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# NONLINEARITIES IN ECONOMIC DYNAMICS*

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Nonlinear models played an important role in modelling economic dynamics during the first part of this century (cf. e.g. Kaldor (1940), Hicks (1950) and Goodwin (1951)). By the 1960s, however, the profession had largely switched to the linear approach making use of Slutzky's (1927) observation that stable low order linear stochastic difference equations could generate cyclic processes that mimicked actual business cycles.

In the context of business cycle modelling there seem to have been at least two reasons that led to the dominance of the linear stochastic difference equations approach. The first one was the fact that the nonlinear systems seemed incapable of reproducing the 'statistical' aspects of actual economic time series. At best, such models were able to produce periodic motion[1] and an examination of the spectra of economic time series showed the absence of the spikes that characterise periodic data. The emphasis on the equilibrium approach to aggregate economic behaviour (cf. Lucas (1986)) would make things even more difficult. The plethora of stability results for models of infinitely lived agents with perfect foresight (Turnpike Theorems) suggested that even the regular fluctuations that had been derived in the literature on endogenous business cycles were incompatible with a theory that had solid microeconomic foundations. Further this indicated that while nonlinearities could be present, the explanations of the fluctuations had to rest solely on the presence of exogenous shocks that working through the equilibrating mechanism would create the observed randomness. If such shocks were absent, the system would tend to a stationary state.

The second reason was the empirical success of the models based on linear stochastic difference equations. Low order autoregressive processes captured some of the features of aggregate time series. In turn these processes can in principle result from an economy with complete markets where production is subject to exogenous shocks. Though the task of confronting these equilibrium models with actual data has not yielded uncontroversial results (see the discussion and references in Section I) there was no obvious gain in the introduction of nonlinearities.[2]

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[1] There were exceptions: Ando and Modigliani (1959) realised that endogenous cycle models could produce more complicated behaviour.

[2] A third perhaps equally compelling reason is that linear or log linear models are much easier to solve and estimate. In the analysis of certain equilibrium models where one must consider explicitly the agents' forecast of the future evolution of endogenous variables the certainty equivalence principle (Simon, 1956; Theil, 1964) that applies to the linear case greatly simplifies things.

Similar conclusions appeared to be warranted from the literature on asset prices. For the most part a random walk seemed to adequately describe stock returns over periods at least as long as a week (Fama, 1970). The equilibrium asset pricing models that followed the work of Rubinstein (1976), Lucas (1978) and others, by linking stock returns to consumption variability provided, in principle, a role for nonlinearities. However, the attempts to implement these models involved parameterisations where, in the absence of shocks, fluctuations would be absent. Reasoning similar to the one concerning macroeconomic fluctuations indicated that this was the general situation.

It is now well known that deterministic systems can generate dynamics that are extremely irregular. A simple example is given by the 'tent map' $f: [0, 1] \to [0, 1]$ such that $f(x) = 2x$ if $0 < x \leqslant 1/2$, $f(x) = 2(1-x)$ if $1/2 < x \leqslant 1$. For a given $x_0$, let $x_t = f(x_{t-1})$, $t = 1, 2, \ldots$. This gives us a time series $\{x_t\}_{t=0}^{\infty}$. For some particular $x_0$ the series $x_t$ is quite well behaved as for example when $x_0 = 2/3$ and consequently $x_t = 2/3$ for all $t = 1, 2, \ldots$. For others (cf. Fig. 1) the $x_t$'s seem to follow a very complicated trajectory. This is the general case. In fact, it can be shown (cf. Sakai and Tokumaru (1980)) that for almost all $x_0 \in [0, 1]$ the autocorrelation at lag $k$ of $\{x_t\}$ will be zero for any $k \neq 0$. Thus its spectra is exactly that of white noise. Many other examples of such 'chaotic' systems can be constructed – some that imitate a stationary AR(1), some involving smooth functions and in higher dimension even invertible functions.[3]

Though there are many, nonequivalent ways of defining chaos, for our

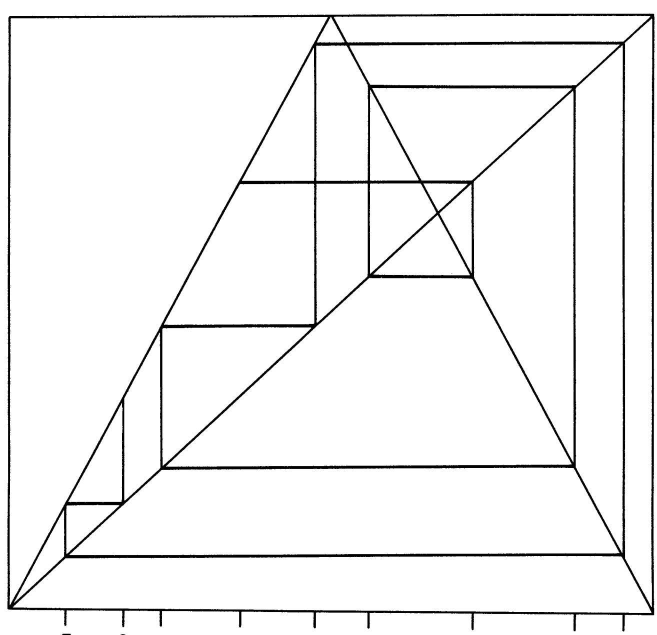


Fig. 1. Text map iterations. In this figure 0 indicates the initial point $x_0$, 1 indicates $x_1 = f(x_0)$, 2 indicates $x_2 = f(x_1)$ etc.

[3] Baumol and Benhabib (1988) provide a simple exposition of how chaos arises in one dimensional difference equations.

purposes a system $x_t = f(x_{t-1})$ where $f$ maps a closed bounded set into itself is chaotic if it exhibits sensitive dependence to initial conditions, that is, small differences in initial conditions tend to be amplified by $f$. It should be obvious that the trajectories of such systems have to be quite complicated. In Section IV below I define what is meant by sensitive dependence and discuss its implications for dynamic economics.

Over the last few years a literature has developed where such complicated dynamics appears in equilibrium models where agents optimise while provided with perfect foresight (cf. Section I below). As I argue in section I, it is extremely unreasonable to believe that purely deterministic models could ever explain the behaviour of aggregate quantities or asset prices in actual economies. None the less the developments discussed in the preceding paragraphs show that nonlinearities could, in theory, explain part of the observed fluctuations in quantities or asset prices.

The tent map example also shows that linear statistical techniques may not be enough to dismiss the influence of nonlinearities. Examples such as these led to the development of measures to distinguish between data generated by a deterministic system from data generated by a 'random' system (cf. Grassberger and Procaccia (1983), Takens (1983)). Section II describes these tests as well as some applications that were made in economic time series. In Section III I expose the distribution theory developed by Brock *et al.* (1987) for statistics based on the Takens–Grassberger–Procaccia measure. Finally, Section V contains some conclusions.

## I. EQUILIBRIUM MODELS THAT EXHIBIT COMPLEX DYNAMICS

Much recent work has been done showing that dynamic models with optimising agents can generate cyclic and/or chaotic trajectories. This work was done under a very strict set of rules. Agents maximise classical, i.e. concave, smooth, often time additive–objective functions subject to well defined constraints and are endowed with perfect foresight.

Benhabib and Nishimura (1979) showed that cyclical paths may arise in the context of continuous time multisector models of optimal growth with an additive, concave objective function in the presence of discounting. The well known connection between competitive equilibrium with perfect foresight and Pareto optimality can be used to reinterpret the paths that solve the Benhabib–Nishimura example as the equilibrium paths of an economy with homogeneous agents. These authors (cf. Benhabib and Nishimura (1985)) also produced an example of an infinite horizon discrete time single capital good model with optimal solutions that were two period cycles that is easy to describe. Consider an economy with a representative individual that possesses in each period an endowment of one unit of labour has a linear utility function for consumption and a discount factor $\delta$. Leisure yields no utility. The single consumption good is produced in each period with the aid of capital ($k$) and labour ($l$) and output at time $t$ equals $(l_t)^{\beta}(k_t)^{\alpha}$ where $l_t(k_t)$ equals the amount of labour (resp. capital) used in the production of the consumption good.

Capital depreciates totally in each period and the amount of capital available at time $t$ equals the amount of labour used in the production of the capital good in period $t-1$. It is easy to see that the maximal utility enjoyed by the representative individual when he starts with an amount of capital $k_t$ and produces an amount of capital $k_{t+1}$ for the next period is $u(k_t, k_{t+1}) = (1-k_{t+1})^\beta (k_t)^\alpha$. Hence we may write the maximisation problem, using this indirect utility function as

$$\text{(P)} \qquad \operatorname{Max} \sum_{t=0}^{\infty} \delta^t u(k_t, k_{t+1}), k_0 \text{ given.}$$

This example with $\alpha = \beta = 0{\cdot}5$ was actually introduced by Weitzman (cf. Samuelson (1973)) to generate cycles with $\delta$ near 1. Weitzman's example is not however stable with respect to perturbations since if $\alpha+\beta < 1$, and $\delta$ is close to one all optimal paths converge to the unique optimal stationary state (cf. Scheinkman (1976)). Benhabib and Nishimura showed that for $\alpha+\beta < 1$, $\alpha > 0{\cdot}5$ and $\delta$ small, a cycle of order two is the optimal solution for an appropriate initial $k_0$.

It is also not difficult to determine that a two period cycle is the most complicated behaviour one can obtain in this class of examples. Associated with a problem as (P) we have a policy function that gives the optimal capital stock at time $t$ as a function of the capital stock at time $t-1$. In the usual Cass–Koopmans one sector model this policy function is nondecreasing and thus no cyclical behaviour can appear. One can readily verify that in the example above, the policy function is monotone and hence either there are no cycles or a cycle of period two arises.[4]

Ivar Ekeland and I (cf. Scheinkman (1984)) suggested that the following modification would generate arbitrarily long cycles for appropriate parameter values. Let

$$u(k_t, k_{t+1}) = (1-k_{t+1})^\beta (k_t - \gamma k_{t+1})^\alpha, \gamma < 1.$$

The story is just as before, except that to produce tomorrow's capital one needs one unit of labour and $\gamma$ units of today's capital, in fixed proportions. Boldrin and Deneckere (1987) showed that this conjecture was in fact correct.

While these examples helped one understand the assumptions on tastes and technologies that would lead to complicated dynamics they did not provide the characterisation of the set of possible types of trajectories that could appear in these models. This question was settled by Boldrin and Montruchio (1986) that established that any (sufficiently smooth) dynamics can arise in the setting of multisector models of optimal growth with an additive, concave objective function by a suitable choice of utility function, technology and discount factor. Again, we may use the link between competitive equilibrium with perfect foresight and Pareto optimality to rephrase this result in terms of equilibria.

Other examples were derived by Benhabib and Day (1982) and Grandmont (1985) for overlapping generations economies and Woodford (1987) in the context of a model with infinitely lived consumers under borrowing constraints.

This research puts the literature on fluctuations arising from a purely

[4] Cf. Scheinkman (1984), Section 3.

deterministic system on an equal theoretical footing with the literature where such fluctuations result from exogenous shocks that constantly impact an intrinsically stable economy that otherwise would converge towards some steady state. Another entirely different matter is whether these systems are capable of fitting observed economic time series or at least of being calibrated to generate data that grossly mimics some of the statistical properties of the actual data.

Kydland and Prescott (1982) showed that a slightly modified neoclassical one good optimal growth model could, in the presence of exogenous technology shocks, generate time series that would mimic a few of the characteristics of some Post War U.S. aggregate economic time series. Further, the free parameters in the model were chosen to be somewhat consistent with microeconomic and growth observations.

There are no comparable results for the literature on deterministic fluctuations. It is extremely unlikely that macroeconomic fluctuations could ever be explained by a purely deterministic model with a manageable number of state variables. There are even theoretical reasons to support this view. The same property that makes chaotic systems look as if they were random – their sensitivity to initial conditions – makes it difficult to forecast future values of the variables that agents take as exogenous. This problem is not dissimilar to that of choosing the correct value of the costate variable in infinite horizon optimisation problems (cf. Hahn (1966)). Rational agents in such a situation would quickly understand the limitations on their ability to predict. At this point such agents may very well act as I do when facing the purely deterministic process of tossing a coin – namely, treat it as if the future values of the variables they are trying to predict are at least in part random.

None the less, nonlinearities may still be responsible for a good part of the apparent fluctuations. Much seems not to be accounted by the log-linear models of real business cycles. One difficulty pointed out by Murphy *et al.* (1989) is with explaining the observed positive correlation of output comovements across sectors. One could always assume that productivity shocks are common to all sectors, but unless that involves the cheapening of some common input or equivalently a productivity shock to a commonly used input, it seems unlikely. Murphy *et al.* argue that the data do not support the hypothesis that productivity shocks to a common input are responsible for the cycle.

All fully worked out examples in the literature on equilibrium fluctuations arising from purely deterministic systems involve single variable systems. This is due to the fact that easily checkable sufficient conditions for chaotic behaviour cover principally the single variable case. The conditions that insure chaos in multi-dimensional systems mostly involve the calculation of Liapunov exponents (cf. Section IV below) although some results by Marotto (1978) may be applicable. None the less, the mathematical intuition indicates that multi-dimensional systems can more easily give rise to chaotic dynamics than single variable ones. Nonlinear systems involving several variables are capable of getting in phase even when coupled by very weak links as in the case of coupled oscillators. Hence adding nonlinearities may improve our ability to explain the comovements.[5] Further, in the log-linear models, an important role is played by sizable and persistent technology shocks for which no good explanations exist. The introduction of nonlinearities may very well reduce the role played by these shocks.

Though some of the empirical tests described in what follows have been applied to macroeconomic time series the shortness of these series makes it hard to distinguish the presence of nonlinearities. Besides the possibility of explaining the observed comovements and diminishing the size and persistence of the exogenous shocks nonlinear models may play an important role in reconciling the existence of a cycle with the apparent unit root behaviour of aggregate time series. This is discussed in Section IV below.

Any equilibrium model has associated with it an asset pricing model. Interpreted as asset pricing models the purely deterministic models with perfect foresight have a strong prediction that is easily rejected by data. If short sales are permitted, or even if assets are held in positive net supply, the returns on all assets must be identical. This follows easily from the no-arbitrage relationships. Hence, once again, one must either assume the presence of some exogenous noise or weaken the notion of perfect foresight.

The difficulty of forecasting future values of a variable generated by chaotic systems seems to make necessary weakening the notion of perfect foresight. The following example is helpful. Consider an economy with homogeneous agents and a single type of machine. This machine outputs at time $t$ an amount of the (non-storable) consumption good that equals $x_t$. The quantity $x_t$ satisfies $x_t = f(x_{t-1})$ where $f$ is the 'tent' map described above. The utility function of each agent is given by

$$U(c_1, c_2, \dots, c_t, \dots) = \mathrm{E} \sum_{t=1}^{\infty} \delta^t (c_t)^{(1-\alpha)}/(1-\alpha)\,; \alpha > 0.$$

The rights to the output of the machine are traded in a competitive market and its price at time $t$ is given by $p_t$. Since, in equilibrium $c_t = x_t$, we must have at each time $t$,

$$p_t(x_t)^{-\alpha} = \delta \mathrm{E}[(x_{t+1})^{-\alpha}(p_{t+1} + x_{t+1})/I_t)], \tag{1}$$

where $I_t$ denotes the information used at time $t$ by each of the homogeneous agents to predict future dividends and prices.

Of course one possible candidate for an equilibrium is the perfect foresight path. In this case agents perfectly predict as of time zero not only $x_t$ but also $p_t$. Suppose however that agents are restricted to linear regressions of vector of variables on their past values. Then by looking at the single variable $x_t$ one could not reject the hypothesis that the $x_t$'s are distributed independently and uniformly in the interval $[0, 1]$. If the $x_t$'s were in fact independently and identically distributed (i.i.d.) and the information that each agent uses at time $t$ is the known distribution of $x_t$ then a solution to (1) is given by

$$p_t = \lambda_\alpha (x_t)^\alpha \text{ where } \lambda_\alpha = [\delta/(1-\delta)]\ \mathrm{E}(x_{t+1})^{1-\alpha}. \tag{2}$$

[5] This idea arose in conversations with Mike Woodford.

If $\alpha = 1$, i.e. the logarithmic utility function, then in fact the $p_t$ given by (2) is a linear function of $x_t$ and hence if we restrict agents to linear regressions of the observables $(x_t, p_t)$ on their past values they could not reject the hypothesis that the dividends and prices are i.i.d. That is, if agents assumed that dividends where i.i.d., then $p_t = \lambda_1 x_t$ would be the candidate equilibrium prices and, in fact, agents could not use past prices and/or dividends to help predict future ones if they were restricted to linear regressions. One could then argue that such an economy would behave as if dividends where i.i.d. For any other value of $\alpha$ however, the prices given by (2) will not constitute an equilibrium even in the case where agents are restricted to linear regressions. The fact that the candidate equilibrium prices given by (2) are a nonlinear function of dividends means that one can use current dividends and prices to predict future prices linearly. In the case $\alpha = 0{\cdot}5$, for instance, if the $p_t$'s are given by (2), a regression of $p_t$ on $p_{t-1}$ and $x_{t-1}$ yields an $R^2$ that typically exceeds 0·8 whereas that of $x_t$ on $p_{t-1}$ and $x_{t-1}$ produces an $R^2$ that typically exceeds 0·7.[6] Hence if the market were to produce prices as if the $x_t$'s are i.i.d., individuals using only linear regression would be able to predict future dividends and prices with some accuracy. Hence the hypothesis that the agent's information at $t$ consists solely of the unconditional distribution of the $x_t$'s would not be correct.

Implicit in this reasoning is a limitation on agents' ability to learn. They start with a prior that dividends are i.i.d. and that the distribution of prices is that implied by (2). They 'learn' by running linear regressions of the observables in their past values. In the logarithmic case this 'learning' would not lead them to alter their priors. This is not true for other values of $\alpha$, but we have not taken the story far enough to exhibit actual equilibria.

In any case, nonlinearities may be responsible for a share of the apparent randomness in asset prices. In the next two sections we exposit some techniques that have been used to examine this question in actual economic data. In Section II we discuss the use of the correlation dimension measure. The basic idea behind the use of this measure is that in a deterministic system given by $x_{t+1} = f(x_t), x_t \in R^n$, the pairs of successive observations $(x_t, x_{t+1})$ lie in the graph of the function $f$ and hence in a lower dimensional set than if the $x_t$'s are 'random'. Section II describes these tests as well as some applications that were made to economic time series. The techniques discussed in Section II have the disadvantage that they are not accompanied by a distribution theory for the relevant statistics. Brock *et al.* (1987) produced an asymptotic theory for statistics based on the correlation dimension and this material is discussed in Section III.

## II. THE CORRELATION DIMENSION

The earlier efforts in applying the ideas of chaotic dynamics in order to uncover nonlinear dependence in economic data (cf. Scheinkman (1985), Brock (1986)) consisted simply of using certain tools developed in the mathematics and

[6] This should not be surprising since the tent can be well approximated by a linear combination of the square root function and a linear function.

physics literature in a rather direct way. The most promising of these practices make use of the correlation dimension.

Let $\mathbf{y}_1, \mathbf{y}_2, \dots$ be a sequence of vectors in $R^p$. For each $\gamma > 0$, let

$$C_m(\gamma) = \frac{2}{m(m-1)} \sum_{1 \leqslant i < j \leqslant m} \theta(\gamma - |y_i - y_j|),$$

where

$$\theta(a) = 0 \quad \text{if } a < 0$$
$$= 1 \quad \text{if } a \geqslant 0.$$

Here,

$$|y_i - y_j| = \max_k |y_i^k - y_j^k|.$$

Intuitively $C_m(\gamma)$ denotes the fraction of the first $m$ vectors $y_i$'s that are within $\gamma$ of each other. For each $\gamma$, let

$$C(\gamma) = \lim_{m \to \infty} C_m(\gamma).$$

The quantity $C(\gamma)$ indicates the fraction of all vectors that are within $\gamma$ of each other. Finally, let us define the correlation dimension of $\{y_t\}_{t=0}^{\infty}$ as

$$d = \lim_{\gamma \to 0} \frac{\log C(\gamma)}{\log \gamma}.$$

Intuition of why $d$ is a measure of 'dimension' can be obtained by considering two examples.

*Example 1:* Let $\mathbf{y}_t = (x_t, x_{t+1})$ where each $x_t$ is an independent draw of a uniform distribution in $[0, 1]$. Here as we double $\gamma$ we expect to multiply by four the number of neighbours of each point. Hence $d = 2$.

*Example 2:* Let $\mathbf{y}_t = (x_t, \alpha x_t)$ where $x_t$ is again an independent draw of a uniform distribution in $[0, 1]$. Here as we double $\gamma$ we expect the number of neighbours to double. Hence $d = 1$.

Suppose now we are given a sequence of real numbers $x_1, x_2, \dots$. For each $N$ let $\mathbf{z}_t^N = (x_t, x_{t+1}, \dots, x_{t+N-1})$, the vector of $N$-histories of the $x$'s. We will write $C_m^N$, $C^N$ and $d^N$ for the quantities corresponding to $C_m$, $C$ and $d$ when $\mathbf{y}_t = \mathbf{z}_t^N$. The length of the histories, $N$, is called the embedding dimension. Note that

$$C_m^N(\gamma) = \frac{2}{m(m-1)} \sum_{1 \leqslant i < j \leqslant m} \prod_{k=0}^{N-1} [\theta(\gamma - |x_{i+k} - x_{j+k}|)].$$

Consider again the tent map discussed in the introduction. For a given $x_0$ let $x_t = f(x_{t-1})$, $t = 1, 2, \dots$ Clearly $d^2 \leqslant 1$, since all $\mathbf{z}_t^2$'s must lie on the graph of the tent map. On the other hand, if the $x_t$'s are independently and uniformly distributed in $[0, 1]$, as we discussed in Example 1 above, $d^2 = 2$. Hence using the correlation dimension we can distinguish between the two possibilities!

This reasoning generalises. Suppose $y_0 \in R^p$ is given and $y_t = F(y_{t-1})$, $t = 1, 2, \dots$. Let $d$ be the correlation dimension of $\{y_t\}_{t=0}^{\infty}$, $h: R^p \to R$ be given and $x_t = h(y_t)$. One may think of the $x_t$'s as observation on the $y_t$'s. Again, let

$\mathbf{z}_t^N = (x_t, x_{t+1}, \dots, x_{t+N-1})$. Then $d^N \leqslant d$, i.e. the 'embedding dimension' does not increase indefinitely with $N$. Further, under some rather general conditions if $N \geqslant 2p+1$ then $d^N = d$.[7] Hence one can even find out the dimension of the unobservable time series $\{y_t\}_{t=0}^{\infty}$. On the other hand, just as in example 1 above, if the $x_t$'s are independently and identically distributed (i.i.d.) on $[0, 1]$ and have a density then $d^N = N$.

These ideas provide a basis for distinguishing among deterministic and random systems by estimating the limit of the ratio $\log C_m^N(\gamma)/\log \gamma$ as $\gamma$ becomes small for each $N$. Here $m$ is the length of your data series minus the length of the longest history you are considering. The estimated $\hat{d}^N$ should stabilise as $N$ increases if your data were generated by a deterministic system.

One can accommodate a mixture of a deterministic system contaminated by a small amount of randomness, i.e. the observed $x_t = h(y_t) + \mu_t$ where the $\mu_t$'s are i.i.d. Looking at deterministic systems were noise distributed uniformly $[-\alpha, \alpha]$ is added to a system of known dimension, many researchers (e.g. Zardecki (1982); Ben-Mizrachi *et al.* (1984); Atten and Caputto (1985)) have found that the graph of log $[C_m^N(\gamma)]$ against log $(\gamma)$ has the slope of the embedding dimension (i.e. $N$) for $\gamma \leqslant \alpha$, and has the slope of the dimension of the deterministic system above that level. Thus at a certain scale one observes behaviour as in a random system, while at a larger scale one sees the deterministic motion.

The application of these techniques to economics present several problems. First the time series in economics tend to be much shorter than it seems necessary to obtain good estimates when the dimensions of the original variable $y_t$ is moderately large (above 2 or 3). There are a few exceptions in finance. More importantly as we argued above it is unlikely that the economic time series of interest are generated by purely deterministic systems. Further the uncertainty is likely to affect the dynamics itself as opposed to merely affecting the observable.

These comments apply specially to estimates of the dimension of the system that cannot be taken as anything but suggestive.[8] Nevertheless these techniques can be useful in detecting the presence of nonlinear dependence on data and inspired the formal statistical tests discussed in the next section.

## III. DISTRIBUTION THEORY

Recall that $C_m^N(\gamma)$ is interpreted as the fraction of the first $m$ $N$-histories that are within $\gamma$ of each other. Let $x_1, x_2, \dots$ be independent with a common distribution $F$. For each $\gamma > 0$, it is expected that, for $m$ large

$$C_m^N(\gamma) \sim [C_m^1(\gamma)]^N. \tag{3}$$

In fact, Brock, Dechert and Scheinkman (1987) (henceforth BDS) established that $\sqrt{m}\{C_m^N(\gamma) - [C_m^1(\gamma)]^N\}$ converges to a normal law. In this section we will

[7] The interested reader may consult Takens (1983) for precise statements.

[8] See Ramsey and Yuan (1987) for a discussion of the effect of the smallness of a data set on the computation of the dimension.

present the BDS result and indicate some extensions. Let us again write $\mathbf{z}_t = (x_t, \dots, x_{t+N-1})$ for the vectors of $N$-histories of the $x$'s. A key point in the BDS approach was the recognition that

$$C_m^N(\gamma) = \frac{2}{m(m-1)} \sum_{1 \leqslant i < j \leqslant m} \prod_{k=0}^{N-1} \theta(\gamma - |x_{i+k} - x_{j+k}|) \tag{4}$$

is a $U$-Statistic in the sense of Hoeffding (1948).[9] Several results concerning Central Limit Theorems for $U$-Statistics exist in the literature. Modern treatments of the theory of $U$-Statistics can be found in Serfling (1980) and Denker and Keller (1983).

In BDS (1987) the following theorem was proved.

THEOREM

*If the $x_j$'s are i.i.d. then,* $\sqrt{m}\{C_m^N(\gamma) - [C_m^1(\gamma)]^N\} \to \mathrm{N}(0, \sigma)$.

Further in the case where the distribution $F$ is continuous a formula to compute a consistent estimator for $\sigma$ was given.

The proof of the theorem as stated here can be exposited in a straightforward manner if we forego the estimates of $\sigma$. We will do so and we will separate the argument in several steps. In what follows we will write $C(\gamma)$ for the probability that two arbitrary $x_j$'s are within $\gamma$ of each other.

*Step 1:* Notice that for any pair $\lambda_1, \lambda_2$ and for any realisation of the histories $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m$ we can write,

$$\lambda_1 C_m^N(\gamma) + \lambda_2 C_m^1(\gamma) = [2/m(m-1)] \sum_{1 \leqslant i < j \leqslant m} h(\mathbf{z}_i, \mathbf{z}_j),$$

where

$$h(\mathbf{z}_i, \mathbf{z}_j) = \lambda_1 \prod_{k=0}^{N-1} \theta(\gamma - |x_{i+k} - x_{j+k}|) + \lambda_2 \theta(\gamma - |x_i - x_j|),$$

and, $\mathbf{z}_i = (x_i, x_{i+1}, \dots, x_{i+N-1})$. The function $h$ is symmetric, i.e. $h(\xi_1, \xi_2) = h(\xi_2, \xi_1)$ for any pair of vectors $(\xi_1, \xi_2) \in R^{2N}$. Hence $\lambda_1 C_m^N(\gamma) + \lambda_2 C_m^1(\gamma)$ is a $U$-statistic and further even though two arbitrary histories $\mathbf{z}_t$ and $\mathbf{z}_\tau$ are not in general independent, they will so be if $|t-\tau| > N$. Hence the theorems of Sen (1963) apply and since $\mathrm{E}h(\mathbf{z}_i, \mathbf{z}_j) = \lambda_1[C(\gamma)]^N + \lambda_2 C(\gamma)$ if $|i-j| > N$, $\sqrt{m}\{\lambda_1 C_m^N(\gamma) + \lambda_2 C_m^1(\gamma) - \lambda_1[C(\gamma)]^N - \lambda_2 C(\gamma)\}$ is asymptotically normal. In the case where $F$ (the distribution of $x_1$) is continuous the formula from Sen (1963) can be used to compute the variance $\sigma(\lambda_1, \lambda_2)$ of the normal distribution.

*Step 2:* Consider now the vector $[C_m^N(\gamma), C_m^1(\gamma)]$. Since a bivariate distribution is normal if (and only if) all linear combinations of its components are normal, we can use Step 1 to conclude immediately that $\sqrt{m}([C_m^N(\gamma), C_m^1(\gamma)] - \{[C(\gamma)]^N, C(\gamma)\})$ is asymptotically $\mathrm{N}(0, \Sigma)$, for a $2 \times 2$ matrix $\Sigma$ with $\Sigma_{11} = \sigma(1, 0)$, $\Sigma_{22} = \sigma(0, 1)$, $\Sigma_{12} = \Sigma_{21} = [\sigma(1, 1) - \sigma(1, 0) - \sigma(0, 1)]/2$.

[9] Originally $U$-statistics were defined for the case where $y_1, y_2, \dots$ are i.i.d. A symmetric function $h: R^N \to R$ is a kernel for $\mu$ if $\mu = \mathrm{E}h(y_1, \dots, y_N)$. Corresponding to the kernel $h$ there is a $U$-statistic $U(y_1, \dots, y_m) = \binom{m}{N}^{-1} \Sigma h(y_{i_1}, \dots, y_{i_N})$ where the summation is over all $\binom{m}{N}$ combinations of $N$ distinct elements $(i_1, \dots, i_N)$ from $\{1, \dots, m\}$.

*Step 3:* Finally consider $C_m^N(\gamma) - [C_m^1(\gamma)]^N = g[C_m^N(\gamma), C_m^1(\gamma)]$, where $g(u,v) = u - v^N$. Since $\sqrt{m}G(X_m)$ is asymptotically normal if $\sqrt{m}X_m \to (0, \Sigma)$ and $G$ has non-zero gradients (this uses what is known in statistics as the delta method) we have our desired result. More precisely,[10]

$$\sqrt{m}\{C_m^N(\gamma) - [C_m^1(\gamma)]^N\} \to \mathrm{N}(0, \sigma),$$

where

$$\sigma = [\partial g/\partial u, \partial g/\partial v]' \boldsymbol{\Sigma} [\partial g/\partial u, \partial g/\partial v] = \{1, N[C(\gamma)]^{N-1}\}' \boldsymbol{\Sigma} \{1, N[C(\gamma)]^{N-1}\}.$$

There are straightforward generalisations of the BDS theorem. Several forms of dependence can be treated with the use of theorems on $U$-statistics established by Denker and Keller (1983). These theorems allow one to deal with a strictly stationary process $x_1, x_2, \ldots$ which satisfies a mixing condition.

Since in this case the probability that two $N$ histories are within $\gamma$ of each other is no longer, in general, equal to the probability that two arbitrary points be no farther than $\gamma$, the statement of the result must change. Let $F_N$ denote the distribution of the vectors of $N$ histories, and

$$C^N(\gamma) = \iint \left[\prod_{k=0}^{N-1} \theta(\gamma - |x_{i+k} - x_{j+k}|)\right] dF_N(\mathbf{z}_i)\, dF_N(\mathbf{z}_j)$$

and

$$\mathbf{z}_l = (x_l, \ldots, x_{l+N-1}).$$

Then,

$$\sqrt{m}(C_m^N(\gamma) - [C_m^1(\gamma)]^N - \{C^N(\gamma) - [C(\gamma)]^N\}) \to \mathrm{N}(0, \sigma).$$

The computation of the variance of the estimator under any specific alternative hypothesis to i.i.d. can also be quite complicated. Hsieh and LeBaron (1988*a*) restate the theorem and propose numerical methods to implement the test under these conditions.

The BDS theorem can be used to test for departures from i.i.d. in data sets where linear tests failed. As such it has power against simple nonlinear deterministic systems that 'look' random from the linear viewpoint – as the 'tent' map mentioned above – as well as related nonlinear stochastic systems. Simulations reported in BDS, Brock *et al.* (1988), and Hsieh and LeBaron (1988*a*, *b*) show that it has good power against many of the favourite nonlinear alternatives. There are of course many other tests for nonlinearities.[11] Further this test has been used to detect departures from random walk behaviour in several economic time series including stock returns (cf. Scheinkman and LeBaron (1989*a*), LeBaron (1988)), foreign exchange rates (Gallant *et al.* 1988; Hsieh, 1989), and some macroeconomic time series (Brock and Sayers, 1988; Scheinkman and LeBaron, 1989*b*).

The fact that $C_m^N(\gamma)$ is a $U$-statistic, and that smooth functions of asymptotically normal variables are themselves asymptotically normal, can also be used to show asymptotically normal behaviour of statistics related to several of the measures discussed earlier as the following examples show.

[10] Cf. Serfling (1980), chapter 3.

[11] Some popular examples are in Engle (1982), Hinich (1982) and Tsay (1986).

*Example 1:* Note that $\lambda_1 C_m^N(\gamma_1) + \lambda_2 C_m^P(\gamma_2)$ is also a $U$-statistic for any vector of parameters $(\lambda_1, \gamma_1, N, \lambda_2, \gamma_2, P)$. For simplicity suppose again that the $x_j$'s are independent. Just as in Steps 1 and 2 of the proof of the theorem above one shows that the vector $\sqrt{m}\{C_m^N(\gamma_1) - [C(\gamma_1)]^N, C_m^N(\gamma_2) - [C(\gamma_2)]^N\}$ is asymptotically a bivariate Normal $\mathrm{N}(\mathbf{0}, \mathbf{\Sigma})$. Let $g(z,y) = \log z - \log y$. Then $g'(x,y) \neq 0$ and hence it follows from the delta method that

$$\sqrt{m}[\log C_m^N(\gamma_1) - \log C_m^N(\gamma_2) - N \log C(\gamma_1) + N \log C(\gamma_2)] \text{ is } \mathrm{N}(0, \sigma),$$

where $$\sigma = \{[C(\gamma_1)]^{-N}, [C(\gamma_2)]^{-N}\}' \mathbf{\Sigma} \{[C(\gamma_1)]^{-N}, [C(\gamma_2)]^{-N}\}.$$

In this way one can in theory compute confidence intervals for measures of the slope of log $[C^N(\gamma)]$ like the ones reported in Scheinkman and LeBaron's (1986). One difficulty lies in estimating the elements of $\mathbf{\Sigma}$. This, in principle, can be done even in the more interesting case where the $x_j$'s exhibit some dependence provided one is willing to assume enough mixing to apply the results from Denker and Keller. Again one must account for the fact that with dependence

$$\lim_{m \to \infty} C_m^N(\gamma) - [C_m^1(\gamma)]^N \neq 0$$

and that the dependence has also to be considered in the calculation of the asymptotic variance.

*Example 2:* Consider the quantity $S_m^N(\gamma) = C_m^{N+1}(\gamma)/C_m^N(\gamma)$ introduced by Scheinkman and LeBaron (1986). The expression

$$S^N(\gamma) = \lim_{m \to \infty} S_m^N(\gamma)$$

gives us the conditional probability that two points are no further than $\gamma$ given that their past histories of length $N$ are at least that close. Just as above, under independence, we can use the fact that the vector $\sqrt{m}\{C_m^{N+1}(\gamma) - [C(\gamma)]^{N+1}, C_m^N(\gamma) - [C(\gamma)]^N\}$ is asymptotically a bivariate Normal to infer that $\sqrt{m}[S_m^N(\gamma) - C(\gamma)]$ is itself asymptotically normal. Scheinkman and LeBaron (1989 *b*) contains a formula for estimating the variance of this distribution.

Frequently one is interested in finding nonlinear dependence on the residuals of particular models fitted to the data. In many macroeconomic time series, for example, low order autoregressive models are known to yield a good fit. In the analyses of foreign exchange rates, ARCH models (cf. Engle (1982)) were used by Hsieh (1989) to pre-filter the data.

In practice one can proceed as in Scheinkman and LeBaron (1989 *a*, *b*) to examine the distribution of the estimated residuals. First the model is estimated and a set of residuals is generated. These residuals are randomly reordered and data sets are then reconstructed using the estimated model. In each of these data sets one reestimates the model and measures the BDS statistics on the residuals. This 'bootstrap' like procedure is then used to determine the significance of the value of the statistics in the original residuals.

Another possibility is to use extensions of the BDS theorem that apply to the case where $x_j$'s are estimated residuals. Some of these are discussed in Brock (1988) and Brock *et al.* (1988).

IV

From the viewpoint of economic dynamics there seem to be two related properties of nonlinear systems of interest. The first one is that such systems can generate the quasi-periodic or even erratic behaviour that characterises some of the economic time series. If the true system exhibits nonlinear dependence then treating the time series as if it was generated simply by a linear stochastic difference equation will lead us to have an exaggerated view of the amount of randomness affecting the system. The second is that such nonlinear systems can generate sensitive dependency to initial conditions, i.e. small initial differences can be magnified by the dynamics. Of course that is a property shared by unstable linear systems but in the nonlinear case this sensitive dependence can occur while the system remains confined to a bounded region which is a necessary requirement in some economic applications. The study of this sensitive dependence to initial conditions is at the heart of nonlinear dynamics and attempts to measure this sensitivity in data generated by dynamical systems are helped by an extremely well developed mathematical theory.

Let us start with a deterministic system $x_{t+1} = f(x_t)$ with $f$ sufficiently smooth. If the initial state is disturbed the characteristic or Liapunov exponents measure the rate at which the initial perturbation increases (decreases). Let us write $x_t(x_0)$ for the solution that starts at $x_0$. Then, to a first order,

$$|x_T(x_0+y_0) - x_T(x_0)| = |f^T(x_0+y_0) - f^T(x_0)| \approx |Df^T(x_0)\, y_0|.$$

Recall that a probability measure $\rho$ is an invariant measure for $f$ if $\rho[f^{-1}(E)] = \rho(E)$ and that an invariant measure is called ergodic if $f^{-1}(E) = E$ implies that either $\rho(E) = 0$ or $\rho(E) = 1$.

Oseledec's theory (cf. Eckman and Ruelle (1985)) implies that under some regularity conditions if $\rho$ is an invariant measure for $f$ that is ergodic, for $\rho$-almost all $x_0$

$$\lambda(x_0, y_0) = \lim_{T\to\infty} [T^{-1} \log |Df^T(x_0)(y_0)|]$$

exists and equals one of possible $N$ values $\lambda^1 \geqslant \ldots \geqslant \lambda^N$. Further for almost all $y_0$ this limit equals $\lambda^1$. In other words for almost all choices of $x_0$ and infinitesimal $y_0$ the change at time $T$, $\delta x_T$ will satisfy

$$\delta x_T \approx y_0\, e^{\lambda^1 T}.$$

In particular, if $\lambda^1$ is positive, small changes in initial conditions will tend to be amplified through time, i.e. the system will exhibit sensitive dependence to initial conditions. If the system lies in a bounded set such amplification cannot go on forever and it is precisely this combination of boundedness and sensitivity that characterises chaotic dynamics. The $\lambda^i$'s are called the characteristic or Liapunov exponents of the map $f$.

These results remain true when one deals with a stochastic difference equation $x_{t+1} = f(x_t, \mu_t)$ where each $t$, $f(\cdot, \mu_t)$ is chosen at random independently and according to a fixed law (cf. Kifer (1986), for details). This, of course, is the case of interest for economic applications.

Eckman *et al.* (1986) construct an algorithm for computing Liapunov exponents from an experimental time series. Eckman *et al.* (1988) applied this algorithm to the CRSP value-weighted portfolio weekly returns and estimated $\lambda^1 = 0{\cdot}15$/week. Since the distribution theory for this estimate is unknown, this value can at this time be taken only as suggestive.

Much attention had been paid recently to the existence of unit roots in macroeconomic time series as well as the implications of the presence of such unit roots. Quah (1987) constructs a stochastic process $y_t$ where the conditional expectation of $y_{t+\tau}$ satisfies $E(y_{t+\tau}|y_t) = \Gamma^\tau y_t$, with $|\Gamma| \geqslant 1$, but none the less possesses a stationary distribution with zero mean. The conditional expectation shows in the case where $|\Gamma| > 1$ a tendency to diverge and in the case where $\Gamma = 1$ no tendency to settle down. This property is of course shared by the usual (linear) unit-root processes, but this has no implications concerning the stationarity of the process. He further argues that this distinction between unit-roots and lack of stationarity – that is missed in much of the macroeconomic literature on unit roots – may be empirically relevant by examining US aggregate output.

Let us consider a system

$$x_t = h(x_{t-1}) + w_t, \tag{5}$$

where $x_t$ lies in a subset of $R^N$, $w_t$ is i.i.d. and such that an ergodic measure $\rho$ exists. A possible definition of 'locally explosive' conditional expectation is exactly that the largest Liapunov exponent is positive. Note that this definition involves changes in the state at time $t+\tau$ in response to a small (infinitesimal) shock $w_t$ at time $t$ as $\tau$ gets large and not simply changes at time $t+1$. Oseledec's theorem roughly states that the magnitude of this change is (with probability one) independent of either the state at time $t$, the direction of the shock or the ensuing history of $w_\tau$'s. In particular the conditional expectation of the magnitude of the change is independent of either the state at time $t$ or the direction of the shock. If the largest Liapunov exponent is positive then the system exhibits sensitive dependence and infinitesimal deviations are amplified. From a local point of view the system behaves as a linear systems with a root outside the unit circle. Obviously, if one considers a finite shock and the support of $\rho$ is compact then the magnitude of the change cannot exceed the diameter of the support of $\rho$. In spite of this sensitive dependence the process $x_t$ is stationary.

## V. CONCLUSION

The research we reviewed in this lecture is clearly in its initial stage. There is no guarantee that yet this attempt to bring nonlinearities to the centre of the study of economic dynamics will succeed. But the vast progress in the mathematics of nonlinear systems has already brought in some interesting dividends in economics. On the theoretical side it has clarified how complicated economic dynamics can be even in the most benign environment. On the empirical side it has led to the development of new tools to detect dependence. To be fair none of these developments are far enough along to bring about a change in the way economic practitioners proceed.

There are at least two directions that could prove specially useful for future work. The first one involves attempts to build explicit computable models that combine small amounts of randomness with nonlinearities and that succeed in generating data that replicate some of the aspects of economic or financial time series. The other is the development of a distribution theory for estimates of Liapunov exponents that would allow one to decide whether sensitive dependence is present on data.

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