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REGULAR ARTICLE

Sentiment-driven limit cycles and chaos

Orlando Gomes[1] · **J. C. Sprott**[2]

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Abstract A recent strand of macroeconomic literature has placed sentiment fluctuations at the forefront of the academic debate about the foundations of business cycles. Waves of optimism and pessimism influence the decisions of investors and consumers, and they might therefore be interpreted as a driving force for the performance of the economy in the short term. In this context, two questions regarding the formation and evolution of psychological moods in an economic setting gain relevance: First, how can we model the process of transmission of sentiments across economic agents? Second, is this process capable of generating endogenous and persistent fluctuations? This paper answers these two questions by proposing a simple and intuitive continuous-time dynamic sentiment spreading model based on the rumor propagation literature. As agents contact with one another, endogenous fluctuations are likely to emerge, with trajectories of sentiment shares potentially exhibiting periodic cycles and chaotic behavior.

Keywords Sentiments · Waves of optimism and pessimism · Endogenous fluctuations · Limit cycles · Chaos

JEL Classification E32 · E03 · C62

✉ Orlando Gomes
omgomes@iscal.ipl.pt

J. C. Sprott
sprott@physics.wisc.edu

[1] Lisbon Accounting and Business School (ISCAL-IPL) and Business Research Unit (BRU-IUL), ISCAL, Av. Miguel Bombarda 20, 1069-035 Lisbon, Portugal

[2] Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, USA

1 Introduction

There is no consensus among macroeconomists about the role that sentiments or "animal spirits" occupy in the context of the forces that effectively drive observed business fluctuations. Although many agree that sentiments or confidence levels are relevant to determine the intensity of economic activity, these terms are often loosely mentioned, and they seldom appear as consistent structural elements of the most popular and widely accepted macro models. The reason for this is simple: as De Grauwe (2011, page 424) pragmatically puts it "the notions of 'animal spirits' and rational expectations do not mix well." In fact, mainstream economic models tend to assume, in a dogmatic way, that agents are rational, markets are complete, and information disseminates widely and freely. These assumptions are convenient for analytical tractability, but leaving no room for sentiments or animal spirits, they contradict the observed behavior of consumers and investors in the economy.[1]

The orthodox interpretation of the foundations of economic behavior naturally ignores local interaction in business relations since this perspective would imply an arbitrary and unjustified confinement of the agents to a limited part of the market (see Durlauf 2012). However, the consensus around such a view is being progressively challenged within the profession, with a growing number of authors embracing the idea that complete and flawless coordination of actions arising from the interaction among millions of agents in the whole market economy is nothing more than a fiction (see Galtier et al. 2012).

The immediate corollary of the observation that complete access to markets is beyond the reach of any individual agent, no matter its dimension and power, is that advancement of the knowledge about socio-economic processes requires approaching the mechanisms through which agents interact locally, coordinate their actions, and decide strategically. Once local interaction is taken seriously and integrated into economic models, the beliefs of agents will surely depart from complete homogeneity, and sentiment waves will probably emerge.

Most of the recent meaningful approaches to business cycles that base their arguments on the notion of sentiments incorporate some kind of informational constraint that limits agents' full access to the market and forces contact at a local level. This is the path followed by the influential studies of Angeletos and La'O (2013) and Milani (2014), Benhabib et al. (2015, 2016) and Chahrour and Gaballo (2015). In all these studies, information frictions that prevent complete, instantaneous, and

[1] At this initial stage, we make a terminological clarification that is important for the discussion that follows. To be rigorous, the concepts of sentiments, understood as a predisposition to be optimistic or pessimistic, and of animal spirits, in the strict Keynesian sense, are not necessarily synonymous. Although some authors refer to these terms interchangeably (e.g., Benhabib et al. 2015, page 549, state that "Fluctuations can be driven by waves of optimism or pessimism, or as in Keynes' terminology, by 'animal spirits' that are distinct from fundamentals"), Keynes' notion of animal spirits has a precise meaning; in the General Theory (Keynes 1936) animal spirits are referred to as *a spontaneous urge to action rather than inaction*. Because our sentiment spreading model deals with behavioral features that do not necessarily comply with this idea of propensity to take actions, we enclose them in the less formal notion of sentiments and refrain from using the term "animal spirits."

global interaction generate sentiment oscillations that give rise to self-fulfilling business fluctuations. Although strictly economic events help propagate the fluctuations, their source is a mix of psychological and sociological drivers that take the form of sentiment shocks.

The above mentioned contributions, although emphasizing the role of interaction and coordination, share a common characteristic with orthodox business cycle theory: fluctuations occur as a consequence of exogenous disturbances; instead of the commonly taken technology, policy, preference or financial shocks, the source of cycles is the occurrence of confidence or sentiment shocks. The ignorance about the sources of sentiment changes that the exogeneity assumption implies is apparently not a strong concern for these theorists since, as Angeletos et al. (2015) assert, any formal model of business cycles must ultimately attribute the causes of fluctuations to an external source. This is a debatable postulate, though, and it is precisely on the endogenous origins of aggregate sentiment oscillations that we will focus our attention.

To the aforementioned literature, we counter with a model that offers a plausible mechanism for endogenously generated systematic sentiment shocks. The evolution of sentiments will be modeled as the outcome of trivial social contact at a local level, but the emergent sentiment fluctuations will have a close association with economic processes in two ways: upstream, we consider that the cadency of contacts among individuals leading to eventual sentiment changes is a by-product of economic relations, with agents choosing to interact given their desire to share ideas, information, and knowledge that, in turn, allow them to enhance productivity and increase output; downstream, sentiment cycles influence the economy through the decisions of a representative investor who will over-invest under generalized optimism and under-invest under pervasive pessimism.

Models of endogenous business cycles, a category which includes our analytical framework, are frequently explored by economic theorists, but they are subject to relevant criticism. Hommes (2013) systematizes in three points the reasons why endogenous nonlinearities do not provide a fully compelling explanation for observable business fluctuations. These are: (*i*) most of the models in this class generate limit cycles that are too regular, thus lacking any meaningful association with the empirical evidence on fluctuations in economic and financial time series; (*ii*) endogenous business cycle models are often built upon 'ad-hoc' dynamic equations, not derived from microfoundations; (*iii*) if the cycles originating from the models' dynamics are regular, then agents will surely be acting irrationally since rational agents would be able to anticipate the trajectories followed by the economic variables, thus revising their expectations accordingly, and consequently causing the fluctuations to fade out.

In light of the above criticism, why is it relevant to search for a framework where sentiment fluctuations, instead of being an external shock, are an intrinsically endogenous component of aggregate behavior? The answer to this question might be structured along the lines suggested by Beaudry et al. (2015), who interpret a limit cycle as a more solid and consistent basis to begin analyzing the impact of exogenous shocks than the common static equilibrium model that, in its deterministic version, typically displays a stable fixed-point equilibrium. In this view, economic activity possesses an underlying cyclical structure, possibly of a regular nature, over which shocks on fundamentals occur and propagate in a more intuitive and natural fashion than what would succeed if the underlying structure were a plain fixed-point steady-state displaying no oscillations whatsoever, case in which the excessive regularity critique becomes even more acute. In the words of Beaudry et al. (2015, page 1), concerning dynamic systems of the macro economy displaying limit cycles, "irregular business cycles can emerge from these underlying regular forces combined with shocks that move the system away from an attracting orbit."

Deterministic cycles of a regular or, possibly, irregular nature might be merged with a stochastic environment, thus weakening the strength of the arguments in the first and third items of Hommes' critique. Endogenous cycles do constitute a plausible interpretation of macro fluctuations, and they may offer a fundamental underlying structure of cyclicality. The view suggested by the model in this paper is that there is an underlying cyclical structure in human behavior, with origins in social interaction, that promotes phases of prevailing optimism that alternate with phases in which pessimism dominates. It is within this structure that shocks over fundamentals will eventually exert their influence.

By embracing the idea that there is an endogenous component in the observed macroeconomic fluctuations, the analysis pursued in this paper shares points in common with two other strands of literature: on one hand, the macro-dynamics disequilibrium theory developed by Asada et al. (2010, 2011), Chiarella and Flaschel (2000), and Chiarella et al. (2005), which associates endogenous fluctuations with wage-price spirals and inventory dynamics; on the other hand, the rational routes to randomness literature, pioneered by Brock and Hommes (1997, 1998), and thoroughly surveyed in Hommes (2006, 2013), where an evolutionary switching process between a fully rational decision rule and a boundedly rational decision rule (an heuristic) conducts to long-term bounded instability in economic and financial time series. Differently from both these approaches, our model associates perpetual cyclical motion with a circular process of meetings among agents that are heterogeneous in the sense of holding different sentiments that nevertheless will change as contact with others occurs.

The adopted model of social interaction is adapted from rumor propagation theory as developed, among many others, by Zanette (2002), Nekovee et al. (2007), Huo et al. (2012), Zhao et al. (2012), and Wang et al. (2013). In rumor spreading models, there are three categories of agents that interact in a complex social network; these are the ignorant or susceptible, the spreaders, and the stiflers. When an ignorant meets a spreader of a rumor, the ignorant becomes a spreader with a given probability; when a spreader meets another spreader or a stifler, the spreader will potentially become a stifler; eventually, the stifler might forget the rumor when in contact with an ignorant and thus might evolve back to the initial state. The adaptation of rumor spreading to sentiment propagation requires transforming ignorant into neutral agents, spreaders into exuberant, and stiflers into non-exuberant individuals (as in Gomes 2015a, b); since sentiments of both optimism and pessimism are considered, there will be five categories of agents: neutral, exuberant optimists, non-exuberant optimists, exuberant pessimists, and non-exuberant pessimists.

At a given time, each agent may occupy any of the sentiment categories, and only local interaction with other individuals will cause a transition from one category to another. The model also allows for the possibility of exuberant individuals, those who more enthusiastically support their sentiment view, to look at the overall sentiment evolution and consider this evolution when forming a subjective transition probability to another sentiment category. As described, the model gives rise to a four-dimensional system of ordinary differential equations (ODEs) that for reasonable parameter values generates persistent cyclical and chaotic trajectories for the various sentiment categories. The chaotic solution is particularly relevant since it indicates that a small and simple set of interaction rules might trigger bounded but irregular waves of optimism and pessimism, even in the absence of external shocks, that resemble the observed empirical time series representing the confidence of agents in the economy (see, e.g., Lemmon and Portniaguina 2006).

In summary, we provide a simple model of fluctuations in agents' confidence, in the context of social contagion, thus offering a rationale for the observed recurrent sentiment shocks. The explanation is focused on direct local interaction among agents, and the resulting chaotic model satisfies Sprott's 2011 criteria for the publication of new chaotic systems: (*i*) the system should credibly model some relevant problem in nature (or, we add, in the economy and in society); (*ii*) the system should display original dynamic behavior; and (*iii*) the system should be as simple as possible. Our model satisfies these requisites, addressing an important social issue with unquestionable economic repercussions using a simple mathematical system. To the best of our knowledge, this is the first continuous-time dynamic model proposed in the literature representing social contact in a simple homogeneous network that has chaotic solutions.

The remainder of the paper is organized as follows. Section 2 describes the model of sentiment spreading and characterizes the underlying dynamics. Section 3 establishes a link between the sentiment propagation mechanism, the economic decisions of agents, and performance of the economy. Section 4 explores the global dynamics of the sentiments' model, highlighting the presence of chaos. Finally, Section 5 gives conclusions.

2 The sentiment propagation model

Consider a large social network, with no particular pre-specified structure, within which economic agents establish interaction relations. Agents are dispersed across five categories that represent different confidence levels about the future performance of the economy. The five categories are: neutrality, exuberant optimism, non-exuberant optimism, exuberant pessimism and non-exuberant pessimism. The respective shares of agents in each category at time t are $x(t)$, $y(t)$, $z(t)$, $v(t)$, and $w(t)$.[2] Since each agent resides in one and only one category at a time, the condition $x + y + z + v + w = 1$, $\forall t$ holds.

[2]Henceforth, to simplify notation and when no ambiguity arises, the time argument in time-dependent variables will be suppressed.

Table 1 Interaction outcomes

Sentiment held by the agent promoting the interaction: / Sentiment held by the agent target of the interaction:	x	y	z	v	w
x	–	y	–	v	–
y	–	z	z	–	–
z	x	–	–	–	–
v	–	–	–	w	w
w	x	–	–	–	–

Start by assuming that sentiment propagation across the network of social relations obeys the following sequence: when a neutral agent meets an exuberant individual, optimist or pessimist, that agent also becomes an exuberant optimist or pessimist, respectively, with a given probability; second, when an exuberant optimist or pessimist meets another optimist or pessimist, exuberant or not, the exuberant individual becomes non-exuberant within the same class of sentiment, with a given probability; third, when a non-exuberant agent, optimist or pessimist, interacts with a neutral agent, that agent becomes neutral, with a given probability. To simplify the analysis, assume the transition probability $\theta \in (0, 1]$ is the same for all the transitions. Table 1 lays out the established interaction rules for an easier perception of the contagion effects. The table elucidates how an agent with the sentiment indicated in each line eventually swap sentiments when meeting an agent with the sentiment displayed in the top of each column. The outcome of the interaction, for the agent that initiates the contact, is presented in the respective cell inside the table.[3] Note that the meetings are one-way contacts: they are triggered by the first agent who is oblivious to the response of the second agent.

Of course one can contest the exact shape of the sequences of interaction and respective outcomes as sketched. As mentioned in the Introduction, they were adopted because they conform to the rumor spreading literature, and sharing a sentiment is probably not too different from sharing a rumor: one passes from neutrality to exuberance, from exuberance to non-exuberance and back to neutrality, given the pattern of established contacts. Furthermore, the established rules have some similarities to the ones Angeletos and La'O (2013) also consider in a related process of dissemination of sentiments. For these authors, agents are split into uninformed, partially informed, and fully informed; the partially informed are also designated, in the

[3] Keep in mind that this outcome occurs with a probability θ; with a probability $1 - \theta$, the contact implies no sentiment change.

mentioned framework, as exuberant, the pivotal class that has some but not complete knowledge of relevant data about the expected evolution of the economy, and thus adopts a particularly active attitude toward the acquisition of information.

In the Angeletos-La'O model, the uninformed are transmuted into partially informed when meeting someone in this second category, and the partially informed become fully informed when meeting someone in the same stage or when meeting a fully-informed agent. In this specific case, the propagation mechanism works as an information ladder where all agents eventually become fully informed, following a process that might be lengthy. Meanwhile, a wave of optimism forms as a progressively larger portion of uninformed economic units become exuberant and, subsequently, it fades out when exuberance is systematically replaced by full information. Therefore, in this view, a sentiment shock is a fad, i.e., a state of less than perfect information that will shrink to zero over time.

The main difference between our proposal and the one underlying the Angeletos-La'O framework is that we consider circularity: once non-exuberant individuals meet those with a neutral attitude, the ones that were not already strong activists for a given sentiment will abandon it and become neutral as well, a state in which they are susceptible to return to their previous sentiment or to shift to the opposite sentiment. The remainder of the process is similar to the one described: a neutral meeting an exuberant is influenced by this second agent with a given probability and might become exuberant about the sentiment as well; an exuberant looses stamina and eventually becomes non-exuberant when meeting a non-exuberant or another exuberant individual (in this second case, the agent realizes that having deep feelings about being optimistic or pessimistic is not an exclusive feature of her own emotional state and such strong feelings are replaced by a more moderate, or non-exuberant, attitude towards the sentiment).

A fundamental innovation of this process is that two polar sentiments – optimism and pessimism – are simultaneously taken, and to go from one to the other, agents must pass through the neutral state. Once, after a meeting, an optimist or a pessimist change their status back to neutrality, the possibility of acquiring the previously held sentiment or, alternatively, changing to the opposite sentiment will be determined by the profile of the agents contacted in subsequent time periods.

Therefore, similar to the Angeletos-La'O framework, cyclical motion is triggered by sentiment dynamics emerging from decentralized and direct contact among agents; different from Angeletos-La'O, though, sentiment switching in this paper is not a process to justify a single impulse-response event; on the contrary, it is a source of everlasting fluctuations where periods of dominant optimism systematically permute with periods of dominant pessimism, given the circular flow assumption.

Although most of the networks considered in various fields of science are complex with a heterogeneous connectivity profile as in the case of the scale-free networks of Barabási and Albert (1999), we consider a homogeneous network in which every agent is equally connected. We do this for analytical tractability and because of an economic rationale explained in Section 3. This rationale is based on the idea that all agents solve the same optimality problem and thus compute the same optimal level of intended connectivity. The degree of connectivity is an integer if we conceive it strictly as the number of links of a node to the other nodes in the network;

alternatively, it can be considered a positive real number if we interpret it as the number of links weighted by the strength of the links. We will adopt the second, broader, interpretation and denote the relevant connectivity parameter by $\kappa > 0$. In what follows it will be convenient to define the product $\zeta \equiv \theta\kappa > 0$.[4]

Applying the law of mass action to the described interaction rules leads to a system of ODEs that can be represented as[5]

$$\begin{cases} \dot{x} = -\zeta x(y+v) + \zeta(z+w)x \\ \dot{y} = \zeta xy - \zeta y(y+z) \\ \dot{v} = \zeta xv - \zeta v(v+w) \\ \dot{z} = \zeta y(y+z) - \zeta zx \\ \dot{w} = \zeta v(v+w) - \zeta wx \\ x(0), y(0), z(0), v(0), w(0) \ \text{given} \end{cases} \tag{1}$$

Note that every term is proportional to ζ, and hence ζ can be eliminated from the equations by re-scaling time according to $t \to t/\zeta$. Observe, also, that the equations in system (1) describe quantitatively the contents of Table 1: the meeting between a neutral (x) and an exuberant (y or v) makes x decrease, and makes each of the shares y and v increase. The meeting between exuberant individuals (y or v) and other agents sharing the same sentiment type ($y+z$ or $v+w$) leads to the flow of a percentage of agents from states y or v to states z or w, respectively. Finally, when non-exuberants (z or w) meet neutrals (x), the x share increases and the shares z and w decrease.

Since the sum of the five shares is equal to unity, the five-dimensional system (1) reduces to a four-dimensional compact form through the suppression of variable x,

$$\begin{cases} \dot{y} = \zeta\,[1-(y+z+v+w)]\,y - \zeta y(y+z) \\ \dot{v} = \zeta\,[1-(y+z+v+w)]\,v - \zeta v(v+w) \\ \dot{z} = \zeta y(y+z) - \zeta z\,[1-(y+z+v+w)] \\ \dot{w} = \zeta v(v+w) - \zeta w\,[1-(y+z+v+w)] \\ y(0), z(0), v(0), w(0) \ \text{given} \end{cases} \tag{2}$$

[4]This product, although convenient for analytical purposes, merges the role of the probability of transition with the role of the degree of connectivity, hiding the individual effects of each parameter when later assessing the model's dynamics. One should note that all that matters for the purpose of the analysis of dynamics is that any specific value of ζ corresponds to an array of values of θ and κ, where a relation of opposite sign exists between the two, i.e., if one considers, e.g., $\zeta = 40$, this might mean a transition probability $\theta = 0.5$ and a connectivity level of $\kappa = 80$ or, alternatively, $\theta = 0.25$ and $\kappa = 160$, or any other combination of parameters such that ζ remains at the specified value.

[5]See Nekovee et al. (2007) for a thorough characterization of a rumor spreading process leading to a set of differential equations with features similar to system (1).

The essential dynamical features of the sentiment variables under system (2) are given by the following two propositions:

Proposition 1 *System* (2) *contains four equilibrium points plus an equilibrium line*, *respectively*,

$$
\begin{aligned}
e_1 &: (y^*, v^*, z^*, w^*) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)\\
e_2 &: (y^*, v^*, z^*, w^*) = \left(\frac{1}{4}, 0, \frac{1}{4}, 0\right)\\
e_3 &: (y^*, v^*, z^*, w^*) = \left(0, \frac{1}{4}, 0, \frac{1}{4}\right)\\
e_4 &: (y^*, v^*, z^*, w^*) = (0, 0, 0, 0)\\
e_5 &: (y^*, v^*, z^*, w^*) = \left(0, 0, z^*, w^*\right), \textit{ with } z^* + w^* = 1.
\end{aligned}
$$

Proof See Appendix. □

Proposition 2 *Equilibrium point* e_1 *is locally stable. All the other equilibria are unstable.*

Proof See Appendix. □

The results in Propositions 1 and 2 indicate that any equilibria where exuberant optimism or exuberant pessimism are absent (which we denote as corner solutions) are unstable. This implies that under condition $y(0) > 0 \wedge v(0) > 0$, the system in the long run will not remain in any of the corner solutions. Thus, the mentioned initial condition will necessarily imply a convergence to the only stable equilibrium point, which is e_1, a point in which agents are equally distributed among neutrality, optimism, and pessimism, and equally distributed among neutrality, exuberance, and non-exuberance.

The symmetry of the stable result (i.e., the coexistence of sentiment states with identical densities) is a straightforward consequence of assuming the same probability of transition for all the transition processes, i.e., the same share of individuals will systematically abandon a sentiment class and will enter another. If we had assumed different probabilities of transition for switching sentiments, the equilibrium state would reflect the existence of a higher percentage of individuals in the state for which the probability of entering is larger and the probability of leaving is smaller relative to the other states. Although there is a constant flow of agents, and no agent remains indefinitely in the same sentiment position, the fully uniform distribution of individuals across states is only possible for a balanced transition where probabilities of change are the same for all switching options.

Figure 1 illustrates the convergence of the four variables toward the only stable long-term solution, e_1, for initial conditions $(y(0), v(0), z(0), w(0)) = (0.01; 0.02; 0; 0)$. The figure confirms the decaying oscillations converging on e_1 as

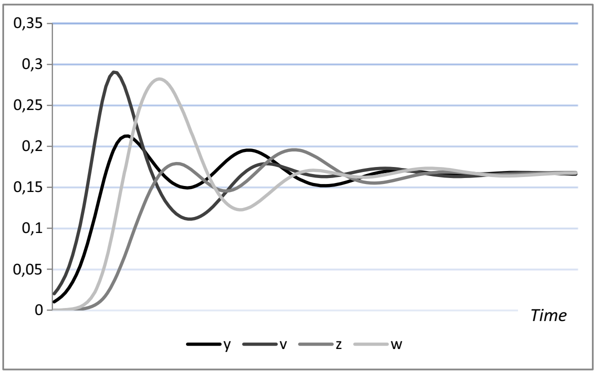

Fig. 1 Convergent sentiment trajectories

predicted by the eigenvalues of the respective Jacobian matrix, which are complex conjugates with a negative real part (see the proof of Proposition 2).

This stability outcome prevails in the relatively simple case described above. However, a reasonable assumption concerning the behavior of exuberant individuals will now be introduced, and this assumption will drastically alter the system's dynamics. Exuberant agents are likely to be the ones with a more pro-active attitude within the network of relations. This means that they will probably search for additional information to decide how to act when they meet another agent.[6] Specifically, we assume that the transition from the exuberant state (y or v) to the non-exuberant state (z or w) will depend on the assessment that exuberant individuals make about the strength of their belief. If they perceive that it is fading (i.e., if the inflow to the sentiment, ζxy or ζxv, is lower than the outflow, ζzx or ζwx, respectively for optimists and pessimists), then the relevant probability of transition θ from exuberance to non-exuberance will assume a higher value than in the opposite case as characterized by the following sigmoidal forms:

$$\theta(y \to z) = \frac{\theta}{2}\left[1 - \tanh\left(\zeta xy - \zeta zx\right)\right] \tag{3}$$

$$\theta(v \to w) = \frac{\theta}{2}\left[1 - \tanh\left(\zeta xv - \zeta wx\right)\right] \tag{4}$$

Function $\theta(y \to z)$ describes the transition probability from optimistic exuberance to optimistic non-exuberance, and function $\theta(v \to w)$ corresponds to the transition probability from pessimistic exuberance to pessimistic non-exuberance. The functions are identical in the sense that in both cases the probability of transition changes according to the trend of the respective sentiment; two boundary values

[6] As in Angeletos and La'O (2013), we attribute a pivotal role to those in a state of exuberance. Neutral and non-exuberant individuals do not search actively for new information and are influenced solely by local interaction. Exuberant people, instead, are pro-active in the sense that besides being influenced by direct contact, they also search for economy wide information on the acceptance about the sentiment they hold (measured by the net inflow of people into the sentiment category). In this case, exuberance is synonymous with a proclivity to make an informed decision.

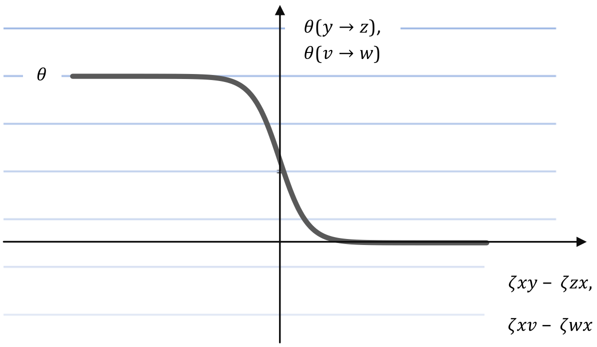

Fig. 2 Exuberance – Non-exuberance transition probability functions

exist: θ is the probability of transition that corresponds to a strong negative change in the sentiment share (this continues to be the transition probability from neutrality to exuberance and from non-exuberance to neutrality, regardless of the overall sentiment changes), while a strong positive change in the corresponding sentiment causes the probability of transition to approach zero. This captures the idea that an exuberant individual who observes an increasing adherence to her sentiment type will not be convinced to shift to the respective non-exuberance state. When sentiments of optimism and pessimism do not vary, the transition probabilities are half of θ: $\theta(y \to z) = \theta(v \to w) = \frac{\theta}{2}$.[7]

Figure 2 shows how the transition probability from exuberance to non-exuberance departs from θ when the flow of agents away from the sentiment (optimism or pessimism) is not large relative to the respective inflow.

The introduced assumption transforms system (2) into a new system of ODEs,

$$\begin{cases} \dot{y} = \zeta xy - \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x(y-z)\right]\right\} y(y+z) \\ \dot{v} = \zeta xv - \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x(v-w)\right]\right\} v(v+w) \\ \dot{z} = \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x(y-z)\right]\right\} y(y+z) - \zeta zx \\ \dot{w} = \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x(v-w)\right]\right\} v(v+w) - \zeta wx \\ y(0), z(0), v(0), w(0) \text{ given; } x = 1 - (y+z+v+w) \end{cases} \tag{5}$$

Section 4 gives an analysis of the global dynamics underlying the above characterized ODE system, revealing the existence of long-term out-of-equilibrium dynamic behavior. For now, we repeat the stability analysis under the new assumption.

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[7]The specific shape of functions (3) and (4) is intended to capture the idea that the transition probability approaches zero when the inflow into the sentiment is stronger than the outflow and approaches $\theta > 0$ when the respective net outflow is a positive value. This is done rather than considering a piecewise function that would be analytically less tractable and arguable less realistic; the continuity is assured by the specific hyperbolic tangent function. This function introduces a convenient S-shaped nonlinearity similar to the one used in some evolutionary switching models where specific relations are defined through the arctangent function (see, e.g., the supply curve (1.2) in Hommes (2013)).

Four equilibrium points and an equilibrium line continue to exist in the new version of the model. However, the changes introduced in the transition probabilities between exuberance and non-exuberance imply a different set of equilibrium points. Equilibria e_4 and e_5 are the same as before, but changes occur for e_1, e_2 and e_3.

Proposition 3 *For system (5), four equilibrium points plus an equilibrium line exist, respectively,*

$$\begin{aligned}
e_1' &: (y^*, v^*, z^*, w^*) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \\
e_2' &: (y^*, v^*, z^*, w^*) = \left(\frac{1}{3}, 0, \frac{1}{3}, 0\right) \\
e_3' &: (y^*, v^*, z^*, w^*) = \left(0, \frac{1}{3}, 0, \frac{1}{3}\right) \\
e_4 &: (y^*, v^*, z^*, w^*) = (0, 0, 0, 0) \\
e_5 &: (y^*, v^*, z^*, w^*) = \left(0, 0, z^*, w^*\right), \textit{ with } z^* + w^* = 1.
\end{aligned}$$

Proof See Appendix. □

Note that by assuming that the transition probability from exuberance to non-exuberance might fall below θ unless the net outflow of agents from the respective category is relatively strong, the first three equilibrium points suffer changes. The first equilibrium is now a point in which agents are equally distributed across the five sentiment categories; the other two points represent equilibrium loci such that excluding one of the sentiments, agents are equally distributed across neutrality, exuberance, and non-exuberance.

Next, we reassess the stability properties of the system with the introduced changes.

Proposition 4 *In the sentiment propagation model with dynamics given by Eq. 5, the equilibria e_2', e_3', e_4 and e_5 are unstable $\forall \zeta > 0$. The equilibrium point e_1' is locally stable under condition $\zeta < \frac{25}{2}$.*

Proof See Appendix. □

The local stability dynamics in the new scenario, with exuberant having different transition probabilities for different global sentiment status, has some similarities with the original framework. In particular, every corner solution for which at least one of the exuberance categories is unpopulated, is unstable. Thus, as before, we can focus the discussion on the remaining equilibrium by imposing the initial condition $y(0) > 0 \wedge v(0) > 0$. For e_1', the universal stability result found for e_1 no longer holds; a supercritical Hopf bifurcation occurs at $\zeta = \frac{25}{2}$, and this bifurcation separates the region of stability for relatively low values of ζ from a region where a stable limit cycle is observed. It is in this region that waves of optimism and pessimism are modeled.

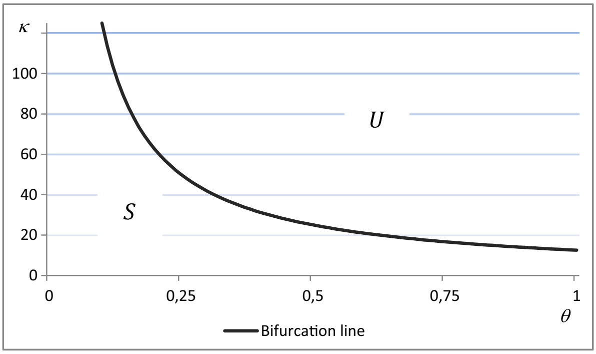

Fig. 3 Stability (S) and instability (U) regions in the sentiment model with the hyperbolic tangent transition probability function

Recall that parameter ζ is the product of the transition probability and the connectivity degree. Hence, another way of presenting the stability condition is through the inequality $\kappa < \frac{25}{2}\frac{1}{\theta}$. This condition is satisfied whenever the degree of connectivity and the probability of transition across sentiment states both have relatively low values. The regions of stability and instability, and the bifurcation line that separates them, are shown in Fig. 3.

Before proceeding with the analysis of the global dynamics underlying the sentiment spreading system that will uncover the formation of endogenous cycles, Section 3 establishes the bridge between the sentiments setup, as described above, and the optimal decisions made by the agents when acting as (boundedly) rational economic players.

3 The economic environment

In this section, two processes are discussed. First, a rationale is proposed for the network homogeneity assumption that underlies the mechanism of sentiment propagation described in the previous section; the degree of connectivity is interpreted as a control variable of a simple intertemporal time allocation planning problem that each individual agent is supposed to solve. In a second part of the section, a straightforward apparatus characterizing the impact of sentiment changes over the economy is presented; on the aggregate, the economy may over-invest (if sentiments of optimism dominate) or under-invest (if sentiments of pessimism prevail), with over-investment and under-investment both implying a departure from the optimal profit condition. As illustrated in the next section, if the interaction framework gives rise to endogenous fluctuations in sentiments, then these will propagate to investment, output, and profits.

The two economic mechanisms described in this section might be integrated under the following interpretation: The first setup, in which a multiplicity of agents solves the same optimal time allocation problem, is the problem of the labor suppliers in an economy who decide how to allocate their effort between participation in the productive activity and interaction in search of new ideas and knowledge that enhance productivity. In a second stage, the labor force will then contribute to a productive activity managed by a representative investor whose decisions are influenced by the overall sentiment level.

We begin the analysis by considering a network with a maximum connectivity degree $\widehat{K} > 0$; this is the degree of connectivity that prevails whenever the market is complete, and hence every agent is in direct contact with everyone else, i.e., $\kappa = \widehat{K}$. In what follows, our argument is that establishing contact with others, although desirable from a knowledge enrichment point of view, requires effort and thus involves economic costs measured in terms of time diverted from productive activities. Consequently, agents will eventually be compelled to choose an optimal degree of interaction lower than $\widehat{K}$. Because agents all face the same time allocation problem, they will all want to set an identical connectivity degree in the steady-state, independent of initial conditions, which justifies the homogeneity across individual agents regarding desired connectivity.

In this scenario, the connectivity degree is the outcome of a decision on the optimal share of time allocated to social interaction with the objective of accumulating additional knowledge. However, this deliberation process has a side effect: the selected connectivity degree will also determine the pace of sentiment spreading, as characterized in Section 2. Sentiment spreading emerges as a by-product of the purposive effort of every agent in searching for new productivity-enhancing knowledge that originates in the process of decentralized contact and communication.

In assembling a framework where workers might allocate time between production and interaction, we essentially follow Lucas and Moll (2014). Here, as in the Lucas-Moll growth model, there is an opportunity cost involving knowledge as an input used to produce goods: time allocated to production is time not spent in fostering contact with others in search for new ideas capable of stimulating productivity. Higher productivity, in turn, allows for the generation of additional output per unit of time. Meetings occur randomly and in a number that is directly related to the fraction of time dedicated to the communication process.

The trade-off between time spent working and time spent searching for ideas that others already hold implies conceiving knowledge as a partially rival input; rival in the short term, because a search resource-consuming effort is required to attain it; non-rival in the long run because it ends up by propagating, sooner or later, throughout the whole of the economy. Knowledge will eventually become disembodied, in the sense of being freely and immediately accessible to all, but only in the long term.

How relevant is the process of acquiring knowledge via systematic social contact as a source of productivity growth? A growing number of economists classify such processes as vital. In Lucas (2009, page 1), an eloquent argument in favour of such a perspective is offered,

> It is widely agreed that the productivity growth of the industrialized economies is mainly an ongoing intellectual achievement, a sustained flow of ideas. Are these ideas the achievements of a few geniuses, Newton, Beethoven and a handful of others, viewed as external to the activities of ordinary people? Are they

> the product of a specialized research sector, engaged in the invention of patent-protected processes over which they have monopoly rights? Both images are based on important features of reality and both have inspired interesting growth theories, but neither seems to me central. What is central, I believe, is the fact that the industrial revolution involved the emergence (or rapid expansion) of a class of educated people, thousands – now many millions – of people who spend entire careers exchanging ideas, solving work-related problems, generating new knowledge.

Besides Lucas (2009), other authors (Staley, 2011; Lafond, 2015) also present compelling arguments to justify the preponderance of social interaction in the diffusion of innovative thinking and in the promotion of economic growth. These studies are inspired in a previous essay by Manski (2000) where it is emphatically argued that the transmission of knowledge occurs many times beyond the boundaries of market relations, entering the domains of a wider sphere of human contact that takes place in the context of different kinds of communities and in the society as a whole.

To encounter the optimal share of time allocated to interaction, we solve a minimal intertemporal optimization problem, with a single state variable (productivity) and a single control variable, the time allocated to contact with others. Each individual agent will select the value of κ over time by solving the following infinite horizon optimal control planning problem:

$$\max_{s(t)} \int_0^{+\infty} U\left[(1-s)A\right] \exp(-\rho t) dt$$

$$\begin{aligned} \text{subject to} \quad &: \\ \dot{A} &= g\frac{\kappa}{\widehat{K}}A \\ \kappa &= f(s) \\ A(0) &= A_0 \text{ given} \end{aligned} \tag{6}$$

In Eq. 6, $A \geq 0$ represents the agent's productivity level, and $1 - s \in (0, 1)$ is the share of time allocated by the agent to production at time t.

Besides production, time can also be allocated in the remaining share s to interaction. Through successful interaction, the agent can increase its connectivity to others. The relation between time spent with interaction and interaction outcome is provided by the continuous and differentiable function $f : [0, 1] \to [0, \widehat{K}]$, with $f' > 0$, $f(0) = 0$ and $f(1) = \widehat{K}$. The conditions that constrain the shape of the function convey the following information: connectivity increases with the time allocated to social interaction activities; if no time is allocated to interaction, the agent will remain in complete isolation; in the opposite case where all the available time is dedicated to interaction, the contacts established by each agent will comprise the whole network.

An additional condition on f in the proposed setting concerns the sign of its second derivative; we assume $f'' > 0$: increasing returns of time allocation for successful interaction are taken in order to capture the intuitive idea that the more extensive the network of contacts the agent already has, the easier it will be to undertake successful new contacts. A functional form that satisfies these properties and that facilitates analytical treatment of the problem is

$$f(s) = \widehat{K}\left[1 - (1 - s)^{1/\tau}\right], \ \ \tau > 1 \tag{7}$$

The dynamic constraint in Eq. 6 on the motion of A reflects the simple notion that productivity grows at a rate that depends on interaction: stronger interaction signifies a wider dissemination of ideas, information, and knowledge, which are the main drivers of individual productivity. As displayed, the state constraint relating productivity evolution is such that in the limiting case $\kappa = \widehat{K}$, the productivity changes at the maximum possible rate $g > 0$. Problem (6) can be interpreted as an analytical structure of endogenous growth since, ultimately, the pace of growth will be endogenously chosen in order to attain the agent's goal. Although a growth ceiling exists, given by rate g, the production-interaction trade-off underlying the optimal control decision will conduct to an intermediate solution where part of the available time goes to production and the other share in trying to approximate the growth rate of productivity to the benchmark level g.

The objective function of the time allocation problem, $U(\cdot) : \mathbb{R}^+ \to \mathbb{R}$, is designed to reflect the utility the agent draws from the generated income. Household income is simply defined as the product of the productivity index and the time allocated to production. The objective function is assumed to be continuous and differentiable, and to exhibit positive and diminishing utility, i.e., $U' > 0$ and $U'' < 0$. The simplest specification that obeys these conditions, and that will be used in the analysis, is a logarithmic function,

$$U\left[(1 - s)A\right] = \ln\left[(1 - s)A\right] \tag{8}$$

Parameter $\rho \geq 0$ is the intertemporal discount rate of future utility.

Once presented with all its components and variables, one might synthesize the agents' decision problem in the following way: agents, who are all alike in terms of the economic choices they face and thus solve an identical problem, wish to maximize, under an infinite horizon, the utility of their income in a setting where the state variable, for which the respective rule of motion is known, is the productivity level, and the control variable is time allocation, or, since there is a direct relation between the two, the degree of social connectivity.

By applying Pontryagin's principle and taking into consideration functions (7) and (8), it is straightforward to solve (6). The current value Hamiltonian function is

$$H(A, \kappa, q) = \ln\left[\left(1 - \frac{\kappa}{\widehat{K}}\right)^{\tau} A\right] + g\frac{\kappa}{\widehat{K}}qA \tag{9}$$

with q representing the co-state variable or shadow-price of the productivity level A. First-order optimality conditions are

$$\frac{\partial H}{\partial \kappa} = 0 \Rightarrow \widehat{K} - \kappa = \frac{\tau\widehat{K}}{gqA} \tag{10}$$

$$\dot{q} = \rho q - \frac{\partial H}{\partial A} \Rightarrow \dot{q} = \left(\rho - g\frac{\kappa}{\widehat{K}}\right)q - \frac{1}{A} \tag{11}$$

and the transversality condition is

$$\lim_{t \to +\infty} A \exp(-\rho t) q = 0 \tag{12}$$

Differentiation of Eq. 10 with respect to time implies that

$$\frac{\dot{\kappa}}{\widehat{K} - \kappa} = \frac{\dot{q}}{q} + \frac{\dot{A}}{A} \tag{13}$$

which, given Eq. 11 and the productivity constraint, is equivalent to

$$\dot{\kappa} = \left(\widehat{K} - \kappa\right) \left[\rho - \frac{g}{\tau \widehat{K}} \left(\widehat{K} - \kappa\right)\right] \tag{14}$$

Equation 14 is a one-dimensional ODE with a single endogenous variable, namely the degree of connectivity. Note that this equation has two equilibrium points; solutions $\kappa^* = \widehat{K}$ and $\kappa^* = \widehat{K}\left(1 - \frac{\rho\tau}{g}\right)$ both of which satisfy $\dot{\kappa} = 0$. The first solution is senseless from an economic viewpoint since it would mean that every agent would be in contact with everyone else in the economy but at an unbearable cost. For the connectivity degree to be equal to its maximum value, all the time must be allocated to interaction, $s^* = 1$, which would imply that no time would be left for production, and thus the long-term steady-state level of utility of the agent would be zero.

The second steady-state is admissible but imposes a constraint on the parameters: $g > \rho\tau$. A lower bound on the growth potential of productivity is required to obtain a feasible solution. Note that this equilibrium point is unstable, $\left.\frac{d\dot{\kappa}(t)}{d\kappa(t)}\right|_{\kappa^* = \widehat{K}\left(1 - \frac{\rho\tau}{g}\right)} = \rho > 0$. Given that κ is a control variable, the instability result suggests that the representative agent will choose to locate at point $\kappa^* = \widehat{K}\left(1 - \frac{\rho\tau}{g}\right)$, $\forall t$. Thus, this value will effectively be the connectivity degree optimally chosen by all the agents in the economy at every instant. The non-completeness of markets is not the outcome of any kind of informational deficiency or other market failure; markets are not complete because agents rationally select an optimal degree of connectivity, lower than $\widehat{K}$, given the trade-off between interaction and participation in the productive activity. Note that for $\kappa = \widehat{K}\left(1 - \frac{\rho\tau}{g}\right)$, productivity grows at a rate $\frac{\dot{A}}{A} = g - \rho\tau$, a positive rate under the imposed condition $g > \rho\tau$, time allocated to the interaction is $s = 1 - \left(\frac{\rho\tau}{g}\right)^{\tau}$, and time allocated to production is $1 - s = \left(\frac{\rho\tau}{g}\right)^{\tau}$.

Consider a simple example. Take $\widehat{K} = 1000$, $\rho = 0.048$, $\tau = 1.25$ and $g = 0.1$. The connectivity degree will be $\kappa = 400$, i.e., each agent will optimally choose to connect with 40% of the whole social network. In this case, the rate of productivity growth is $\frac{\dot{A}}{A} = 0.04$, and time will be allocated in the following way: $s = 0.4719$ and $1 - s = 0.5281$, i.e., the time spent in interaction activities is a smaller fraction than the time allocated to production, under the assumed parameterization.

The meaningful point that we once again stress is that engaging in interaction has a well defined purpose, which is to enhance productivity, but it has a by-product: as agents interact to share ideas, information, and knowledge and, as a consequence, raise their ability to increase the value of what they produce, they also share their sentiments about the future performance of the economy, which will put them at each instant in one of the sentiment categories of the previous section. How fast sentiments spread in this setting is then an indirect consequence of the effort agents make to share ideas for promoting productivity growth.

A pertinent question is whether sentiments can somehow enter the optimal decision of the agent. The answer in the proposed setting is no for two reasons. First, sentiment switching, as characterized in Section 2, is a completely state-driven process; no agent has the ability to choose the respective sentiment state. Second, if agents wanted to control the rate of sentiment change, for instance, avoiding all variation and thus suppressing sentiments as a source of fluctuations from the economy, they could set κ to zero, but this would not be a rational decision because zero fluctuations would come at an unbearable cost: no productivity growth would occur. Therefore, in the proposed scenario, agents solve a long-term growth problem that, nonetheless, by shaping how these agents interact, impacts the short-run aggregate behavior, generating more or less intense waves of optimism and pessimism.

In the second part of this section, we discuss how sentiments, which evolve under the social contact process characterized in Section 2, eventually spread into the economy. The presented framework is intentionally plain and simple, since it will show that any imbalance between optimism and pessimism generates an economic outcome that is less favorable than what occurs if the mass of agents is neutral on average.

We assume a representative entrepreneur who wishes to maximize her profits. [8] Profits are defined as the difference between output revenues and costs, both investment and labor costs. The output of the entrepreneur is given by a Cobb-Douglas function with Harrod-neutral technological progress,

$$F(K, AL) = K^{\alpha}\left[\left(\frac{\rho\tau}{g}\right)^{\tau} AL\right]^{1-\alpha} \tag{15}$$

In the production function (15), $K \geq 0$ represents the stock of capital, $L \geq 0$ is a fixed amount of labor that corresponds to the workforce in the economy that the entrepreneur can hire, and $\left(\frac{\rho\tau}{g}\right)^{\tau} A$ is the contribution of each worker to the productive process, a value that corresponds to the output of the agents' time allocation problem addressed above. The parameter $\alpha \in (0, 1)$ is the output-capital elasticity.

The representative firm faces two types of costs: (i) labor costs, which correspond to the wage (marginal productivity of labor) times the amount of labor,

$$wL = \frac{\partial F}{\partial L} L = (1-\alpha) F(K, AL) \tag{16}$$

[8] The assumption of a profit maximizing firm is not, as we shall see below, essential for the analysis, since the only requirement is to consider some benchmark value for investment over which waves of optimism or pessimism might exert some effect. Nevertheless, by solving this problem, expressions for the investment, profits and output will be explicitly derived, and these will be important for the simulation exercise to be performed in the next section.

and (*ii*) investment costs, which are the rental price for a unit of capital, $R > 0$, times the amount of acquired capital stock,

$$I = RK \tag{17}$$

Then the profit function is

$$\Pi = F(K, AL) - wL - I = \alpha F(K, AL) - I \tag{18}$$

The entrepreneur will want to maximize profits for a given level of investment. Straightforward maximization of Eq. 18 yields

$$\frac{\partial \Pi}{\partial I} = 0 \Rightarrow I = \alpha^{2/(1-\alpha)} \left(\frac{1}{R}\right)^{\alpha/(1-\alpha)} \left(\frac{\rho\tau}{g}\right)^{\tau} AL \tag{19}$$

Condition (19) furnishes the optimal level of investment; this investment level will grow over time at the same rate as productivity, i.e., $\frac{\dot{I}}{I} = \frac{\dot{A}}{A} = g - \rho\tau$. To write an expression for optimal profits, one substitutes investment, as displayed in Eq. 19, into Eq. 18, giving the result

$$\Pi = (1-\alpha)\alpha^{(1+\alpha)/(1-\alpha)} \left(\frac{1}{R}\right)^{\alpha/(1-\alpha)} \left(\frac{\rho\tau}{g}\right)^{\tau} AL \tag{20}$$

Profits, as investment, grow over time at the same rate as labor productivity. This observation also applies to the entrepreneur's output; the production level that satisfies the optimal profits condition is

$$F(.) = \alpha^{2\alpha/(1-\alpha)} \left(\frac{1}{R}\right)^{\alpha/(1-\alpha)} \left(\frac{\rho\tau}{g}\right)^{\tau} AL \tag{21}$$

Having presented a benchmark for the analysis of optimal production decisions, we now add to this framework the impact of sentiments by taking the following function,

$$a(y, v, z, w) = \frac{1 + (\overline{a} - 1)(y + z)}{1 + (\overline{a} - 1)(v + w)}, \ \overline{a} > 1 \tag{22}$$

Function (22) is designed to evaluate sentiment imbalances; whenever $y + z = v + w$, $a(\cdot) = 1$; the value of $a(\cdot)$ increases above 1 when the number of optimists exceeds the number of pessimists and falls below 1 in the opposite case. Figure 4 illustrates the intuition behind Eq. 22; three lines are presented in the referential $(y + z; v + w)$. The line that separates the quadrant in two equal parts represents case $a(.) = 1$; any line to the left of $a(.) = 1$ is such that $a(.) < 1$ (pessimism is dominant); any line to the right of $a(.) = 1$ is such that $a(.) > 1$ (optimism is dominant).

Next, we make investment depend on the sentiments of the population. The intuition is that sentiments exert influence on the confidence of consumers, thus distorting demand decisions. The representative firm reacts to the distortion on aggregate demand by raising investment above the optimal level when optimism dominates and by lowering investment below the optimal level when optimism is overpowered by pessimism. Let $\widetilde{I}$ be the effective investment level, which in our specification will relate to optimal investment by

$$\widetilde{I}(t) = a(\cdot)I(t) \tag{23}$$

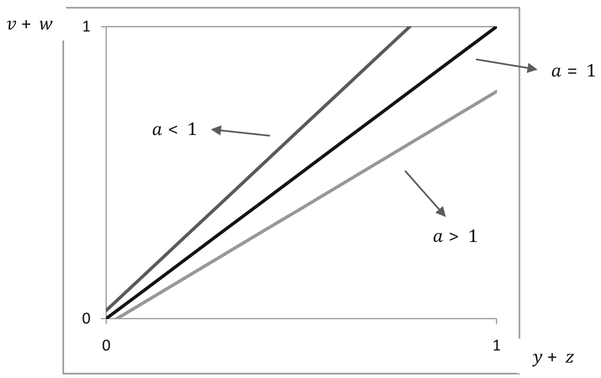

Fig. 4 Function $a(y, v, z, w)$

In a sense, $I(t)$ reflects fundamentals while $a(\cdot)$ is a distorting systematic shock that has its origins in the constantly changing dominant aggregate sentiment.

According to condition (23), effective investment will coincide with optimal investment only when the numbers of optimists and pessimists in the economy are balanced; departures from this coincidence occur for $y + z > v + w$, the case in which $\widetilde{I} > I$, and for $y + z < v + w$, the case in which $\widetilde{I} < I$. Therefore, under the proposed assumption, when optimism is dominant, the entrepreneur will over-invest; in the opposite circumstance, she will under-invest relative to the benchmark optimal investment level. Substituting the new notion of investment into the profit expression gives

$$\widetilde{\Pi} = \alpha^{(1+\alpha)/(1-\alpha)} a(\cdot)^{\alpha} \left[1 - \alpha a(\cdot)^{1-\alpha}\right] \left(\frac{1}{R}\right)^{\alpha/(1-\alpha)} \left(\frac{\rho\tau}{g}\right)^{\tau} AL \tag{24}$$

Any value of $a(\cdot)$ lower or higher than unity will make profits fall below the optimal level, (20). As the next section will reveal, for reasonable parameter values, the local interaction framework will trigger endogenous fluctuations that represent episodes of departure from optimal profits, with extreme periods of prevailing optimism or pessimism implying stronger reductions in profit. Output, as given by

$$\widetilde{F}(.) = a(\cdot)^{\alpha} \alpha^{2\alpha/(1-\alpha)} \left(\frac{1}{R}\right)^{\alpha/(1-\alpha)} \left(\frac{\rho\tau}{g}\right)^{\tau} AL \tag{25}$$

will also display fluctuations, simulating the persistence of business cycles.

4 Long-term dynamics and chaos

The global dynamics of the sentiment interaction model outlined in Section 2, namely the dynamics concerning system (5), is now examined with regard to the single bifurcation parameter ζ. Figure 5 shows the four Lyapunov exponents, the Kaplan–Yorke dimension, and the local maxima of $y(t)$ as a function of ζ over the range $0 < \zeta < 200$. For calculation of the Lyapunov exponents, time has been scaled according to $1/\zeta$ to avoid a linear dependence of the numerical values on ζ. As indicated in Section 2, the coexisting equilibrium point e_1' at $y^* = v^* = z^* = w^* = \frac{1}{5}$ is stable for $\zeta < \frac{25}{2}$ with all four eigenvalues equal to $-0.1 + \zeta/125$, which agree with the four Lyapunov exponents in this range.

At $\zeta = \frac{25}{2}$, a supercritical Hopf bifurcation occurs and spawns a stable limit cycle with a dominant angular frequency of $\sqrt{19}/10$. The limit cycle grows rapidly in size and undergoes a period-doubling route to chaos, which onsets around $\zeta = 100$ and persists to arbitrarily large values of ζ except for a number of small windows of periodicity. At $\zeta = 150$, the chaos is fully developed with Lyapunov exponents

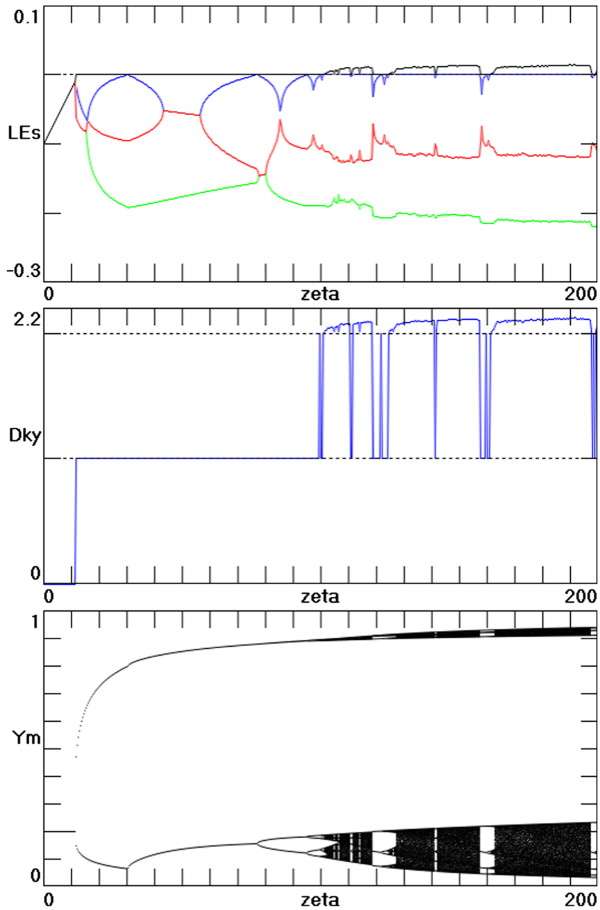

Fig. 5 Lyapunov exponents, Kaplan-Yorke dimension and local maxima of $y(t)$

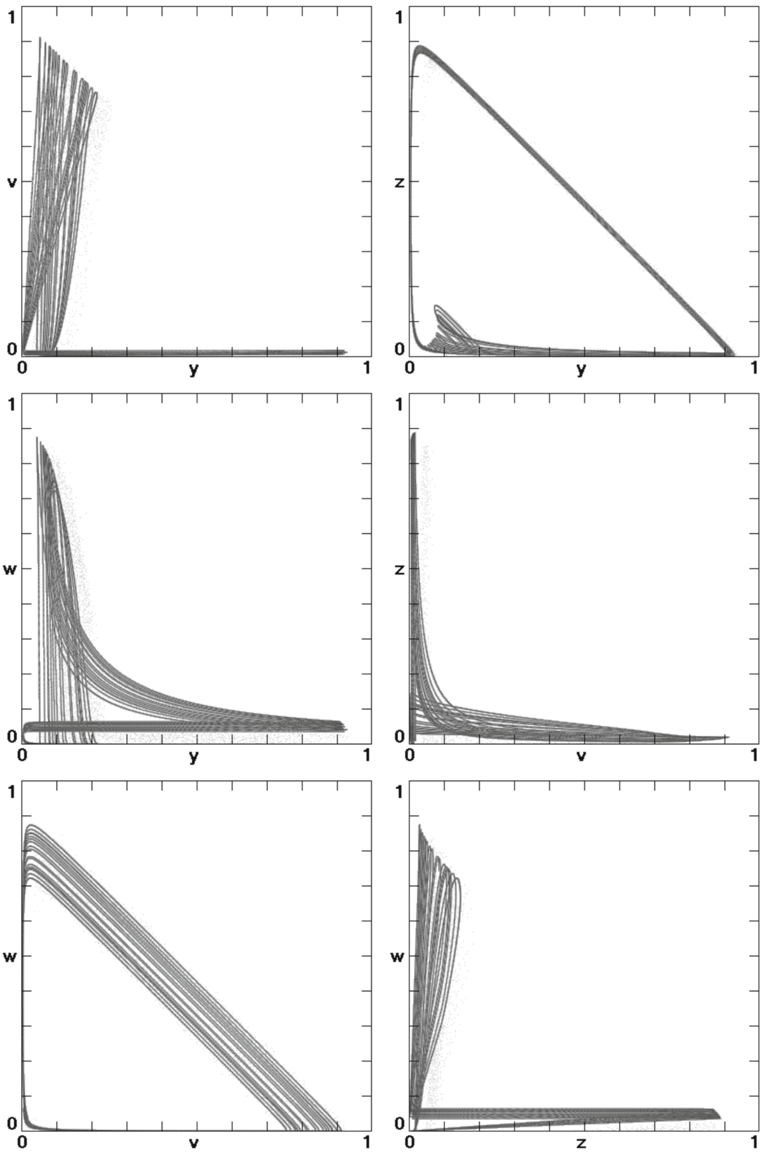

Fig. 6 Strange attractors ($\zeta = 150$)

of (0.0133, 0, −0.1196, −0.2038), and the strange attractor shown in Fig. 6 has a Kaplan–Yorke dimension of 2.1109.[9]

[9]Given that the system is four-dimensional, it involves four Lyapunov characteristic exponents (LCE). The identification of chaos, based on the evidence of sensitive dependence on initial conditions, requires the existence of at least one positive LCE, as is observed in this particular case.

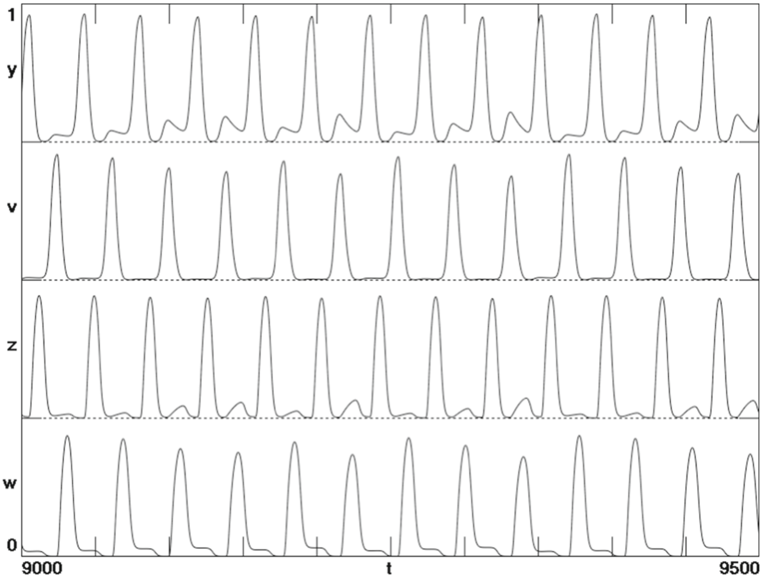

Fig. 7 Sentiment time trajectories ($\zeta = 150$)

There is no evidence of hysteresis or multistability (coexisting attractors). The strange attractor is self-excited rather than hidden in the sense of Leonov et al. (2011), and it has a Class 3 basin of attraction as described by Sprott and Xiong (2015). No evidence of hyperchaos (two positive Lyapunov exponents) was found. This system is relatively stiff at large values of ζ since the argument of the hyperbolic tangent function is large, necessitating an integrator with an adaptive step size.

In Fig. 7, the occurrence of endogenous fluctuations in every sentiment category is highlighted, given the chosen parameter value ($\zeta = 150$); the respective time series are presented after a time interval for which the respective transients have vanished. Alternate periods of low sentiment intensity and peaks of optimism and pessimism are evident.

As emphasized in Section 3, the irregular cycles of animal spirits can be transposed in a straightforward way to the economic aggregates, namely the entrepreneur's investment and profits and the economy's output, which, under the influence of sentiments are given, respectively, by Eqs. 23, 24, and 25. Figure 8 shows long-term trajectories for the variables of effective investment, profits, and output, with each of these variables presented as a ratio with respect to productivity.[10] The trajectories are drawn after selecting specific values for the relevant parameters. Namely,

[10]Note that productivity grows over time at a constant rate $g - \rho\tau$ (productivity growth is not affected by sentiment oscillations). Effective investment, profits and output, in turn, suffer the influence of sentiment changes, and hence they will grow at a rate that fluctuates around the productivity growth rate (which is also the growth rate of the optimal levels of investment, profits and output). Therefore, a straightforward way of displaying graphically the oscillations underlying each of the variables is presenting them as ratios with respect to productivity. In this way, the values of the variables will fluctuate around a constant level.

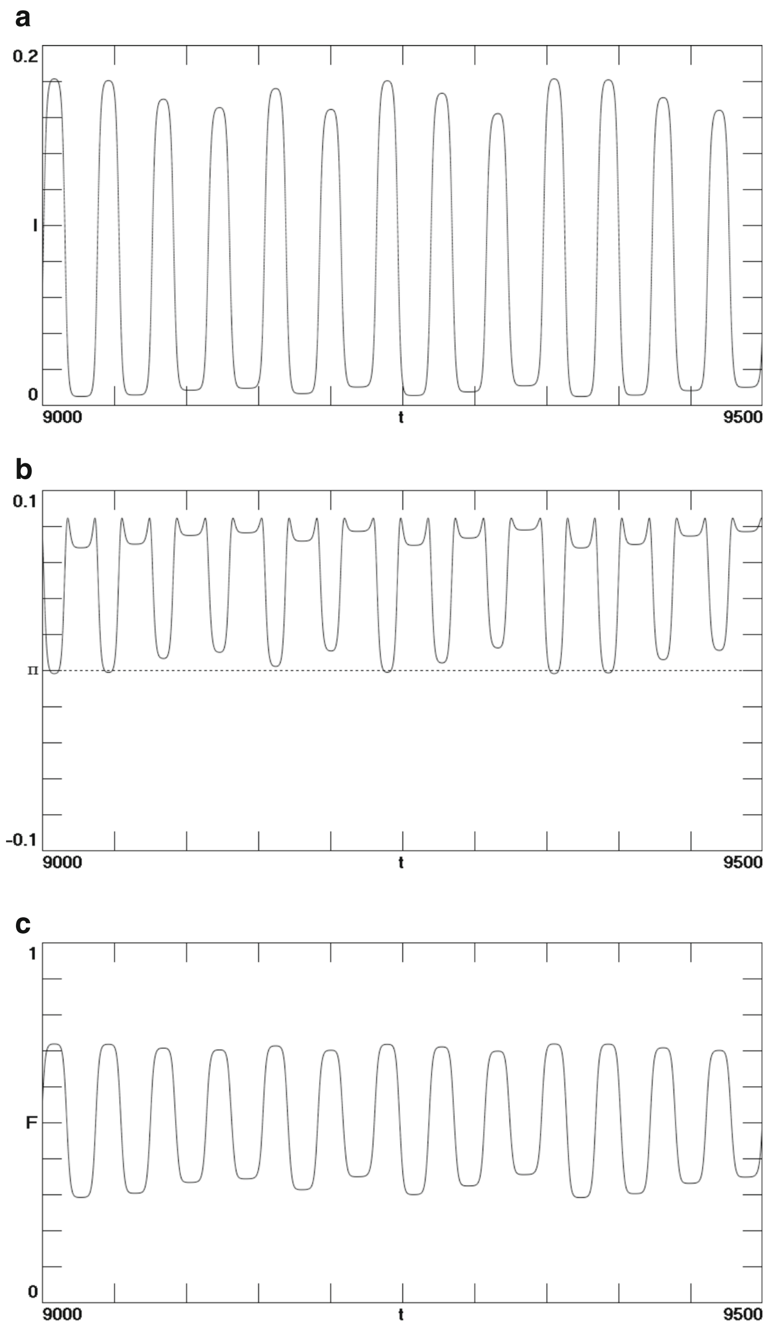

Fig. 8 **a** Long-term investment-productivity ratio under sentiment waves. **b** Long-term profits-productivity ratio under sentiment waves. **c** Long-term output-productivity ratio under sentiment waves

in the example we consider $\widehat{K} = 1000$, $\rho = 0.048$, $\tau = 1.25$, $g = 0.1$ (as in Section 3, which implies that $\kappa = 400$); in order to maintain $\zeta = 150$, we also let $\theta = 0.375$. Furthermore, for production conditions, we assume $\alpha = 0.25$, $R = 0.1$, and $L = 1$. Finally, the parameter in the sentiment function (22) is taken as $\overline{a} = 10$.[11] For the selected values, the sentiment-free levels of investment, profits and output, per productivity unit, are

$$\frac{I}{A} = 0.0282; \frac{\Pi}{A} = 0.0847; \frac{F}{A} = 0.4515$$

and, by adding sentiments in the proposed form, these constant values give rise to the trajectories exhibiting endogenous fluctuations in the figure. As discussed above, note that investment and output will fluctuate about their sentiment-free levels, and profits will remain below the optimal level, with periods of stronger sentiment imbalances leading to lower profits.

Observe that in the case of full optimism ($y + z = 1$), $\widetilde{\frac{I}{A}} = 0.2822$ and $\widetilde{\frac{F}{A}} = 0.8029$, while in the case of full pessimism ($v + w = 1$), $\frac{I}{A} = 0.0028$ and $\frac{F}{A} = 0.2539$. The presented values are the virtual boundaries for the fluctuations displayed in Fig. 8, panels a and b. Relative to profits, the highest potential value is the one implied by $y + z = v + w$; if $y + z = 1$, then $\widetilde{\frac{\Pi}{A}} = -0.0815$, and if $v + w = 1$, then 0.0606. The extreme scenarios indicate that excessive optimism is, in this case, more harmful than excessive pessimism, leading to an eventual result of negative profits.

Finally, observe that a change in the value of the parameter of the sentiment function, $\overline{a}$, will not disturb the sentiment-free levels of investment, profits and output; however, the change will have impact on the amplitude of the fluctuations; namely, whenever $\overline{a} > 10$, the cycles will be more pronounced, and the opposite when $\overline{a} < 10$, with relation to the benchmark scenario. Take, for instance, $\overline{a} = 12$ and $\overline{a} = 7$. In the first case, if $y + z = 1$, then $\widetilde{\frac{I}{A}} = 0.3386$; and if $v + w = 1$, then $\widetilde{\frac{I}{A}} = 0.0023$. For $\overline{a} = 7$, if $y + z = 1$, then $\frac{I}{A} = 0.1975$; and if $v + w = 1$, then $\frac{I}{A} = 0.0040$. It is notorious that fluctuations are more pronounced for larger values of the parameter. A similar outcome would result from analyzing profits and output oscillations. Hence one might interpret this parameter as a measure of the impact of consumer sentiments on the intensity of business fluctuations.

5 Conclusion

Recent macroeconomic literature has revived sentiments as a driver of aggregate fluctuations by suggesting that the economy is constantly suffering confidence shocks. However, this literature is basically silent about the sources of such shocks. This

[11] Below, we consider alternate values for this parameter.

paper devised a local interaction framework with the goal of explaining how sentiments spread and how this dissemination might lead to waves of optimism and pessimism.

Agents in the economy were separated into five categories, and they interact locally and change from one category to another. The specific formulation of the interaction process leads to the possibility of limit cycles and aperiodic chaotic cycles, suggesting that regular and irregular sentiment waves may be endogenously triggered as a result of the way individual agents establish contact with one another. The single condition that is necessary to ignite the process is that there must be at least one exuberant individual in each of the sentiment states, optimism and pessimism. Exuberance is the seed that triggers everlasting sentiment oscillations. Furthermore, the requisites for persistent sentiment waves are far from being demanding. All that is required is for the probability of transition across sentiment states and the connectivity degree not be unreasonably low.

Although sentiments are modeled as a purely social contact process, two links were established between the sentiment framework and economic decisions. First, connectivity among agents emerges as a direct consequence of the need to enhance productivity through the exchange of ideas and knowledge. As agents establish contact, they will end up sharing sentiments as well, and thus the degree of connectivity that is determined in the context of the optimization of time allocation between interaction and participation in the productive activity is the same degree of connectivity that will determine the social interaction that drives potential sentiment changes. Second, pure rationality is abandoned once we introduce the concepts of over-investment and under-investment relative to a benchmark optimal investment level. When optimism dominates, the investment will exceed optimal levels, and when pessimism dominates, the investment will fall below optimal. In both cases, the net income generated by the assumed representative firm will be lower than the one obtained when sentiments of optimism and pessimism are balanced.

Business cycles are illustrated in this setting when merging the framework concerning investment decisions with the sentiment propagation apparatus, where the latter involves nonlinear dynamics and thus allows for endogenous waves of optimism and pessimism.

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Appendix

Proof of Proposition 1 Applying the equilibrium condition $\dot{y} = \dot{v} = \dot{z} = \dot{w} = 0$ to system (2) gives

$$\begin{cases} \left[1 - (y^* + z^* + v^* + w^*)\right] y^* = y^*(y^* + z^*) \\ \left[1 - (y^* + z^* + v^* + w^*)\right] v^* = v^*(v^* + w^*) \\ y^*(y^* + z^*) = z^* \left[1 - (y^* + z^* + v^* + w^*)\right] \\ v^*(v^* + w^*) = w^* \left[1 - (y^* + z^* + v^* + w^*)\right] \end{cases}$$

The solutions of the above system are the four equilibrium points and the equilibrium line in the proposition. □

Proof of Proposition 2 The Jacobian matrix for system (2) has the general form

$$J = \zeta \begin{bmatrix} 1-4y^*-2z^*-v^*-w^* & -y^* & -2y^* & -y^* \\ -v^* & 1-y^*-z^*-4v^*-2w^* & -v^* & -2v^* \\ 2y^*+2z^* & z^* & -(1-2y^*-2z^*-v^*-w^*) & z^* \\ w^* & 2v^*+2w^* & w^* & -(1-y^*-z^*-2v^*-2w^*) \end{bmatrix}$$

For each of the equilibria,

$$e_1 : J = \frac{\zeta}{6}\begin{bmatrix} -2 & -1 & -2 & -1 \\ -1 & -2 & -1 & -2 \\ 4 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$e_2 : J = \frac{\zeta}{4}\begin{bmatrix} -2 & -1 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ 4 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$e_3 : J = \frac{\zeta}{4}\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & -2 & -1 & -2 \\ 0 & 0 & -2 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$e_4 : J = \zeta\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$e_5 : J = \zeta\begin{bmatrix} -z^* & 0 & 0 & 0 \\ 0 & -(1-z^*) & 0 & 0 \\ 2z^* & z^* & z^* & z^* \\ 1-z^* & 2(1-z^*) & 1-z^* & 1-z^* \end{bmatrix}$$

From the Jacobian matrices, the respective eigenvalues are

$$\begin{gathered} e_1 : \lambda_{1,2} = \left(-1 \pm i\sqrt{3}\right)\tfrac{\zeta}{6};\ \lambda_{3,4} = \left(-1 \pm i\sqrt{11}\right)\tfrac{\zeta}{6} \\ e_2, e_3 : \lambda_{1,2} = \left(-1 \pm i\sqrt{7}\right)\tfrac{\zeta}{4};\ \lambda_3 = -\tfrac{\zeta}{2};\ \lambda_4 = \tfrac{\zeta}{2} \\ e_4 : \lambda_{1,2} = -\zeta;\ \lambda_{3,4} = \zeta \\ e_5 : \lambda_1 = -z^*\zeta;\ \lambda_2 = -(1-z^*)\zeta;\ \lambda_3 = 0;\ \lambda_4 = \zeta \end{gathered}$$

The signs of the eigenvalues show that the equilibrium points e_2, e_3, e_4 and e_5 are all unstable since at least one of the eigenvalues has a non-negative sign. Point e_1 is locally stable because the respective eigenvalues are two pairs of complex conjugates with negative real parts. □

Proof of Proposition 3 Equilibrium condition $\dot{y} = \dot{v} = \dot{z} = \dot{w} = 0$ is now applied to system (5) with the result,

$$\begin{cases} x^*y^* = \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x^*(y^* - z^*)\right]\right\} y^*(y^* + z^*) \\ x^*v^* = \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x^*(v^* - w^*)\right]\right\} v^*(v^* + w^*) \\ \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x^*(y^* - z^*)\right]\right\} y^*(y^* + z^*) = z^*x^* \\ \frac{\zeta}{2}\left\{1 - \tanh\left[\zeta x^*(v^* - w^*)\right]\right\} v^*(v^* + w^*) = w^*x^* \end{cases}$$

Given that $\tanh(0) = 0$, the solution of the above system gives the set of equilibrium points claimed in the proposition. □

Proof of Proposition 4 The Jacobian matrix of system (5) is a modified version of the one presented in the proof of Proposition 2 given by

$$\widehat{J} = \zeta \begin{bmatrix} 1 - 3y^* - \frac{3}{2}z^* - v^* - w^* + \frac{\chi(y,y)}{2}y^*(y^*+z^*) \\ -v^* + \frac{\chi(v,y)}{2}v^*(v^*+w^*) \\ y^* + \frac{3}{2}z^* - \frac{\chi(y,y)}{2}y^*(y^*+z^*) \\ w^* - \frac{\chi(v,y)}{2}v^*(v^*+w^*) \end{bmatrix}$$

$$\begin{matrix} -y^* + \frac{\chi(y,v)}{2}y^*(y^*+z^*) \\ 1 - y^* - z^* - 3v^* - \frac{3}{2}w^* + \frac{\chi(v,v)}{2}v^*(v^*+w^*) \\ z^* - \frac{\chi(y,v)}{2}y^*(y^*+z^*) \\ v^* + \frac{3}{2}w^* - \frac{\chi(v,v)}{2}v^*(v^*+w^*) \end{matrix}$$

$$\begin{matrix} -\frac{3}{2}y^* + \frac{\chi(y,z)}{2}y^*(y^*+z^*) \\ -v^* + \frac{\chi(v,z)}{2}v^*(v^*+w^*) \\ -\left(1 - \frac{3}{2}y^* - 2z^* - v^* - w^*\right) - \frac{\chi(y,z)}{2}y^*(y^*+z^*) \\ w^* - \frac{\chi(v,z)}{2}v^*(v^*+w^*) \end{matrix}$$

$$\begin{matrix} -y^* + \frac{\chi(y,w)}{2}y^*(y^*+z^*) \\ -\frac{3}{2}v^* + \frac{\chi(v,w)}{2}v^*(v^*+w^*) \\ z^* - \frac{\chi(y,w)}{2}y^*(y^*+z^*) \\ -\left(1 - y^* - z^* - \frac{3}{2}v^* - 2w^*\right) - \frac{\chi(v,w)}{2}v^*(v^*+w^*) \end{matrix} \Bigg]$$

with

$$\chi(y,y) = \left.\frac{\partial \tanh\left[\zeta x(y-z)\right]}{\partial y}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(y-z)]}{\partial y}}{\cosh^2\left[\zeta x(y-z)\right]}\right|_{(y^*,z^*,v^*,w^*)}$$
$$= \zeta(1 - 2y^* - v^* - w^*)$$

$$
\begin{aligned}
\chi(y,v) &= \left.\frac{\partial \tanh\left[\zeta x(y-z)\right]}{\partial v}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(y-z)]}{\partial v}}{\cosh^2\left[\zeta x(y-z)\right]}\right|_{(y^*,z^*,v^*,w^*)} \\
&= -\zeta\left(y^*-z^*\right) \\
\chi(y,z) &= \left.\frac{\partial \tanh\left[\zeta x(y-z)\right]}{\partial z}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(y-z)]}{\partial z}}{\cosh^2\left[\zeta x(y-z)\right]}\right|_{(y^*,z^*,v^*,w^*)} \\
&= -\zeta\left(1-2z^*-v^*-w^*\right) \\
\chi(y,w) &= \left.\frac{\partial \tanh\left[\zeta x(y-z)\right]}{\partial w}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(y-z)]}{\partial w}}{\cosh^2\left[\zeta x(y-z)\right]}\right|_{(y^*,z^*,v^*,w^*)} \\
&= -\zeta\left(y^*-z^*\right) \\
\chi(v,y) &= \left.\frac{\partial \tanh\left[\zeta x(v-w)\right]}{\partial y}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(v-w)]}{\partial y}}{\cosh^2\left[\zeta x(v-w)\right]}\right|_{(y^*,z^*,v^*,w^*)} \\
&= -\zeta\left(v^*-w^*\right) \\
\chi(v,v) &= \left.\frac{\partial \tanh\left[\zeta x(v-w)\right]}{\partial v}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(v-w)]}{\partial v}}{\cosh^2\left[\zeta x(v-w)\right]}\right|_{(y^*,z^*,v^*,w^*)} \\
&= \zeta\left(1-y^*-z^*-2v^*\right) \\
\chi(v,z) &= \left.\frac{\partial \tanh\left[\zeta x(v-w)\right]}{\partial z}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(v-w)]}{\partial z}}{\cosh^2\left[\zeta x(v-w)\right]}\right|_{(y^*,z^*,v^*,w^*)} \\
&= -\zeta\left(v^*-w^*\right) \\
\chi(v,w) &= \left.\frac{\partial \tanh\left[\zeta x(v-w)\right]}{\partial w}\right|_{(y^*,z^*,v^*,w^*)} = \left.\frac{\frac{\partial[\zeta x(v-w)]}{\partial w}}{\cosh^2\left[\zeta x(v-w)\right]}\right|_{(y^*,z^*,v^*,w^*)} \\
&= -\zeta\left(1-y^*-z^*-2w^*\right)
\end{aligned}
$$

For e_1', observe that $\chi(y,y)=\chi(v,v)=\frac{\zeta}{5}$, $\chi(y,v)=\chi(y,w)=\chi(v,y)=\chi(v,z)=0$, and $\chi(y,z)=\chi(v,w)=-\frac{\zeta}{5}$. Given the values of the derivatives of the hyperbolic tangent functions evaluated in the equilibrium, it is straightforward to compute the respective Jacobian matrix,

$$
e_1' : \widehat{J} = \frac{\zeta}{10}\begin{bmatrix}
-3+\frac{2}{25}\zeta & -2 & -3-\frac{2}{25}\zeta & -2 \\
-2 & -3+\frac{2}{25}\zeta & -2 & -3-\frac{2}{25}\zeta \\
5-\frac{2}{25}\zeta & 2 & 1+\frac{2}{25}\zeta & 2 \\
2 & 5-\frac{2}{25}\zeta & 2 & 1+\frac{2}{25}\zeta
\end{bmatrix}
$$

The eigenvalues of $\widehat{J}$ for e_1' are

$$\lambda_{1,2} = \left[-\left(1-\frac{2}{25}\zeta\right) \pm \sqrt{\left(1-\frac{2}{25}\zeta\right)^2 - 4}\right]\frac{\zeta}{10};$$

$$\lambda_{3,4} = \left[-\left(1-\frac{2}{25}\zeta\right) \pm \sqrt{\left(1-\frac{2}{25}\zeta\right)^2 - 20}\right]\frac{\zeta}{10}$$

Relative to the above eigenvalues, four different cases are identifiable (excluding the border cases that imply the existence of bifurcation points): (i) if $\zeta > \frac{25}{2}\left(1+\sqrt{20}\right)$ then the four eigenvalues have positive real values; (ii) if $\frac{75}{2} < \zeta < \frac{25}{2}\left(1+\sqrt{20}\right)$ then two of the eigenvalues are positive real roots, while the other two are a pair of complex conjugate eigenvalues with a positive real part; (iii) if $\frac{25}{2} < \zeta < \frac{75}{2}$, then the eigenvalues are two pairs of complex conjugates with positive real parts; (iv) if $\zeta < \frac{25}{2}$ then the eigenvalues are two pairs of complex conjugates with negative real parts. Only in the last case will stability hold, and thus the condition for stability is the one claimed in the proposition.

Next, we analyze the stability of the other equilibrium points and confirm that they are also unstable. Observe that:

• For e_2' and e_3' : $\chi(y,y) = \chi(v,v) = \frac{\zeta}{3}$, $\chi(y,v) = \chi(y,w) = \chi(v,y) = \chi(v,z) = 0$, $\chi(y,z) = \chi(v,w) = -\frac{\zeta}{3}$;

• For e_4 : $\chi(y,y) = \chi(v,v) = \zeta$, $\chi(y,v) = \chi(y,w) = \chi(v,y) = \chi(v,z) = 0$, $\chi(y,z) = \chi(v,w) = -\zeta$;

• For e_5 : $\chi(y,y) = \chi(y,z) = z^*\zeta$, $\chi(y,v) = \chi(y,w) = \chi(v,y) = \chi(v,z) = 0$, $\chi(v,v) = \chi(v,w) = (1-z^*)\zeta$.

With these derivatives, matrix $\widehat{J}$ in each case will be

$$e_2' : \widehat{J} = \frac{\zeta}{6}\begin{bmatrix} -3+\frac{2}{9}\zeta & -2 & -3-\frac{2}{9}\zeta & -2 \\ 0 & 2 & 0 & 0 \\ 5-\frac{2}{9}\zeta & 2 & 1+\frac{2}{9}\zeta & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$e_3' : \widehat{J} = \frac{\zeta}{6}\begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & -3+\frac{2}{9}\zeta & -2 & -3-\frac{2}{9}\zeta \\ 0 & 0 & -2 & 0 \\ 2 & 5-\frac{2}{9}\zeta & 2 & 1+\frac{2}{9}\zeta \end{bmatrix}$$

$$e_4 : \widehat{J} = \zeta\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$e_5 : \widehat{J} = \zeta\begin{bmatrix} -\frac{1}{2}z^* & 0 & 0 & 0 \\ 0 & -\frac{1}{2}(1-z^*) & 0 & 0 \\ \frac{3}{2}z^* & z^* & z^* & z^* \\ 1-z^* & \frac{3}{2}(1-z^*) & 1-z^* & 1-z^* \end{bmatrix}$$

As in the original model, the eigenvalues of the pair of equilibrium points e_2' and e_3' are identical and are given in this case by

$$e_2', e_3' : \lambda_{1,2} = \left[-\left(1 - \frac{2}{9}\zeta\right) \pm \sqrt{\left(1 - \frac{2}{9}\zeta\right)^2 - 12} \right] \frac{\zeta}{6}; \lambda_3 = -\frac{\zeta}{3}; \lambda_4 = \frac{\zeta}{3}$$

At least one of the eigenvalues of $\widehat{J}$ for e_2' and e_3' has a positive sign, regardless of the value of ζ, and therefore the instability of the respective equilibria is confirmed. For e_4, the Jacobian matrices J and $\widehat{J}$ are identical. Thus the corresponding eigenvalues are also the same, namely $\lambda_{1,2} = -\zeta; \lambda_{3,4} = \zeta$, and local stability is absent in this case as well. Finally, note that the eigenvalues for e_5 are $\lambda_1 = -\frac{1}{2}z^*\zeta; \lambda_2 = -\frac{1}{2}(1 - z^*)\zeta; \lambda_3 = 0; \lambda_4 = \zeta$; for the equilibrium line, one of the eigenvalues is positive and one other is zero, which again implies instability. □

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