Existence, uniqueness and computation of solutions to dynamic models with occasionally binding constraints.

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Outline of the research agenda

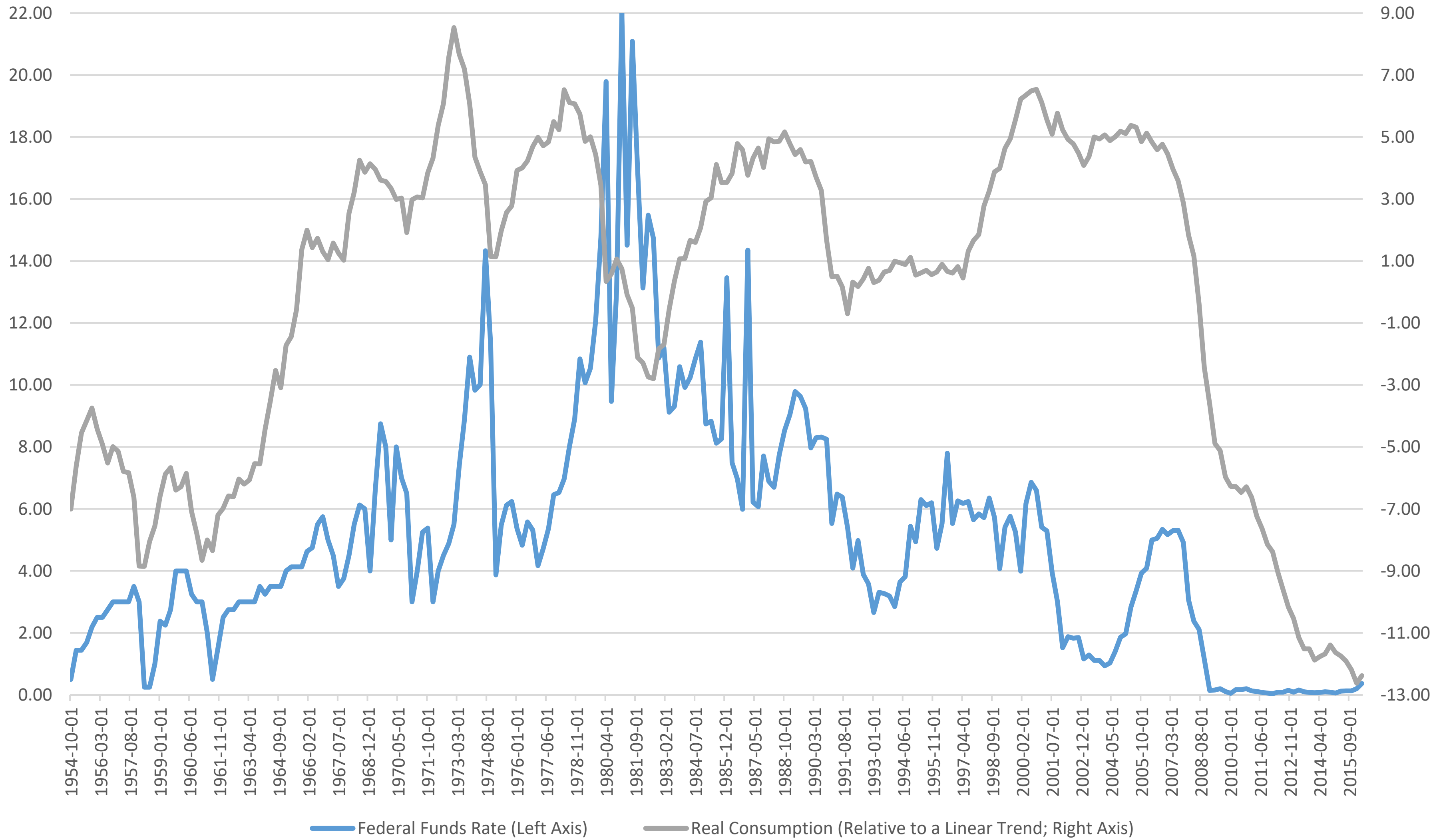
- "Blanchard and Kahn (1980) with occasionally binding constraints."

- Paper 1 (Theory): Existence and uniqueness results for otherwise linear models under perfect foresight.
- Paper 2 (Computational): A new computational algorithm designed to be robust, accurate and scalable.
- Paper 3 (Estimation; work in progress): Efficient estimation, filtering and smoothing of non-linear models, including those with occasionally binding constraints.

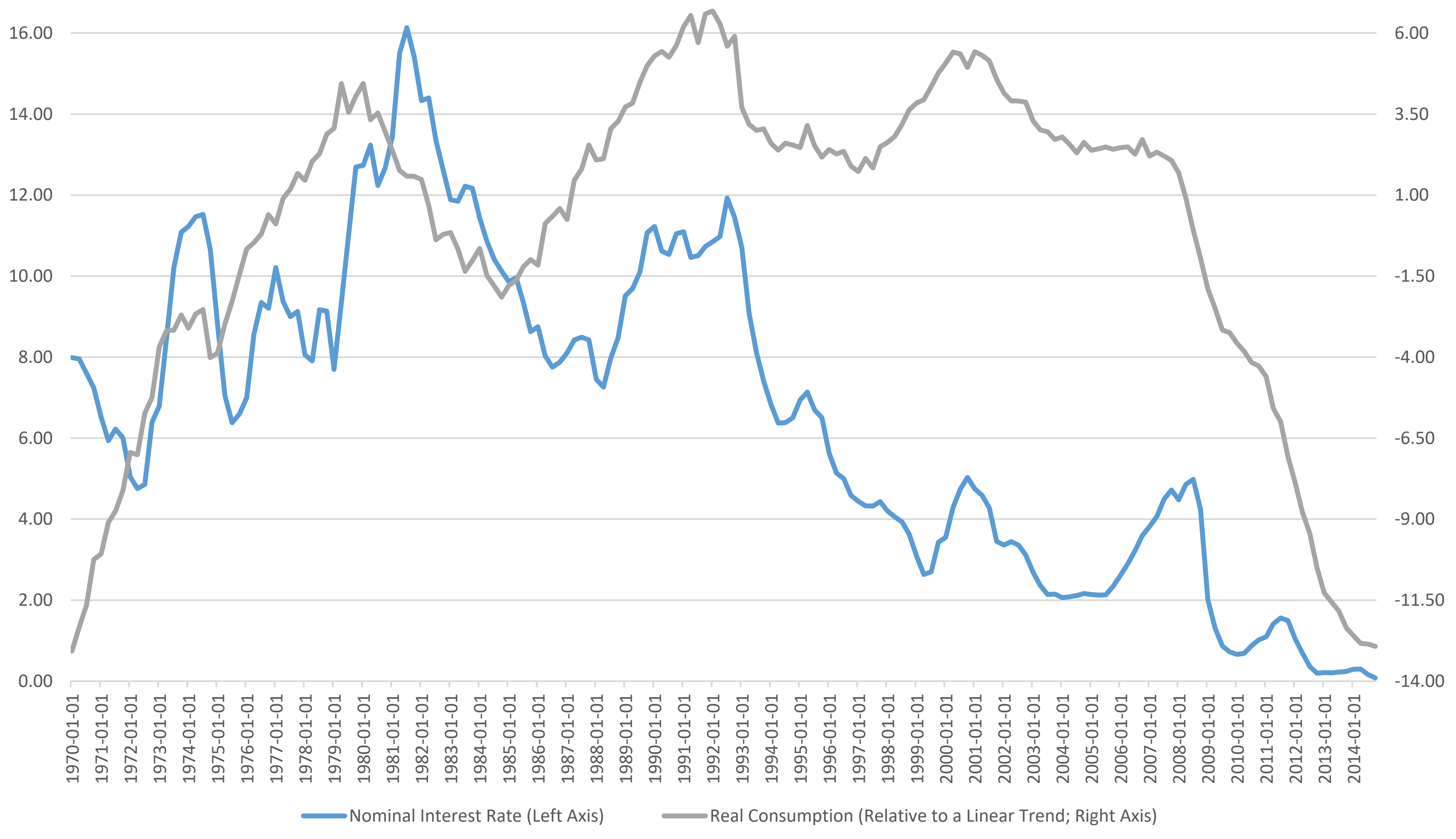
- All procedures implemented in my DynareOBC toolkit, which is one stop shop for all things OBC.
  - Designed to be super easy to use.
  - I will show many examples today.
  - Available from https://github.com/tholden/dynareOBC.

Why should we care about occasionally binding constraints?

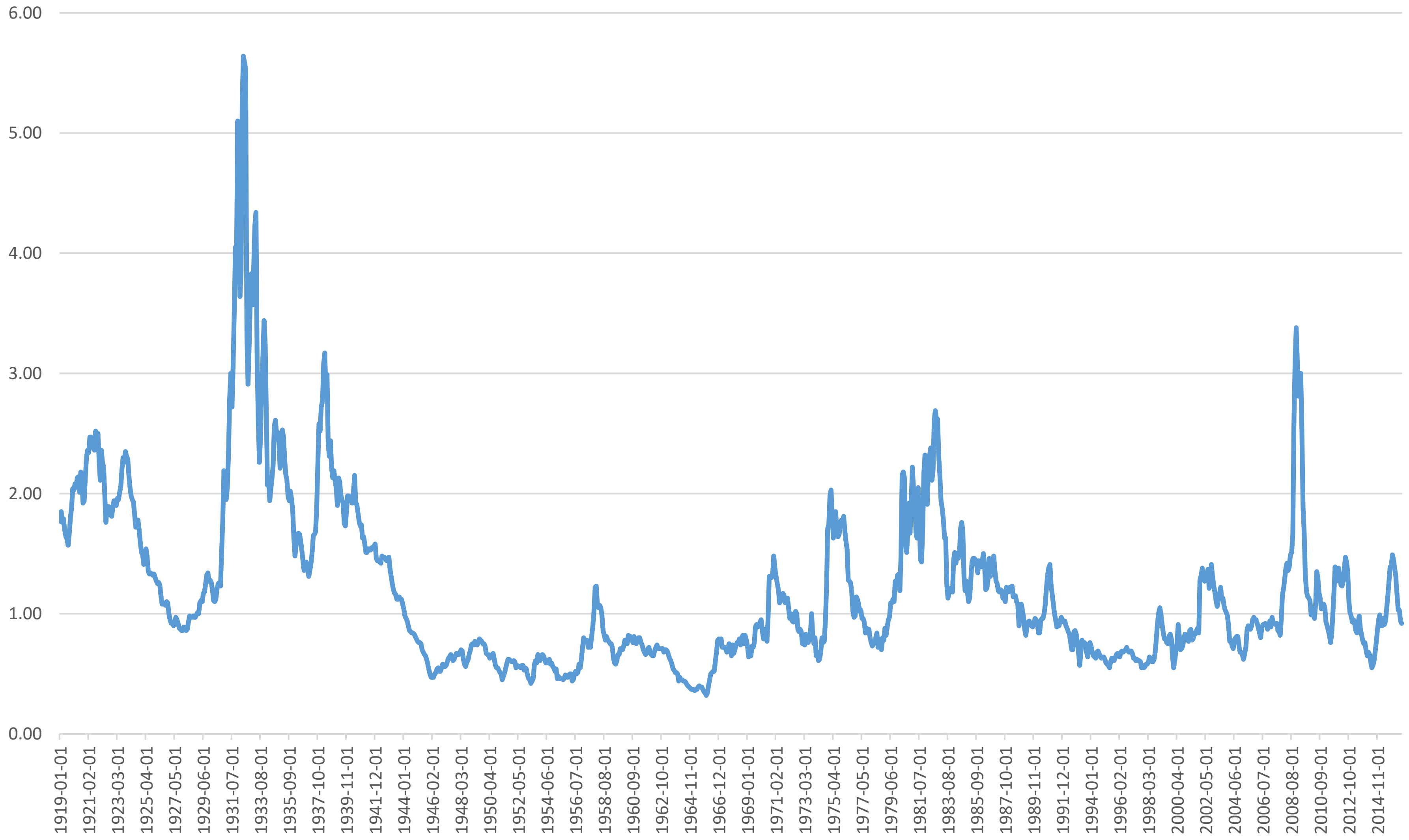
US Federal Funds Rate and Real Consumption (Relative to a Linear Trend)



Euro Area Nominal Interest Rates and Real Consumption (Relative to a Linear Trend)

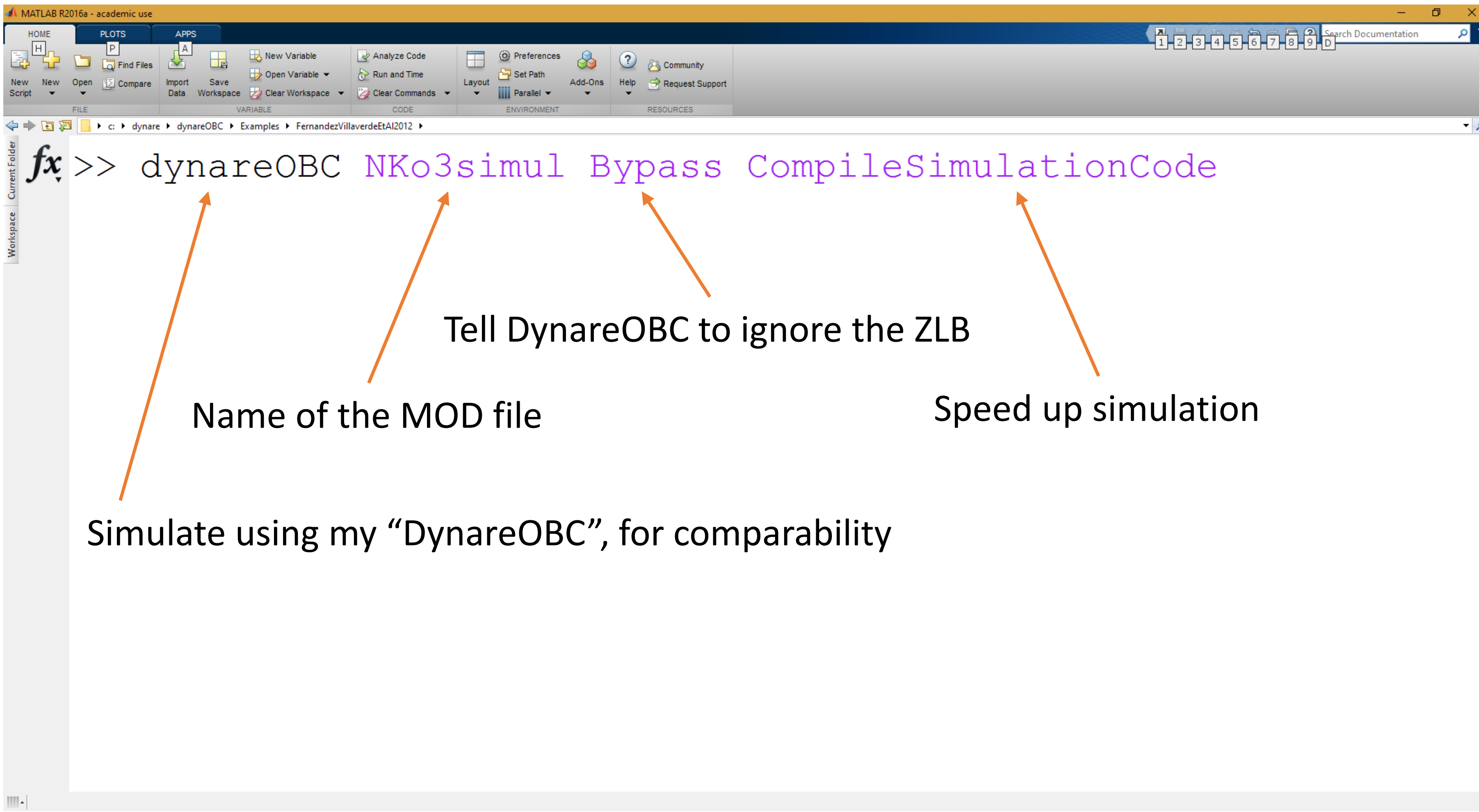


Moody's BAA-AAA Spread

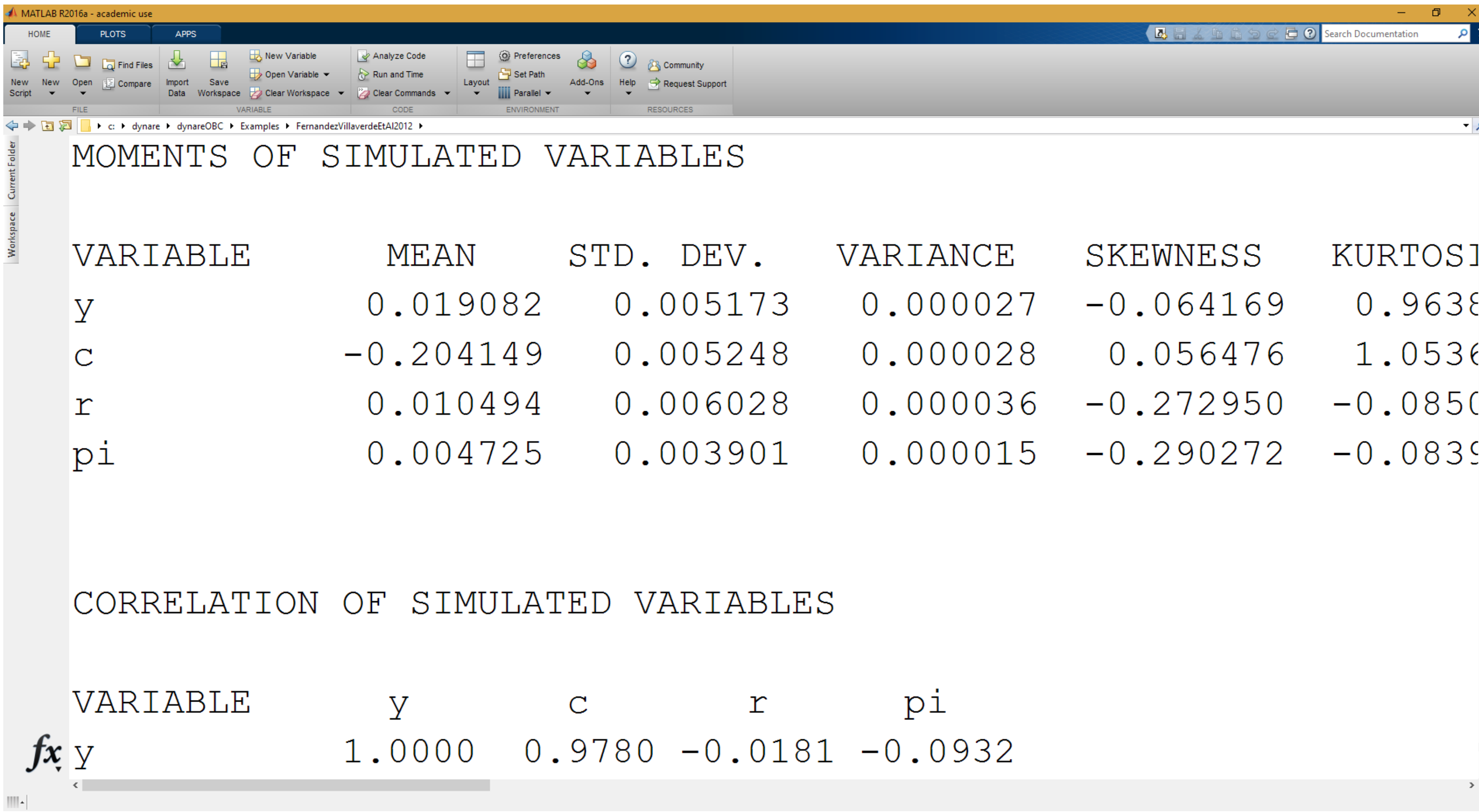


What do models tell us about the effect of the ZLB?

Simulating the Fernández-Villaverde et al. (2015) model at order 3, without the ZLB



Moments exhibit minimal skewness and kurtosis



MOMENTS OF SIMULATED VARIABLES

VARIABLE	MEAN	STD. DEV.	VARIANCE	SKEWNESS	KURTOSI
y	0.019082	0.005173	0.000027	-0.064169	0.963
c	-0.204149	0.005248	0.000028	0.056476	1.053
r	0.010494	0.006028	0.000036	-0.272950	-0.085
pi	0.004725	0.003901	0.000015	-0.290272	-0.083

CORRELATION OF SIMULATED VARIABLES

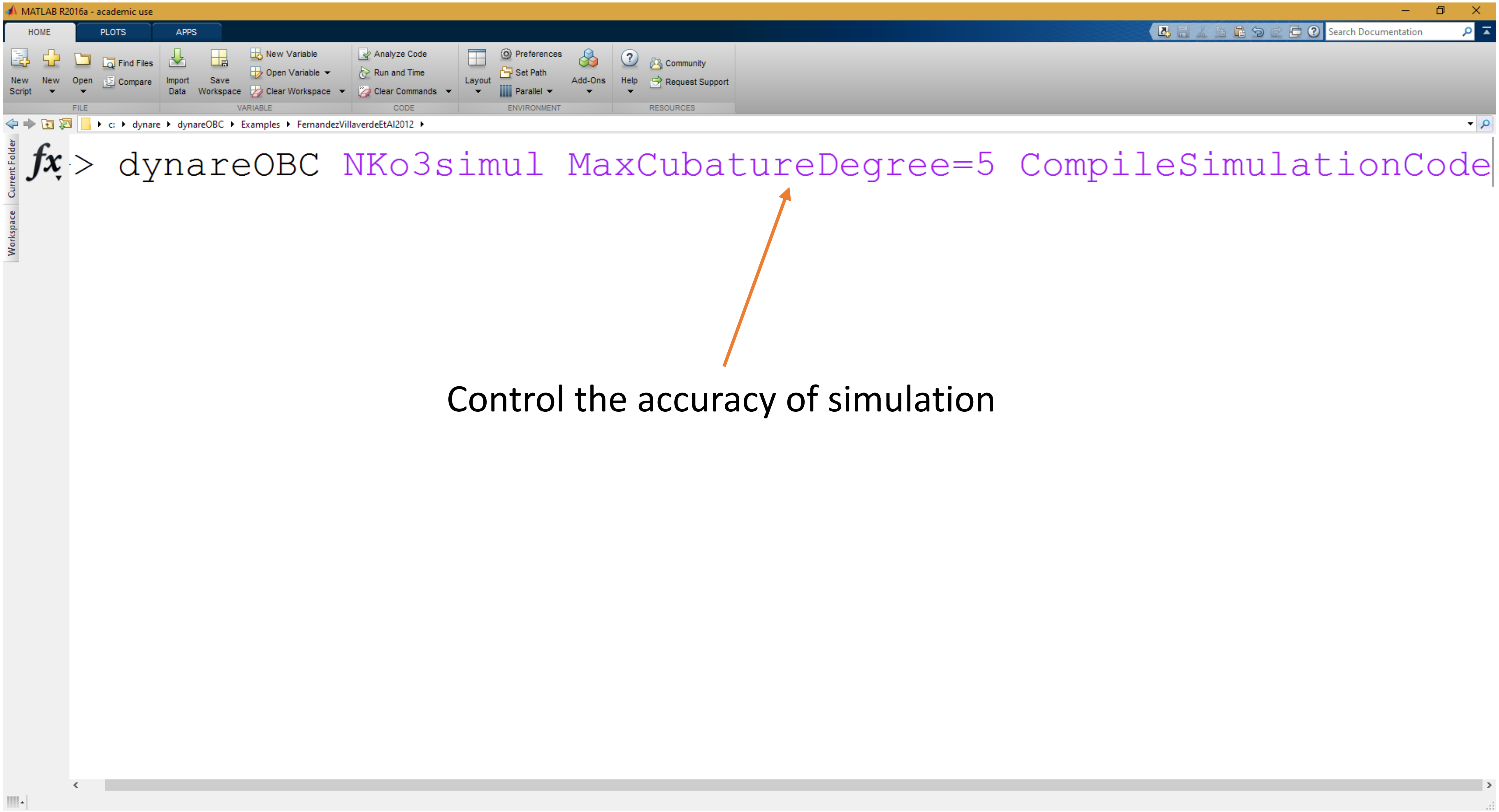
VARIABLE	y	c	r	pi
y	1.0000	0.9780	-0.0181	-0.0932

The MOD file has a "max" in it, giving the OBC

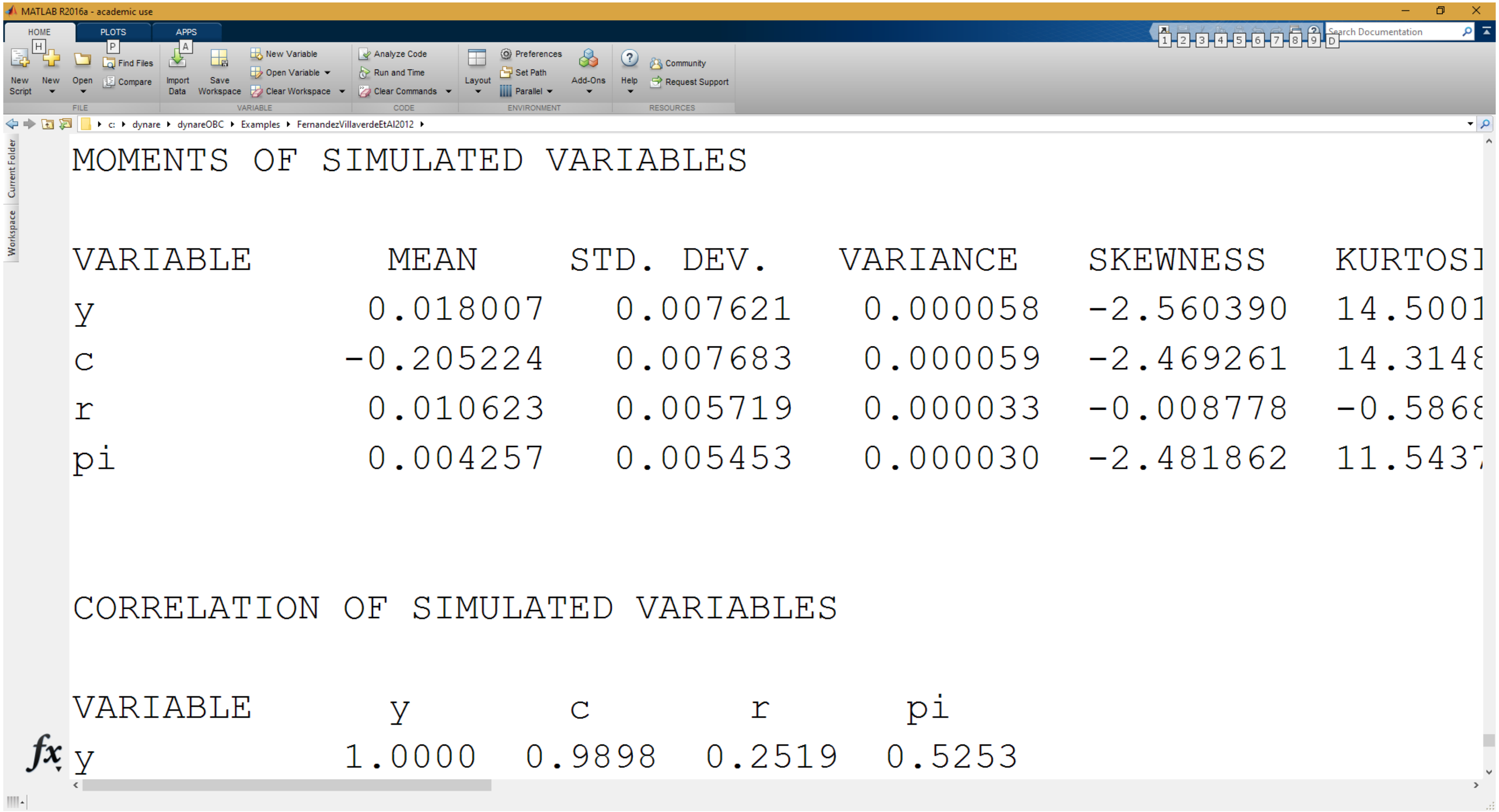
Editor - C:\dynare\dynareOBC\Examples\FernandezVillaverdeEtAl2012\NKo3simul.mod

```
    #M = exp(-sigma_m * epsilon_m);
    #R = exp( max( 0, log( ( PI_STEADY / beta_STEADY ) * ( ((P
    #AUX2 = varepsilon / (varepsilon - 1) * AUX1;
    #AUX2_LEAD = varepsilon / (varepsilon - 1) * AUX1_LEAD;
    1 = R * beta_LEAD * ( C / C_LEAD ) / PI_LEAD;
    AUX1 = MC * (Y/C) + theta * beta_LEAD * PI_LEAD^(varepsilo
    AUX2 = PI_STAR * ((Y/C) + theta * beta_LEAD * ((PI_LEAD^(v
    log( NU ) = log( theta * (PI^varepsilon) * NU_LAG + (1 - t
    y = log( Y );
    c = log( C );
    r = log( R );
    pi = log( PI );
end;
```

DynareOBC can handle mod files with “max”, “min” or “abs”



In the presence of the ZLB, there are occasionally terrible outcomes



MOMENTS OF SIMULATED VARIABLES

VARIABLE	MEAN	STD. DEV.	VARIANCE	SKEWNESS	KURTOSI
y	0.018007	0.007621	0.000058	-2.560390	14.5001
c	-0.205224	0.007683	0.000059	-2.469261	14.3148
r	0.010623	0.005719	0.000033	-0.008778	-0.5868
pi	0.004257	0.005453	0.000030	-2.481862	11.5437

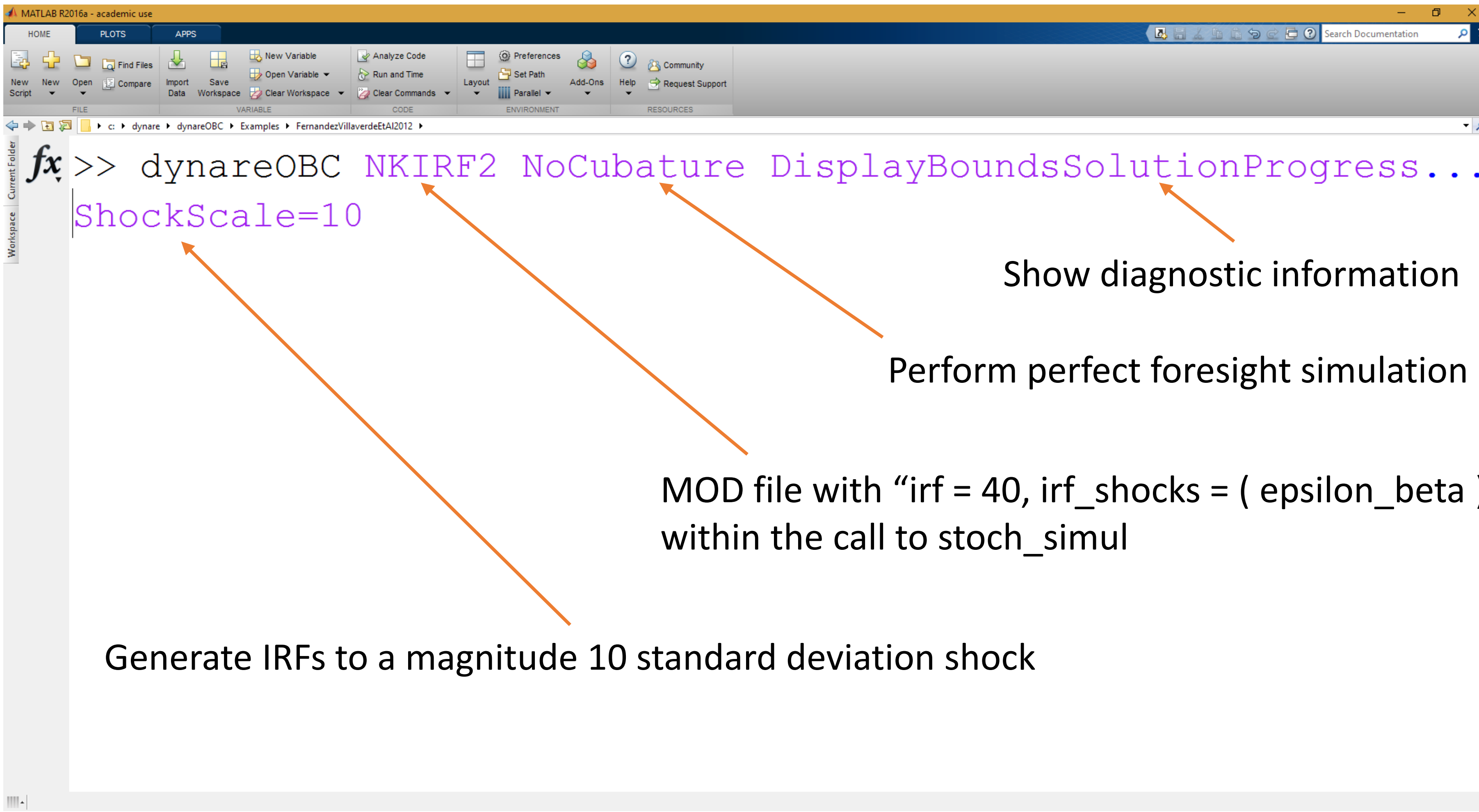
CORRELATION OF SIMULATED VARIABLES

VARIABLE	y	c	r	pi
y	1.0000	0.9898	0.2519	0.5253

What is DynareOBC doing?

Idea is that OBCs provide endogenous news about the bounded variable

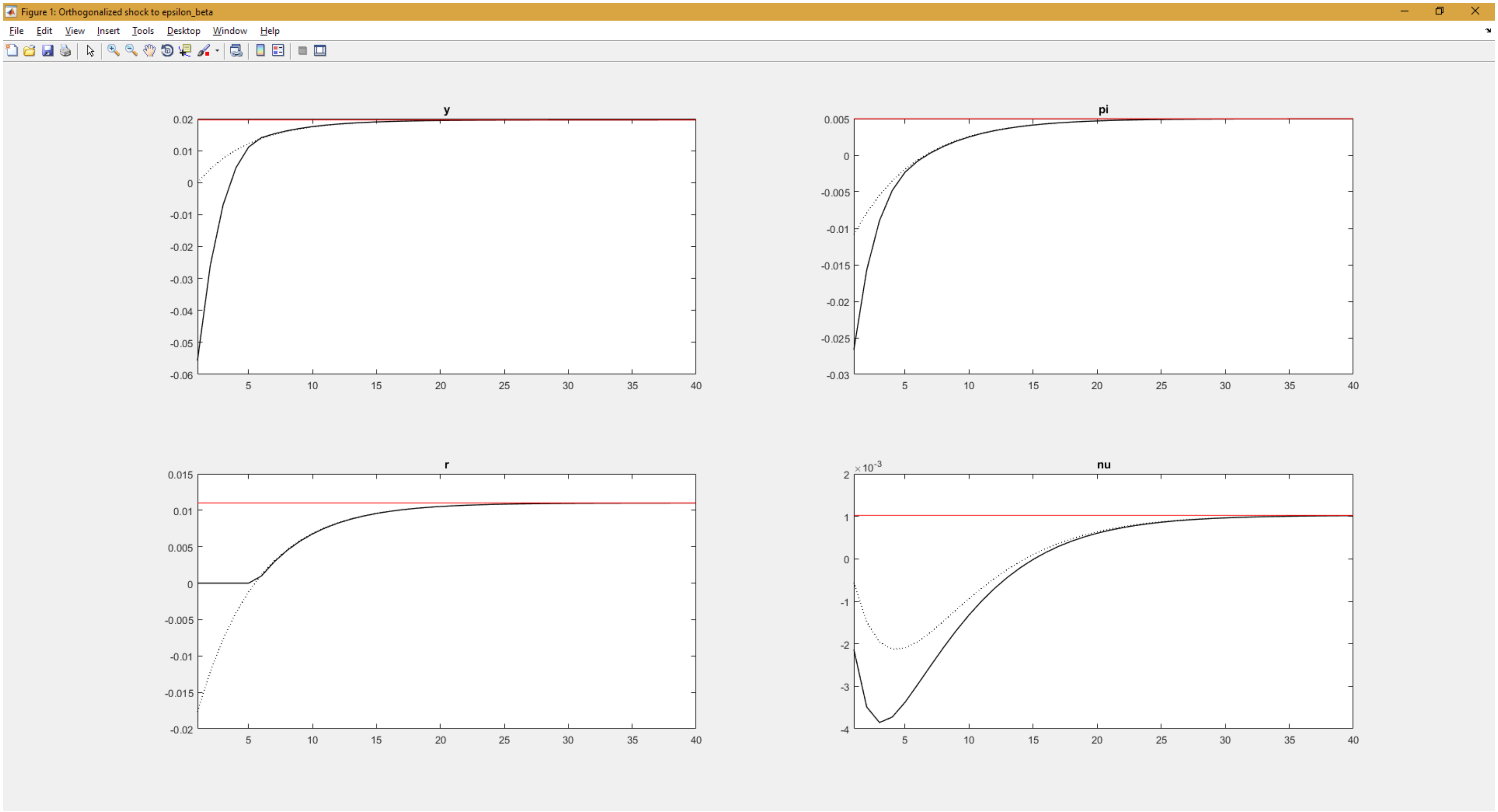
Generating a perfect foresight IRF with diagnostic information, using DynareOBC



Of course DynareOBC will output IRFs

Output

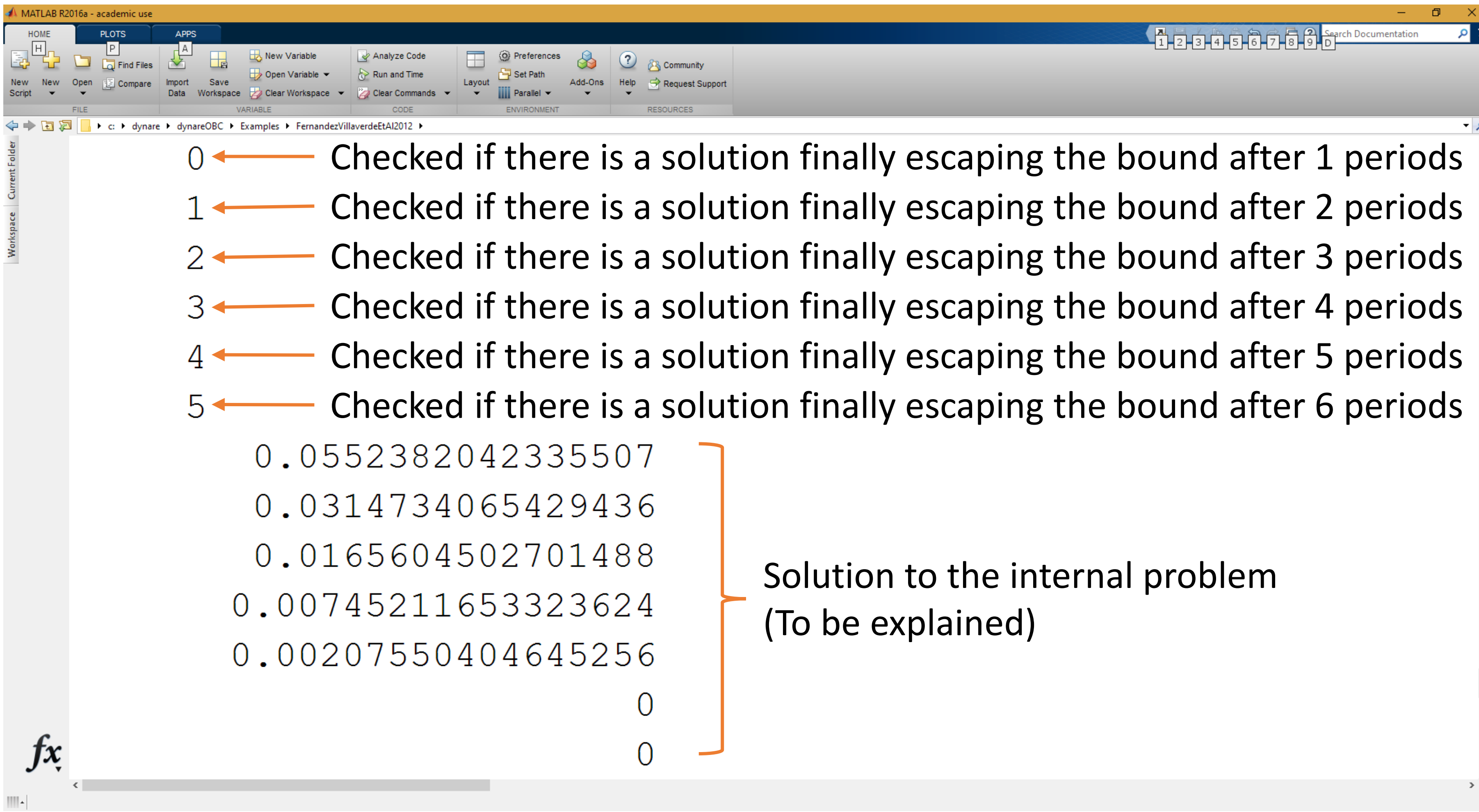
Inflation



Nominal Interest Rates

Price dispersion

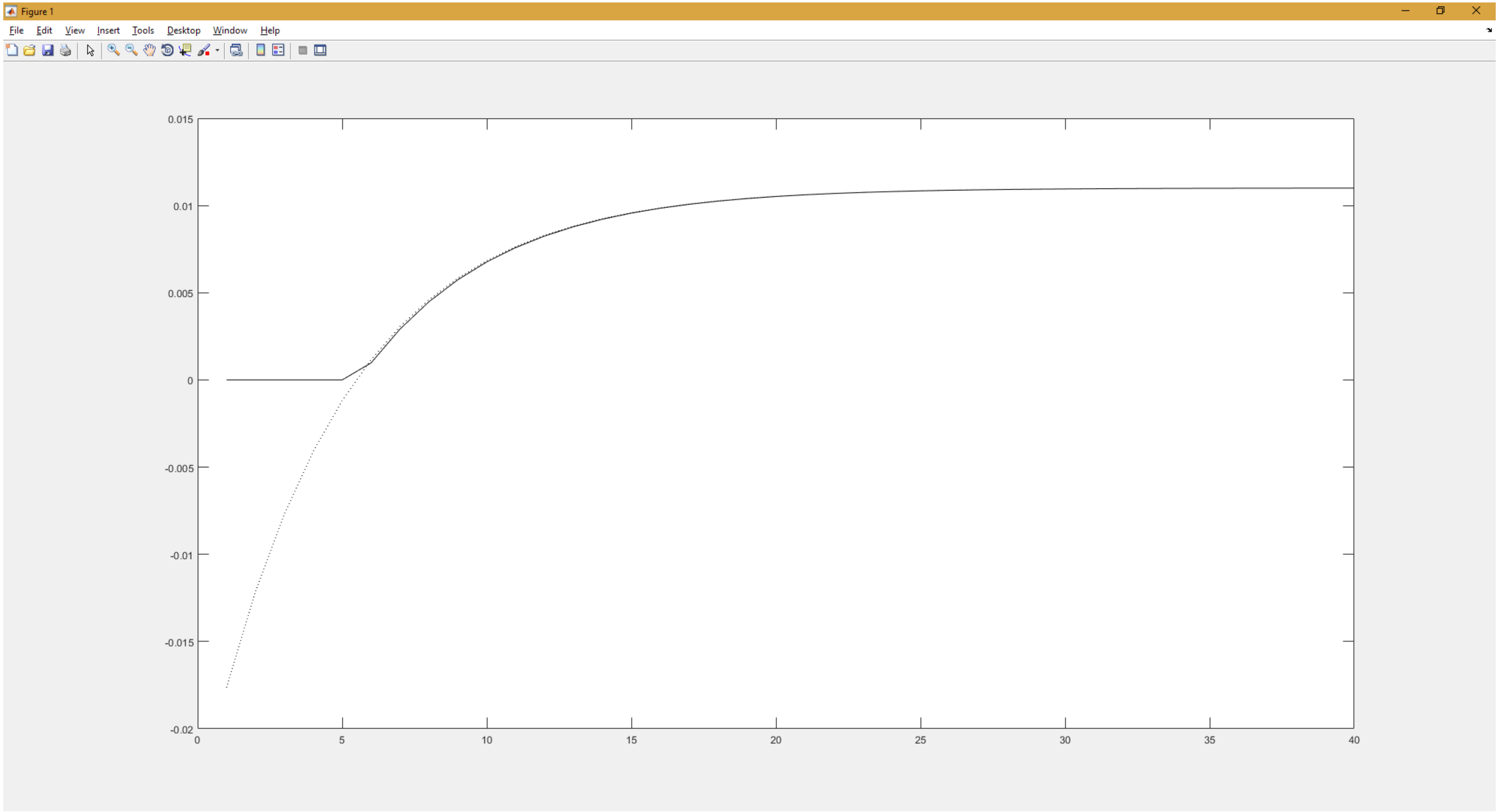
But will also output diagnostic information



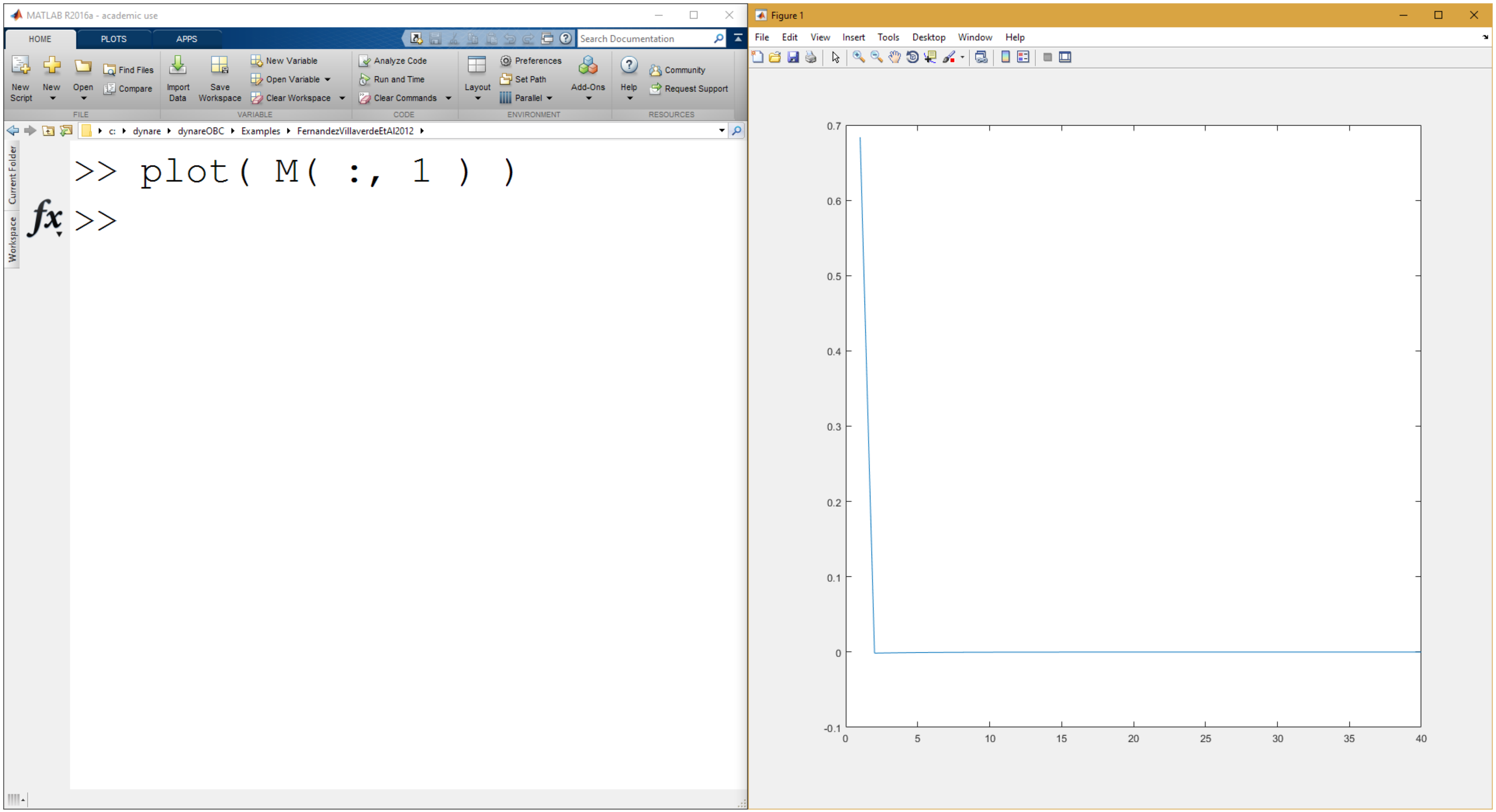
Relationship between the internal problem solution and the IRFs

```
>> r_steady = log( 1.005 / 0.994 );
   q = dynareOBC_.IRFsWithoutBounds.r_epsilon_beta' + r_steady;
   M = dynareOBC_.MMatrix( 1:40, 1:5 );
   y = [ 0.055238204233507
         0.031473406542943
         0.016560450270148
        0.00745211653323624
        0.00207550404645256 ];
   plot( 1:40, q, ':k', 1:40, q + M * y, '-k' );
>>
```

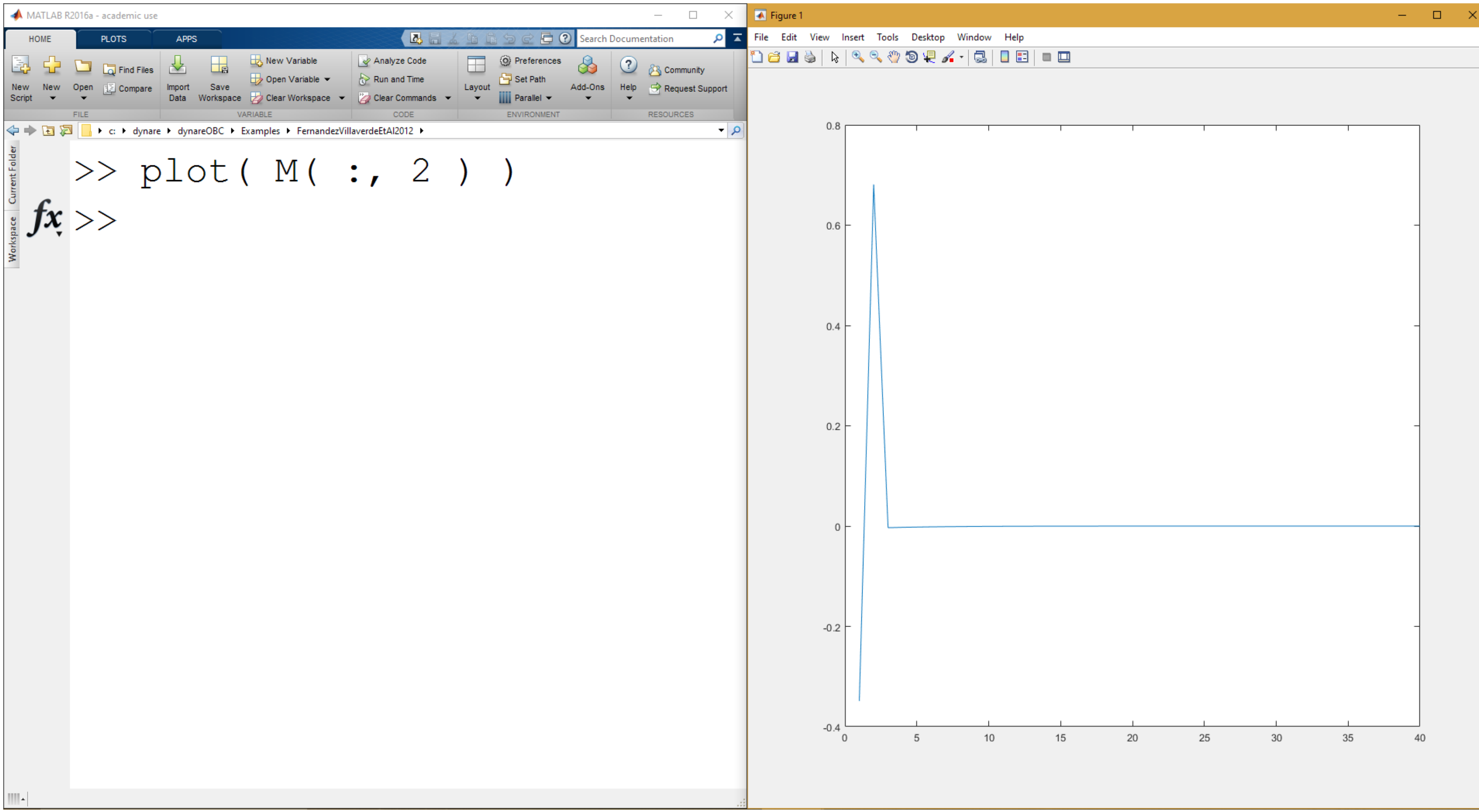
$q + My$ is the IRF imposing the bound



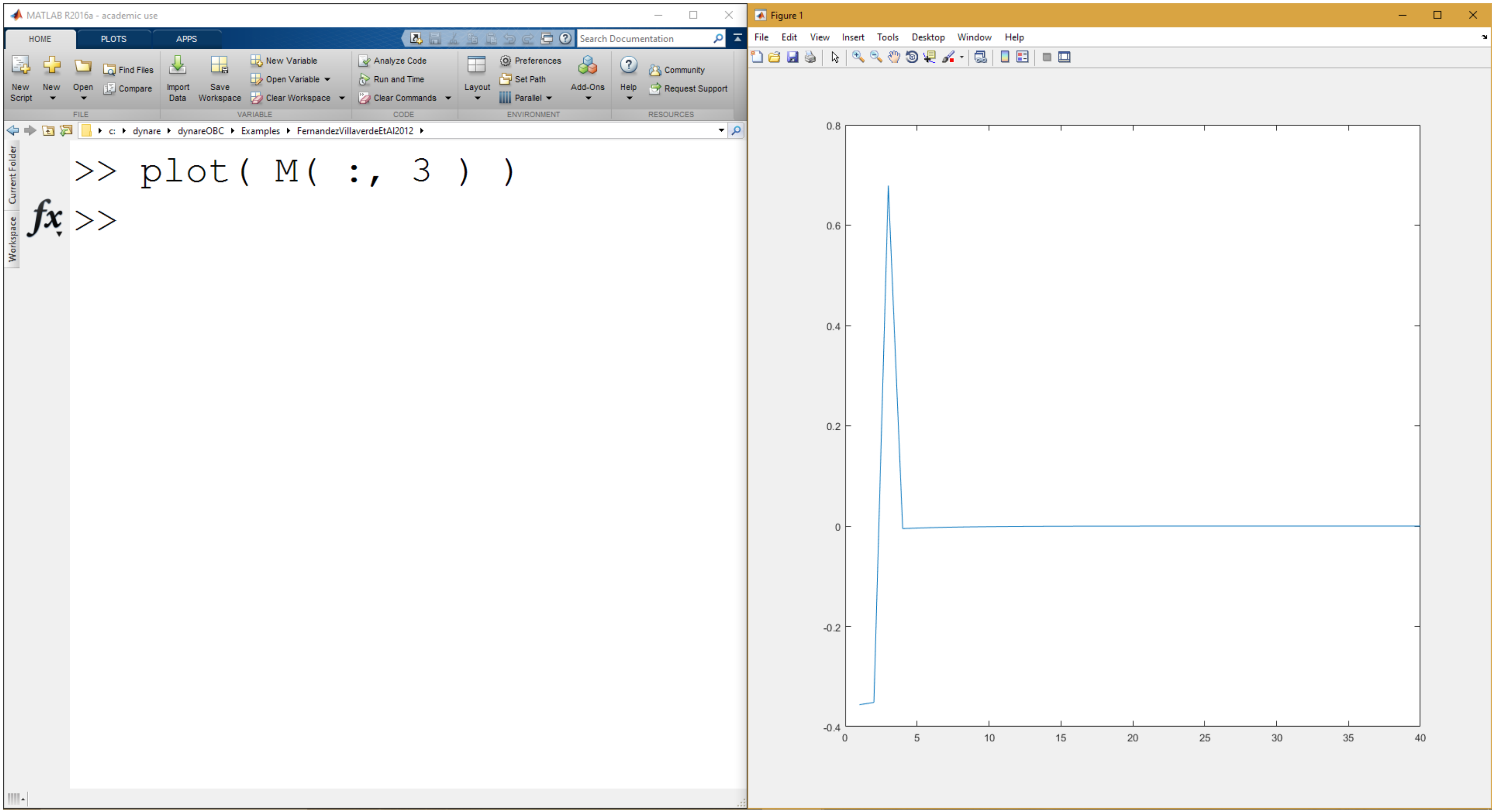
The 1st column of M is the impulse response to a monetary policy shock



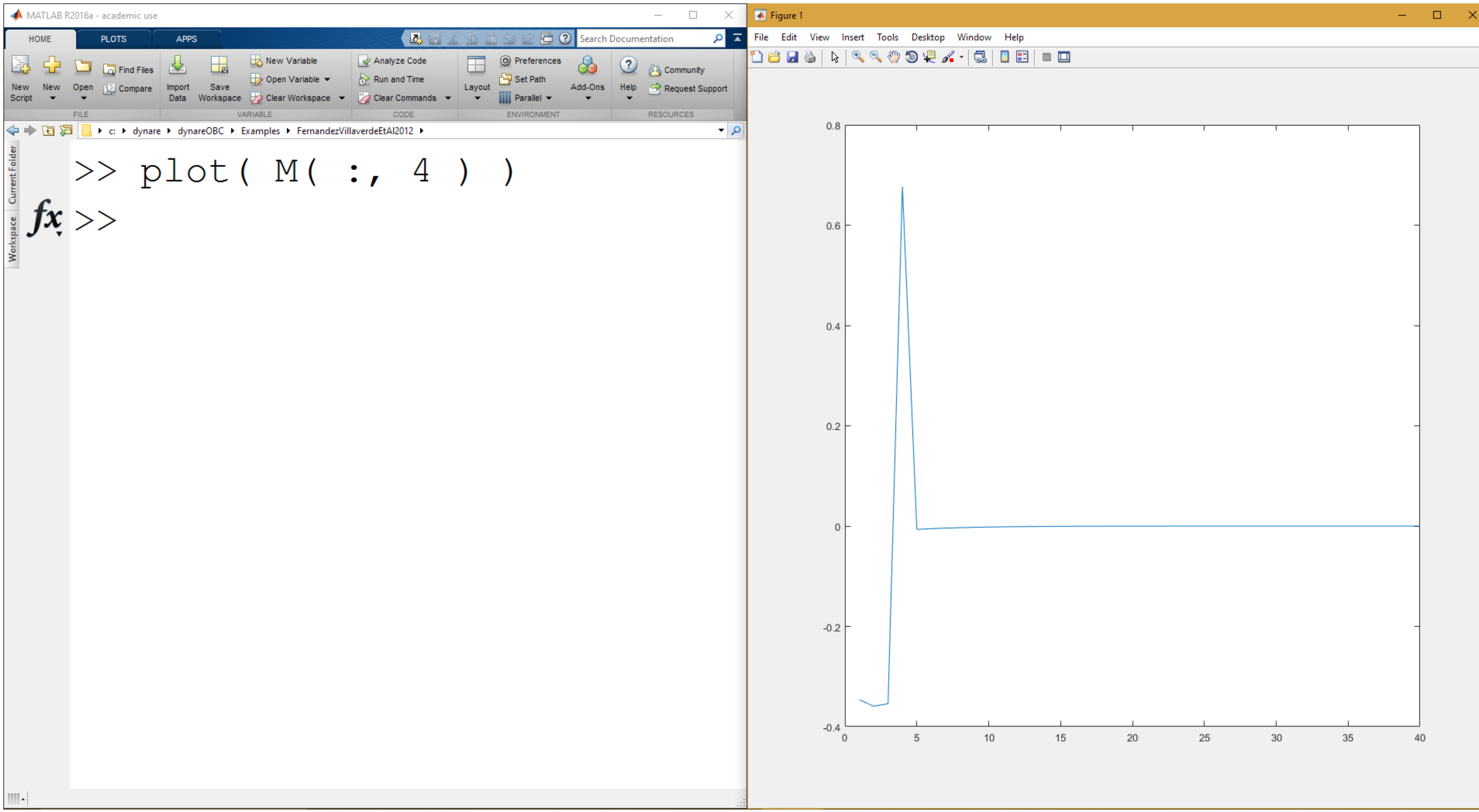
The 2nd column of M is the impulse response to the news that a monetary policy shock will hit in 1 period



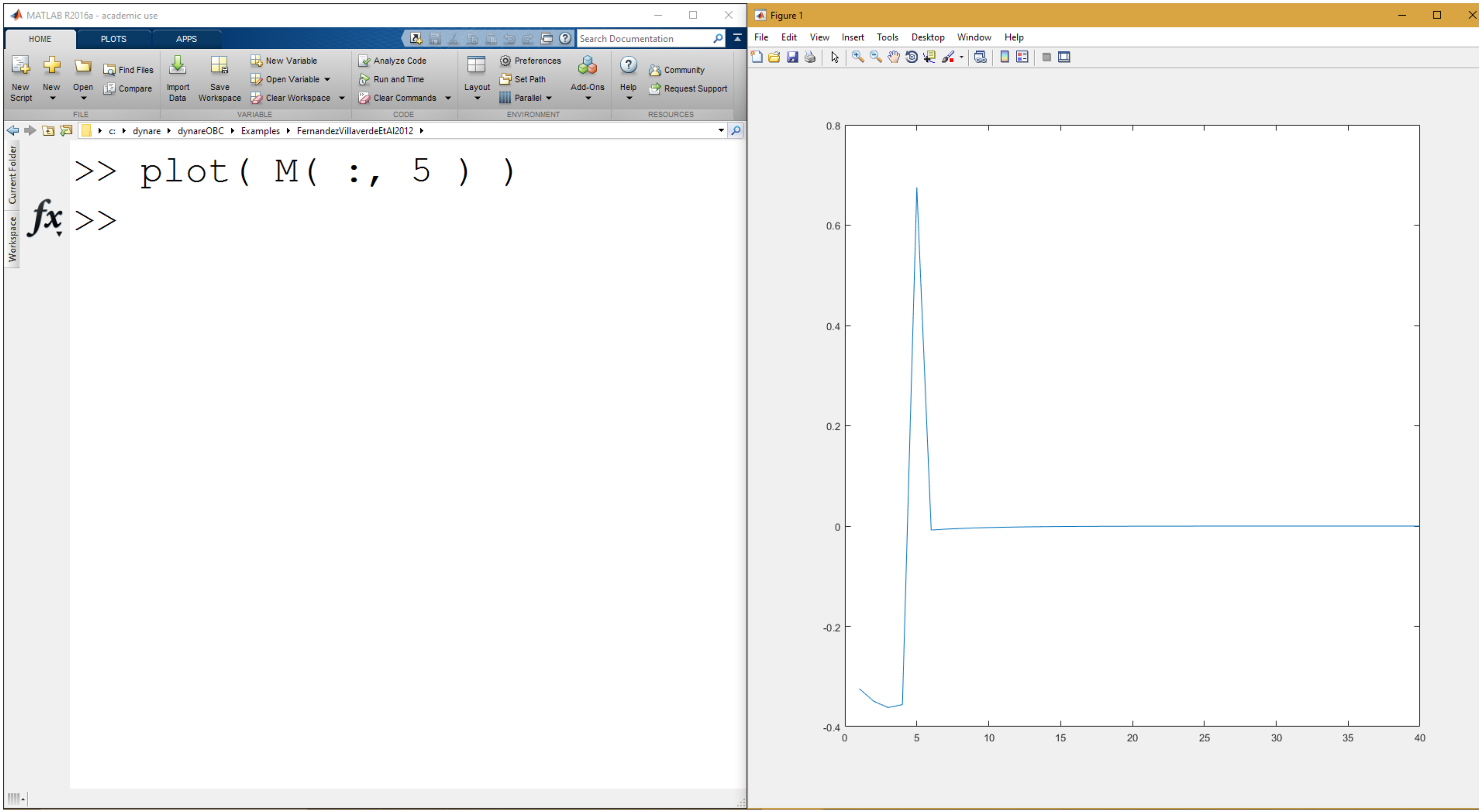
The 3rd column of M is the impulse response to the news that a monetary policy shock will hit in 2 periods



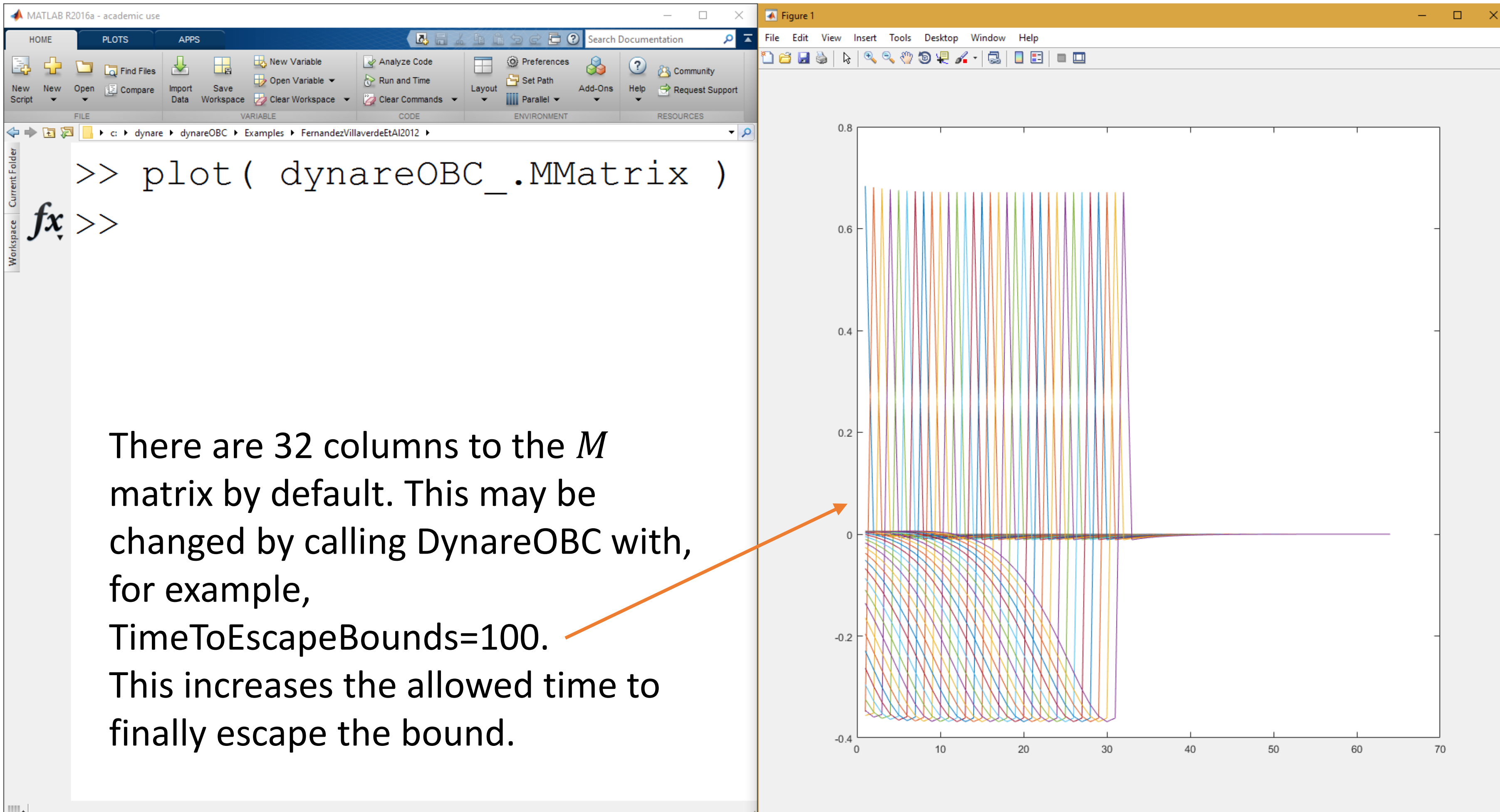
The 4th column of M is the impulse response to the news that a monetary policy shock will hit in 3 periods



The 5th column of M is the impulse response to the news that a monetary policy shock will hit in 4 periods

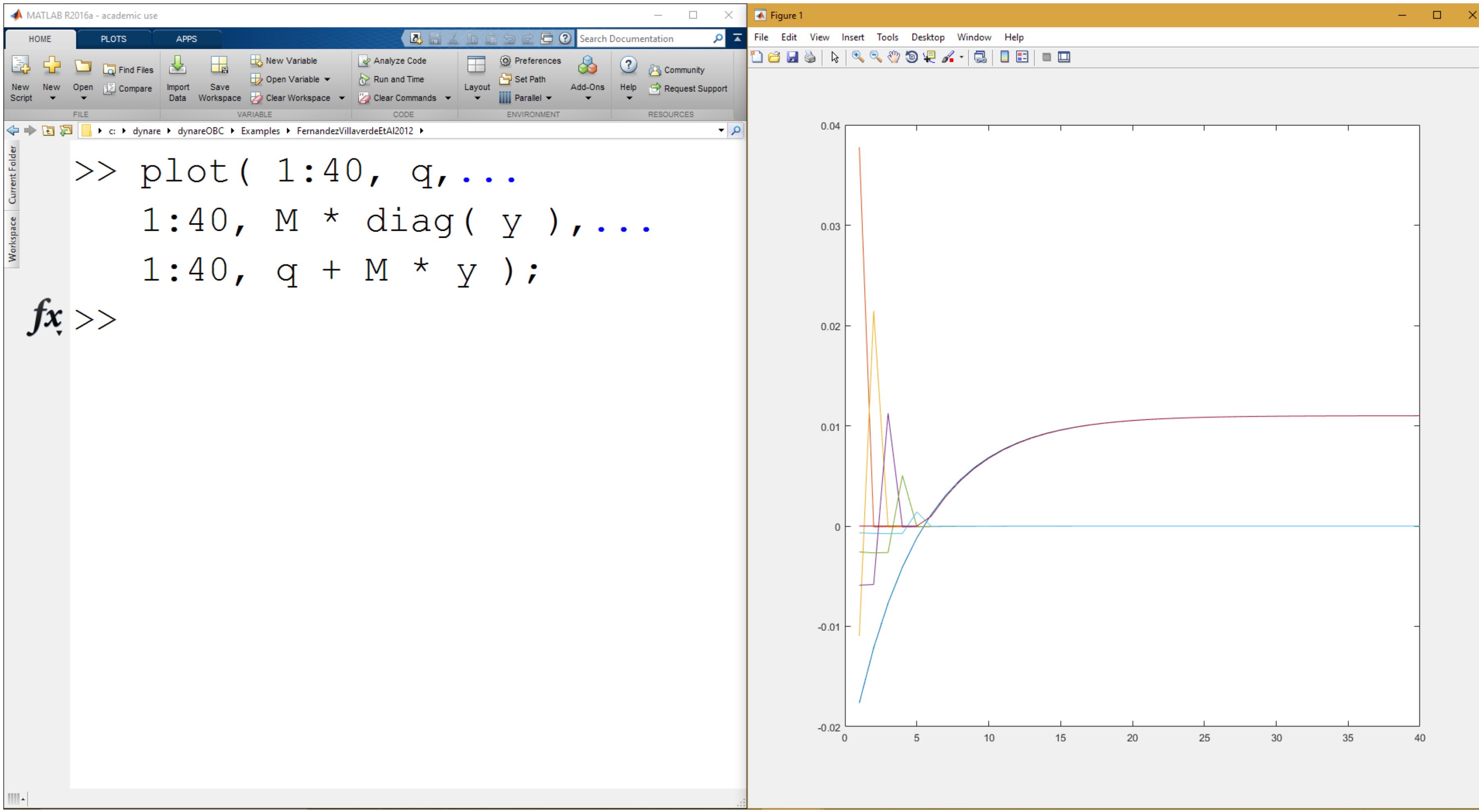


And so on…



There are 32 columns to the M matrix by default. This may be changed by calling DynareOBC with, for example, TimeToEscapeBounds=100. This increases the allowed time to finally escape the bound.

Constructing the solution



A key proposition from Holden (2016a)

- Suppose the model is linear apart from a single ZLB constraint.
- And suppose we are only interested in perfect foresight solutions that return to the standard steady-state.

- Let q be the impulse response of the bounded variable, ignoring the bound.
- Let M be the matrix stacking the impulse responses to news shocks to the bounded equation.

- If there is a y such that: $y \geq 0$, $y'(q + My) = 0$ and $q + My \geq 0$, then $q + My$ gives the impulse response of the bounded variable in a solution to the perfect foresight problem.

- Conversely, if the perfect foresight problem has a solution, then there is a y, with properties as before, which implements that solution.

Dangers of the ZLB

Multiplicity

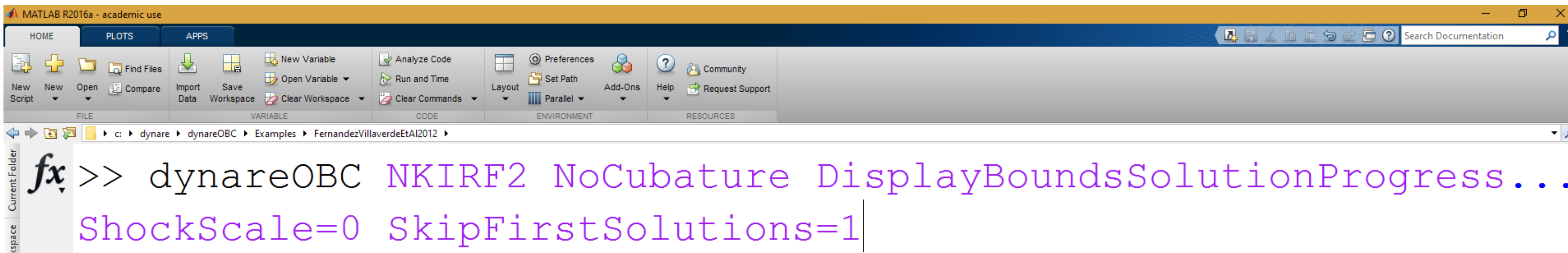
- Suppose that there is no “impulse” at all, so q is constant at the steady-state value.

- One solution is $y = 0$, so the bounded variable remains at its steady-state.

- However, there may be other solutions.

- If everyone believes that some combination of news shocks would hit the bounded variable, and that combination drives the economy to the ZLB in the periods they hit, then a temporary jump to the ZLB can be self fulfilling.

Finding additional solutions with DynareOBC



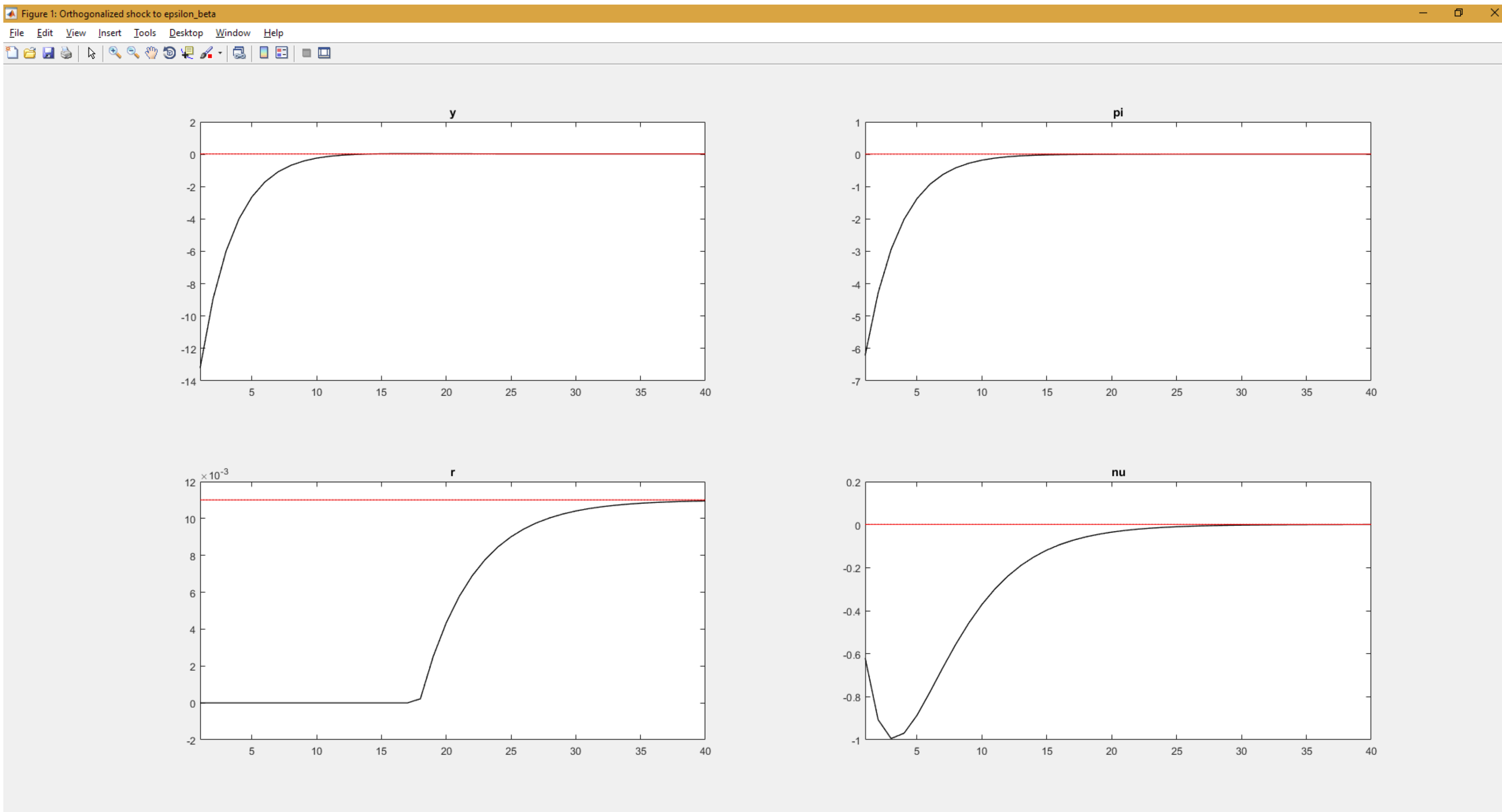
Always ignore the solution in which the bound is finally escaped fastest, unless it is the only one.

Use a magnitude zero shock, so any movement is purely self-fulfilling.

Self-fulfilling jumps to the ZLB may feature huge drops in output and inflation

Output

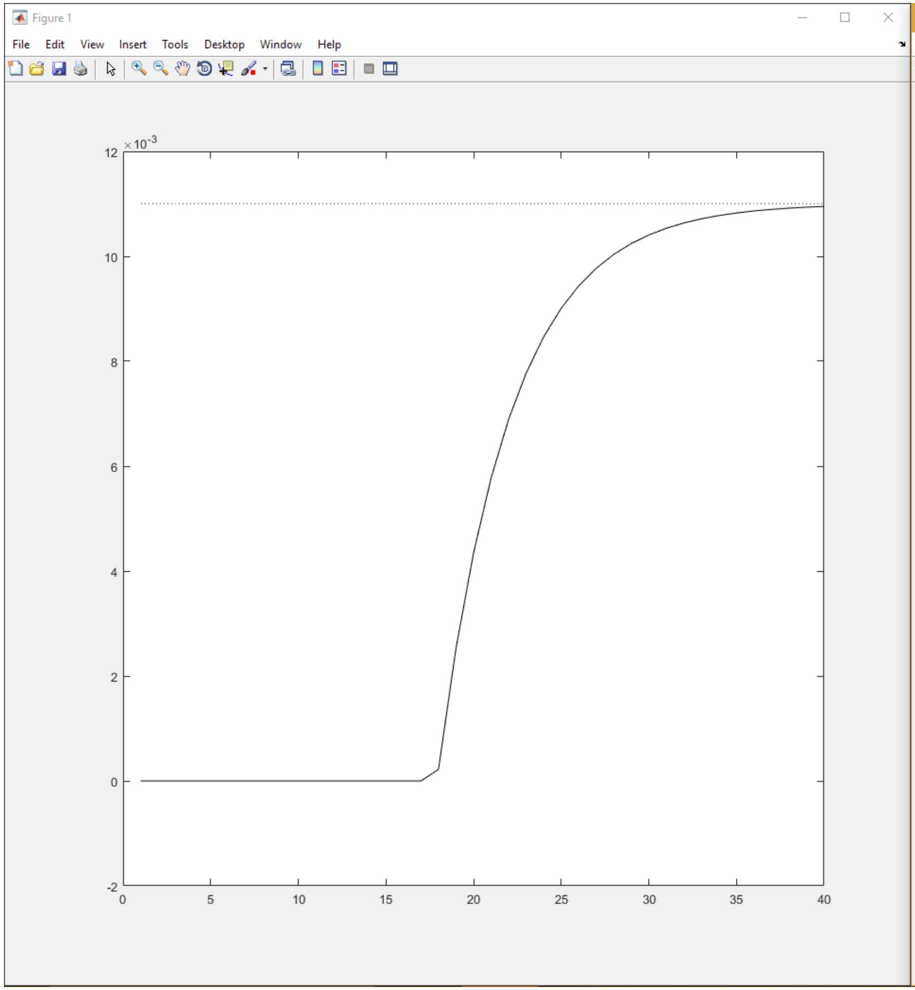
Inflation

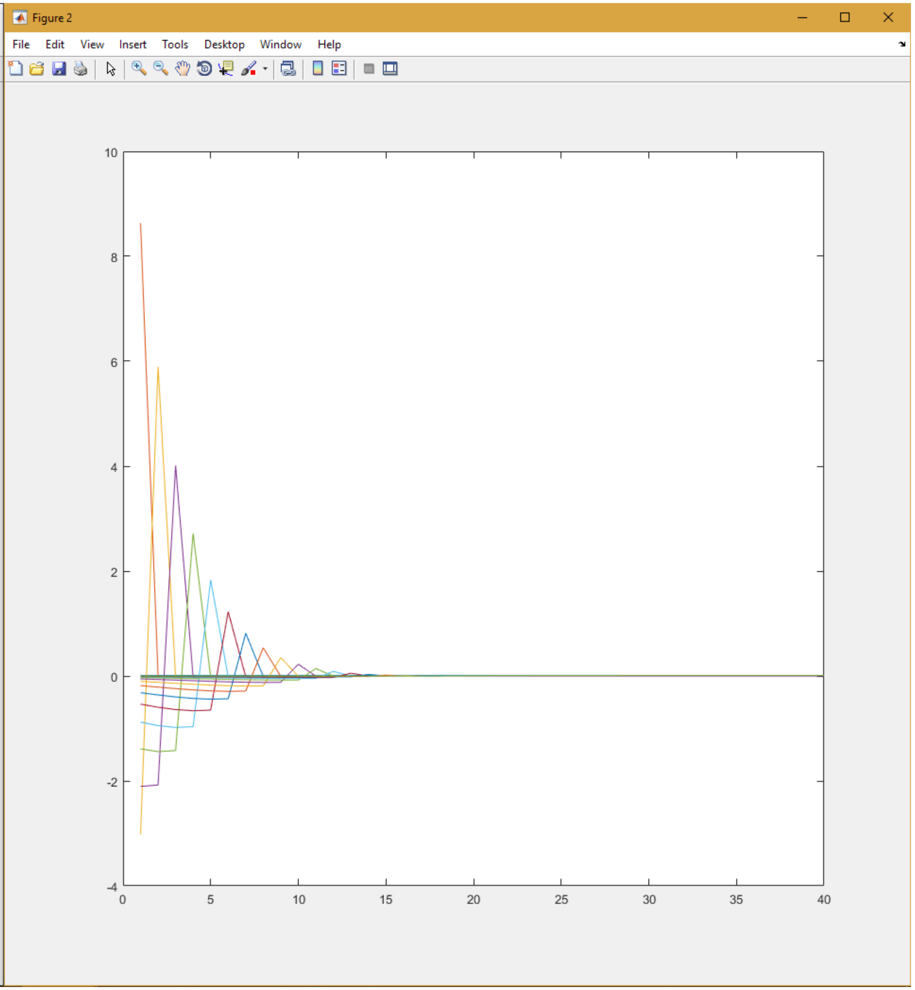


Nominal Interest Rates

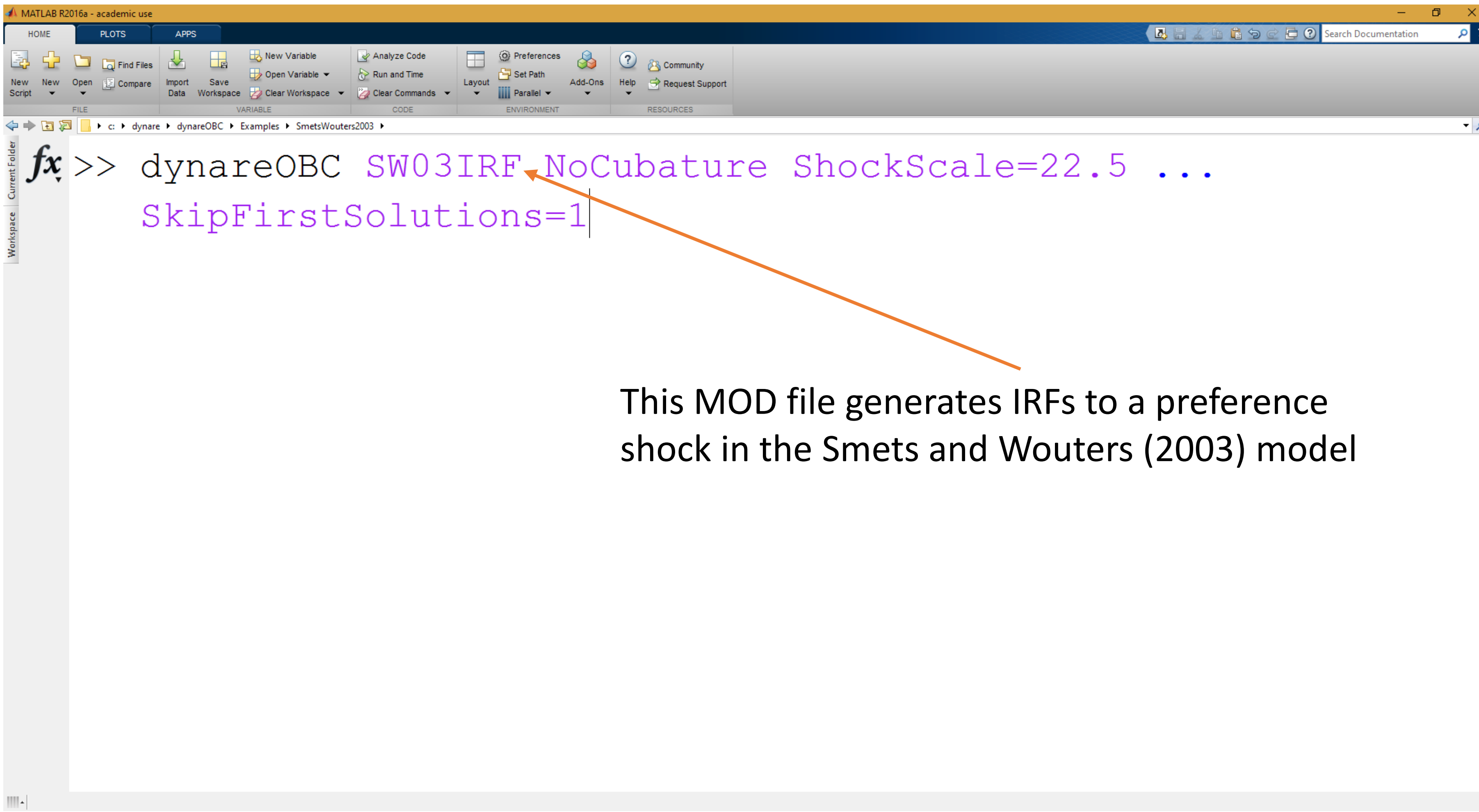
Price dispersion

Construction of a self-fulfilling jump (analogous plots to before)





Self-fulfilling jumps to the ZLB need not feature crazy behaviour if interest rates are already low

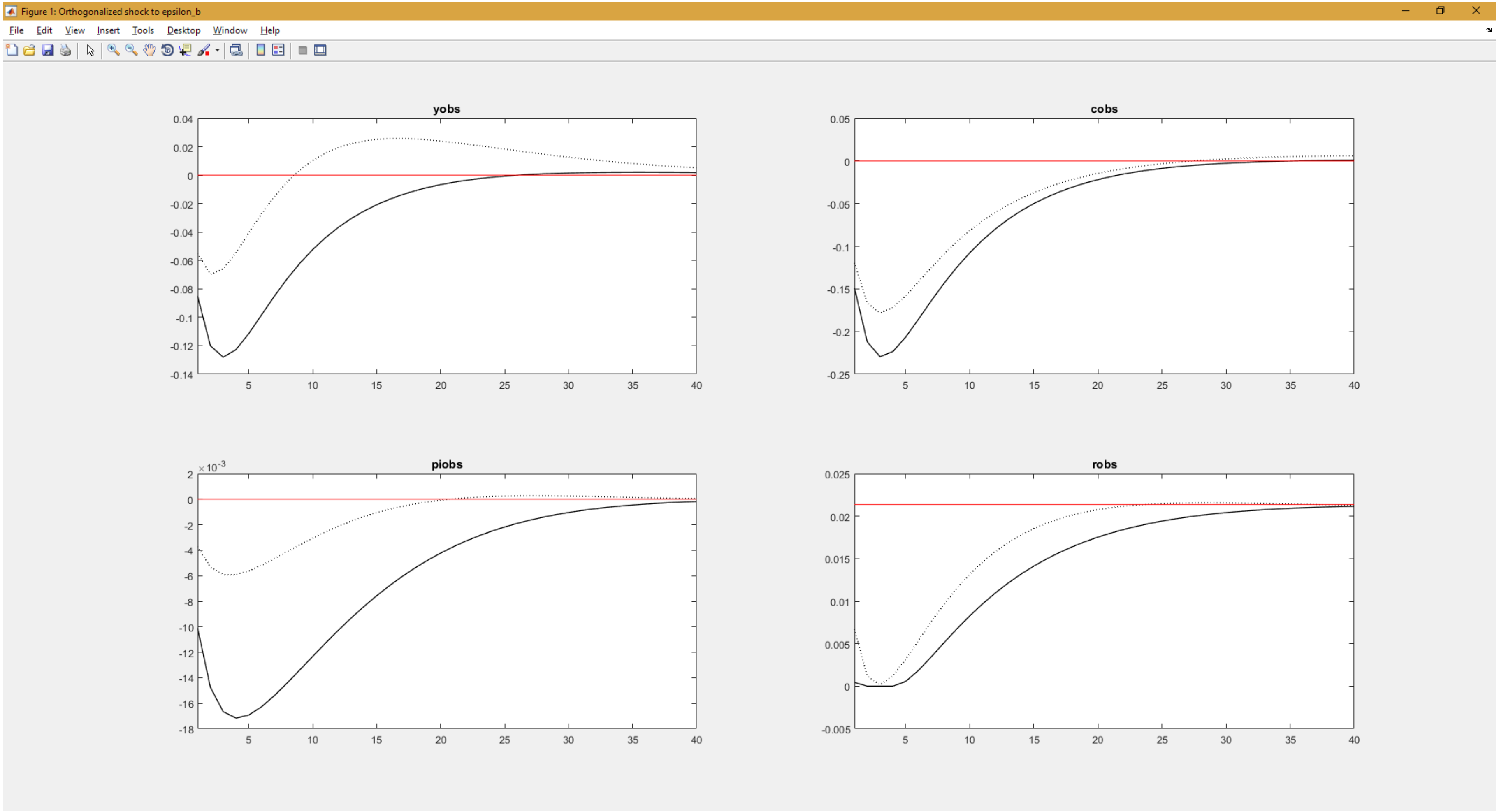


This MOD file generates IRFs to a preference shock in the Smets and Wouters (2003) model

Generated self fulfilling jump to the ZLB is in line with the magnitude of the crisis in the Euro area

Output

Consumption



Inflation

Nominal Interest Rates

Existence and uniqueness conditions

Holden (2016a) proves necessary and sufficient conditions for uniqueness.

It also proves some necessary and some sufficient conditions for existence.

All results are based on examining the properties of the model's M matrix.

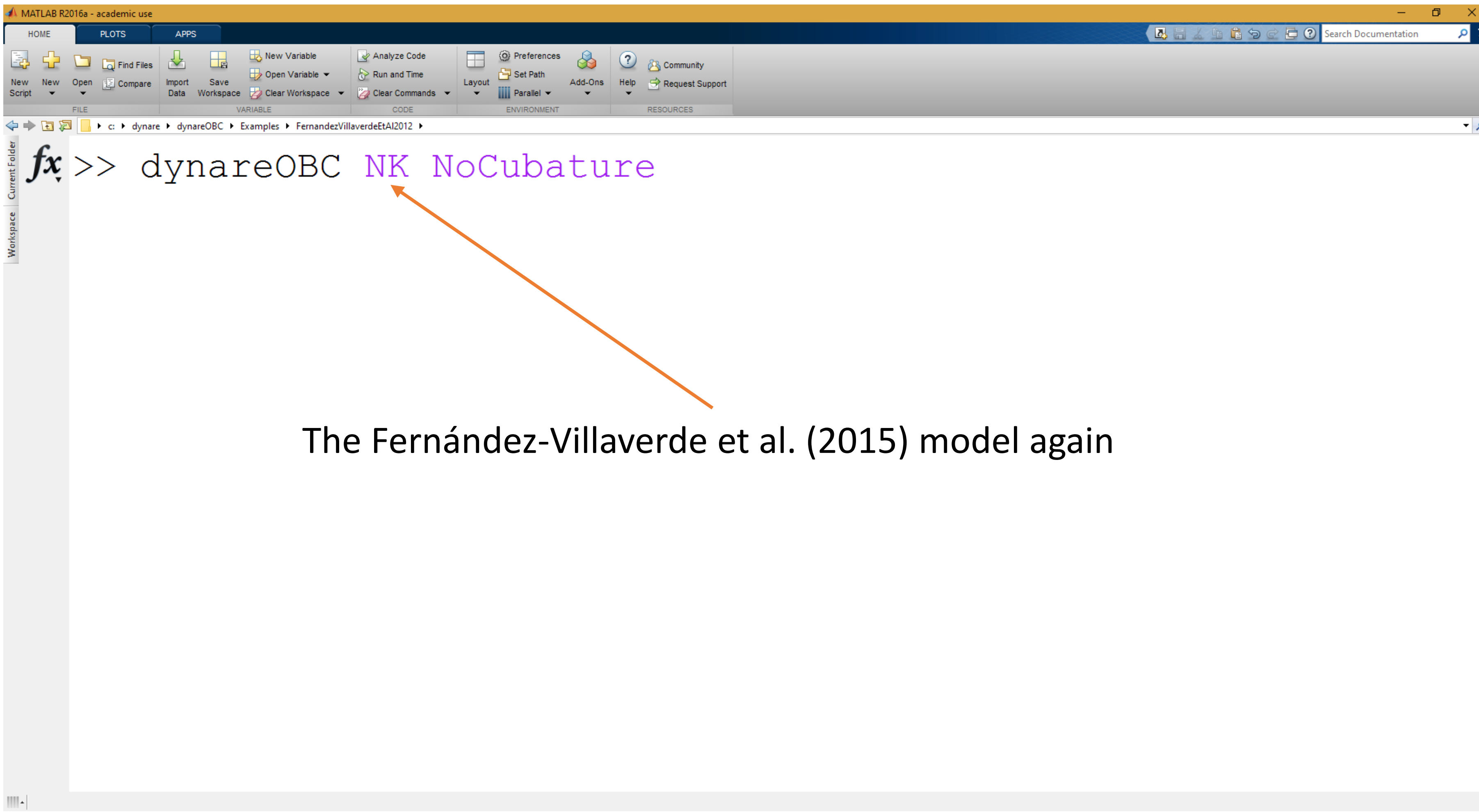
Terminal conditions

- All results are conditional on agents believing that the economy will eventually return to a given (locally-determinate) steady-state.
  - In line with the approach of Brendon, Paustian, and Yates (2013; 2016).

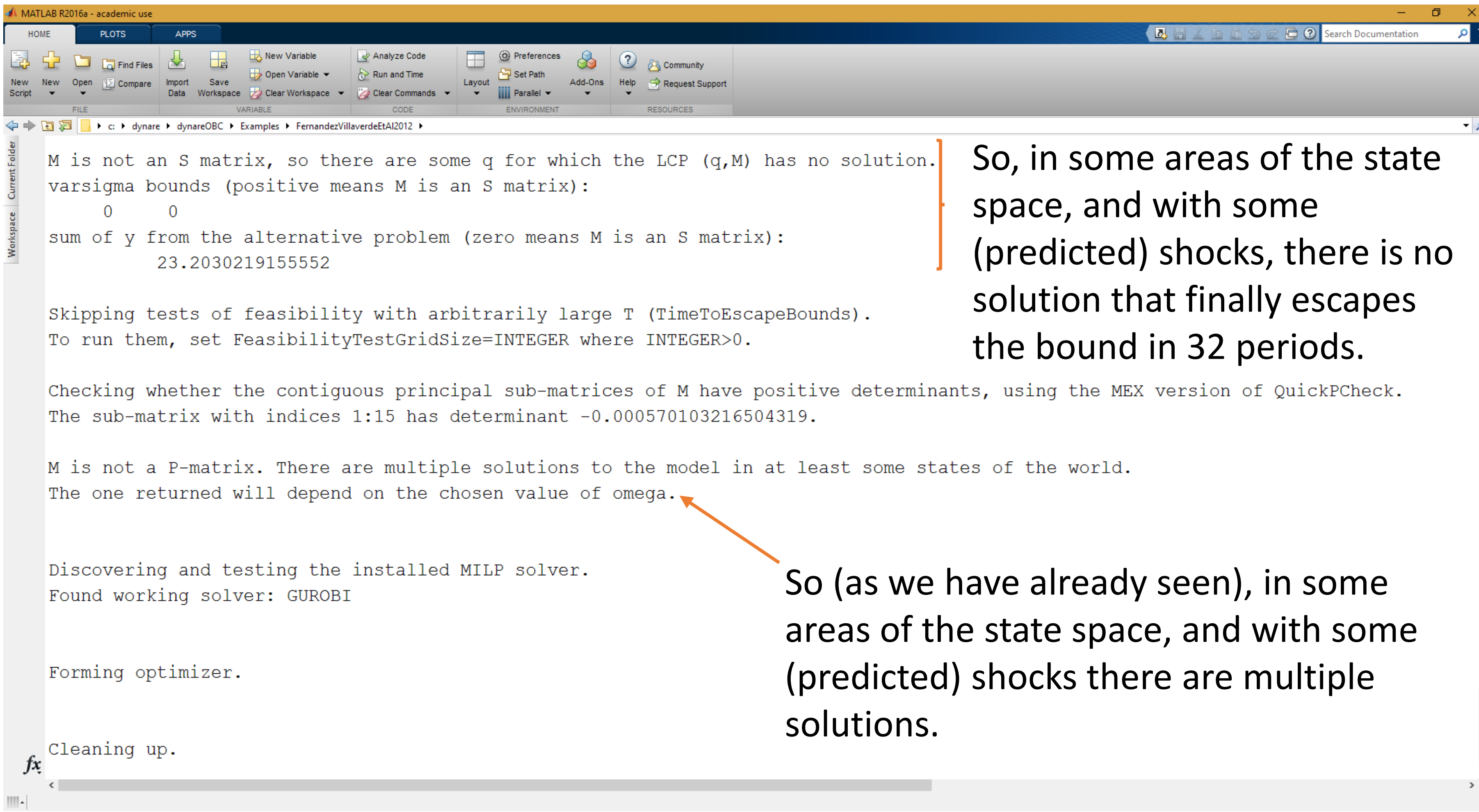
- Accords with a belief in the credibility of the long-run inflation target.
  - Christiano and Eichenbaum (2012) argue deflationary equilibria may be ruled out by switching to a money growth rule following severe deflation, along the lines of Christiano and Rostagno (2001).
  - Belief in credibility of the long-run target is in line with the evidence of Gürkaynak, Levin, and Swanson (2010).

- Contrary to the approach of, e.g. Benhabib, Schmitt-Grohe, and Uribe (2001a,b), Schmitt-Grohe and Uribe (2012), Mertens and Ravn (2014), Aruoba, Cuba-Borda and Schorfheide (2013).

Testing theoretical existence and uniqueness conditions using DynareOBC



DynareOBC's default existence and uniqueness output



So, in some areas of the state space, and with some (predicted) shocks, there is no solution that finally escapes the bound in 32 periods.

So (as we have already seen), in some areas of the state space, and with some (predicted) shocks there are multiple solutions.

Other tests

- DynareOBC also implements tests of existence without any constraint on how long it takes to finally escape the bound.

- It can also test for whether there are times when one solution of multiple hits the bound, and the other does not.

- It can also prove there is a unique solution by checking the full necessary and sufficient conditions, rather than just checking some weaker necessary conditions.

- Please see the ReadMe.pdf and Holden (2016a) for full details.

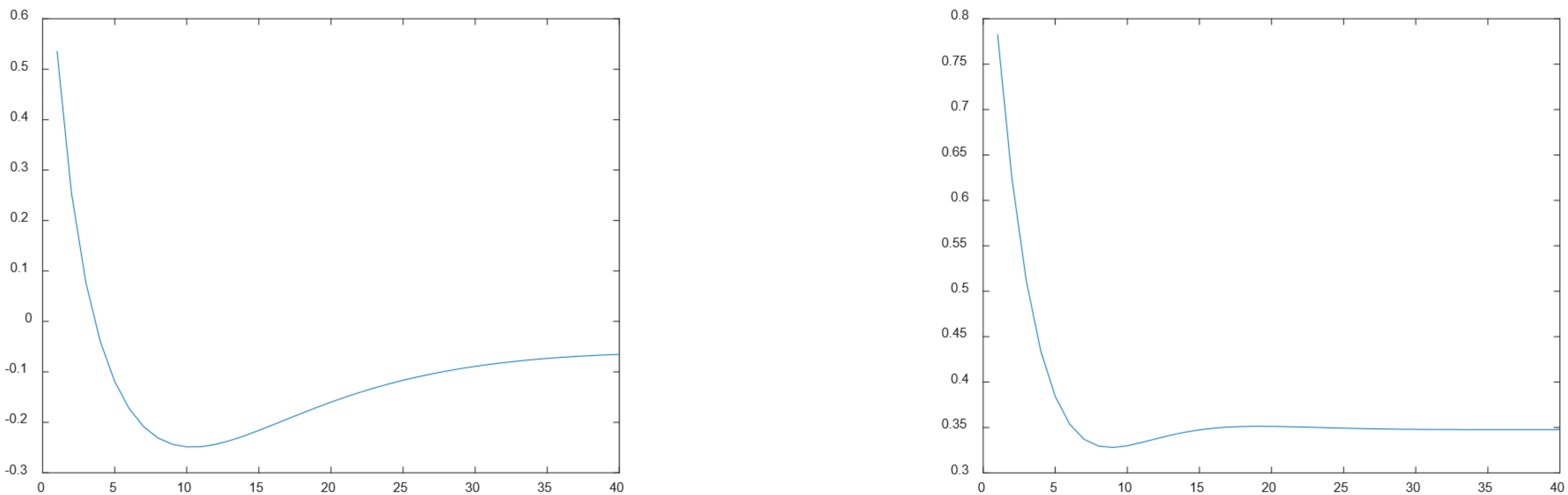
Avoiding multiplicity and non-existence in New Keynesian models

- Across a variety of models and specifications, I have discovered only one type of monetary policy rule that can restore uniqueness and global existence.
  - Necessary to rule out self-fulfilling jumps to the ZLB, and getting stuck there permanently.

- Large Taylor rule coefficients do not help.
- Nor does responding to expected output or inflation.

- However, in all models encountered, a response (however weak) to the price level does produce a unique solution, and global existence.
  - The implied promised future inflation following deflation today prevents inflation from falling far enough to generate a self-fulfilling jump to the ZLB.

Large Taylor rule coefficients do make multiplicity harder to sustain (though not impossible)

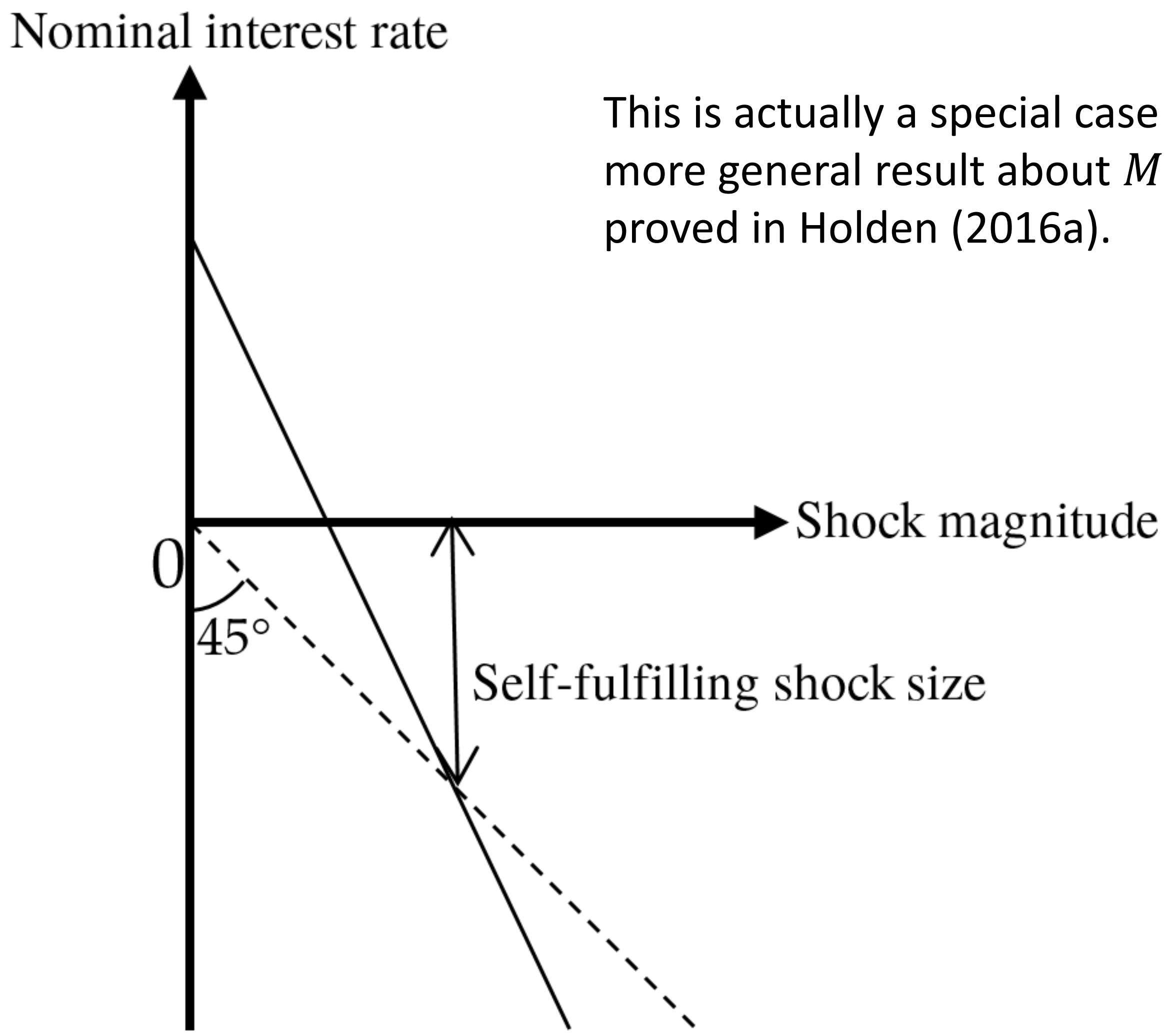


The Smets and Wouters (2003) model | The Smets and Wouters (2007) model

Figure 6: The diagonals of the M matrices for the Smets and Wouters (2003; 2007) models

With a sufficiently large coefficient on inflation, the (2003) model can produce a positive diagonal of M, as in the (2007) model.

Why negative diagonals of M lead to multiplicity.



This is actually a special case of a much more general result about M matrices that is proved in Holden (2016a).

Tractable, accurate, simulation of large models

Solving the internal problem

- DynareOBC has to repeatedly solve the following problem, called a "linear complementarity problem" or LCP.
  - Given q and M, find $y \geq 0$ such that $y'(q + My) = 0$ and $q + My \geq 0$.
- I show in Holden (2016b) that this LCP can be encoded as a mixed integer linear programming problem (MILP), in such a way that the MILP always has a solution, even when the LCP does not.
  - When the LCP has no solution, the MILP's solution will reveal this.
- Since MILPs may be solved in finite time, this implies that my algorithm will always find a solution in finite time.
- No other general solution procedure for models with OBCs has this property.
  - All other methods may fail to converge without that implying non-existence.

Broad idea behind simulation in DynareOBC (See Holden 2016b for the details)

- Extended path type simulation is based on solving each period as if all agents believed that no shocks would arrive in future.

- Stochastic extended path type simulation (Adjemian and Juillard 2013) averages over draws of future shocks.
- It is as if all agents believed that there are shocks in S future periods, but that they will learn the value of *all* these shocks next period.

- We take the stochastic extended path approach, but exploit properties of our solution procedure to enable us to consider S periods of future uncertainty with only polynomial in S evaluations.
  - Rather than exponential in both S and the number of shocks, as in Adjemian and Juillard (2013).

Accuracy results

- To convince you that DynareOBC works in practice, we give some accuracy results.

- We examine three simple models, to ensure accuracy tests are reliable.

- These are:
  - A very simple model with an analytic solution.
  - A model for which log-linearization gives the exact answer in the absence of bounds.
  - An otherwise linear open-economy model.

A simple model with an analytic solution

- Closed economy, no capital, inelastic unit labour supply.
- Households maximise:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

- Subject to the budget constraint:

$$A_t + R_{t-1} B_{t-1} = C_t + B_t$$

  - A_t is productivity. R_t is the real interest rate.
  - B_t is the household's holdings of zero net supply bonds.

- Define $g_t \coloneqq \log A_t - \log A_{t-1}$. Evolves according to:

$$g_t = \max\{0, (1-\rho)\bar{g} + \rho g_{t-1} + \sigma \varepsilon_t\},$$

- $\varepsilon_t \sim \text{NIID}(0,1)$, $\beta \coloneqq 0.99$, $\gamma \coloneqq 5$, $\bar{g} \coloneqq 0.05$, $\rho \coloneqq 0.95$, $\sigma \coloneqq 0.07$.

Accuracy in the simple model, along simulated paths of length 1000 (after 100 periods dropped)

Bound in Model	*Order*	*Cubature*	*Seconds*	*Log_{10} Mean Abs Error*	*Log_{10} Root M.S.E.*	*Log_{10} Max Abs Error*	*Log_{10} Mean Abs Error at Bound*
No	*1*	*N/A*	66	-3.213	-3.213	-3.213	
No	*2*	*N/A*	62	-16.82	-16.63	-15.78	
No	*3*	*N/A*	53	-16.70	-16.57	-15.95	
Yes	1	No	141	-2.435	-2.218	-1.882	-1.882
Yes	2	No	139	-2.425	-2.194	-1.862	-1.862
Yes	3	No	140	-2.425	-2.194	-1.862	-1.862
Yes	1	Monomial, Degree 3	274	-3.136	-3.073	-2.725	-3.131
Yes	2	Monomial, Degree 3	1537	-3.378	-3.172	-2.706	-3.893
Yes	3	Monomial, Degree 3	1397	-3.378	-3.172	-2.706	-3.893
Yes	2	Sparse, Degree 3	1794	-3.016	-2.777	-2.415	-2.415
Yes	2	Sparse, Degree 5	1840	-3.016	-2.777	-2.415	-2.415
Yes	2	Sparse, Degree 7	2009	-3.280	-3.032	-2.663	-2.663
Yes	2	QMC, 15 Points	1965	-3.040	-2.895	-2.664	-2.664
Yes	2	QMC, 63 Points	3184	-3.394	-3.260	-3.020	-3.020
Yes	2	QMC, 1023 Points	5197	-3.804	-3.638	-3.351	-3.351

[1] All timings are "wall" time, and include time spent starting the parallel pool, time spent compiling code (although written in MATLAB, DynareOBC generates and compiles C code for key routines), and time spent calculating accuracy. Code was run on one of the following (very similar) twenty core machines: 2x E5-2670 v2 2.5GHz, 64GB RAM; 2x E5-2660 v3 2.6GHz, 128GB RAM. Use of machines with network attached storage means that there may be some additional variance in these timings.

[2] Errors conditional on the bounded variable being less than *0.0001*. The numbers for this column would be identical had we used root mean squared errors or maximum absolute errors, conditional on being at the bound.

A model for which log-linearization is exact without bounds

- The social planner chooses consumption, C_t, L_t, and K_t, to maximise:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left[\log C_{t+k} - \frac{L_{t+k}^{1+\nu}}{1+\nu}\right],$$

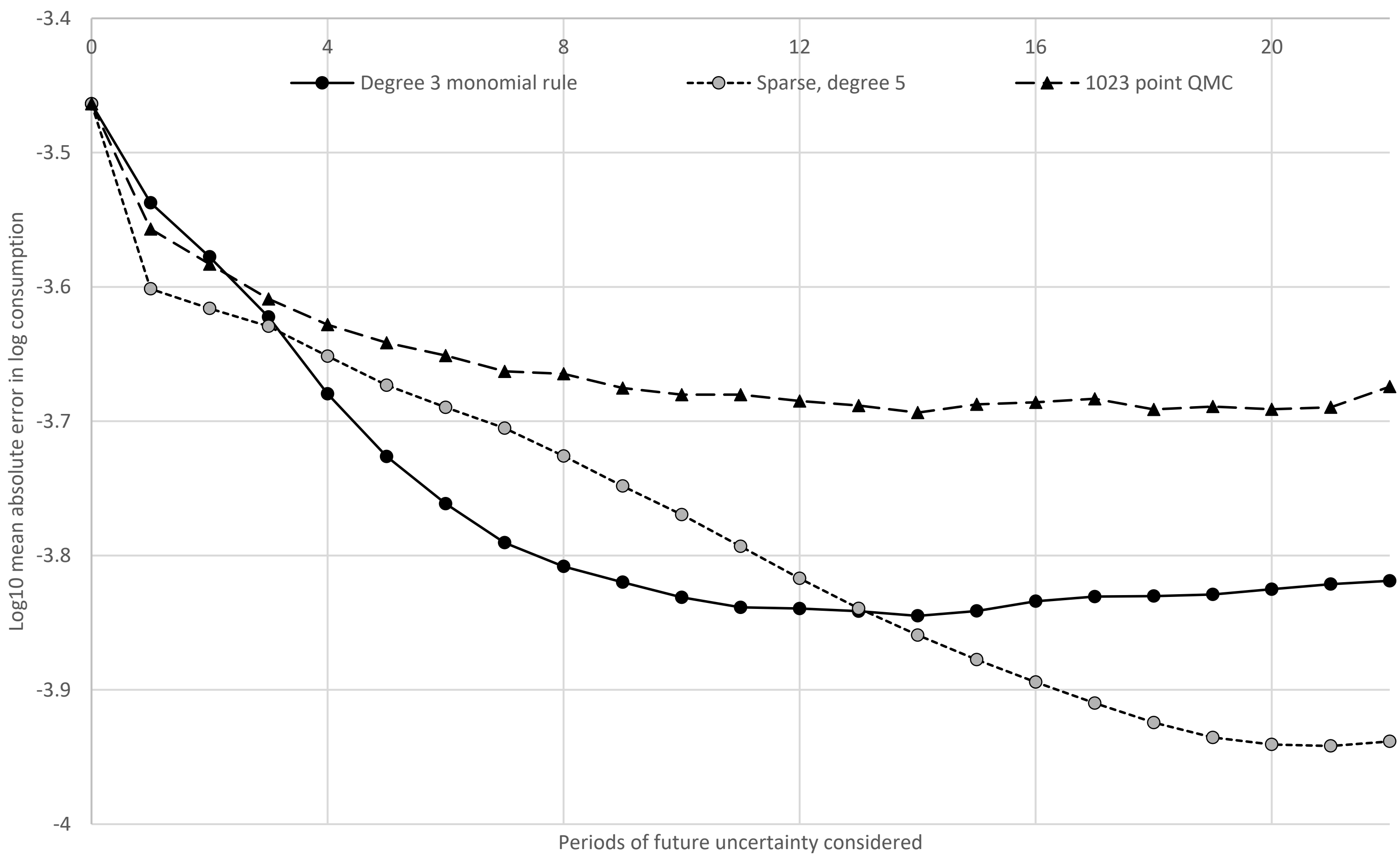
- subject to the capital constraint: $K_t \geq \theta K_{t-1}$,
- and the budget constraint $C_t + K_t = Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$.

- Productivity, A_t, evolves according to $A_t = A_{t-1}^{\rho} \exp \varepsilon_t$, where $\varepsilon_t \sim \mathrm{N}(0, \sigma^2)$.

- Set $\alpha = 0.3$, $\beta = 0.99$, $\nu = 2$, $\theta = 0.99$, $\rho = 0.95$ and $\sigma = 0.01$.

- Compare to a full global solution.

Effect of increasing periods of uncertainty on accuracy, along simulated paths



An otherwise linear open-economy model

- The social planner chooses C_t, D_t and B_t to maximise:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left[-\frac{1}{2}(1-C_t)^2 - \frac{\phi}{2} B_t^2 \right],$$

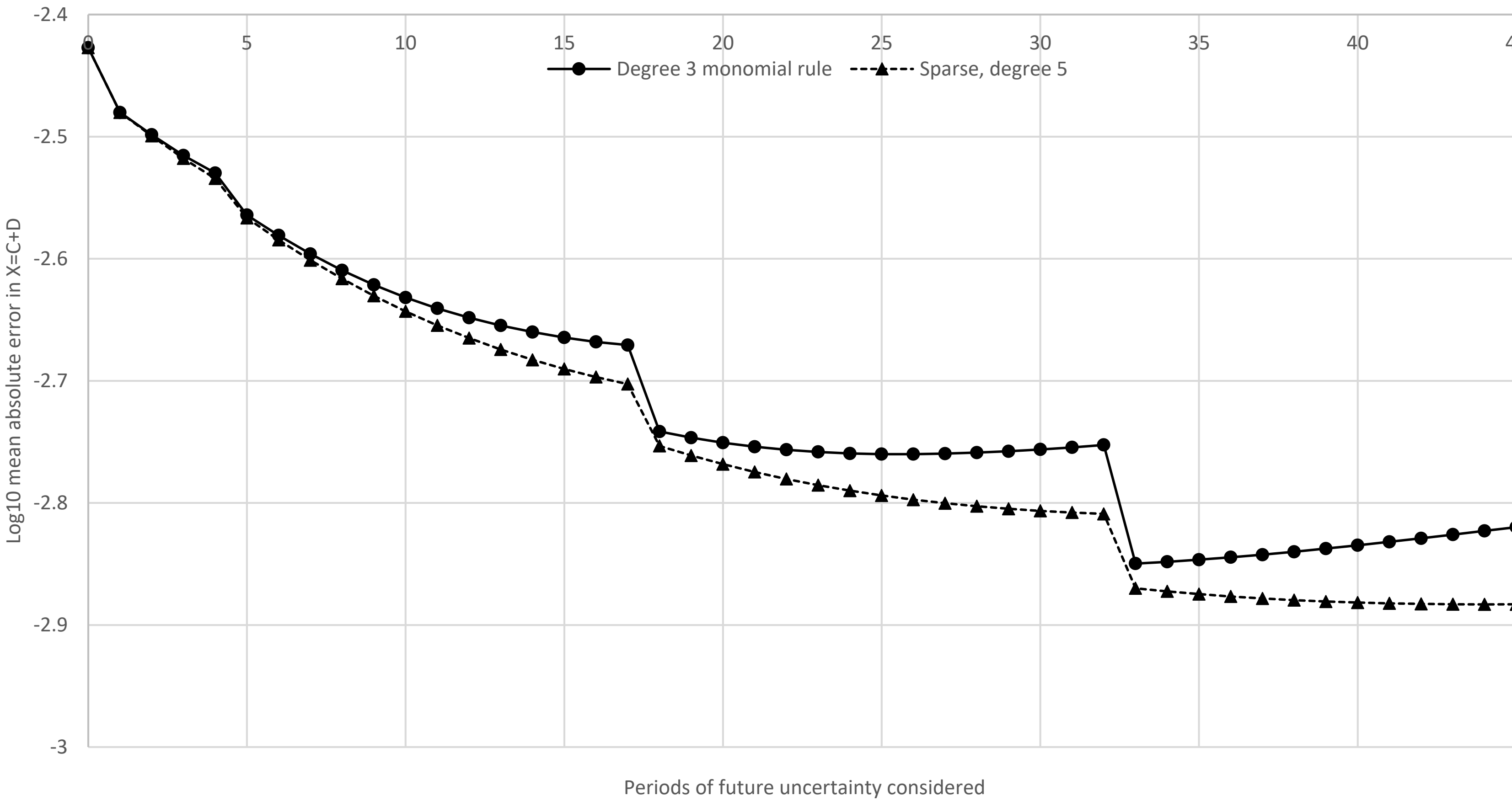
- subject to the budget constraint: $C_t + D_t + B_t - RB_{t-1} = Y_t = \max\{\underline{Y}, A_t\}$,
- the positivity constraints: $0 \leq C_t,\ 0 \leq D_t$,
- and the certain repayment of interest constraint:

$$\forall k \in \mathbb{N}^+, \qquad \Pr_t\big((R-1)B_t \leq Y_{t+k}\big) = 1.$$

- Productivity evolves according to $A_t = (1-\rho)\mu + \rho A_{t-1} + \sigma\varepsilon_t$, where $\varepsilon_t \sim \text{NIID}(0,1)$.

- Set $\beta = 0.99,\ \mu = 0.5,\ \rho = 0.95,\ \sigma = 0.05,\ \underline{Y} = 0.25,\ R = \beta^{-1}$ and $\phi = R - 1$.

Effect of increasing periods of uncertainty on accuracy, along simulated paths (as before)



The DynareOBC toolkit (1/2)

- Complete code to test your own models is available under an open source license from:

  https://github.com/tholden/dynareOBC

- As we've seen, to use it, just include a max, min or abs in your mod file, then type "dynareOBC modfilename".
  - You can have as many bounds as you like.

- Assorted command line options are documented on the home page and in the ReadMe.pdf.

- Provides accurate simulation under rational expectations even for large models, as documented in the computational companion paper.

- Also supports estimation.

The DynareOBC toolkit (2/2)

- Even if you do not have OBCs in your model, DynareOBC may be useful since it can:
  - Simulate MLVs, including integrating over ones with +1 terms, which makes checking accuracy very easy.
  - Perform exact, faster, average IRF calculation without Monte Carlo.
  - Estimate non-linear models at 3^{rd} order using the cubature Kalman filter.

- DynareOBC is covered in the advanced DSGE modelling summer course run at the University of Surrey.

- I’m also happy to run courses at your institutions.
  - Your colleagues at the Bundesbank survived one such course!

Conclusion

- Theory paper (Holden 2016a) proves completely general “Blanchard Kahn conditions” for models with occasionally binding constraints, and shows that multiplicity is to be expected in models with a ZLB.

- The theory paper presents a powerful argument for level targeting rules. If you think the Taylor principle should be followed to rule out indeterminacy, then you should support a level target in the presence of the ZLB.

- Computational paper (Holden 2016b) provides an efficient perfect foresight solver that is guaranteed to finish in finite time, and which can detect solution non-existence.

- Computational paper also gives an accurate solver for even large models with occasionally binding constraints.

Slides with technical details for questions

A note on rational expectations solutions

- The theory paper focusses on results for perfect-foresight solutions, i.e. solutions to models without future uncertainty.

- Nonetheless, these results with have implications for solutions under rational expectations.
  - E.g. if a solution satisfying the terminal condition does not exist in all states of the world, then 100% belief in the terminal condition cannot be consistent with rationality.
  - Thus, the backwards induction arguments used by e.g. Mertens and Ravn (2014) imply global indeterminacy.

- Additionally, I show that:
  - Under mild assumptions, with sufficiently small shocks, there are at least as many solutions to the rational expectations model as there are to the model without future uncertainty.

A note on linearization

- Braun, Körber, and Waki (2012) (BKW) show multiplicity in a non-linear New Keynesian model.

- However, the linearized version of the model with the bound reintroduced posses a unique equilibrium.

- The multiplicity we find is strictly in addition to the BKW type multiplicity.

- Perhaps lucky given that:
  - Christiano and Eichenbaum (2012) brand the BKW multiple-equilibria as mere "mathematical curiosities" due to learnability considerations.
  - The BKW approach to finding multiple equilibria does not readily generalize to models with endogenous state variables.

The set-up without bounds (1/3)

- Suppose for $t \in \mathbb{N}^+$:

$$\left(\hat{A} + \hat{B} + \hat{C}\right)\hat{\mu} = \hat{A}\hat{x}_{t-1} + B\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \widehat{D}\varepsilon_t,$$

  - where $\mathbb{E}_{t-1}\varepsilon_t = 0$ for all $t \in \mathbb{N}^+$,
  - $\varepsilon_t = 0$ for $t > 1$, (impulse response/perfect foresight simulation).

- $\hat{x}_0$ is given as an initial condition.
- Terminal condition: $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$.

The set-up without bounds (2/3)

- For $t \in \mathbb{N}^+$, define:

$$x_t := \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}, \qquad \mu := \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}, \qquad A := \begin{bmatrix} \hat{A} & \widehat{D} \\ 0 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}, \qquad C := \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$$

- then, for $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1},$$

- and $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, $x_t \to \mu$ as $t \to \infty$.

- Take this as the form of our problem without bounds in the following.

The set-up without bounds (3/3)

- **Problem 1**
- Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}.$$

- **Assumption:** For any given $x_0 \in \mathbb{R}^n$, Problem 1 has a unique solution, of the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in \mathbb{N}^+$, where $F = -(B + CF)^{-1}A$, and all of the eigenvalues of F are weakly inside the unit circle.

- **Assumption:** $\det(A + B + C) \neq 0$.

The set-up with bounds

- **Problem 2**
- Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\left\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + \left(B_{1,\cdot} + I_{1,\cdot}\right)(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\right\},$$

$$\left(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot}\right)\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

- and such that $x_{1,t} > 0$ for $t > T$. (Existence of such a T is WLOG.)
- Ruling out solutions that get stuck at another steady-state by assumption.
- Note: KKT type conditions e.g. $y \geq 0$, $\lambda \geq 0$, $y\lambda = 0$ may be encoded as: $0 = \min\{y, \lambda\}$, so $y = \max\{0, y - \lambda\}$.

The news shock set-up

- **Problem 3**
- Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$, $y_t \to 0$, as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$\begin{gathered}(A+B+C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1}y_{1,t-1},\\ \forall i \in \{1,\dots,T-1\}, \qquad y_{i,t} = y_{i+1,t-1},\\ y_{T,t} = 0.\end{gathered}$$

- A version of Problem 1 with news shocks up to horizon T added to the first equation.
  - The value of $y_{t,0}$ gives the news shock that hits in period t.
  - I.e. $y_{1,t-1} = y_{t,0}$ for $t \le T$, and $y_{1,t-1} = 0$ for $t > T$.

A representation of solutions to Problem 3

- **Lemma**: There is a unique solution to Problem 3 that is linear in x_0 and y_0.

- Let $x_t^{(3,k)}$ be the solution to Problem 3 when $x_0 = \mu$, $y_0 = I_{\cdot,k}$.
- Let $M \in \mathbb{R}^{T \times T}$ satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t, k \in \{1, .., T\},$$

  - i.e. M horizontally stacks the (column-vector) relative impulse responses to the news shocks.

- Let $x_t^{(1)}$ be the solution to Problem 1 for some given x_0.
- Then the solution to Problem 3 for given x_0, y_0 satisfies:

$$\left(x_{1,1\dots T}\right)' = q + M y_0,$$

  - where $q \coloneqq \left(x_{1,1\dots T}^{(1)}\right)'$, i.e. the path of the first variable in the absence of the bound.

The links between the solutions to Problem 2 and the solution to Problem 3 (1/2)

- Let $x_t^{(2)}$ be a solution to Problem 2 given an arbitrary x_0.
- Define:

$$e_t := \begin{cases} -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + \left(B_{1,\cdot} + I_{1,\cdot}\right)\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right] & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}$$

- **Lemma:** The following statements hold:
  - $e_{1\ldots T} \geq 0$, $x_{1,1\ldots T}^{(2)} \geq 0$ and $x_{1,1\ldots T}^{(2)} \circ e_{1\ldots T} = 0$,
  - $x_t^{(2)}$ is the unique solution to Problem 3 when started with $x_0 = x_0^{(2)}$ and with $y_0 = e_{1\ldots T}'$.
  - If $x_t^{(2)}$ solves Problem 3 when started with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e_{1\ldots T}'$.

The links between the solutions to Problem 2 and the solution to Problem 3 (2/2)

- **Proposition:** The following statements hold:
  - Let $x_t^{(3)}$ be the unique solution to Problem 3 when initialized with some x_0, y_0. Then $x_t^{(3)}$ is a solution to Problem 2 when initialized with x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$ with $t > T$.
  - Let $x_t^{(2)}$ be any solution to Problem 2 when initialized with x_0. Then there exists a $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, such that $x_t^{(2)}$ is the unique solution to Problem 3 when initialized with x_0, y_0.

Linear complementarity problems (LCPs)

- The previous proposition establishes that solving the model with occasionally binding constraints is equivalent to solving the following "linear complementarity problem".

- **Problem 4**
- Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. Find $y \in \mathbb{R}^T$ such that: $y \geq 0$, $y \circ (q + My) = 0$ and $q + My \geq 0$.

- We call this the linear complementarity problem (q, M).

Generalisations

- For multiple bounds:
  - We stack the impulse responses of the bounded variables ignoring bounds into q.
  - We stack the vectors of news shocks to each variable into y.
  - M is a block matrix of each bounded variable's responses to each bounded variable's news shocks.
  - Then the stacked solution for the paths of the bounded variables is $q + My$, and we again have an LCP, so results go through as before.

- For bounds not at zero:
  - If $z_{1,t} = \max\{z_{2,t}, z_{3,t}\}$, then $z_{1,t} - z_{2,t} = \max\{0, z_{3,t} - z_{2,t}\}$.

- For minimums:
  - If $z_{1,t} = \min\{z_{2,t}, z_{3,t}\}$, then $-z_{1,t} = \max\{-z_{2,t}, -z_{3,t}\}$.

Is our M matrix special?

- The properties of solutions to LCPs (existence, uniqueness, computational difficulty) are determined by the properties of the M matrix.
  - One might think that ours would have "nice" properties because of where it came from.

- Unfortunately:

- **Proposition:** For any matrix $\mathcal{M} \in \mathbb{R}^{T \times T}$, there exists a model in the form of Problem 2 with a number of state variables given by a quadratic in T, such that $M = \mathcal{M}$ for that model.

Efficient computation of solutions (1/2)

- If M is unrestricted, or M is a "P_0-matrix", then finding a single solution to the LCP (q, M) is "strongly NP complete".

- If we could do this efficiently (i.e. in polynomial time), we could also solve in polynomial time any problem whose solution could be efficiently verified.
  - This includes, for example, breaking all standard forms of cryptography.

- Since there is a model corresponding to any M matrix, with quadratic in T states, if there were a solution algorithm for DSGE models with OBCs that worked in time polynomial in the number of states, then it could also be used to defeat all known forms of cryptography.
  - So there almost certainly can't be such an algorithm!

Efficient computation of solutions (2/2)

- Polynomial time algorithms exist for special cases, but checking whether the relevant ones apply is not possible in polynomial time.

- This means that there cannot be an algorithm for checking if a model e.g. has a unique solution, that runs in time polynomial in the number of states.

Our computational approach to the perfect foresight problem

- **Problem 7**
- Suppose $\widetilde{\omega} > 0$, $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given.
- Find $\alpha \in \mathbb{R}$, $\hat{y} \in \mathbb{R}^T$, $z \in \{0,1\}^T$ to maximise α subject to the following constraints: $\alpha \geq 0$, $0 \leq \hat{y} \leq z$, $0 \leq \alpha q + M\hat{y} \leq \widetilde{\omega}(1_{T\times 1} - z)$.

- **Proposition**: If $\alpha, \hat{y}, z$ solve Problem 7, then if $\alpha = 0$, the LCP (q, M) has no solution, and if $\alpha > 0$, then $y := \frac{\hat{y}}{\alpha}$ solves it. (Partial converse in paper.)

- As $\widetilde{\omega} \to 0$, the solution to Problem 7 is the solution to the LCP which minimises $\|q + My\|_\infty$.
- As $\widetilde{\omega} \to \infty$, the solution to Problem 7 is the solution to the LCP which minimises $\|y\|_\infty$.

Application to models with uncertainty

- To convert the perfect foresight solver into a solver for stochastic models, we start by using a variant of the extended path algorithm of Fair and Taylor (1983).
  - Each period we draw a shock, and then solve for the expected future path of the model, ignoring the impact of the OBC on expectations (for now).
  - From this expected path, we can solve for the news shocks necessary to impose the bound.
  - We then add those news shocks to today's variables, and step the model forward using the model's transition matrix.
  - Not consistent with rational expectations but this will be (partially) rectified.

The non-linear problem (1/2)

- **Problem 6**
- Suppose that $x_0 \in \mathbb{R}^n$ is given and that $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^n$, $g, h: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^c$ are given continuously $d \in \mathbb{N}^+$ times differentiable functions.

- Find $x_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^c$ for $t \in \mathbb{N}^+$ such that for all $t \in \mathbb{N}^+$:

$$
\begin{aligned}
0 &= \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t), \\
r_t &= \mathbb{E}_t \max\{h(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t), g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\}
\end{aligned}
$$

- where $\varepsilon_t \sim$NIID$(0, \Sigma)$, where the $\max$ operator acts elementwise on vectors, and where the information set is such that for all $t \in \mathbb{N}^+$, $\mathbb{E}_{t-1}\varepsilon_t = 0$ and $\mathbb{E}_t\varepsilon_t = \varepsilon_t$.

The non-linear problem (2/2)

- **Assumption:** There exists $\mu_x \in \mathbb{R}^n$ and $\mu_r \in \mathbb{R}^c$ such that:

$$0 = f(\mu_x, \mu_x, \mu_x, \mu_r, 0),$$

$$\mu_r = \max\{h(\mu_x, \mu_x, \mu_x, \mu_r, 0), g(\mu_x, \mu_x, \mu_x, \mu_r, 0)\},$$

- and such that for all $a \in \{1, \dots, c\}$:

$$\big(h(\mu_x, \mu_x, \mu_x, \mu_r, 0)\big)_a \neq \big(g(\mu_x, \mu_x, \mu_x, \mu_r, 0)\big)_a.$$

Application via linearization (1/2)

- Without loss of generality, suppose our model is:

$$
\begin{aligned}
0 &= \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t), \\
r_t &= \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\},
\end{aligned}
$$

  - where $g(\mu_x, \mu_x, \mu_x, \mu_r, 0) \gg 0$.

- Linearizing around the steady-state gives:

$$r_t = \mu_r + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(r_t - \mu_r) + g_5 \varepsilon_t.$$

- We replace this with the more accurate:

$$r_t = \max\{0, \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(r_t - \mu_v) + g_5 \varepsilon_t\}.$$

Application via linearization (2/2)

- For our algorithm, we replace this in turn with:

$$r_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)},$$

- for all $a \in \{1, \dots, c\}$, where, for all $a \in \{1, \dots, c\}$:

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \eta_{i,t}^{(a)}$$
$$y_{T,t}^{(a)} = \eta_{T,t}^{(a)}.$$

Application via higher order pruned perturbation

- We first take a pruned perturbation approximation to the source non-linear model.

- A convenient property of pruned perturbation solutions of order d is that they are linear in additive shocks of the form η_t^d.
  - So using shocks of this form preserves the tractable linearity.
  - In fact the M matrix we get at second or higher order is equal to the M matrix at first order, (at least in the limit as the variance of the news shocks goes to zero).

- While this is still treating the bound in a perfect-foresight manner (for now), by taking a higher order approximation we at least capture other risk channels.

Accounting for the risk of hitting the bound

- In period t, our approach approximates the value of x_t in the model of Problem 6 by the solution to the system:

$$
\begin{aligned}
0 &= \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t),\\
r_t &= \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\},\\
\forall s \in \mathbb{N}^+, \qquad 0 &= f(x_{t+s-1}, x_{t+s}, x_{t+s+1}, r_{t+s}, \kappa_s \varepsilon_{t+s}),\\
\forall s \in \mathbb{N}^+, \qquad r_t &= \max\{0, g(x_{t+s-1}, x_{t+s}, x_{t+s+1}, r_{t+s}, \kappa_s \varepsilon_{t+s})\},
\end{aligned}
$$

  - where $\kappa_1, \kappa_2, \ldots \in [0,1]$ control the degree of future uncertainty considered.

- Equivalent to supposing that in period t agents believe that in period $t+1$ they will be told the value of all future shocks (i.e. $\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots$).
  - From the perspective of period t, all future shocks are uncertain, meaning that this should capture well the effect of risk.
  - If the model is linear then by the law of iterated expectations, there is no approximation at all.

Integrating over future uncertainty (1/3)

- Following Adjemian and Juillard (2013) the procedure is as follows:
  - Draw shocks for a certain number of future periods, $t+1,\dots,t+S$.
  - Solve for the perfect foresight path assuming they were known at t.
  - Repeat many times to get expectations.

- In their very general non-linear set-up, doing this integration requires p^{mS} solutions of the perfect foresight problem,
  - for some $p>1$, m is the number of shocks, S is the integration horizon.

- Solving their general perfect foresight problem is also orders of magnitude slower than solving our LCP.

Integrating over future uncertainty (2/3)

- Let $w_{t,s}$ be the value the bounded variables would take at s if the constraints did not apply from period t onwards.

- By the properties of pruned perturbation solutions, we can evaluate $\operatorname{cov}_t(w_{t,t+i}, w_{t,t+j})$, for $t, i, j \in \mathbb{N}$ in closed form.
  - So we can take a Gaussian approximation to the joint distribution of $w_{t,t}, w_{t,t+1}, \ldots$, and efficiently integrate over these variables via Gaussian cubature techniques.
  - Rather than exponential in both m and S evaluations, we just need polynomial in S evaluations.

- For each draw of $w_{t,t}, w_{t,t+1}, \ldots$, we solve the bounds problem to get the cumulated news shocks (i.e. y).

Integrating over future uncertainty (3/3)

- Unlike Adjemian and Juillard (2013) we do not just consider full variance shocks up to some horizon, and then nothing beyond.

- Instead, we apply a windowing function to the shock variances, to ensure that the covariance is a smooth function of time.
  - This reduces artefacts caused by the sudden change at horizon S.

- In particular, we scale the shock variance at horizon k by:

$$\kappa_k^2 = \frac{1}{2}\left(1 + \cos\left(\pi \frac{k-1}{S}\right)\right).$$

- The cosine form has some desirable frequency domain properties.

Three alternative Gaussian cubature methods

- With $\hat{S} \leq S$ the integration dimension, these are:
  - A degree 3 monomial rule with $2\hat{S} + 1$ nodes and positive weights.
    - Positive weights give robustness. Evaluates far from steady-state though.
  - The Genz and Keister (1996) Gaussian cubature rules with $\mathrm{O}\left(\hat{S}^K\right)$ nodes.
    - $2K + 1$ is the degree of monomial integrated exactly.
    - Since the rules are nested, adaptive degree is possible.
  - Quasi-Monte Carlo.
    - Much less efficient than the others on well behaved functions, but is much better behaved on non-differentiable ones.