Lecture 5
Approximating Distribution

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Approaches

- Continuum of households makes the wealth distribution an infinite dimensional object in the state space
- Need approximation

- Several approaches beyond Monte-Carlo simulation
  1. Discretization of the density function
  2. Eigenvector method
  3. Parametrizing in exponential family

Discretization of the Invariant Density Function

- The grid should be finer than the one we used to compute the optimal savings rule
- Choose some initial density function $\lambda^0\left(a_k, y_i\right)$
- For every $\left(a_k, y_j\right)$ on the grid:

$$\lambda^1\left(a_k, y_j\right) = \sum_{y_i \in Y} \pi_{ij} \sum_{m \in M_i} \frac{a_{k+1} - g\left(a_m, y_i\right)}{a_{k+1} - a_k} \lambda^0\left(a_m, y_i\right)$$

$$\lambda^1\left(a_{k+1}, y_j\right) = \sum_{y_i \in Y} \pi_{ij} \sum_{m \in M_i} \frac{g\left(a_m . y_i\right) - a_k}{a_{k+1} - a_k} \lambda^0\left(a_m, y_i\right)$$

where $M_i = \{m = 1, \ldots, N | a_k \leqslant g\left(a_m, y_i\right) \leqslant a_{k+1}\}$

Lottery Rule

- We can think of this way of handling the discrete approximation to the density function as forcing the agents in the economy to play a lottery.

- If the optimal policy is to save $a' \in [a_k, a_{k+1}]$, then with probability $\frac{a_{k+1}-a'}{a_{k+1}-a_k}$ you go to a_k, and with probability $\frac{a'-a_k}{a_{k+1}-a_k}$ you go to a_{k+1}.

Eigenvector Method

- The invariant pdf for a Markov transition matrix Q is λ^* that satisfies $\lambda^* Q = \lambda^*$
- Perron-Frobenius Theorem: Q has a unique dominant eigenvalue $\epsilon = 1$ such that
  - its associated eigenvector has all positive entries
  - all other eigenvalues are smaller than ϵ in absolute value
  - Q has no other eigenvector with all non-negative entries
- This eigenvector (renormalized so that it sums to one) is the unique invariant distribution

Parameterized Distribution

- For smooth and unimodal distributions we can approximate with parameterized functions in the exponential families
- Central moments are sufficient statistics of the distribution

Parameterized Distribution

- To approximate a one-dimensional distribution on k, assume the density function form

$$P(k,\rho) = \rho_0 \exp\left(\begin{array}{c} \rho_1 (k - m_1) \\ +\rho_2 \left[(k - m_1)^2 - m_2\right] \\ +\rho_3 \left[(k - m_1)^3 - m_3\right] \\ \cdots \\ +\rho_N \left[(k - m_1)^N - m_N\right] \end{array}\right)$$

- ρ – parameters to pin down, m – central moments

Parameterized Distribution

- We choose ρ such that moment condition holds

$$
\begin{aligned}
\int (k - m_1) \, P(k, \rho) \, \mathrm{d}k &= 0 \\
\int \left[(k - m_1)^2 - m_2 \right] P(k, \rho) \, \mathrm{d}k &= 0 \\
&\dots \\
\int \left[(k - m_1)^N - m_N \right] P(k, \rho) \, \mathrm{d}k &= 0
\end{aligned}
$$

- These condition happen to be the optimal condition of

$$
\min_{\rho_1, \ldots, \rho_N} \int P(k, \rho) \, \mathrm{d}k
$$

- And the distribution has to be normalized to unity

$$
\frac{1}{\rho_0} = \int P(k, \rho) \, \mathrm{d}k
$$

Parameterized Distribution

Steps to compute invariant distribution:

1. Choose capital grid k_i and guess initial central moments $m_{j,0}$
2. Solve the minimization problem for ρ with $m_{j,0}$

$$\frac{1}{\rho_0} = \min_{\rho_1,\dots,\rho_N} \int P\left(k,\rho\right) \mathrm{d}k$$

3. With current distribution $P\left(k,\rho\right)$, compute moments using policy function $g(k)$

$$m_{1,1} = \int g\left(k\right) P\left(k,\rho\right) \mathrm{d}k$$

$$m_{2,1} = \int \left(g\left(k\right) - m_{1,1}\right)^2 P\left(k,\rho\right) \mathrm{d}k$$

$$\dots$$

$$m_{N,1} = \int \left(g\left(k\right) - m_{1,1}\right)^N P\left(k,\rho\right) \mathrm{d}k$$

4. Check convergence of m