GMM, Indirect Inference and Bootstrap

The three classical tests

Willi Mutschler

TU Dortmund

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The three classical tests

- Wald test, Lagrange multiplier test and likelihood ratio test (W, LM, LR)
- Hypotheses

$$H_0 : r(\theta) = 0 \quad \text{vs} \quad H_1 : r(\theta) \neq 0$$

- Often, r is a scalar-valued function and θ is a scalar
- The function r may be non-linear!

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Basic test ideas:

- Wald test: If $r(\theta) = 0$ is true, then $r(\hat{\theta}_{ML})$ will be close to 0
- Likelihood ratio test: If $r(\theta) = 0$ is true, then $\ln L(\hat{\theta}_R)$ will not be far below $\ln L(\hat{\theta}_{ML})$
- Lagrange multiplier test: If $r(\theta) = 0$ is true, the score function $g(\hat{\theta}_R) = \partial \ln L(\hat{\theta}_R)/\partial\theta$ will be close to 0

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Example:

- Let $X_1, \ldots, X_n$ be a random sample from $X \sim Exp(\lambda)$
- Test $H_0 : \lambda = 4$ against $H_1 : \lambda \neq 4$
- Different notation:
$$H_0 : r(\lambda) = 0$$
where $r(\lambda) = \lambda - 4$
- See `threetests.R`

Wald test

- Hypotheses

$$\begin{aligned} H_0 &: \quad r(\theta) = 0 \\ H_1 &: \quad r(\theta) \neq 0 \end{aligned}$$

  with functions $r = (r_1, \ldots, r_m)$
- m is the number of restrictions
- Wald test: If $r(\theta) = 0$ is true, then $r(\hat{\theta}_{ML})$ will be close to 0

- Asymptotically, under H_0 (by delta method!)

$$r(\hat{\theta}_{ML}) \sim N\left(0, Cov(r(\hat{\theta}_{ML}))\right)$$

with

$$Cov(r(\hat{\theta}_{ML})) = \frac{\partial r(\hat{\theta}_{ML})}{\partial \theta'} \cdot Cov(\hat{\theta}_{ML}) \cdot \frac{\partial r(\hat{\theta}_{ML})}{\partial \theta}$$

- Remember: If $X \sim N(\mu, \Sigma)$, then $(X-\mu)' \, \Sigma^{-1} \, (X-\mu) \sim \chi^2_m$
- Wald test statistic

$$W = r(\hat{\theta}_{ML})' \left[Cov(r(\hat{\theta}_{ML}))\right]^{-1} r(\hat{\theta}_{ML}) \overset{asy}{\sim} \chi^2_m$$

Remarks:

- Reject H_0 if W is larger than the $(1-\alpha)$-quantile of the χ^2_m-distribution
- Usually, $Cov(r(\hat{\theta}_{ML}))$ must be replaced by $\widehat{Cov}(r(\hat{\theta}_{ML}))$
- The Wald test is not invariant with respect to re-parametrizations
- The Wald test only requires the unrestricted ML estimator
- Ideal, if $\hat{\theta}_{ML}$ is much easier to calculate than $\hat{\theta}_R$

Likelihood ratio test

- Is $\ln L(\hat{\theta}_{ML})$ significantly larger than $\ln L(\hat{\theta}_R)$?
- LR test statistic

$$
\begin{aligned}
LR &= -2\ln\left(\frac{L(\hat{\theta}_R)}{L(\hat{\theta}_{ML})}\right) \\
&= -2\left(\ln L(\hat{\theta}_R) - \ln L(\hat{\theta}_{ML})\right)
\end{aligned}
$$

- Asymptotic distribution: $LR \overset{asy}{\sim} \chi^2_m$

Remarks:

- Reject H_0 if LR is larger than the $(1-\alpha)$-quantile of the χ^2_m-distribution
- To compute LR, one requires both the unrestricted estimator $\hat{\theta}_{ML}$ and the restricted estimator $\hat{\theta}_R$
- Ideal, if both $\hat{\theta}_{ML}$ and $\hat{\theta}_R$ are easy to calculate
- The LR test is often used to compare different models to each other

Lagrange multiplier test

- Is $g(\hat{\theta}_R)$ significantly different from 0?
- The test is based on the restricted estimator $\hat{\theta}_R$
- Lagrange approach: $\max_\theta \ln L(\theta)$ s.t. $r(\theta) = 0$
- LM test statistic

$$LM = g(\hat{\theta}_R)' \cdot \left[\mathcal{I}(\hat{\theta}_R)\right]^{-1} \cdot g(\hat{\theta}_R) \overset{asy}{\sim} \chi^2_m$$

with

$$\mathcal{I}(\hat{\theta}_R) = -E\left(\frac{\partial^2 \ln L(\hat{\theta}_R)}{\partial\theta\partial\theta'}\right)$$

Remarks:

- Reject H_0 if LM is larger than the $(1-\alpha)$-quantile of the χ^2_m-distribution
- The LM test only requires the restricted estimator
- Ideal, if $\hat{\theta}_R$ is much easier to calculate than $\hat{\theta}_{ML}$
- The LM test is often used to test misspecifications (heteroskedasticity, autocorrelation, omitted variables etc.)
- Asymptotically, the three tests are equivalent

The three classical tests

Multivariate case

- Example: Production function

$$Y_i = X_{i1}^{a_1} \cdot X_{i2}^{a_2} + u_i$$

  where $u_i \sim N(0, 0.05^2)$
- Log-likelihood function $\ln L(a_1, a_2)$
- ML estimators $\hat{a}_1$ and $\hat{a}_2$
- Hypothesis test of $a_1 + a_2 = 1$ or $a_1 + a_2 - 1 = 0$
- See `classtest.R`