7.2 Seasonal ARMA

Let us assume that there is seasonality in the data, but no trend. Then we could model the data as

$$X_t = s_t + Y_t, \tag{7.3}$$

where Y_t is a stationary process. The seasonality component is such that

$$s_t = s_{t-h},$$

where h denotes the length of the period and

$$\sum_{k=1}^{h} s_k = 0.$$

In Section 2.2.3 we have discussed removing the seasonal effect from the data by differencing at lag h. We have introduced the lag-h operator

$$\nabla_h X_t = X_t - X_{t-h} = X_t - B^h X_t = (1 - B^h)X_t,$$

which, for (7.3), gives

$$\nabla_h X_t = s_t + Y_t - s_{t-h} - Y_{t-h} = \nabla_h Y_t. \tag{7.4}$$

Hence, this operation removes the seasonality effect. This fact leads to introducing the **seasonal ARMA model**, denoted by $ARMA(P,Q)_h$, which is of the form

$$\Phi(B^h)X_t = \Theta(B^h)Z_t, \tag{7.5}$$

where

$$\Phi(B^h) = 1 - \Phi_1 B^h - \Phi_2 B^{2h} - \ldots - \Phi_P B^{Ph},$$

and

$$\Theta(B^h) = 1 + \Theta_1 B^h + \Theta_2 B^{2h} + \ldots + \Theta_Q B^{Qh}$$

are, respectively, the seasonal AR operator and the seasonal MA operator, with seasonal period of length h.

Remark 7.4. Analogously to $ARMA(p,q)$, the $ARMA(P,Q)_h$ model is causal only when the roots of $\Phi(z^h)$ lie outside the unit circle, and it is invertible only when the roots of $\Theta(z^h)$ lie outside the unit circle.

Example 7.1. Seasonal $ARMA(1,1)_{12}$.

Such a model can be written as

$$(1 - \Phi B^{12})X_t = (1 + \Theta B^{12})Z_t,$$

or

$$X_t - \Phi X_{t-12} = Z_t + \Theta Z_{t-12},$$

which is a generalization of (7.4).

When written as

$$X_t = \Phi X_{t-12} + Z_t + \Theta Z_{t-12},$$

and compared to $ARMA(1, 1)$

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$$

we see that the seasonal ARMA presents the series in terms of its past values at lag equal to the length of the period (here h=12), while the non-seasonal ARMA does it in terms of its past values at lag 1. Seasonal ARMA incorporates the seasonality into the model.

Similarly as for the non-seasonal ARMA, here too, we require $|\Phi| < 1$ for the causality and $|\Theta| < 1$ for invertibility of the model.

Remark 7.5. Note that seasonal $ARMA(0, Q)_h$ is a seasonal $MA(Q)_h$, and seasonal $ARMA(P, 0)_h$ is a seasonal $AR(P)_h$.

Example 7.2. ACF of $MA(1)_{12}$
A seasonal MA model with the period length $h = 12$ can be written as

$$X_t = Z_t + \Theta Z_{t-12}.$$

It is a zero mean stationary model and it is easy to calculate its autocovariance, namely

$$\begin{aligned}
\gamma(\tau) &= \operatorname{cov}[Z_t + \Theta Z_{t-12}, Z_{t+\tau} + \Theta Z_{t+\tau-12}] \\
&= \mathrm{E}[(Z_t + \Theta Z_{t-12})(Z_{t+\tau} + \Theta Z_{t+\tau-12})] \\
&= \mathrm{E}(Z_t Z_{t+\tau}) + \Theta\,\mathrm{E}(Z_t Z_{t+\tau-12}) + \Theta\,\mathrm{E}(Z_{t-12} Z_{t+\tau}) + \Theta^2\,\mathrm{E}(Z_{t-12} Z_{t+\tau-12}) \\
&= \begin{cases} (1+\Theta^2)\sigma^2 & \text{for } \tau = 0, \\ \Theta\sigma^2 & \text{for } \tau = \pm 12, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Thus, the only non-zero correlations are $\rho(0) = 1$ and

$$\rho(\pm 12) = \frac{\Theta}{1+\Theta^2},$$

which is of the same form as $\rho(\pm 1)$ for a non-seasonal MA(1).

Example 7.3. ACF of $AR(1)_h$
Using the techniques for calculating ACVF and ACF of the non-seasonal AR(1) we obtain

$$\gamma(\tau) = \begin{cases} \frac{\sigma^2}{1-\Phi^2} & \text{for } \tau = 0, \\ \frac{\sigma^2\Phi^k}{1-\Phi^2} & \text{for } \tau = \pm hk, \ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

This give the ACF similar to the ACF of a non-seasonal AR(1), namely

$$\rho(\tau) = \begin{cases} 1 & \text{for } \tau = 0, \\ \Phi^k & \text{for } \tau = \pm hk, \ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

The following table summarizes the behaviour of the ACF and PACF of the causal and invertible seasonal ARMA models (see R.H.Shumway and Stoffer (2000)).

	$AR(P)_h$	$MA(Q)_h$	$ARMA(P,Q)_h$
ACF	Tails off at lags kh,	Cuts off after lag Qh	Tails off at lags kh
PACF	Cuts off after lag Ph	Tails off at lags kh	Tails off at lags kh

where h is the length of the seasonal period, $k = 1, 2, \ldots$ and the values of ACF and PACF are zero at non-seasonal lags $\tau \neq kh$.

7.2.1 Mixed Seasonal ARMA

When we combine seasonal and non-seasonal operators we obtain a model

$$\Phi(B^h)\phi(B)X_t = \Theta(B^h)\theta(B)Z_t,$$

which is called **mixed seasonal ARMA** and it is denoted by

$$ARMA(p,q) \times (P,Q)_h.$$

The behavior of the ACF and PACF for such models is a combination of behavior of the seasonal and nonseasonal parts of the model.

Example 7.4. $ARMA(0,1) \times (1,0)_{12}$

Such a model has the following form

$$X_t - \Phi X_{t-12} = Z_t + \theta Z_{t-1},$$

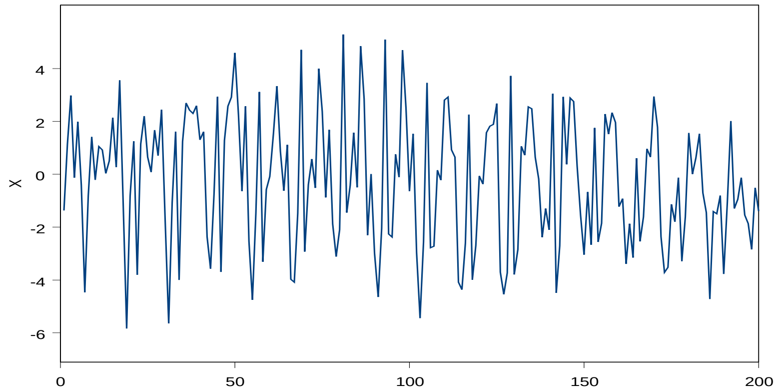

Figure 7.1: Simulated $ARMA(0,1)(1,0)_{12}$ process.

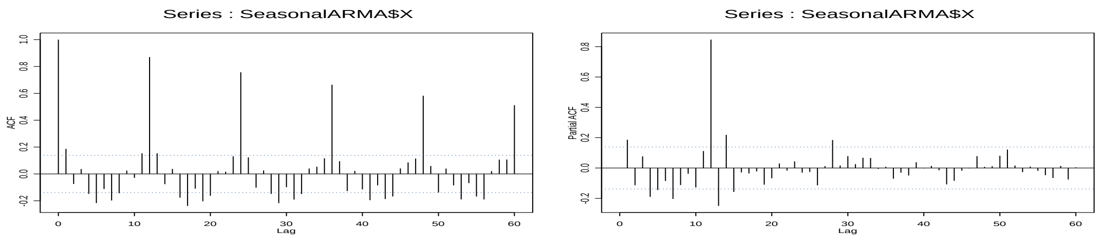

Figure 7.2: ACF and PACF of the above $ARMA(0,1)(1,0)_{12}$ process.

where $|\Phi| < 1$ and $|\theta| < 1$. Here we obtain

$$\rho(\tau) = \begin{cases} \Phi^k & \text{for } \tau = 12k, \; k = 1, 2, \dots, \\ \frac{\theta}{1+\theta^2}\Phi^k & \text{for } \tau = 12k \pm 1, \; k = 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

7.2.2 Seasonal ARIMA

Mixed seasonal ARMA is a stationary process. In practice however we often have nonstationary processes. Seasonal nonstationarity can occur when the process is nearly periodic in the season and the seasonal component varies slowly from period to period (say from year to year) according to a random walk, that is

$$s_t = s_{t-h} + V_t,$$

where V_t is a white noise. We can subtract the effect of the season (say month) using the backshift operator B^h to obtain seasonal stationarity

$$X_t - X_{t-h} = (1 - B^h)X_t.$$

This is a seasonal difference of order 1. In general we define a seasonal difference of order D as

$$\nabla_h^D X_t = (1 - B^h)^D X_t,$$

where $D = 1, 2, \ldots$. Usually $D = 1$ is sufficient to obtain seasonal stationarity. This leads to a very general **seasonal autoregressive integrated moving average (SARIMA)** model written as follows

$$\Phi(B^h)\phi(B)\nabla_H^D \nabla^d X_t = \alpha + \Theta(B^h)\theta(B)Z_t, \tag{7.6}$$

and denoted by $ARIMA(p, d, q) \times (P, D, Q)_h$.

Example 7.5. The model $ARIMA(0, 1, 1)\times(0, 1, 1)_{12}$ with $\alpha = 0$ is often applied for various economic data. Using formula (7.6) we obtain

$$(1 - B^{12})(1 - B)X_t = (1 + \Theta B^{12})(1 + \theta B)Z_t,$$

or, when expanded, we get the following form

$$(1 - B - B^{12} + B^{13})X_t = (1 + \theta B + \Theta B^{12} + \Theta\theta B^{13})Z_t,$$

or

$$X_t = X_{t-1} + X_{t-12} - X_{t-13} + Z_t + \theta Z_{t-1} + \Theta Z_{t-12} + \Theta\theta Z_{t-13}.$$

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