Online appendices to: "Existence and uniqueness of solutions to dynamic models with occasionally binding constraints."

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Contents

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Appendix A: Getting started with DynareOBC

DynareOBC is a MATLAB toolbox designed to simulate and analyse models with occasionally binding constraints. It relies on Dynare (Adjemian et al. 2011) internally. To get started with DynareOBC, first download the latest release (`dynareOBCRelease.zip`) from:

https://github.com/tholden/dynareOBC/releases

Extract the zip archive into a sub-folder. You should also install the latest stable version of Dynare from:

http://www.dynare.org/download/dynare-stable

While DynareOBC contains a MILP solver, for best results, at this point, you should install a commercial MILP solver. Many of these are free for academics. We have had good results with Gurobi, which is available for academics by following the steps here:

http://www.gurobi.com/academia/for-universities

Other MILP solvers which are available for free to academics are documented in DynareOBC's `ReadMe.pdf`.

If you do not have administrative rights on your machine, you will also need to get your administrator to install a few minor dependencies for you, which otherwise DynareOBC would install itself. Full instructions for this are given in DynareOBC's `ReadMe.pdf`.

Next, open MATLAB, reset the MATLAB path (to be on the safe side) and then add only the following folders to your path. In each case, you should not click "add with subfolders". Only the folders specified need adding:

1) The "`matlab`" folder within Dynare.
2) The root folder of DynareOBC, i.e. the folder containing "`dynareOBC.m`".
3) The "`matlab`" folder within whichever MILP solver you installed (if any).

You can now test your set-up of DynareOBC by typing:

```
dynareOBC TestSolvers
```

at the MATLAB command prompt. The first time you run DynareOBC it will install various dependencies, and it may restart MATLAB several times. Note that if you have not installed a commercial MILP solver, you should say “yes” when offered the choice to install “SCIP”, otherwise DynareOBC’s performance will be severely compromised. When DynareOBC has installed everything necessary, it will run the solver tests. Double check in particular that the LP and MILP tests are passing. (Results for the other tests, e.g. semi-definite programming are not relevant.)

If everything has worked up to this point, then you now have a fully functioning install of DynareOBC. To see it in action, you could start by running DynareOBC’s included examples. Most of these examples can be run by changing to DynareOBC’s “`Examples`” directory in MATLAB, and then executing the script “`RunAllExamples`”. This iterates over the various sub-directories of the “`Examples`” directory, running the script “`RunExample`” within each.

Developing your own models for use with DynareOBC is easy. You can include one or more occasionally binding constraints directly within your `MOD` file. For example, to include a zero lower bound on nominal interest rates, your `MOD` file might contain the line:

```
i = max( 0, 1.5 * pi + 0.25 * y );
```

DynareOBC supports both `max` and `min` (with two arbitrary arguments) and `abs` (with one arbitrary argument). There are no restrictions on what is contained within the brackets. You do not have to have a `0` term, and it does not matter which of the arguments of `max` or `min` is bigger or smaller in steady state. The only limitation is that the two arguments of `max` or `min` cannot be identical in steady state (likewise, the argument of `abs` cannot be zero in steady state). For a work-around of this limitation in a financial frictions context, see the approach of Swarbrick, Holden & Levine (2016).

Once you have included an OBC in you `MOD` file, you can run it with DynareOBC by typing:

```
dynareOBC ModFileName.mod
```

where “ModFileName.mod” is the name of your MOD file. Just as with standard Dynare, if you have specified e.g. “irf=40” within your stoch_simul command, then DynareOBC will produce impulse responses. Likewise, if you have specified e.g. “periods=1000” within your stoch_simul command, then DynareOBC will produce a stochastic simulation.

As an example, the file “bbw2016.mod” in the “Examples/BonevaBraunWaki2016” directory of DynareOBC contains the line:

```
r = max( 0, re + phi_pi * (pi - pi_STEADY) + phi_y * (gdp - gdp_STEADY) );
```

in its model block, and has the following stoch_simul command:

```
stoch_simul( order = 1, periods = 0, irf = 40 );
```

We can run the MOD file (implementing the model of Boneva, Braun & Waki (2016)) by typing:

```
dynareOBC bbw2016.mod
```

from within the “Examples/BonevaBraunWaki2016” directory. Doing this produces two sets of impulse responses, however none of them hit the zero lower bound, as the shock is too small. To produce impulse responses to a larger shock, we can run DynareOBC with the ShockScale command line option. This increases the size of the initial impulse in IRF generation, without altering the standard deviations of the model’s shocks, or otherwise changing the behaviour of stochastic simulation. For example, if we run:

```
dynareOBC bbw2016.mod ShockScale=5
```

then DynareOBC produces IRFs to a 5 standard deviation shock to each of the model’s exogenous variables. This produces the two plots shown in Figure 1. In all DynareOBC plots, the solid line shows the economy’s path imposing the bound(s), and the dotted line shows the path the economy would have taken were it not for the bound(s).

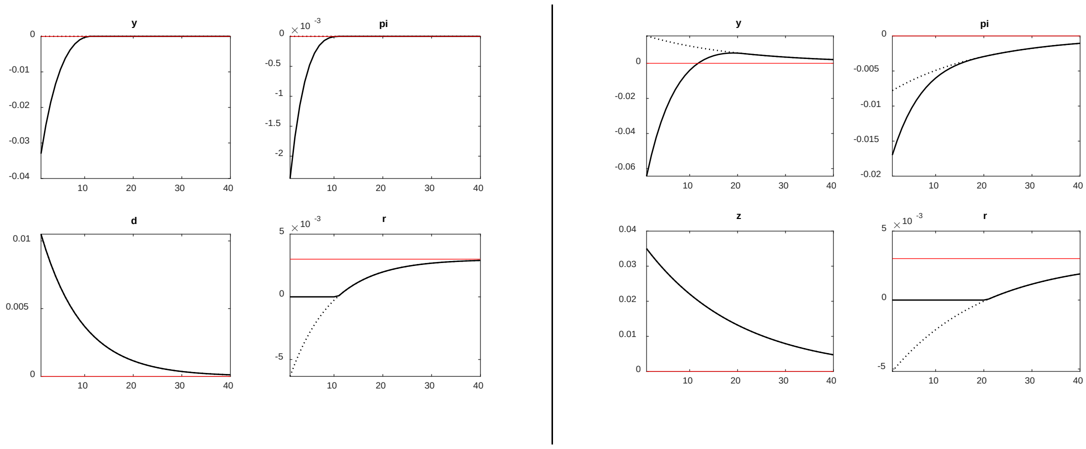

Figure 1: Sample output from running "`dynareOBC bbw2016.mod ShockScale=5`". The left 4 panels show the response to a 5 standard deviation demand shock. The right 4 panels show the response to a 5 standard deviation productivity shock.

All variables are in logarithms. In all cases, the dotted lines show the path the economy would have followed were it not for the ZLB.

DynareOBC always also outputs diagnostic information about the model. For example, for this model, DynareOBC outputs the following, after it has made its final internal call to Dynare. Here we have made the most important lines bold to highlight them, and we have removed some additional white space:

`Beginning to solve the model.`

`Solving the model for specific parameters.`

`Saving NLMA parameters.`

`Retrieving IRFs to shadow shocks.`

`Preparing normalized sub-matrices.`

`Largest P-matrix found with a simple criterion included elements up to horizon 32 periods.`

`The search for solutions will start from this point.`

`Pre-calculating the augmented state transition matrices and possibly conditional covariances.`

`Performing initial checks on the model.`

`M is an S matrix, so the LCP is always feasible. This is a necessary condition for there to always be a solution.`

`varsigma bounds (positive means M is an S matrix):`

```
            6.7600451299659          6.76004512996604
sum of y from the alternative problem (zero means M is an S matrix):
     0
Skipping tests of feasibility with infinite T
(TimeToEscapeBounds).
To run them, set FeasibilityTestGridSize=INTEGER where INTEGER>0.
Skipping further P tests, since we have already established that
M is a P-matrix.
The M matrix with T (TimeToEscapeBounds) equal to 32 is a P-matrix.
There is a unique solution to the model, conditional on the bound
binding for at most 32 periods.
This is a necessary condition for M to be a P-matrix with
arbitrarily large T (TimeToEscapeBounds).
A weak necessary condition for M to be a P-matrix with arbitrarily
large T (TimeToEscapeBounds) is satisfied.
Discovering and testing the installed MILP solver.
Found working solver: GUROBI
Forming optimizer.
Preparing to simulate the model.
Simulating IRFs.
Cleaning up.
```

We see that DynareOBC's fast default diagnostics already identified that the M matrix for this model was a P-matrix and an S-matrix, as well as providing some weak evidence that M is a P-matrix for arbitrarily high T. (Discussion of this uniqueness in light of the results of Boneva, Braun & Waki (2016) is contained in Appendix D.4.)

Note that DynareOBC refers to T as "`TimeToEscapeBounds`". This is the name DynareOBC gives to the command line option to control the size of the linear complementarity problems DynareOBC solves internally. To see why this may be necessary, try running the command:

```
dynareOBC bbw2016.mod ShockScale=10
```

Now DynareOBC does not complete successfully. Instead it reports:

```
Error using SolveBoundsProblem (line 241)
Impossible problem encountered. Try increasing TimeToEscapeBounds,
or reducing the magnitude of shocks.
```

To avoid this problem, we just need to follow the advice of the error message and run with a higher value for "`TimeToEscapeBounds`". For example, if we run:

`dynareOBC bbw2016.mod ShockScale=10 TimeToEscapeBounds=64`

then DynareOBC completes successfully. In this case the response to the productivity shock stays at the ZLB for more than 32 periods, which is significant as 32 is the default number of periods for "`TimeToEscapeBounds`".

"`TimeToEscapeBounds`" and "`ShockScale`" are two of DynareOBC's command line options. There is a full list of these options in the "`ReadMe.pdf`" contained in DynareOBC's root directory, along with details on what each option does. Since the full list of options may be somewhat bewildering though, we conclude this getting started guide with details of those options most relevant to the analysis of a model's properties and those which impact perfect foresight simulation. Note that all options accepting a number must be entered without a space between the name of the option, the equals sign and the number.

- `TimeToEscapeBounds=INTEGER` (default: `32`)

  The number of periods after which the model is expected to be away from any occasionally binding constraints. If there is no solution which finally escapes within this time, DynareOBC will produce an error.

- `TimeToReturnToSteadyState=INTEGER` (default: `64`)

  The number of periods in which to verify that the constraints are not being violated.

- `ReverseSearch`

By default, DynareOBC finds a solution in which the last period at the bound is as soon as possible. This option makes DynareOBC find a solution in which the last period at the bound is as remote as possible, subject to being less than the longest horizon (i.e. `TimeToEscapeBounds`).

- `FullHorizon`

  By default, DynareOBC finds a solution in which the last period at the bound is as soon as possible. This option makes DynareOBC just solve the bounds problem at the longest horizon.

- `Omega=FLOAT` (default: `1000`)

  The tightness of the constraint on the news shocks. If this is large, solutions with news shocks close to zero will be returned when there are multiple solutions. It is often helpful to combine this option with `FullHorizon` so that DynareOBC does not just choose the solution which escapes the bound first.

- `SkipFirstSolutions=INTEGER` (default: `0`)

  If this is greater than `0`, then DynareOBC ignores the first `INTEGER` solutions it finds, unless no other solutions are found, in which case it takes the last found one. Thus, without `ReverseSearch`, this tends to find solutions at the bound for longer. With `ReverseSearch`, this tends to find solutions at the bound for less time.

- `FeasibilityTestGridSize=INTEGER` (default: `0`)

  Specifies the number of points in each of the two axes of the grid on which a test of a sufficient condition for feasibility with $T = \infty$ is performed. Setting a larger number increases the chance of finding feasibility, but may be slow.

  If `FeasibilityTestGridSize=0` then the test is disabled.

- `SkipQuickPCheck`

  Disables the "quick" check to see if the M matrix has any contiguous principal sub-matrices with non-positive determinants.

- `PTest=INTEGER` (default: `0`)

Runs a fast as possible test to see if the top `INTEGERxINTEGER` submatrix of M is a P-matrix. Set this to `0` to disable these tests.

- `AltPTest=INTEGER` (default: `0`)

  Uses a slower, more verbose procedure to test if the top `INTEGERxINTEGER` submatrix of M is a P-matrix. Set this to 0 to disable these tests.

- `FullTest=INTEGER` (default: `0`)

  Runs very slow tests to see if the top `INTEGERxINTEGER` submatrix of M is a $P_{(0)}$ and/or (strictly) semi-monotone matrix.

- `UseVPA`

  Enables more accurate evaluation of determinants using the symbolic toolbox.

- `ShockScale=FLOAT` (default: `1`)

  Scale of shocks for IRFs. This allows the calculation of IRFs to shocks larger or smaller than one standard deviation.

- `IRFsAroundZero`

  By default at first order, IRFs are centred around the steady state. This option instead centres IRFs around `0`.

Appendix B: Additional matrix properties and their relationships

The following definitions help us state our additional results: [2]

Definition 6 (Principal sub-matrix, Principal minor) For a matrix $M \in \mathbb{R}^{T\times T}$, the **principal sub-matrices** of M are the matrices $\left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}$, where $S, k_1, \dots, k_S \in \{1,\dots,T\}, k_1 < k_2 < \cdots < k_S$, i.e. the **principal sub-matrices** of M are formed by deleting the same rows and columns. The **principal minors** of M are the determinants of M's principal sub-matrices.

Definition 7 (P(₀)-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is called a **P-matrix (P_0-matrix)** if the principal minors of M are all strictly (weakly) positive.[3]

Definition 8 (General positive (semi-)definite) A matrix $M \in \mathbb{R}^{T\times T}$ is called **general positive (semi-)definite** if $M + M'$ is positive (semi-)definite (p.(s.)d.).

Definition 9 ((Non-)Degenerate matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is called a **non-degenerate matrix** if the principal minors of M are all non-zero. M is called a **degenerate matrix** if it is not a non-degenerate matrix.

Definition 10 (Sufficient matrices) $M \in \mathbb{R}^{T\times T}$ is called **column sufficient** if M is a P_0-matrix, and for each principal sub-matrix $W \coloneqq \left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}$ of M with zero determinant, and for each proper principal sub-matrix $\left[W_{i,j}\right]_{i,j=l_1,\dots,l_R}$ of W $(R < S)$ with zero determinant, the columns of $\left[W_{i,j}\right]_{\substack{i=1,\dots,S \\ j=l_1,\dots,l_R}}$ are not a basis for the column space of W.[4] M is called **row sufficient** if M' is column sufficient. M is called **sufficient** if it is column and row sufficient.

---

[2] In each case, we give the definitions in a constructive form which makes clear both how the property might be verified computationally, and the links between definitions. For the original definitions, and the proofs of equivalence between the ones below and the originals, see Cottle, Pang & Stone (2009a) and Xu (1993).
[3] This is equivalent to our original definition of a P-matrix in Definition 3 (P-matrix). (Cottle, Pang & Stone 2009a)
[4] This may be checked via the singular value decomposition.

Definition 11 ((Strictly) Copositive) A matrix $M \in \mathbb{R}^{T\times T}$ is called (**strictly) copositive** if $M + M'$ is (strictly) semi-monotone.[5]

Definition 12 (Adequate matrices) $M \in \mathbb{R}^{T\times T}$ is called **column adequate** if M is a P_0-matrix, and for each principal sub-matrix $W := [M_{i,j}]_{i,j=k_1,\dots,k_S}$ of M with zero determinant, the columns of $[M_{i,j}]_{\substack{i=1,\dots,T\\ j=k_1,\dots,k_S}}$ are linearly dependent. M is called **row adequate** if M' is column adequate. M is called **adequate** if it is column adequate and row adequate.

Definition 13 (S(0)-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is called an **S-matrix** (**S_0-matrix)** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$ ($My \geq 0$). [6]

Definition 14 ((Strictly) Semi-monotone) A matrix $M \in \mathbb{R}^{T\times T}$ is called (**strictly) semi-monotone** if each of its principal sub-matrices is an **S_0-matrix** (**S-matrix)**.

For example, consider the $T = 3$ case with $M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$. Then M is a P-matrix if and only if $M_{11} > 0$, $M_{22} > 0$, $M_{33} > 0$, $\det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} > 0$, $\det \begin{bmatrix} M_{11} & M_{13} \\ M_{31} & M_{33} \end{bmatrix} > 0$, $\det \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} > 0$ and $\det M > 0$.

Cottle, Pang & Stone (2009a) note the following relationships between these classes (amongst others):

Lemma 1 The following hold:

1) All general positive semi-definite matrices are copositive, sufficient and P_0.
2) All general positive definite matrices are P matrices.

---

[5] Väliaho (1986) contains an alternative characterisation which avoids solving any linear programming problems.

[6] These conditions may be rewritten as $\sup\{\varsigma \in \mathbb{R} | \exists y \geq 0 \text{ s.t. } \forall t \in \{1,\dots,T\}, (My)_t \geq \varsigma \wedge y_t \leq 1\} > 0$, and $\sup\left\{\sum_{t=1}^{T} y_t \,\middle|\, y \geq 0, My \geq 0 \wedge \forall t \in \{1,\dots,T\}, y_t \leq 1\right\} > 0$, respectively. As linear programming problems, these may be solved in time polynomial in T using the methods of e.g. Roos, Terlaky, and Vial (2006). Alternatively, by Ville's Theorem of the Alternative (Cottle, Pang & Stone 2009b), M is not an S_0-matrix if and only if $-M'$ is an S-matrix.

3) P_0 includes skew-symmetric, general positive semi-definite, sufficient and P matrices.
4) All P_0-matrices, and all copositive matrices are semi-monotone, and all P-matrices, and all strictly copositive matrices are strictly semi-monotone.
5) All column (row) adequate matrices are column (row) sufficient.

A common "intuition" is that in models without state variables, M must be both a P matrix, and an S matrix. In fact, this is not true. Indeed, there are even purely static models for which M is in neither of these classes, as we prove the following result in Appendix L.2.

Proposition 3 There is a purely static model for which $M_{1:\infty,1:\infty} = -I_{\infty\times\infty}$, which is neither a P-matrix, nor an S-matrix, for any T.

Appendix C: Supplemental results

Appendix C.1: Uniqueness

The following corollary of Theorem 2 gives more easily verified necessary conditions for uniqueness.

Corollary 4 If for all $q \in \mathbb{R}^T$, the LCP (q, M) has a unique solution, then:

1. All of the principal sub-matrices of M are P-matrices, S-matrices and strictly semi-monotone. (Cottle, Pang & Stone 2009a)
2. M has a strictly positive diagonal. (Immediate from definition.)
3. All of the eigenvalues of M have complex arguments in the interval $\left(-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T}\right)$. (Fang 1989)

The following corollary of Theorem 2 gives more easily verified sufficient conditions for uniqueness.

Corollary 5 For an arbitrary matrix A, denote the spectral radius of A by $\rho(A)$, and its largest and smallest singular values by $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$, respectively. Let $|A|$ be the matrix with $|A|_{ij} = |A_{ij}|$ for all i, j. Then, for any matrix $M \in \mathbb{R}^{T\times T}$, if there exist diagonal matrices $D_1, D_2 \in \mathbb{R}^{T\times T}$ with positive diagonals, such that $W := D_1 M D_2$ satisfies one of the following conditions, then for all $q \in \mathbb{R}^T$, the LCP (q, M) has a unique solution:

1. W is general positive definite. (Cottle, Pang & Stone 2009a)
2. W has a positive diagonal, and $\langle W\rangle^{-1}$ is a nonnegative matrix, where $\langle W\rangle$ is the matrix with $\langle W\rangle_{ij} = -|W_{ij}|$ for $i \neq j$ and $\langle W\rangle_{ii} = |W_{ii}|$. (Bai & Evans 1997)
3. $\rho(|I - W|) < 1$. (Li & Wu 2016)
4. $(I + W)'(I + W) - \sigma_{\max}(|I - W|)^2 I$ is positive definite. (Li & Wu 2016)
5. $\sigma_{\max}(|I - W|) < \sigma_{\min}(I + W)$. (Li & Wu 2016)
6. $\sigma_{\min}\left((I - W)^{-1}(I + W)\right) > 1$. (Li & Wu 2016)
7. $\sigma_{\max}\left((I + W)^{-1}(I - W)\right) < 1$. (Li & Wu 2016)

8. $\rho(|(I+W)^{-1}(I-W)|) < 1$. (Li & Wu 2016)

In our experience, whenever M is a P-matrix, it will usually satisfy one of these conditions when D_1 and D_2 are chosen so that all rows and columns of $|W|$ have maximum equal to 1, using the algorithm of Ruiz (2001).

We also have necessary conditions for uniqueness with arbitrary T. In particular:

Proposition 4 Given an otherwise linear model with an OBC, the limit, d_k of the k^{th} diagonal[7] of M with $T = \infty$ exists, is finite, and is computable in time polynomial in k and the number of state variables of the model. If for all finite T, M is a P-matrix, then for all $S > 0$, the $S \times S$ Toeplitz matrix with k^{th} diagonal d_k is a P_0-matrix.

The properties of the limits of the diagonals of M are established in Appendix L.1 as part of the proof of Proposition 2. The rest of the claim follows from the continuity of determinants.

Since some classes of models almost never possess a unique solution when at the zero lower bound, we might reasonably require a lesser condition, namely that at least when the solution to the model without a bound is a solution to the model with the bound, then it ought to be the unique solution. This is equivalent to requiring that when q is non-negative, the LCP (q, M) has a unique solution. Conditions for this are given in the following proposition:

Proposition 5 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \gg 0$ ($q \geq 0$) if and only if M is (strictly) semi-monotone. (Cottle, Pang & Stone 2009a)

Hence, by verifying that M is semi-monotone, we can reassure ourselves that introducing the bound will not change the solution away from the bound. When this condition is violated, even when the economy is a long way from the bound, there may be solutions which jump to the bound. Since principal sub-matrices of (strictly)

[7] We take diagonal indices to be increasing as one moves up and right in M.

semi-monotone are (strictly) semi-monotone, a failure of (strict) semi-monotonicity for some T implies a failure for all larger T.

Where there are multiple solutions, we might like to select one via some objective function. This is tractable when either the number of solutions is finite, or the solution set is convex:

Proposition 6 The LCP (q, M) has a finite (possibly zero) number of solutions for all $q \in \mathbb{R}^T$ if and only if M is non-degenerate. (Cottle, Pang & Stone 2009a)

Proposition 7 The LCP (q, M) has a convex (possibly empty) set of solutions for all $q \in \mathbb{R}^T$ if and only if M is column sufficient. (Cottle, Pang & Stone 2009a)

Finally, conditions for uniqueness of the path of the bounded variable is given in the following proposition:

Proposition 8 There exists w such that for any solution y of the LCP (q, M), $q + My = w$ if and only if M is column adequate. (Cottle, Pang & Stone 2009a)

Appendix C.2: Existence

We now turn to sufficient conditions for existence of a solution for finite T.

Proposition 9 The LCP (q, M) is solvable if it is feasible and, either:

1. M is row-sufficient, or,
2. M is copositive and for all non-singular principal sub-matrices W of M, all non-negative columns of W^{-1} possess a non-zero diagonal element.

(Cottle, Pang & Stone 2009a; Väliaho 1986)

If either condition 1 or condition 2 of Proposition 9 is satisfied, then to check existence for any particular q, we only need to solve a linear programming problem. As this will be faster than solving the particular LCP, this may be helpful in practice. Moreover:

Proposition 10 The LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if at least one of the following conditions holds: (Cottle, Pang & Stone 2009a)

1. M is an S-matrix, and either condition 1 or 2 of Proposition 9 is satisfied.
2. M is copositive and non-degenerate.
3. M is a P-, a strictly copositive or strictly semi-monotone matrix.

If condition 1, 2 or 3 of Proposition 10 is satisfied, then the LCP will always have a solution. Therefore, for any path of the bounded variable in the absence of the bound, we will also be able to solve the model when the bound is imposed. Finally, in the special case of nonnegative M matrices we can derive conditions for existence that are both necessary and sufficient:

Proposition 11 If M is a nonnegative matrix, then the LCP (q, M) is solvable for all $q \in \mathbb{R}^T$ if and only if M has a positive diagonal. (Cottle, Pang & Stone 2009a)

Appendix D: Additional discussion

Appendix D.1: On the relevance of our terminal condition

Most of our results are conditional on the economy returning to a given steady state about which the economy is locally determinate. For NK models, this means the steady state with positive inflation, unless the model is augmented with a sunspot equation following Farmer, Khramov & Nicolò (2015). Using the positive inflation steady state contrasts with the prior literature, beginning with Benhabib, Schmitt-Grohé & Uribe (2001a; 2001b), and further developed by Schmitt-Grohé & Uribe (2012), Mertens & Ravn (2014) and Aruoba, Cuba-Borda & Schorfheide (2018), amongst others. In this prior literature, indeterminacy comes from the fact that agents may place positive probability on the economy converging towards the deflationary steady state.

A priori, it does not seem obvious that agents should place positive probability on the economy converging to the deflationary steady state. Firstly, the central banks of most major economies have announced (positive) inflation targets. Thus, convergence to a deflationary steady state would represent a spectacular failure to hit the target. As argued by Christiano and Eichenbaum (2012), a central bank may rule out the deflationary equilibria in practice by switching to a money growth rule following severe deflation, along the lines of Christiano & Rostagno (2001).[8] Furthermore, Richter & Throckmorton (2015) and Gavin et al. (2015) present evidence that the deflationary equilibrium is unstable[9] under rational expectations if shocks are large enough, making it much harder for agents to coordinate upon it. Finally, a belief that inflation will eventually return to the vicinity of its target appears to be in line with the empirical evidence of Gürkaynak, Levin & Swanson (2010). It is an important question, then, whether there are still multiple equilibria even when all agents believe

[8] See also Christiano & Takahashi (2018).

[9] They show that policy function iteration is not stable near the deflationary equilibria.

that in the long-run, the economy will return to a particular steady state. It is on such equilibria that we focus in this paper.

However, our results have important consequences even if one is not convinced that agents should expect a return to the inflationary steady state. In particular, we also show that for standard NK models with endogenous state variables, there is a positive probability of ending up in a state of the world (i.e. with certain state variables and shock realisations) in which there is no perfect foresight path returning to the "good" steady state.[10] Hence, if we suppose that in the stochastic model, agents deal with uncertainty by integrating over the space of possible future shock sequences, as in the original stochastic extended path algorithm of Adjemian & Juillard (2013),[11] then such agents would always put positive probability on tending to the "bad" steady state, rationalising the beliefs needed to sustain multiplicity in the prior literature. Interestingly, since we show that switching to a price level target would remove the non-existence problem, it could also help ensure beliefs about long-run inflation remain positive, avoiding this extra source of indeterminacy.

Appendix D.2: On the relevance of results which are conditional on an arbitrary path for the bounded variable

In e.g. Theorem 2, for some of our results we suppose that the model's state space is rich enough such that for any $\tilde{q}$, there exists x_0 such that $q(x_0) = \tilde{q}$ with that x_0, where $q(x_0)$ gives the q from Definition 1 for the given value of x_0. In most models,

[10] If the LCP (q, M) is not feasible, then for any $\hat{q} \le q$ and $y \ge 0$, since (q, M) is not feasible there exists $t \in \{1, \dots, T\}$ such that $0 > (q + My)_t \ge (\hat{q} + My)_t$, so the LCP $(\hat{q}, M)$ is also not feasible. Consequently, if q is viewed as a draw from an absolutely continuous distribution (e.g. because the space of shocks is sufficiently rich), then if there are some q for which the model has no solution satisfying the terminal condition, even with arbitrarily large T, then the model will have no solution satisfying the terminal condition with positive probability.

[11] Strictly, this is not fully rational, as it is equivalent to assuming that agents act as if the uncertainty in all future periods would be resolved next period. However, in practice this appears to be a close approximation to full rationality, as demonstrated by Holden (2016). The authors of the original stochastic path method now have a more complicated version that is fully consistent with rationality (Adjemian & Juillard 2016).

one way to achieve this is to augment equation (1) with an exogenous forcing process, so:

$$i_t = \max\{0, f(x_{t-1}, x_t, x_{t+1}) + \nu_t\}$$

where $\nu_t = 0$ for $t > T$, and where the entire path of ν_t is known in period 1. I.e. ν_t acts like news shocks. This is equivalent to a model without such a forcing process but with T more state variables which track the arrival of these shocks (see Appendix E). For this approach to satisfy the condition, M must be full rank so that it can be inverted to find the shocks required to produce the desired q.

In the monetary policy context, such "news shocks" may be justified as reflecting forward guidance. A more general justification for the presence of "news shocks" is that they provide a way of dealing with future uncertainty during stochastic simulation. If we simulate a non-linear stochastic model by repeated perfect-foresight simulation, in each period assuming that there are no future shocks, then we are ignoring future uncertainty. As previously mentioned, one way to allow for future uncertainty is to follow the original stochastic extended path approach of Adjemian & Juillard (2013) by drawing lots of samples of future shocks for periods $1, \dots, T$, calculating the perfect-foresight paths conditional on those future shocks, and then averaging over these realised paths.[12] In a linear model with shocks with unbounded support, providing at least one shock has an impact on i_t for each $t \in \{1, \dots, T\}$, the distribution of future paths of $(i_t)_{t=1}^{\infty}$ will have positive support over the entirety of $\mathbb{R}^T$. This justifies looking for results that hold for any possible q.

Appendix D.3: Discussion of Brendon, Paustian & Yates (2013; 2016)

The most relevant prior work for ours is that of Brendon, Paustian & Yates (2013; 2016), henceforth abbreviated to BPY. Like us, these authors examined perfect foresight equilibria of NK models with terminal conditions. In BPY (2013), the authors show analytically that in a very simple NK model, featuring a response to the growth

[12] See Footnote 11 for caveats to this procedure.

rate in the Taylor rule, there are multiple perfect-foresight equilibria when all agents believe that with probability one, in one period's time, they will escape the bound and return to the neighbourhood of the "good" steady state. Furthermore, in BPY (2013; 2016), the authors show numerically that in some select other models, there are multiple perfect-foresight equilibria when the economy begins at the steady state, and all agents believe that the economy will jump to the bound, remain there for some number of periods, before leaving it endogenously, after which they believe they will never hit the bound again.

Relative to these authors, we provide far more general theoretical results, and these permit numerical analysis that is both more robust and less restrictive. This robustness and generality will prove crucial in showing multiplicity even in simple NK models, with entirely standard Taylor rules. For example, whereas BPY (2016) write that price-dispersion "does not have a strong enough impact on equilibrium allocations for the sort of propagation that we need", we show that the presence of price dispersion is sufficient for multiplicity. Likewise, whereas BPY (2013; 2016) find a much weaker role for multiplicity when the monetary rule does not include a response to the growth rate of output, our findings of multiplicity will not be at all dependent on such a response, implying very different policy prescriptions.

Appendix D.4: Uniqueness and multiplicity

We have presented necessary and sufficient conditions for uniqueness in otherwise linear models with terminal conditions. Some caveats are in order though.

Bodenstein (2010) showed that linearization can exclude equilibria. Additionally, Boneva, Braun & Waki (2016) show that there may be multiple perfect-foresight solutions to a non-linear NK model with ZLB, converging to the non-deflationary steady state, even though the linearized version of their model (with a ZLB) has a unique equilibrium.[13] Thus, the multiplicity we find is strictly in addition to the multiplicity found by those authors. While the theoretical and computational methods used by Boneva, Braun & Waki (2016) have the great advantage that they can cope with fully non-linear models, it appears that they cannot cope with endogenous state variables, which limits their applicability. By producing tools for analysing otherwise linear models including state variables, our tools and results provide a complement to those of Boneva, Braun & Waki (2016). For one piece of evidence of the continued relevance of our results in a non-linear setting, note that the multiplicity found in a simple linearized model in BPY (2013) is also found in the equivalent non-linear model in BPY (2016).

Additionally, the tools of this paper can be used to analyse the properties of perfect-foresight models with nonlinearities other than an occasionally binding constraint. Recall that we showed $i(y) = q + My + \mathrm{O}(y'y)$ as $y'y \to 0$, where M is defined in terms of partial derivatives of the path (see Definition 1), and that we did not need to impose linearity to derive the complementary slackness constraints on y. Thus, in a fully non-linear perfect foresight context, we can still use the tools we develop here to look at the (first order approximate) properties of perfect foresight problems in which y does not become too large in the solution (which usually means that q does not go too negative). In particular, we do not need to linearize before deriving q or M, so we can preserve accuracy even though only large shocks might drive us to the bound. (In this non-linear case, M will be a function of the initial state.) In future work, we hope to examine how the M matrix changes with the state of the economy in a fully non-linear model, potentially allowing us to place the results of Boneva, Braun & Waki (2016) in a broader context.

[13] To see the uniqueness of the linearized version, navigate to the "`Examples/BonevaBraunWaki2016`" folder within DynareOBC and then run "`RunExample`". The "`bbw2016.mod`" file in that folder implements the Boneva, Braun & Waki (2016) model for DynareOBC use.

Furthermore, we maintain that studying multiplicity in otherwise linear models is an independently important exercise. Firstly, macroeconomists have long relied on existence and uniqueness results based on linearization of models without occasionally binding constraints, even though this may produce spurious uniqueness in some circumstances.[14] Secondly, it is nearly impossible to find all perfect foresight solutions in general non-linear models, since this is equivalent to finding all the solutions to a huge system of non-linear equations, when even finding all the solutions to large systems of quadratic equations is computationally intractable. At least if we have the full set of solutions to the otherwise linear model, we may use homotopy continuation methods to map these solutions into solutions of the non-linear model. Furthermore, finding all solutions under uncertainty is at least as difficult in general, as the policy function is also defined by a large system of non-linear equations. Thirdly, Christiano and Eichenbaum (2012) argue that e-learnability considerations render the additional equilibria of Boneva, Braun & Waki (2016) mere "mathematical curiosities", suggesting that the equilibria that exist in the linearized model are of independent interest, whatever one's view on this debate. Finally, our main results for NK models imply non-uniqueness, so concerns of spurious uniqueness under linearization will not be relevant in these cases.

From the preceding discussion, we see that our choice to focus on otherwise-linear models under perfect-foresight, with fixed terminal conditions, has biased our results in favour of uniqueness for three distinct reasons. Firstly, because there are potentially more solutions under rational expectations than under perfect-foresight, as we prove in Appendix I. Secondly, because there are potentially other solutions returning to alternate steady states. Thirdly, because the original fully non-linear model may

[14] Only comparatively recently have all the formal properties of higher order perturbation solutions been established. E.g. Lan & Meyer-Gohde (2014) show that the first order determinacy conditions are sufficient for the existence and uniqueness of the higher-order terms in a standard perturbation, and Lan & Meyer-Gohde (2013) show that a "moving average" type perturbation is stable if and only if the first order determinacy conditions hold. However, any such perturbation solution is only valid within some domain of convergence, so even these results do not mean that first order determinacy implies global determinacy.

possess yet more solutions. This means our results on the multiplicity of solutions to New Keynesian models are all the more surprising, and that it is all the more likely that multiplicity of equilibria is an important factor in explaining actual economies' spells at the ZLB.

Duarte (2016) considers how a central bank might ensure determinacy in a simple continuous time new Keynesian model. Like us, he finds that the Taylor principle is not sufficient in the presence of the ZLB. He shows that determinacy may be produced by using a rule that holds interest rates at zero for a history dependent amount of time, before switching to a $\max\{0, \dots\}$ Taylor rule. While we do not allow for such switches in central bank behaviour, we do find an important role for history dependence, through price targeting.

Hebden, Lindé & Svensson (2011) propose a simple way to find multiplicity: namely, hit the model with a large shock which pushes it towards the bound, and see if one can find more than one set of periods such that being at the bound during those periods is an equilibrium. In practice, this suggests first looking if there is a solution which finally escapes the bound after one period, then looking to see if there is one which finally escapes the bound after two periods, and so on. DynareOBC actually implements this method for finding multiple solutions, as it is easy to constrain the MILP representation of the LCP problem to be at the bound in the final period.[15] Often, this procedure will succeed in finding an example of multiplicity, and thus proving that the original model does not possess a unique solution. However, it cannot work completely generally as the multiplicity may only arise for very particular paths of the bounded variable in the absence of the constraints.

Jones (2015) also presents a uniqueness result for models with occasionally binding constraints. He shows that if one knows the set of periods at which the constraint binds, then under standard assumptions, there is a unique path. However,

[15] This is controlled with the `SkipFirstSolutions=INTEGER` option.

there is no reason there should be a unique set of periods at which the constraint binds, consistent with the model. The multiplicity for models with occasionally binding constraints precisely stems from there being multiple sets of periods at which the model could be at the bound. Our results are not conditional on knowing in advance the periods at which the constraint binds.

Finally, uniqueness results have also been derived in the Markov switching literature, see e.g. Davig & Leeper (2007) and Farmer, Waggoner & Zha (2010; 2011), though the assumed exogeneity of the switching in these papers limits their application to endogenous OBCs such as the ZLB. Determinacy results with endogenous switching were derived by Marx & Barthelemy (2013), but they only apply to forward looking models that are sufficiently close to ones with exogenous switching, and there is no reason e.g. a standard NK model with a ZLB should have this property. Our results do not have this limitation.

Appendix D.5: Existence and non-existence

We also produced conditions for the existence of any perfect-foresight solution to an otherwise linear model with a terminal condition. These results provide new intuition for the prior literature on existence under rational expectations, which has found that NK models with a ZLB might have no solution at all if the variance of shocks is too high. For example, Mendes (2011) derived analytic results on existence as a function of the variance of a demand shock, and Basu & Bundick (2015) showed the potential quantitative relevance of such results. Furthermore, conditions for the existence of an equilibrium in a simple NK model with discretionary monetary policy are derived in close form for a model with a two-state Markov shock by Nakata & Schmidt (2014). They show that the economy must spend a small amount of time in the bad state for the equilibrium to exist, which again links existence to variance.

While our results are not directly related to the variance of shocks, as we work under perfect foresight, they are nonetheless related. We showed that whether a perfect foresight solution exists depends on the perfect-foresight path taken by nominal interest rates in the absence of the bound. Most of our results assumed that this path was arbitrary. However, in a model with a small number of shocks, all of bounded support, and no information about future shocks, clearly not all paths are possible for nominal interest rates in the absence of the bound. The more shocks are added (e.g. news shocks), and the wider their support, the greater will be the support of the space of possible paths for nominal interest rates in the absence of the ZLB, and hence, the more likely will be non-existence of a solution for a positive measure of paths. This helps to explain the literature's prior results.

There has also been some prior work by Richter & Throckmorton (2015) and Gavin et al. (2015; Appendix B) that has related a kind of eductive stability (the convergence of policy function iteration) to other properties of the model. Non-convergence of policy function iteration is suggestive of non-existence, though not definitive evidence. While the procedure of the cited authors has the advantage of working with the fully non-linear model under rational expectations, this limitation means that it cannot directly address the question of existence. By contrast, our results are theoretical and directly address existence. Thus, both procedures should be viewed as complementary; while ours definitively answers the question of existence in the slightly limited world of perfect foresight, otherwise linear models, the Richter & Throckmorton results give answers on stability in a richer setting.

Another approach to establishing the existence of an equilibrium is to produce it to satisfactory accuracy, by solving the model in some way. Under perfect foresight, the method described in Holden (2016) is a possibility, and the method of Guerrieri & Iacoviello (2015) is a prominent alternative. Under rational expectations, policy function iteration methods have been used by Fernández-Villaverde et al. (2015) and Richter & Throckmorton (2015), amongst others. However, this approach cannot establish non-existence or prove uniqueness. As such it is of little use to the policy maker who wants policy guidance to ensure existence and/or uniqueness. Furthermore, if the problem is solved globally, one cannot in general rule out that there is not an area of non-existence outside of the grid on which the model was solved. Similarly, if the model is solved under perfect foresight for a given initial state, then the fact that a solution exists for that initial point gives no guarantees that a solution should exist for other initial points. Thus, there is an essential role for more general results on global existence, as we have produced here.

Appendix D.6: Economic significance of multiplicity at the ZLB

There are two reasons why one might be sceptical about the economic significance of the multiple equilibria caused by the presence of the ZLB that we find. Firstly, as with any non-fundamental equilibrium, the coordination of beliefs needed to sustain the equilibrium may be difficult. Secondly, self-fulfilling jumps to the ZLB may feature implausibly large falls in output and inflation. This reflects the implausibly large response to news about future policy innovations, a problem that has been termed the "forward guidance puzzle" in the literature (Carlstrom, Fuerst & Paustian 2015; Del Negro, Giannoni & Patterson 2015).[16] However, if the economy is already in a recession, then both problems are substantially ameliorated. If interest rates are already low, then it does not seem too great a stretch to suggest that a drop in confidence may lead people to expect to hit the ZLB. Even more plausibly, if the economy is already at the ZLB, then small changes in confidence could easily select an equilibrium featuring a longer spell at the ZLB than in the equilibrium with the shortest time there. Indeed, there is no good reason people should coordinate on the equilibrium with the shortest time at the ZLB. Moreover, with interest rates already low, the size of the required self-fulfilling news shock is much smaller, meaning that

[16] McKay, Nakamura & Steinsson (2016) point out that these implausibly large responses to news are muted in models with heterogeneous agents, and give a simple "discounted Euler" approximation that produces similar results to a full heterogeneous agent model. While including a discounted Euler equation makes it harder to generate multiplicity (e.g. reducing the parameter space with multiplicity in the Brendon, Paustian & Yates (2013) model), when there is multiplicity, the resulting responses are much larger, as the weaker response to news means the required endogenous news shocks need to be much greater in order to drive the model to the bound.

the additional drop in output and inflation caused by a jump to the ZLB will be much more moderate.

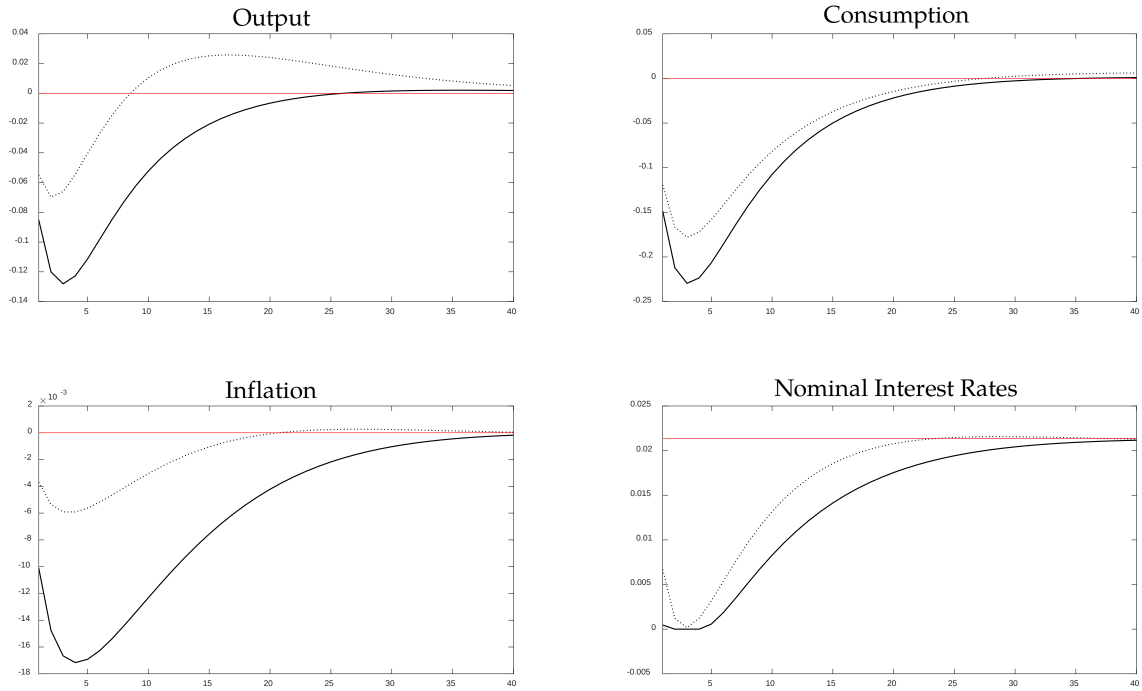

Figure 2: Two solutions following a preference shock in the Smets & Wouters (2003) model.
All variables are in logarithms. The dotted line is a solution which does not hit the bound. The solid line is an alternative solution which does hit the bound.

As an illustration, in Figure 2 we plot the impulse response to a large magnitude preference shock (scaling utility), in the Smets & Wouters (2003) model.[17,18] The shock is not quite large enough to send the economy to the ZLB[19] in the standard solution, shown with a dotted line. However, there is an alternative solution in which the economy jumps to the bound one period after the initial shock, remaining there for three periods. While the alternative solution features larger drops in output and

[17] The shock is 22.5 standard deviations. While this is implausibly large, the economy could be driven to the bound with a run of much smaller shocks. It is also worth recalling that the model was estimated on the great moderation period, and so the estimated standard deviations may be too low. Finally, recent evidence (Cúrdia, Del Negro & Greenwald 2014) suggests that the shocks in DSGE models should be fat tailed, making large shocks more likely.

[18] This figure is one of those that may be generated by running “`GeneratePlots`” within the “`Examples/SmetsWouters2003`” directory of DynareOBC.

[19] Since the Smets & Wouters (2003) model does not include trend growth, it is impossible to produce a steady state value for nominal interest rates that is consistent with both the model and the data. We choose to follow the data, setting the steady state of nominal interest rates to its mean level over the same sample period used by Smets & Wouters (2003), using data from the same source (Fagan, Henry & Mestre 2005).

inflation, the falls are broadly in line with the magnitude of the crisis, with Eurozone GDP and consumption now being about 20% below a pre-crisis log-linear trend, and the largest drop in Eurozone consumption inflation from 2008q3 to 2008q4 being around 1%.[20] Considering this, we view it as plausible that multiplicity of equilibria was a significant component of the explanation for the great recession.

Appendix D.7: Price level targeting

Our results suggest that given belief in an eventual return to inflation, a determinate equilibrium may be produced in standard NK models if the central bank switches to targeting the price level, rather than the inflation rate. As the previous figure made clear, the welfare benefits to this could be substantial.[21] There is of course a large literature advocating price level targeting already. Vestin (2006) made an important early contribution by showing that its history dependence mimics the optimal rule, a conclusion reinforced by Giannoni (2010). Eggertsson & Woodford (2003) showed the particular desirability of price level targeting in the presence of the ZLB, since it produces inflation after the bound is escaped. A later contribution by Nakov (2008) showed that this result survived taking a fully global solution, and Coibion, Gorodnichenko & Wieland (2012) showed that it still holds in a richer model. More recently, Basu & Bundick (2015) have argued that a response to the price level ensures equilibria exists even when shocks have large standard deviations, avoiding the problems stressed by Mendes (2011), while also solving the contractionary bias caused by the ZLB. Our argument is distinct from these; we showed that in the presence of the ZLB, inflation targeting rules are indeterminate, even conditional on an eventual return to inflation, whereas price level targeting rules produce determinacy, in the sense of the existence of a unique perfect-foresight path returning to the inflationary steady state.

[20] Data was again from the area-wide model database (Fagan, Henry & Mestre 2005).

[21] We look more formally at welfare in a model very similar to the Smets & Wouters (2003) model in Appendix F.5.

Our results are also distinct from those of Adão, Correia & Teles (2011) who showed that if the central bank is not constrained to respect the ZLB out of equilibrium (i.e. for non-market-clearing prices),[22] and if the central bank uses a rule that responds to the right hand side of the Euler equation, then a globally unique equilibrium may be produced, even without ruling out explosive beliefs about prices. Their rule has the flavour of a (future) price-targeting rule, due to the presence of future prices in the right-hand side of the Euler equation. Here though, we are assuming that the central bank must satisfy the ZLB even out of equilibrium (i.e. for all prices), which makes it harder to produce uniqueness. Additionally, we do not require that the central bank can choose a knife-edge value for its response to the (future) price-level, or that it knows the precise form of agents' utility functions, both of which are apparently required by the rule of Adão, Correia & Teles and which may be difficult in practice. However, in line with the New Keynesian literature, we maintain the standard assumption that explosive paths for inflation are ruled out, an assumption which the knife-edge rules of Adão, Correia & Teles do not require.[23]

Somewhat contrary to our results, Armenter (2016) shows that in a simple otherwise linear NK model, if the central bank pursues Markov (discretionary) policy subject to an objective targeting inflation, nominal GDP or the price level, then the presence of a ZLB produces additional equilibria quite generally. This difference between our results and those of Armenter (2016) is driven both by the fact that we rule out getting stuck in the neighbourhood of the deflationary steady state by assumption, and since we assume commitment to a rule.

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[22] The precise meaning of this is most clearly expounded in Bassetto (2004). The distinction is between constraints that hold for any prices, such as agent first order conditions, and constraints that hold only for the market clearing prices, such as market clearing conditions. The contention of Bassetto (2004) is that the ZLB is in the latter category—the central bank can promise negative nominal interest rates off the equilibrium path, which will give determinacy under appropriate timing assumptions without negative rates actually being required. (A commitment to negative nominal rates provides an infinite nominal transfer in those states, entirely devaluing nominal wealth, so pushing up prices and preventing negative rates ever being called for.) Bassetto notes however how dangerous it would be to rely on such infinite transfers given the possibility of misspecification.

[23] Note that the unstable solutions under price level targeting feature exponential growth in the logarithm of the price level, which also implies explosions in inflation rates.

Appendix D.8: Checking the existence and uniqueness conditions in practice

The paper has presented many results, but the practical details of what one should test and in what order may still be unclear. Luckily, a lot of the decisions are automated by the author's DynareOBC toolkit, but we present a suggested testing procedure here in any case. This also serves to give an overview of our results and their limitations.

For checking feasibility and existence, the most powerful result is Proposition 2 and Corollary 3. If the lower bound from Proposition 2 is positive, for all sufficiently high T, the LCP is always feasible. If further conditions are satisfied for a given T, (see Proposition 9 and Proposition 10) then this guarantees existence for that particular T. However, since the additional conditions are sufficient and not necessary, in practice it may not be worth checking them, as we have never encountered a problem without a solution that was nonetheless feasible. Finding a T for which Proposition 2 produces a positive lower bound on ς requires a bit of trial and error. T will need to be big enough that the asymptotic approximation is accurate, which usually requires T to be bigger than the time it takes for the model's dynamics to die out. However, if T is too large, then DynareOBC's conservative approach to handling numerical error means that it can be difficult to reject $\underline{\varsigma} = 0$. Usually though, an intermediary value for T can be found at which we can establish $\underline{\varsigma} > 0$, even with a conservative approach to numerical error.

For checking non-existence, Proposition 2 and Corollary 3 can still be useful, though in this case, it does not provide definitive proof of non-feasibility, due to inescapable numerical inaccuracies. For a particular T, we may test if M is not an S-matrix in time polynomial in T by solving a simple linear programming problem. If M is not an S-matrix, then by Proposition 1 and Corollary 2, there are some q for which there is no path which does not violate the bound in the first T periods. With T larger than the time it takes for the model's dynamics to die out, this provides further evidence of non-existence for arbitrarily large T. In any case, given that only having a solution that stays at the bound for 250 years is arguably as bad as having no solution at all, for medium scale models, we suggest to just check if M is an S-matrix with $T = 1000$.

For checking uniqueness vs multiplicity, it is important to remember that while we can prove uniqueness for a given finite T by proving that the M matrix is a P-matrix, once we have found one T for which M is not a P-matrix (so there are multiple solutions, by Theorem 2 and Corollary 1), we know the same is true for all higher T. If we wish to prove that there is a unique solution up to some horizon T, then the best approach is to begin by testing the sufficient conditions from Corollary 5, with our suggested D_1 and D_2. If none of these conditions pass, then it is probable that M is not a P-matrix. In any case, checking that an M which fails the conditions of Corollary 5 is a P-matrix for very large T may not be computationally feasible, though finding a counter-example usually is.

If we wish to establish multiplicity, then Corollary 4 provides a guide. It is trivial to check if M has any nonpositive elements on its diagonal, in which case it cannot be a P-matrix. We can also check whether the expression derived in Appendix L.1 for the limit of the diagonal of M is non-negative, which is a necessary condition for M to be a P-matrix for all large T (this is a special case of Proposition 4). It is also trivial to check the eigenvalue condition given in Corollary 4, and that M is an S-matrix. If none of these checks established that M is not a P-matrix, then a search for a principal sub-matrix with negative determinant is the obvious next step. It is sensible to begin by checking the contiguous principal sub-matrices.[24] These correspond to a single spell at the ZLB which is natural given that impulse responses in DSGE models tend to be single peaked. This is so reliable a diagnostic (and so fast) that DynareOBC reports it automatically for all models. Continuing, one could then check all the 2×2 principal sub-matrices, then the 3×3 ones, and so on. With T around the half-life of the model's dynamics, usually one of these tests will quickly produce the required counter-

[24] Some care must be taken though as checking the signs of determinants of large matrices is numerically unreliable.

example. A similar search strategy can be used to rule out semi-monotonicity, implying multiplicity when away from the bound, by Proposition 5.

Given the computational challenge of verifying whether M is a P-matrix, without Corollary 5, it may be tempting to wonder if our results really enable one to accomplish anything that could not have been accomplished by a naïve brute force approach. For example, it has been suggested that given T and an initial state, one could check for multiple equilibria by considering all of the 2^T possible combinations of periods at which the model could be at the bound and testing if each guess is consistent with the model, following, for example, the solution algorithms of Fair and Taylor (1983) or Guerrieri & Iacoviello (2015). Since there are 2^T principal sub-matrices of M, it might seem likely that this will be computationally very similar to checking if M is a P-matrix. However, our uniqueness results are not conditional on q or the initial state, rather they give conditions under which there is a unique solution for any possible path that the economy would take in the absence of the bound. Thus, while the brute force approach may tell you about uniqueness given an initial state in a reasonable amount of time, using our results, in a comparable amount of time you will learn whether there are multiple solutions for any possible q. A brute force approach to checking for all possible initial conditions would require one to solve a linear programming problem for each pair of possible sets of periods at the bound, of which there are $2^{2T-1} - 2^{T-1}$.[25] This is far more computationally demanding than our approach, and becomes intractable for even very small T. Additionally, our approach is numerically more robust, allows the easy management of the effects of numerical error to avoid false positives and false negatives, and requires less work in each step. Finally, we stress that in most cases, thanks to Corollary 4 and Corollary 5, no such

[25] Given the periods in the constrained regime, the economy's path is linear in the initial state. Excepting knife edge cases of rank deficiency, any multiplicity must involve two paths each at the bound in a different set of periods. Consequently, a brute force approach to finding multiplicity unconditional on the initial state is to guess two different sets of periods at which the economy is at the bound, then solve a linear programming problem to find out if there is a value of the initial state for which the regimes on each path agree with their respective guesses.

search of the sub-matrices of M is required under our approach, and a proof or counter-example may be produced in time polynomial in T, just as it may be when checking for existence with our results.

Appendix E: Formal treatment of our equivalence result

Appendix E.1: Problem set-ups

In the absence of occasionally binding constraints, calculating an impulse response or performing a perfect foresight simulation exercise in a linear DSGE model is equivalent to solving the following problem:[26]

Problem 1 (Linear) Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that:

1) $x_t \to \mu$ as $t \to \infty$,
2) for all $t \in \mathbb{N}^+$:

$$0 = A(x_{t-1} - \mu) + B(x_t - \mu) + C(x_{t+1} - \mu), \tag{3}$$

We make the following assumption throughout the paper and these appendices:

Assumption 1 For any given $x_0 \in \mathbb{R}^n$, Problem 1 (Linear) has a unique solution, which (without loss of generality) takes the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in \mathbb{N}^+$, where $0 = A + BF + CFF$ (so $F = -(B + CF)^{-1}A$), and where the eigenvalues of F are strictly inside the unit circle.

Conditions (A′) and (B) from Sims's (2002) generalisation of the standard Blanchard-Kahn (1980) conditions are necessary and sufficient for Assumption 1 to hold. Further, to avoid dealing specially with the knife-edge case of exact unit eigenvalues in the part of the model that is solved forward, here we rule it out with the subsequent assumption, which is, in any case, a necessary condition for perturbation to produce a consistent approximation to a non-linear model, and which is also necessary for the linear model to have a unique steady state:

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[26] The absence of shocks and expectations here is without loss of generality. For suppose $(\hat{A} + \hat{B} + \hat{C})\hat{\mu} = \hat{A}\hat{x}_{t-1} + \hat{B}\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \widehat{D}\varepsilon_t$, with $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$, and that $\varepsilon_t = 0$ for $t > 1$, as in an impulse response or perfect foresight simulation exercise. Then, if we define $x_t := \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}$, $\mu := \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}$, $A := \begin{bmatrix} \hat{A} & \widehat{D} \\ 0 & 0 \end{bmatrix}$, $B := \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}$, $C := \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$, then we are left with a problem in the form of Problem 1 (Linear), with the extended initial condition $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, and the extended terminal condition $x_t \to \mu$ as $t \to \infty$. Expectations disappear as there is no uncertainty after period 0.

Assumption 2 $\det(A + B + C) \neq 0$.

We are interested in models featuring occasionally binding constraints. We will concentrate on models featuring a single ZLB type constraint in their first equation, which does not bind in steady state, and which we treat as defining the first element of x_t. Generalising from this special case to models with one or more fully general bounds is straightforward and is discussed in Appendix H. First, let us write $x_{1,t}$, $I_{1,\cdot}$, $A_{1,\cdot}$, $B_{1,\cdot}$, $C_{1,\cdot}$ for the first row of x_t, I, A, B, C (respectively) and $x_{-1,t}$, $I_{-1,\cdot}$, $A_{-1,\cdot}$, $B_{-1,\cdot}$, $C_{-1,\cdot}$ for the remainders. Likewise, we write $I_{\cdot,1}$ for the first column of the identity, I, and so on. Then, from adding $x_{1,t}$ to both sides of the first equation within the system (3), then incorporating a max, we produce the system of interest:

Problem 2 (OBC) Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that:

1) $x_t \to \mu$ as $t \to \infty$,
2) for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\},$$
$$0 = A_{-1,\cdot}(x_{t-1} - \mu) + B_{-1,\cdot}(x_t - \mu) + C_{-1,\cdot}(x_{t+1} - \mu),$$

3) $x_{1,t} > 0$ for $t > T$.

given:

Assumption 3 $\mu_1 > 0$, where μ_1 is the first element of μ.

Were it not for the max, this problem would be identical to Problem 1 (Linear), providing that Assumption 3 holds, as the existence of a $T \in \mathbb{N}$ such that $x_{1,t} > 0$ for $t > T$ is guaranteed by the fact that $x_{1,t} \to \mu_1$ as $t \to \infty$.

We will analyse Problem 2 (OBC) with the help of solutions to the auxiliary problem:

Problem 3 (News) Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that:

1) $x_t \to \mu, y_t \to 0$, as $t \to \infty$,
2) for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1}y_{1,t-1},$$
$$y_{T,t} = 0,$$
$$\forall i \in \{1, \dots, T-1\},\ y_{i,t} = y_{i+1,t-1}.$$

This is a version of Problem 1 (Linear) with a forcing process ("news") up to horizon T added to the first equation. We use this representation in which the forcing process enters via an augmented state to make clear that this is also a special case of Problem 1 (Linear). By construction, the value of $y_{i,t}$ gives the shock that in period t is expected to arrive in i periods. (To be clear: the first index of $y_{i,t}$ indexes over the elements of the vector $y_t \in \mathbb{R}^T$; the second index of $y_{i,t}$ indexes over periods.) Hence, as there is no uncertainty, $y_{t,0}$ gives the shock that will hit in period t, i.e. $y_{1,t-1} = y_{2,t-2} = \cdots = y_{t,0}$ for $t \le T$, and $y_{1,t-1} = 0$ for $t > T$. Thus, the first equation of the first block could be rewritten:

$$x_{1,t} = I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0},$$

which is in the form of equation (2) from the main paper.

Appendix E.2: Relationships between the problems

Since $y_{1,t-1} = 0$ for $t > T$, by Assumption 1, $(x_{T+1} - \mu) = F(x_T - \mu)$. Now define $s_{T+1} := 0$. Then with $t = T$, we have that $(x_{t+1} - \mu) = s_{t+1} + F(x_t - \mu)$. Proceeding now by backwards induction on t, note that:

$$0 = A(x_{t-1} - \mu) + B(x_t - \mu) + CF(x_t - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0},$$

so:

$$\begin{aligned}(x_t - \mu) &= -(B + CF)^{-1}[A(x_{t-1} - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0}] \\ &= F(x_{t-1} - \mu) - (B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0}),\end{aligned}$$

i.e., if we define: $s_t := -(B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0})$, then $(x_t - \mu) = s_t + F(x_{t-1} - \mu)$. By induction then, this holds for all $t \in \{1, \dots, T\}$, establishing:[27]

Lemma 2 There is a unique solution to Problem 3 (News) that is linear in x_0 and y_0.

For future reference, let $x_t^{(3,k)}$ be the solution to Problem 3 (News) when $x_0 = \mu$, $y_0 = I_{\cdot,k}$ (i.e. a vector which is all zeros apart from a 1 in position k). Then, by linearity, for arbitrary y_0 the solution to Problem 3 (News) when $x_0 = \mu$ is given by:

$$x_t - \mu = \sum_{k=1}^{T} y_{k,0}\left(x_t^{(3,k)} - \mu\right).$$

Now, let $M \in \mathbb{R}^{T\times T}$ satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t, k \in \{1, \dots, T\}, \tag{4}$$

i.e. M horizontally stacks the (column-vector) relative impulse responses of the first variable to the news shocks, with the first column giving the response to a contemporaneous shock, the second column giving the response to a shock anticipated by one period, and so on. Then, this result implies that for arbitrary x_0 and y_0, the path of the first variable in the solution to Problem 3 (News) is given by:

$$(x_{1,1:T})' = q + My_0, \tag{5}$$

where $q := \left(x_{1,1:T}^{(1)}\right)'$ and where $x_t^{(1)}$ is the unique solution to Problem 1 (Linear), for the given x_0, i.e. q is the path of the first variable in the absence of news shocks or bounds. [28] Since M is not a function of either x_0 or y_0, equation (5) gives a highly convenient representation of the solution to Problem 3 (News).

Now let $x_t^{(2)}$ be a solution to Problem 2 (OBC) given some x_0. Since $x_t^{(2)} \to \mu$ as $t \to \infty$, there exists $T' \in \mathbb{N}$ such that for all $t > T'$, $x_{1,t}^{(2)} > 0$. We assume without loss of generality that $T' \leq T$. We seek to relate the solution to Problem 2 (OBC) with the one to Problem 3 (News) for an appropriate choice of y_0. First, for all $t \in \mathbb{N}^+$, let:

$$\hat{e}_t := -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + \left(B_{1,\cdot} + I_{1,\cdot}\right)\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right],$$

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[27] This representation of the solution to Problem 3 (News) was inspired by that of Anderson (2015).

[28] This representation was also exploited by Holden (2010) and Holden and Paetz (2012).

$$e_t := \begin{cases} \hat{e}_t & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}, \tag{6}$$

i.e. e_t is the shock that would need to hit the first equation for the positivity constraint on $x_{1,t}^{(2)}$ to be enforced. Note that by the definition of Problem 2 (OBC), $e_t \geq 0$ and $x_{1,t}^{(2)} e_t = 0$, for all $t \in \mathbb{N}^+$.

Now, from the definition of Problem 2 (OBC), we also have that for all $t \in \mathbb{N}^+$,

$$0 = A\left(x_{t-1}^{(2)} - \mu\right) + B\left(x_t^{(2)} - \mu\right) + C\left(x_{t+1}^{(2)} - \mu\right) + I_{\cdot,1} e_t.$$

Furthermore, if $t > T$, then $t > T'$, and hence $e_t = 0$. Hence, by Assumption 1, $\left(x_{T+1}^{(2)} - \mu\right) = F\left(x_T^{(2)} - \mu\right)$. Thus, much as before, if we define $\tilde{s}_{T+1} := 0$, then with $t = T$, $\left(x_{t+1}^{(2)} - \mu\right) = \tilde{s}_{t+1} + F\left(x_t^{(2)} - \mu\right)$. Consequently,

$$0 = A\left(x_{t-1}^{(2)} - \mu\right) + B\left(x_t^{(2)} - \mu\right) + CF\left(x_t^{(2)} - \mu\right) + C\tilde{s}_{t+1} + I_{\cdot,1} e_t,$$

so:

$$\left(x_t^{(2)} - \mu\right) = F\left(x_{t-1}^{(2)} - \mu\right) - (B + CF)^{-1}\left(C\tilde{s}_{t+1} + I_{\cdot,1} e_t\right),$$

i.e., if we define: $\tilde{s}_t := -(B + CF)^{-1}\left(C\tilde{s}_{t+1} + I_{\cdot,1} e_t\right)$, then $\left(x_t^{(2)} - \mu\right) = \tilde{s}_t + F\left(x_{t-1}^{(2)} - \mu\right)$. As before, by induction this must hold for all $t \in \{1, \dots, T\}$. By comparing the definitions of s_t and $\tilde{s}_t$, and the laws of motion of x_t under both problems, we then immediately have that if Problem 3 (News) is started with $x_0 = x_0^{(2)}$ and $y_0 = e'_{1:T}$, then $x_t^{(2)}$ solves Problem 3 (News). Conversely, if $x_t^{(2)}$ solves Problem 3 (News) for some y_0, then from the laws of motion of x_t under both problems it must be the case that $\tilde{s}_t = s_t$ for all $t \in \mathbb{N}$, and hence from the definitions of s_t and $\tilde{s}_t$, we have that $y_0 = e'_{1:T}$. This establishes the following result:

Lemma 3 For any solution, $\left(T, x_t^{(2)}\right)$ to Problem 2 (OBC):

1) With $e_{1:T}$ as defined in equation (6), $e_{1:T} \geq 0$, $x_{1,1:T}^{(2)} \geq 0$ and $x_{1,1:T}^{(2)} \circ e_{1:T} = 0$, where $\circ$ denotes the Hadamard (entry-wise) product.
2) $x_t^{(2)}$ is also the unique solution to Problem 3 (News) with $x_0 = x_0^{(2)}$ and $y_0 = e'_{1:T}$.
3) If $x_t^{(2)}$ solves Problem 3 (News) with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e'_{1:T}$.

To use the easy solution to Problem 3 (News) to assist us in solving Problem 2 (OBC) just requires one more result. In particular, we need to show that if $y_0 \in \mathbb{R}^T$ is such that $y_0 \geq 0$, $x^{(3)}_{1,1:T} \circ y_0' = 0$ and $x^{(3)}_{1,t} \geq 0$ for all $t \in \mathbb{N}$, where $x^{(3)}_t$ is the unique solution to Problem 3 (News) when started at x_0, y_0, then $x^{(3)}_t$ must also be a solution to Problem 2 (OBC).

So, suppose that $y_0 \in \mathbb{R}^T$ is such that $y_0 \geq 0$, $x^{(3)}_{1,1:T} \circ y_0' = 0$ and $x^{(3)}_{1,t} \geq 0$ for all $t \in \mathbb{N}$, where $x^{(3)}_t$ is the unique solution to Problem 3 (News) when started at x_0, y_0. We would like to prove that in this case $x^{(3)}_t$ must also be a solution to Problem 2 (OBC). I.e., we must prove that for all $t \in \mathbb{N}^+$:

$$x^{(3)}_{1,t} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x^{(3)}_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x^{(3)}_t - \mu) + C_{1,\cdot}(x^{(3)}_{t+1} - \mu)\}, \tag{7}$$

$$0 = A_{-1,\cdot}(x^{(3)}_{t-1} - \mu) + B_{-1,\cdot}(x^{(3)}_t - \mu) + C_{-1,\cdot}(x^{(3)}_{t+1} - \mu).$$

By the definition of Problem 3 (News), the latter equation must hold with equality. Hence, we just need to prove that equation (7) holds for all $t \in \mathbb{N}^+$. So, let $t \in \mathbb{N}^+$. Now, if $x^{(3)}_{1,t} > 0$, then $y_{t,0} = 0$, by the complementary slackness type condition ($x^{(3)}_{1,1:T} \circ y_0' = 0$). Thus, from the definition of Problem 3 (News):

$$\begin{aligned} x^{(3)}_{1,t} &= I_{1,\cdot}\mu + A_{1,\cdot}(x^{(3)}_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x^{(3)}_t - \mu) + C_{1,\cdot}(x^{(3)}_{t+1} - \mu) \\ &= \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x^{(3)}_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x^{(3)}_t - \mu) + C_{1,\cdot}(x^{(3)}_{t+1} - \mu)\}, \end{aligned}$$

as required. The only remaining case is that $x^{(3)}_{1,t} = 0$ (since $x^{(3)}_{1,t} \geq 0$ for all $t \in \mathbb{N}$, by assumption), which implies that:

$$\begin{aligned} x^{(3)}_{1,t} &= 0 = A_{1,\cdot}(x_{t-1} - \mu) + B_{1,\cdot}(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0} \\ &= I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0}, \end{aligned}$$

by the definition of Problem 3 (News). Thus:

$$I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) = -y_{t,0} \leq 0.$$

Consequently, equation (7) holds in this case too, completing the proof.

Together with Lemma 2, Lemma 3, and our representation of the solution of Problem 3 (News) from equation (5), this completes the proof of the following key theorem:

Theorem 1 (Restated) The following hold:

1) Let $x_t^{(3)}$ be the unique solution to Problem 3 (News) given $T \in \mathbb{N}^+$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$. Then $\left(T, x_t^{(3)}\right)$ is a solution to Problem 2 (OBC) given x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t > T$.
2) Let $\left(T, x_t^{(2)}\right)$ be any solution to Problem 2 (OBC) given x_0. Then there exists a unique $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, and such that $x_t^{(2)}$ is the unique solution to Problem 3 (News) given T, x_0 and y_0.

Appendix F: Example applications to New Keynesian models

In the first subsections here, we examine the simple Brendon, Paustian & Yates (BPY) (2013) model, before going on to consider a variant of it with price targeting, which we show to produce determinacy. In the BPY (2013) model, multiplicity and non-existence stem from a response to growth rates in the Taylor rule. However, we do not want to give the impression that multiplicity and non-existence are only caused by such a response, or that they are only a problem in carefully constructed theoretical examples. Thus, in Appendix F.4, we show that a standard NK model with positive steady state inflation and a ZLB possesses multiple equilibria in some states, and no solutions in others, even with an entirely standard Taylor rule. We also show that here too price level targeting is sufficient to restore determinacy. Finally, we show that these conclusions also carry through to the posterior-modes of the Smets & Wouters (2003; 2007) models.

Appendix F.1: The simple Brendon, Paustian & Yates (BPY) (2013) model

Brendon, Paustian & Yates (2013), henceforth BPY, provide a simple New Keynesian model that we can use to illustrate and better understand these cases. Its equations follow:[29]

$$x_{i,t} = \max\{0,1-\beta+\alpha_{\Delta y}(x_{y,t}-x_{y,t-1})+\alpha_\pi x_{\pi,t}\},$$

$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t}+\beta-1-\mathbb{E}_t x_{\pi,t+1}),$$

$$x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t},$$

where $x_{i,t}$ is the nominal interest rate, $x_{y,t}$ is the deviation of output from steady state, $x_{\pi,t}$ is the deviation of inflation from steady state, and $\beta \in (0,1)$, $\gamma, \sigma, \alpha_{\Delta y} \in (0,\infty)$, $\alpha_\pi \in (1,\infty)$ are parameters. The model's only departure from the textbook three equation NK model is the presence of an output growth rate term in the Taylor rule.

[29] An implementation of this model is contained within DynareOBC in the file "`Examples/BrendonPaustianYates2013/BPYModel.mod`".

This introduces an endogenous state variable in a tractable manner. In Appendix L.3, below, we prove the following:

Proposition 12 The BPY model is in the form of Problem 2 (OBC), and satisfies Assumptions 1, 2 and 3. With $T = 1$, $M < 0$ ($M = 0$) if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$ ($\alpha_{\Delta y} = \sigma\alpha_\pi$).

Hence, by Theorem 1 (Restated) , when all agents believe the bound will be escaped after at most one period, if $\alpha_{\Delta y} < \sigma\alpha_\pi$, the model has a unique solution for all q, i.e. no matter what the nominal interest rate would be that period were no ZLB. If $\alpha_{\Delta y} = \sigma\alpha_\pi$, then the model has a unique solution whenever $q > 0$, infinitely many solutions when $q = 0$, and no solutions leaving the ZLB after one period when $q < 0$. Finally, if $\alpha_{\Delta y} > \sigma\alpha_\pi$ then the model has two solutions when $q > 0$, one solution when $q = 0$ and no solution escaping the ZLB next period when $q < 0$.

The mechanism here is as follows. The stronger the response to the growth rate, the more persistent is output, as the monetary rule implies additional stimulus if output was high last period. Suppose then that there was an unexpected positive shock to nominal interest rates. Then, due to the persistence, this would lower not just output and inflation today, but also output and inflation next period. With low expected inflation, real interest rates are high, giving consumers an additional reason to save, and thus further lowering output and inflation this period and next. With sufficiently high $\alpha_{\Delta y}$, this additional amplification is so strong that nominal interest rates fall this period, despite the positive shock, explaining why M may be negative.[30] Now, consider varying the magnitude of the original shock. For a sufficiently large

[30] Note that this cannot happen in the canonical 3 equation NK model in which the central bank responds to the output gap, not output growth. For, without state variables, in the period after the shock's arrival, inflation will be at steady state. Thus, in the period of the shock, real interest rates move one for one with nominal interest rates. Were the positive shock to the nominal interest rate to produce a fall in its level, then the Euler equation would imply high consumption today, also implying high inflation today via the Phillips curve. But, with consumption, inflation, and the shock all positive, the nominal interest rate must be above steady state, contradicting our assumption that it had fallen.

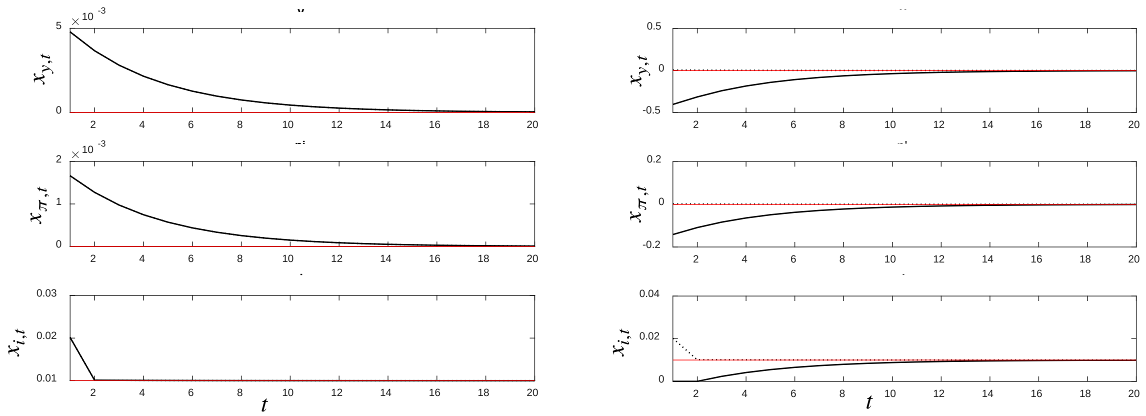
shock, interest rates would hit zero. At this point, there is no observable evidence that a shock has arrived at all, since the ZLB implies that given the values of output and inflation, nominal interest rates should be zero even without a shock. Such a jump to the ZLB must then be a self-fulfilling prophecy. Agents expect low inflation, so they save, which, thanks to the monetary rule, implies low output tomorrow, rationalising the expectations of low inflation.

We finish this section with an example of multiplicity in the BPY (2013) model. This serves to illustrate the potential economic consequences of multiplicity in NK models. We present impulse responses to a shock to the Euler equation under two different solutions. With the shock added to the Euler equation, it now takes the form:

$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1} - (0.01)\varepsilon_t).$$

The rest of the BPY model's equations remain as they were given above. We take the parameterisation $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, following BPY, and we additionally set $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.6$, to ensure we are in the region with multiple solutions. In Figure 3, we show two alternative solutions to the impulse response to a magnitude 1 shock to ε_t. The solid line in the left plot gives the solution which minimises $\|y\|_\infty$. This solution never hits the bound, and is moderately expansionary. The solid line in the right plot gives the solution which minimises $\|q + My\|_\infty$. (The dotted line there repeats the left plot, for comparison.) This solution stays at the bound for two periods, and is strongly contractionary, with a magnitude around 100 times larger than the other solution.[31]

[31] The plots in Figure 3 may be generated by navigating to the "`Examples/BrendonPaustianYates2013`" folder within DynareOBC, and then running "`GeneratePlots`".



Minimum $\|y\|_\infty$ solution **Minimum $\|q + My\|_\infty$ solution**

Figure 3: Alternative solutions following a magnitude 1 impulse to ε_t in the BPY model.

Appendix F.2: The BPY model with shadow interest rate persistence

We showed that if $\alpha_{\Delta y} > \sigma\alpha_\pi$ in the BPY (2013) model, then with $T = 1$, $M < 0$. When $T > 1$, this implies that M is neither P_0, general positive semi-definite, semi-monotone, co-positive, nor sufficient, since the top-left 1×1 principal sub-matrix of M is the same as when $T = 1$. Thus, if anything, when $T > 1$, the parameter region in which there are multiple solutions (when away from the bound or at it) is larger. However, numerical experiments suggest that this parameter region in fact remains the same as T increases, which is unsurprising given the weak persistence of this model. Thus, if we want more interesting results with higher T, we need to consider a model with a stronger persistence mechanism.

One obvious possibility is to consider models with either persistence in the interest rate, or persistence in the "shadow" rate that would hold were it not for the ZLB. Following BPY (2013), we introduce persistence in the shadow interest rate by replacing the previous Taylor rule with $x_{i,t} = \max\{0, x_{d,t}\}$, where $x_{d,t}$, the shadow nominal interest rate is given by: [32]

$$x_{d,t} = (1-\rho)\big(1 - \beta + \alpha_{\Delta y}(x_{y,t} - x_{y,t-1}) + \alpha_\pi x_{\pi,t}\big) + \rho x_{d,t-1}.$$

[32] An implementation of this model is contained within DynareOBC in the file "`Examples/BrendonPaustianYates2013/BPYModelPersistent.mod`".

It is easy to verify that this may be put in the form of Problem 2 (OBC), and that with $\beta \in (0,1)$, $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$, $\alpha_\pi \in (1, \infty)$, $\rho \in (-1,1)$, Assumption 2 is satisfied. For our numerical exercise, we again set $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, $\rho = 0.5$, following BPY.

In Figure 4, we plot the regions in $(\alpha_{\Delta y}, \alpha_\pi)$ space in which M is a P-matrix (P_0-matrix) when $T = 2$ or $T = 4$. In the smaller T case, the P-matrix region is much larger. This relationship appears to continue to hold for both larger and smaller T, with the equivalent $T = 1$ plot being almost entirely shaded, and the large T plot tending to the equivalent plot from the model without monetary policy persistence. Intuitively, the persistence in the shadow nominal interest rate dampens the immediate response of nominal interest rates to inflation and output growth, making it harder to induce a ZLB episode over short-horizons.

Further evidence that the long-horizon behaviour is the same as in the model without persistence is provided by the fact that with $T = 20$, $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.05$, [33] M is a P-matrix. Moreover, from Proposition 2 with $T = 50$, we have that $\varsigma > 6.385 \times 10^{-8}$, so the model always has a feasible path, in the sense of Definition 5 (Feasibility), by Corollary 3.[34]

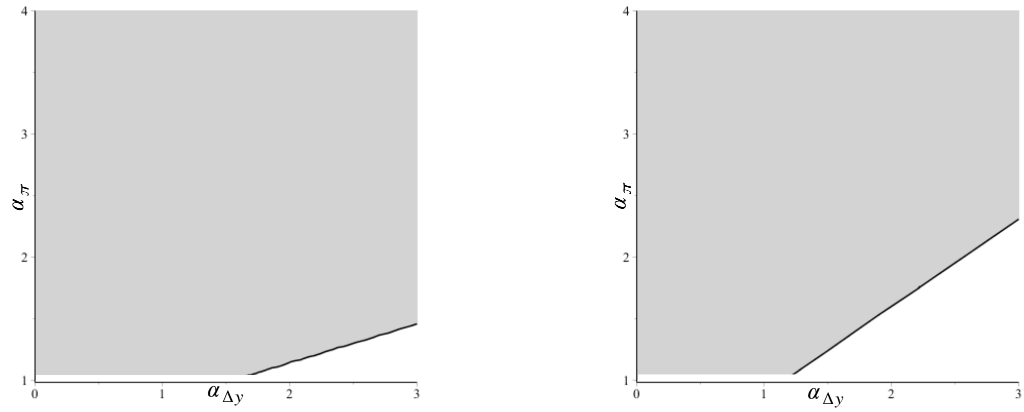
On the other hand, with $T = 200$, $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.51$, then M is not an S-matrix,[35] meaning that for all sufficiently large T, M is not a P-matrix, so there are sometimes multiple solutions. Additionally, from Proposition 2 with $T = 200$, $\varsigma \leq 0 +$ numerical error, meaning that it is likely that the model does not have a solution for all possible paths of $x_{i,t}$.[36]

[33] Results for larger $\alpha_{\Delta y}$ were impossible due to numerical errors.

[34] This result is one of those produced by the "`GenerateDeterminacyResults`" script within the "`Examples/BrendonPaustianYates2013`" folder of DynareOBC.

[35] This was verified a second way by checking that $-M'$ was an S_0-matrix, as discussed in footnote 6.

[36] These results are also among those produced by the "`GenerateDeterminacyResults`" script within the "`Examples/BrendonPaustianYates2013`" folder of DynareOBC.



$T = 2$ **$T = 4$**

Figure 4: Regions in which M is a P-matrix (shaded grey) or a P₀-matrix (shaded grey, plus the black line), when $T = 2$ (left) or $T = 4$ (right).

Appendix F.3: The BPY model with price level targeting

We may also introduce persistence in shadow interest rates by setting:

$$x_{d,t} = (1 - \rho)(1 - \beta) + \left(\alpha_{\Delta y}\left(x_{y,t} - x_{y,t-1}\right) + \alpha_\pi x_{\pi,t}\right) + \rho x_{d,t-1},$$

where $x_{i,t} = \max\{0, x_{d,t}\}$. If the second bracketed term was multiplied by $(1 - \rho)$, then this would be entirely standard, however as written here, in the limit as $\rho \to 1$, this tends to:

$$x_{d,t} = 1 - \beta + \alpha_{\Delta y} x_{y,t} + \alpha_\pi x_{p,t}$$

where $x_{p,t}$ is the price level, so $x_{\pi,t} = x_{p,t} - x_{p,t-1}$. This is a level targeting rule, with nominal GDP targeting as a special case with $\alpha_{\Delta y} = \alpha_\pi$. Note that the omission of the $(1 - \rho)$ coefficient on $\alpha_{\Delta y}$ and α_π is akin to having a "true" response to output growth of $\frac{\alpha_{\Delta y}}{1-\rho}$ and a "true" response to inflation of $\frac{\alpha_\pi}{1-\rho}$, so in the limit as $\rho \to 1$, we effectively have an infinitely strong response to these quantities. It turns out that this is sufficient to produce determinacy for all $\alpha_{\Delta y}, \alpha_\pi \in (0, \infty)$. In particular, given the model: [37]

$$x_{i,t} = \max\{0{,}1 - \beta + \alpha_{\Delta y} x_{y,t} + \alpha_\pi x_{p,t}\},$$

$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{p,t+1} + x_{p,t}\right),$$

$$x_{p,t} - x_{p,t-1} = \beta \mathbb{E}_t x_{p,t+1} - \beta x_{p,t} + \gamma x_{y,t},$$

we prove in Appendix L.4, below, that the following proposition holds:

[37] An implementation of this model is contained within DynareOBC in the file "`Examples/BrendonPaustianYates2013/BPYModelPriceLevelTargeting.mod`".

Proposition 13 The BPY model with price targeting is in the form of Problem 2 (OBC), and satisfies Assumptions 1, 2 and 3. With $T = 1$, $M > 0$ for all $\alpha_\pi \in (0, \infty)$, $\alpha_{\Delta y} \in [0, \infty)$.

Furthermore, with $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2 + \sigma)$, as before, and $\alpha_{\Delta y} = 1$, $\alpha_\pi = 1$, if we check our lower bound on ς with $T = 20$, we find that $\varsigma > 0.042$. Hence, this model always has a feasible path, in the sense of Definition 5 (Feasibility). Given that $d_0 > 0$ for this model, and that for $T = 1000$, M is a P-matrix by our sufficient conditions from Corollary 5, this is strongly suggestive of the existence of a unique solution for any q and for arbitrarily large T.[38]

Appendix F.4: The linearized Fernández-Villaverde et al. (2015) model

The discussion of the BPY (2013) model might lead one to believe that multiplicity and non-existence is solely a consequence of overly aggressive monetary responses to output growth, and overly weak monetary responses to inflation. However, it turns out that basic NK models without indexation to a positive steady-state inflation rate by non-optimising firms (and hence price dispersion in the steady state), still imply multiple equilibria in some states of the world (i.e. for some q) and no solutions in others, even with extremely aggressive monetary responses to inflation and without any monetary response to output growth. Price level targeting again fixes these problems though.

We show these results in the Fernández-Villaverde et al. (2015) model, which is a basic non-linear New Keynesian model without capital or price indexation of non-resetting firms, but featuring (non-valued) government spending and steady-state inflation (and hence price-dispersion).[39] The model's equilibrium conditions follow:

[38] These results are also among those produced by the "`GenerateDeterminacyResults`" script within the "`Examples/BrendonPaustianYates2013`" folder of DynareOBC.

[39] An implementation of this model is contained within DynareOBC in the file "`Examples/FernandezVillaverdeEtAl2015/NK.mod`".

$$\frac{1}{C_t} = R_t \mathbb{E}_t \left[\frac{\beta_{t+1}}{\Pi_{t+1} C_{t+1}} \right] \qquad (*)$$

$$\psi L_t^{\vartheta} C_t = W_t$$

$$\varepsilon X_{1,t} = (\varepsilon - 1) X_{2,t}$$

$$X_{1,t} = \frac{Y_t}{C_t} \frac{W_t}{A_t} + \theta \mathbb{E}_t \beta_{t+1} \Pi_{t+1}^{\varepsilon} X_{1,t+1} \qquad (*)$$

$$X_{2,t} = \Pi_t^* \left(\frac{Y_t}{C_t} + \theta \mathbb{E}_t \beta_{t+1} \frac{\Pi_{t+1}^{\varepsilon-1}}{\Pi_{t+1}^*} X_{2,t+1} \right) \qquad (*)$$

$$\log R_t = \max \left\{ 0, \log R + \phi_\pi \log \left(\frac{\Pi_t}{\Pi} \right) + \phi_y \log \left(\frac{Y_t}{Y} \right) + \sigma_m \varepsilon_{m,t} \right\}$$

$$G_t = S_t Y_t$$

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta) \Pi_t^{*\,1-\varepsilon}$$

$$\nu_t = \theta \Pi_t^{\varepsilon} \nu_{t-1} + (1-\theta) \Pi_t^{*\,-\varepsilon} \qquad (*)$$

$$C_t + G_t = Y_t = \frac{A_t}{\nu_t} L_t$$

$$\log \beta_t = \left(1 - \rho_\beta\right) \log \beta + \rho_\beta \log \beta_{t-1} + \sigma_\beta \varepsilon_{\beta,t}$$

$$\log A_t = (1 - \rho_A) \log A + \rho_A \log A_{t-1} + \sigma_A \varepsilon_{A,t}$$

$$\log S_t = (1 - \rho_S) \log S + \rho_S \log S_{t-1} + \sigma_S \varepsilon_{S,t}$$

Welfare in the model in period t is given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=0}^{s} \beta_{t+s} \right] \left[\log C_{t+s} - \frac{\psi}{1+\vartheta} L_t^{1+\vartheta} \right].$$

After substitutions, the model can be reduced to just the four non-linear equations marked with $(*)$ above (plus the three shock laws of motion) which are functions of gross inflation, Π_t, labour supply, L_t, price dispersion, ν_t, and an auxiliary variable introduced from the firms' price-setting first order condition, $X_{1,t}$, (plus the shocks). Of these variables, only price dispersion enters with a lag. We linearize the model around its steady state, and then reintroduce the "max" operator which linearization removed from the Taylor rule. [40] All parameters are set to the values given in

[40] Before linearization, we transform the model's variables so that the transformed variables take values on the entire real line. I.e. we work with the logarithms of labour supply, price dispersion and the auxiliary variable. For inflation, we note that inflation is always less than $\theta^{\frac{1}{1-\varepsilon}}$. Thus, we work with a logit transformation of inflation over $\theta^{\frac{1}{1-\varepsilon}}$.

Fernández-Villaverde et al. (2015). There is no response to output growth in the Taylor rule, so any multiplicity cannot be a consequence of the mechanism highlighted by BPY (2013).

For this model, numerical calculations reveal that with $T \le 14$, M is a P-matrix. However, with $T \ge 15$, M is not a P matrix, and thus there are certainly some states of the world (some q) in which the model has multiple solutions. Furthermore, with $T = 1000$, our upper bound on ς from Proposition 2 implies that $\varsigma \le 0 +$ numerical error, suggesting that the model does not have a solution for all possible paths of interest rates.[41]

To make the mechanism behind these results clear, we will compare the Fernández-Villaverde et al. (2015) model to an altered version of it with full indexation to steady-state inflation of prices that are not set optimally. To a first order approximation, the model with full indexation never has any price dispersion, and thus has no endogenous state variables. It is thus a purely forwards looking model, and so it is perhaps unsurprising that it should have a unique equilibrium given a terminal condition, even in the presence of the ZLB.

[41] These results are among those that may be generated by running "`GenerateDeterminacyResults`" within the "`Examples/FernandezVillaverdeEtAl2015`" directory of DynareOBC.

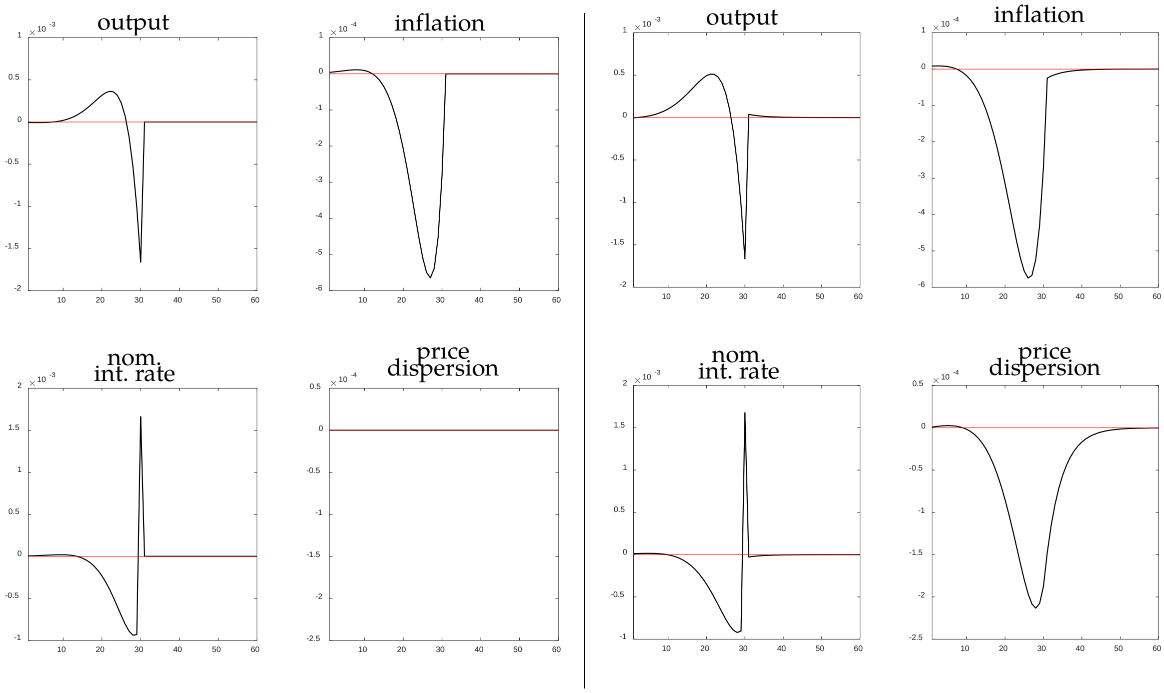


Figure 5: Impulse responses to a shock announced in period 1, but hitting in period 30, in basic New Keynesian models with (left 4 panels) and without (right 4 panels) indexation to steady-state inflation. All variables are in logarithms. In both cases, the model and parameters are taken from Fernández-Villaverde et al. (2015), the only change being the addition of complete price indexation to steady-state inflation for non-updating firms in the left hand plots.

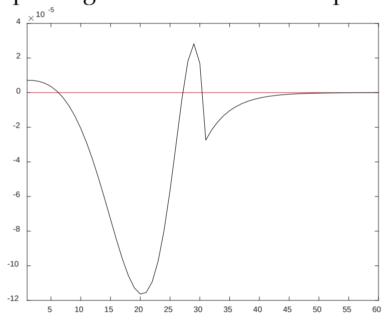


Figure 6: Difference between the IRFs of nominal interest rates from the two models shown in Figure 5. Negative values imply that nominal interest rates are lower in the model without indexation.

In Figure 5 we plot the impulse responses of first order approximations to both models to a shock to nominal interest rates that is announced in period one but that does not hit until period thirty.[42] For both models, the shape is similar, however, in the model without indexation, the presence of price dispersion reduces inflation both before and after the shock hits. This is because the predicted fall in inflation compresses the price distribution, reducing dispersion, and thus reducing the number of firms making large adjustments. The fall in price dispersion also increases output,

[42] This figure is one of those that may be generated by running "`GeneratePlots`" within the "`Examples/FernandezVillaverdeEtAl2015`" directory of DynareOBC.

due to lower efficiency losses from miss-pricing. However, the effect on interest rates is dominated by the negative inflation effect, as the Taylor-rule coefficient on output cannot be too high if there is to be determinacy.[43] For reference, the difference between the IRFs of nominal interest rates in each model is plotted in Figure 6, making clear that interest rates are on average lower following the shock in the model without indexation.

Remarkably, this small difference in the impulse responses between models is enough that the linearized model without indexation has multiple equilibria given a ZLB, but the linearized model with full indexation is determinate. This illustrates just how fragile is the uniqueness in the linearized purely forward-looking model. Informally, what is needed for multiplicity is that the impulse responses to positive news shocks to interest rates are sufficiently negative for a sufficiently high amount of time that a linear combination of them could be negative in every period in which a shock arrives. Here, price dispersion is providing the required additional reduction to nominal interest rates following a news shock.

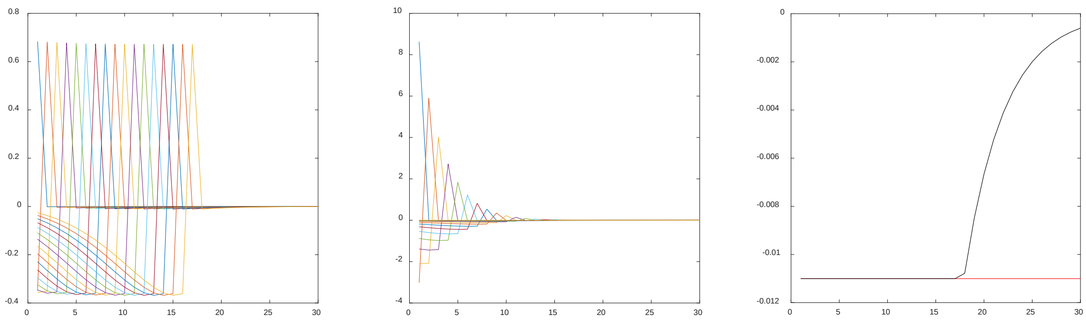


Figure 7: Construction of multiple equilibria in the Fernández-Villaverde et al. (2015) model.
The left plot shows the IRFs to news shocks arriving zero to sixteen quarters after becoming known. The middle plot shows the same IRFs scaled appropriately. The right plot shows the sum of the scaled IRFs shown in the central figure, where the red line gives the ZLB's location, relative to steady state.

We illustrate how multiplicity emerges in the model without indexation by showing, in Figure 7, the construction of an additional equilibrium which jumps to the

[43] One might think the situation would be different if the response to output was high enough that the rise in output after the shock produced a rise in interest rates. However, as observed by Ascari and Ropele (2009), the determinacy region is smaller in the presence of price dispersion than would be suggested by the Taylor criterion. Numerical experiments suggest that in all the determinate region, interest rates are below steady state following the shock.

ZLB for seventeen quarters.[44] If the economy is to be at the bound for seventeen quarters, then for those seventeen quarters, the nominal interest rate must be higher than it would be according to the Taylor rule, meaning that we need to consider seventeen endogenous news shocks, at horizons from zero to sixteen quarters into the future. The impulse responses to unit shocks of this kind are shown in the leftmost plot. Each impulse response has broadly the same shape as the one shown for nominal interest rates in the right of Figure 5. The central figure plots the same impulse responses again, but now each line is scaled by a constant so that their sum gives the line shown in black in the rightmost plot. In this rightmost plot, the red line gives the ZLB's location, relative to steady state, thus the combined impulse response spends seventeen quarters at the ZLB before returning to steady state. Since there are only "news shocks" in the periods in which the economy is at the ZLB, this gives a perfect foresight rational expectations equilibrium which makes a self-fulfilling jump to the ZLB.

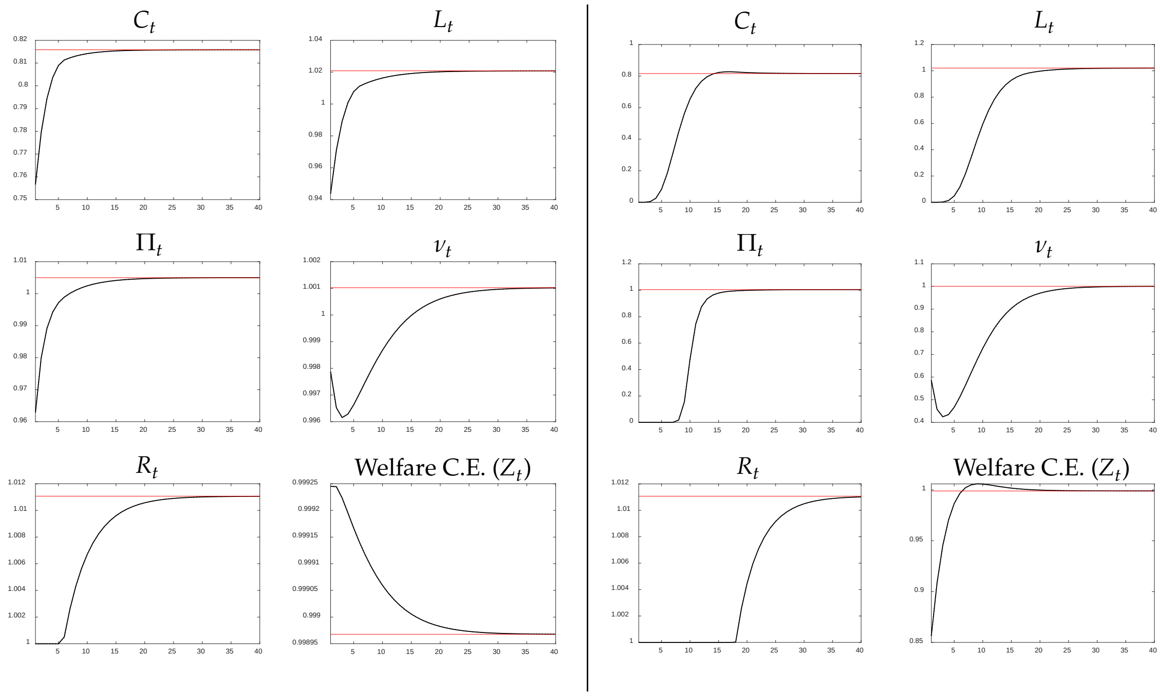


Figure 8: A "good" solution (left 6 panels) and a "bad" solution (right 6 panels), following a 10 standard deviation demand shock in the Fernández-Villaverde et al. (2015) model.
All variables are in levels. The calculation of the welfare consumption equivalent is detailed in the text.

[44] Seventeen quarters was the minimum span for which an equilibrium of this form could be found.

Figure 8 illustrates the potential consequences of this multiplicity.[45] It shows two solutions following a 10 standard deviation demand shock (i.e. a positive shock to β_t). For purely illustrative purposes, we also include a consumption equivalent measure of welfare. This is the quantity Z_t which solves:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=0}^{s} \beta_{t+s} \right] \left[\log C_{t+s} - \frac{\psi}{1+\vartheta} L_t^{1+\vartheta} \right]$$
$$= \mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=0}^{s} \beta_{t+s} \right] \left[\log(\tilde{C}_{t+s} Z_t) - \frac{\psi}{1+\vartheta} \tilde{L}_t^{1+\vartheta} \right],$$

where $\tilde{C}_t$ and $\tilde{L}_t$ are the values consumption and labour supply would take were prices flexible. Z_t will be less than one in steady-state due to the distortion of price-dispersion. However, these welfare calculations come with two caveats. Firstly, all our calculations here are under perfect foresight, so our welfare measure is not capturing any of the effects of uncertainty. Secondly, our welfare measure is based on an underlying first order approximation, which is likely to be unreliable given such big shocks. To mitigate this, we calculate welfare and other variables in a way which introduces no further error beyond the approximation error coming from the four endogenous variables, inflation, labour supply, price dispersion and the firms' auxiliary variable. Thus, all equations except the four marked with $(*)$ will hold exactly, e.g. it will always be true that $C_t + G_t = Y_t = \frac{A_t}{\nu_t} L_t$ ensuring that consumption levels are feasible given labour supply and price dispersion. Despite this, approximation error is likely to be substantial. With these caveats in mind, we see that while welfare actually improves in the "fundamental" solution (due to the reduction in price dispersion), in the second solution consumption equivalent welfare falls by about 12%.

[45] This figure is one of those that may be generated by running "`GeneratePlots`" within the "`Examples/FernandezVillaverdeEtAl2015`" directory of DynareOBC.

The situation is quite different under price level targeting. In particular, if we replace inflation in the monetary rule with the price level relative to its linear trend, which evolves according to:

$$\log P_t = \log P_{t-1} + \log\left(\frac{\Pi_t}{\Pi}\right), \tag{8}$$

then with $T = 200$, the lower bound from Proposition 2 implies that $\varsigma > 0.003$, and hence that the model is always feasible, in the sense of Definition 5 (Feasibility). Furthermore, even with $T = 1000$, M is a P-matrix by our sufficient conditions from Corollary 5. [46] This is strongly suggestive of uniqueness even for arbitrarily large T, given the reasonably short-lived dynamics of the model.

Appendix F.5: The Smets & Wouters (2003; 2007) models

Smets & Wouters (2003) [47] and Smets & Wouters (2007) [48] are the canonical medium-scale linear DSGE models, featuring assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation and general monetary policy reaction functions. The former model is estimated on Euro area data, while the latter is estimated on US data. The latter model also contains trend growth (permitting its estimation on non-detrended data), and a slightly more general aggregator across industries. However, overall, they are quite similar models, and any differences in their behaviour chiefly stems from differences in the estimated parameters. Since both models are incredibly well known in the literature, we omit their equations here, referring the reader to the original papers for further details.

To assess the likelihood of multiple equilibria at or away from the zero lower bound, we augment each model with a ZLB on nominal interest rates, and evaluate

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[46] These results are among those that may be generated by running "`GenerateDeterminacyResults`" within the "`Examples/FernandezVillaverdeEtAl2015`" directory of DynareOBC.

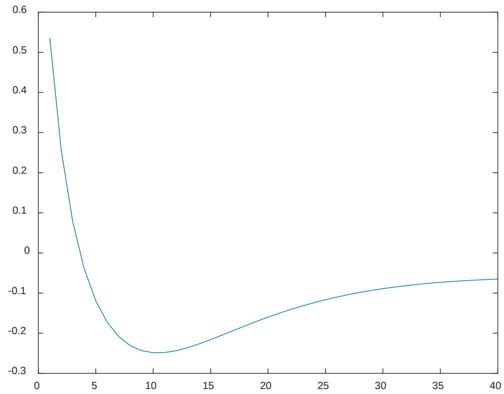
[47] An implementation of this model is contained within DynareOBC in the file "`Examples/SmetsWouters2003/SW03.mod`". This MOD file was derived from the Macro Model Database (Wieland et al. 2012).

[48] An implementation of this model is contained within DynareOBC in the file "`Examples/SmetsWouters2007/SW07.mod`". This MOD file was derived from files provided by Johannes Pfeifer here: http://goo.gl/CP53x5.

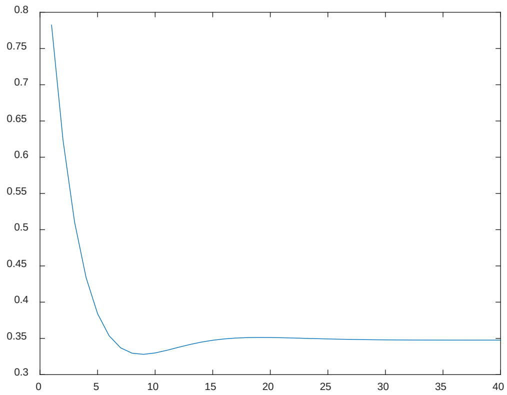
the properties of each model's M matrix at the estimated posterior-modes from the original papers. To minimise the deviation from the original papers, we do not introduce an auxiliary variable for shadow nominal interest rates, so the monetary rules take the form of $i_t = \max\{0, \rho_i i_{t-1} + (1 - \rho_i)(\cdots) + \cdots\}$, in both cases. Our results would be essentially identical with a shadow nominal interest rate though.

If the diagonal of the M matrix ever goes negative, then the M matrix cannot be semi-monotone, or P_0, and hence the model will sometimes have multiple solutions even when away from the zero lower bound (i.e. for some positive q), by Proposition 5. In Figure 9, we plot the diagonal of the M matrix for each model in turn, i.e. the impact on nominal interest rates in period t of news in period 1 that a positive, magnitude one shock will hit nominal interest rates in period t.[49] Immediately, we see that while in the US model, these impacts remain positive at all horizons, in the Euro area model, these impacts turn negative after just a few periods, and remain so at least up to period 40. Therefore, in the ZLB augmented Smets & Wouters (2003) model, there is not always a unique equilibrium. Furthermore, if a run of future shocks was drawn from a distribution with unbounded support, then the value of these shocks was revealed to the model's agents (as in the stochastic extended path), then there would be a positive probability that the model without the ZLB would always feature positive interest rates, but that the model with the ZLB could hit zero.

[49] This figure is one of those that may be generated by running "`GeneratePlots`" within both the "`Examples/SmetsWouters2003`" and "`Examples/SmetsWouters2007`" directories of DynareOBC.



The Smets & Wouters (2003) model



The Smets & Wouters (2007) model

Figure 9: The diagonals of the M matrices for the Smets & Wouters (2003; 2007) models

It remains for us to assess whether M is a $\mathrm{P}_{(0)}$-matrix or (strictly) semi-monotone for the Smets & Wouters (2007) model. Numerical calculations reveal that for $T < 9$, M is a P-matrix, and hence is strictly semi-monotone. However, with $T \geq 9$, the top-left 9×9 sub-matrix of M has negative determinant and is not an S or $\mathrm{S}_{(0)}$ matrix. Thus, for $T \geq 9$, M is not a $\mathrm{P}_{(0)}$-matrix or (strictly) semi-monotone, and hence this model also has multiple equilibria, even when away from the bound. While placing a larger coefficient on inflation in the Taylor rule can make the Euro area picture more like the US one, with a positive diagonal to the M matrix, even with incredibly large coefficients, M remains a non-P-matrix for both models. This is driven by the fact that both the real and nominal rigidities in the model help reduce the average value of the impulse response to a positive news shock to the monetary rule. Following such a shock's arrival, they help ensure that the fall in output is persistent. Prior to its arrival, consumption habits and capital or investment adjustment costs help produce a larger anticipatory recession. Hence, in both the Euro area and the US, we ought to take seriously the possibility that the existence of the ZLB produces non-uniqueness.

As an example of such non-uniqueness, in Figure 10 we plot two different solutions following the most likely combination of shocks to the Smets & Wouters (2007) model that would produce negative interest rates for a year in the absence of a

ZLB.[50,51] In both cases, the dotted line shows the response in the absence of the ZLB. Particularly notable is the flip in sign, since the shocks most likely to take the model to the ZLB for a year are expansionary ones reducing prices (i.e. positive productivity and negative mark-up shocks). Appendix D.6 shows an example of multiplicity in the Smets & Wouters (2003) model, and discusses the economic relevance of such multiplicity.

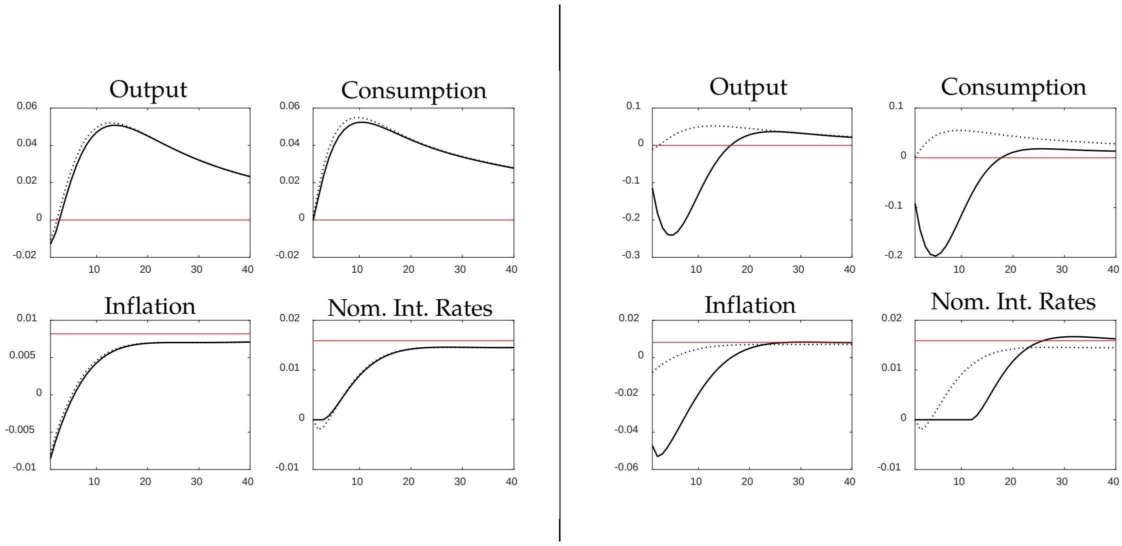


Figure 10: A "good" solution (left 4 panels) and a "bad" solution (right 4 panels), following a combination of shocks to the Smets & Wouters (2007) model

All variables are in logarithms. The precise combination of shocks is detailed in Footnote 50. In all cases, the dotted lines show the path the economy would have followed were it not for the ZLB.

In addition, it turns out that for neither model is M an S-matrix even with $T = 1000$, and thus for both models there are some $q \in \mathbb{R}^{1000}$ for which no solution exists. This is strongly suggestive of non-existence for some q even for arbitrarily large T. This is reinforced by the fact that for the Smets & Wouters (2007) model, with $T = 1000$, Proposition 2 gives that $\varsigma \leq 0 +$ numerical error.[52]

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[50] We find the vector w that minimises $w'w$ subject to $\bar{r} + Zw \leq 0$, where $\bar{r}$ is the steady state interest rate, and columns of Z give four periods of the impulse response of interest rates to the given shocks. This gives the following shock magnitudes: productivity, 3.56 s.d.; risk premium, -2.70 s.d.; government, -1.63 s.d.; investment, -4.43 s.d.; monetary, -2.81 s.d.; price mark-up, -3.19 s.d.; wage mark-up, -4.14 s.d..

[51] This figure is one of those that may be generated by running "`GeneratePlots`" within the "`Examples/SmetsWouters2007`" directory of DynareOBC.

[52] These results are among those that may be generated by running "`GenerateDeterminacyResults`" within both the "`Examples/SmetsWouters2003`" and "`Examples/SmetsWouters2007`" directories of DynareOBC.

With a response to the price level, the picture is very different. Consider first the Smets & Wouters (2003) model, and suppose we add a response to the price level to the monetary rule, so it becomes:

$$i_t = \max\{0, \rho_i i_{t-1} + (1-\rho_i)\log(P_t) + \text{other terms from the original model}\}, \qquad (9)$$

where ρ_i is as in the original model, and where the price level P_t again evolves per equation (8). Then, with $T = 1000$, for the Euro area model we have that $\varsigma > 0.0001$, so Corollary 3 implies that the model is always feasible, in the sense of Definition 5 (Feasibility).[53]

For the Smets & Wouters (2007) model we can get even stronger results if we use a simpler rule that just responds to nominal GDP, i.e.:

$$i_t = \max\{0, \rho_i i_{t-1} + (1-\rho_i)\log(P_t Y_t)\},$$

where again ρ_i is as in the original model, P_t is as before and Y_t is real GDP. Then, with $T = 1000$, our sufficient conditions from Corollary 5 imply that M is a P-matrix. Hence, the model with a nominal GDP response always has a unique solution conditional on escaping after at most 250 years. Furthermore, from Proposition 2, with $T = 1000$, we have that $\varsigma > 0.0063$, so Corollary 3 implies that the model is always feasible, in the sense of Definition 5 (Feasibility).[54] As one would expect, this result is also robust to departures from equal, unit, coefficients. Thus, price level targeting again appears to be sufficient for determinacy in the presence of the ZLB.

We can get a sense of the potential welfare benefits of a switch to price level targeting by comparing equilibria with and without a response to the price level in a closely related model, that of Adjemian, Darracq Pariès & Moyen (2007).[55] This is essentially a re-estimated version of the Smets & Wouters (2003) model.[56] It is convenient for our purposes though because whereas the original Smets & Wouters

[53] This result are among those that may be generated by running "`GenerateDeterminacyResults`" within the "`Examples/SmetsWouters2003`" directory of DynareOBC.

[54] These results are among those that may be generated by running "`GenerateDeterminacyResults`" within both the "`Examples/SmetsWouters2003`" and "`Examples/SmetsWouters2007`" directories of DynareOBC.

[55] An implementation of this model is contained within DynareOBC in the file "`Examples/AdjemianDarracqPariesMoyen2007/SWNLWCD.mod`".

[56] The only significant difference is that habits are internal, not external.

(2003) model was hand-linearized, with some ad hoc changes made only to the linearized equations, the Adjemian, Darracq Pariès & Moyen (2007) model is presented in its fully non-linear form, and welfare measures are derived. The measure of consumption equivalent welfare we use here is much as in the previous section. It is the amount of extra consumption services flow you would have to give to an inhabitant of the flexible price version of the model to make them indifferent between their economy and that of the model.[57] Unlike in the Fernández-Villaverde et al. (2015) model though, here it is assumed that non-updating prices are indexed to a combination of lagged inflation, and the steady-state level of inflation. Thus, there is no price-dispersion in steady-state, so steady-state welfare equals that of the flexible price economy. Our impulse response exercise in Figure 11 follows that of Figure 2,[58] and without a response to the price level, the responses of other variables are very similar to those in that figure.[59] However, with a response to the price level, introduced as in equation (9), the second solution no longer exists, so the welfare outcome is much improved (a 0.6% drop rather than a 5% drop). As in the previous section though, this is again subject to the same caveats on accuracy.[60]

[57] Habits slightly complicate this. Following Adjemian, Darracq Pariès & Moyen (2007), we assume that it is the habit adjusted consumption flow that is adjusted in the flexible price economy to derive the consumption equivalent welfare. I.e. $(C_{t+s} - hC_{t+s-1})$ in the utility function is replaced with $(C_{t+s} - hC_{t+s-1})Z_t$, where Z_t captures the consumption equivalent welfare.

[58] This figure may be generated by running "`RunExample`" within the "`Examples/AdjemianDarracqParieMoyen2007`" directory of DynareOBC.

[59] In this case, we need a slightly larger shock for a comparable exercise. It is now 24.5 standard deviations rather than 22.5 standard deviations in Figure 2.

[60] While the economy is moving less far from its steady-state following this shock than in the example from the previous section, here all variables, including welfare, are in first order approximations.

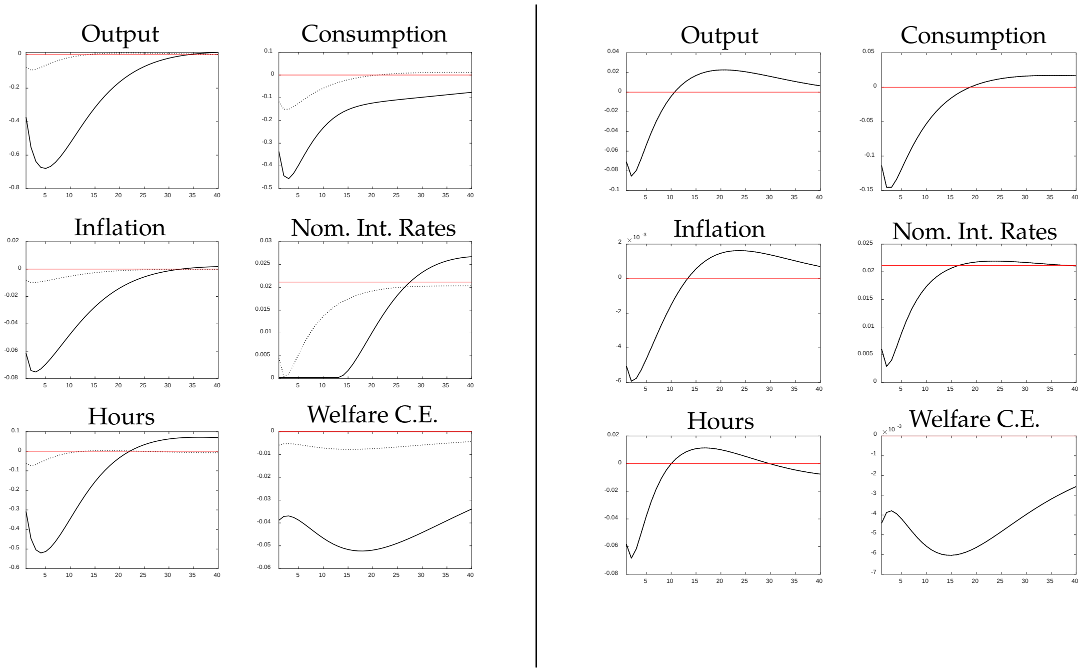

Figure 11: Two solutions following a preference shock in the Adjemian, Darracq Pariès & Moyen (2007) model, without (left 6 panels) and with (right 6 panels) a response to the price level.
All variables are in logarithms. The dotted line is a solution which does not hit the bound. The solid line is an alternative solution which does hit the bound in the absence of price level targeting. The two solutions are identical with a response to the price level. The calculation of the welfare consumption equivalent is detailed in the text.

Appendix G: Small LCPs

Appendix G.1: LCPs of size 1

When $T = 1$, it is particularly easy to characterise the properties of LCPs. This amounts to considering the behaviour of an economy in which everyone believes there will be at most one period at the bound. In this case, y gives the "shock" to the bounded equation necessary to impose the bound, and M gives the contemporaneous response of the bounded variable to an unanticipated shock: i.e. in a ZLB context, M gives the initial jump in nominal interest rates following a standard monetary policy shock.

First, suppose that M (a scalar as $T = 1$ for now) is positive. Then, if $q > 0$, for any $y \geq 0$, $q + My > 0$, so by the complementary slackness condition, in fact $y = 0$. Conversely, if $q \leq 0$, then there is a unique y satisfying the complementary slackness condition given by $y = -\frac{q}{M} \geq 0$. Thus, with $M > 0$, there is always a unique solution to the $T = 1$ LCP. With $M = 0$, $q + My = q$, so a solution to the LCP exists if and only if $q \geq 0$. It will be unique providing $q > 0$ (by the complementary slackness condition), but when $q = 0$, any $y \geq 0$ gives a solution. Finally, suppose that $M < 0$. Then, if $q > 0$, there are precisely two solutions. The "standard" solution has $y = 0$, but there is an additional solution featuring a jump to the bound in which $y = -\frac{q}{M} > 0$. If $q = 0$, then there is a unique solution ($y = 0$) and if $q < 0$, then with $y \geq 0$, $q + My < 0$, so there is no solution at all. Hence, the $T = 1$ LCP already provides examples of cases of uniqueness, non-existence and multiplicity.

Appendix G.2: LCPs of size 2

We now consider the $T = 2$ special case, where we can again easily derive results from first principles. Recall that a solution $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ to the LCP $\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right)$ satisfies $y_1 \geq 0$, $y_2 \geq 0$, $q_1 + M_{11}y_1 + M_{12}y_2 \geq 0$, $q_2 + M_{21}y_1 + M_{22}y_2 \geq 0$, $y_1(q_1 + M_{11}y_1 + M_{12}y_2) = 0$, and $y_2(q_2 + M_{21}y_1 + M_{22}y_2) = 0$. With two quadratics, there are up to four generic solutions, given by:

1) $y_1 = y_2 = 0$. Exists if $q_1 \geq 0$ and $q_2 \geq 0$.

2) $y_1 = -\frac{q_1}{M_{11}}, y_2 = 0$. Exists if $\frac{q_1}{M_{11}} \leq 0$ and $M_{11}q_2 \geq M_{21}q_1$.

3) $y_1 = 0, y_2 = -\frac{q_2}{M_{22}}$. Exists if $\frac{q_2}{M_{22}} \leq 0$ and $M_{22}q_1 \geq M_{12}q_2$.

4) $y_1 = \frac{M_{12}q_2 - M_{22}q_1}{M_{11}M_{22} - M_{12}M_{21}}, y_2 = \frac{M_{21}q_1 - M_{11}q_2}{M_{11}M_{22} - M_{12}M_{21}}$. Exists if $y_1 \geq 0$ and $y_2 \geq 0$.

Additionally, there are extra solutions in knife-edge cases:

5) If $q_1 = 0, M_{11} = 0$ and $q_2 \geq 0$ then any $y_1 \geq 0$ is a solution with $y_2 = 0$.

6) If $q_2 = 0, M_{22} = 0$ and $q_1 \geq 0$ then any $y_2 \geq 0$ is a solution with $y_1 = 0$.

7) If $q_1 = 0$, $q_2 = 0$, $M_{11}M_{22} = M_{12}M_{21}$, then any $y_1 \geq 0$ and $y_2 \geq 0$ with $M_{21}y_1 = -M_{22}y_2$ is a solution.

8) If $q_1 = 0$, $q_2 = 0$, $M_{11} = M_{12} = M_{21} = M_{22}$, then any $y_1 \geq 0$ and $y_2 \geq 0$ are a solution.

Appendix H: Generalizations

It is straightforward to generalise the results of this paper to less restrictive otherwise linear models with occasionally binding constraints.

Firstly, if the constraint is on a variable other than $x_{1,t}$, or in another equation than the first, then all of the results go through as before, just by relabelling and rearranging. Furthermore, if the constraint takes the form of $z_{1,t} = \max\{z_{2,t}, z_{3,t}\}$, where $z_{1,t}$, $z_{2,t}$ and $z_{3,t}$ are linear expressions in the contemporaneous values, lags and leads of x_t, then, assuming without loss of generality that $z_{3,\cdot} > z_{2,\cdot}$ in steady state, we have that $z_{1,t} - z_{2,t} = \max\{0, z_{3,t} - z_{2,t}\}$. Hence, adding a new auxiliary variable $x_{n+1,t}$, with the associated equation $x_{n+1,t} = z_{1,t} - z_{2,t}$, and replacing the constrained equation with $x_{n+1,t} = \max\{0, z_{3,t} - z_{2,t}\}$, we have a new equation in the form covered by our results. Moreover, if rather than a max we have a min, we just use the fact that if $z_{1,t} = \min\{z_{2,t}, z_{3,t}\}$, then $-z_{1,t} = \max\{-z_{2,t}, -z_{3,t}\}$, which is covered by the generalisation just established. The easiest encoding of the complementary slackness conditions, $z_t \geq 0$, $\lambda_t \geq 0$ and $z_t \lambda_t = 0$, is $0 = \min\{z_t, \lambda_t\}$, which is of this form.

To deal with multiple occasionally binding constraints, we use the representation from Holden and Paetz (2012). Suppose there are c constrained variables in the model. For $a \in \{1, \dots, c\}$, let $q^{(a)}$ be the path of the a^{th} constrained variable in the absence of all constraints. For $a, b \in \{1, \dots, c\}$, let $M^{(a,b)}$ be the matrix whose k^{th} column is the impulse response of the a^{th} constrained variable to magnitude 1 news shocks at horizon $k - 1$ to the equation defining the b^{th} constrained variable. For example, if $c = 1$ so there is a single constraint, then we would have that $M^{(1,1)} = M$ as defined in equation (4). Finally, let:

$$q := \begin{bmatrix} q^{(1)} \\ \vdots \\ q^{(c)} \end{bmatrix}, \qquad M := \begin{bmatrix} M^{(1,1)} & \cdots & M^{(1,c)} \\ \vdots & \ddots & \vdots \\ M^{(c,1)} & \cdots & M^{(c,c)} \end{bmatrix},$$

and let y be a solution to the LCP (q, M). Then the vertically stacked paths of the constrained variables in a solution which satisfies these constraints is given by $q + My$, and Theorem 1 (Restated) goes through as before.

Appendix I: Relationship between multiplicity under perfect-foresight, and multiplicity under rational expectations

By augmenting the state-space appropriately, the first order conditions of a general, non-linear, rational expectations, DSGE model may always be placed in the form:

$$0 = \mathbb{E}_t \hat{f}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \sigma\varepsilon_t),$$

for all $t \in \mathbb{Z}$, where $\sigma \in [0,1]$, $\hat{f}: (\mathbb{R}^{\hat{n}})^3 \times \mathbb{R}^m \to \mathbb{R}^{\hat{n}}$, and where for all $t \in \mathbb{Z}$, $\hat{x}_t \in \mathbb{R}^{\hat{n}}$, $\varepsilon_t \in \mathbb{R}^m$, $\mathbb{E}_{t-1}\varepsilon_t = 0$, and $\mathbb{E}_t \hat{x}_t = \hat{x}_t$. Since $\hat{f}$ is arbitrary, without loss of generality we may further assume that $\varepsilon_t \sim \mathrm{NIID}(0, I)$. We further assume:

Assumption 4 $\hat{f}$ is everywhere continuous.

The continuity of $\hat{f}$ does rule out some models, but all models in which the only source of non-differentiability is a max or min operator will have a continuous $\hat{f}$.

Now, by further augmenting the state space, we can then find a continuous function $f: (\mathbb{R}^n)^3 \times \mathbb{R}^m \to \mathbb{R}^n$ such that for all $t \in \mathbb{Z}$:

$$0 = f(x_{t-1}, x_t, \mathbb{E}_t x_{t+1}, \sigma\varepsilon_t),$$

where for all $t \in \mathbb{Z}$, $x_t \in \mathbb{R}^n$ and $\mathbb{E}_t x_t = x_t$.[61] (Note that this f is not intended to be the f from Section 3 of the paper.) A solution to this model is given by a policy function. Given f is continuous, it is natural to restrict attention to continuous policy functions.[62] Furthermore, given the model's transversality conditions, we are usually only interested in stationary, Markov solutions, so the policy function will not be a function of t or of lags of the state. Additionally, in this paper we are only interested in solutions in which the deterministic model converges to some particular steady state μ. Thus, we make the following assumption:

[61] For example, we may use the equations: $\hat{x}_t^\circ = \hat{x}_{t-1}$, $\hat{\varepsilon}_t = \varepsilon_t$, $z_t = \hat{f}(\hat{x}_{t-1}^\circ, \hat{x}_{t-1}, \hat{x}_t, \sigma\hat{\varepsilon}_{t-1})$, $0 = \mathbb{E}_t z_{t+1}$, with $x_t := [\hat{x}_t' \quad \hat{x}_t^{\circ\prime} \quad \hat{\varepsilon}_t' \quad z_t']'$.

[62] Note also that in standard dynamic programming applications, the policy function will be continuous. See e.g. Theorem 9.8 of Stokey, Lucas, and Prescott (1989).

Assumption 5 The policy function is given by a continuous function: $g: [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, such that for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g(\sigma, g(\sigma, x, e), \sigma\varepsilon), e),$$

where $\varepsilon \sim \mathrm{N}(0, I)$ and $\mathbb{E}_\varepsilon$ denotes an expectation with respect to ε. Furthermore, for all $x_0 \in \mathbb{R}^n$, the recurrence $x_t = g(0, x_{t-1}, 0)$ satisfies $x_t \to \mu$ as $t \to \infty$.

To produce a lower bound on the number of policy functions satisfying Assumption 5, we need two further assumptions. The first assumption just gives the existence of the "time iteration" (a.k.a. "policy function iteration") operator $\mathcal{T}$, and ensures that it has a fixed point.

Assumption 6 Let $\mathcal{G}$ denote the space of all continuous functions $[0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We assume there exists a function $\mathcal{T}: \mathcal{G} \to \mathcal{G}$ such that for all $(g, \sigma, x, e) \in \mathcal{G} \times [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f\big(x, \mathcal{T}(g)(\sigma, x, e), \mathbb{E}_\varepsilon g\big(\sigma, \mathcal{T}(g)(\sigma, x, e), \sigma\varepsilon\big), e\big).$$

We further assume that if there exists some $(g, \sigma) \in \mathcal{G} \times [0,1]$ such that for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f\big(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g\big(\sigma, g(\sigma, x, e), \sigma\varepsilon\big), e\big),$$

then for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $\mathcal{T}(g)(\sigma, x, e) = g(\sigma, x, e)$.

The second assumption ensures that time iteration always converges when started from a solution to the model with no uncertainty after the current period. This is a weak assumption since the policy functions under uncertainty are invariably close to the policy function in the absence of uncertainty.

Assumption 7 Let $h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function giving a solution to the model in which there is no future uncertainty, i.e. for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, h(x, e), h(h(x, e), 0), e).$$

Further, define $g_{h,0} \in \mathcal{G}$ by $g_{h,0}(\sigma, x, e) = h(x, e)$ for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$, and define $g_{h,k} \in \mathcal{G}$ inductively by $g_{h,k+1} = \mathcal{T}(g_{h,k})$ for all $k \in \mathbb{N}$. Then there exists some $g_{h,\infty} \in \mathcal{G}$ such that $g_{h,\infty} = \mathcal{T}(g_{h,\infty})$ and for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,k}(\sigma, x, e) \to g_{h,\infty}(\sigma, x, e)$ as $k \to \infty$.

Note, by construction, if h is as in Assumption 7, then for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, g_{h,0}(0, x, e), \mathbb{E}_\varepsilon g_{h,0}(0, g_{h,0}(0, x, e), 0\varepsilon), e).$$

Hence, by Assumption 6, for all $k \in \mathbb{N}$, all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,k}(0, x, e) = g_{h,0}(0, x, e)$. Consequently, for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,\infty}(0, x, e) = g_{h,0}(0, x, e) = h(x, e)$.

Now suppose that h_1 and h_2 were as in Assumption 7, and that there exists $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, such that $h_1(x, e) \neq h_2(x, e)$. Then, by the continuity of $g_{h_1,\infty}$ and $g_{h_2,\infty}$, there is some $\mathcal{S} \subseteq [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$ of positive measure, with $(0, x, e) \in \mathcal{S}$, such that for all $(\sigma, x, e) \in \mathcal{S}$, $g_{h_1,\infty}(\sigma, x, e) \neq g_{h_2,\infty}(\sigma, x, e)$. Hence, the rational expectations policy functions differ, at least for small σ. Thus, if Assumption 6 and Assumption 7 are satisfied, there are at least as many policy functions satisfying Assumption 5 as there are solutions to the model in which there is no future uncertainty.

Appendix J: Results from and for dynamic programming

Appendix J.1: The linear-quadratic case

Alternative existence and uniqueness results for the infinite T problem can be established via dynamic programming methods, under the assumption that Problem 2 (OBC) comes from the first order conditions solution of a social planner problem. These have the advantage that their conditions are potentially much easier to evaluate, though they also have somewhat limited applicability. We focus here on uniqueness results, since these are of greater interest.

Suppose that the social planner in some economy solves the following problem:

Problem 4 (Linear-Quadratic) Suppose $\mu \in \mathbb{R}^n$, $\Psi^{(0)} \in \mathbb{R}^{c\times 1}$ and $\Psi^{(1)} \in \mathbb{R}^{c\times 2n}$ are given, where $c \in \mathbb{N}$. Define $\tilde{\Gamma}: \mathbb{R}^n \to \mathbb{P}(\mathbb{R}^n)$ (where $\mathbb{P}$ denotes the power-set operator) by:

$$\tilde{\Gamma}(x) = \left\{ z \in \mathbb{R}^n \middle| 0 \le \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix} \right\}, \tag{10}$$

for all $x \in \mathbb{R}^n$. *(Note: $\tilde{\Gamma}(x)$ will give the set of feasible values for next period's state if the current state is x. Equality constraints may be included by including an identical lower bound and upper bound.)* Define:

$$\widetilde{X} \coloneqq \{ x \in \mathbb{R}^n | \tilde{\Gamma}(x) \ne \emptyset \}, \tag{11}$$

and suppose without loss of generality that for all $x \in \mathbb{R}^n$, $\tilde{\Gamma}(x) \cap \widetilde{X} = \tilde{\Gamma}(x)$. *(Note: this means that the linear inequalities bounding $\widetilde{X}$ are already included in those in the definition of $\tilde{\Gamma}(x)$. It is without loss of generality as the planner will never choose an $\tilde{x} \in \tilde{\Gamma}(x)$ such that $\tilde{\Gamma}(\tilde{x}) = \emptyset$.)* Further define $\tilde{\mathcal{F}}: \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ by:

$$\tilde{\mathcal{F}}(x,z) = u^{(0)} + u^{(1)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}' \tilde{u}^{(2)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}, \tag{12}$$

for all $x, z \in \widetilde{X}$, where $u^{(0)} \in \mathbb{R}$, $u^{(1)} \in \mathbb{R}^{1\times 2n}$ and $\tilde{u}^{(2)} = \tilde{u}^{(2)\prime} \in \mathbb{R}^{2n\times 2n}$ are given. Finally, suppose $x_0 \in \widetilde{X}$ is given and $\beta \in (0,1)$, and choose $x_1, x_2, \ldots$ to maximise:

$$\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \tag{13}$$

subject to the constraints that for all $t \in \mathbb{N}^+$, $x_t \in \tilde{\Gamma}(x_{t-1})$.

To ensure the problem is well behaved, we make the following assumption:

Assumption 8 $\tilde{u}^{(2)}$ is negative-definite.

In Appendix L.5, below, we establish the following (unsurprising) result:

Proposition 14 If either $\widetilde{X}$ is compact, or, $\tilde{\Gamma}(x)$ is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \widetilde{X}$, then for all $x_0 \in \widetilde{X}$, there is a unique path $(x_t)_{t=0}^{\infty}$ which solves Problem 4 (Linear-Quadratic).

We wish to use this result to establish the uniqueness of the solution to the first order conditions. The Lagrangian for our problem is given by:

$$\sum_{t=1}^{\infty} \beta^{t-1} \left[\widetilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda'_{\Psi,t} \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] \right], \tag{14}$$

for some KKT-multipliers $\lambda_t \in \mathbb{R}^c$ for all $t \in \mathbb{N}^+$. Taking the first order conditions leads to the following necessary KKT conditions, for all $t \in \mathbb{N}^+$:

$$0 = u^{(1)}_{\cdot,2} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,2} + \lambda'_t \Psi^{(1)}_{\cdot,2} + \beta \left[u^{(1)}_{\cdot,1} + \begin{bmatrix} x_t - \mu \\ x_{t+1} - \mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,1} + \lambda'_{t+1} \Psi^{(1)}_{\cdot,1} \right], \tag{15}$$

$$0 \leq \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}, \qquad 0 \leq \lambda_t, \qquad 0 = \lambda_t \circ \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right], \tag{16}$$

where subscripts 1 and 2 refer to blocks of rows or columns of length n. Additionally, for μ to be the steady state of x_t and $\overline{\lambda}$ to be the steady state of λ_t, we require:

$$0 = u^{(1)}_{\cdot,2} + \overline{\lambda}' \Psi^{(1)}_{\cdot,2} + \beta \left[u^{(1)}_{\cdot,1} + \overline{\lambda}' \Psi^{(1)}_{\cdot,1} \right], \tag{17}$$

$$0 \leq \Psi^{(0)}, \qquad 0 \leq \overline{\lambda}, \qquad 0 = \overline{\lambda} \circ \Psi^{(0)}. \tag{18}$$

In Appendix L.6, below, we prove the following result:

Proposition 15 Suppose that for all $t \in \mathbb{N}$, $(x_t)_{t=1}^{\infty}$ and $(\lambda_t)_{t=1}^{\infty}$ satisfy the KKT conditions given in equations (15) and (16), and that as $t \to \infty$, $x_t \to \mu$ and $\lambda_t \to \overline{\lambda}$, where μ and λ satisfy the steady state KKT conditions given in equations (17) and (18). Then $(x_t)_{t=1}^{\infty}$ solves Problem 4 (Linear-Quadratic). If, further, either condition of Proposition 14 is satisfied, then $(x_t)_{t=1}^{\infty}$ is the unique solution to Problem 4 (Linear-Quadratic), and there can be no other solutions to the KKT conditions given in equations (15) and (16) satisfying $x_t \to \mu$ and $\lambda_t \to \overline{\lambda}$ as $t \to \infty$.

Now, it is possible to convert the KKT conditions given in equations (15) and (16) into a problem in the form of the multiple-bound generalisation of Problem 2 (OBC) quite generally. To see this, first note that we may rewrite equation (15) as:

$$0 = u_{\cdot,2}^{(1)\prime} + \tilde{u}_{2,1}^{(2)}(x_{t-1} - \mu) + \tilde{u}_{2,2}^{(2)}(x_t - \mu) + \Psi_{\cdot,2}^{(1)\prime}\lambda_t \\ + \beta\left[u_{\cdot,1}^{(1)\prime} + \tilde{u}_{1,1}^{(2)}(x_t - \mu) + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{\cdot,1}^{(1)\prime}\lambda_{t+1}\right].$$

Now, $\tilde{u}_{2,2}^{(2)} + \beta u_{1,1}^{(2)}$ is negative definite, hence we may define $\mathcal{V} := \Psi_{\cdot,2}^{(1)}\left[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\right]^{-1}$, so:

$$\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \\ = \Psi^{(0)} + \left(\Psi_{\cdot,1}^{(1)} - \mathcal{V}\tilde{u}_{2,1}^{(2)}\right)(x_{t-1} - \mu) - \mathcal{V}\left[u_{\cdot,2}^{(1)\prime} + \beta\left[u_{\cdot,1}^{(1)\prime} + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{\cdot,1}^{(1)\prime}\lambda_{t+1}\right]\right] \\ -\Psi_{\cdot,2}^{(1)}\left[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\right]^{-1}\Psi_{\cdot,2}^{(1)\prime}\lambda_t. \tag{19}$$

Moreover, equation (16) implies that if the k^{th} element of $\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$ is positive, then the k^{th} element of λ_t is zero, so:

$$\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} = \max\{0, z_t\}, \tag{20}$$

where:

$$z_t := \Psi^{(0)} + \left(\Psi_{\cdot,1}^{(1)} - \mathcal{V}\tilde{u}_{2,1}^{(2)}\right)(x_{t-1} - \mu) \\ - \mathcal{V}\left[u_{\cdot,2}^{(1)\prime} + \beta\left[u_{\cdot,1}^{(1)\prime} + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{\cdot,1}^{(1)\prime}\lambda_{t+1}\right]\right] \\ - \left[\Psi_{\cdot,2}^{(1)}\left[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\right]^{-1}\Psi_{\cdot,2}^{(1)\prime} + \mathcal{W}\right]\lambda_t,$$

and $\mathcal{W} \in \mathbb{R}^{c \times c}$ is an arbitrary, positive diagonal matrix. A natural choice is:

$$\mathcal{W} := -\operatorname{diag}^{-1}\operatorname{diag}\left[\Psi_{\cdot,2}^{(1)}\left[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\right]^{-1}\Psi_{\cdot,2}^{(1)\prime}\right],$$

providing this is positive (it is nonnegative at least as $\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}$ is negative definite), where the diag operator maps matrices to a vector containing their diagonal, and diag^{-1} maps vectors to a matrix with the given vector on the diagonal, and zeros elsewhere.

We claim that we may replace equation (16) with equation (20) without changing the model. We have already shown that equation (16) implies equation (20), so we just have to prove the converse. We continue to suppose equation (15) holds, and thus, so too does equation (19). Then, from subtracting equation (19) from equation (20), we have that:

$$\mathcal{W}\lambda_t = \max\{-z_t, 0\}.$$

Hence, as $\mathcal{W}$ is a positive diagonal matrix, and the right-hand side is nonnegative, $\lambda_t \geq 0$. Furthermore, the kth element of λ_t is non-negative if and only if the kth element of z_t is non-positive (as $\mathcal{W}$ is a positive diagonal matrix), which in turn holds if and only if the kth element of $\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$ is equal to zero, by equation (20). Thus equation (16) is satisfied.

Combined with our previous results, this gives the following proposition:

Proposition 16 Suppose we are given a problem in the form of Problem 4 (Linear-Quadratic). Then, the KKT conditions of that problem may be placed into the form of the multiple-bound generalisation of Problem 2 (OBC). Let $\left(q_{x_0}, M\right)$ be the infinite LCP corresponding to this representation, given initial state $x_0 \in \widetilde{X}$. Then, if y is a solution to the LCP, $q_{x_0} + My$ gives the stacked paths of the bounded variables in a solution to Problem 4 (Linear-Quadratic). If, further, either condition of Proposition 14 is satisfied, then this LCP has a unique solution for all $x_0 \in \widetilde{X}$, which gives the unique solution to Problem 4 (Linear-Quadratic), and, for sufficiently large T^*, the finite LCP $\left(q_{x_0}^{(T^*)}, M^{(T^*)}\right)$ has a unique solution $y^{(T^*)}$ for all $x_0 \in \widetilde{X}$, where $q_{x_0}^{(T^*)} + M^{(T^*)} y^{(T^*)}$ gives the first T^* periods of the stacked paths of the bounded variables in a solution to Problem 4 (Linear-Quadratic).

This proposition provides some evidence that the LCP will have a unique solution when it is generated from a dynamic programming problem with a unique solution. In the next subsection, we derive similar results for models with more general constraints and objective functions. The proof of this proposition also showed an alternative method for converting KKT conditions into equations of the form handled by our methods.

Appendix J.2: The general case

Here we consider non-linear dynamic programming problems with general objective functions. Consider then the following generalisation of Problem 4 (Linear-Quadratic):

Problem 5 (Non-linear) Suppose $\Gamma: \mathbb{R}^n \to \mathbb{P}(\mathbb{R}^n)$ is a given compact, convex valued continuous function. Define $X := \{x \in \mathbb{R}^n | \Gamma(x) \neq \emptyset\}$, and suppose without loss of generality that for all $x \in \mathbb{R}^n$, $\Gamma(x) \cap X = \Gamma(x)$. Further suppose that $\mathcal{F}: X \times X \to \mathbb{R}$ is a given twice continuously differentiable, concave function, and that $x_0 \in X$ and $\beta \in (0,1)$ are given.

Choose $x_1, x_2, \dots$ to maximise:

$$\liminf_{T \to \infty} \sum_{t=1}^{T} \beta^{t-1} \mathcal{F}(x_{t-1}, x_t),$$

subject to the constraints that for all $t \in \mathbb{N}^+$, $x_t \in \Gamma(x_{t-1})$.

For tractability, we make the following additional assumption, which enables us to uniformly approximate Γ by a finite number of inequalities:

Assumption 9 X is compact.

Then, by Theorem 4.8 of Stokey, Lucas, and Prescott (1989), there is a unique solution to Problem 5 (Non-linear) for any x_0. We further assume the following to ensure that there is a natural point to approximate around:[63]

Assumption 10 There exists $\mu \in X$ such that for any given $x_0 \in X$, in the solution to Problem 5 (Non-linear) with that x_0, as $t \to \infty$, $x_t \to \mu$.

Having defined μ, we can let $\tilde{\mathcal{F}}$ be a second order Taylor approximation to $\mathcal{F}$ around μ, which will take the form of equation (12). Assumption 8 will be satisfied

[63] If X is convex, then the existence of a fixed point of the policy function is a consequence of Brouwer's Fixed Point Theorem, but there is no reason the fixed point guaranteed by Brouwer's Theorem should be even locally attractive.

for this approximation thanks to the concavity of $\mathcal{F}$. To apply the previous results, we also then need to approximate the constraints.

Suppose first that the graph of Γ is convex, i.e. the set $\{(x,z)|x \in X, z \in \Gamma(x)\}$ is convex. Since it is also compact, by Assumption 9, for any $\epsilon > 0$, there exists $c \in \mathbb{N}$, $\Psi^{(0)} \in \mathbb{R}^{c\times 1}$ and $\Psi^{(1)} \in \mathbb{R}^{c\times 2n}$ such that with $\tilde{\Gamma}$ defined as in equation (10) and $\widetilde{X}$ defined as in equation (11):

1) $\mu \in \widetilde{X} \subseteq X$,
2) for all $x \in X$, there exists $\tilde{x} \in \widetilde{X}$ such that $\|x - \tilde{x}\|_2 < \epsilon$,
3) for all $x \in \widetilde{X}$, $\tilde{\Gamma}(x) \subseteq \Gamma(x)$,
4) for all $x \in \widetilde{X}$, and for all $z \in \Gamma(x)$, there exists $\tilde{z} \in \tilde{\Gamma}(x)$ such that $\|z - \tilde{z}\|_2 < \epsilon$.

(This follows from standard properties of convex sets.) Then, by our previous results, the following proposition is immediate:

Proposition 17 Suppose we are given a problem in the form of Problem 5 (Non-linear) (and which satisfies Assumption 9 and Assumption 10). If the graph of Γ is convex, then we can construct a problem in the form of the multiple-bound generalisation of Problem 2 (OBC) which encodes a local approximation to the original dynamic programming problem around $x_t = \mu$. Furthermore, the LCP corresponding to this approximation will have a unique solution for all $x_0 \in \widetilde{X}$. Moreover, the approximation is consistent for quadratic objectives in the sense that as the number of inequalities used to approximate Γ goes to infinity, the approximate value function converges uniformly to the true value function.

Unfortunately, if the graph of Γ is non-convex, then we will not be able to derive similar results. To see the best we could do along similar proof lines, here we merely sketch the construction of an approximation to the graph of Γ in this case. We will need to assume that there exists $z \in \operatorname{int}\Gamma(x)$ for all $x \in X$, which precludes the existence of equality constraints.[64] We first approximate the graph of Γ by a polytope

[64] This is often not too much of a restriction, since equality constraints may be substituted out.

(i.e. n dimensional polygon) contained in the graph of Γ such that all points in the graph of Γ are within $\frac{\epsilon}{2}$ of a point in the polytope. Then, providing ϵ is sufficiently small, for each simplicial surface element of the polytope, indexed by $k \in \{1, \dots, c\}$, we can find a quadratic function $q_k: X \times X \to \mathbb{R}$ with:

$$q_k = \Psi_k^{(0)} + \Psi_{k,\cdot}^{(1)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix} + \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}' \Psi_k^{(2)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}$$

for all $x, z \in X$ and such that q_k is zero at the corners of the simplicial surface element, such that q_k is nonpositive on its surface, such that $\Psi_k^{(2)}$ is symmetric positive definite, and such that all points in the polytope are within $\frac{\epsilon}{2}$ of a point in the set:

$$\{(x,z) \in X \times X | \forall k \in \{1, \dots, S\}, 0 \le q_k(x,z)\}.$$

This gives a set of quadratic constraints that approximate Γ. If we then define:

$$\tilde{u}^{(2)} := u^{(2)} + \sum_{k=1}^{c} \bar{\lambda}'_{\Psi,k} \Psi_k^{(2)},$$

where $u^{(2)}$ is the Hessian of $\mathcal{F}$, then the Lagrangian in equation (14) is the same as what would be obtained from taking a second order Taylor approximation to the Lagrangian of the problem of maximising our non-linear objective subject to the approximate quadratic constraints, suggesting it may perform acceptably well for x near μ, along similar lines to the results of Levine, Pearlman, and Pierse (2008) and Benigno & Woodford (2012). However, existence of a unique solution to the original problem cannot be used to establish even the existence of a solution of the approximated problem, since only linear approximations to the quadratic constraints would be imposed by our algorithm, giving a reduced choice set (as the quadratic terms are positive definite).

Appendix K: Price level targeting example calculations

To recap, the model with "news" shocks is:

$$r + p_{t+1} - p_t = r + \phi(p_t - p_{t-1}) + \chi p_t + \nu_t,$$

with $\chi > 0$ and $\phi > 1$, so:

$$p_{t+1} = (1 + \phi + \chi)p_t - \phi p_{t-1} + \nu_t.$$

We fix $p_0 = 0$.

We look for a solution in the form $p_t = \sum_{j=-\infty}^{\infty} G_j \nu_{t+j}$, where $\nu_t = 0$ for all $t < 0$. Substituting in, we have:

$$\sum_{j=-\infty}^{\infty} G_{j-1}\nu_{t+j} = (1 + \phi + \chi) \sum_{j=-\infty}^{\infty} G_j \nu_{t+j} - \phi \sum_{j=-\infty}^{\infty} G_{j+1}\nu_{t+j} + \nu_t,$$

so:

$$G_{-1} = (1 + \phi + \chi)G_0 - \phi G_1 + 1,$$
$$\forall j \neq 0, \qquad G_{j-1} = (1 + \phi + \chi)G_j - \phi G_{j+1}.$$

We conjecture that $G_j = G_0 \zeta^j$ for $j \geq 0$ and $G_j = G_0 \eta^{-j}$ for $j \leq 0$, for some $G_0 \in \mathbb{R}$ and $\zeta, \eta \in (-1,1)$. Then:

$$1 = (1 + \phi + \chi)\zeta - \phi\zeta^2,$$
$$\eta^2 = (1 + \phi + \chi)\eta - \phi.$$

Thus:

$$\eta = \frac{1 + \phi + \chi - \sqrt{(1 + \phi + \chi)^2 - 4\phi}}{2} = 1 - \frac{\chi}{\phi - 1} + \mathrm{O}(\chi^2),$$

$$\zeta = \frac{\eta}{\phi} = \frac{1 + \phi + \chi - \sqrt{(1 + \phi + \chi)^2 - 4\phi}}{2\phi} = \frac{1}{\phi}\left(1 - \frac{\chi}{\phi - 1}\right) + \mathrm{O}(\chi^2),$$

$$G_0 = -\frac{1}{\sqrt{(1 + \phi + \chi)^2 - 4\phi}} = -\frac{1}{\phi - 1} + \frac{\phi + 1}{(\phi - 1)^3}\chi + \mathrm{O}(\chi^2),$$

where, here and in the following, the $\mathrm{O}(\chi^2)$ terms are taken as $\chi \to 0$. Note that:

$$(1 + \phi + \chi)^2 - 4\phi = (\phi - 1)^2 + 2\chi(1 + \phi) + \chi^2 > 0$$

providing that $\chi \geq 0$, so this solution is real as required. Additionally:

$$\eta = \frac{1 + \phi + \chi - \sqrt{(1 + \phi + \chi)^2 - 4\phi}}{2} > \frac{1 + \phi + \chi - \sqrt{(1 + \phi + \chi)^2}}{2} = 0,$$

and for $\chi > 0$:

$$\eta = \frac{1+\phi+\chi-\sqrt{(\phi-1)^2+2\chi(1+\phi)+\chi^2}}{2}$$

$$< \frac{1+\phi+\chi-\sqrt{(\phi-1)^2+2\chi(\phi-1)+\chi^2}}{2} = \frac{1+\phi+\chi-\sqrt{(\phi-1+\chi)^2}}{2}$$

$$= 1,$$

again as required.

Substituting back in, we have:

$$\begin{aligned}
i_t &= r + p_{t+1} - p_t \\
&= r + G_0\left[\sum_{j=-\infty}^{-1}\eta^{-j}\nu_{t+1+j} + \sum_{j=0}^{\infty}\zeta^j\nu_{t+1+j} - \sum_{j=-\infty}^{0}\eta^{-j}\nu_{t+j} - \sum_{j=1}^{\infty}\zeta^j\nu_{t+j}\right] \\
&= r + G_0\left[\sum_{j=-\infty}^{0}\eta^{-j+1}\nu_{t+j} + \sum_{j=1}^{\infty}\zeta^{j-1}\nu_{t+j} - \sum_{j=-\infty}^{0}\eta^{-j}\nu_{t+j} - \sum_{j=1}^{\infty}\zeta^j\nu_{t+j}\right] \\
&= r + G_0\left[(1-\zeta)\sum_{j=1}^{\infty}\zeta^{j-1}\nu_{t+j} - (1-\eta)\sum_{j=-\infty}^{0}\eta^{-j}\nu_{t+j}\right] \\
&= r - \sum_{j=1}^{\infty}\frac{\nu_{t+j}}{\phi^j} + G_0\left[\sum_{j=1}^{\infty}\left((1-\zeta)\zeta^{j-1} + \frac{1}{G_0\phi^j}\right)\nu_{t+j} - (1-\eta)\sum_{j=-\infty}^{0}\eta^{-j}\nu_{t+j}\right] \\
&= r - \sum_{j=1}^{\infty}\frac{\nu_{t+j}}{\phi^j} + \left[\frac{1}{(\phi-1)^2}\left(\sum_{j=-\infty}^{0}\nu_{t+j} + \sum_{j=1}^{\infty}\frac{\nu_{t+j}}{\phi^{j-1}}\right) + \frac{1}{\phi-1}\sum_{j=1}^{\infty}\frac{(j-1)\nu_{t+j}}{\phi^j}\right]\chi + \mathrm{O}(\chi^2).
\end{aligned}$$

Since the partial derivative of the term in square brackets here with respect to ν_s is strictly positive for all $s \in \mathbb{N}^+$, at least for small χ, all of the elements of M must be strictly monotonically increasing in χ. In particular, at least for small χ, the elements of M with $\chi > 0$ are strictly greater than the elements of M with $\chi = 0$.

In fact, this holds for all χ. Given that $G_0 < 0$ and $\eta < 1$, from examining the square bracketed term of the penultimate equation above, we just need that $(1-\zeta)\zeta^{j-1} + \frac{1}{G_0\phi^j} < 0$ for $j \geq 1$. With $j = 1$ this holds as:

$$
\begin{aligned}
(1-\zeta)\zeta^{1-1}+\frac{1}{G_0\phi^1} &= \frac{2\phi-1-\phi-\chi+\sqrt{(1+\phi+\chi)^2-4\phi}}{2\phi}-\frac{2\sqrt{(1+\phi+\chi)^2-4\phi}}{2\phi} \\
&= \frac{-\chi+(\phi-1)-\sqrt{(1+\phi+\chi)^2-4\phi}}{2\phi} \\
&= \frac{-\chi+(\phi-1)-\sqrt{(\phi-1)^2+2\chi(1+\phi)+\chi^2}}{2\phi}<0.
\end{aligned}
$$

So, using the fact that $\zeta=\frac{\eta}{\phi}<\eta<1$:

$$0<1-\zeta<-\frac{1}{G_0\phi}.$$

Thus as $0<\frac{\eta}{\phi}=\zeta=\frac{\eta}{\phi}<\frac{1}{\phi}$, in fact for all $j\geq 1$:

$$(1-\zeta)\zeta^{j-1}<-\frac{1}{G_0\phi^j},$$

as required. Thus, for all $\chi>0$, the elements of M with $\chi>0$ are strictly greater than the elements of M with $\chi=0$. Although our proof in the text is "for sufficiently small χ", this is at least suggestive that M will still be a P-matrix even for large χ.

Appendix L: Further proofs

Appendix L.1: Proof of Proposition 2

We first establish the following Lemma:

Lemma 4 The (time-reversed) difference equation $A\hat{d}_{k+1} + B\hat{d}_k + C\hat{d}_{k-1} = 0$ for all $k \in \mathbb{N}^+$ has a unique solution satisfying the terminal condition $\hat{d}_k \to 0$ as $k \to \infty$, given by $\hat{d}_k = H\hat{d}_{k-1}$, for all $k \in \mathbb{N}^+$, for some H with eigenvalues in the unit circle.

First, define $G := -C(B + CF)^{-1}$, and note that if L is the lag (right-shift) operator, the model from Problem 1 (Linear) can be written as:

$$L^{-1}(ALL + BL + C)(x - \mu) = 0.$$

Furthermore, by the definitions of F and G:

$$(L - G)(B + CF)(I - FL) = ALL + BL + C,$$

so, the stability of the model from Problem 1 (Linear) is determined by the solutions for $z \in \mathbb{C}$ of the polynomial:

$$0 = \det(Az^2 + Bz + C) = \det(Iz - G)\det(B + CF)\det(I - Fz).$$

Now by Assumption 1, all of the roots of $\det(I - Fz)$ are strictly outside of the unit circle, and all of the other roots of $\det(Az^2 + Bz + C)$ are weakly inside the unit circle (else there would be indeterminacy), thus, all of the roots of $\det(Iz - G)$ are weakly inside the unit circle. Therefore, if we write $\rho_{\mathcal{M}}$ for the spectral radius of some matrix $\mathcal{M}$, then, by this discussion and Assumption 2, $\rho_G < 1$.

Now consider the time reversed model:

$$L(AL^{-1}L^{-1} + BL^{-1} + C)d = 0,$$

subject to the terminal condition that $d_k \to 0$ as $k \to \infty$. Now, let $z \in \mathbb{C}$, $z \neq 0$ be a solution to:

$$0 = \det(Az^2 + Bz + C),$$

and define $\tilde{z} = z^{-1}$, so:

$$\begin{aligned}0 = \det(A + B\tilde{z} + C\tilde{z}^2) &= z^{-2}\det(Az^2 + Bz + C)\\ &= \det(I - G\tilde{z})\det(B + CF)\det(I\tilde{z} - F).\end{aligned}$$

By Assumption 1, all the roots of $\det(I\tilde{z} - F)$ are inside the unit circle, thus they cannot contribute to the dynamics of the time reversed process, else the terminal condition would be violated. Thus, the time reversed model has a unique solution satisfying the terminal condition with a transition matrix with the same eigenvalues as G. Consequently, this solution can be calculated via standard methods for solving linear DSGE models, and it will be given by $d_k = Hd_{k-1}$, for all $k > 0$, where $H = -(B + AH)^{-1}C$, and $\phi_H = \phi_G < 1$, by Assumption 2. This completes the proof of Lemma 4.

Now let $s_t^*, x_t^* \in \mathbb{R}^{n \times \mathbb{N}^+}$ be such that for any $y \in \mathbb{R}^{\mathbb{N}^+}$, the k^{th} columns of $s_t^* y$ and $x_t^* y$ give the value of s_t and x_t following a magnitude 1 news shock at horizon k, i.e. when $x_0 = \mu$ and y_0 is the k^{th} row of $I_{\mathbb{N}^+ \times \mathbb{N}^+}$. Then:

$$\begin{aligned} s_t^* &= -(B + CF)^{-1}[I_{\cdot,1} I_{t,1:\infty} + G I_{\cdot,1} I_{t+1,1:\infty} + G^2 I_{\cdot,1} I_{t+2,1:\infty} + \cdots] \\ &= -(B + CF)^{-1} \sum_{k=0}^{\infty} (GL)^k \, I_{\cdot,1} I_{t,1:\infty} \\ &= -(B + CF)^{-1} (I - GL)^{-1} I_{\cdot,1} I_{t,1:\infty}, \end{aligned}$$

where the infinite sums are well defined as $\rho_G < 1$, and where $I_{t,1:\infty} \in \mathbb{R}^{1 \times \mathbb{N}^+}$ is a row vector with zeros everywhere except position t where there is a 1. Thus:

$$s_t^* = [0_{n \times (t-1)} \quad s_1^*] = L^{t-1} s_1^*.$$

Furthermore,

$$(x_t^* - \mu^*) = \sum_{j=1}^{t} F^{t-j} s_k^* = \sum_{j=1}^{t} F^{t-j} L^{j-1} s_1^*,$$

i.e.:

$$(x_t^* - \mu^*)_{\cdot,k} = \sum_{j=1}^{t} F^{t-j} s_{1,\cdot,k+1-j}^* = -\sum_{j=1}^{\min\{t,k\}} F^{t-j} (B + CF)^{-1} G^{k-j} I_{\cdot,1},$$

and so, the k^{th} offset diagonal of M ($k \in \mathbb{Z}$) is given by the first row of the k^{th} column of:

$$L^{-t}(x_t^* - \mu^*) = L^{-1} \sum_{j=1}^{t} (FL^{-1})^{t-j} s_1^* = L^{-1} \sum_{j=0}^{t-1} (FL^{-1})^{j} s_1^*,$$

where we abuse notation slightly by allowing L^{-1} to give a result with indices in $\mathbb{Z}$ rather than $\mathbb{N}^+$, with padding by zeros. Consequently, for all $k \in \mathbb{N}^+$, $M_{t,k} = \mathrm{O}(t^n \rho_F^t)$, as $t \to \infty$, for all $t \in \mathbb{N}^+$, $M_{t,k} = \mathrm{O}\big(k^n \rho_G^k\big)$, as $k \to \infty$, and for all $k \in \mathbb{Z}$, $M_{t,t+k} - \lim_{\tau\to\infty} M_{\tau,\tau+k} = \mathrm{O}(t^{n-1}(\rho_F \rho_G)^t)$, as $t \to \infty$.

Hence,

$$\sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y$$

exists and is well defined. We need to provide conditions under which $\sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y > 0.$[65]

To produce such conditions, we need constructive bounds on M, even if they have slightly worse convergence rates. For any matrix, $\mathcal{M} \in \mathbb{R}^{n \times n}$ with $\rho_{\mathcal{M}} < 1$, and any $\phi \in (\rho_{\mathcal{M}}, 1)$, let:

$$C_{\mathcal{M},\phi} := \sup_{k \in \mathbb{N}} \big\|(\mathcal{M}\phi^{-1})^k\big\|_2.$$

Furthermore, for any matrix, $\mathcal{M} \in \mathbb{R}^{n \times n}$ with $\rho_{\mathcal{M}} < 1$, and any $\epsilon > 0$, let:

$$\rho_{\mathcal{M},\epsilon} := \max\{|z| | z \in \mathbb{C}, \sigma_{\min}(\mathcal{M} - zI) = \epsilon\},$$

where $\sigma_{\min}(\mathcal{M} - zI)$ is the minimum singular value of $\mathcal{M} - zI$, and let $\epsilon^*(\mathcal{M}) \in (0, \infty]$ solve:

$$\rho_{\mathcal{M},\epsilon^*(\mathcal{M})} = 1.$$

(This has a solution in $(0, \infty]$ by continuity as $\rho_{\mathcal{M}} < 1$.) Then, by Theorem 16.2 of Trefethen and Embree (2005), for any $K \in \mathbb{N}$ and $k > K$:

$$\big\|(\mathcal{M}\phi^{-1})^k\big\|_2 \le \big\|(\mathcal{M}\phi^{-1})^K\big\|_2 \big\|(\mathcal{M}\phi^{-1})^{k-K}\big\|_2 \le \frac{\big\|(\mathcal{M}\phi^{-1})^K\big\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})}.$$

[65] We might ideally have liked a lower bound on $\sup_{y \in \ell_1 \cap [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y$ since by the Moore-Osgood theorem, these would imply a lower bound on $\lim_{T\to\infty} \max_{y \in [0,1]^T} \min_{t \in \{1,\dots,T\}} M_{t,1:T} y$ and thus imply that M was an S-matrix for all sufficiently large T. However, we have not managed to obtain a non-trivial lower bound on $\sup_{y \in \ell_1 \cap [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y$.

Now, $\left\|(\mathcal{M}\phi^{-1})^K\right\|_2 \to 0$ as $K \to \infty$, hence, there exists some $K \in \mathbb{N}$ such that:

$$\sup_{k=0,\dots,K}\left\|(\mathcal{M}\phi^{-1})^k\right\|_2 \geq \frac{\left\|(\mathcal{M}\phi^{-1})^K\right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})} \geq \sup_{k>K}\left\|(\mathcal{M}\phi^{-1})^k\right\|_2,$$

meaning $\mathcal{C}_{\mathcal{M},\phi} = \sup_{k=0,\dots,K}\left\|(\mathcal{M}\phi^{-1})^k\right\|_2$. The quantity $\rho_{\mathcal{M},\epsilon}$ (and hence $\epsilon^*(\mathcal{M})$) may be efficiently computed using the methods described by Wright and Trefethen (2001), and implemented in their EigTool toolkit[66]. Thus, given $\mathcal{M}$ and ϕ, $\mathcal{C}_{\mathcal{M},\phi}$ may be calculated in finitely many operations by iterating over $K \in \mathbb{N}$ until a K is found which satisfies:

$$\sup_{k=0,\dots,K}\left\|(\mathcal{M}\phi^{-1})^k\right\|_2 \geq \frac{\left\|(\mathcal{M}\phi^{-1})^K\right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})}.$$

From the definition of $\mathcal{C}_{\mathcal{M},\phi}$, we have that for any $k \in \mathbb{N}$ and any $\phi \in (\rho_{\mathcal{M}}, 1)$:

$$\left\|\mathcal{M}^k\right\|_2 \leq \mathcal{C}_{\mathcal{M},\phi}\phi^k.$$

Now, fix $\phi_F \in (\rho_F, 1)$ and $\phi_G \in (\rho_G, 1)$,[67] and define:

$$\mathcal{D}_{\phi_F,\phi_G} := \mathcal{C}_{F,\phi_F}\mathcal{C}_{G,\phi_F}\left\|(B+CF)^{-1}\right\|_2,$$

then, for all $t, k \in \mathbb{N}^+$:

$$\left|M_{t,k}\right| = \left|(x_t^* - \mu^*)_{1,k}\right| \leq \left\|(x_t^* - \mu^*)_{\cdot,k}\right\|_2 \leq \sum_{j=1}^{\min\{t,k\}} \left\|F^{t-j}\right\|_2 \left\|(B+CF)^{-1}\right\|_2 \left\|G^{k-j}\right\|_2$$

$$\leq \mathcal{D}_{\phi_F,\phi_G} \sum_{j=1}^{\min\{t,k\}} \phi_F^{t-j}\phi_G^{k-j} = \mathcal{D}_{\phi_F,\phi_G}\phi_F^t\phi_G^k \frac{(\phi_F\phi_G)^{-\min\{t,k\}} - 1}{1 - \phi_F\phi_G}.$$

Additionally, for all $t \in \mathbb{N}^+, k \in \mathbb{Z}$:

$$\begin{aligned}\left|M_{t,t+k} - \lim_{\tau\to\infty} M_{\tau,\tau+k}\right| &= \left|\left(L^{-t}(x_t^* - \mu^*)\right)_{1,k} - \left(\lim_{\tau\to\infty} L^{-t}(x_t^* - \mu^*)\right)_{1,k}\right| \\ &\leq \left\|\left(L^{-1}\sum_{j=0}^{t-1}(FL^{-1})^j s_1^* - L^{-1}\sum_{j=0}^{\infty}(FL^{-1})^j s_1^*\right)_{\cdot,k}\right\|_2 \\ &= \left\|\left(\sum_{j=\max\{t,-k\}}^{\infty} F^j s_{1,\cdot,j+k+1}^*\right)_{\cdot,0}\right\|_2\end{aligned}$$

[66] This toolkit is available from https://github.com/eigtool/eigtool, and is included in DynareOBC.

[67] In practice, we try a grid of values, as it is problem dependent whether high ϕ_F and low $\mathcal{C}_{F,\phi_F}$ is preferable to low ϕ_F and high $\mathcal{C}_{F,\phi_F}$.

$$
\begin{aligned}
&= \left\| \sum_{j=\max\{t,-k\}}^{\infty} F^j (B+CF)^{-1} G^{j+k} I_{\cdot,1} \right\|_2 \\
&\le \sum_{j=\max\{t,-k\}}^{\infty} \left\|F^j\right\|_2 \left\|(B+CF)^{-1}\right\|_2 \left\|G^{j+k}\right\|_2 \\
&\le \mathcal{D}_{\phi_F,\phi_G} \sum_{j=\max\{t,-k\}}^{\infty} \phi_F^j \phi_G^{j+k} = \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{\max\{t,-k\}} \phi_G^{\max\{0,t+k\}}}{1-\phi_F\phi_G},
\end{aligned}
$$

so, for all $t,k \in \mathbb{N}^+$:

$$
\left| M_{t,k} - \lim_{\tau\to\infty} M_{\tau,\tau+k-t} \right| \le \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G}.
$$

To evaluate $\lim_{\tau\to\infty} M_{\tau,\tau+k-t}$, note that this limit is the top element from the $(k-t)^{\text{th}}$ column of:

$$
\begin{aligned}
d &:= \lim_{\tau\to\infty} L^{-\tau}(x_\tau^* - \mu^*) = L^{-1}(I - FL^{-1})^{-1} s_1^* \\
&= -(I - FL^{-1})^{-1}(B+CF)^{-1}(I-GL)^{-1} I_{\cdot,1} I_{0,-\infty:\infty},
\end{aligned}
$$

where $I_{0,-\infty:\infty} \in \mathbb{R}^{1\times\mathbb{Z}}$ is zero everywhere apart from index 0 where it equals 1. Hence, by the definitions of F and G:

$$
AL^{-1}d + Bd + CLd = -I_{\cdot,1} I_{0,-\infty:\infty}.
$$

In other words, if we write d_k in place of $d_{\cdot,k}$ for convenience, then, for all $k \in \mathbb{Z}$:

$$
Ad_{k+1} + Bd_k + Cd_{k-1} = -\begin{cases} I_{\cdot,1} & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}
$$

I.e. the homogeneous part of the difference equation for d_{-t} is identical to that of $x_t - \mu$. The time reversal here is intuitive since we are indexing diagonals such that indices increase as we move up and to the right in M, but time is increasing as we move down in M.

Exploiting the possibility of reversing time is the key to easy evaluating d_k. First, note that for $k < 0$, it must be the case that $d_k = Fd_{k+1}$, since the shock has already "occurred" (remember, that we are going forwards in "time" when we reduce k). Likewise, since $d_k \to 0$ as $k \to \infty$, as we have already proved that the first row of M converges to zero, by Lemma 4, it must be the case that $d_k = Hd_{k-1}$, for all $k > 0$, where $H = -(B + AH)^{-1}C$, and $\phi_H < 1$.

It just remains to determine the value of d_0. By the previous results, this must satisfy:

$$-I_{\cdot,1} = Ad_1 + Bd_0 + Cd_{-1} = (AH + B + CF)d_0.$$

Hence:

$$d_0 = -(AH + B + CF)^{-1} I_{\cdot,1}.$$

This gives a readily computed solution for the limits of the diagonals of M. Lastly, note that:

$$|d_{-t,1}| \le \|d_{-t}\|_2 = \|F^t d_0\|_2 \le \|F^t\|_2 \|d_0\|_2 \le \mathcal{C}_{F,\phi_F} \phi_F^t \|d_0\|_2,$$

and:

$$|d_{t,1}| \le \|d_t\|_2 = \|H^t d_0\|_2 \le \|H^t\|_2 \|d_0\|_2 \le \mathcal{C}_{H,\phi_H} \phi_H^t \|d_0\|_2.$$

We will use these results in producing our bounds on ς.

First, fix $T \in \mathbb{N}^+$, and define a new matrix $\underline{M}^{(T)} \in \mathbb{R}^{\mathbb{N}^+ \times \mathbb{N}^+}$ by $\underline{M}^{(T)}_{1:T,1:T} = M_{1:T,1:T}$, and for all $t, k \in \mathbb{N}^+$, with $\min\{t, k\} > T$, $\underline{M}^{(T)}_{t,k} = d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G}$, then:

$$\begin{aligned}
\varsigma &\ge \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} \begin{bmatrix} y \\ y_\infty 1_{\infty \times 1} \end{bmatrix} \ge \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \inf_{t \in \mathbb{N}^+} \underline{M}^{(T)}_{t,1:\infty} \begin{bmatrix} y \\ y_\infty 1_{\infty \times 1} \end{bmatrix} \\
&= \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{gathered} \min_{t=1,\dots,T} \left[M_{t,1:T} y + \sum_{k=T+1}^{\infty} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_\infty \right], \\
\inf_{t \in \mathbb{N}^+, t > T} \left[\sum_{k=1}^{T} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_k + \sum_{k=T+1}^{\infty} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_\infty \right] \end{gathered} \right\} \\
&\ge \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{gathered} \min_{t=1,\dots,T} \left[M_{t,1:T} y + \left((I - H)^{-1} d_{T+1-t} \right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1 - \phi_F \phi_G)(1 - \phi_G)} y_\infty \right], \\
\min_{t=T+1,\dots,2T} \left[\begin{gathered} \sum_{k=1}^{T} \left(d_{-(t-k),1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_k + \left((I - F)^{-1} (d_{-1} - d_{-(t-T)}) \right)_1 y_\infty \\
+ \left((I - H)^{-1} d_0 \right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1 - \phi_F \phi_G)(1 - \phi_G)} y_\infty \end{gathered} \right], \\
\inf_{t \in \mathbb{N}^+, t > 2T} \left[\begin{gathered} \sum_{k=1}^{T} d_{-(t-k),1} y_k + \left((I - F)^{-1} (d_{-1} - d_{-(t-T)}) \right)_1 y_\infty \\
+ \left((I - H)^{-1} d_0 \right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{2T+1} \phi_G}{(1 - \phi_F \phi_G)(1 - \phi_G)} \end{gathered} \right] \end{gathered} \right\}.
\end{aligned}$$

Now, for $t \ge T$:

$$\begin{aligned}
\left| \left((I - F)^{-1} d_{-(t-T)} \right)_1 \right| &\le \left\| (I - F)^{-1} d_{-(t-T)} \right\|_2 \le \left\| (I - F)^{-1} \right\|_2 \left\| d_{-(t-T)} \right\|_2 \\
&\le \mathcal{C}_{F,\phi_F} \phi_F^{t-T} \left\| (I - F)^{-1} \right\|_2 \|d_0\|_2,
\end{aligned}$$

so:

$$\begin{aligned}
&\sum_{k=1}^{T} d_{-(t-k),1} y_k - ((I-F)^{-1} d_{-(t-T)})_1 y_\infty \\
&\qquad \geq -\sum_{k=1}^{T} C_{F,\phi_F} \phi_F^{t-k} \|d_0\|_2 - C_{F,\phi_F} \phi_F^{t-T} \left\|(I-F)^{-1}\right\|_2 \|d_0\|_2 y_\infty \\
&\qquad = -C_{F,\phi_F} \frac{\phi_F^t(\phi_F^{-T}-1)}{1-\phi_F} \|d_0\|_2 - C_{F,\phi_F} \phi_F^{t-T} \left\|(I-F)^{-1}\right\|_2 \|d_0\|_2 y_\infty .
\end{aligned}$$

Thus $\varsigma \geq \underline{\varsigma}_T$, where:

$$\underline{\varsigma}_T := \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{gathered}
\min_{t=1,\dots,T} \left[M_{t,1:T} y + \left((I-H)^{-1} d_{T+1-t}\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \right], \\
\min_{t=T+1,\dots,2T} \left[\begin{gathered} \sum_{k=1}^{T} \left(d_{-(t-k),1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_k + \left((I-F)^{-1}(d_{-1} - d_{-(t-T)})\right)_1 y_\infty \\ + \left((I-H)^{-1} d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \end{gathered} \right], \\
\left[\begin{gathered} -C_{F,\phi_F} \frac{\phi_F^{2T+1}(\phi_F^{-T}-1)}{1-\phi_F} \|d_0\|_2 - C_{F,\phi_F} \phi_F^{T+1} \left\|(I-F)^{-1}\right\|_2 \|d_0\|_2 y_\infty + \left((I-F)^{-1} d_{-1}\right)_1 y_\infty \\ + \left((I-H)^{-1} d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{2T+1} \phi_G}{(1-\phi_F\phi_G)(1-\phi_G)} \end{gathered} \right]
\end{gathered} \right\}.$$

The final minimand in this expression is less than (but converges to):

$$((I-F)^{-1} d_{-1})_1 y_\infty + ((I-H)^{-1} d_0)_1 y_\infty,$$

i.e. a weakly positive multiple of the doubly infinite sum of $d_{1,k}$ over all $k \in \mathbb{Z}$. If this expression is negative, then the optimum will have $y_\infty = 0$ giving (uninformatively) $\underline{\varsigma}_T \leq 0$.

To construct an upper bound on ς, fix $T \in \mathbb{N}^+$, and define a new matrix $\overline{M}^{(T)} \in \mathbb{R}^{\mathbb{N}^+ \times \mathbb{N}^+}$ by $\overline{M}^{(T)}_{1:T,1:T} = M_{1:T,1:T}$, and for all $t,k \in \mathbb{N}^+$, with $\min\{t,k\} > T$, $\overline{M}^{(T)}_{t,k} = |d_{k-t,1}| + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G}$. Then:

$$\begin{aligned}
\varsigma &= \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y \leq \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} \overline{M}^{(T)}_{t,1:\infty} y \leq \sup_{y \in [0,1]^{\mathbb{N}^+}} \min_{t=1,\dots,T} \overline{M}^{(T)}_{t,1:\infty} y \\
&\leq \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \overline{M}^{(T)}_{t,1:\infty} \begin{bmatrix} y \\ 1_{\infty \times 1} \end{bmatrix} \\
&\leq \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T} y + \sum_{k=T+1}^{\infty} |d_{k-t,1}| + \sum_{k=T+1}^{\infty} \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right] \\
&\leq \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T} y + \sum_{k=T+1-t}^{\infty} |d_{k,1}| + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{1-\phi_F\phi_G} \sum_{k=0}^{\infty} \phi_G^k \right]
\end{aligned}$$

$$\leq \max_{y\in[0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T}y + \mathcal{C}_{H,\phi_H}\|d_0\|_2 \phi_H^{T+1-t} \sum_{k=0}^{\infty} \phi_H^k + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} \right]$$

$$= \overline{\varsigma}_T := \max_{y\in[0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T}y + \frac{\mathcal{C}_{H,\phi_H}\|d_0\|_2 \phi_H^{T+1-t}}{1-\phi_H} + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} \right].$$

Note that if $M_{1:T,1:T}$ is an S-matrix, $\overline{\varsigma}_T > 0$.

Appendix L.2: Proof of Proposition 3

Consider the model:

$$a_t = \max\{0, b_t\}, \qquad a_t = 1 - c_t, \qquad c_t = a_t - b_t.$$

The model has steady state $a = b = 1$, $c = 0$. Furthermore, in the model's Problem 3 (News) type equivalent, in which for $t \in \mathbb{N}^+$:

$$a_t = \begin{cases} b_t + y_{t,0} & \text{if } t \leq T \\ b_t & \text{if } t > T' \end{cases}$$

where $y_{\cdot,\cdot}$ is defined as in Problem 3 (News), we have that:

$$c_t = \begin{cases} y_{t,0} & \text{if } t \leq T \\ 0 & \text{if } t > T' \end{cases}$$

so:

$$b_t = \begin{cases} 1 - 2y_{t,0} & \text{if } t \leq T \\ 1 & \text{if } t > T' \end{cases}$$

implying:

$$a_t = \begin{cases} 1 - y_{t,0} & \text{if } t \leq T \\ 1 & \text{if } t > T \end{cases}.$$

thus, $M = -I$ for this model.

Appendix L.3: Proof of Proposition 12

Defining $x_t = [x_{i,t} \quad x_{y,t} \quad x_{\pi,t}]'$, the BPY model is in the form of Problem 2 (OBC), with:

$$A := \begin{bmatrix} 0 & -\alpha_{\Delta y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\dfrac{1}{\sigma} & -1 & 0 \\ 0 & \gamma & -1 \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \dfrac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A+B+C)=\det\begin{bmatrix}-1 & 0 & \alpha_\pi\\ -\dfrac{1}{\sigma} & 0 & \dfrac{1}{\sigma}\\ 0 & \gamma & -1\end{bmatrix}\neq 0$$

as $\alpha_\pi\neq 1$ and $\gamma\neq 0$. Let $f:=F_{2,2}$, where F is as in Assumption 1. Then:

$$F=\begin{bmatrix}0 & \alpha_{\Delta y}(f-1)+\alpha_\pi\dfrac{\gamma f}{1-\beta f} & 0\\ 0 & f & 0\\ 0 & \dfrac{\gamma f}{1-\beta f} & 0\end{bmatrix}.$$

Hence:

$$f=f^2-\frac{1}{\sigma}\left(\alpha_{\Delta y}(f-1)+\alpha_\pi\frac{\gamma f}{1-\beta f}-\frac{\gamma f^2}{1-\beta f}\right),$$

i.e.:

$$\beta\sigma f^3-\Big(\big(\alpha_{\Delta y}+\sigma\big)\beta+\gamma+\sigma\Big)f^2+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)f-\alpha_{\Delta y}=0. \qquad (21)$$

When $f\leq 0$, the left-hand side is negative, and when $f=1$, the left-hand side equals $(\alpha_\pi-1)\gamma>0$ (by assumption on α_π), hence equation (21) has either one or three solutions in $(0,1)$, and no solutions in $(-\infty,0]$. We wish to prove there is a unique solution in $(-1,1)$. First note that when $\alpha_\pi=1$, the discriminant of the polynomial is:

$$\Big((1-\beta)\big(\alpha_{\Delta y}-\sigma\big)-\gamma\Big)^2\Big(\big(\beta\alpha_{\Delta y}\big)^2+2\beta(\gamma-\sigma)\alpha_{\Delta y}+(\gamma+\sigma)^2\Big).$$

The first multiplicand is positive. The second is minimised when $\sigma=\beta\alpha_{\Delta y}-\gamma$, at the value $4\beta\gamma\alpha_{\Delta y}>0$, hence this multiplicand is positive too. Consequently, at least for small α_π, there are three real solutions for f, so there may be multiple solutions in $(0,1)$.

Suppose for a contradiction that there were at least three solutions to equation (21) in $(0,1)$ (double counting repeated roots), even for arbitrary large $\beta\in(0,1)$. Let $f_1,f_2,f_3\in(0,1)$ be the three roots. Then, by Vieta's formulas:

$$3>f_1+f_2+f_3=\frac{\big(\alpha_{\Delta y}+\sigma\big)\beta+\gamma+\sigma}{\beta\sigma},$$

$$3 > f_1 f_2 + f_1 f_3 + f_2 f_3 = \frac{(1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma}{\beta\sigma},$$

$$1 > f_1 f_2 f_3 = \frac{\alpha_{\Delta y}}{\beta\sigma},$$

so:

$$(2\beta - 1)\sigma > \beta\alpha_{\Delta y} + \gamma > \gamma > 0$$

$$\beta > \frac{1}{2}, \qquad (2\beta-1)\sigma > \gamma,$$

$$\beta\sigma > \beta\alpha_{\Delta y} + \gamma + \sigma(1-\beta),$$

$$2\beta\sigma > (1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma(1-\beta),$$

$$\beta\sigma > \alpha_{\Delta y}.$$

Also, the first derivative of equation (21) must be positive at $f = 1$, so:

$$(1-\beta)\left(\alpha_{\Delta y} - \sigma\right) + (\alpha_\pi - 2)\gamma > 0.$$

Combining these inequalities gives the bounds:

$$0 < \alpha_{\Delta y} < 2\sigma - \frac{\gamma + \sigma}{\beta},$$

$$2 + \frac{(1-\beta)\left(\sigma - \alpha_{\Delta y}\right)}{\gamma} < \alpha_\pi < \frac{(3\beta - 1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma}.$$

Furthermore, if there are multiple solutions to equation (21), then the discriminant of its first derivative must be nonnegative, i.e.:

$$\left(\left(\alpha_{\Delta y} + \sigma\right)\beta + \gamma + \sigma\right)^2 - 3\beta\sigma\left((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\right) \geq 0.$$

Therefore, we have the following bounds on α_π:

$$2 + \frac{(1-\beta)\left(\sigma - \alpha_{\Delta y}\right)}{\gamma} < \alpha_\pi \leq \frac{\left(\left(\alpha_{\Delta y} + \sigma\right)\beta + \gamma + \sigma\right)^2 - 3\beta\sigma\left((1+\beta)\alpha_{\Delta y} + \sigma\right)}{3\beta\sigma\gamma}$$

since,

$$\frac{(3\beta - 1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma} - \frac{\left(\left(\alpha_{\Delta y} + \sigma\right)\beta + \gamma + \sigma\right)^2 - 3\beta\sigma\left((1+\beta)\alpha_{\Delta y} + \sigma\right)}{3\beta\sigma\gamma}$$

$$= \frac{\left((2\sigma - \alpha_{\Delta y})\beta - \gamma - \sigma\right)\left((4\sigma + \alpha_{\Delta y})\beta + \gamma + \sigma\right)}{3\beta\gamma\sigma} > 0$$

as $\alpha_{\Delta y} < 2\sigma - \frac{\gamma+\sigma}{\beta}$. Consequently, there exists $\lambda, \mu, \kappa \in [0,1]$ such that:

$$\alpha_\pi = (1-\lambda)\left[2+\frac{(1-\beta)(\sigma-\alpha_{\Delta y})}{\gamma}\right]$$
$$+\lambda\left[\frac{\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)^2-3\beta\sigma\Big((1+\beta)\alpha_{\Delta y}+\sigma\Big)}{3\beta\sigma\gamma}\right],$$
$$\alpha_{\Delta y} = (1-\mu)[0]+\mu\left[2\sigma-\frac{\gamma+\sigma}{\beta}\right],$$
$$\gamma = (1-\kappa)[0]+\kappa[(2\beta-1)\sigma]$$

These simultaneous equations have unique solutions for α_π, $\alpha_{\Delta y}$ and γ in terms of λ, μ and κ. Substituting these solutions into the discriminant of equation (21) gives a polynomial in $\lambda,\mu,\kappa,\beta,\sigma$. As such, an exact global maximum of the discriminant may be found subject to the constraints $\lambda,\mu,\kappa\in[0,1]$, $\beta\in\left[\frac{1}{2},1\right]$, $\sigma\in[0,\infty)$, by using an exact compact polynomial optimisation solver, such as that in the Maple computer algebra package. Doing this gives a maximum of 0 when $\beta\in\left\{\frac{1}{2},1\right\}$, $\kappa=1$ and $\sigma=0$. But of course, we actually require that $\beta\in\left(\frac{1}{2},1\right)$, $\kappa<1$, $\sigma>0$. Thus, by continuity, the discriminant is negative over the entire domain. This gives the required contradiction to our assumption of three roots to the polynomial, establishing that Assumption 1 holds for this model.

Now, when $T=1$, M is equal to the top left element of the matrix $-(B+CF)^{-1}$, i.e.:

$$M=\frac{\beta\sigma f^2-\big((1+\beta)\sigma+\gamma\big)f+\sigma}{\beta\sigma f^2-\Big((1+\beta)\sigma+\gamma+\beta\alpha_{\Delta y}\Big)f+\sigma+\alpha_{\Delta y}+\gamma\alpha_\pi}.$$

Now, multiplying the denominator by f gives:

$$\beta\sigma f^3-\Big((1+\beta)\sigma+\gamma+\beta\alpha_{\Delta y}\Big)f^2+\big(\sigma+\alpha_{\Delta y}+\gamma\alpha_\pi\big)f$$
$$=\left[\beta\sigma f^3-\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f^2+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)f-\alpha_{\Delta y}\right]$$
$$-\left[\beta\alpha_{\Delta y}f-\alpha_{\Delta y}\right]=(1-\beta f)\alpha_{\Delta y}>0,$$

by equation (21). Hence, the sign of M is that of $\beta\sigma f^2-\big((1+\beta)\sigma+\gamma\big)f+\sigma$. I.e., M is negative if and only if:

$$\frac{((1+\beta)\sigma+\gamma)-\sqrt{((1+\beta)\sigma+\gamma)^2-4\beta\sigma^2}}{2\beta\sigma}<f$$
$$<\frac{((1+\beta)\sigma+\gamma)+\sqrt{((1+\beta)\sigma+\gamma)^2-4\beta\sigma^2}}{2\beta\sigma}.$$

The upper limit is greater than 1, so only the lower is relevant. To translate this bound on f into a bound on $\alpha_{\Delta y}$, we first need to establish that f is monotonic in $\alpha_{\Delta y}$.

Totally differentiating equation (21) gives:

$$\left[3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)\right]\frac{df}{d\alpha_{\Delta y}}=(1-\beta f)(1-f)$$
$$>0.$$

Thus, the sign of $\frac{df}{d\alpha_{\Delta y}}$ is equal to that of:

$$3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big).$$

Note, however, that this expression is just the derivative of the left-hand side of equation (21) with respect to f.

To establish the sign of $\frac{df}{d\alpha_{\Delta y}}$, we consider two cases. First, suppose that equation (21) has three real solutions. Then, the unique solution to equation (21) in (0,1) is its lowest solution. Hence, this solution must be below the first local maximum of the left-hand side of equation (19). Consequently, at the $f\in(0,1)$, which solves equation (21), $3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)>0$. Alternatively, suppose that equation (21) has a unique real solution. Then the left-hand side of this equation cannot change sign in between its local maximum and its local minimum (if it has any). Thus, at the $f\in(0,1)$ at which it changes sign, we must have that $3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)>0$. Therefore, in either case $\frac{df}{d\alpha_{\Delta y}}>0$, meaning that f is monotonic increasing in $\alpha_{\Delta y}$.

Consequently, to find the critical $(f, \alpha_{\Delta y})$ at which M changes sign, it is sufficient to find the lowest solution with respect to both f and $\alpha_{\Delta y}$ of the pair of equations:

$$\beta\sigma f^2 - \left((1+\beta)\sigma + \gamma\right)f + \sigma = 0,$$

$$\beta\sigma f^3 - \Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)f - \alpha_{\Delta y} = 0.$$

The former implies that:

$$\beta\sigma f^3 - \left((1+\beta)\sigma + \gamma\right)f^2 + \sigma f = 0,$$

so, by the latter:

$$\alpha_{\Delta y}\beta f^2 - \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi\Big)f + \alpha_{\Delta y} = 0.$$

If $\alpha_{\Delta y} = \sigma\alpha_\pi$, then this equation holds if and only if:

$$\sigma\beta f^2 - \left((1+\beta)\sigma + \gamma\right)f + \sigma = 0.$$

Therefore, the critical $(f, \alpha_{\Delta y})$ at which M changes sign are given by:

$$\alpha_{\Delta y} = \sigma\alpha_\pi,$$

$$f = \frac{\left((1+\beta)\sigma + \gamma\right) - \sqrt{\left((1+\beta)\sigma + \gamma\right)^2 - 4\beta\sigma^2}}{2\beta\sigma}.$$

Thus, M is negative if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$, and M is zero if and only if $\alpha_{\Delta y} = \sigma\alpha_\pi$.

Appendix L.4: Proof of Proposition 13

Defining $x_t = [x_{i,t} \quad x_{y,t} \quad x_{p,t}]'$, the price targeting model from Appendix F.3 is in the form of Problem 2 (OBC), with:

$$A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & -1 & -\frac{1}{\sigma} \\ 0 & \gamma & -1-\beta \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A + B + C) = \det \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & 0 & 0 \\ 0 & \gamma & -1 \end{bmatrix} \neq 0$$

as $\alpha_{\Delta y} \neq 0$ and $\alpha_\pi \neq 0$. Let $f := F_{3,3}$, where F is as in Assumption 1.

Then:

$$F = \begin{bmatrix} 0 & 0 & \dfrac{f(1-f)(\sigma\alpha_\pi - \alpha_{\Delta y})}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & \dfrac{f(1-f-\alpha_\pi)}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & f \end{bmatrix},$$

and so:

$$\beta\sigma f^3 - \Big((1+2\beta)\sigma + \beta\alpha_{\Delta y} + \gamma\Big)f^2 + \Big((2+\beta)\sigma + (1+\beta)\alpha_{\Delta y} + (1+\alpha_\pi)\gamma\Big)f \\ - (\sigma + \alpha_{\Delta y}) = 0.$$

Now define:

$$\hat{\alpha}_{\Delta y} := \sigma + \alpha_{\Delta y}, \qquad \hat{\alpha}_\pi := 1 + \alpha_\pi$$

so:

$$\beta\sigma f^3 - \Big((\hat{\alpha}_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\hat{\alpha}_{\Delta y} + \gamma\hat{\alpha}_\pi + \sigma\Big)f - \hat{\alpha}_{\Delta y} = 0.$$

This is identical to the equation for f in Appendix L.3, apart from the fact that $\hat{\alpha}_{\Delta y}$ has replaced $\alpha_{\Delta y}$ and $\hat{\alpha}_\pi$ has replaced α_π. Hence, by the results of Appendix L.3, Assumption 1 holds for this model as well.

Finally, for this model, with $T = 1$, we have that:

$$M = \frac{(1-f)(1+(1-f)\beta)\sigma^2 + \Big((1+(1-f)\beta)\alpha_{\Delta y} + \big((1-f) + \alpha_\pi f\big)\gamma\Big)\sigma + (1-f)\gamma\alpha_{\Delta y}}{\Big((1-f)(1+(1-f)\beta)\sigma + (1+(1-f)\beta)\alpha_{\Delta y} + \big((1-f) + \alpha_\pi\big)\gamma\Big)(\sigma + \alpha_{\Delta y})} > 0.$$

Appendix L.5: Proof of Proposition 14

If $\widetilde{X}$ is compact, then Γ is compact valued. Furthermore, $\widetilde{X}$ is clearly convex, and Γ is continuous. Thus assumption 4.3 of Stokey, Lucas, and Prescott (1989) (henceforth: SLP) is satisfied. Since the continuous image of a compact set is compact, $\widetilde{\mathcal{F}}$ is bounded above and below, so assumption 4.4 of SLP is satisfied as well. Furthermore, $\widetilde{\mathcal{F}}$ is concave and Γ is convex, so assumptions 4.7 and 4.8 of SLP are satisfied too. Thus, by Theorem 4.6 of SLP, with $\mathcal{B}$ defined as in equation (22) and v^* defined as in equation (23), $\mathcal{B}$ has a unique fixed point which is continuous and equal to v^*. Moreover, by Theorem 4.8 of SLP, there is a unique policy function which attains the supremum in the definition of $\mathcal{B}(v^*) = v^*$.

Now suppose that $\widetilde{X}$ is possibly non-compact, but $\tilde{\Gamma}(x)$ is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \widetilde{X}$. We first note that for all $x, z \in \widetilde{X}$:

$$\widetilde{\mathcal{F}}(x,z) \leq u^{(0)} - \frac{1}{2} u^{(1)} \tilde{u}^{(2)^{-1}} u^{(1)\prime},$$

thus, our objective function is bounded above without additional assumptions. For a lower bound, we assume that for all $x \in \widetilde{X}$, $x \in \tilde{\Gamma}(x)$, so holding the state fixed is always feasible. This is true in very many standard applications. Then, the value of setting $x_t = x_0$ for all $t \in \mathbb{N}^+$ provides a lower bound for our objective function.

More precisely, we define $\mathbb{V} := \{v | v: \widetilde{X} \to [-\infty, \infty)\}$ and $\underline{v}, \overline{v} \in \mathbb{V}$ by:

$$\underline{v}(x) = \frac{1}{1-\beta} \widetilde{\mathcal{F}}(x_0, x_0),$$

$$\overline{v}(x) = \frac{1}{1-\beta} \left[u^{(0)} - \frac{1}{2} u^{(1)} \tilde{u}^{(2)^{-1}} u^{(1)\prime} \right],$$

for all $x \in \widetilde{X}$.

Finally, define $\mathcal{B}: \mathbb{V} \to \mathbb{V}$ by:

$$\mathcal{B}(v)(x) = \sup_{z \in \tilde{\Gamma}(x)} \left[\widetilde{\mathcal{F}}(x,z) + \beta v(z) \right] \tag{22}$$

for all $v \in \mathbb{V}$ and for all $x \in \widetilde{X}$. Then $\mathcal{B}(\underline{v}) \geq \underline{v}$ and $\mathcal{B}(\overline{v}) \leq \overline{v}$. Furthermore, if some sequence $(x_t)_{t=1}^{\infty}$ satisfies the constraint that for all $t \in \mathbb{N}^+$, $x_t \in \tilde{\Gamma}(x_{t-1})$, and the objective in (8) is finite for that sequence, then it must be the case that $\|x_t\|_\infty t \beta^{\frac{t}{2}} \to 0$ as $t \to \infty$ (by the comparison test), so:

$$\liminf_{t \to \infty} \beta^t \underline{v}(x_t) = 0.$$

Additionally, for any sequence $(x_t)_{t=1}^{\infty}$:

$$\limsup_{t \to \infty} \beta^t \overline{v}(x_t) = 0.$$

Thus, our dynamic programming problem satisfies the assumptions of Theorem 2.1 of Kamihigashi (2014), and so $\mathcal{B}$ has a unique fixed point in $[\underline{v}, \overline{v}]$ to which $\mathcal{B}^k(\underline{v})$ converges pointwise, monotonically, as $k \to \infty$, and which is equal to the function $v^*: \widetilde{X} \to \mathbb{R}$ defined by:

$$v^*(x_0) = \sup \left\{ \textstyle\sum_{t=1}^{\infty} \beta^{t-1} \widetilde{\mathcal{F}}(x_{t-1}, x_t) \,\middle|\, \forall t \in \mathbb{N}^+, x_t \in \Gamma(x_{t-1}) \right\}, \tag{23}$$

for all $x_0 \in \widetilde{X}$.

Furthermore, if we define:

$$\mathbb{W} := \{v \in V | v \text{ is continuous on } \widetilde{X}, v \text{ is concave on } \widetilde{X}\},$$

then as $\tilde{u}^{(2)}$ is negative-definite, $\underline{v} \in \mathbb{W}$. Additionally, under the assumption that $\tilde{\Gamma}(x)$ is compact valued, if $v \in \mathbb{W}$, then $\mathcal{B}(v) \in \mathbb{W}$, by the Theorem of the Maximum,[68] and, furthermore, there is a unique policy function which attains the supremum in the definition of $\mathcal{B}(v)$. Moreover, $v^* = \lim_{k\to\infty} \mathcal{B}^k(\underline{v})$ is concave and lower semi-continuous on $\widetilde{X}$.[69] We just need to prove that v^* is upper semi-continuous.[70] Suppose for a contradiction that it is not, so there exists $x^* \in \widetilde{X}$ such that:

$$\limsup_{x\to x^*} v^*(x) > \lim_{k\to\infty} v^*(x^*).$$

Then, there exists $\delta > 0$ such that for all $\epsilon > 0$, there exists $x_0^{(\epsilon)} \in \widetilde{X}$ with $\|x^* - x_0^{(\epsilon)}\|_\infty < \epsilon$ such that:

$$v^*\big(x_0^{(\epsilon)}\big) > \delta + v^*(x^*).$$

Now, by the definition of a supremum, for all $\epsilon > 0$, there exists $\big(x_t^{(\epsilon)}\big)_{t=1}^\infty$ such that for all $t \in \mathbb{N}^+$, $x_t^{(\epsilon)} \in \Gamma\big(x_{t-1}^{(\epsilon)}\big)$ and:

$$v^*\big(x_0^{(\epsilon)}\big) < \delta + \sum_{t=1}^\infty \beta^{t-1} \tilde{\mathcal{F}}\big(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}\big).$$

Hence:

$$\sum_{t=1}^\infty \beta^{t-1} \tilde{\mathcal{F}}\big(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}\big) > v^*\big(x_0^{(\epsilon)}\big) - \delta > v^*(x^*).$$

Now, let $\mathcal{S}_0 := \{x \in \widetilde{X} | \|x^* - x\|_\infty \le 1\}$, and for $t \in \mathbb{N}^+$, let $\mathcal{S}_t := \Gamma(\mathcal{S}_{t-1})$. Then, since we are assuming Γ is compact valued, for all $t \in \mathbb{N}$, $\mathcal{S}_t$ is compact by the continuity of Γ. Furthermore, for all $t \in \mathbb{N}$ and $\epsilon \in (0,1)$, $x_t^{(\epsilon)} \in \mathcal{S}_t$. Hence, $\prod_{t=0}^\infty \mathcal{S}_t$ is sequentially compact in the product topology. Thus, there exists a sequence $(\epsilon_k)_{k=1}^\infty$ with $\epsilon_k \to 0$ as $k \to \infty$ and such that $x_t^{(\epsilon_k)}$ converges for all $t \in \mathbb{N}$. Let $x_t := \lim_{k\to\infty} x_t^{(\epsilon_k)}$, and note that

[68] See e.g. Theorem 3.6 and following of Stokey, Lucas, and Prescott (1989).
[69] See e.g. Lemma 2.41 of Aliprantis and Border (2013).
[70] In the following, we broadly follow the proof of Lemma 3.3 of Kamihigashi and Roy (2003).

$x^* = x_0 \in \mathcal{S}_0 \subseteq \widetilde{X}$, and that for all $t, k \in \mathbb{N}^+$, $x_t^{(\epsilon_k)} \in \Gamma\left(x_{t-1}^{(\epsilon_k)}\right)$, so by the continuity of Γ, $x_t \in \Gamma(x_{t-1})$ for all $t \in \mathbb{N}^+$. Thus, by Fatou's Lemma:

$$v^*(x^*) \geq \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \geq \limsup_{k \to \infty} \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}\left(x_{t-1}^{(\epsilon,k)}, x_t^{(\epsilon,k)}\right) > v^*(x^*),$$

which gives the required contradiction. Thus, v^* is continuous and concave, and there is a unique policy function attaining the supremum in the definition of $\mathcal{B}(v^*) = v^*$.

Appendix L.6: Proof of Proposition 15

Suppose that $(x_t)_{t=1}^{\infty}$, $(\lambda_t)_{t=1}^{\infty}$ satisfy the KKT conditions given in equations (15) and (16), and that $x_t \to \mu$ and $\lambda_t \to \overline{\lambda}$ as $t \to \infty$. Let $(z_t)_{t=0}^{\infty}$ satisfy $z_0 = x_0$ and $z_t \in \tilde{\Gamma}(z_{t-1})$ for all $t \in \mathbb{N}^+$. Then, by the KKT conditions and the concavity of:

$$(x_{t-1}, x_t) \mapsto \tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right],$$

we have that for all $T \in \mathbb{N}^+$:[71]

$$\begin{aligned}
&\sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \\
&= \sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \\
&\geq \sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right. \\
&\qquad\qquad \left. - \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} z_{t-1} - \mu \\ z_t - \mu \end{bmatrix} \right] \right] \\
&\geq \sum_{t=1}^{T} \beta^{t-1} \left[\left[u_{\cdot,2}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,2}^{(2)} + \lambda_t' \Psi_{\cdot,2}^{(1)} \right] (x_t - z_t) \right. \\
&\qquad\qquad \left. + \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_t' \Psi_{\cdot,1}^{(1)} \right] (x_{t-1} - z_{t-1}) \right]
\end{aligned}$$

[71] Here, we broadly follow the proof of Theorem 4.15 of Stokey, Lucas, and Prescott (1989).

$$= \sum_{t=1}^{T} \beta^{t-1} \left[\left[u_{\cdot,2}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,2}^{(2)} + \lambda_t' \Psi_{\cdot,2}^{(1)} \right.\right.$$

$$\left.\left. + \beta \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_t - \mu \\ x_{t+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{t+1}' \Psi_{\cdot,1}^{(1)} \right] \right] (x_t - z_t) \right]$$

$$+ \beta^T \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{T+1}' \Psi_{\cdot,1}^{(1)} \right] (z_T - x_T)$$

$$= \beta^T \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{T+1}' \Psi_{\cdot,1}^{(1)} \right] (z_T - x_T).$$

Thus:

$$\sum_{t=1}^{\infty} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right]$$

$$\geq \lim_{T\to\infty} \beta^T \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{T+1}' \Psi_{\cdot,1}^{(1)} \right] (z_T - x_T)$$

$$= \lim_{T\to\infty} \beta^T \left[u_{\cdot,1}^{(1)} + \bar{\lambda}' \Psi_{\cdot,1}^{(1)} \right] (z_T - \mu) = \lim_{T\to\infty} \beta^T \left[u_{\cdot,1}^{(1)} + \bar{\lambda}' \Psi_{\cdot,1}^{(1)} \right] z_T.$$

Now, suppose $\lim_{T\to\infty} \beta^T z_T \neq 0$, then since $\tilde{u}^{(2)}$ is negative definite:

$$\sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(z_{t-1}, z_t) = -\infty,$$

so $(z_t)_{t=0}^{\infty}$ cannot be optimal. Hence, regardless of the value of $\lim_{T\to\infty} \beta^T z_T$:

$$\sum_{t=1}^{\infty} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \geq 0,$$

which implies that $(x_t)_{t=1}^{\infty}$ solves Problem 4 (Linear-Quadratic).

References

Adão, Bernardino, Isabel Correia & Pedro Teles. 2011. 'Unique Monetary Equilibria with Interest Rate Rules'. *Review of Economic Dynamics* 14 (3) (July): 432–442.

Adjemian, Stéphane, Houtan Bastani, Fréderic Karamé, Michel Juillard, Junior Maih, Ferhat Mihoubi, George Perendia, Johannes Pfeifer, Marco Ratto & Sébastien Villemot. 2011. *Dynare: Reference Manual Version 4*. CEPREMAP.

Adjemian, Stéphane, Matthieu Darracq Pariès & Stéphane Moyen. 2007. *Optimal Monetary Policy in an Estimated DSGE for the Euro Area*. Working Paper Series. European Central Bank.

Adjemian, Stéphane & Michel Juillard. 2013. 'Stochastic Extended Path Approach'.

———. 2016. 'The Stochastic Extended Path Approach'. National Bank of Belgium.

Aliprantis, C.D. & K. Border. 2013. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Studies in Economic Theory. Springer Berlin Heidelberg.

Anderson, Gary. 2015. 'A Solution Strategy for Occasionally Binding Constraints'. Unpublished.

Armenter, Roc. 2016. *The Perils of Nominal Targets*. Federal Reserve Bank of Philadelphia.

Aruoba, S. Borağan, Pablo Cuba-Borda & Frank Schorfheide. 2018. 'Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries'. *The Review of Economic Studies* 85 (1) (January 1): 87–118.

Ascari, Guido & Tiziano Ropele. 2009. 'Trend Inflation, Taylor Principle, and Indeterminacy'. *Journal of Money, Credit and Banking* 41 (8). Journal of Money, Credit and Banking: 1557–1584.

Bai, Zhong-Zhi & D.J. Evans. 1997. 'Matrix Multisplitting Relaxation Methods for Linear Complementarity Problems'. *International Journal of Computer Mathematics* 63 (3–4) (January 1): 309–326.

Bassetto, Marco. 2004. 'Negative Nominal Interest Rates'. *American Economic Review* 94 (2): 104–108.

Basu, Susanto & Brent Bundick. 2015. 'Endogenous Volatility at the Zero Lower Bound: Implications for Stabilization Policy'. *National Bureau of Economic Research Working Paper Series* No. 21838.

Benhabib, Jess, Stephanie Schmitt-Grohé & Martín Uribe. 2001a. 'Monetary Policy and Multiple Equilibria'. *American Economic Review* 91 (1). American Economic Review: 167–186.

———. 2001b. 'The Perils of Taylor Rules'. *Journal of Economic Theory* 96 (1–2): 40–69.

Benigno, Pierpaolo & Michael Woodford. 2012. 'Linear-Quadratic Approximation of Optimal Policy Problems'. *Journal of Economic Theory* 147 (1). Journal of Economic Theory: 1–42.

Blanchard, Olivier Jean & Charles M. Kahn. 1980. 'The Solution of Linear Difference Models under Rational Expectations'. *Econometrica* 48 (5): 1305–1311.

Bodenstein, Martin. 2010. 'Trade Elasticity of Substitution and Equilibrium Dynamics'. *Journal of Economic Theory* 145 (3) (May): 1033–1059.

Boneva, Lena Mareen, R. Anton Braun & Yuichiro Waki. 2016. 'Some Unpleasant Properties of Loglinearized Solutions When the Nominal Rate Is Zero'. *Journal of Monetary Economics* 84 (December 1): 216–232.

Brendon, Charles, Matthias Paustian & Tony Yates. 2013. 'The Pitfalls of Speed-Limit Interest Rate Rules at the Zero Lower Bound'. Bank of England.

———. 2016. 'Self-Fulfilling Recessions at the Zero Lower Bound'.

Carlstrom, Charles T., Timothy S. Fuerst & Matthias Paustian. 2015. 'Inflation and Output in New Keynesian Models with a Transient Interest Rate Peg'. *Journal of Monetary Economics* 76 (November): 230–243.

Christiano, Lawrence J. & Martin Eichenbaum. 2012. 'Notes on Linear Approximations, Equilibrium Multiplicity and e-Learnability in the Analysis of the Zero Lower Bound'.

Christiano, Lawrence J. & Massimo Rostagno. 2001. *Money Growth Monitoring and the Taylor Rule*. Working Paper. National Bureau of Economic Research.

Christiano, Lawrence J. & Yuta Takahashi. 2018. 'Discouraging Deviant Behavior in Monetary Economics'. *National Bureau of Economic Research Working Paper Series* No. 24949.

Coibion, Olivier, Yuriy Gorodnichenko & Johannes Wieland. 2012. 'The Optimal Inflation Rate in New Keynesian Models: Should Central Banks Raise Their Inflation Targets in Light of the Zero Lower Bound?' *The Review of Economic Studies* 79 (4) (October 1): 1371–1406.

Cottle, Richard W., Jong-Shi Pang & Richard E. Stone. 2009a. '3. Existence and Multiplicity'. In *The Linear Complementarity Problem*, 137–223. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.

———. 2009b. '2. Background'. In *The Linear Complementarity Problem*, 43–136. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.

Cúrdia, Vasco, Marco Del Negro & Daniel L. Greenwald. 2014. 'Rare Shocks, Great Recessions'. *Journal of Applied Econometrics* 29 (7): 1031–1052.

Davig, Troy & Eric M. Leeper. 2007. 'Generalizing the Taylor Principle'. *American Economic Review* 97 (3): 607–635.

Del Negro, Marco, Marc Giannoni & Christina Patterson. 2015. *The Forward Guidance Puzzle*. Staff Reports. Federal Reserve Bank of New York.

Duarte, Fernando. 2016. *How to Escape a Liquidity Trap with Interest Rate Rules*. Staff Reports. Federal Reserve Bank of New York.

Eggertsson, Gauti B. & Michael Woodford. 2003. 'The Zero Bound on Interest Rates and Optimal Monetary Policy'. *Brookings Papers on Economic Activity* 34 (1): 139–235.

Fagan, Gabriel, Jerome Henry & Ricardo Mestre. 2005. 'An Area-Wide Model for the Euro Area'. *Economic Modelling* 22 (1): 39–59.

Fair, Ray C & John B Taylor. 1983. 'Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models'. *Econometrica* 51 (4) (July): 1169–85.

Fang, Li. 1989. 'On the Spectra of P- and P0-Matrices'. *Linear Algebra and Its Applications* 119 (July): 1–25.

Farmer, Roger E. A., Daniel F. Waggoner & Tao Zha. 2010. 'Generalizing the Taylor Principle: Comment'. *American Economic Review* 100 (1): 608–17.

Farmer, Roger E.A., Vadim Khramov & Giovanni Nicolò. 2015. 'Solving and Estimating Indeterminate DSGE Models'. *Journal of Economic Dynamics and Control* 54 (May 1): 17–36.

Farmer, Roger E.A., Daniel F. Waggoner & Tao Zha. 2011. 'Minimal State Variable Solutions to Markov-Switching Rational Expectations Models'. *Frontiers in Structural Macroeconomic Modeling* 35 (12) (December): 2150–2166.

Fernández-Villaverde, Jesús, Grey Gordon, Pablo Guerrón-Quintana & Juan F. Rubio-Ramírez. 2015. 'Nonlinear Adventures at the Zero Lower Bound'. *Journal of Economic Dynamics and Control* 57 (C): 182–204.

Gavin, William T., Benjamin D. Keen, Alexander W. Richter & Nathaniel A. Throckmorton. 2015. 'The Zero Lower Bound, the Dual Mandate, and Unconventional Dynamics'. *Journal of Economic Dynamics and Control* 55 (June): 14–38.

Giannoni, Marc P. 2010. *Optimal Interest-Rate Rules in a Forward-Looking Model, and Inflation Stabilization versus Price-Level Stabilization*. NBER Working Papers. National Bureau of Economic Research, Inc.

Guerrieri, Luca & Matteo Iacoviello. 2015. 'OccBin: A Toolkit for Solving Dynamic Models with Occasionally Binding Constraints Easily'. *Journal of Monetary Economics* 70 (March): 22–38.

Gürkaynak, Refet S., Andrew Levin & Eric Swanson. 2010. 'Does Inflation Targeting Anchor Long-Run Inflation Expectations? Evidence from the U.S., UK, and Sweden'. *Journal of the European Economic Association* 8 (6) (December): 1208–1242.

Hebden, James, Jesper Lindé & Lars Svensson. 2011. 'Optimal Monetary Policy in the Hybrid New Keynesian Model under the Zero Lower Bound'. Unpublished.

Holden, Tom D. 2010. *Products, Patents and Productivity Persistence: A DSGE Model of Endogenous Growth*. Economics Series Working Paper. University of Oxford, Department of Economics.

———. 2016. 'Computation of Solutions to Dynamic Models with Occasionally Binding Constraints.' Unpublished.

Holden, Tom D. & Michael Paetz. 2012. 'Efficient Simulation of DSGE Models with Inequality Constraints'. School of Economics Discussion Papers.

Jones, Callum. 2015. 'Unanticipated Shocks and Forward Guidance at the Zero Lower Bound'. *New York University*.

Kamihigashi, Takashi. 2014. 'Elementary Results on Solutions to the Bellman Equation of Dynamic Programming: Existence, Uniqueness, and Convergence'. *Economic Theory* 56 (2) (June 1): 251–273.

Kamihigashi, Takashi & Santanu Roy. 2003. *A Nonsmooth, Nonconvex Model of Optimal Growth*. Discussion Paper Series. Research Institute for Economics & Business Administration, Kobe University.

Lan, Hong & Alexander Meyer-Gohde. 2013. 'Solving DSGE Models with a Nonlinear Moving Average'. *Journal of Economic Dynamics and Control* 37 (12) (December 1): 2643–2667.

———. 2014. 'Solvability of Perturbation Solutions in DSGE Models'. *Journal of Economic Dynamics and Control* 45 (August 1): 366–388.

Levine, Paul, Joseph Pearlman & Richard Pierse. 2008. 'Linear-Quadratic Approximation, External Habit and Targeting Rules'. *Journal of Economic Dynamics and Control* 32 (10). Journal of Economic Dynamics and Control: 3315–3349.

Li, Cui-Xia & Shi-Liang Wu. 2016. 'A Note on the Unique Solution of Linear Complementarity Problem'. *Cogent Mathematics* 3 (1) (December 20).

Marx, Magali & Jean Barthelemy. 2013. *State-Dependent Probability Distributions in Non Linear Rational Expectations Models*. Society for Economic Dynamics.

McKay, Alisdair, Emi Nakamura & Jón Steinsson. 2016. *The Discounted Euler Equation: A Note*. National Bureau of Economic Research, Inc.

Mendes, Rhys R. 2011. *Uncertainty and the Zero Lower Bound: A Theoretical Analysis*. University Library of Munich, Germany.

Mertens, Karel RSM & Morten O. Ravn. 2014. 'Fiscal Policy in an Expectations-Driven Liquidity Trap'. *The Review of Economic Studies*: rdu016.

Nakata, Taisuke & Sebastian Schmidt. 2014. *Conservatism and Liquidity Traps*. Board of Governors of the Federal Reserve System (U.S.).

Nakov, Anton. 2008. 'Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate'. *International Journal of Central Banking* 4 (2): 73–127.

Richter, Alexander W. & Nathaniel A. Throckmorton. 2015. 'The Zero Lower Bound: Frequency, Duration, and Numerical Convergence'. *The B.E. Journal of Macroeconomics* 15 (1): 157–182.

Roos, C., T. Terlaky & J. P. Vial. 2006. *Interior Point Methods for Linear Optimization*. Springer.

Ruiz, Daniel. 2001. *A Scaling Algorithm to Equilibrate Both Rows and Columns Norms in Matrices*. CM-P00040415.

Schmitt-Grohé, Stephanie & Martín Uribe. 2012. *The Making Of A Great Contraction With A Liquidity Trap and A Jobless Recovery*. National Bureau of Economic Research, Inc.

Sims, ChristopherA. 2002. 'Solving Linear Rational Expectations Models'. *Computational Economics* 20 (1–2) (October 1): 1–20.

Smets, Frank & Rafael Wouters. 2003. 'An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area'. *Journal of the European Economic Association* 1 (5): 1123–1175.

———. 2007. 'Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach'. *American Economic Review* 97 (3) (June): 586–606.

Stokey, N., R. J. Lucas & E. Prescott. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press.

Swarbrick, Jonathan, Tom D. Holden & Paul Levine. 2016. 'Endogenous Financial Crises in a Model of Occasionally Binding Constrained Borrowing'.

Trefethen, L.N. & M. Embree. 2005. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*. Princeton University Press.

Väliaho, Hannu. 1986. 'Criteria for Copositive Matrices'. *Linear Algebra and Its Applications* 81 (September 1): 19–34.

Vestin, David. 2006. 'Price-Level versus Inflation Targeting'. *Journal of Monetary Economics* 53 (7) (October): 1361–1376.

Wieland, Volker, Tobias Cwik, Gernot J. Müller, Sebastian Schmidt & Maik Wolters. 2012. 'A New Comparative Approach to Macroeconomic Modeling and Policy Analysis'. *The Great Recession: Motivation for Re-Thinking Paradigms in Macroeconomic Modeling* 83 (3) (August): 523–541.

Wright, Thomas G & Lloyd N Trefethen. 2001. 'Large-Scale Computation of Pseudospectra Using ARPACK and Eigs'. *SIAM Journal on Scientific Computing* 23 (2): 591–605.

Xu, Song. 1993. 'Notes on Sufficient Matrices'. *Linear Algebra and Its Applications* 191 (September 15): 1–13.