BGP Model
Some Derivations

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1 Model

Consider the following model with two states variables. The first equation describes the optimal choice of investment I_t,

$$I_t = \phi I_{t-1} + \psi K_t + s_t^I \tag{1}$$

while the second equation describes the law of motion of capital K_t

$$K_t = I_{t-1} + (1-\delta)K_{t-1} + s_t^K. \tag{2}$$

Notice that s_t^I and s_t^K are two exogenous shocks such that $s_t^I \perp s_\tau^K$ for all t and τ. In addition, we also assume that $s_t^j \perp s_\tau^j$ for all t and τ for $j \in \{I, K\}$. Moving from the structural form to the reduced form,

$$\begin{aligned} I_t &= (\phi + \psi)I_{t-1} + \psi(1-\delta)K_{t-1} + \psi s_t^K + s_t^I \\ K_t &= I_{t-1} + (1-\delta)K_{t-1} + s_t^K. \end{aligned} \tag{3}$$

Under mild full-rank conditions, it can be easily derived that in steady state, both investment and capital are equal to $I_{ss} = K_{ss} = 0$. The system is linear and it can be written more compactly as

$$\begin{pmatrix} I_t \\ K_t \end{pmatrix} = \begin{pmatrix} \phi+\psi & \psi(1-\delta) \\ 1 & (1-\delta) \end{pmatrix} \begin{pmatrix} I_{t-1} \\ K_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_t^I \\ s_t^K \end{pmatrix} \tag{4}$$

which is

$$X_t = BX_{t-1} + As_t \tag{5}$$

For the rest of the document, we will assume that both ϕ and δ are real, positive and smaller than one.[1]

2 Stability Conditions

In order to study the stability conditions, we parametrically evaluate the eigenvalues λ_1 and λ_2 of matrix A. In other words, we need to solve the following problem,

$$\det(B - \lambda I) = 0$$

[1] The economic interpretation of this assumption is straightforward. A positive δ between 0 and 1 means that a part of capital in the previous period is lost in the form of depreciation. Instead, ϕ positive suggests that Equation 1 is the linearized form of a model with investment-adjustment cost.

which can be rewritten as,

$$\det \begin{pmatrix} \phi+\psi-\lambda & \psi(1-\delta) \\ 1 & 1-\delta-\lambda \end{pmatrix} = 0$$

which is,

$$\begin{aligned}
0 &= (\phi+\psi-\lambda)(1-\delta-\lambda) - \psi(1-\delta) \\
&= (\phi+\psi)(1-\delta) + \lambda^2 - (\phi+\psi+1-\delta)\lambda - \psi(1-\delta) \\
&= \lambda^2 - (\phi+\psi+1-\delta)\lambda + \phi(1-\delta)
\end{aligned}$$

Solving over λ yields,

$$\lambda_{1,2} = \frac{1}{2}\left[(\phi+\psi+1-\delta) \pm \sqrt{(\phi+\psi+1-\delta)^2 - 4\phi(1-\delta)}\right] \tag{6}$$

3 Shock Dependent Cyclical responses to I_t

Isolate K_t from Equation 2 and get,

$$K_t = \frac{1}{1-(1-\delta)L}(I_{t-1} + s_t^K)$$

which lagged of one period is

$$K_{t-1} = \frac{1}{1-(1-\delta)L}(I_{t-2} + s_{t-1}^K)$$

substitute is into reduced form investment equation in System 3 which is

$$I_t = (\phi+\psi)I_{t-1} + \psi(1-\delta)\left[\frac{1}{1-(1-\delta)L}(I_{t-2} + s_{t-1}^K)\right] + \psi s_t^K + s_t^I$$

which can be rewritten as

$$I_t = (\phi+\psi+1-\delta)I_{t-1} - \phi(1-\delta)I_{t-2} + \psi s_t^K + s_t^I - (1-\delta)s_{t-1}^I$$

3.1 Spectrum Conditional on Preference Shocks

$$\begin{aligned}
& I_t - (\phi+\psi+1-\delta)I_{t-1} + \phi(1-\delta)I_{t-2} = s_t^I - (1-\delta)s_{t-1}^I \\
\Rightarrow \quad & [1-(\phi+\psi+1-\delta)L + \phi(1-\delta)L^2]I_t = (1-(1-\delta)L)s_t^I
\end{aligned}$$

which is

$$I_t = \frac{1-(1-\delta)L}{1-(\phi+\psi+1-\delta)L+\phi(1-\delta)L^2}s_t^I \tag{7}$$

Spectrum of I_t is

$$\begin{aligned}
g_I(e^{-i\omega}) &= \left(\frac{1-(1-\delta)e^{-i\omega}}{1-(\phi+\psi+1-\delta)e^{-i\omega}+\phi(1-\delta)e^{-2i\omega}}\right)\left(\frac{1-(1-\delta)e^{i\omega}}{1-(\phi+\psi+1-\delta)e^{i\omega}+\phi(1-\delta)e^{2i\omega}}\right)\sigma_I^2 \\
&= \frac{(1-(1-\delta)e^{-i\omega})(1-(1-\delta)e^{i\omega})}{(1-(\phi+\psi+1-\delta)e^{-i\omega}+\phi(1-\delta)e^{-2i\omega})(1-(\phi+\psi+1-\delta)e^{i\omega}+\phi(1-\delta)e^{2i\omega})}\sigma_I^2 \\
&= \frac{1+(1-\delta)-(1-\delta)(e^{i\omega}+e^{-i\omega})}{1+(\phi+\psi+1-\delta)^2+\phi^2(1-\delta)^2+(\phi+\psi+1-\delta)(1-\phi^2(1-\delta))(e^{i\omega}+e^{-i\omega})-\phi(1-\delta)(e^{2i\omega}+e^{-2i\omega})}\sigma_I^2 \\
&= \frac{1+(1-\delta)-2(1-\delta)\cos(\omega)}{1+(\phi+\psi+1-\delta)^2+\phi^2(1-\delta)^2+2(\phi+\psi+1-\delta)(1-\phi^2(1-\delta))\cos(\omega)-2\phi(1-\delta)\cos(2\omega)}\sigma_I^2 \\
&= \frac{h_1(\omega)}{h_2(\omega)}
\end{aligned} \tag{8}$$

First order condition is,

$$\frac{\partial g_I(e^{-i\omega})}{\partial \omega} = \frac{2(1-\delta)\sin(\omega)h_2(\omega) - h_1(\omega)[4\phi(1-\delta)\sin(2\omega) - 2(\phi+\psi+1-\delta)(1-\phi(1-\delta))\sin(\omega)]}{h_2(\omega)^2}\sigma_I^2 \tag{9}$$