Lecture 5
Approximating Distribution

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Approaches

- Continuum of households makes the wealth distribution an infinite dimensional object in the state space
- Need approximation

- Several approaches beyond Monte-Carlo simulation
  1. Discretization of the density function
  2. Eigenvector method
  3. Parametrizing in exponential family

Discretization of the Invariant Density Function

- The grid should be finer than the one we used to compute the optimal savings rule
- Choose some initial density function $\lambda^0 \left(a_k, y_i\right)$
- For every $\left(a_k, y_j\right)$ on the grid:

$$\lambda^1 \left(a_k, y_j\right) = \sum_{y_i \in Y} \pi_{ij} \sum_{m \in M_i} \frac{a_{k+1} - g\left(a_m, y_i\right)}{a_{k+1} - a_k} \lambda^0 \left(a_m, y_i\right)$$

$$\lambda^1 \left(a_{k+1}, y_j\right) = \sum_{y_i \in Y} \pi_{ij} \sum_{m \in M_i} \frac{g\left(a_m . y_i\right) - a_k}{a_{k+1} - a_k} \lambda^0 \left(a_m, y_i\right)$$

where $M_i = \left\{m = 1, \ldots, N | a_k \leqslant g\left(a_m, y_i\right) \leqslant a_{k+1}\right\}$

Lottery Rule

- We can think of this way of handling the discrete approximation to the density function as forcing the agents in the economy to play a lottery.

- If the optimal policy is to save $a' \in [a_k, a_{k+1}]$, then with probability $\frac{a_{k+1}-a'}{a_{k+1}-a_k}$ you go to a_k, and with probability $\frac{a'-a_k}{a_{k+1}-a_k}$ you go to a_{k+1}.

Eigenvector Method

- The invariant pdf for a Markov transition matrix Q is λ^* that satisfies $\lambda^* Q = \lambda^*$
- Perron-Frobenius Theorem: Q has a unique dominant eigenvalue $\epsilon = 1$ such that
  - its associated eigenvector has all positive entries
  - all other eigenvalues are smaller than ϵ in absolute value
  - Q has no other eigenvector with all non-negative entries
- This eigenvector (renormalized so that it sums to one) is the unique invariant distribution

Parameterized Distribution

- For smooth and unimodal distributions we can approximate with parameterized functions in the exponential families

- Central moments are sufficient statistics of the distribution

Parameterized Distribution

- To approximate a one-dimensional distribution on k, assume the density function form

$$P\left(k,\rho\right)=\rho_0\exp\left(\begin{array}{c}\rho_1\left(k-m_1\right)\\ +\rho_2\left[\left(k-m_1\right)^2-m_2\right]\\ +\rho_3\left[\left(k-m_1\right)^3-m_3\right]\\ \cdots\\ +\rho_N\left[\left(k-m_1\right)^N-m_N\right]\end{array}\right)$$

- ρ – parameters to pin down, m – central moments

Parameterized Distribution

- We choose ρ such that moment condition holds

$$\int (k - m_1) P(k, \rho) \, \mathrm{d}k = 0$$
$$\int \left[(k - m_1)^2 - m_2 \right] P(k, \rho) \, \mathrm{d}k = 0$$
$$\dots$$
$$\int \left[(k - m_1)^N - m_N \right] P(k, \rho) \, \mathrm{d}k = 0$$

- These condition happen to be the optimal condition of

$$\min_{\rho_1, \dots, \rho_N} \int P(k, \rho) \, \mathrm{d}k$$

- And the distribution has to be normalized to unity

$$\frac{1}{\rho_0} = \int P(k, \rho) \, \mathrm{d}k$$

Parameterized Distribution

Steps to compute invariant distribution:

1. Choose capital grid $\{k_i\}$ and guess initial central moments $m_{j,0}$
2. Solve the minimization problem for ρ with $m_{j,0}$

$$\frac{1}{\rho_0} = \min_{\rho_1,\dots,\rho_N} \int P(k,\rho)\,\mathrm{d}k$$

3. With current distribution $P(k,\rho)$, compute moments using policy function $g(k)$

$$m_{1,1} = \int g(k)\,P(k,\rho)\,\mathrm{d}k$$

$$m_{2,1} = \int (g(k) - m_{1,1})^2\,P(k,\rho)\,\mathrm{d}k$$

$$\dots$$

$$m_{N,1} = \int (g(k) - m_{1,1})^N\,P(k,\rho)\,\mathrm{d}k$$

4. Check convergence of m