Existence and uniqueness of solutions to dynamic models with occasionally binding constraints

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Abstract: Policy makers would like to prevent self-fulfilling fluctuations. Given the prevalence of occasionally binding constraints (OBCs) such as the zero lower bound (ZLB), this requires understanding the determinacy of models with OBCs. To this end, we derive existence and uniqueness conditions for otherwise linear models with OBCs. Our main result gives necessary and sufficient conditions for such a model to have a unique perfect foresight solution returning to a given steady state, for any initial condition. We show that while standard New Keynesian models with a ZLB possess multiple perfect-foresight paths eventually escaping the ZLB, price level targeting restores determinacy.

Keywords: *occasionally binding constraints, zero lower bound, existence, uniqueness, price targeting*

JEL Classification: *C62, E3, E4, E5*

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The author would like to thank all those who have commented on the paper and related work, including but not limited to: Gary Anderson, Martin Andreasen, Paul Beaurdry, Saroj Bhattarai, Florin Bilbiie, Charles Brendon, Zsolt Csizmadia, Oliver de Groot, Michael Devereux, Bill Dupor, Roger Farmer, Michael Funke, William Gavin, Fabio Ghironi, Andy Glover, Luca Guerrieri, Pablo Guerrón-Quintana, Matteo Iacoviello, Tibor Illés, Michel Juillard, Hong Lan, Paul Levine, Vivien Lewis, Jesper Lindé, Albert Marcet, Enrique Martinez-Garcia, Antonio Mele, Alexander Meyer-Gohde, Virgiliu Midrigan, Adrienn Nagy, Matthias Paustian, Jesse Perla, Frank Portier, Søren Ravn, Alexander Richter, Chris Sims, Jonathan Swarbrick, Harald Uhlig, Simon Wren-Lewis and Carlos Zarazaga. Financial support provided to the author by the ESRC and the EC is also greatly appreciated. Furthermore, the author gratefully acknowledges the University of Washington for providing the office space in which part of this paper was written.
The research leading to these results has received funding from the European Commission's Seventh Framework Programme (FP7/2007-2013) under grant agreement "Integrated Macro-Financial Modelling for Robust Policy Design" (MACFINROBODS, grant no. 612796).

1. Introduction

Consider an otherwise linear model with occasionally binding constraints (OBCs) and a fixed terminal condition. This paper provides the first necessary and sufficient conditions for there to be a unique perfect foresight solution to such a model, for any shock sequence and any value of the initial state. This gives determinacy conditions for models with occasionally binding constraints, much as Blanchard & Kahn (1980) provided for the linear case. Our conditions relate uniqueness to the positivity of the impact on the bounded variable(s) of positive news shocks to those variable at different horizons.

A key application of the results of Blanchard & Kahn (1980) was showing that in New Keynesian (NK) models without a zero lower bound (ZLB), determinacy requires the Taylor principle to be satisfied, meaning there is a sufficiently aggressive response to inflation.[1] A key application of our results shows that this is insufficient in the presence of the ZLB, even if an eventual escape from the ZLB is guaranteed.

It is now well known that the ZLB can lead to multiplicity of steady states (see e.g. Benhabib, Schmitt-Grohé & Uribe 2001a; 2001b). However, multiple steady states do not automatically imply multiple dynamic solutions, as agents' beliefs may rule out paths converging to deflation.[2] We apply our results to examine the circumstances under which NK models admit multiple solutions when agents have such beliefs.[3] We find NK models with a ZLB generally have multiple solutions which eventually escape the ZLB, even with a monetary rule that satisfies the

[1] See e.g. Clarida, Galí & Gertler (1997; 2000).

[2] This is in line with the evidence of Gürkaynak, Levin & Swanson (2010). Christiano and Eichenbaum (2012) argue that deflation can be escaped by switching to a money growth rule. This may justify the observed beliefs. See Section 6.1.1 for further related discussion.

[3] Hebden, Lindé & Svensson (2011) and Brendon, Paustian & Yates (2013; 2019) provide some specific examples of NK models with multiple transition paths to the standard steady state in certain states. See Appendix D.1 for further discussion of Brendon, Paustian & Yates (2013; 2019).

Taylor principle. However, a weak response to the price level in the monetary rule is sufficient to restore determinacy.

To see how multiplicity is possible with the terminal condition fixed, suppose the model's agents knew that from next period onwards, the economy would be away from the bound. Then, in an otherwise linear model, expectations of next period's outcomes would be linear in today's variables. However, substituting out these expectations does not leave a linear system in today's variables, due to the OBC. For some models, this non-linear system will have two solutions, with one featuring a slack constraint, and the other having a binding constraint. Thus, even though the rule for forming expectations is pinned down, multiple outcomes may be possible. Without the assumption that next period the economy is away from the bound, the scope for multiplicity is even richer, and there may be infinitely many solutions.

We prove that under mild assumptions there are at least as many solutions under rational expectations as under perfect foresight.[4] Thus, our results imply lower bounds on the number of solutions under rational expectations, even though the two solution concepts differ due to the OBC's non-linearity.

In additional results, we give conditions for the global[5] existence of perfect foresight solutions returning to a given steady state for otherwise linear models with OBCs. Non-existence of solutions returning to the "standard" steady state may rationalise the beliefs needed to sustain indeterminacy driven by multiple steady states, e.g. a belief in the possibility of converging to deflation.[6]

The next section presents simple examples of multiplicity and non-

[4] Proven in Appendix I.

[5] I.e. independent of the value of state variables and shock sequences.

[6] The consequences of indeterminacy of this kind has been explored by Schmitt-Grohé & Uribe (2012), Mertens & Ravn (2014) and Aruoba, Cuba-Borda & Schorfheide (2018), amongst others.

existence. In Section 3, we provide the key equivalence result enabling us to examine existence and uniqueness in models with OBCs via examining the properties of linear complementarity problems. Section 4 provides our main results on existence and uniqueness, which we use to examine select examples in Section 5. Finally, Section 6 places our results in the context of the broader literature and discusses key assumptions. It also argues that equilibrium multiplicity can help explain observed outcomes during the great recession.

2. Multiplicity in simple models

We begin by supplying simple examples of multiplicity and solution non-existence, of the type on which we focus in this paper. This will make clear why these problems are so common in models with OBCs and will illustrate the idea behind our results.

2.1. A simple first example

Consider the simplest possible "NK" model: the flexible price limit. The model consists of the Fisher equation[7] and the Taylor rule:

$$i_t = r_t + \pi_{t+1},$$
$$i_t = \max\{0, r_t + \phi\pi_t\},$$

where the real interest rate $r_t = r + \varepsilon_t$, with $\varepsilon_t = 0$ for all $t > 1$, and where i_t is the nominal rate, π_t is inflation and $\phi > 1$. We seek to solve for π_t for $t = 1,2,\dots$. Although the model has two steady states (the "standard" one with $i = r$ and $\pi = 0$, plus the deflationary one with $i = 0$ and $\pi = -r$), we assume that the economy returns to the standard steady state away from the ZLB. This implies that $\pi_t = 0$ for $t > 1$.[8] Hence, in period 1:

$$r + \varepsilon_1 = i_1 = \max\{0, r + \varepsilon_1 + \phi\pi_1\}.$$

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[7] There is no expectation operator in the Fisher equation as there is no uncertainty for $t > 1$.

[8] Suppose $i_t = 0$ for some $t > 1$, so by the Fisher equation $\pi_{t+1} = -r$, meaning $i_{t+1} = 0$ by the Taylor rule. By induction, $i_s = 0$ for all $s \geq t$, contradicting our assumption of a return to the standard steady state. Thus, $i_t > 0$ for all $t > 1$, so $\pi_t = 0$ for $t > 1$.

If $\varepsilon_1 > -r$, then $i_1 = r + \varepsilon_1 > 0$, so $\pi_1 = 0$. However, if $\varepsilon_1 < -r$, then $i_1 < 0$ according to the Fisher equation, which is never consistent with the Taylor rule. Thus, the model has no solution returning to the standard steady state in this case.[9] Finally, if $\varepsilon_1 = -r$, then $i_1 = 0$ and any $\pi_1 \leq 0$ is consistent with the model. To summarise: with $\varepsilon_1 > -r$, the model has a unique solution returning to the standard steady state; with $\varepsilon_1 = -r$, the model has multiple such solutions; and with $\varepsilon_1 < -r$, the model has no such solutions.

2.2. An example with robust multiplicity

The previous example only has multiplicity in a knife-edge case, but in richer models, multiplicity is more widespread. For example, suppose the central bank responds to lagged as well as current inflation.[10] This is the easiest way of generating some endogenous persistence, but almost any state variable would have a similar effect.[11] Assuming r_t is now constant, the model is then:

$$r + \pi_{t+1} = i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1}\},$$

where $\phi - \psi > 1$ and $\psi > 0$.[12] The initial state, π_0, is given. To further simplify presentation, we set $\phi := 2$, so $\psi < 1$. Our results are not specific to this special case, however.

Away from the ZLB, the model's solution must take the form $\pi_t = A\pi_{t-1}$. Substituting this back into the model's equations gives that $A = 1 - \sqrt{1 - \psi}$, so the persistence is increasing in ψ. Note that this "standard" solution is away from the ZLB at t when $0 < r + \pi_{t+1} = r + A^2\pi_{t-1}$, i.e. if and only if $\pi_{t-1} > -\frac{r}{A^2}$.

[9] Indeed, the model has no bounded solution in this case. $r + \varepsilon_1 < 0$ means we must have $\pi_2 > 0$ to ensure $i_1 \geq 0$. Thus, since $\phi > 1$, $\pi_t \to \infty$ and $i_t \to \infty$ as $t \to \infty$.

[10] This may be justified as responding negatively to lagged inflation is optimal in the presence of inflation inertia coming from indexation to past inflation. See e.g. Giannoni & Woodford (2003).

[11] Brendon, Paustian & Yates (2013; 2019) consider an Euler + Phillips curve set-up in which the monetary policy maker responds to output growth in order to introduce an endogenous state.

[12] These assumptions are sufficient for a real determinate solution in the absence of the ZLB.

Now, suppose that in period 1 the economy was at the ZLB, but that it was expected to escape next period, meaning that $\pi_2 = A\pi_1$. The Fisher equation then implies that $0 = i_1 = r + A\pi_1$, so $\pi_1 = -\frac{r}{A}$. This outcome is an equilibrium only if it is consistent with the monetary rule in period 1 and 2, which is true if and only if $\pi_0 \geq -\frac{r}{A^2}$.[13]

To recap, the standard solution is always away from the ZLB when $\pi_0 > -\frac{r}{A^2}$ and being at the ZLB today but escaping next period is an equilibrium when $\pi_0 \geq -\frac{r}{A^2}$. So, if $\pi_0 > -\frac{r}{A^2}$, then there are two solutions: the standard one with $\pi_t = A\pi_{t-1}$ and $i_t > 0$ for all $t > 0$, plus an alternative solution in which $\pi_1 = -\frac{r}{A}$ (so $\pi_1 < A\pi_0$) and $i_1 = 0$. This additional solution jumps to the bound in period 1 but escapes it next period, before gradually returning to the standard steady state. Crucially, the additional solution does not require any change in beliefs about the steady state to which the economy will converge.

Conversely, if $\pi_0 < -\frac{r}{A^2}$, the only remaining possibility is that the model is at the ZLB for more than one period. But if $i_{t+1} = 0$ with $i_{t+2} > 0$ for some $t > 0$, then by the Fisher equation, $\pi_{t+1} = -\frac{r}{A}$ and $i_t = r - \frac{r}{A} < 0$ which is inconsistent with the monetary rule. So, there cannot be a solution path returning to the standard steady state when $\pi_0 < -\frac{r}{A^2}$.

As we approach the canonical model with $\psi \to 0$ (but $\psi \neq 0$), the region of non-existence shrinks but the multiplicity region grows until it encompasses the entire state space.[14] Given that the Fisher equation and Taylor rule are the core of all NK models, it should then be unsurprising that there is non-knife-edge multiplicity in all NK models with endogenous state variables that we have analysed. Even price dispersion

[13] I.e. only if $r + \phi\pi_1 - \psi\pi_0 \leq 0$ and $r + \phi\pi_2 - \psi\pi_1 \geq 0$ with $\pi_1 = -\frac{r}{A}$ and $\pi_2 = -r$. The former holds if and only if $\pi_0 \geq \frac{r}{\psi}\left(1 - \frac{2}{A}\right) = -\frac{r}{A^2}$. The latter is equivalent to $0 \leq \left(\frac{\psi}{A} - 1\right)r = (1 - A)r$, which always holds.

[14] With $\psi = 0$ and constant r, there is a unique solution returning to the standard steady state (as with $\psi = 0$, if $i_t = 0$ for some $t > 0$, then $\pi_{t+1} = -r$, so $i_{t+1} = 0$ as well). This no longer holds once a shock is introduced, as seen above.

suffices as a state. We show examples in Section 5 and Appendix F.

2.3. The mechanics of our main results

Even in such a simple model, deriving these pen and paper results on multiplicity and non-existence is cumbersome. Our general theoretical results provide a convenient alternative. To understand how they work, it is helpful to begin by looking at the impact of a monetary policy shock in the previous model. I.e. consider the model:

$$r + \pi_{t+1} = i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + \nu_t\},$$

where $\nu_t = 0$ for $t > 1$ and π_0 is again given. The solution away from the ZLB must take the form $\pi_t = A\pi_{t-1} + F\nu_t$, with A as before and $F = -\frac{1}{\phi - A} < 0$. Thus, away from the ZLB, $i_1 = r + A^2\pi_0 + AF\nu_1$. With $\psi > 0$, $AF < 0$, so a positive monetary policy shock actually lowers nominal interest rates.

Now suppose that we choose $\nu_1 = -\frac{r + A^2\pi_0}{AF}$. Since $F < 0$, this is a positive innovation if and only if $\pi_0 > -\frac{r}{A^2}$. With this value of ν_1, $\pi_1 = A\pi_0 + F\nu_1 = -\frac{r}{A}$ and $i_1 = r + A^2\pi_0 + AF\nu_1 = 0$, so this shock is just the right magnitude to drive the economy to touch the ZLB. Observe too that the outcome for inflation is identical to that in the alternative solution to the model without a shock considered previously. This coincidence is explained by the fact that if $\pi_0 > -\frac{r}{A^2}$, then:

$$0 = i_1 = r + \phi\pi_1 - \psi\pi_0 + \nu_1 = \max\{0, r + \phi\pi_1 - \psi\pi_0\}.$$

Given the ZLB and the positivity of ν_1, there is no observable evidence that a shock has arrived at all, since the ZLB implies that with these values of output and inflation, nominal interest rates should be zero even without a shock. Such a jump to the ZLB must then be a self-fulfilling prophecy: agents' beliefs and equilibrium outcomes are as if such a monetary policy shock had hit, whether or not it did in reality. Given $\psi > 0$, the condition for multiplicity ($\pi_0 > -\frac{r}{A^2}$) here is then precisely the same as the condition for there to be a positive shock that drives interest rates to zero in the absence of the ZLB ($\pi_0 > -\frac{r}{A^2}$). Likewise, the condition for there to be multiplicity for some π_0 ($\psi > 0$) is precisely the condition for a positive shock to have a negative effect ($\psi > 0$), which is what permits this censoring away of positive shocks.

This reveals a tight connection between multiplicity and positive shocks having negative effects. Indeed, our key condition for uniqueness will require that positive shocks to the bounded variable have positive effects.[15] It will also require that news today about a future positive shock to the bounded variable will result in the bounded variable being higher in the period the shock arrives. This is the natural generalisation for models in which the bound may be hit in future periods. More than this, it requires that the impact of news shocks to the bounded variable at different horizons be "jointly" positive, in a sense to be made clear.

3. Equivalence result

We now present the result that establishes an equivalence between solutions of a DSGE model with OBCs, and solutions of a linear complementarity problem (LCP). This result will enable us to leverage existing work on existence and uniqueness for LCPs.

For now, we assume that there is a single OBC of the form $i_t = \max\{0, \dots\}$, where i_t is the constrained variable (not necessarily interest rates). This covers all OBCs one encounters in practice, possibly via a transformation.[16] For example, the Karush-Kuhn-Tucker type constraints $i_t \geq 0$, $\lambda_t \geq 0$, $i_t \lambda_t = 0$ hold if and only if $0 = \min\{i_t, \lambda_t\}$ which in turn holds if and only if $i_t = \max\{0, i_t - \lambda_t\}$. It is also straightforward to generalize to multiple constraints.[16] We continue to look for perfect

[15] The condition requires strict positivity precisely so cases like $\psi = 0$ are treated correctly as cases with multiple solutions. We will always assume that the shock and/or state space is sufficiently rich that the path in the absence of the bound is arbitrary. See Section 6.1.2 for discussion of this assumption.

[16] See Appendix H.

foresight solutions converging to a steady state at which $i_t > 0$,[17] taking as given the value of the initial state of the model's endogenous variables. We assume throughout that without the bound, the model would be determinate around a unique steady state.

Without loss of generality then, the equation containing the bound is of the form:

$$i_t = \max\{0, f(x_{t-1}, x_t, x_{t+1})\}, \tag{1}$$

where x_t contains the model's period t endogenous variables, including i_t, and f is some differentiable function (later restricted to be linear). The model's other equations are of the form:

$$0 = g(x_{t-1}, x_t, x_{t+1}),$$

for some differentiable function g (also later restricted to be linear). Now define:

$$y_t := \max\{0, f(x_{t-1}, x_t, x_{t+1})\} - f(x_{t-1}, x_t, x_{t+1}).$$

By construction, $y_t \geq 0$. Also:

$$i_t = f(x_{t-1}, x_t, x_{t+1}) + y_t. \tag{2}$$

Despite its simplicity (we have just added and subtracted a term), this result turns out to be crucial. It states that the value of the bounded variable is given by its value in the absence of the constraint (but given other endogenous variables), plus an additional positive "forcing" term capturing the effect of the constraint. Furthermore, by construction, if $i_t > 0$, then $y_t = 0$ and if $y_t > 0$, then $i_t = 0$. Thus, for all t, the bounded variable i_t and the forcing term y_t satisfy the complementary slackness condition, $i_t y_t = 0$. For further intuition, note that when the constraint originally came from the Karush-Kuhn-Tucker (KKT) conditions $i_t \geq 0$, $\lambda_t \geq 0$, $i_t \lambda_t = 0$ (so $i_t = \max\{0, i_t - \lambda_t\}$), then $y_t = \max\{0, i_t - \lambda_t\} - i_t + \lambda_t = \lambda_t$, meaning y_t recovers the original KKT multiplier. Finally, note that since we are assuming the model returns to a steady state where $i_t >$

[17] Constraints that bind in steady state may be handled via a transformation. See Appendix H.

0, there must be some period T such that for all $t > T$, $y_t = 0$.

In order to understand the behaviour of the model with OBCs, it is helpful to first consider the behaviour of a model without OBCs but with an exogenous forcing process in one equation. In particular, we consider replacing equation (1) with equation (2), where for now we treat y_t as an exogenous forcing process. Since we are working under perfect-foresight, we assume that the entire path of y_t is known in period 1. We also assume that there exists some period T such that for $t > T$, $y_t = 0$, as this always holds when y_t arises endogenously from an OBC.

We now make the following key definitions:

Definition 1 Under the setup of the preceding text:

- $y \coloneqq [y_1, \dots, y_T]'$ is a vector giving the path of the forcing variable.
- $i: \mathbb{R}^T \to \mathbb{R}^T$ is a function, where for all y, $i(y)$ is a vector containing the first T elements of the path of i_t, for the given path of the forcing variable y, as determined by equation (2).
- $q \coloneqq i(0)$ is a vector giving the first T elements of the path of i_t when $y_t = 0$ for all t, i.e. q gives the path i_t would follow were there no bound in the model.
- M is a $T \times T$ matrix where the 1st column equals $\left.\frac{\partial i(y)}{\partial y_1}\right|_{y=0}$, the 2nd equals $\left.\frac{\partial i(y)}{\partial y_2}\right|_{y=0}$, and so on.

Then, by Taylor's theorem $i(y) = q + My + \mathrm{O}(y'y)$ for small y. Henceforth, we restrict f and g to be linear, in which case this approximation is exact and $i(y) = q + My$, with only q, not M, depending on the initial state. We prove this and establish expressions for the elements of M in Appendix E. The proof proceeds by backwards induction, starting from the known transition matrix in period $T + 1$ from which point on the economy is away from the bound.

Note that with f and g linear, the first column of M gives the impulse response to a contemporaneous shock to i_t, the second column of M gives the impulse response to a one period ahead news shock to i_t, and so on.[18] Hence the path of i_t is given by its path in the absence of constraints or a forcing process, plus a linear combination of impulse responses to the "news" contained in y.

When y arises endogenously from an OBC, we show in Appendix E that $i(y) = q + My$ still holds. Effectively then, the OBC provides "endogenous news" that i_t will be higher than it would be without the bound, in periods in which the bound is hit. Given the complementary slackness conditions for y_t already established, and the positivity of the path of the bounded variable i_t, we then have that $y \geq 0$, $q + My \geq 0$ and $y'(q + My) = 0$. It turns out that these conditions completely characterise the solution in the presence of OBCs, as shown in the following key theorem:

Theorem 1

1) Suppose x_t is a solution to the model without an OBC in which equation (1) is replaced with equation (2), with y_t as an exogenous driving process. Suppose there is some $T \geq 0$ such that $y_t = 0$ for $t > T$. Then x_t is also a solution to the original model with an OBC, permanently escaping the bound after at most T periods, if and only if $y \geq 0$, $q + My \geq 0$, $y'(q + My) = 0$, $f(x_{t-1}, x_t, x_{t+1}) \geq 0$ for $t > T$.
2) Suppose x_t is a solution to the model with an OBC which eventually escapes the bound. Then there exists $T \geq 0$ and a unique $T \times 1$ vector y such that: $y \geq 0$, $q + My \geq 0$, $y'(q + My) = 0$, $f(x_{t-1}, x_t, x_{t+1}) \geq 0$ for $t > T$ and such that x_t is the unique solution to the model without

[18] The idea of imposing an OBC by adding news shocks is also present in Holden (2010), Hebden, Lindé & Svensson (2011), Holden & Paetz (2012) and Bodenstein, Guerrieri & Gust (2013). Laséen & Svensson (2011) use a similar technique to impose a path of nominal interest rates, in a non-ZLB context. None of these papers formally establish our equivalence result. News shocks were introduced by Beaudry & Portier (2006).

an OBC in which equation (1) is replaced with equation (2), with y_t exogenous.

The proof (in Appendix E) again relies on backward induction arguments. This theorem establishes that in order to solve for the perfect-foresight solution of the model with OBCs, we just need to guess a sufficiently high T, then find a forcing process y which solves the following "linear complementarity problem" (LCP):

Definition 2 (LCP) We say $y \in \mathbb{R}^T$ solves the **LCP** (q, M) if and only if $y \geq 0$, $q + My \geq 0$ and $y'(q + My) = 0$.

LCPs have been extensively studied in mathematics. See Cottle (2009) for a brief introduction, and Cottle, Pang & Stone (2009a) for a definitive survey.[19] LCPs can be solved via mixed-integer linear programming (MILP), for which optimised solvers exist. This approach is developed into a solution algorithm for models with OBCs in Holden (2016).

Note that if y solves the LCP (q, M), then for any $\kappa > 0$, κy solves the LCP $(\kappa q, M)$. Thus, the properties (existence, uniqueness, difficulty, etc.) of an LCP cannot depend on the magnitude of q, i.e. how close q is to the bound. For example, this means that raising the inflation target is unlikely to affect the determinacy properties of a model with a ZLB.[20]

4. Existence and uniqueness results

This section gives our main theoretical results on the existence and uniqueness of perfect foresight solutions to models that are linear apart from an OBC. Supplemental results are contained in Appendices C and J, with the latter relating our findings to models solvable via dynamic programming. Our results exploit the bijection between solutions of the model with an OBC and solutions to the LCP, which permits us to import

[19] Also see Appendix G, for direct results on the properties of small LCPs.
[20] There may be indirect effects as changing the inflation target may change M.

the conclusions of the LCP literature. The LCP results all rest on the properties of the M matrix. Here we will focus on just two: that of being a P-matrix and that of being an S-matrix. The former will be key for uniqueness, and the latter for existence.

The results of the prior LCP literature consider existence and uniqueness for any possible q. For this to be relevant in our context, the state space must be rich enough that for some initial condition, any given path of i_t in the absence of the bound can be produced. One easy way to ensure this holds is to augment equation (1) with an exogenous forcing process, so:

$$i_t = \max\{0, f(x_{t-1}, x_t, x_{t+1}) + \nu_t\}$$

where $\nu_t = 0$ for $t > T$, and where the entire path of ν_t is known in period 1. Thus, ν_t acts like a series of news shocks. This is equivalent to a model without such a forcing process but with T more state variables tracking the arrival of these shocks (see Appendix E).[21] It is plausible in many contexts that there should be news shocks to the bounded variable. For example, news shocks to interest rates may reflect forward guidance.

4.1. Uniqueness results

We start by looking at uniqueness. The main definition follows:

Definition 3 (P-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is a **P-matrix** if and only if for all $z \in \mathbb{R}^{T\times 1}$ with $z \neq 0$, there exists $t \in \{1, \ldots, T\}$, such that $z_t(Mz)_t > 0$. (Cottle, Pang & Stone 2009b)

Clearly, all symmetric positive definite matrices are P-matrices, so this definition captures a broader notion of positivity for an arbitrary matrix. Additionally, the diagonal of any P-matrix must be positive. In the context of models with a ZLB, this means that if M is a P-matrix then positive monetary policy shocks must increase nominal interest rates.

[21] To always be able to find the shocks needed to produce the desired q, M must be full rank.

Additionally, news about future positive monetary shocks must lead to higher nominal interest rates in the period the shock actually hits. Recall that in Section 2.3 we found that multiplicity was driven by positive monetary policy shocks having negative effects. Thus, it is unsurprising that some type of positivity of the responses of the bounded variable to shocks is key for uniqueness. In fact:

Theorem 2 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix. If M is not a P-matrix, then for some q the LCP (q, M) has multiple solutions.
(Samelson, Thrall & Wesler 1958; Cottle, Pang & Stone 2009b)

Applied to models with an OBC, this becomes:

Corollary 1 Consider an otherwise linear model with an OBC. Let $T > 0$. Then:

1) If M is a P-matrix, then for any x_0 there exists a unique path $(x_t)_{t=1}^{\infty}$ with x_t satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T + 1$ on.
2) [Implied by 1.] If M is a P-matrix, and $(x_t)_{t=1}^{\infty}$ is a solution to the model with an OBC that is away from the bound from period $T + 1$ onwards, then $(x_t)_{t=1}^{\infty}$ is the unique solution that is away from the bound from period $T + 1$ on.

Furthermore, suppose that the model's state space is rich enough such that for any path $\tilde{q} \in \mathbb{R}^T$, there exists x_0 such that $q(x_0) = \tilde{q}$ (making explicit the dependency of q on x_0), then:

3) If M is not a P-matrix then there exists x_0 such that there are multiple paths $(x_t)_{t=1}^{\infty}$ with x_t satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T + 1$ onwards.

This result is the equivalent for models with OBCs of the key theorem of Blanchard & Kahn (1980). Its proof is immediate from Theorem 1. Note that if M is not a P-matrix for some T, then M will also not be a P-matrix for any larger T,[22] so to show general multiplicity it suffices to show that M is not a P-matrix for a small T.

Parts 1) and 3) of Corollary 1 are of practical relevance despite the non-imposition of the bound from period $T+1$ onwards for several reasons. Firstly, with large T, we expect there to be at least one solution that is away from the bound by period $T+1$. Secondly, it stretches the plausibility of rational expectations to suppose that outcomes today depend on whether the economy is expected to be at the ZLB in (say) 250 years. Thus, we may only be interested in equilibria that escape the bound within (say) $T=1000$ quarters. Finally, technological developments are likely to make many OBCs eventually obsolete. For example, a move to electronic cash will mean the ZLB is no longer a constraint. If agents believe this will happen within 250 years, then taking $T=1000$ quarters would be appropriate. Thus, for practical purposes we might consider $T=1000$ equivalent to $T=\infty$.

To see why being a P-matrix is the correct notion of positivity, suppose that y and $\tilde{y}$ both solved the LCP (q,M). Thus, for all $t\in\{1,\dots,T\}$, $0=y_t(q+My)_t=\tilde{y}_t(q+M\tilde{y})_t$, so:

$$\begin{aligned}(y-\tilde{y})_t\big(M(y-\tilde{y})\big)_t &= (y-\tilde{y})_t\big((q+My)-(q+M\tilde{y})\big)_t\\ &= y_t(q+My)_t+\tilde{y}_t(q+M\tilde{y})_t-y_t(q+M\tilde{y})_t-\tilde{y}_t(q+My)_t\le 0\end{aligned}$$

as y_t, $\tilde{y}_t$, $q+My$ and $q+M\tilde{y}$ must all be weakly positive. Hence, if we define $z=y-\tilde{y}$, then we have that for all $t\in\{1,\dots,T\}$, $z_t(Mz)_t\le 0$. If M is a P-matrix, this implies that $z=0$ so $y=\tilde{y}$, meaning the solution is unique.[23] Informally, M being a P-matrix guarantees positive shocks to i_t increase i_t enough on average that one cannot have the kinds of self-

[22] Immediate from the alternative definition in Appendix B. See also Cottle, Pang & Stone (2009b).
[23] This argument just follows that of Cottle, Pang & Stone (2009b).

fulfilling jumps to the bound we saw in Section 2.

One approach to assessing whether M is a P-matrix involves checking the positivity of the determinants of all M's 2^T principal sub-matrices (see Appendix B). Since this is rather onerous, in Appendix C.1 we present both easier to verify necessary conditions, and easier to verify sufficient conditions. These give a fast answer one way or the other in most cases. See Appendix D.2 for a practical guide to checking the various conditions.

4.2. Revisiting our first example

Section 2.1 showed that a model with flexible prices and a standard Taylor rule had multiple equilibria. To put this result into the context of our general theory, we derive the M matrix for this model.[24] Since the M matrix stacks the impulse responses to news shocks at different horizons (ignoring the bound), we start by augmenting the model without bound by an exogenous forcing process, ν_t, giving:

$$r + \pi_{t+1} = i_t = r + \phi\pi_t + \nu_t.$$

Given the entire path of ν_t is known in period 1, the solution must take the infinite moving-average form $\pi_t = \sum_{j=0}^{\infty} F_j \nu_{t+j}$. Matching coefficients implies that $F_j = -\phi^{-(j+1)}$ for all $j \in \mathbb{N}$, so $i_t = r - \sum_{j=1}^{\infty} \phi^{-j} \nu_{t+j}$. From this, we can read off the columns of the M matrix. The first column is the path of $i_t - r$ when $\nu_1 = 1$ and $\nu_t = 0$ for $t \neq 1$, which is $0,0,\ldots$. The second column is the path of $i_t - r$ when $\nu_2 = 1$ and $\nu_t = 0$ for $t \neq 2$, which is $\phi^{-1}, 0,0,\ldots$. The third is $\phi^{-2}, \phi^{-1}, 0,0,\ldots$, and so on. Thus, for any T, the M matrix has a zero diagonal, a strictly negative upper triangle, and a zero lower triangle. Consequently, all M's principal sub-matrices have zero determinant, so M cannot be a P-matrix (by the condition from Appendix B). Thus, as we already saw, this model does not always have a unique solution when augmented with appropriate

[24] In the notation of Section 2.2, this is the $\psi = 0$ case.

shocks (in this case, a shock to the real interest rate).

4.3. Existence results

We now turn to existence conditions. In this case, the key property is being an S-matrix:

Definition 4 (S-matrix) A matrix $M \in \mathbb{R}^{T \times T}$ is called an **S-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$.[25] *Note: all P-matrices are S-matrices.*

Again, this captures a type of positivity of M. It is considerably weaker than the condition of being a P-matrix required for uniqueness. In a model with a ZLB it would be satisfied, for example, if raising rates today raised rates at all horizons due to the model's persistence. (This corresponds to taking $y = [1,0,0, \dots]'$.)

The property of being an S-matrix is closely related to the feasibility of an LCP:

Definition 5 (Feasibility) We say $y \in \mathbb{R}^T$ is **feasible** for the LCP (q, M) if and only if $y \geq 0$ and $q + My \geq 0$. We say a path $(x_t)_{t=1}^{\infty}$ is **feasible** for a model with an OBC given initial state x_0, if when equation (1) is replaced by equation (2), with y_t exogenous, there is some $(y_t)_{t=1}^{\infty}$ with $y_t \geq 0$ for all t, such that $(x_t)_{t=1}^{\infty}$ solves the model with equation (2), and $i_t \geq 0$ for all t.

By definition, if an LCP has a solution, then it is feasible. Likewise, if a model with an OBC has a solution, then it is feasible. If a monetary policy maker could make credible promises about (positive) future monetary policy shocks, then feasibility would be sufficient to allow the policy maker to ensure a solution.

If M is an S-matrix then feasibility is guaranteed:

[25] This may be tested by solving a linear programming problem. See Appendix B.

Proposition 1 The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. If the LCP (q, M) has a solution for all $q \in \mathbb{R}^T$, then M is an S-matrix. (Cottle, Pang & Stone 2009b)

Moreover, in most cases one encounters in practice, an LCP is solvable whenever it is feasible, i.e. whenever M is an S-matrix.[26] This has immediate practical consequences: if M is an S-matrix for some T then we are likely to be able to solve the size T LCPs we encounter in simulating the model, whatever the model's path without the bound, q.

Additionally, from Theorem 1, we have:

Corollary 2 Let $T > 0$. Consider an otherwise linear model with an OBC where the model's state space is rich enough such that for any path $\tilde{q} \in \mathbb{R}^T$, there exists x_0 such that $q(x_0) = \tilde{q}$. Then if M is not an S-matrix, there exists x_0 such that:

1) There is no path $(x_t)_{t=1}^{\infty}$ with x_t satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T + 1$ onwards.
2) [Implied by 1.] There is no path $(x_t)_{t=1}^{\infty}$ satisfying the model's equations which escapes the bound after at most T periods.

Since large T (e.g. 1000) may be equivalent to $T = \infty$ for all practical purposes, this result is already a helpful guide to the non-existence of relevant solutions.

Nonetheless, we can also directly obtain results on the existence or feasibility of solutions when the constraint is imposed for all periods (i.e. $T = \infty$). Proposition 1 implies that the infinite LCP (q, M) is feasible for all $q \in \mathbb{R}^{\mathbb{N}^+}$ if and only if $\varsigma := \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} (My)_t > 0$. Furthermore, in Appendix L.1 we prove:

[26] Formal sufficient conditions for existence are provided in Appendix C.2.

Proposition 2 Given an otherwise linear model with an OBC, there exist potentially informative bounds $\underline{\varsigma}_S$, $\overline{\varsigma}_S$, computable in time polynomial in S, such that $\underline{\varsigma}_S \leq \varsigma \leq \overline{\varsigma}_S$.[27]

This enables us to derive results despite the infeasible infinite dimensional problem that defines ς. Relating this to our situation gives:

Corollary 3 Suppose that for some S, $\underline{\varsigma}_S > 0$. Then for any x_0 the model with an OBC has a feasible path (a necessary condition for existence of a solution). Conversely, suppose $\overline{\varsigma}_S = 0$. Then there is some path for the bounded variable, $\left(\tilde{q}_t\right)_{t=1}^{\infty}$, such that if x_0 satisfies $q(x_0) = \tilde{q}$ (i.e. in a version of the model without a bound $i_t = \tilde{q}_t$, for all t), then the model has no solution.

This result is important as it gives existence conditions without any dependence on T. It tells us if there are situations in which there is no solution that eventually escapes the bound even if we allow the constraint to bind for an arbitrarily long (but finite) amount of time.

We note that the proof of Proposition 2 may be of independent interest for two reasons. Firstly, as it derives closed form expressions for the limits of the diagonals of M, via novel expressions for the impulse response to a news shock as the horizon goes to infinity. Secondly, because it derives constructive bounds on the elements of M using results on pseudospectra from Trefethen and Embree (2005).

5. Application of our results to the zero lower bound

This section presents examples of the application of our results to NK models with a ZLB. We start by using them to show analytically that a response to the price level produces uniqueness in the model of Section 2.1. We then apply our results to the Smets & Wouters (2003; 2007)

[27] The practical informativeness of these bounds is made clear by the results for NK models in Section 5.2 and Appendix F.

models. Appendix F contains further NK examples.

5.1. Again revisiting our first example

Suppose we modify the model of Section 2.1 to include a response to the price level, p_t, in the Taylor rule, so:

$$r + p_{t+1} - p_t = i_t = \max\{0, r + \phi(p_t - p_{t-1}) + \chi p_t\},$$

where $p_0 = 0$. In this case, to find M we need to solve the model:

$$r + p_{t+1} - p_t = i_t = r + \phi(p_t - p_{t-1}) + \chi p_t + \nu_t,$$

which must have a solution in the form $p_t = \sum_{j=-\infty}^{\infty} G_j \nu_{t+j}$, where $\nu_t = 0$ for all $t \le 0$. By matching coefficients, we can derive closed form expressions for G_j, given in Appendix K. Furthermore, we show there that for any T, all of the elements of M are strictly increasing in χ for small χ. Thus, by Jacobi's formula, for any principal sub-matrix W of M with $W \in \mathbb{R}^{S\times S}$ ($S \le T$), if $\chi = 0$:

$$\frac{d \det W}{d\chi} = \frac{dW_{S,1}}{d\chi}(-1)^{S-1} \det W_{1:(S-1),2:S} = \frac{dW_{S,1}}{d\chi} \prod_{s=1}^{S-1} (-W_{s,s+1}) > 0,$$

as with $\chi = 0$, W must be strictly upper triangular with negative elements in the upper triangle. Thus, for any T, there exists $\overline{\chi}_T \in (0, \infty]$ such that for all $\chi \in (0, \overline{\chi}_T)$, M is a P-matrix. Consequently, a weak but positive response to the price level restores determinacy in this model. Since the Fisher equation and the Taylor rule are present in all NK models, it is natural to expect that this result should generalize across all NK models. As an example, in the next section we examine price targeting within two medium scale DSGE models.

5.2. The Smets & Wouters (2003; 2007) models

Smets & Wouters (2003) and Smets & Wouters (2007) are prototypical medium-scale linear DSGE models, featuring assorted shocks, habits, price and wage indexation, capital (with adjustment costs and variable utilisation) and general monetary policy reaction functions. The former model is estimated on Euro area data, while the latter is estimated on US data. The latter model also contains trend growth (permitting its estimation on non-detrended data), and a slightly more general aggregator across industries. However, they are broadly similar models, and any differences in their behaviour chiefly stems from differences in the estimated parameters. Since both models are well known in the literature, we omit their equations here, referring the reader to the original papers for further details.

To assess the likelihood of multiple equilibria in the presence of the ZLB, we augment each model with a ZLB on nominal interest rates, and evaluate the properties of each model's M matrix at the estimated posterior-modes from the original papers. To minimise the deviation from the original papers, we do not introduce an auxiliary variable for shadow nominal interest rates, so the monetary rules take the form of $i_t = \max\{0, \rho_i i_{t-1} + (1-\rho_i)(\cdots) + \cdots\}$, in both cases. Our results would be similar with a shadow nominal interest rate.

Recall that for ZLB models, the diagonal of the M matrix captures whether positive news shocks to monetary policy raise nominal interest rates in the period in which the shock hits. If this diagonal ever goes negative, then the M matrix cannot be a P-matrix, and hence the model will have multiple solutions in at least some states. In Figure 1,[28] we plot the diagonal of the M matrix for the two models. We see that while in the US model, these impacts remain positive at all horizons, in the Euro area model, these impacts turn negative after just a few periods, and remain so at least up to period 40. Therefore, in the ZLB augmented Smets & Wouters (2003) model, there is not always a unique equilibrium.

[28] Details on replicating all of the results in this section are contained in Appendix F.5.

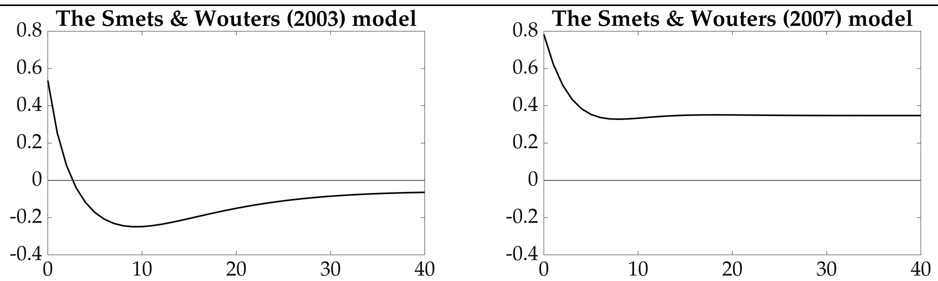

Figure 1: The diagonals of the M matrices for the Smets & Wouters (2003; 2007) models

For the Smets & Wouters (2007) model, numerical calculations reveal that for $T < 9$, M is a P-matrix. However, with $T \geq 9$, the top-left 9×9 sub-matrix of M has negative determinant, so for $T \geq 9$, M is not a P-matrix.[29] Thus, this model also has multiple equilibria. While placing a larger coefficient on inflation in the Taylor rule can make the Euro area picture more like the US one, with a positive diagonal to the M matrix, even with incredibly large coefficients, M remains a non-P-matrix for both models. This is driven by the real and nominal rigidities in the model reducing the average value of the impulse response to a positive news shock to the monetary rule. Following such a shock's arrival, the rigidities help ensure that the fall in output is persistent. Prior to its arrival, consumption habits and capital/investment adjustment costs help produce a larger anticipatory recession. Hence, in both the Euro area and the US, we ought to take seriously the possibility that the existence of the ZLB produces non-uniqueness.

[29] The value of T at which the model switches from M being a P-matrix to M not being a P-matrix is parameter dependent. It reflects the strength of the model's endogenous persistence.

[30] We find the vector w that minimises $w'w$ subject to $\bar{r} + Zw \leq 0$, where $\bar{r}$ is the steady state interest rate, and columns of Z give four periods of the IRF of interest rates to the given shocks. This gives: productivity, 3.56 s.d.; risk premium, -2.70 s.d.; government, -1.63 s.d.; investment, -4.43 s.d.; monetary, -2.81 s.d.; price mark-up, -3.19 s.d.; wage mark-up, -4.14 s.d..

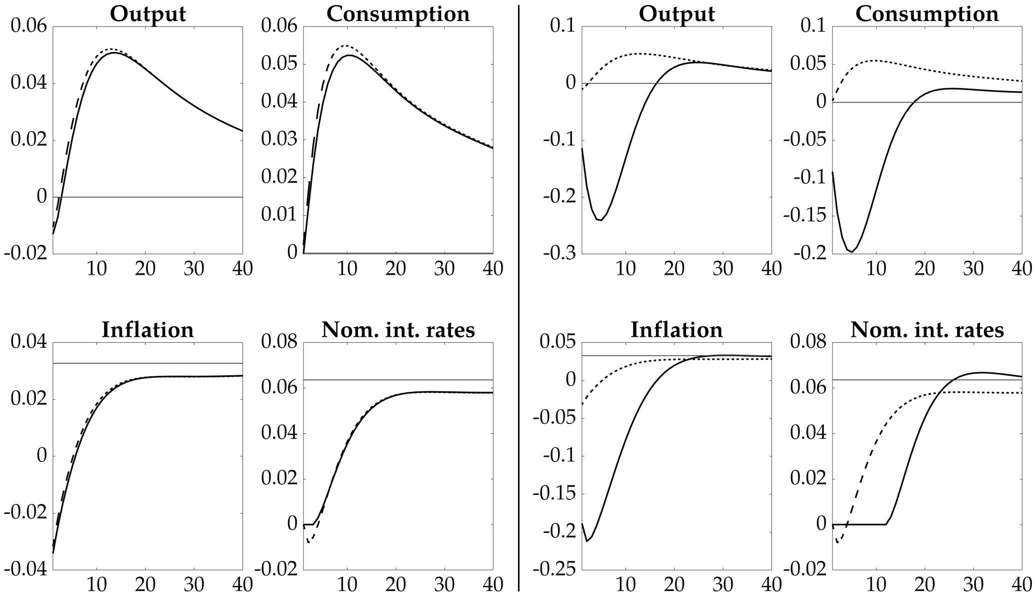

Figure 2: A "good" solution (left 4 panels) and a "bad" solution (right 4 panels), following a combination of shocks to the Smets & Wouters (2007) model

All variables are in logarithms. Inflation and nominal interest rates are annualized.
The precise combination of shocks is detailed in Footnote 30.
In all plots, dashed lines show the path the economy would have followed without the ZLB.

As an example of multiplicity, in Figure 2 we plot two different solutions following the combination of shocks to the Smets & Wouters (2007) model that are most likely to produce negative interest rates for a year in the absence of a ZLB.[30] In both solutions, the dashed line shows the response in the absence of the ZLB, for reference. Particularly notable is the flip in sign, since the shocks most likely to take the model to the ZLB for a year are expansionary ones reducing prices (positive productivity and negative mark-up). Section 6.3.1 shows an example of multiplicity in the Smets & Wouters (2003) model, and discusses the economic relevance of such multiplicity.

In addition, it turns out that for neither model is M an S-matrix even with $T = 1000$, and thus for both models there are some initial states (possibly augmented with monetary policy news shocks) for which no solution exists which escapes the bound after at most 250 years. This is strongly suggestive of non-existence for some initial states even for arbitrarily large T. This is reinforced by the fact that for the Smets & Wouters (2007) model, with $T = 1000$, Proposition 2 gives that $\varsigma \leq 0 +$ numerical error (with ς as defined in Section 4.3), which is suggestive of non-existence even for infinite T.

With a response to the price level, the situation is very different. Suppose that in both models we replace the monetary rule by a simple rule responding to the price level and output growth, so it becomes:

$$i_t = \max\left\{0, \rho_i i_{t-1} + (1-\rho_i)\log\left(P_t \frac{Y_t}{Y_{t-1}}\right)\right\},$$

where ρ_i is as in the original model, Y_t is real GDP and where the price level P_t evolves according to $\log P_t = \log P_{t-1} + \log\left(\frac{\Pi_t}{\Pi}\right)$. Then, with $T = 1000$, for either model the sufficient conditions we introduce in Appendix C.1 imply that M is a P-matrix. Hence, the models have a unique solution conditional on escaping after at most 250 years. Additionally, we have that $\varsigma > 0.036$ for the Euro area model with this monetary rule, and that $\varsigma > 0.009$ for the US one (with ς as defined in Section 4.3). Hence, Corollary 3 implies that the model always has a feasible path. This is a necessary condition for existence of a solution for any initial state. As one would expect, these results are also robust to departures from equal, unit, coefficients on prices and output growth. Thus, price level targeting again appears to be sufficient for determinacy in the presence of the ZLB.

In Appendix F we show that this result further generalises to other models. As expected, given the analytic results of the previous section, a response to the price level ensures determinacy in the presence of the ZLB across a wide range of NK models. The intuition again comes down to the sign of the response to monetary policy (news) shocks. With the price level in the Taylor rule, the reduction in prices brought about by a positive monetary policy (news) shock must be followed eventually by a counter-balancing increase. But if inflation is higher in future, then real rates are lower today, meaning that consumption, output, inflation and nominal rates will all be relatively higher today. This ensures that positive monetary policy (news) shocks have sufficiently positive effects on nominal rates to prevent self-fulfilling jumps to the bound. Thus, in the presence of the ZLB, a positive response to the price level is the equivalent of the Taylor principle. We discuss this in the context of the existing literature on price level targets in Section 6.3.2.

6. Further discussion

To see the broader relevance of our results, in this section we further examine them in the context of the prior literature. We start by providing further justification for our assumptions. We then provide additional context for our general results. We finish with a discussion of the application to the zero lower bound, including examining the potential contribution of multiplicity to observed outcomes in the great recession.

6.1. Our assumptions

In this subsection, we discuss the relevance of our assumptions: first, the imposition of a terminal condition; next, the need for a sufficiently rich state space in some results.

6.1.1. Our terminal condition

Our results are conditional on the economy returning to a given steady state about which the economy is locally determinate. For models with a ZLB, this means the steady state with positive inflation, unless the model is augmented with a sunspot equation following Farmer, Khramov & Nicolò (2015). This approach contrasts with the prior literature, beginning with Benhabib, Schmitt-Grohé & Uribe (2001a; 2001b), and further developed by Schmitt-Grohé & Uribe (2012), Mertens & Ravn (2014) and Aruoba, Cuba-Borda & Schorfheide (2018), amongst others. In this literature, indeterminacy comes from the fact that agents place positive probability on the economy converging towards the deflationary steady state.

A priori, it is unclear whether agents should place positive probability on the economy converging to deflation. Firstly, the central banks of most major economies have announced (positive) inflation targets. Thus, convergence to a deflationary steady state would represent a spectacular failure to hit the target. As argued by Christiano and Eichenbaum (2012), a central bank may rule out the deflationary equilibria in practice by switching to a money growth rule following severe deflation, along the lines of Christiano & Rostagno (2001).[31] Furthermore, Richter & Throckmorton (2015) and Gavin et al. (2015) present evidence that the deflationary equilibrium is unstable[32] under rational expectations if shocks are large enough, making it much harder for agents to coordinate upon it. Finally, a belief that inflation will eventually return to the vicinity of its target appears to be in line with the empirical evidence of Gürkaynak, Levin & Swanson (2010). It is thus an important question whether there are still multiple equilibria even when all agents believe that in the long run the economy will return to the standard steady state.

However, our results have important consequences even if one is not convinced that agents should expect a return to the standard steady state. Our examples in Appendix F show that for standard NK models with endogenous state variables, there is a positive probability of arriving in a state of the world (with certain values of state variables and shocks) in which there is no perfect foresight path returning to the non-deflationary steady state.[33] Hence, if we suppose that in the presence of

[31] See also Christiano & Takahashi (2018).

[32] They show that policy function iteration is not stable near the deflationary equilibria.

[33] If the LCP (q, M) is not feasible, then for any $\hat{q} \le q$ and $y \ge 0$, since (q, M) is not feasible there exists $t \in \{1, \dots, T\}$ such that $0 > (q + My)_t \ge (\hat{q} + My)_t$, so the LCP $(\hat{q}, M)$ is also not feasible.

risk, agents deal with uncertainty by integrating over the space of possible future shock sequences, as in the original stochastic extended path algorithm of Adjemian & Juillard (2013),[34] then such agents would likely place positive probability on tending to the "bad" steady state.[35] This rationalises the beliefs needed to sustain multiplicity in the prior literature. As switching to a price level target would remove the non-existence problem, it could also help ensure beliefs about long-run inflation remain positive, avoiding this extra source of indeterminacy.

6.1.2. Rich state spaces

In e.g. Theorem 2, for some of our results we suppose that the model's state space is rich enough such that for any $\tilde{q}$, there exists x_0 such that $q(x_0) = \tilde{q}$, where $q(x_0)$ gives the q from Definition 1 (the path in the absence of the bound) for the given value of x_0 (the initial state). As we mentioned, one way to ensure this condition is satisfied is by adding news shocks to the bounded variable. News shocks are plausible in many contexts. One general justification for the presence of news shocks is that they capture future uncertainty, following the original stochastic extended path of Adjemian & Juillard (2013). As discussed above, this posits that agents draw multiple samples of future shocks for periods $1, \dots, T$, calculate the perfect-foresight paths conditional on

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Consequently, if q is viewed as a draw from an absolutely continuous distribution, then if there are some q for which the model has no solution satisfying the terminal condition, then there is no solution with positive probability.

[34] This is not fully rational, as it is equivalent to assuming that agents act as if the uncertainty in all future periods would be resolved next period. However, in practice this appears to be a close approximation to full rationality, as demonstrated by Holden (2016). The authors of the original stochastic path method now have a version that is fully consistent with rationality (Adjemian & Juillard 2016).

[35] The non-existence of a solution returning to the standard steady state does not necessarily imply the existence of a solution returning to the deflationary one. However, given the indeterminacy of the deflationary steady state, it is easier to find a solution returning there in general. With no solution returning to the standard steady state, if there is a solution to the model at all, it must be one converging to the deflationary steady state.

those future shocks, and then average over these realised paths.[36] In a linear model with shocks with unbounded support, providing at least one shock has an impact on i_t for each $t \in \{1, \dots, T\}$, the distribution of future paths of $(i_t)_{t=1}^{\infty}$ will have positive support over the entirety of $\mathbb{R}^T$. This justifies looking for results that hold for any possible q.

6.2. Our general results

We now further discuss our results on uniqueness/multiplicity and existence/non-existence with respect to the prior literature.

6.2.1. Uniqueness and multiplicity

We have presented uniqueness results for otherwise linear models with terminal conditions. We argue here for the importance of these results despite their limitation to otherwise linear models.

Bodenstein (2010) showed that linearization can exclude equilibria. Additionally, Boneva, Braun & Waki (2016) show that there may be multiple perfect-foresight solutions to a non-linear NK model with ZLB, converging to the standard steady state, even though the linearized version of their model (with a ZLB) has a unique equilibrium. Thus, the multiplicity we find is strictly in addition to the multiplicity found by those authors. Our results complement those of Boneva, Braun & Waki (2016), since we are able to handle endogenous state variables, while their methods permit the analysis of fully non-linear models without endogenous states. Additionally, note that the multiplicity found in a simple linearized model in Brendon, Paustian & Yates (2013) is also found in the equivalent non-linear model in Brendon, Paustian & Yates (2019). This is suggestive evidence for the continued relevance of our results in the fully non-linear case.

In fact, the tools of this paper can be used to analyse the properties of perfect-foresight models with nonlinearities other than an

[36] See Footnote 34 for caveats to this procedure.

occasionally binding constraint. Recall that we showed $i(y) = q + My + \mathrm{O}(y'y)$ as $y'y \to 0$, where M is defined in terms of partial derivatives of the path (see Definition 1). We did not need to impose linearity to derive the complementary slackness constraints on y. Thus, in a fully non-linear perfect foresight context, we can still use the tools we develop here to look at the (first order approximate) properties of perfect foresight problems in which y does not become too large in the solution (which usually means that q does not go too negative). In particular, we do not need to linearize before deriving q or M, so we can preserve accuracy even though only large shocks might drive us to the bound. In this fully non-linear case, M will be a function of the initial state.

Furthermore, studying multiplicity in otherwise linear models is an independently important exercise. Firstly, macroeconomists have long relied on existence and uniqueness results based on linearization of models without occasionally binding constraints, even though this may produce spurious uniqueness in some circumstances.[37] Secondly, it is nearly impossible to find all perfect foresight solutions in general non-linear models, since this is equivalent to finding all the solutions to a huge system of non-linear equations, when even finding all the solutions to large systems of quadratic equations is computationally intractable. At least if we have the full set of solutions to the otherwise linear model, we may use homotopy continuation methods to map these solutions into solutions of the non-linear model. Furthermore, finding all solutions under uncertainty is at least as difficult in general, as the policy functions are also defined by a large system of non-linear equations. Thirdly, Christiano and Eichenbaum (2012) argue that the additional equilibria of Boneva, Braun & Waki (2016) may be mere "mathematical curiosities"

[37] Perturbation solutions are only valid within some domain of convergence, so even the results of e.g. Lan & Meyer-Gohde (2013; 2014) do not mean that first order determinacy implies global determinacy.

due to their non-e-learnability. This suggests that the equilibria that exist in the linearized model are of independent interest, whatever one's view on this debate. Finally, our main results for NK models imply non-uniqueness, so concerns of spurious uniqueness under linearization will not be relevant in these cases.

Indeed, our choice to focus on otherwise-linear models under perfect-foresight, with fixed terminal conditions, has biased our results in favour of uniqueness for three distinct reasons. Firstly, because there at least as many solutions under rational expectations as under perfect-foresight, as we prove in Appendix I. Secondly, because there are potentially other solutions returning to alternate steady states. Thirdly, because the original fully non-linear model may possess yet more solutions. It is thus all the more surprising that we still find multiplicity under perfect foresight in otherwise linear NK models with a ZLB.

However, we are certainly not the first to look at multiplicity in otherwise linear models with OBCs. Hebden, Lindé & Svensson (2011) propose a simple way to find multiplicity: namely, hit the model with a large shock which pushes it towards the bound, and see if one can find more than one set of periods such that being at the bound during those periods is an equilibrium. In practice, this suggests first looking if there is a solution which finally escapes the bound after one period, then looking to see if there is one which finally escapes the bound after two periods, and so on.[38] This procedure may succeed in finding an example of multiplicity, and thus proving that the original model does not possess a unique solution. However, it cannot work completely generally as the multiplicity may only arise in very particular states, or may feature multiple spans at the bound.

[38] This is tractable in our context, as it is easy to constrain the MILP representation of the LCP problem to be at the bound in the final period. The "DynareOBC" toolkit takes this approach. See Holden (2016) for further details.

Like us, Jones (2015) presents a uniqueness result for models with occasionally binding constraints. He shows that if one knows the set of periods in which the constraint binds, then under standard assumptions, there is a unique path in which the constraint binds in those periods. However, the multiplicity for models with OBCs precisely stems from there being multiple sets of periods at which the model could be at the bound. Our results are not conditional on knowing in advance the periods at which the constraint binds.

Finally, uniqueness results have also been derived in the Markov switching literature. Examples include Davig & Leeper (2007), Farmer, Waggoner & Zha (2010; 2011) and Barthélemy & Marx (2019). These papers assume regime switching is exogenous. This prevents their application to OBCs, which generate endogenous regime switches. Determinacy results with endogenous switching were derived by Barthélemy & Marx (2017) assuming regime transition probabilities are a smooth function of the state. These results are not directly applicable to OBCs as OBCs produce jumps in regime transition probabilities.

6.2.2. Existence and non-existence

We also produced conditions for the existence of a perfect-foresight solution to an otherwise linear model with a terminal condition. These results provide new intuition for the prior literature on existence under rational expectations, which has found that NK models with a ZLB might have no solution at all if the variance of shocks is too high. For example, Mendes (2011) derived analytic results on existence as a function of the variance of a demand shock, and Basu & Bundick (2015) showed the quantitative relevance of such results. Furthermore, conditions for the existence of an equilibrium in a simple NK model with discretionary monetary policy are derived in close form for a model with a two-state Markov shock by Nakata & Schmidt (2019). They show that the economy must spend a small amount of time in the bad state for the equilibrium to exist, which again links existence to variance.

While our results are not directly related to the variance of shocks, as we work under perfect foresight, they are nonetheless linked. We showed that whether a perfect foresight solution exists depends on the perfect-foresight path taken by nominal interest rates in the absence of the bound. Many of our results assumed that this path was arbitrary. However, in a model with a small number of shocks, all of bounded support, and no information about future shocks, clearly not all paths are possible for nominal interest rates in the absence of the bound. The more shocks are added (e.g. news shocks), and the wider their support, the greater will be the support of the space of possible paths for nominal interest rates in the absence of the ZLB, and hence, the more likely will be non-existence of a solution for a positive measure of paths. This helps to explain the literature's prior results.

There has also been some prior work by Richter & Throckmorton (2015) and Gavin et al. (2015; Appendix B) that has related a kind of eductive stability (the convergence of policy function iteration) to other properties of the model. Non-convergence of policy function iteration is suggestive of non-existence, though not definitive evidence. These results on stability for small, fully non-linear models under rational expectations are complementary to our results on existence for arbitrarily large, otherwise linear models under perfect foresight.

It is also possible to establish existence by finding a solution to the model, perhaps conditional on the initial state. Under perfect foresight, the methods described in Holden (2010; 2016) are a possibility, and the method of Guerrieri & Iacoviello (2015) (extending Jung, Teranishi & Watanabe (2005)) is a prominent alternative. Under rational expectations, policy function iteration methods have been used by

Fernández-Villaverde et al. (2015) and Richter & Throckmorton (2015), amongst others. However, solution algorithms cannot help us establish non-existence: non-convergence of a solution algorithm does not imply non-existence.[39] Furthermore, if the problem is solved globally, there could still be an area of non-existence outside of the grid on which the model was solved. Similarly, if a solution is found under perfect foresight for a given initial state, then there is still no guarantee of solution existence for other initial points. If we wish to guide policy makers in how they should act to ensure existence in any state, then there is an essential role for results on global existence, as we have produced here.

6.3. The application to the zero lower bound

We finish this section by discussing the relevance of our application to the ZLB: first by examining the plausibility of the multiple equilibria we find; next by looking further at price level targets.

6.3.1. Plausibility of multiplicity at the ZLB

We need to answer two key questions to establish the economic relevance of self-fulfilling spells at the ZLB. Firstly, is the coordination of beliefs needed to sustain the equilibrium plausible? Secondly, do such equilibria feature reasonable movements in macroeconomic variables? It is true that self-fulfilling jumps to the ZLB may feature implausibly large falls in output and inflation. This is closely related to the so-called "forward guidance puzzle" (Carlstrom, Fuerst & Paustian 2015; Del Negro, Giannoni & Patterson 2015).[40] However, if interest rates are

[39] The algorithm of Holden (2016) is a partial exception. This algorithm always converges, either producing a solution, or a proof of non-existence.

[40] McKay, Nakamura & Steinsson (2017) point out that these implausibly large responses to news are muted in models with heterogeneous agents, and give a simple "discounted Euler" approximation that produces similar results to a full heterogeneous agent model. While including a discounted Euler equation makes it harder to generate multiplicity (e.g. reducing the parameter space with multiplicity in the Brendon, Paustian & Yates (2013) model), when there is multiplicity, the resulting responses are much larger, as the weaker response to news means the required endogenous "news" needs to be much greater in order to drive the model to the bound.

already low (due to a recession), then a much smaller self-fulfilling "news" shock is needed to produce a jump to the ZLB. Thus, there will be a much more moderate drop in output and inflation. Furthermore, with interest rates low, it takes a smaller movement in confidence for people to expect to hit the ZLB. Even more plausibly, if the economy is already at the ZLB, then small changes in confidence could easily select an equilibrium featuring a longer spell there than in the equilibrium that leaves fastest. Indeed, there is no good reason people should coordinate on the equilibrium with the shortest time at the ZLB.

As an illustration, in Figure 3 we plot the impulse response to a large magnitude preference shock (scaling felicity), in the Smets & Wouters (2003) model.[41] The shock is not quite large enough to send the economy to the ZLB[42] in the standard solution, shown with a dashed line. However, there is an alternative solution in which the economy jumps to the bound one period after the initial shock, remaining there for three periods. While the alternative solution features larger drops in output and inflation, the falls are broadly in line with the magnitude of the crisis, with Eurozone GDP and consumption falling about 20% below a pre-crisis log-linear trend, and the largest drop in annualized Eurozone consumption inflation from 2008q3 to 2008q4 being around 4.4%.[43] Considering this, we view it as plausible that multiplicity of equilibria was a significant component of the explanation for the great recession.

[41] The shock is 22.5 standard deviations. While this is implausibly large, the economy could be driven to the bound with a run of smaller shocks. It is also worth recalling that the model was estimated on the great moderation period, so the estimated standard deviations may be too low, and the real interest rate too high. Finally, recent evidence (Cúrdia, Del Negro & Greenwald 2014) suggests that the shocks in DSGE models should be fat tailed, making large shocks more likely.
[42] Since the Smets & Wouters (2003) model does not include trend growth, it is impossible to produce a steady state value for nominal interest rates that is consistent with both the model and the data. We choose to follow the data, setting the steady state of nominal interest rates to its mean level over the same sample period used by Smets & Wouters (2003), using data from the same source (Fagan, Henry & Mestre 2005).
[43] Data was again from the area-wide model database (Fagan, Henry & Mestre 2005).

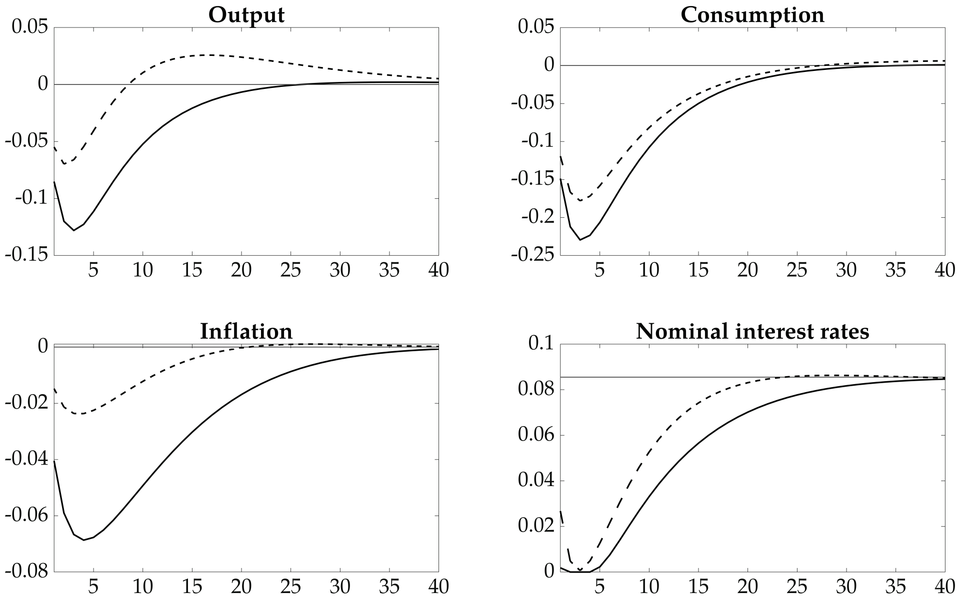

Figure 3: Two solutions following a preference shock in the Smets & Wouters (2003) model.
All variables are in logarithms. Inflation and nominal interest rates are annualized.
The dashed line is a solution which does not hit the bound.
The solid line is an alternative solution which does hit the bound.

6.3.2. Price level targeting

Our results suggest that given belief in an eventual return to inflation, a determinate equilibrium may be produced in standard NK models if the central bank switches to targeting the price level, rather than the inflation rate. As the previous figure made clear, the benefits to this could be substantial.[44]

There is of course a large literature advocating price level targeting already. Vestin (2006) made an important early contribution by showing that its history dependence mimics the optimal rule, a conclusion reinforced by Giannoni (2014). Eggertsson & Woodford (2003) showed the particular desirability of price level targeting in the presence of the ZLB, since it produces inflation after the bound is escaped. A later contribution by Nakov (2008) showed that this result survived taking a

[44] We look more formally at welfare in a model very similar to the Smets & Wouters (2003) model in Appendix F.5.

fully global solution, and Coibion, Gorodnichenko & Wieland (2012) showed that it still holds in a richer model. More recently, Basu & Bundick (2015) have argued that a response to the price level ensures equilibria exists even when shocks have large variances, avoiding the problems stressed by Mendes (2011). Our argument is distinct from these; we showed that in the presence of the ZLB, inflation targeting rules are indeterminate, even conditional on an eventual return to inflation, whereas price level targeting rules produce determinacy, in the sense of the existence of a unique perfect-foresight path returning to the standard steady state.

Our results are also distinct from those of Adão, Correia & Teles (2011) who showed that if the central bank is not constrained to respect the ZLB out of equilibrium (i.e. for non-market-clearing prices),[45] and if the central bank uses a rule that responds to the right hand side of the Euler equation, then a globally unique equilibrium may be produced, even without ruling out explosive beliefs about prices. Their rule has the flavour of a (future) price-targeting rule, due to the presence of future prices in the right-hand side of the Euler equation. We assume though that the central bank must satisfy the ZLB even out of equilibrium (i.e. for all prices), which makes it harder to produce uniqueness. However, in line with the bulk of the NK literature, we maintain the standard assumption that explosive paths for inflation are ruled out, an assumption which the rules of Adão, Correia & Teles (2011) do not

[45] Bassetto (2004) gives a precise definition of this. The distinction is between constraints that hold for any prices, such as agent first order conditions, and constraints that hold only for the market clearing prices, such as market clearing conditions. The contention of Bassetto (2004) is that the ZLB is in the latter category—the central bank can promise negative nominal interest rates off the equilibrium path, which will give determinacy without negative rates actually being required. (Negative rates provide an infinite nominal transfer, entirely devaluing nominal wealth, so pushing up prices and preventing negative rates ever being called for.) Bassetto notes how dangerous it would be to rely on such infinite transfers given the possibility of misspecification.

require.[46] As such, our results are complementary to those of Adão, Correia & Teles (2011).

Somewhat contrary to our results, Armenter (2017) shows that in a simple otherwise linear NK model, if the central bank pursues Markov (discretionary) policy subject to an objective targeting inflation, nominal GDP or the price level, then the presence of a ZLB produces additional equilibria quite generally. This difference between our results and those of Armenter (2017) is chiefly driven by the fact that we rule out getting stuck in the neighbourhood of the deflationary steady state by assumption. We also assume commitment to a rule.

In other related work, Duarte (2016) considers how a central bank might ensure determinacy in a simple continuous time new Keynesian model. Like us, he finds that the Taylor principle is not sufficient in the presence of the ZLB. He shows that determinacy may be produced by using a rule that holds interest rates at zero for a history dependent amount of time, before switching to a $\max\{0, \dots\}$ Taylor rule. While we do not allow for such switches in central bank behaviour, we do find an important role for history dependence, through price targeting.

7. Conclusion

Determinacy conditions are crucial for understanding the behaviour of the models we work with in macroeconomics. This paper provides the first general theoretical results on existence and uniqueness for otherwise linear models with occasionally binding constraints, given terminal conditions. As such, it may be viewed as doing for models with OBCs what Blanchard & Kahn (1980) did for linear models. Applying our results, we showed that multiplicity is the norm in New Keynesian models, but that a response to the price level can restore determinacy.

[46] Note that the unstable solutions under price level targeting feature exponential growth in the logarithm of the price level, which also implies explosions in inflation rates.

Our conditions may be easily checked numerically using the "DynareOBC" toolkit we provide.[47]

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