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EQUILIBRIUM MODELS DISPLAYING ENDOGENOUS FLUCTUATIONS AND CHAOS
A Survey*

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This paper surveys current developments in dynamic general equilibrium theory concerned with conditions under which endogenous fluctuations are possible. Existing contributions are divided into two classes: the larger one consists of models with a unique perfect-foresight equilibrium, while the second includes models where equilibrium is indeterminate and the set of equilibria involving endogenous fluctuations includes 'sunspots equilibria'. While the two classes of models involve oscillations of different natures and with different policy implications, we show that both are consistent with optimizing behavior and competitive equilibrium.

1. Introduction

The idea that market mechanisms are *inherently dynamically unstable* has played a minor role in studies of aggregate fluctuations over the past quarter century. Instead, the dominant strategy, both in equilibrium business-cycle theory and in econometric modelling of aggregate fluctuations, has been to assume model specifications for which equilibrium is determinate and intrinsically stable, so that in the absence of continuing exogenous shocks the economy would tend toward a steady-state growth path.

Recent work, however, has seen a revival of interest in the hypothesis that aggregate fluctuations might represent an endogenous phenomenon that would persist even in the absence of stochastic 'shocks' to the economy.

The endogenous cycle hypothesis is not new. Indeed, the earliest formal models of business cycles were largely of this type, including most notably the business-cycle models proposed by John Hicks, Nicholas Kaldor, and Richard

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Goodwin.[1] By the late 1950's, however, this way of attempting to model aggregate fluctuations had largely fallen out of favor, the dominant approach having become instead the Slutsky–Frisch–Tinbergen methodology of exogenous stochastic 'impulses' that are transformed into a characteristic pattern of oscillations through the filtering properties of the economy's 'propagation mechanism'. There were probably three main reasons for the overwhelming popularity of the latter methodology, apart from whatever comfort may have been provided by a vision of the market process as fundamentally self-stabilizing.

First of all, endogenous cycle models are essentially nonlinear while the linear specifications that were possible in the case of the exogenous shock models were extremely convenient, both from the analytical point of view and from the point of view of empirical testing.

Second, the endogenous cycle hypothesis was thought to have been empirically refuted. Actual business cycles, it was easily shown, are far from being regular periodic motions.[2] The spectrum, for example, of an aggregate time series typically exhibits no pronounced peaks, let alone actual spikes as one would expect in the case of deterministic cycles. And econometric models were estimated that, when simulated with repeated exogenous stochastic shocks, produced data that looked like actual business cycles, but that when simulated without the exogenous shocks converged to a steady state [Adelman and Adelman (1959)]. Demonstrations of this kind appeared to show that the true structural relations implied an intrinsically stable economy.

These sorts of considerations have less force today than they must have seemed to around 1960. We have come to understand that the simple empirical 'refutations' of the endogenous cycle hypothesis do not prove as much as they might have seemed to. It is now understood that deterministic dynamical systems can generate *chaotic* dynamics, that can look very irregular and that can have autocorrelation functions and spectra that exactly mimic those of a 'stable' linear stochastic model, such as a stationary AR(1) model [Sakai and Tokumaru (1980)]. Furthermore, it is now recognized that the fact that a stable model gives the best fit within the class of models considered is no proof that the true (or more accurate) model may not be an unstable one that generates endogenous cycles. John Blatt (1978) showed that when a linear autoregression was fit to periodic data from a simulation of the Hicks cycle model, the parameter estimates implied a stable second-order autoregressive process for output, of the kind that is in fact obtained from autoregressions of actual GNP data.

[1] For overviews of the early (nonoptimizing, nonequilibrium) literature, see, e.g., Blatt (1983) or Lorenz (1989). For recent extensions of Goodwin's model, see Goodwin and Punzo (1988).

[2] Sir John Hicks, in private communication, has indicated that this was the reason for his loss of interest in endogenous cycle models.

Techniques that can, in principle, distinguish certain stochastic fluctuations from deterministic or even 'noisy' chaotic data have been developed in the natural sciences, especially among physicists [see the excellent discussion in Eckmann and Ruelle (1985)]. They have been refined, improved, and recently applied to economic data by Brock, Scheinkman, and their coauthors. We will not review this literature here, but instead refer the reader to Brock (1988) and Scheinkman (1990).

In general, the types of nonparametric tests for nonlinearity and endogenous instability that have been proposed seem to require quite large samples if reliable results are to be obtained [on this point see Ramsey and Yuan (1989) and Ramsey, Sayers, and Rothman (1988)], and this may well mean that definitive conclusions will not be possible in the case of economic time series, proceeding in this fashion. The question of the relative empirical validity of the exogenous and endogenous cycle hypotheses is likely to be decided only by comparing the predictions of theoretical models from both classes, whose parameters are either econometrically estimated or 'calibrated' after the fashion of Kydland and Prescott (1982). Thus far, no tests of this kind have been performed using endogenous cycle models (and admittedly, none of the available theoretical models appear likely to fare well under such a test – some reasons for which are discussed below). The development of theoretical models that could be tested in this way should be a major object of further research.

Another reason for the decline from favor of the endogenous cycle hypothesis concerns the inadequate behavioral foundations of the early models of this kind. The stability results obtained for many simple equilibrium models based upon optimizing behavior – in particular the celebrated 'turnpike theorems' for optimal growth models (discussed below) – doubtless led many economists to suppose that the endogenous cycle models were not only lacking in explicit foundations in terms of optimizing behavior, but depended upon behavioral assumptions that were necessarily inconsistent with optimization. This latter issue is the central focus of the present paper.

We survey the literature that shows that endogenous fluctuations (either periodic or chaotic) can persist in the absence of exogenous shocks, in rigorously formulated equilibrium models in which agents optimize with perfect foresight. We find it useful to divide the known examples into two categories. On the one hand (sections 2, 3, and 4) are models with a unique perfect-foresight equilibrium which involves perpetual fluctuations for most initial conditions. In such cases it is clear how the forces that bring about a competitive equilibrium also require the economy to exhibit endogenous fluctuations. On the other hand (sections 5 and 6) are models in which perfect-foresight equilibrium is indeterminate and among the large set of possible equilibria are ones in which the state of the economy oscillates forever. In cases of this sort, the forces that bring about competitive equilibrium do not require that perpetual fluctuations occur. While we regard the indeterminacy in cases of this sort to indicate a type of instability of the competitive process – in that arbitrary events can determine which equilibrium occurs – it is of qualitatively a different sort than in the case of models of the first type, which are the primary focus of the present survey.

We begin (section 2) with a brief discussion of an early nonoptimizing model of complex economic dynamics, because it allows us to raise some issues regarding the consistency of the postulated behavior with optimization that are then resolved in the more recent literature on optimizing models. We also discuss in the context of this simple example some of the techniques that can be used to demonstrate the existence of endogenous fluctuations. For reasons of brevity we have minimized the number of mathematical definitions and theorems used. The reader who wishes to know more about the mathematics of endogenous cycles and chaos may wish to consult with standard references as Collet and Eckmann (1980), Devaney (1986), Guckenheimer and Holmes (1983), Chow and Hale (1982), Lasota and Mackey (1985), and Ruelle (1989).

2. A simple example of complex economic dynamics[3]

Day (1982) considers a one-sector, neoclassical growth model in which the dynamics of capital accumulation has the form:

$$k_{t+1} = [s(k_t) \cdot f(k_t)]/(1+\lambda) = h(k_t), \tag{2.1}$$

where s is the saving function, f the production function, and $\lambda > 0$ is the exogenous population's growth rate. This is a discrete-time version of the famous Solow (1956) growth model. In the discrete-time form (2.1) Solow's assumption of a constant, exogenous saving rate and of a neoclassical, concave production function gives rise to a map $h(k_t)$ which is monotonically increasing and has one and only one interior steady state $k^* = h(k^*)$. A typical case is represented in fig. 1a. For example, in the case of a constant saving ratio σ and a Cobb–Douglas form for f, (2.1) becomes $k_{t+1} = \sigma B k_t^{\beta}/(1+\lambda)$ which is monotonic and therefore stable. Day argues that this result is not robust. The first modification he suggests is to the production function. By introducing a 'pollution effect' in it, one obtains

$$k_{t+1} = \sigma B k_t^{\beta}(m - k_t)^{\gamma}/(1+\lambda), \tag{2.2}$$

which gives chaotic accumulation paths at certain parameter values. Another

[3] While we concentrate here only on the growth model of Day (1982), it is worth mentioning that recent literature has also investigated the cyclic and chaotic properties of Keynes–Kaldor- and Goodwin-type models. The bibliography to this survey includes some of the relevant contributions in this area.

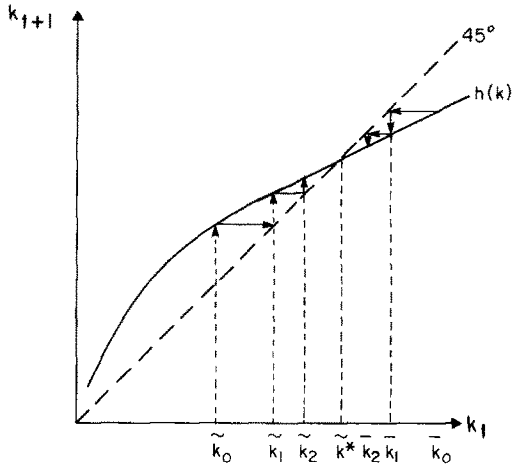

Fig. 1a

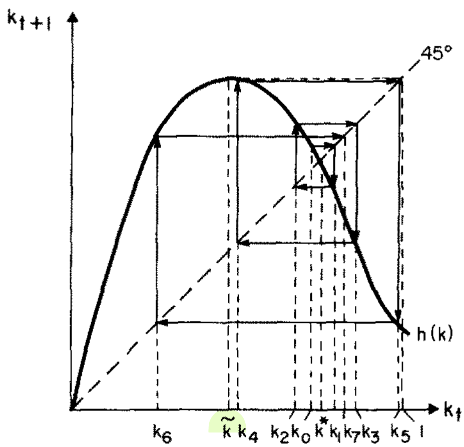

Fig. 1b

'complicated' accumulation rule is suggested in Day (1982) by returning to the Cobb–Douglas form for the production function and allowing instead for a variable saving rate, $s(k) = a(1 - b/r)k/y$,

$$k_{t+1} = \left[a/(1+\lambda)\right] k_t\left[1 - (b/\beta B)k_t^{1-\beta}\right]. \tag{2.3}$$

What is special in (2.2) and (2.3)? If one tries to picture the graphs of the $h(k_t)$ functions given by (2.2) and (2.3) something like fig. 1b will typically obtain. As the trajectory we have depicted there suggests, orbits need not converge to the steady state. The function is not monotonic, but hump-shaped. It maps some interval, say [0, 1], into itself and satisfies $h(0)=0$, $h(1)=0$, and, if $\tilde{k}$ denotes the unique critical point, $h(\tilde{k}) \leq 1$. Maps of the real line into itself that satisfy these properties are called 'unimodal'. They have been extensively studied by mathematicians and natural scientists as they represent the simplest kind of dynamical system that displays endogenous cycles and 'chaotic' behavior. We will provide here a few elementary properties of these maps that are needed to understand our discussion. A typical, and much studied, example is the quadratic map $h(x)=\mu x(1-x)$ for $\mu \in [1, 4]$, which we will also use here as an explanatory device because of its extreme simplicity.[4] We are considering difference equations

$$x_{t+1}=h(x_t), \tag{2.4}$$

with $h: I \to I$, where $I \subset \mathbb{R}$ is some bounded interval that we can identify with [0, 1] without loss of generality. For given $x_0 \in I$, (2.4) generates a sequence $\{x_t\}_{t=0}^{\infty}$ of points in I as $x_t = h^t(x_0)$;[5] we are interested in understanding where 'most' of these sequences will go when $t \to \infty$. Here 'most' is intended with respect to the Lebesgue measure on the set I of initial condition x_0. The 'destination' (if there exists one) of the iterative process $x_{t+1}=h(x_t)$ will be, for the time being, vaguely named an 'attractor'. Most likely our readers will be already familiar with the simplest ones, i.e., fixed points and periodic orbits of some finite period N. The first are those $x \in I$ such that $x=h(x)$, this set is denoted with $Fix(h)$. If, for some $N \geq 1$, we have $x=h^N(x)$, then x is periodic of period N and belongs to a set that we denote $Per_N(h)$. We can then define $Per(h)=\cup_{N=1}^{\infty} Per_N(h)$. Obviously $Fix(h) \subset Per(h)$. Then $x^* \in Per_N(h)$ is attractive if there exists a neighborhood B of x^* such that $\lim h^{NT}(x)=x^*$ as $T \to +\infty$ for all $x \in B$. A map h that has only periodic attractors of this kind is said to have 'simple attractors'. An extremely powerful theorem by Sarkovskij (1964) indicates how complex the set $Per(h)$ may be. He shows that if h is continuous, the set $Per(h)$ can be organized according to a precise ordering of the natural numbers such that, if h has a cycle of period N, then h has cycles of all periods m that precede N in Sarkovskij's ordering. The startling feature of this ordering is that while 1 is the smallest number and 2 is the one immediately following it, 3 is the largest. This in particular means that when a period-3 cycle occurs, then cycles of period N, for N *any natural* number, are simultaneously present (even if they need not, and will not in general, attract nearby orbits). This makes it clear

[4] While both eqs. (2.2) and (2.3) have qualitative properties very much similar to $\mu x(1-x)$, they are analytically more cumbersome to handle.

[5] Here $h^t(x)$ for $t=1,2,\ldots$ is defined as $h^t(x)=h(h^{t-1}(x))$ and $h^1(x)=h(x)$.

that h needs not be very complicated in order to generate very complicated dynamics. But one can show more: the existence of a period-3 cycle also implies the existence of a subset S of the domain of h (called the 'scrambled' set) on which 'chaotic dynamics' occurs. This is the celebrated 'period-3 implies chaos' theorem of Li and Yorke (1975). It has been widely used by economists because of its simplicity as it requires only checking the existence of a period-3 orbit in order to deduce the existence of 'chaos'.[6] Explaining the meaning of 'chaos' requires a few more concepts and definitions.

$Per(h)$ may not exhaust the set of 'attractors' for the orbits of h, in particular because an orbit may in fact have no clear destination and just move around I (or a subset of I) returning infinitely often to visit certain neighborhoods. This type of asymptotic recurrence is captured by the notion of the 'nonwandering' set.

Definition 1. The point $x \in I$ is *nonwandering* if there exists a sequence $x_n \to x$, $x_n \in I$, and a sequence $t_n \to \infty$ such that $h^{t_n}(x_n) \to x$. The collection of all such x's is called the nonwandering set and is denoted by $\Omega(h)$.

For monotonic maps the set $\Omega(h)$ turns out to be extremely simple. In fact if $h: I \to I$ is monotonic increasing, then $\Omega(h) = Fix(h)$, and if it is monotonic decreasing, then $\Omega(h) = \{Fix(h) \cup Per_2(h)\}$. This should be obvious from graphical inspection, plus the fact that when h is monotonic decreasing, h^2 is monotonic increasing. This also implies that we need h to be nonmonotonic in order to get trajectories more complicated than $Per_2(h)$.

Even if $h(x) = \mu x(1-x)$ is not monotonic it is not difficult to see that for $\mu < 3$, $\Omega(h) = Fix(h)$, while at $\mu = 3$ a first cycle of period 2 emerges that is attractive for $3 < \mu < 3.449499$. At $\mu = 3.449499$ a cycle of period 4 emerges. Similarly, attracting cycles of period 2^n emerge (following the Sarkovskij's order!) at the values μ_n, where $\mu_3 = 3.549090$, $\mu_4 = 3.564407$, $\mu_5 = 3.568759$, $\mu_6 = 3.569692$, etc. We describe later the 'bifurcation' through which a period 2^n cycle emerges from one of period 2^{n-1}.

But cycles of higher order are not the most complicated kind of possible asymptotic behavior. Intuitively we may say that a trajectory $x_{t+1} = h(x_t)$ is 'complicated' if: (a) it does not converge to any point in $Fix(h)$ or $Per(h)$, (b) it does not explode either to $\pm\infty$ (i.e., it stays within some bounded interval), (c) trajectories $x'_{t+1} = h(x'_t)$ starting from x'_0 'nearby' x_0 tend to 'move away' from $x_{t+1} = h(x_t)$ after enough iterations. This is only an intuitive

[6] Moreover there are a few simple, qualitative properties that guarantee the existence of a period 3. In fact let $h: I \to I$ be continuous, with I an interval, if there exist disjoint subintervals $I_1 \subset I$ and $I_2 \subset I$ such that $h(I_1) \supset I_2$ and $h(I_2) \supset (I_1 \cup I_2)$, then there is a period 3 for h. See Devaney (1986) for more details.

description to which one may attach different formal 'interpretations'. A simple one is:

Definition 2. Let $h: I \to I$ define a dynamical system as in (2.4). We say that h exhibits *topological chaos* if:

(i) for every period N, there exist points $x_N \in I$ such that $h^N(x_N) = x_N$;

(ii) there exists a nondenumerable set $S \subset I$ and an $\varepsilon > 0$ such that for every pair x and y in S with $x \neq y$:

$$\lim_{n\to\infty} \sup |h^n(x) - h^n(y)| \geq \varepsilon, \qquad \lim_{n\to\infty} \inf |h^n(x) - h^n(y)| = 0,$$

and for every $y \in Per(h)$ and $x \in S$:

$$\lim_{n\to\infty} \sup |h^n(x) - h^n(y)| \geq \varepsilon.$$

Li and Yorke (1975) prove that when h has a period-3 point, then it will have topological chaos in the sense just defined. Nevertheless this kind of chaos may not be very interesting to economists. The reason is that the scrambled set, S, may turn out to be of (Lebesgue) measure zero, i.e., the probability of starting in S is zero, while any initial condition outside S results in an orbit converging to a cycle of finite period. For the quadratic map this occurs, for example, at $\mu = 3.828427$, where topological chaos exists but almost all initial conditions lead asymptotically to a period-3 cycle. In the technical literature a notion of '*observable chaos*' or '*ergodic chaos*' has emerged. Ergodic chaos (loosely speaking) means: (a) S has positive (Lebesgue) measure (it has full measure, for example, for the quadratic map with $\mu = 4$) so that aperiodic trajectories are in fact observable, and (b) asymptotically the sequence $\{x_t\}_{t=0}^{\infty}$ obtained by iterating $h(x_t)$ approximates an ergodic and absolutely continuous distribution which is invariant under h and that summarizes the limiting statistical properties of the (deterministic) chaotic trajectories. Again, for the quadratic map with $\mu = 4$, an invariant distribution for $\mu x(1-x)$ exists, namely $f(x) = \{\pi[x(1-x)]^{1/2}\}^{-1}$. Unfortunately to prove that 'ergodic chaos' exists is not trivial. A relatively simple result, in the one-dimensional case, is the following:

Theorem [Lasota and Yorke (1973)]. Let h: $I \to I$ be piecewise C^2 and expansive, i.e., such that $\inf_{x \in I} |h'(x)| > 1$. Then h has an absolutely continuous invariant measure, whose support accordingly has positive Lebesgue measure. Moreover, if h is unimodal, then the measure is ergodic, so that for almost all

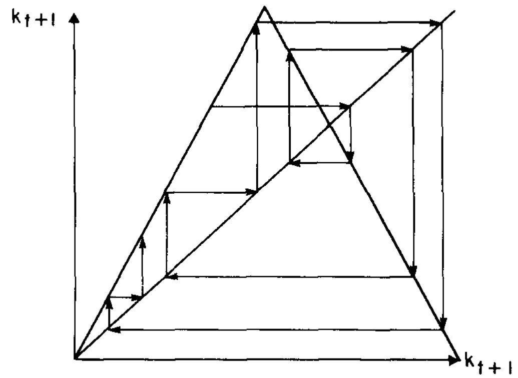

Fig. 2

initial conditions this measure describes the long-run frequency with which different neighborhoods are visited.

An example, indeed a classical one, of a map satisfying the conditions of the Lasota–Yorke theorem is given by the so-called 'tent map' (fig. 2) which has the general form:

$$h(x) = \begin{cases} \lambda x, & x \in \left[0, \dfrac{1}{\lambda}\right], \\ \dfrac{\lambda}{1-\lambda}(1-x), & x \in \left[\dfrac{1}{\lambda}, 1\right], \end{cases} \qquad \lambda > 1. \tag{2.5}$$

In this case it is easy to understand why orbits starting from a generic point in [0, 1] cannot converge to any cycle of a finite period: the slope of h is everywhere larger than 1 and therefore no fixed point of $h^N(x)$, for any $N \geq 1$, can possibly be attractive. On the other hand, any orbit has to stay bounded forever in [0, 1], so that it has to 'wander around' the unit interval and, eventually, visit every neighborhood of it, hence the ergodic behavior. The quadratic map for $\mu = 4$ also exhibits this behavior, since under a change of variables it can be shown to be equivalent to a tent map with $\lambda = 2$.

A (technical) word of caution should be added here. Point (c) in our 'intuitive' definition of chaos could be expressed, in the jargon of the field, as 'sensitive dependence on initial conditions'. For many mathematicians, this property (technically, the existence of 'positive Liapunov exponents') defines 'chaos'. We will not treat here this difficult technical concept; the curious reader is invited to consult Eckmann and Ruelle (1985). It should be pointed out, though, that neither 'topological' nor 'ergodic' chaos as defined here necessarily implies the existence of a positive Liapunov exponent. The map $4x(1-x)$ is, however, chaotic also in this third sense as it has a Liapunov exponent equal to $\log 2$.

To complete our survey of discrete-time dynamical systems, we need to say a few words about 'bifurcation theory'. Consider a continuous family of systems of the form (2.4) indexed by a parameter $\mu \in \mathbb{R}$, i.e., $x_{t+1} = h_\mu(x_t)$. Assume that for, say, $\mu < a$ the system h_μ has an attractor A of a certain kind (e.g., a stable fixed point). Assume that at $\mu = a$ the attractor A 'loses stability' (nearby orbits will not converge to A any more) and that a 'new attractor' (e.g., periodic cycle) $A' \neq A$ appears which is stable for $\mu \in [a, a+\varepsilon]$, where $\varepsilon > 0$, and that orbits near A now converge to A'. Then $\mu = a$ is a *bifurcation point* for the system and, in the jargon, we say that A' bifurcates from A at $\mu = a$. In the case of the family $h_\mu(x) = \mu x(1-x)$, $\mu = 3$ is a first bifurcation point as explained before. Obviously there are many different types of bifurcations, even in the case of simple one-dimensional maps; each different bifurcation produces a different new attractor A' from a given preexisting attractor A. The theory is in fact very general; it applies in high dimensions, to both continuous- and discrete-time dynamical systems and to the study of singularities as well. A general reference is Chow and Hale (1982); simpler discussions may be found in the references given at the end of section 1.

A special, albeit rather pervasive, type of bifurcation for one-dimensional maps has often been used in economics; this is the *flip* or *period-doubling* bifurcation. Loosely speaking [see, e.g., Devaney (1986) for a precise statement], it says that if $x^* = h_\mu(x^*)$, $\mathrm{d}h_\mu(x^*)/\mathrm{d}x = -1$, and $\partial(\mathrm{d}h_\mu(x^*)/\mathrm{d}x)/\partial\mu \neq 0$ at $\mu = \bar{\mu}$, then (given a generically valid condition on higher derivatives) a stable cycle of period 2 will bifurcate from x^* for μ in a (right or left) neighborhood of $\bar{\mu}$. Notice that, replacing h with h^N in the former, one gets a period $2N$ cycle out of one of period N.

The period-2 orbit emerging at $\mu = 3$ for $\mu x(1-x)$ is created by means of a flip bifurcation, as are all the other subsequent ones of period 2^n at the parameter values μ_n listed before. A cascade of period-doubling bifurcations (following along the 'power of two' portion of the Sarkovskij ordering) is a frequent (albeit not the unique) pattern observed for unimodal maps in the 'transition to chaos'. Once again $\mu x(1-x)$ perfectly respects this pattern. It is now time to go back to economics.

Day's examples generate families of unimodal maps that illustrate these propositions. Thus, they show that extremely simple behavioral hypotheses and model structures can produce very complicated dynamics. However, one may nevertheless question whether the sort of behavior assumed is in fact consistent with optimization within the assumed environment. For example, the assumption of a constant saving ratio was often used in the early 'descriptive' growth models and can indeed be derived from intertemporal utility maximization under certain hypotheses, but it becomes especially implausible when a production function of the type embodied in (2.2) is proposed. Why should a maximizing agent ever save up to the point at which marginal returns to capital are negative if he can obtain the same output level with much less capital stock?

Although it is less obvious, Day's case of a variable saving ratio and a monotonic production function [i.e. eq. (2.3)] is equally inconsistent with intertemporal utility maximization at least in the case of a representative consumer model. This was pointed out (in a general form) in Dechert (1984). If the consumer-producer maximizes $\sum_{t=0}^{\infty} u(c_t)\delta^t$, where u is concave and δ is in (0, 1), even if the production function is not concave, the optimal program $\{k_0, k_1, k_2, \dots\}$ can be expressed by a policy function $k_{t+1} = \tau(k_t)$ which is monotonically increasing. The economic prediction is accordingly that such a society will asymptotically converge to some stationary position. The latter is unique when f is concave. From this we have to conclude that the chaotic examples derived from a one-sector growth model would not pass the rationality critique. Such a critique turns out to be rather weak itself, as it holds true only for the special type of growth model considered above. This will be illustrated in the next two sections.

3. Optimal growth models

The kind of strong characterization given above of the qualitative properties of equilibrium dynamics turns out not to be possible even for slightly more general optimal growth models, perfectly competitive dynamic economies in which all agents are identical and optimize over the entire infinite horizon of the economy.[7] We will fully describe only the discrete-time representation of such economies, though we also discuss a continuous-time version of the same model the translation should be immediate.[8]

In every period $t = 0, 1, 2, \dots$, a representative agent derives satisfaction from a 'consumption' vector $c_t \in \mathbb{R}^m_+$, according to a utility function $u(c_t)$ which is taken to be increasing, concave, and as smooth as needed. The state of the world is described by a vector $x_t \in \mathbb{R}^n_+$ of stocks and by a feasible set $F \subset \mathbb{R}^{2n}_+ \times \mathbb{R}^m_+$ composed of all the triples of today's stocks, today's consumptions, and tomorrow's stocks that are technologically compatible, i.e., a point

[7]See Bewley (1982) and the literature quoted therein for a reconciliation of the abstraction of a single representative agent that controls both consumption and production decisions with the case of many independent consumers and producers.

[8]The reader is referred to Cass and Shell (1976), Bewley (1982), Becker and Majumdar (1989), and especially McKenzie (1986, 1987) for more complete treatments.

in F has the form (x_t, c_t, x_{t+1}). Now define:

$$V(x, y) = \max_c u(c) \quad \text{s.t.} \quad (x, c, y) \in F, \tag{3.1}$$

and let $D \subset \mathbb{R}_+^{2n}$ be the projection of F along the c coordinates. Then V, which is called the short-run or instantaneous return function, will give the maximum utility achievable at time t if the state is x and we have chosen to go into state y by tomorrow. It should be easy to see that to maximize the discounted sum $\sum_{t=0}^{\infty} u(c_t)\delta^t$ s.t. $(x_t, c_t, x_{t+1}) \in F$ is equivalent to $\max \sum_{t=0}^{\infty} V(x_t, x_{t+1})\delta^t$ s. t. $(x_t, x_{t+1}) \in D$. [Here the parameter δ indicates the rate at which future utilities are discounted from today's standpoint (impatience) and takes values in $(0,1)$.]

It is mathematically simpler to consider the problem in the latter (reduced) form. Standard neoclassical assumptions on u and F would typically yield a pair V and D satisfying the following properties. The return function $V(x, y)$ is strictly concave, increasing in x and decreasing in y. The technology set $D \subset X \times X \subset \mathbb{R}_+^{2n}$ is convex and compact. X is the feasible set which is also convex and compact. The initial state x_0 is given.

The optimization problem we are facing can be equivalently described as one of dynamic programming [see Stokey et al. (1989) for the details of this derivation]. The value function for such a problem is defined by the Bellman equation:

$$W(x) = \max_y \{ V(x, y) + \delta W(y) \quad \text{s.t.} \quad (x, y) \in D \}. \tag{3.2}$$

A solution to (3.2) will be a map $\tau_\delta : X \to X$ describing the optimal sequence of states $\{x_0, x_1, x_2, \dots\}$ as a dynamical system $x_{t+1} = \tau_\delta(x_t)$ on X. The time evolution described by τ_δ contains all the relevant information about the dynamic behavior of our model economy. In particular, the price vectors p_t of the stocks x_t that realize the optimal program as a competitive equilibrium over time follow a dynamic process that (when the solution $\{x_t\}$ is interior to X and V is smooth) is homeomorphic to the one for the stocks. In other words, $p_{t+1} = \theta(p_t)$ with $\theta = \delta W' \circ \tau \circ (W'\delta)^{-1}$, where W' is the first derivative of the value function.

Accordingly, the question that concerns us is: what are the predictions of the theory about the asymptotic behavior of the dynamical system τ_δ? In particular, do competitive equilibrium and perfect foresight imply convergence to a stationary state as in the one-sector growth model? A first, remarkable answer is given by the following:

Turnpike theorem (discrete time). *Under the maintained assumptions, for given V and D there exists a level $\bar{\delta}$ of the discount factor such that, for all δ in the* *nonempty interval* $[\bar{\delta}, 1)$, *the function* τ_δ *that solves (3.2) has a unique globally attractive fixed point* $x^* = \tau_\delta(x^*)$. *Such an* x^* *is also interior to* X *under additional mild restrictions.*

This means that under a set of fairly general hypotheses we are able to predict that, if people are not 'too impatient' relative to the given V and D, then they should move toward a stationary state where history repeats itself indefinitely and no surprises ever arise.[9]

As remarkable as it is, the turnpike property requires rather special conditions. In particular, how close must $\bar{\delta}$ be to one, and what happens when δ is smaller than $\bar{\delta}$? These are important questions. It is hard to rely heavily on a property that may depend critically on such a volatile and not directly observable factor as 'society's average degree of impatience'.

It turns out in fact that, as the discount factor moves away from $\bar{\delta}$ toward zero, τ_δ may become 'practically anything'. This was proved in Boldrin and Montrucchio (1986b) [but see also Boldrin and Montrucchio (1984) and (1986a) and Montrucchio (1986) for additional results]:

Theorem (anti-turnpike). *Let* $\theta: X \to X$ *be any* C^2*-map describing a dynamical system on the compact, convex set* $X \subset \mathbb{R}^n$. *Then there exist a technology set* D, *a return function* V, *and a discount factor* $\delta \in (0,1)$ *satisfying the maintained hypotheses and such that* θ *is the policy function* τ_δ *that solves (3.2) for the given* D, V, *and* δ.

The proof is of a constructive type, so that one may effectively compute a fictitious economy for any desired dynamics. This 'general possibility theorem' makes clear that any kind of strange dynamic behavior is fully compatible with competitive markets, perfect foresight, decreasing returns, etc. Independently Deneckere and Pelikan (1986) also presented some one-dimensional examples of models satisfying our assumptions and having the quadratic map $4x(1-x)$ as their optimal policy function for appropriately selected values of δ. Also, Neumann et al. (1988) provide a technical improvement on the Boldrin–Montrucchio construction that, after a slight modification, enables one to derive the classical chaotic map of Henon (1976): $\tau_\delta(x, y) = (y, 1 + 0.3x - 1.4y^2)$ for $\delta = 0.20$, which is twenty times larger than the initial estimates [see Boldrin and Montrucchio (1989, ch. 3) for this and other examples].

These results may contrast with the common intuition according to which infinite-horizon, concave programming problems should yield optimal policies

[9] In the form given here the Turnpike Theorem is due to Scheinkman (1976). McKenzie (1976) and Rockafellar (1976) proved it for the continuous-time version (on this point see the discussion below), and Bewley (1982) and Yano (1984) generalized it to the many-agents case [but see McKenzie (1986) for a more careful attribution of credit].

that are 'dynamically regular' as concavity typically penalizes oscillations. In order to see the economic reasons for this it is worth looking at some specific and simple examples. Let us begin with the one-sector model we briefly introduced at the end of section 2, and which was used by Dechert to prove that cycles and chaos are not optimal in that framework. This is a special case of the general model we are considering, with $V(x_t, x_{t+1}) = u[f(x_t) - x_{t+1}]$ and $D = \{(x_t, x_{t+1}) \text{ s.t. } 0 \leq x_{t+1} \leq f(x_t)\}$. For that model the Turnpike Theorem holds independently of the discount factor as τ_δ is always monotonic increasing. Unfortunately, such a nice feature does not persist even if the simplest generalization of the one-sector model is taken into account.

This was first proved by Benhabib and Nishimura (1985). They considered a model with two goods – consumption and capital – which are produced by two different sectors by means of capital and labor. Given the two concave and homogeneous of degree-1 production functions, one can then define a Production Possibility Frontier (PPF) $T(x_t, x_{t+1}) = c_t$, that gives the producible amount of consumption when the aggregated capital stock is x_t (a scalar), labor is efficiently and fully employed, and tomorrow's stock must be x_{t+1}. The return function is now $V(x_t, x_{t+1}) = u[T(x_t, x_{t+1})]$ and $D = \{(x_t, x_{t+1}) \text{ s.t. } 0 \leq x_{t+1} \leq F(x_t, 1)\}$, where F is the production function of the capital good sector and labor has been normalized to one. In such a case τ_δ is not always upward-sloping. If the consumption sector uses a capital–labor ratio higher than the one used by the capital sector it will be downward-sloping. Let x^* be the (unique) interior fixed point [i.e., $\tau_\delta(x^*) = x^*$]. This is the candidate for the turnpike. Assume, for simplicity, that τ_δ is differentiable in a neighborhood of x^*. The derivative $\tau'_\delta(x^*)$ at the steady state changes as δ moves in $(0,1)$, everything else equal. Benhabib and Nishimura show that it may take up the value -1 for admissible δ's, in such a way that the conditions for a flip (period-doubling) bifurcation are realized. In this case an optimal cycle of period 2 will exist which also is attractive: no more turnpike! One may provide examples of this phenomenon showing that such an outcome is by no means due to 'pathological' technologies and preferences.

One may go even further and show that cycles of every period as well as chaos (in the sense of topological chaos) may arise in the same class of two-sector optimal growth models. A theoretical analysis is provided in Boldrin (1986). It is proved that the policy function $x_{t+1} = \tau_\delta(x_t)$ is unimodal when (for example) factor-intensity reversal occurs between the two sectors (this is not strictly necessary). Suppose that there is a level, say k^*, of the aggregate capital stock such that when k_t is in $[0, k^*)$ the capital sector has a higher capital–labor ratio, whereas the opposite is true when k_t is in $(k^*, \bar{k}]$, where $\bar{k}$ is the maximum level of capital that the economy can sustain. This technological feature provides the underlying reason for the unimodal shape for τ_δ (i.e., τ_δ is as in fig. 1b). Variations in the level of the discount factor δ then can produce a cascade of period-doubling bifurcations that (technicalities aside) leads to period-3 orbits and chaos. By means of a simple example where the two production functions are respectively CES (consumption sector) and Leontief (investment sector) it can be shown that even very standard technologies allow for period-doubling bifurcations at certain parameter values. This example is fully worked out in Boldrin and Deneckere (1987) for the case in which the production function for the consumption sector is Cobb–Douglas.[10] Assuming, in order to simplify the algebra, that the utility function is linear, the short-run return function $V(k_t, k_{t+1})$ may be written as $(1 - k_{t+1} + \mu k_t)^{\alpha} \cdot (k_t(1 + \gamma\mu) - \gamma k_{t+1})^{1-\alpha}$, where α is the Cobb–Douglas coefficient, $0 < \gamma < 1$ is the capital–labor ratio in the investment sector and $(1 - \mu)$ is the capital depreciation rate. The value $k_t = \gamma$ corresponds to the critical point of the policy function; for all $k_t < \gamma$, the investment sector is more capital–labor-intensive and, therefore, any increase in the capital stock today will make it optimal to further increase capital tomorrow (τ_δ is upward-sloping), for $k_t > \gamma$ the opposite is true and the substitution effect along the production possibility frontier makes it optimal to have smaller stocks of capital k_{t+1} associated to larger stocks k_t (τ_δ is downward-sloping). The reader may notice that the basic logic behind this is the same as with the Rybczynski theorem. Once properly parameterized the economy displays various types of dynamic behavior, from the simple convergence to a stationary state, to cycles of different finite periods, to 'chaos'. In particular, for any given level of discounting δ, a stable period 2 can always appear for values of α, γ, and μ in the unit interval. The same in fact seems to be true for cycles of orders 2^n. It should also be noticed that the technological parameter values at which this occurs are rather extreme and they become even more so when chaos is obtained. A typical chaotic triple would have $\alpha = 0.03$, $\gamma = 0.09$, $\delta \simeq 0.1$ ($\mu = 0$ here, but similar sets of parameters work for μ next to 1). If one goes back to the general CES formulation things improve only slightly. For levels of the elasticity of substitution in production that are not too extreme, aperiodic motions appear when the discount factor is in the range (0.2, 0.3) (this still implies an interest rate around 400%). When the elasticity of substitution becomes extremely small (i.e., in the range 0.01, 0.02), then chaos is present also for less unreasonable levels of discounting, such as 0.7 or 0.8.

Cycles are not special to the discrete-time version of such models. In a very early work Magill (1979) had pointed out that cyclical (albeit converging to a steady state) motions were possible for solutions to undiscounted continuous-time optimization problems. He was able to show that the origin of oscillations along optimal trajectories is directly related to the existence of asymmetries in the Hessian function of the short-run maximand evaluated at the steady state. The use of a model without discounting prevented him from making these

[10] Ivar Ekeland and José Scheinkman had conjectured earlier on that such a parametric form may lead to irregular trajectories [Scheinkman (1984)].

oscillatory motions persistent and from proving the existence of limit cycles. This was achieved in Benhabib and Nishimura (1979). The two authors use bifurcation theory for ordinary differential equations to prove that limit cycles can occur [consult again Chow and Hale (1982) under 'Hopf bifurcation' for the details]. Let us show very briefly how this can happen. In continuous time we face an optimal control problem of the form:

$$\max \int_0^\infty V(x, \dot{x}) \exp(-\rho t) \mathrm{d}t \quad \text{s.t.} \quad (x, \dot{x}) \in D, \quad x(0) \text{ given}. \tag{3.3}$$

Here $x(t)$ is a vector depending on time, $\dot{x}$ is its time derivative, D is again the convex feasible set, and ρ is the discount factor in $[0, \infty)$. (Note that $\rho = 0$ is equivalent to $\delta = 1$ in discrete time.) Using the Maximum Principle one defines a Hamiltonian

$$H(x, q) = \max_{\dot{x}} \{ V(x, \dot{x}) + \langle q, \dot{x} \rangle \quad \text{s.t.} \quad (x, \dot{x}) \in D \}, \tag{3.4}$$

which can be interpreted as the current value of national income evaluated at the (shadow) prices q [on this point see Cass and Shell (1976)].

The solution to (3.3) can then be characterized by the dynamical system:

$$\begin{aligned} \dot{x} &= \partial H(x, q)/\partial q, \\ \dot{q} &= -\partial H(x, q)/\partial x + \rho q. \end{aligned} \tag{3.5}$$

Linearization of (3.5) around the steady state will yield, after some manipulations, a Jacobian matrix J that can be written as $J = \tilde{J} + (\rho/2)I$, where I is the $2n \times 2n$ identity matrix. As $\tilde{J}$ is a Hamiltonian matrix, we may consider how its eigenvalues will change with the discount factor ρ, and then add $\rho/2$ to obtain those of J. If $\rho = 0$, it is a well known result that under strict concavity in x and strict convexity in q of H the $2n$ eigenvalues of $\tilde{J}$ will split into n positive and n negative ones. The steady state will be a saddle point with a stable manifold of dimension n and the optimal program will steer the system on the stable manifold thereby guaranteeing convergence to the turnpike. For $\rho > 0$, this is not necessarily true; the saddle-point property may be lost as some of the negative eigenvalues become positive.[11] The turnpike theorems give sufficient conditions under which the stability property of the

[11]It remains true, and global convergence is assured, in the special case in which $\partial^2 H(x, q)/\partial^2 q = [-\partial^2 H(x, q)/\partial^2 x]^T$. This was proved in Magill and Scheinkman (1979). When this symmetry condition does not hold, there is room for oscillations.

saddle point is preserved for small ρ. For the purposes of this discussion a particularly useful form of turnpike theorem is the one proved by Rockafellar (1976) (see also the related Cass and Shell paper in the same issue of the *Journal of Economic Theory*). As we pointed out, the Hamiltonian $H(x, q)$ is concave in x and convex in q; we say it is α-concave in x if $H(x, q) + (\alpha/2)\|x\|^2$ is still concave on its domain of definition for all feasible q and $\alpha > 0$, and that it is β-convex in q if $H(x, q) - (\beta/2)\|q\|^2$ is convex in q on its domain for all admissible x and $\beta > 0$. Then one has:

Turnpike theorem (continuous time). *Suppose the Hamiltonian given in (3.4) is α-concave and β-convex in a convex neighborhood of $(\bar{x},\bar{q}) \in \mathbb{R}^n \times \mathbb{R}^n$, where $(\bar{x},\bar{q})$ is a rest point for (3.5) [i.e., $\partial H(\bar{x},\bar{q})/\partial q = 0$ and $-\partial H(\bar{x},\bar{q})/\partial x + \rho\bar{q} = 0$]. Assume that the discount rate satisfies $\rho^2 < 4\alpha\beta$. Then (under a few additional technical conditions) for every initial condition (x_0, q_0), the unique solution $(x(t), q(t))$ to (3.5) that maximizes (3.3) converges to $(\bar{x}, \bar{q})$ as $t \to +\infty$.*

This version of the turnpike property is useful because it relates the level of discounting to the 'curvature' of the Hamiltonian which in turn depends (albeit in a very complicated way) on the curvatures of the technology and the preferences. The more concave–convex is H, the higher is the level of impatience compatible with regular dynamic behaviors. But as Benhabib and Nishimura (1979) showed, when ρ grows for fixed α and β, a pair (or more than a pair) of eigenvalues may change the sign of their real part by crossing the imaginary axis. In such a case (care taken for the technical details) a Hopf bifurcation occurs. The limit cycle associated with it turns out to be an attractor for the system (3.5). Once again the turnpike property is lost as people become more impatient.

Some characteristics of the oscillatory or chaotic paths so obtained in both versions of the optimal growth model need to be stressed. First of all they are realized as equilibrium paths, in the sense that all markets are continuously clearing at each point in time, prices adjust completely and no productive resource is unemployed. Moreover, they are Pareto-efficient in the sense that it is impossible to modify the allocation of resources that they imply, in order to increase the welfare of some agent without making somebody else worse off. The economic-policy implications of these facts are obvious and we do not intend to elaborate further on them. Secondly, oscillations here are strictly market-driven: it is the existence of certain factor–intensity relations across sectors that make it profitable for the producers (and the consumers alike) to invest, produce (and consume) in an oscillatory form. Even if all the prices are the 'right ones' (i.e., no conditions for profitable arbitrage exist), still the pure seeking of individual profits will bring out cyclic behavior.

We should point out here that an 'anti-turnpike' theorem is available also for continuous-time models. This was proved in Montrucchio (1987).[12] The proof proceeds essentially as in the discrete case and therefore permits the construction of fictitious economies that optimally evolve according to any prescribed law of motion $\dot{x} = f(x)$.

The extension to the continuous-time case is particularly useful in clarifying the extent to which a high rate of time discount is required for the existence of complex dynamics. In Boldrin and Montrucchio (1986a,b), Boldrin (1986), and Deneckere and Pelikan (1986), all of the parametric examples of chaotic optimal accumulation paths require very small discount factors, or equivalently high rates of time preference. Some have therefore concluded that chaotic oscillations are mere mathematical curiosa (at least in optimal growth models) and not a plausible line of research in business-cycle theory.

In fact, it is clear that appropriate choice of the technology and the single-period utility function can allow chaos to exist for as low a rate of time preference as one likes. This is clearest in the case of the continuous-time examples. In such examples, arbitrary rescaling of the time unit gives economies in which the rate of time preference (in, say, percent per year) can be arbitrarily low. Suppose that the economy characterized by V, ρ, and D exhibits endogenous cycles or chaos. Then consider a new economy with an objective function $\tilde{V}(x, \dot{x}) = V(x, \varepsilon^{-1}\dot{x})$, a discount factor $\tilde{\rho} = \varepsilon\rho$, and a feasible set $\tilde{D} = \{(x, \dot{x}) \mid (x, \varepsilon^{-1}\dot{x}) \in D\}$. This will also be a standard optimal growth model, and if the equilibrium law of motion for the original economy was $\dot{x} = f(x)$, the law of motion for the new economy will be $\dot{x} = \tilde{f}(x) = \varepsilon f(x)$. The new economy will also exhibit cycles or chaos, since the dynamical systems are equivalent, but by choosing $\varepsilon \ll 1$, we can make $\tilde{\rho}$ arbitrarily small. It is straightforward to conclude that any kind of dynamics can be made optimal at any level of discounting, no matter how small the latter is [Boldrin and Montrucchio (1989, ch. 3)].

This does not contradict the turnpike theorem of Rockafellar quoted above, because the Hamiltonian of the new economy is given by $\tilde{H}(x, q) = H(x, \varepsilon q)$, so that if H was α-concave and β-convex, $\tilde{H}$ is α-concave but only $\beta\varepsilon^2$-convex. Thus as ε is made small, the degree of convexity of the Hamiltonian becomes small as well. Before it is possible to conclude that empirically realistic values for the rate of time preference are 'too low' to allow endogenous cycles, one must discuss what is an empirically realistic degree of curvature for the Hamiltonian.[13]

[12] A similar result was obtained, independently, by Sorger (1988).

[13] We are indebted to David Levine for discussion of this point. A similar point is made by Benhabib and Rustichini (1989), who show that for any discount rate ρ greater than the rate of depreciation μ of capital stocks, one can find parameter values for a three-sector Cobb–Douglas production technology that results in an optimal growth path attracted to a limit cycle. Their construction assumes a linear utility function, which helps to keep the curvature of the Hamiltonian low.

4. Models with market imperfections and determinate equilibrium dynamics

In the event that markets are incomplete, imperfectly competitive, or otherwise less than fully efficient, the conditions under which endogenous equilibrium fluctuations can occur are less stringent. There is no general 'turnpike' theorem for this case, in fact, it is possible to construct economies in which borrowing constraints permit one to make consumers' rate of time preference arbitrarily small while continuing to have endogenous cycles or chaos. Nor is the kind of intersectoral relations in the production technology considered earlier necessary in order for endogenous fluctuations to occur; for example, endogenous cycles and chaos can occur even in the case of a one-sector production technology.

A simple type of market imperfection is an assumption that agents are unable to borrow against all types of future incomes. The first demonstration that borrowing constraints could make endogenous cycles possible even in the case of a finite number of infinite-lived consumer types and a one-sector production technology was due to Bewley (1986).[14] Bewley showed how borrowing constraints could result in equilibrium dynamics in such a model formally analogous to the capital-accumulation paths that could occur in the overlapping-generations model of Diamond (1965). Bewley's result depends upon specifications that typically also imply indeterminacy of perfect-foresight equilibrium. Hence further discussion of this example is deferred to section 6.

Another model based on borrowing constraints is examined by Woodford (1988b). In this economy, there are two types of infinite-lived consumers – workers, who supply labor inputs to the production process, and entrepreneurs, who own the capital stock and organize production and hence who make the investment decisions. Workers are assumed to be unable to save by accumulating physical capital and organizing production themselves. There is also a limitation upon the extent to which workers can indirectly invest in productive capital by lending to entrepreneurs. In the simplest case loan contracts are assumed to be completely unenforceable. In this case workers must consume each period exactly the wage bill, and entrepreneurs must finance investment entirely out of retained earnings from that period. The capital stock in each period will then be equal to the previous period's gross returns to capital times the fraction of their wealth that entrepreneurs do not wish to consume. This can easily result in a unimodal map $k_{t+1}=f(k_t)$ of the form shown in fig. 1b of section 2, since gross returns to capital will be a decreasing function of the capital stock if labor supply is sufficiently inelastic at high levels of labor supply, and capital is not too easily substituted for labor. This is the case when, for example, the single-period utility function of the entrepreneurs is logarithmic in consumption. Then entrepreneurs will consume a constant fraction $(1-\delta)$ (δ being again the discount factor) of

[14]A somewhat similar type of borrowing constraint is considered in Scheinkman and Weiss (1986), but there exogenous stochastic shocks play a major role in sustaining oscillations.

their wealth in each period. If the preferences of workers are additively separable between periods, equilibrium labor supply will depend only upon the current real wage, so that it can be represented by a function $s(w_t)$. Assume that the latter is monotonically increasing. With a constant returns to scale production function $Y_t = F(k_t, l_t)$, the equilibrium real wage w_t will be determined by the current stock k_t as the unique solution $w(k_t)$ to the relation $F_l(k_t, s(w_t)) = w_t$. The unique equilibrium solution for the following period's capital stock will then be

$$k_{t+1} = f(k_t) \equiv \beta k_t F_k(k_t, s(w(k_t))). \tag{4.1}$$

The function $f(k)$ in (4.1) can easily be unimodal. For example, if workers' preferences are such that the labor-supply function is linear, $s(w) = mw$, and, in addition, the technology is Leontief, with $a > 0$ units of output being produced per unit of capital using $b > 0$ units of labor, then (4.1) becomes

$$k_{t+1} = \beta\left[ak_t - (b^2/m)k_t^2\right]. \tag{4.2}$$

Setting $\mu = \beta a^2 m/b^2$, (4.2) becomes the quadratic map discussed earlier in section 2, and chaos, both 'topological' and 'ergodic', will occur for $\mu \in [3.57, 4]$. (Note that for appropriate a, b, and m chaos may persist as β is made to approach 1. Hence there is no 'turnpike' property.)

The existence of ergodic chaotic dynamics can be assured for open sets of parameter values by constructing an example in which the map $f(k)$ in (4.1) is everywhere expanding. This requires a kink in $f(k)$ at the peak, but this can easily come about if, for example, the elasticity of labor supply is discontinuous at this point. If the elasticity of labor supply falls sufficiently greatly at the kink, it is possible for $f(k)$ to be sharply increasing before the peak and sharply decreasing thereafter.

Endogenous cycles and chaos in this type of economy do not depend upon the assumption of a Leontief technology, but it is important that the substitutability between capital and labor not be too great. For the fall in k_{t+1} for large values of k_t obviously depends upon total gross returns to capital being a decreasing function of the size of the current capital stock at that point, which is only possible if it is not possible to easily substitute capital for labor when the real wage rises. In fact, Hernandez (1988) proves a 'turnpike' theorem for a class of economies in which consumers cannot borrow against future labor income, under the assumption that the production technology allows sufficient substitutability between factors for total returns to capital to be a monotonically increasing function of the capital stock. It is not clear how far this result can be generalized, but it suggests that low factor substitutability may be important for the existence of endogenous instability even in more complicated examples. See also the discussion by Becker and Foias (1989).

Similar dynamics are also possible in the case of a much weaker restriction upon financial intermediation, namely, in the case that workers can lend to entrepreneurs, but debt contracts contingent upon firm-specific technology shocks are unenforceable. In such a case, if there is a continuum of firms with independent realizations of the technology shock, and the technology shock takes an appropriate form, the equilibrium real wage and aggregate production are deterministic functions of the aggregate capital stock despite the existence of a stochastic technology for each individual entrepreneur/firm. If entrepreneurs' preferences are homothetic, each entrepreneur will choose levels of consumption, investment, and borrowing that are proportional to the amount by which his gross returns in the current period exceed his debt commitment. As a result the deterministic (perfect-foresight) dynamics of the aggregate capital stock are independent of the distribution of capital holdings across entrepreneurs with different histories of technology shocks. In this model, if the technology shock occurring in the worst state that cannot be insured against is sufficiently bad, a low level of gross returns to capital will result in a low capital stock in the following period, even though total current income is high. Thus k_{t+1} can again be a decreasing function of k_t, for high values of k_t, and all of the phenomena discussed above can occur. The distribution of income continues to have a significant effect upon capital accumulation, despite the existence of a competitive market for (noncontingent) loans.

Financial constraints are not the only kind of market imperfections that can give rise to endogenous fluctuations even with an arbitrarily low rate of time discount. Deneckere and Judd (1986) demonstrate the possibility of endogenous fluctuations in the rate of introduction of new products in an economy in which the creation of a new product involves a one-time fixed cost and allows the innovator a one-period monopoly of production of the new product. The model is a simplified discrete-time variant of that of Judd (1985). Each period, new products are introduced to the point where the monopoly rents from the production of each new product are no greater than the fixed cost of creation of a new product. The total number of products produced in a given period (N_t) is a determinate function of the number of old (nonmonopolized) products in existence, and hence of N_{t-1}. Deneckere and Judd show that for a certain parametric class of preferences, N_t is a function of N_{t-1} similar to the one shown in fig. 2, only turned upside down.

Like the Wodford example, this one in no way relies upon high rates of time preference. Producers' decisions about whether to introduce new products do not depend upon the rate at which profits are discounted, because profits are obtained only for a single period and the fixed cost is paid in that same period. The level of monopoly rents depends only upon consumers' elasticity of substitution between different products within a given period, not upon their preferences regarding present as opposed to future consumption.

Neither of these examples can easily be parameterized for comparison with actual data, since each makes a number of very special assumptions in order to obtain equilibrium dynamics that can be described by a first-order nonlinear difference equation for a single state variable. This is, as of now, the only case in which a reasonably thorough analytical characterization of the types of possible asymptotic dynamics is available. Further progress in evaluation of the empirical relevance of the endogenous cycle hypothesis will doubtless depend upon the use of numerical simulations to determine whether chaotic fluctuations occur for realistic parameter values in more complicated models.

5. Indeterminacy and endogenous fluctuations in overlapping-generations models

Endogenous cycles and chaos again occur as equilibrium phenomena in the presence of complete and perfectly competitive intertemporal markets, also in the case of an economy made up of overlapping generations of finite-lived consumers. In this case, as noted earlier, the equilibria involving perpetual deterministic fluctuations are only some members of a large set of rational-expectations equilibria, which also includes equilibria converging to a stationary state. Nonetheless, this class of examples has been crucial for the development of modern interest in the endogenous cycle hypothesis, since it provided the first general equilibrium examples of the possibility of chaotic economic dynamics, through the work of Benhabib and Day (1982) and Grandmont (1985).

Consider the simple overlapping-generations model treated by Gale (1973). The economy consists of a sequence of generations of two-period-lived consumers, each identical in number. There is a single perishable consumption good each period of which each consumer has an endowment w_1 in the first period of life and w_2 in the second period of life. Each consumer born in period t seeks to maximize $U(c_{1t}, c_{2t+1})$, where c_{jt} is consumption in period t by consumers in the jth period of life and where U is a concave function increasing in both arguments. Finally, there exists a single asset, fiat money, in constant supply $M > 0$, all of which is initially held by a group of consumers who are already in their final period of life in the first period of the model.

Let p_t be the price of the consumption good in terms of money in period t. A perfect-foresight equilibrium price sequence $\{p_t\}$ must solve the difference equation

$$U_1\left(w_1 - \frac{M}{p_t}, w_2 + \frac{M}{p_{t+1}}\right) = U_2\left(w_1 - \frac{M}{p_t}, w_2 + \frac{M}{p_{t+1}}\right)\frac{p_t}{p_{t+1}}. \qquad (5.1)$$

The relation between M/p_t and M/p_{t+1} required by eq. (5.1) can be graphed

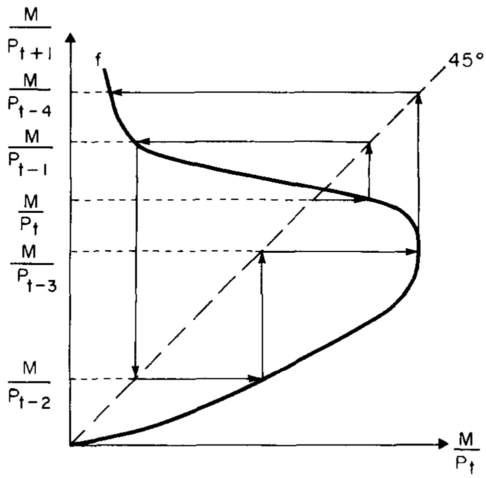

Fig. 3

as in fig. 3. When both first- and second-period consumption are normal goods, there will be a unique positive solution for M/p_t for each positive value for M/p_{t+1}, that we may write $M/p_t = f(M/p_{t+1})$. However, as indicated by the figure, the function f need not be invertible. The sort of unimodal function shown occurs if preferences are such that desired saving in youth is a decreasing function of the expected real return on money, for high enough levels of that return.

It is apparent that, when the unimodal map is steep enough [which Grandmont (1985) shows to require simply that the marginal utility of second-period consumption fall sharply enough with increases in second-period consumption, near the level of consumption that occurs in the stationary monetary equilibrium], the (backwards) perfect-foresight trajectories so traced can involve complex oscillations. As was first noted by Gale, deterministic cycles are possible. Cass, Okuno, and Zilcha (1979) showed that the deterministic cycles could be of arbitrary period, and Grandmont discusses in detail the order in which cycles of various periods occur as the map is made progressively steeper, applying the theory of unimodal maps set out in Collet and Eckmann (1980). All such cycles obviously represent possible equilibria of the forward perfect-foresight dynamics as well; i.e., they can be extended indefinitely into the future as well as indefinitely into the past.

Grandmont also discusses conditions under which the backward perfect-foresight dynamics would exhibit topological chaos. This result is of less obvious significance for the forward perfect-foresight dynamics, since the 'chaotic' property of a trajectory can be defined only in terms of its asymptotic behavior as it is continued indefinitely, and the existence of such a property for trajectories extended backward indefinitely from a date T does not necessarily imply anything about the kind of trajectories that exist going *forward* from a date T. Furthermore, it is clear that not just chaotic but *genuinely random* perfect-foresight trajectories do exist for the forward dynamics, albeit for reasons that do not require the use of the theory of nonlinear maps of the interval. For it is evident from fig. 3 that for many values of p_t there would be two different values of p_{t+1}, expectation of either of which would result in a market-clearing price of p_t. So we can construct a (forward) perfect-foresight equilibrium trajectory by starting at some arbitrary $p_0 \geq \underline{P}$ [where $M/\underline{P} \equiv \sup_{p>0} f(M/p)$], and proceeding recursively, for each value of t, choosing p_{t+1}, given p_t, so that (5.2) is satisfied, and also so that $p_{t+1} \geq \underline{P}$. It is clear that this iteration can be continued forever.

Let us suppose furthermore that the map f is so steep that $f(M/\underline{P}) < f^2(M/\underline{P}) < f^3(M/\underline{P}) < M/\underline{P}$, like the one in fig. 1b. Then, for any choice of p_0 so that M/p_0 lies within the open interval $I = (f(M/\underline{P}), M/\underline{P})$, it is also possible to continue the iteration forever, always choosing p_t so that M/p_t lies within I. Furthermore, each time M/p_t lies within the interval $I_2 = (f^2(M/\underline{P}), M/\underline{P})$, there are *two* possible choices of p_{t+1} which will continue the series. One may use a randomizing device (e.g., a coin toss) to chose between them. And each time M/p_t lies within the interval $I_1 = (f(M/\underline{P}), f^2(M/\underline{P}))]$, the unique available choice for p_{t+1} places M/p_{t+1} within I_2. Hence there is never more than one consecutive period in which one does not get to use the randomizing device. The resulting time series for the price level is accordingly quite random.

It is also known from the work of Shell (1977), Azariadis (1983), Azariadis and Guesnerie (1986), and many subsequent authors that genuinely stochastic ('sunspot') equilibria exist in this model in which consumers do not know the following period's price level with certainty. These equilibria correspond to stochastic processes for $\{p_t\}$ that satisfy the stochastic generalization of (5.1) obtained by applying the expectation operator E_t to both sides. The existence of such stochastic equilibria is closely related to the indeterminacy of perfect-foresight equilibrium, as is discussed further in Woodford (1984).

Benhabib and Day (1982) present a variant of the model in which the existence of chaotic perfect-foresight equilibria is of greater interest. Their model is the same, except that instead of assuming the existence of a fixed positive supply of fiat money, they assume the existence of government lending to the finite-lived private consumers. If the initial old consumers owe to the government real indebtedness of d_0 and if the government's lending policy each period is to lend (at a market-clearing real interest rate) a quantity exactly equal to the amount currently being repaid by the old consumers, then each period the government will lend $d_{t+1}/r_t = d_t$ to the young at a gross real interest rate of r_t. The obvious counterpart to the first-order conditions (5.1) implies that a sequence $\{d_t\}$ corresponds to a perfect-foresight equilibrium if and only if it satisfies

$$U_1(w_1 + d_t, w_2 - d_{t+1}) = U_2(w_1 + d_t, w_2 - d_{t+1})\frac{d_{t+1}}{d_t}, \tag{5.2}$$

for all $t \geq 0$, given the initial obligation d_0. Now eq. (5.2) is identical to (5.1) except that d_t replaces $-M/p_t$, and so the equilibrium dynamics can be studied using the same offer curve diagram as in fig. 3, except that now (assuming that the government is a net creditor) we are interested in the lower left rather than the upper right quadrant. Rotating the diagram by 180° we get back to fig. 1b. If consumption is a normal good in both periods of life, (5.2) can be solved for d_{t+1} as a unique function of d_t, which we may write $d_{t+1} = f(d_t)$. Furthermore, the new function f is exactly the same as the function f in Grandmont's model, if one constructs a Benhabib–Day economy by reversing the time pattern of endowments and preferences in the Grandmont economy [i.e., one replaces (w_1, w_2) by (w_2, w_1), and $U(c_1, c_2)$ by $U(c_2, c_1)$]. Hence Benhabib and Day's results on the conditions under which endogenous cycles and chaos are possible are parallel to Grandmont's. The map f can be sharply downward-bending, as shown in fig. 1b, if desired borrowing does not increase very much as the real rate of interest on loans falls, which in turn occurs if the marginal utility of first-period consumption falls rapidly with increases in first-period consumption. Again, equilibrium cycles and chaotic dynamics are possible. In Benhabib and Day's model, the chaotic paths relate to the *forward* perfect-foresight equilibrium dynamics, so they do indicate the possibility of chaotic equilibrium trajectories starting from a given initial condition. Furthermore, in the Benhabib–Day model, perfect-foresight equilibrium is unique, so that for certain initial conditions, a cyclic or chaotic trajectory will be the *only* possible equilibrium (assuming that the initial private-sector obligation d_0 is fixed in real rather than nominal terms).[15] It can however be objected that Benhabib and Day only give sufficient conditions for the existence of 'topological chaos', i.e., for a map f such that a chaotic equilibrium occurs for some initial values d_0, but perhaps only for d_0 in a set of zero measure. Hence their example does not clearly provide an explanation for the observation of apparently stochastic fluctuations. It should be pointed out, on the other hand, that 'ergodic chaos' can occur in their model, for some particular choices of preferences and endowments, but it is not clear how robust their examples are.

[15]One could reduce the degree of indeterminacy in the Grandmont model as well by treating it as a model in which outstanding government debt is rolled over forever, assuming that the initial outstanding government debt is given in real terms. Then the initial value of $-d_0 = M/p_0$ would be predetermined rather than arbitrary. But there would still exist stochastic equilibria because of the noninvertibility of the map f in that case.

Some have argued that the sharply backward-bending supply of savings (or demand for loans) as a function of the expected real return, upon which these examples depend, is not empirically plausible. Kehoe et al. (1986) show that deterministic cycles and chaotic equilibria are both impossible in a general stationary overlapping-generations model (allowing for an arbitrary finite number of goods per period, an arbitrary number of consumer types per operation, and an arbitrary finite number of periods of life per consumer), if all goods are *gross substitutes.*

However, the Kehoe et al. result applies only to endowment economies, i.e., economies in which there is no production. Reichlin (1986) shows that endogenous cycles are possible in an overlapping-generations economy with production, even when consumer demands for all goods satisfy the gross substitutes condition (savings are an increasing function of the real rate of return, labor supply is an increasing function of the real wage). Reichlin (1987) also shows that chaotic equilibrium dynamics are possible in an overlapping-generations economy with production. Further results relating to endogenous cycles and chaos in overlapping-generations models with production are presented by Farmer (1986), Benhabib and Laroque (1988), and Jullien (1988).

Sims (1986) has also criticized the relevance of the Benhabib–Day/Grandmont examples for business-cycle theory on the ground that, since the deterministic cycles must last for two or more periods and since in the OLG model the lifetime of consumers is only two periods, the examples show only that endogenous cycles are possible with *periods equal to or greater than a human lifetime.*

The conditions under which endogenous cycles might exist in overlapping-generations models with long-lived consumers have not yet been much studied. Aiyagari (1989) considers a family of overlapping-generations models in which the number of periods that each generation lives is made progressively longer, while the rate of time preference and the elasticity of substitution of consumption between periods remain constant and while each period's endowment continues to fall between the same upper and lower bounds. He shows that for any integer k, k-period deterministic cycles eventually cease to exist, within such a family of economies, once the lifetime T is made large enough. This provides some support for the view that short-period cycles are not likely to occur in economies with long-lived consumers. However, it should be noted that Aiyagari's result does not actually show that the cycles that are possible must always be long compared to the lifetime of consumers. Furthermore, the result is obtained only under relatively special assumptions: a special class of intertemporal preferences, lifetime endowment patterns that involve endowments bounded both above and away from zero, and no production.

Reichlin (1989) shows that deterministic cycles of period 2 are possible for arbitrarily long-lived consumers and an arbitrarily low rate of time preference, in a pure endowment economy and for preferences of the kind considered by

Aiyagari, if one abandons Aiyagari's assumption that endowments must remain bounded away from zero. Reichlin's results are similar to those of Benhabib and Day and of Grandmont in that a sufficiently low degree of intertemporal substitutability of consumption is shown to be necessary in order for the equilibrium cycles to exist. Whitesell (1986) provides some numerical examples, in the context of a continuous-time model with stochastic lifetimes for individuals, but a deterministic rate of death for each generation in aggregate, and a one-sector production technology with endogenous labor supply, showing that relatively short period cycles are possible in overlapping-generations models with relatively long-lived consumers.

The objection raised earlier to the Grandmont example of endogenous cycles, that there exist many other equilibria as well (including an equilibrium with a constant price level), can be answered if one asserts that perfect-foresight equilibrium is a relevant concept only when viewed as the eventual limit of a disequilibrium 'learning' process. In this case, it is possible that the stationary equilibrium would be *unstable* under the learning dynamics, which would instead converge, from most initial conditions, to one of the equilibrium cycles. Grandmont (1985) and Grandmont and Laroque (1986) give conditions upon the 'learning' process for this to occur. These results provide some reason to suppose that the deterministic cycles treated by Grandmont could represent observable phenomena. But no such results exist in the case of the chaotic perfect-foresight dynamics shown to be possible in the Grandmont model.

6. Indeterminacy and endogenous fluctuations in models with infinite-lived consumers and market imperfections

The overlapping-generations examples are also relevant as an indication of what can happen in economies with long-lived consumers facing various sorts of financial constraints. As noted in section 4, Bewley (1986) has shown that borrowing constraints in an economy with a finite number of infinite-lived consumer types can result in dynamics formally analogous to those in an overlapping-generations economy. An economy with equilibrium dynamics like those of the Grandmont (1985) example is discussed by Bewley (1980). [See also Sargent (1987, ch. 6).] Suppose that the economy is made up of equal numbers of two types of infinite-lived consumers, each of whom has preferences of the form $\sum_{t=0}^{\infty}\delta^t u(c_t)$, where $0<\delta<1$ and u is an increasing, concave function. Type-A consumers have an endowment of w_1 in all even periods and w_2 in all odd periods (where $w_1>w_2$), while the endowment pattern of type-B consumers is exactly the reverse. Suppose furthermore that debt contracts are unenforceable, so that neither consumer type is able to borrow against future income. Consumers can save only by holding fiat money, which exists in a fixed positive supply $M>0$. An equilibrium is possible in which the entire money supply is held at the end of each period by consumers of the type with endowment w_1 if the price level sequence $\{p_t\}$ satisfies the following sequence of conditions:

$$u'\left(w_1 - \frac{M}{p_t}\right) = \delta u'\left(w_2 + \frac{M}{p_{t+1}}\right)\frac{p_t}{p_{t+1}}, \tag{6.1}$$

$$u'\left(w_2 + \frac{M}{p_t}\right) \geq \delta u'\left(w_1 - \frac{M}{p_{t+1}}\right)\frac{p_t}{p_{t+1}}. \tag{6.2}$$

Condition (6.1) shows how p_t is determined in period t as a function of expectations regarding the value of p_{t+1}. This equation is of the form (5.1), and again cyclic or chaotic trajectories are among its solutions if the function u is sufficiently concave. It is necessary, however, in the present case also to check that the fluctuating solutions never violate the bound (6.2). This will necessarily be satisfied (give $\delta < 1$) in the case that $\{p_t\}$ fluctuates over a sufficiently small range, but large price fluctuations resulting in large variations in the marginal utility of consumption (as must occur in the case of chaotic dynamics) are consistent with (6.2) only if δ is significantly less than one.

If consumers smooth their consumption path by accumulating capital (rather than fiat money) in their high-endowment periods and one interprets the endowments as being of labor rather than of the consumption good, a similar model of infinite-lived consumers subject to borrowing constraints can mimic the dynamics of capital accumulation in the Diamond (1965) overlapping-generations model. In this model, endogenous equilibrium cycles are possible in the case that aggregate savings are a backward-bending function of the expected real return. In this way Bewley (1986) shows that endogenous cycles are possible. This example again depends upon a high rate of time discount ($\delta = 0.5$).

Woodford (1988a) shows that cash-in-advance constraints can result in dynamics similar to those of an overlapping-generations model with short lifetimes even when all agents are infinite-lived. Consider, for example, the cash-in-advance model studied by Wilson (1979), in which the infinite-lived representative consumer seeks to maximize

$$\sum_{t=0}^{\infty} \delta^t U(c_t, \bar{n} - n_t), \tag{6.3}$$

where c_t is consumption in period t and n_t is output supplied, $0 < \delta < 1$, and U is increase and concave in both arguments, subject to the sequence of budget constraints

$$p_t c_t \leq M_t, \tag{6.3a}$$

$$M_{t+1} = M_t + p_t(n_t - c_t), \tag{6.3b}$$

where M_t is money balances carried into period t. In the case of a constant money supply $M > 0$, perfect-foresight equilibria in which the cash-in-advance constraint (6.3a) always binds correspond to price sequences $\{p_t\}$ satisfying the conditions

$$U_2\left(\frac{M}{p_t}, \bar{n} - \frac{M}{p_t}\right) = \delta U_1\left(\frac{M}{p_{t+1}}, \bar{n} - \frac{M}{p_{t+1}}\right)\frac{p_t}{p_{t+1}}, \tag{6.4}$$

$$U_1\left(\frac{M}{p_t}, \bar{n} - \frac{M}{p_t}\right) \geq U_2\left(\frac{M}{p_t}, \bar{n} - \frac{M}{p_t}\right), \tag{6.5}$$

in all periods. Condition (6.4) indicates how the equilibrium price level p_t is determined by expectations regarding p_{t+1}. It is of the same general form as (5.1), and again cyclic or chaotic trajectories are among its solutions if U is a sufficiently concave function of its first argument. But, as in the case just discussed, it is also necessary to checks that the fluctuating solutions never violate the bound (6.5). Again this is necessarily satisfied (given $\delta < 1$) if $\{p_t\}$ fluctuates over a sufficiently small range, but δ much less than one is required in order for large fluctuations in the price level to be consistent with this condition.

Similar results are also possible in representative consumer monetary models of the Sidrausky–Brock type, in which services from real money balances are directly an argument of the utility function. Indeed, the Lucas–Stokey model is known to be formally analogous to a particular case of such a model. Matsuyama (1989a, b, c) demonstrates the possibility of deterministic equilibrium cycles, chaotic equilibrium trajectories, and 'sunspot' equilibria in a parametric class of models of this type.

Woodford (1986) also shows that equilibrium dynamics similar to those occurring in an overlapping-generations model with production can result in a model with infinite-lived workers and entrepreneurs, from the existence of a cash-in-advance constraint upon workers' consumption purchases, together with a constraint of the sort discussed in section 4, according to which entrepreneurs must finance investment entirely out of the returns to capital. In this case the equilibrium dynamics are very close to those characteristic of the Reichlin (1986) model, and as a result endogenous cycles are possible under circumstances similar to those discussed by Reichlin. In particular, in this case (unlike the Bewley example of a production economy) endogenous cycles can occur even when savings are an increasing function of the expected return.

These examples show that it is not entirely fair to criticize the examples of endogenous fluctuations in economies with overlapping generations of two-period-lived consumers as involving mechanisms that could generate cycles only at very low frequencies. The same sort of intertemporal substitution can also generate endogenous fluctuations in models where consumers are represented as infinite-lived, and in which the period of the fluctuations bears no relation to any time scale relating to human biology. Indeed, the 'periods' in the models with financial constraints just described must surely be interpreted as relatively short, so that if anything these models should be criticized for only generating cycles of frequency too *high* to correspond to actual 'business cycles'.[16] Reichlin's overlapping-generations model yields cycles that are approximately six 'periods' in length, or three times the lifetime of consumers, in the life-cycle interpretation of his model. This is a time scale much longer than that on which 'business cycles' occur. But in the Woodford reinterpretation of this model, six 'periods' is still too short for a 'business cycle', under the most reasonable interpretation of the length of a 'period' in a cash-in-advance model.[17] Hence construction of examples that allow endogenous cycles at 'business cycle' frequencies in the case of empirically realistic parameter specifications remains an important challenge for this line of research.[18]

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[16] The possibility in these models of equilibrium cycles of arbitrarily long periodicity, as discussed by Grandmont (1985), does not imply the possibility of 'long' business cycles in the sense that would usually be given this term. In any model that is formally analogous to the Grandmont model, real money balances can never increase for more than one consecutive 'period', so that fluctuations are necessarily high-frequency even if they do not exactly repeat for a very long time.

[17] However, Woodford (1988a) shows that if one considers stationary 'sunspot' equilibria rather than only the limiting case of purely deterministic cycles, it is possible to obtain cycles that last for several quarters on average, regardless of how short 'periods' are taken to be.

[18] Perfect-foresight equilibrium is also indeterminate in many other kinds of models with infinite-lived agents in the case of various market imperfections that need not involve financial constraints. In many such cases endogenous cycles or 'chaos' are among the possible types of equilibrium dynamics [Shleifer (1986), Murphy et al. (1988), Diamond and Fudenberg (1989), Howitt and McAfee (1988), Hammour (1988)]. Perhaps some of these mechanisms can produce cycles at 'business-cycle' frequencies, although none of the papers listed discuss this. The Deneckere and Judd (1986) cycles could be at 'business-cycle' frequencies, if firms that introduce new products can exploit their monopoly for something like a two-year period.

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