BGP Model
Some Derivations

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1 Model

Consider the following model with two states variables. The first equation describes the law of motion of investment I_t,

$$I_t = \alpha_{1,1} I_{t-1} + \alpha_{1,2} K_t + s_t^I \tag{1}$$

while the second equation describes the law of motion of capital K_t

$$K_t = I_{t-1} + \alpha_{2,2} K_{t-1} + s_t^K. \tag{2}$$

Notice that s_t^I and s_t^K are two disturbances such that $s_t^I \perp s_\tau^K$ for all t and τ. In addition, we also assume that $s_t^j \perp s_\tau^j$ for all t and τ for $j \in \{I, K\}$.[1]

The system is linear and it can be written in a more compactly as

$$\begin{pmatrix} I_t \\ K_t \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} I_{t-1} \\ K_{t-1} \end{pmatrix} + \begin{pmatrix} s_t^I \\ s_t^K \end{pmatrix} \tag{3}$$

which is

$$X_t = AX_{t-1} + s_t \tag{4}$$

For the rest of the document, we will assume that both $\alpha_{1,1}$ and $\alpha_{2,2}$ are positive.[2]

2 Stability Conditions

In order to study the stability conditions, we parametrically evaluate the eigenvalues λ_1 and λ_2 of matrix A. In other words, we need to solve the following problem,

$$\det(A - \lambda I) = 0$$

which can be rewritten as,

$$\det \begin{pmatrix} \alpha_{1,1} - \lambda & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} - \lambda \end{pmatrix} = 0$$

[1] It can be easily derived that in steady state, both investment and capital are equal to $I_{ss} = K_{ss} = 0$. Without loss of generality, this result simplifies the analysis because levels outside of steady state are also deviations from steady state.

[2] The economic interpretation of this assumption is straightforward. A positive value of $\alpha_{2,2}$ means that a part of capital in the previous period is not fully depreciated and survived as an endowment in the subsequent period. Instead, $\alpha_{1,1}$ positive suggests that Equation 1 is the reduced form of a setting with dynamic strategic complementary of investment. If investment has been positive in the past then it is convenient to invest more also today.

which is,

$$\begin{aligned} 0 &= (\alpha_{1,1} - \lambda)(\alpha_{2,2} - \lambda) - \alpha_{1,2}\alpha_{2,1} \\ &= \alpha_{1,1}\alpha_{2,2} + \lambda^2 - (\alpha_{1,1} + \alpha_{2,2})\lambda - \alpha_{1,2}\alpha_{2,1} \\ &= \lambda^2 - (\alpha_{1,1} + \alpha_{2,2})\lambda + \alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1} \end{aligned}$$

Solving over λ yields,

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left[(\alpha_{1,1} + \alpha_{2,2}) \pm \sqrt{(\alpha_{1,1} + \alpha_{2,2})^2 - 4(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1})}\right] \\ &= \frac{1}{2}\left[(\alpha_{1,1} + \alpha_{2,2}) \pm \sqrt{(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1}}\right] \end{aligned} \tag{5}$$

Proposition 1 *Sufficient condition to have complex eigenvalues is to have $\alpha_{1,2}$ and $\alpha_{2,1}$ with a different sign.*

Proof Proof follows from Equation 5. In particular, $(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}\alpha_{2,1}$ might be negative if and only if the sign of $\alpha_{1,2}$ is different than the sign of $\alpha_{2,1}$.

Proposition 2 *Sufficient condition to have real eigenvalues is to have $\alpha_{1,2}$ and $\alpha_{2,1}$ with the same sign.*

Proof Proof straightforwardly follows from Proposition 1.

Proposition 3 *If $\alpha_{1,2}\alpha_{2,1} = 0$, then the system is stable (both $|\lambda_{1,2}| < 1$) if and only if both $\alpha_{1,1}$ and $\alpha_{2,2}$ are smaller than one.*[3]

Proof Since $\lambda_{1,2} = \frac{1}{2}[\alpha_{1,1} + \alpha_{2,2} \pm (\alpha_{1,1} - \alpha_{2,2})]$ then we have that $\lambda_1 = \alpha_{1,1}$ and $\lambda_2 = \alpha_{2,2}$. If both $\alpha_{1,1}$ and $\alpha_{2,2}$ are smaller than one then the system is automatically stable. On the other hand, if the system is stable - $|\lambda_{1,2}| < 1$ - then it must be the case that both $\alpha_{1,1}$ and $\alpha_{2,2}$ are smaller than one.

3 Shock Dependent Cyclical responses

From now on we will assume that $\alpha_{1,2}\alpha_{2,1} < 0$, $\lambda_{1,2}$ are real and their module is smaller than one. In particular, I will assume that $\alpha_{1,2} < 0$ and $\alpha_{2,1} > 0$.

3.1 Investment-Specific Shock

Assume in period t, when the system is in steady state, that $s_t^I = 1$. Impact responses at t of both variables are,

$$\begin{pmatrix} I_t \\ K_t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

At time $t+1$, dynamic responses are,

$$\begin{pmatrix} I_{t+1} \\ K_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} \\ \alpha_{2,1} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

At time $t+2$, dynamic responses are,

$$\begin{pmatrix} I_{t+2} \\ K_{t+2} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1}^2 + \alpha_{1,2}\alpha_{2,1} \\ \alpha_{2,1}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_{1,1} \\ \alpha_{2,1} \end{pmatrix} \tag{8}$$

[3]Notice that we assumed both $\alpha_{1,1}$ and $\alpha_{2,2}$ to be non negative in Section 1.

At time $t+3$, dynamic responses are,

$$\begin{aligned}\begin{pmatrix} I_{t+3} \\ K_{t+3} \end{pmatrix} &= \begin{pmatrix} \alpha_{1,1}(\alpha_{1,1}^2 + \alpha_{1,2}\alpha_{2,1}) + \alpha_{1,2}(\alpha_{2,1}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1}) \\ \alpha_{2,1}(\alpha_{1,1}^2 + \alpha_{1,2}\alpha_{2,1}) + \alpha_{2,2}(\alpha_{2,1}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1}) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_{1,1}^2 + \alpha_{1,2}\alpha_{2,1} \\ \alpha_{2,1}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1} \end{pmatrix}\end{aligned} \tag{9}$$

3.2 Capital-Specific Shock

Assume in period t, when the system is in steady state, that $s_t^K = 1$. Impact responses at t of both variables are,

$$\begin{pmatrix} I_t \\ K_t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

At time $t+1$, dynamic responses are,

$$\begin{pmatrix} I_{t+1} \\ K_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha_{1,2} \\ \alpha_{2,2} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

At time $t+2$, dynamic responses are,

$$\begin{pmatrix} I_{t+2} \\ K_{t+2} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2} \\ \alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2 \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_{1,2} \\ \alpha_{2,2} \end{pmatrix} \tag{12}$$

At time $t+3$, dynamic responses are,

$$\begin{aligned}\begin{pmatrix} I_{t+3} \\ K_{t+3} \end{pmatrix} &= \begin{pmatrix} \alpha_{1,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{1,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2) \\ \alpha_{2,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{2,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2} \\ \alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2 \end{pmatrix}\end{aligned} \tag{13}$$

At time $t+4$, dynamic responses are,

$$\begin{aligned}&\begin{pmatrix} I_{t+4} \\ K_{t+4} \end{pmatrix} = \\ &\begin{pmatrix} \alpha_{1,1}[\alpha_{1,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{1,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2)] + \alpha_{1,2}[\alpha_{2,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{2,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2)] \\ \alpha_{2,1}[\alpha_{1,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{1,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2)] + \alpha_{2,2}[\alpha_{2,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{2,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2)] \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_{1,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{1,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2) \\ \alpha_{2,1}(\alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2}) + \alpha_{2,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2) \end{pmatrix}\end{aligned} \tag{14}$$

Proposition 4 *Investment I_t displays a shock-dependent cyclical dynamics if*

1. $\alpha_{1,1}^2 < |\alpha_{1,2}\alpha_{2,1}|$
2. $\alpha_{2,2}^2 > |\alpha_{1,2}\alpha_{2,1}|$

Proof Notice that after an investment-specific shock s_t^I, responses of I_t are,

$$\begin{aligned} I_t &= 1 \\ I_{t+1} &= \alpha_{1,1} \\ I_{t+2} &= \alpha_{1,1}^2 + \alpha_{1,1}\alpha_{2,1} \\ I_{t+3} &= \alpha_{1,1}(\alpha_{1,1}^2 + \alpha_{1,2}\alpha_{2,1}) + \alpha_{1,2}(\alpha_{2,1}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1}) \end{aligned}$$

which implies that both I_t and I_{t+1} are positive by construction. However, I_{t+2} and I_{t+3} are negative. In particular, I_{t+2} is negative because $\alpha_{1,1}^2 < |\alpha_{1,2}\alpha_{2,1}|$. Moreover, I_{t+3} is negative because both $\alpha_{1,1}(\alpha_{1,1}^2 + \alpha_{1,2}\alpha_{2,1})$ and $\alpha_{1,2}(\alpha_{2,1}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1})$ are negative. This is already sufficient to show that investment I_t displays a cyclical dynamics after an investment-specific shock, s_t^I.

Conversely, after a capital-specific shock s_t^K, responses of I_t are,

$$\begin{aligned}
I_t &= 0 \\
I_{t+1} &= \alpha_{1,2} \\
I_{t+2} &= \alpha_{1,1}\alpha_{1,2} + \alpha_{1,2}\alpha_{2,2} \\
I_{t+3} &= \alpha_{1,1}(\alpha_{1,1}^2 + \alpha_{1,2}\alpha_{2,2}) + \alpha_{1,2}(\alpha_{1,2}\alpha_{2,1} + \alpha_{2,2}^2)
\end{aligned}$$