Occasionally Binding Constraints using Anticipated Shocks in a Constant-Parameter DSGE Model

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Agenda

- The DSGE model with anticipated information
- Conditional forecasting
- Occasionally-binding constraints with anticipated shocks

The DSGE model with anticipated information

The generic problem

$$\mathbb{E}f\left(x_{t+1}, x_t, x_{t-1}, \eta_t\right) = 0$$

- x_t is the vector of endogenous variables
- $f(.)$ is vector of potentially nonlinear functions of their arguments
- η_t is the vector of shocks with, without loss of generality,

$$\underbrace{\eta_t}_{n_\eta \times 1} \sim \mathbb{N}\left(0, I\right)$$

Solution with News shocks I

We modify the model to include "news shocks"

$$\mathbb{E}f\left(\tilde{x}_{t+1},\tilde{x}_t,\tilde{x}_{t-1},\varepsilon_t,\varepsilon_{t-1}^{(1)},...,\varepsilon_{t-h}^{(h)}\right)=0$$

The solution takes the form

$$\tilde{x}_t=T\left(\tilde{x}_{t-1},\varepsilon_t,\varepsilon_{t-1}^{(1)},...,\varepsilon_{t-h}^{(h)}\right)$$

Solution with News shocks II

without loss of generality, the problem and the solution can be re-written as

$$\mathbb{E}f\left(x_{t+1}, x_t, x_{t-1}, \eta_t\right) = 0$$

$$x_t = T\left(x_{t-1}, \eta_t\right)$$

by defining

$$x_t \equiv \left[\begin{array}{ccccc} \tilde{x}_t' & \varepsilon_t' & \varepsilon_{t-1}^{(1)\prime} & \cdots & \varepsilon_{t-h+1}^{(h)\prime} \end{array}\right]'$$

and

$$\eta_t \equiv \varepsilon_t$$

Solution under anticipated shocks

$$x_t = T\left(x_{t-1}, \eta_t, \eta_{t+1}, ..., \eta_{t+h}\right)$$

Finding the solution T amounts to solving the problem

$$\mathbb{E}f\left(\begin{array}{c} T\left(x_t, \eta_{t+1}, \eta_{t+2}, ..., \eta_{t+h+1}\right), \\ T\left(x_{t-1}, \eta_t, \eta_{t+1}, ..., \eta_{t+h}\right), \\ x_{t-1}, \eta_t \end{array}\right) = 0$$

or

$$\mathbb{E}f\left(\begin{array}{c} T\left(T\left(x_{t-1}, \eta_t, \eta_{t+1}, ..., \eta_{t+h}\right), \eta_{t+1}, \eta_{t+2}, ..., \eta_{t+h+1}\right), \\ T\left(x_{t-1}, \eta_t, \eta_{t+1}, ..., \eta_{t+h}\right), \\ x_{t-1}, \eta_t \end{array}\right) = 0$$

We do not modify the model

Solution under anticipated shocks: the linear case I

Abstracting from constant terms the problem to solve is

$$A^{+}\mathbb{E}x_{t+1} + A^{0}x_t + A^{-}x_{t-1} + B\eta_t = 0$$

and its (exact) solution is

$$x_t = T_x x_{t-1} + T_{\eta,0}\eta_t + T_{\eta,1}\eta_{t+1} +, ..., + T_{\eta,h}\eta_{t+h}$$

with special case

$$x_t = T_x x_{t-1} + T_{\eta,0}\eta_t$$

which applies when $E_t \eta_{t+s} = 0$ for $s > 0$

Solution under anticipated shocks: the linear case II

The general solution implies

$$\begin{aligned}
0 &= A^{+}\mathbb{E}\left[T_x x_t + T_{\eta,0}\eta_{t+1} + T_{\eta,1}\eta_{t+2} +, ..., + T_{\eta,h}\eta_{t+h+1}\right] \\
&\quad + A^{0}x_t + A^{-}x_{t-1} + B\eta_t \\
&= A^{+}\left[T_{\eta,0}\eta_{t+1} + T_{\eta,1}\eta_{t+2} +, ..., + T_{\eta,h-1}\eta_{t+h}\right] \\
&\quad + \left(A^{+}T_x + A^{0}\right)\left[T_x x_{t-1} + T_{\eta,0}\eta_t + T_{\eta,1}\eta_{t+1} +, ..., + T_{\eta,h}\eta_{t+h}\right] \\
&\quad + A^{-}x_{t-1} + B\eta_t
\end{aligned}$$

Solution under anticipated shocks: the linear case III

or

$$
\begin{aligned}
0 \quad = \quad & \left[\left(A^{+}T_x + A^{0}\right)T_x + A^{-}\right]x_{t-1} \\
& + \left[\left(A^{+}T_x + A^{0}\right)T_{\eta,0} + B\right]\eta_t \\
& \left[+A^{+}T_{\eta,0} + \left(A^{+}T_x + A^{0}\right)T_{\eta,1}\right]\eta_{t+1} \\
& \dots \\
& + \left[A^{+}T_{\eta,h-1} + \left(A^{+}T_x + A^{0}\right)T_{\eta,h}\right]\eta_{t+h}
\end{aligned}
$$

- Hence the system can be solve in a recursive fashion.
- Once we have solved for T_x (using Sims, Klein, AIM, Uhlig, etc.) we can solve for $T_{\eta,0}$, then $T_{\eta,1}$, ..., then $T_{\eta,h}$

Solution under anticipated shocks: the linear case IV

In particular,

$$T_{\eta,0} = -\left(A^{+}T_x + A^{0}\right)^{-1} B$$

$$\begin{aligned} T_{\eta,1} &= -\left(A^{+}T_x + A^{0}\right)^{-1} A^{+}T_{\eta,0} \\ T_{\eta,2} &= -\left(A^{+}T_x + A^{0}\right)^{-1} A^{+}T_{\eta,1} \end{aligned}$$

$$\ldots$$

$$T_{\eta,h} = -\left(A^{+}T_x + A^{0}\right)^{-1} A^{+}T_{\eta,h-1}$$

Solution under anticipated shocks: the linear case V

- The signs of the impacts alternate in powers of $\left(A^{+}T_{x}+A^{0}\right)^{-1}A^{+}$
- A discounting occurs if the eigenvalues of $\left(A^{+}T_{x}+A^{0}\right)^{-1}A^{+}$ are inside the unit circle:
  - future shocks have lower impact than current ones.
  - Or alternatively, future shocks have to be big for them to matter for today's decisions.

Conditional forecasting

The future system I

Consider the system k periods ahead

$$x_{t+1} = T_x x_t + T_{\eta,0}\eta_{t+1} + T_{\eta,1}\eta_{t+2} +, ..., + T_{\eta,h}\eta_{t+1+h}$$

$$x_{t+2} = T_x x_{t+1} + T_{\eta,0}\eta_{t+2} + T_{\eta,1}\eta_{t+3} +, ..., + T_{\eta,h}\eta_{t+2+h}$$

$$x_{t+k} = T_x^k x_t + \sum_{j=1}^{k} \sum_{s=0}^{h} T_x^{k-j} T_{\eta,s}\eta_{t+j+s}$$

The future system II

Stacking all future data

$$
\underbrace{\begin{bmatrix} x_{t+1} \\ x_{t+2} \\ \vdots \\ x_{t+k} \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} T_x \\ T_x^2 \\ \vdots \\ T_x^k \end{bmatrix}}_{\bar{X}} x_t + \underbrace{\begin{bmatrix} \Phi_{1,0} & \cdots & \Phi_{1,h} & 0 & \cdots & 0 \\ \Phi_{2,0} & \cdots & \Phi_{2,h} & \Phi_{2,h+1} & & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \Phi_{k,0} & \cdots & \Phi_{k,h} & \Phi_{k,h+1} & \cdots & \Phi_{k,k+h} \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} \eta_{t+1} \\ \vdots \\ \eta_{t+h+1} \\ \eta_{t+h+2} \\ \vdots \\ \eta_{t+h+k+1} \end{bmatrix}}_{\eta} \tag{1}
$$

The future system III

or

$$X = \bar{X} + \Phi\eta$$

The future system IV

if $h = 0$

$$x_{t+k} = T_x^k x_t + \sum_{j=1}^{k} T_x^{k-j} T_{\eta,0} \eta_{t+j}$$

$$\begin{bmatrix} x_{t+1} \\ x_{t+2} \\ \vdots \\ x_{t+k} \end{bmatrix} = \begin{bmatrix} T_x \\ T_x^2 \\ \vdots \\ T_x^k \end{bmatrix} x_t + \begin{bmatrix} \Phi_{1,0} & 0 & \cdots & 0 \\ \Phi_{2,0} & \Phi_{2,1} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots & \Phi_{k,k} \end{bmatrix} \begin{bmatrix} \eta_{t+1} \\ \eta_{t+2} \\ \vdots \\ \eta_{t+k+1} \end{bmatrix}$$

where

$$\begin{bmatrix} \Phi_{1,0} & 0 & \cdots & 0 \\ \Phi_{2,0} & \Phi_{2,1} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots & \Phi_{k,k} \end{bmatrix} = \begin{bmatrix} T_{\eta,0} & 0 & \cdots & 0 \\ T_x T_{\eta,0} & T_{\eta,0} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_x^{k-1} T_{\eta,0} & T_x^{k-2} T_{\eta,0} & \cdots & T_{\eta,0} \end{bmatrix}$$

The future system V

We have

$$X \sim \mathbb{N}\left(\bar{X}, \Phi\Phi'\right)$$

Restrictions on the future I

Suppose we are given the restriction

$$DX \sim \mathbb{TN}\left(\mu, \Omega, [L, H]\right)$$

- In practice, D is usually a selection matrix but it can also be a set of linear combinations on the rows of X.
- Maih (2010) develops algorithms for finding solutions with anticipated shocks in linear(ized) rational expectations models.
- Juillard and Maih (2010) use these algorithms to estimate models with observed real-time expectations data.

The restriction implies that

$$\underbrace{D\Phi}_{R}\eta \sim \mathbb{TN}\left(\mu - D\bar{X}, \Omega, \left[L - D\bar{X}, H - D\bar{X}\right]\right)$$

where rectangular matrix R is of size $q \times (h + k + 1)\, n_\eta$ and of rank $q \leq m_\eta \equiv (h + k + 1)\, n_\eta$

Restrictions on the future II

Consider the decomposition

$$\eta = M_1\gamma_1 + M_2\gamma_2 \tag{2}$$

with

$$\gamma_1 \sim \mathbb{N}\left(0, I_{m_\eta - q}\right)$$

where

- $\underset{m_\eta \times (m_\eta - q)}{M_1}$ is chosen to be an orthonormal basis for the null space of R
- and $\underset{m_\eta \times q}{M_2}$ either an orthonormal basis for the null space of M_1' or an orthonormal basis for the column space of R'

Restrictions on the future III

Then

$$R\eta \sim \mathbb{TN}\left(\mu - D\bar{X}, \Omega, [\underline{\mathsf{r}}, \bar{r}]\right) \Longrightarrow RM_2\gamma_2 \sim \mathbb{TN}\left(\mu - D\bar{X}, \Omega, [\underline{\mathsf{r}}, \bar{r}]\right)$$

with RM_2 invertible!!!

Hard conditions

$$RM_2\gamma_2 = \left(\mu - D\bar{X}\right) \iff \gamma_2 = \left[RM_2\right]^{-1}\left(\mu - D\bar{X}\right)$$

and the shocks that are needed to satisfy the restrictions are

$$\eta = M_1\gamma_1 + M_2\gamma_2$$

Occasionally-binding constraints with anticipated shocks

Principle

Apply conditional forecasting whenever a restriction is violated

Shock selection and the ZLB

Not all shocks need to have the same anticipation horizon e.g.

- the government can announce future taxes,
- the central bank can announce interest rates can remain low for an extended period of time

A forward-back shooting algorithm I

- The original forward-back shooting algorithm by Hebden et al. (2011) is based on news shocks.
- In Linde et al. (2016) and Linde, Maih and Wouters (2017) it is based on anticipated shocks

A forward-back shooting algorithm II

- The steps for any particular simulation period t

  1. Simulate the system
  2. if the ZLB is not violated move to $t + 1$, go to step 1
  3. if the ZLB is violated let $n = 1$

     3.1 assume it is going to last n periods and find the anticipated monetary policy shocks that are necessary for keeping the interest rate at the ZLB for n periods

     3.2 if period $t + n - 1$ does not violate go to step 1 with t set to $t + n - 1$

     3.3 if $t + n - 1$ violates, set $n = n + 1$ and go to 3a

A forward-back shooting algorithm III

- At each step, the number of anticipated shocks is equal to the number of violations.
- In this particular case, $M_1 = 0$ and $M_2 = I$, i.e. there is a unique combination of shocks that satisfy the restrictions

Sign reversals and forward guidance

- Implicit in the max operator is the idea that interest rates cannot go below the floor when they should. This would translate into positive monetary policy shocks at the ZLB.
- When the ZLB binds for too long, however, it may occur that some of the future monetary policy shocks that are required to satisfy the ZLB become positive, implying an expansionary rather than a contractionary ZLB
- Does Sign reversals imply forward guidance?

Comparison to OCCBIN

- The piece-wise linear strategy considers a sequence of time-varying matrices
- No special consideration for the explicit role of policy
- No possibility of modeling green shoots (e.g. staying longer at the ZLB)
- Poor handling of complementary slackness conditions (e.g. sign of lagrange multipliers)
- No (direct) possibility of multiple regimes (e.g. high, medium, low)
- It would be equivalent to using all shocks in the system to satisfy the restrictions without having a control over the exact combination of those shocks

Handling multiple constraints

- The theory for conditional forecasting applies to multiple constraints
- Hence it is straightforward to apply it to multiple occasionally binding constraints.

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