$

- Let $X$ be a random variable with density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right),$$

  then $X$ is called **normally (Gaussian) distributed**
- The distribution function is implemented in R etc., but it is not tabulated
- Moments: $E(X) = \mu$ and $Var(X) = \sigma^2$

# Multivariate normal distribution

Connections

- Let $U \sim N(0,1)$, then $X = \mu + \sigma U \sim N(\mu, \sigma^2)$
- Let $X \sim N\left(\mu, \sigma^2\right)$, then $U = (X - \mu)/\sigma \sim N(0,1)$
- Distribution functions

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- Quantile functions

$$x_p = \mu + \sigma u_p$$

- A $K$-dimensional random vector $X = (X_1, \ldots, X_K)'$ is called **multivariate normal** with parameters

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_K \end{bmatrix} \quad \text{und} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1K} \\ \vdots & \ddots & \vdots \\ \sigma_{K1} & \cdots & \sigma_K^2 \end{bmatrix}$$

if, for all $x = (x_1, \ldots, x_K)' \in \mathbb{R}^K$, the density function is

$$f(x) = (2\pi)^{-\frac{K}{2}} (\det \Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)\right)$$

# Multivariate normal distribution

K-dimensional normal distribution

- Notation

$$X \sim N(\mu, \Sigma)$$

- $\mu$ is a column vector of length $K$
- $\Sigma$ is a non-singular, positive definite $(K \times K)$ matrix (we exclude singular matrices for simplicity)
- Moments

$$\begin{aligned} E(X) &= \mu \\ Cov(X) &= \Sigma \end{aligned}$$

- Marginal distributions of $X$ are normal: If

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$$

  then

$$\begin{aligned} X_1 &\sim N(\mu_1, \Sigma_{11}) \\ X_2 &\sim N(\mu_2, \Sigma_{22}) \end{aligned}$$

- **Attention**: Even if all elements are normal, the vector need not be multivariate normal!

- Conditional distributions are normal,

$$
\begin{aligned}
X_1|\left(X_2 = x_2\right) &\sim N\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}\left(x_2 - \mu_2\right), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right) \\
X_2|\left(X_1 = x_1\right) &\sim N\left(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}\left(x_1 - \mu_1\right), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)
\end{aligned}
$$

- Linear transformations are normal: If $X \sim N(\mu, \Sigma)$, then

$$AX + b \sim N\left(A'\mu + b, A'\Sigma A\right)$$

# Multivariate normal distribution

## Properties

- Let $X \sim N(\mu, \Sigma)$ with $\Sigma$ positive definite, then there is a matrix $V$, such that
$$\Sigma = VV'$$
and
$$X = \mu + VU$$
where
$$U \sim N(0, I)$$

- Let $X \sim N(\mu, \Sigma)$ be the vector of returns of $K$ assets
- Let $A$ be a $(1 \times K)$ row vector of portfolio weights
- Portfolio return $Y = AX$ (scalar)
- The portfolio return is normally distributed,

$$Y \sim N\left(A\mu, A\Sigma A'\right)$$