

# Existence and uniqueness of solutions to dynamic models with occasionally binding constraints.

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**Abstract:** We present the first necessary and sufficient conditions for there to be a unique perfect-foresight solution to an otherwise linear dynamic model with occasionally binding constraints, given a fixed terminal condition. We derive further results on the existence of a solution in the presence of such terminal conditions. These results give determinacy conditions for models with occasionally binding constraints, much as Blanchard and Kahn (1980) did for linear models. In an application, we show that widely used New Keynesian models with endogenous states possess multiple perfect foresight equilibrium paths when there is a zero lower bound on nominal interest rates, even when agents believe that the central bank will eventually attain its long-run, positive inflation target. This illustrates that a credible long-run inflation target does not render the Taylor principle sufficient for determinacy in the presence of the zero lower bound. However, we show that price level targeting does restore determinacy providing agents believe that inflation will eventually be positive.

**Keywords:** *occasionally binding constraints, zero lower bound, existence, uniqueness, price targeting, Taylor principle, linear complementarity problem*

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**The latest version of this paper may be downloaded from:**

<https://github.com/tholden/dynareOBC/raw/master/TheoryPaper.pdf>

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# 1. Introduction

Since the financial crisis of 2007-2008, many central banks around the world have chosen to keep their nominal interest rate close to 0%. While in a few countries, rates on some assets have gone slightly negative, central banks are unable to push their target rate to the level a Taylor rule might suggest since agents always have the option of holding cash. In practice, then, central banks face a zero lower bound (ZLB) on their policy rate, which limits their ability to provide stimulus in severe recessions. Furthermore, during the crisis, both households, firms and banks have hit their borrowing constraints, which has limited their ability to smooth out its effects. However, the theoretical results on determinacy which justify the Taylor principle do not apply to models with occasionally binding constraints (OBCs), such as the zero lower bound, or a borrowing constraint, meaning that the profession still lacks all the necessary tools for understanding the behaviour of such models.

In this paper, we develop theoretical tools for understanding the behaviour of otherwise linear models with occasionally binding constraints.<sup>2</sup> Much as the seminal paper of Blanchard and Kahn (1980) provided necessary and sufficient conditions for the existence of a unique perfect foresight solution to a linear model that returns to a given steady-state, we will provide the first necessary and sufficient conditions for there to be a unique perfect foresight solution, returning to a given steady-state, in otherwise linear models with occasionally binding constraints. We further show that these conditions are not satisfied in standard New Keynesian (NK) models.

We will also provide both necessary conditions and sufficient conditions for there to exist any such solutions. When no solution always returning to the given steady-state exists, as we show to be the case in standard NK models, this implies that the model must converge to some alternative steady-state. We note that while in the fully linear case, rational expectations and perfect-foresight solutions coincide, in the otherwise linear case considered here, this will not be the case. However, since under quite mild assumptions, there are weakly more solutions under rational expectations than under perfect foresight (as proven in online appendix A), our results imply lower bounds on the number of solutions under rational expectations.

As was observed by Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b), in the presence of OBCs, there are often multiple steady-states. For example, a model with a zero lower bound on nominal interest rates and Taylor rule monetary policy when away from the bound will have an additional “bad” deflationary steady-state in which nominal interest rates are zero. The presence of such multiple steady-states means that there can be sunspot equilibria which jump between the neighbourhoods of the two

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<sup>2</sup> A companion paper (Holden 2016), develops computational tools for understanding the same thing.

steady-states. Furthermore, if agents put a positive probability on being in the neighbourhood of the “bad” steady-state in future, then since this “bad” steady-state is indeterminate, then by a backwards induction argument, there is indeterminacy now. The consequences of indeterminacy of these kinds have been explored by Schmitt-Grohé and Uribe (2012), Mertens and Ravn (2014) and Aruoba, Cuba-Borda, and Schorfheide (2014), amongst others. In all cases, the key to generating indeterminacy are switches in agents’ beliefs about the point to which the economy would converge without future uncertainty.

However, the central banks of most major economies have announced (positive) inflation targets. Thus, convergence to a deflationary steady-state would represent a spectacular failure to hit the target. As argued by Christiano and Eichenbaum (2012), a central bank may rule out the deflationary equilibria in practice by switching to a money growth rule following severe deflation, along the lines of Christiano and Rostagno (2001). Furthermore, Richter and Throckmorton (2015) and Appendix B of Gavin et al. (2015) present evidence that the deflationary equilibrium is unstable<sup>3</sup> under rational expectations if shocks are large enough, making it much harder for agents to coordinate upon it. This suggests that at least in the long-run, agents ought to believe that we will return to the vicinity of the inflation target, and they ought to place zero probability on paths converging to deflation. Such beliefs also appear to be in line with the empirical evidence of Gürkaynak, Levin, and Swanson (2010). If agents’ beliefs satisfy these restrictions, then the kinds of multiplicity studied by the authors cited in the previous paragraph are ruled out. It is an important question, then, whether there are still multiple equilibria even when all agents believe that in the long-run, the economy will return to a particular steady-state.

It is on such equilibria that we focus on in this paper, providing necessary and sufficient conditions for the existence of a unique perfect-foresight path, and examining whether it is consistent with rationality for agents to believe that the economy will eventually return to the given steady state. In an application, we show that many otherwise linear New Keynesian models featuring endogenous state variables (e.g. price dispersion), do not possess a unique perfect-foresight path. This means that even when agents’ long-run expectations are pinned down, there is still multiplicity of equilibria. Thus, the Taylor principle is not sufficient for determinacy in the presence of occasionally binding constraints. Indeed, we show that in these models, there are some initial states from which the economy has one return path that never hits the ZLB, and another that does hit it, so there may be multiplicity even when away from the bound. However, we show that under a price-targeting regime, there is a unique equilibrium path even when we impose the ZLB. Thus, if the standard

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<sup>3</sup> They show that policy function iteration is not stable near the deflationary equilibria.

arguments for the Taylor principle convince policy makers, then, given they face the zero lower bound, they ought to consider adopting a price level target.

Our paper also provides both necessary and sufficient conditions for the existence of any perfect-foresight solutions which return to the original (“good”) steady-state. These results include both global (non-)existence results (i.e. ones independent of the initial state and/or shock realisation), and particular (non-)existence results (i.e. ones that are conditional on the current realisations). When no perfect-foresight equilibrium returning to the “good” steady-state exists, agents must switch their beliefs to the other (“bad”) steady-state, where they will remain in the absence of any way for agents to coordinate back on the “good” steady-state.

We show that for standard New Keynesian models with endogenous state variables, there is a positive probability of ending up in a state of the world (i.e. with certain state variables and shock realisations) in which there is no perfect foresight path returning to the “good” steady-state. Furthermore, if we suppose that in the stochastic model, agents deal with uncertainty by integrating over the space of possible future shock sequences, as in the original stochastic extended path algorithm of Adjemian and Juillard (2013),<sup>4</sup> then such agents would always put positive probability on tending to the “bad” steady-state. Since the second steady-state is indeterminate in NK models, then this implies global indeterminacy by a backwards induction argument. Once again though, price level targeting would be sufficient to restore determinacy.

### **1.1. Further related literature on multiplicity**

A growing body of work has already examined multiplicity in New Keynesian models subject to the ZLB, even aside from the literature started by Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b) and discussed previously. In this section, we will discuss these other strands to the literature. We conclude this section by arguing that our approach here is a conservative one that makes it as unlikely as possible that we should find multiplicity. That we still do find multiplicity in standard models is thus even more surprising. We consider the prior literature on existence and non-existence in the following section.

The most relevant prior work is that of Brendon, Paustian, and Yates (2013; 2016), henceforth abbreviated to BPY. Like us, these authors examined perfect foresight equilibria of New Keynesian models with terminal conditions. In BPY (2013), the authors show analytically that in a very simple NK model, featuring a response to the growth rate in the Taylor rule, there are multiple perfect-foresight equilibria when all agents believe that with probability one, in one period’s time, they will escape the

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<sup>4</sup> Strictly, this is not fully rational, as it is equivalent to assuming that agents act as if the uncertainty in all future periods would be resolved next period. However, in practice this appears to be a close approximation to full rationality, as demonstrated by Holden (2016). The authors of the original stochastic path method now have a more complicated version that is fully consistent with rationality (Adjemian and Juillard 2016).

bound and return to the neighbourhood of the “good” steady-state. Furthermore, in the aforementioned paper, and in BPY (2016), the authors show numerically that in some select other models, there are multiple perfect-foresight equilibria when the economy begins at the steady-state, and all agents believe that the economy will jump to the bound, remain there for some weakly positive number of periods, before leaving it endogenously, after which they believe they will never hit the bound again.

Relative to these authors, we will provide far more general theoretical results, and these will permit numerical analysis that is both more robust and less restrictive. This robustness and generality will prove crucial in showing multiplicity even in simple NK models, with entirely standard Taylor rules. For example, whereas BPY (2016) write that price-dispersion “does not have a strong enough impact on equilibrium allocations for the sort of propagation that we need”, we will show that in fact the presence of price dispersion is sufficient for multiplicity. Likewise, whereas BPY (2013; 2016) find a much weaker role for multiplicity when the monetary rule does not include a response to the growth rate of output, our findings of multiplicity will not be at all dependent on such a response, implying very different policy prescriptions.

All the other work on multiplicity at the zero lower bound that we are aware of does not impose a terminal condition. This of course started with Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b) as already discussed, but there are several other key threads to the literature that are related to our work. One branch looks at Markov switching policy rules. Key papers include Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2010; 2011). While this literature has been able to derive tight theoretical results, the usually assumed exogeneity of the switching means that the model’s shocks cannot drive the model into the ZLB state, limiting its application to the present context. Some determinacy results with endogenous switching were derived by Marx and Barthelemy (2013), but unfortunately they only apply to forward looking models that are sufficiently close to ones with exogenous switching, and there is no reason to think that a standard NK model with a ZLB should satisfy this property.

In other related work, Armenter (2016) shows that in a simple otherwise linear New Keynesian model with a ZLB, if the central bank pursues Markov (discretionary) policy subject to an objective targeting inflation, nominal GDP or the price level, then there are multiple equilibria quite generally. The existence of multiple discretionary equilibria in models of monetary policy without a ZLB is well established (see e.g. Albanesi, Chari, and Christiano (2003) and the other papers cited by Armenter (2016)), but Armenter shows that even in a model that would have a unique equilibrium in the absence of the ZLB, its introduction produces additional Markov equilibria. By contrast, we will find that adopting a Taylor rule including a term in the price level leads to there being a unique perfect-foresight path returning to the inflationary

steady-state. This difference between our results and those of Armenter (2016) is driven both by our assumption of commitment to a rule, and by the fact that we rule out getting stuck in the neighbourhood of the deflationary steady-state by assumption.

In further related results, Braun, Körber, and Waki (2012) show that there may be multiple perfect-foresight solutions to a non-linear New-Keynesian model, converging to the non-deflationary steady-state. However, it turns out that the linearized version of their model has a unique equilibrium, even when the ZLB is imposed. Thus, the multiplicity we find is strictly in addition to the multiplicity found by those authors. While the theoretical and computational methods used by Braun, Körber, and Waki (2012) have the great advantage that they can cope with fully non-linear models, it appears that they cannot cope with endogenous state variables, which limits their applicability. By producing tools for analysing otherwise linear models including state variables, our tools and results provide a complement to those of Braun, Körber, and Waki (2012). Evidence of their continued relevance in a non-linear setting is provided by the fact that the multiplicity found in a simple linearized model by Brendon, Paustian, and Yates (2013) (and by us in this paper) is also found in the equivalent non-linear model by Brendon, Paustian, and Yates (2016).

Of course, ideally, we would have liked to analyse models with other nonlinearities apart from the occasionally binding constraint(s). Linearization can artificially exclude equilibria (see e.g. Bodenstein (2010)), and, as just mentioned, Braun, Körber, and Waki (2012) show examples of linearization excluding equilibria in the presence of the ZLB. Nonetheless, we maintain that studying multiplicity in otherwise linear models is still an important exercise. Firstly, macroeconomists have long relied on existence and uniqueness results based on linearization of models without occasionally binding constraints, even though this may produce spurious uniqueness in some circumstances. Secondly, it is nearly impossible to find all perfect foresight solutions in general non-linear models, since this is equivalent to finding all the solutions to a huge system of non-linear equations, when even finding all the solutions to large systems of quadratic equations is computationally intractable. At least if we have the full set of solutions to the otherwise linear model, we may use homotopy continuation methods to map these solutions into solutions of the non-linear model. Furthermore, finding all solutions under uncertainty is at least as difficult in general, as the policy function is also defined by a large system of non-linear equations. Thirdly, Christiano and Eichenbaum (2012) argue that e-learnability considerations render the additional equilibria of Braun, Körber, and Waki (2012) mere “mathematical curiosities”, suggesting that the equilibria that exist in the linearized model are of independent interest, whatever one’s view on this debate. Finally, our main results for New-

Keynesian models will imply non-uniqueness, so concerns of spurious uniqueness under linearization will not be relevant in these cases.

From the preceding discussion, we see that our choice to focus on otherwise-linear models under perfect-foresight, with fixed terminal conditions, may bias our results in favour of uniqueness for four distinct reasons. Firstly, because there are potentially more solutions under rational expectations than under perfect-foresight, as we demonstrate in online appendix A; secondly, because there are potentially other solutions returning to alternate steady-states; thirdly, because the original fully non-linear model may possess yet more solutions; and fourthly because there may be further equilibria under discretionary policy. This means that our results on the multiplicity of solutions to New Keynesian models are all the more surprising, and that it is all the more likely that multiplicity of equilibria is an important factor in explaining actual economies' spells at the zero lower bound.

## **1.2. Prior literature on existence and non-existence**

Turning now to existence and non-existence results, several papers have found that New Keynesian models with a zero lower bound might have no solution at all if the variance of shocks is too high. Mendes (2011) derived analytic results on existence as a function of the variance of a demand shock, and Basu and Bundick (2015) showed the potential quantitative relevance of such results. Furthermore, conditions for the existence of an equilibrium in a simple NK model with discretionary monetary policy are derived in close form for a model with a two-state Markov shock by Nakata and Schmidt (2014). They show that the economy must spend a small amount of time in the bad state for the equilibrium to exist, which again links existence to variance.

Our results will not be directly related to the variance of shocks, as we work under perfect foresight. Nonetheless, our theoretical results under perfect foresight may help explain some of the prior results in the stochastic case. We will show that whether a perfect foresight solution exists will depend on the perfect-foresight path taken by nominal interest rates in the absence of the bound. Throughout, we will assume that this path is arbitrary, as there is always some information about future shocks that could be revealed today to produce a given path. However, in a model with a small number of shocks, all of bounded support, and no information about future shocks, clearly not all paths are possible for nominal interest rates in the absence of the bound. The more shocks are added (e.g. news shocks), and the wider their support, the greater will be the support of the space of possible paths for nominal interest rates in the absence of the ZLB, and hence, the more likely will be non-existence of a solution for a positive measure of paths. This gives new intuition for the prior results.

There has also been some prior work by Richter and Throckmorton (2015) and Gavin et al. (2015; Appendix B) that has related a kind of eductive stability (the convergence

of policy function iteration) to other properties of the model. Non-convergence of policy function iteration is suggestive of non-existence, though not definitive evidence. While the procedure of the cited authors has the advantage of working with the fully non-linear model under rational expectations, this limitation means that it cannot directly address the question of existence. By contrast, our results are theoretical and directly address existence. Thus, both procedures should be viewed as complementary; while ours definitively answers the question of existence in the slightly limited world of perfect foresight, otherwise linear models, the Richter and Throckmorton results give answers on stability in a richer setting.

Another approach to establishing the existence of an equilibrium is to produce it to satisfactory accuracy, by solving the model in some way. Under perfect foresight, the procedure outlined in this paper's companion is a possibility (Holden 2016), and the method of Guerrieri and Iacoviello (2012) is a prominent alternative. Under rational expectations, policy function iteration methods have been used by Fernández-Villaverde et al. (2015) and Richter and Throckmorton (2015), amongst others. However, this approach cannot establish non-existence or prove uniqueness. As such it is of little use to the policy maker who wants policy guidance to ensure existence and/or uniqueness. Furthermore, if the problem is solved globally, one cannot in general rule out that there is not an area of non-existence outside of the grid on which the model was solved. Similarly, if the model is solved under perfect foresight for a given initial state, then the fact that a solution exists for that initial point gives no guarantees that a solution should exist for other initial points. Thus, it is essential to produce more general results on global existence, as we do here.

### **1.3. Intuition for the multiplicity mechanism**

The key idea behind all our proofs is that an OBC provides a source of endogenous news about the future. When a shock hits, driving the economy to the bound in some future periods, that tells us that in those future periods, the (lower) bounded variable will be higher than it would be otherwise.<sup>5</sup> Hence, any shock that causes the ZLB to be hit may be thought of as providing a source of endogenous news about future innovations to the monetary rule, of just the right magnitude needed to impose the ZLB. For example, if the magnitude of a productivity shock is such that in the absence of the ZLB, nominal interest rates would be negative a year after the original shock, then, in the presence of the ZLB, the shock is providing endogenous news that nominal interest rates will be higher than normal in one year's time.

Thinking in terms of endogenous news shocks also helps to provide intuition for the presence of multiple equilibria. As a first step towards such intuition, consider a New

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<sup>5</sup> The idea of imposing the zero lower bound by adding news shocks is also present in Holden (2010), Hebden et al. (2011), Holden and Paetz (2012) and Bodenstein et al. (2013). News shocks were introduced to the literature by Beaudry and Portier (2006).



Keynesian model in which learning about a future positive shock to nominal interest rates actually leads to lower rates in the period in which the shock arrives. This is true, for example, in the model of Smets and Wouters (2003) at the posterior mode, in part since that model features a strong response to the growth rate of output.<sup>6</sup>

To explain why this may happen in the Smets and Wouters (2003) model, it helps first to explain why it will not happen in a basic three equation NK model without state variables. Without state variables, the period after the shock's arrival, inflation will be at steady state. Thus, in the period of the shock, real interest rates move one for one with nominal interest rates. Were the positive shock to the nominal interest rate to produce a fall in its level, then the Euler equation would imply high consumption today, also implying high inflation today via the Phillips curve. But, with consumption, inflation, and the shock all positive, the nominal interest rate must be above steady-state, contradicting our assumption that it had fallen. However, a response to the growth rate of output breaks this proof by contradiction, since the bigger is the contraction the period of the shock, the higher growth and interest rates will be in future, and so the lower will be future inflation. Thus, real rates may rise even if the nominal rate falls, something that may be brought about by a sufficiently large response to growth rates.

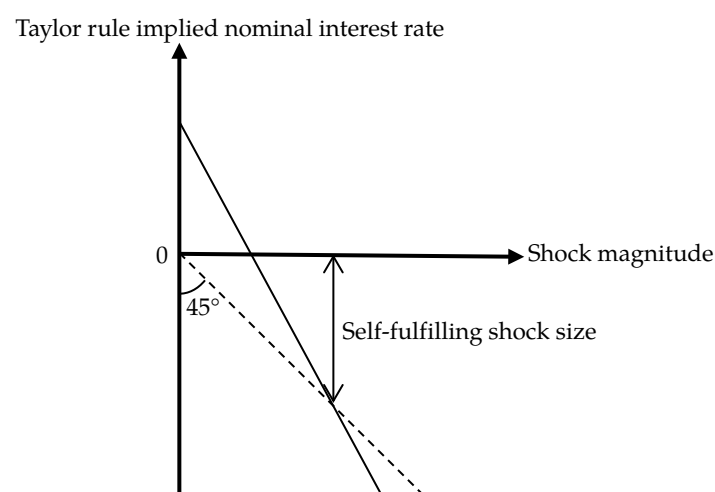


Figure 1: Self-fulfilling news shocks

Now, in such a model, there will be some magnitude of news shock to nominal interest rates today at which the news is of precisely the correct magnitude to bring the negative interest rates implied by the Taylor rule up to zero, in that period. Thus, in the presence of the ZLB, a news shock of this magnitude becomes a self-fulfilling prophecy. If agents believed such a shock would occur, then Taylor rule implied interest rates would be negative in future, so such a shock would indeed “occur” in

<sup>6</sup> This is the mechanism stressed by Brendon, Paustian, and Yates (2013; 2016).

order to impose the ZLB. This as illustrated in Figure 1. If the news shock magnitude is zero, then nominal interest rates do not move from their positive steady-state. However, the larger is the magnitude of the news shock, the lower are nominal interest rates according to the Taylor rule (not including the shock), as shown by the solid black line. The actual nominal interest rate is the sum of the black line and the size of the shock (i.e. the distance from the horizontal axis to the dashed line). At the marked point, the magnitude of news shock needed to impose the ZLB is precisely the magnitude of shock needed to generate the fall in Taylor ruled implied interest rates, giving a self-fulfilling jump to the bound.

A similar channel can produce multiplicity even in models without a response to growth rates in the Taylor rule, and even when news about positive shocks to the monetary rules always results in increases in interest rates in the period in which the shock arrives. The requirement is that there is some combination of future periods such that with appropriate positive shocks anticipated to arrive in each, interest rates are below steady-state in each of the given periods, allowing for a similar self-fulfilling prophecy. Informally, what is needed is that the impulse responses to positive news shocks to interest rates are sufficiently negative for a sufficiently high amount of time that a linear combination of them could be negative in every period in which a shock arrives. Since non-persistent shocks cannot have persistent effects without endogenous state variables, in practice endogenous state variables are likely to be necessary for multiplicity.<sup>7</sup>

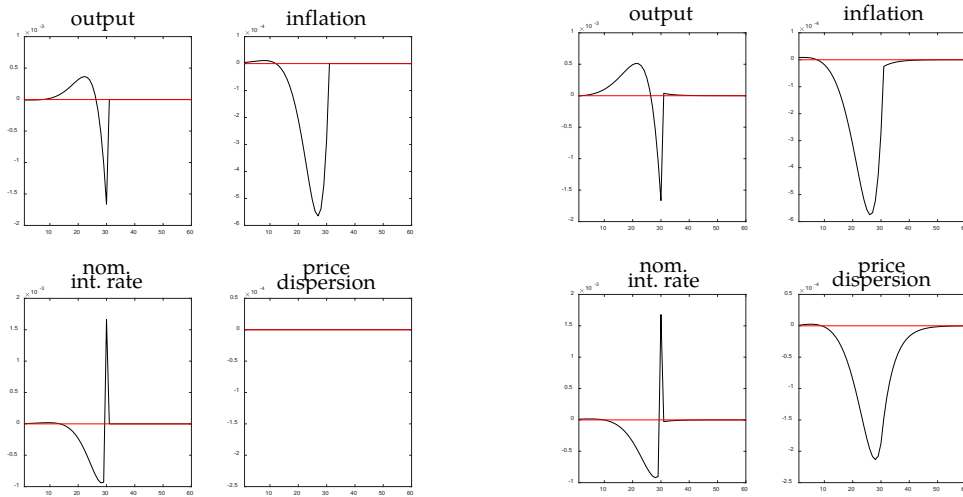
To make this clearer, compare a standard Calvo (1983)-pricing New Keynesian model without capital or price indexation, but with positive trend inflation, to one with full indexation to steady-state inflation. More precisely, the models considered are that of Fernández-Villaverde et al. (2015) with the ZLB removed, and a version of the same model with indexation. Further details are given in section 3.3.

To a first order approximation, the model with full indexation never has any price dispersion, and thus has no endogenous state variables. It is thus a purely forwards looking model, and so it is unsurprising that it should have a unique equilibrium given a terminal condition, even in the presence of the zero lower bound. In Figure 2 we plot the impulse responses of first order approximations to both models to a shock to nominal interest rates that is announced in period one but that does not hit until period thirty. For both models, the shape is similar, however, in the model without indexation, the presence of price dispersion reduces inflation both before and after the shock hits. This is because the predicted fall in inflation compresses the price distribution, reducing dispersion, and thus reducing the number of firms making very large adjustments. The fall in price dispersion also increases output, due to lower

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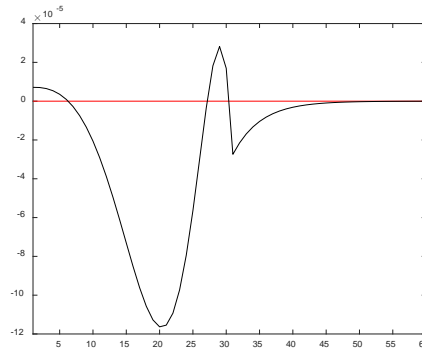
<sup>7</sup> Esoteric examples of multiplicity without endogenous state variables may be constructed; one is given in online appendix B.

efficiency losses from miss-pricing. However, the effect on interest rates is dominated by the negative inflation effect, as the Taylor-rule coefficient on output cannot be too high if there is to be determinacy.<sup>8</sup> For reference, the difference between the IRFs of nominal interest rates in each model is plotted in Figure 3, making clear that interest rates are on average lower following the shock in the model without indexation.



**Figure 2: Impulse responses to a shock announced in period 1, but hitting in period 30, in basic New Keynesian models with (left) and without (right) indexation to steady-state inflation.**

All variables are in logarithms. In both cases, the model and parameters are taken from Fernández-Villaverde et al. (2015), the only change being the addition of complete price indexation to steady-state inflation for non-updating firms in the left hand plots.



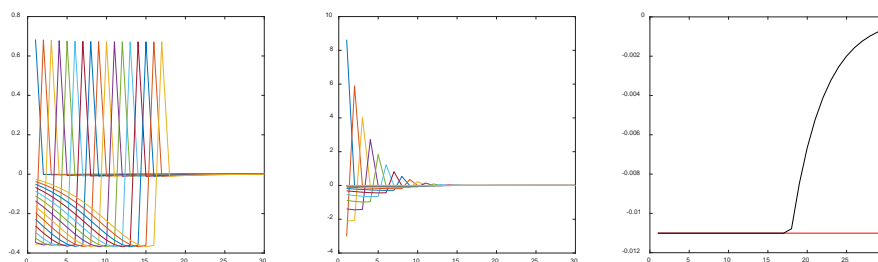
**Figure 3: Difference between the impulse responses of nominal interest rates from the two models shown in Figure 2.**

Negative values imply that nominal interest rates are lower in the model without indexation.

Remarkably, this small difference in the impulse responses between models is enough that the linearized model without indexation has multiple equilibria given a ZLB, but the linearized model with full indexation is determinate. This illustrates just how fragile is the uniqueness in the linearized purely forward looking model. We

<sup>8</sup> One might think that the situation would be different if the coefficient on output was high, so that the rise in output after the shock produced a rise in interest rates. However, as observed by Ascari and Ropele (2009), the determinacy region is much smaller in the presence of price dispersion than would be suggested by the standard Taylor criterion. Numerical experiments suggest that in all the determinate region, interest rates are below steady-state following the shock.

illustrate how multiplicity emerges in the model without indexation by showing, in Figure 4, the construction of an additional equilibrium which jumps to the ZLB for seventeen quarters.<sup>9</sup> If the economy is to be at the bound for seventeen quarters, then for those seventeen quarters, the nominal interest rate must be higher than it would be according to the Taylor rule, meaning that we need to consider seventeen endogenous news shocks, at horizons from zero to sixteen quarters into the future. The impulse responses to unit shocks of this kind are shown in the leftmost plot. Each impulse response has broadly the same shape as the one shown for nominal interest rates in the right of Figure 2. The central figure plots the same impulse responses again, but now each line is scaled by a constant so that their sum gives the line shown in black in the rightmost plot. In this rightmost plot, the red line gives the ZLB's location, relative to steady-state, thus the combined impulse response spends seventeen quarters at the ZLB before returning to steady-state. Since there are only “news shocks” in the periods in which the economy is at the ZLB, this gives a perfect foresight rational expectations equilibrium which makes a self-fulfilling jump to the ZLB.



**Figure 4: Construction of multiple equilibria in the basic NK model with price dispersion.**

The left-hand plot shows the impulse responses to news shocks arriving zero to sixteen quarters after becoming known. The middle plot shows the same impulse responses scaled appropriately. The right-hand plot shows the sum of the scaled impulse responses shown in the central figure, where the red line gives the ZLB's location, relative to steady-state.

In richer New Keynesian models, both real and nominal rigidities help reduce the average value of the impulse response to a positive news shock to the monetary rule. Following the shock's arrival, they help ensure that the fall in output is persistent. Prior to the arrival, consumption smoothing (aided by internal habits) and capital or investment adjustment costs help produce a larger anticipatory recession. Given these mechanisms, we will see that multiplicity is the rule in NK models. Its absence in the basic model with price indexation is a knife-edge result, as evidenced by how small a change to impulse responses is sufficient to produce multiplicity.

#### 1.4. Outline of the following

Our paper is structured as follows. In the following section, section 2, we present our key theoretical results on otherwise linear perfect foresight models. We then

<sup>9</sup> Seventeen quarters was the minimum span for which an equilibrium of this form could be found.

discuss the application of these results to New Keynesian models in section 3. Section 4 concludes. All files needed for the replication of this paper's numerical results are included in the "Examples" directory of the author's DynareOBC toolkit,<sup>10</sup> which implements an algorithm for simulating models with occasionally binding constraints that we discuss in a companion paper (Holden 2016), as well as checking the existence and uniqueness conditions that we will discuss here.

## 2. Theoretical results on occasionally binding constraints in otherwise linear models under perfect foresight

In this section, we present our main theoretical results on existence and uniqueness of perfect foresight solutions to models which are linear apart from an occasionally binding constraint. We start by defining the problem to be solved, and examining its relationship both to the problem without OBCs, and to a related problem with news shocks to the bounded variable. Using the news shock representation, we demonstrate that solving the model with OBCs is equivalent to solving a linear complementarity problem. We then discuss some theoretical background on these problems, along with a special case and motivating example, before presenting the main existence and uniqueness results. Further results are contained in the appendix, with section 6.4 giving additional definitions, 6.5 presenting results on the number and convexity of solutions, and 6.6 relating our results to the properties of models solvable via dynamic programming. We conclude this section with a practical guide to checking the existence and uniqueness conditions.

### 2.1. Problem set-ups

We start by describing the problem set-up without bounds. Suppose that for  $t \in \mathbb{N}^+$ , (i.e.  $t \in \mathbb{N}, t > 0$ ), the first order conditions of some model may be represented as:

$$(\hat{A} + \hat{B} + \hat{C})\hat{\mu} = \hat{A}\hat{x}_{t-1} + \hat{B}\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \hat{D}\varepsilon_t,$$

where  $\hat{\mu} \in \mathbb{R}^{\hat{n}}$  and  $\hat{x}_t \in \mathbb{R}^{\hat{n}}, \varepsilon_t \in \mathbb{R}^m, \mathbb{E}_{t-1}\varepsilon_t = 0$  for all  $t \in \mathbb{N}^+$ , and suppose that  $\hat{x}_0$  is given as an initial condition. Throughout this paper, we will refer to first order conditions such as these as "the model", conflating them with the optimisation problem(s) which gave rise to them.

Furthermore, suppose that  $\varepsilon_t = 0$  for  $t > 1$ , as in an impulse response or perfect foresight simulation exercise. Additionally, we assume the existence of a terminal condition of the form  $\hat{x}_t \rightarrow \hat{\mu}$  as  $t \rightarrow \infty$ , coming, for example, from the source model's transversality constraints. For  $t \in \mathbb{N}^+$ , we then define  $x_t := \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}$ ,  $\mu := \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}$ ,  $A := \begin{bmatrix} \hat{A} & \hat{D} \\ 0 & 0 \end{bmatrix}$ ,  $B := \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}$ ,  $C := \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$ , so, for  $t \in \mathbb{N}^+$ :

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}, \quad (1)$$

<sup>10</sup> These files may be viewed online at <https://github.com/tholden/dynareOBC/tree/master/Examples>.

and we have the extended initial condition  $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$ , and the extended terminal condition  $x_t \rightarrow \mu$  as  $t \rightarrow \infty$ . Expectations have disappeared since there is no uncertainty after period 0. Thus, the problem of solving the original model has the same form as that given in:

---

**Problem 1** Suppose that  $x_0 \in \mathbb{R}^n$  is given. Find  $x_t \in \mathbb{R}^n$  for  $t \in \mathbb{N}^+$  such that  $x_t \rightarrow \mu$  as  $t \rightarrow \infty$ , and such that for all  $t \in \mathbb{N}^+$ , equation (1) holds.

---

We make the following assumption in all the following:

---

**Assumption 1** For any given  $x_0 \in \mathbb{R}^n$ , Problem 1 has a unique solution, which takes the form  $x_t = (I - F)\mu + Fx_{t-1}$ , for  $t \in \mathbb{N}^+$ , where  $F = -(B + CF)^{-1}A$ , and where all of the eigenvalues of  $F$  are weakly inside the unit circle.

---

Sims's (2002) generalisation of the standard Blanchard-Kahn (1980) conditions is necessary and sufficient for this. Further, to avoid dealing specially with the knife-edge case of exact unit eigenvalues (even if they are constrained to the part of the model that is solved forward), in the following we rule it out with the subsequent assumption, which is, in any case, a necessary condition for perturbation to produce a consistent approximation to a source non-linear model, and which is also necessary for the linear model to have a unique steady-state:

---

**Assumption 2**  $\det(A + B + C) \neq 0$ .

---

The combination of Assumption 1 and Assumption 2 imply that all the eigenvalues of  $F$  are strictly inside the unit circle.

We are interested in models featuring occasionally binding constraints. We will concentrate on models featuring a single zero lower bound type constraint in their first equation, which we treat as defining the first element of  $x_t$ . Generalising from this special case to models with one or more fully general bounds is straightforward, and is discussed in the companion paper (Holden 2016). All the results below go through in the more general case with minimal effort.

First, let us write  $x_{1,t}, I_{1,\cdot}, A_{1,\cdot}, B_{1,\cdot}, C_{1,\cdot}$  for the first row of  $x_t, I, A, B, C$  (respectively) and  $x_{-1,t}, I_{-1,\cdot}, A_{-1,\cdot}, B_{-1,\cdot}, C_{-1,\cdot}$  for the remainders. Likewise, we write  $I_{\cdot,1}$  for the first column of  $I$ , and so on. Then, we are interested in the solution to:

---

**Problem 2** Suppose that  $x_0 \in \mathbb{R}^n$  is given. Find  $T \in \mathbb{N}$  and  $x_t \in \mathbb{R}^n$  for  $t \in \mathbb{N}^+$  such that  $x_t \rightarrow \mu$  as  $t \rightarrow \infty$ , and such that for all  $t \in \mathbb{N}^+$ :

$$x_{1,t} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\},$$

$$(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot})\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

and such that  $x_{1,t} > 0$  for  $t > T$ .

---

Note that in this problem we are implicitly ruling out any solutions which get permanently stuck at an alternative steady-state, by assuming that the terminal condition remains as before. In the monetary policy context, this amounts to assuming that the central banks' inflation target is credible. Since  $x_{1,t} \rightarrow \mu_1$  as  $t \rightarrow \infty$ , it is without loss of generality to assume the existence of a  $T \in \mathbb{N}$  such that  $x_{1,t} > 0$  for  $t > T$ , but this  $T$  will play an important role in the below, so we introduce it now. We continue to assume that there is no uncertainty after period 0, so, in this non-linear model, the path of the endogenous variables will not necessarily match up with the path of their expectation in a richer model in which there was uncertainty.

In many models, the occasionally binding constraint comes from the KKT conditions of an optimisation problem, which take the form  $y \geq 0, \lambda \geq 0$  and  $y\lambda = 0$ . These may be converted into the max/min form since they are equivalent to the single equation  $0 = \min\{y, \lambda\}$ , which holds if and only if  $y = \max\{0, y - \lambda\}$ , which is in the form of that of Problem 2. Additionally, in the appendix, section 6.6, we give a more natural, alternative procedure for converting KKT conditions into a problem in the form of that Problem 2. The idea is that one can use the model's equations to find the value the (lower) constrained variable would take were there no constraint and were the Lagrange multiplier on the constraint equal to zero today. This gives a "shadow" value of the constrained variable, and the actual value it takes will be the maximum of the bound and this shadow value.

We will analyse Problem 2 with the help of solutions to the auxiliary problem:

---

**Problem 3** Suppose that  $T \in \mathbb{N}$ ,  $x_0 \in \mathbb{R}^n$  and  $y_0 \in \mathbb{R}^T$  is given. Find  $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$  for  $t \in \mathbb{N}^+$  such that  $x_t \rightarrow \mu, y_t \rightarrow 0$ , as  $t \rightarrow \infty$ , and such that for all  $t \in \mathbb{N}^+$ :

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{.,1}y_{1,t-1},$$

$$y_{T,t} = 0, \quad \forall i \in \{1, \dots, T-1\}, y_{i,t} = y_{i+1,t-1}.$$


---

This may be thought of as a version of Problem 1 with news shocks up to horizon  $T$  added to the first equation. By construction, the value of  $y_{i,t}$  gives the shock that in period  $t$  is expected to arrive in  $i$  periods. Hence, as all information is known in period 0,  $y_{t,0}$  gives the shock that will hit in period  $t$ , i.e.  $y_{1,t-1} = y_{t,0}$  for  $t \leq T$ , and  $y_{1,t-1} = 0$  for  $t > T$ .

## 2.2. Relationships between the problems

A straightforward backwards induction argument (given in appendix 6.1) gives the following helpful result:

---

**Lemma 1** There is a unique solution to Problem 3 that is linear in  $x_0$  and  $y_0$ .

---

For future reference, let  $x_t^{(3,k)}$  be the solution to Problem 3 when  $x_0 = \mu$ ,  $y_0 = I_{\cdot,k}$  (i.e. a vector which is all zeros apart from a 1 in position  $k$ ). Then, by linearity, for arbitrary  $y_0$  the solution to Problem 3 when  $x_0 = \mu$  is given by:

$$x_t - \mu = \sum_{k=1}^T y_{k,0} (x_t^{(3,k)} - \mu).$$

Now, let  $M \in \mathbb{R}^{T \times T}$  satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \quad \forall t, k \in \{1, \dots, T\}, \quad (2)$$

i.e.  $M$  horizontally stacks the (column-vector) relative impulse responses of the first variable to the news shocks, with the first column giving the response to a contemporaneous shock, the second column giving the response to a shock anticipated by one period, and so on. Then, this result implies that for arbitrary  $y_0$ , the path of the first variable in the solution to Problem 3 when  $x_0 = \mu$  is given by:  $(x_{1,1:T})' = \mu_1 + My_0$ , where  $x_{1,1:T}$  is the row vector of the first  $T$  values of the first component of  $x_t$ . Furthermore, for both arbitrary  $x_0$  and  $y_0$ , the path of the first variable in the solution to Problem 3 is given by:

$$(x_{1,1:T})' = q + My_0, \quad (3)$$

where  $q := (x_{1,1:T}^{(1)})'$  and  $x_t^{(1)}$  is the unique solution to Problem 1, for the given  $x_0$ , i.e.  $q$  is the path of the first variable in the absence of news shocks or bounds.<sup>11</sup> Since  $M$  is not a function of either  $x_0$  or  $y_0$ , equation (3) gives a highly convenient representation of the solution to Problem 3.

Now let  $x_t^{(2)}$  be a solution to Problem 2 given an arbitrary  $x_0$ . Since  $x_t^{(2)} \rightarrow \mu$  as  $t \rightarrow \infty$ , there exists  $T' \in \mathbb{N}$  such that for all  $t > T'$ ,  $x_{1,t}^{(2)} > 0$ . We assume without loss of generality that  $T' \leq T$ . We seek to relate the solution to Problem 2 with the solution to Problem 3 for an appropriate choice of  $y_0$ . First, for all  $t \in \mathbb{N}^+$ , let:

$$e_t := \begin{cases} -[I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(2)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(2)} - \mu) + C_{1,\cdot}(x_{t+1}^{(2)} - \mu)] & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}, \quad (4)$$

i.e.  $e_t$  is the shock that would need to hit the first equation for the positivity constraint on  $x_{1,t}^{(2)}$  to be enforced. Note that by the definition of Problem 2,  $e_t \geq 0$  and  $x_{1,t}^{(2)}e_t = 0$ , for all  $t \in \mathbb{N}^+$ . With a little more work, again using a backwards induction argument, (given in appendix 6.2) we then have the following:

---

**Lemma 2** For any solution,  $(T, x_t^{(2)})$  to Problem 2:

- 1) With  $e_{1:T}$  as defined in equation (3),  $e_{1:T} \geq 0$ ,  $x_{1,1:T}^{(2)} \geq 0$  and  $x_{1,1:T}^{(2)} \circ e_{1:T} = 0$ , where  $\circ$  denotes the Hadamard (entry-wise) product.
  - 2)  $x_t^{(2)}$  is also the unique solution to Problem 3 with  $x_0 = x_0^{(2)}$  and  $y_0 = e'_{1:T}$ .
  - 3) If  $x_t^{(2)}$  solves Problem 3 with  $x_0 = x_0^{(2)}$  and with some  $y_0$ , then  $y_0 = e'_{1:T}$ .
- 

<sup>11</sup> This representation was also exploited by Holden (2010) and Holden and Paetz (2012).



To use the easy solution to Problem 3 to assist us in solving Problem 2 just requires one more result. In particular, we show in appendix 6.3 that if  $y_0 \in \mathbb{R}^T$  is such that  $y_0 \geq 0$ ,  $x_{1,1:T}^{(3)} \circ y'_0 = 0$  and  $x_{1,t}^{(3)} \geq 0$  for all  $t \in \mathbb{N}$ , where  $x_t^{(3)}$  is the unique solution to Problem 3 when started at  $x_0, y_0$ , then  $x_t^{(3)}$  must also be a solution to Problem 2. Together with Lemma 1, Lemma 2, and our representation of the solution of Problem 3 from equation (3), this completes the proof of the following proposition:

---

**Proposition 1** The following hold:

- 1) Let  $x_t^{(3)}$  be the unique solution to Problem 3 given  $T \in \mathbb{N}^+$ ,  $x_0 \in \mathbb{R}^n$  and  $y_0 \in \mathbb{R}^T$ . Then  $(T, x_t^{(3)})$  is a solution to Problem 2 given  $x_0$  if and only if  $y_0 \geq 0$ ,  $y_0 \circ (q + My_0) = 0$ ,  $q + My_0 \geq 0$  and  $x_{1,t}^{(3)} \geq 0$  for all  $t \in \mathbb{N}$  with  $t > T$ .
  - 2) Let  $(T, x_t^{(2)})$  be any solution to Problem 2 given  $x_0$ . Then there exists a  $y_0 \in \mathbb{R}^T$  such that  $y_0 \geq 0$ ,  $y_0 \circ (q + My_0) = 0$ ,  $q + My_0 \geq 0$ , such that  $x_t^{(2)}$  is the unique solution to Problem 3 given  $T$ ,  $x_0$  and  $y_0$ .
- 

### 2.3. The linear complementarity representation

Proposition 1 establishes that we may solve Problem 2 by conjecturing a (sufficiently high) value for  $T$  and then solving the following problem:

---

**Problem 4** Suppose  $T \in \mathbb{N}^+$ ,  $q \in \mathbb{R}^T$  and  $M \in \mathbb{R}^{T \times T}$  are given. Find  $y \in \mathbb{R}^T$  such that  $y \geq 0$ ,  $y \circ (q + My) = 0$  and  $q + My \geq 0$ . We call this the **linear complementarity problem (LCP)**  $(q, M)$ . (Cottle 2009)

---

These problems have been extensively studied, and so we can import results on the properties of LCPs to derive results on the properties of models with OBCs.

All the results in the literature on LCPs rest on properties of the matrix  $M$ , thus we would like to establish if the structure of our particular  $M$  implies it has any special properties. Unfortunately, we prove the following result in this paper's companion paper (Holden 2016), which means that  $M$ 's origin implies no particular properties:

---

**Proposition 2** For any  $T \in \mathbb{N}^+$  and any matrix  $M \in \mathbb{R}^{T \times T}$ , there exists a model in the form of Problem 2 with a number of state variables given by a quadratic in  $T$ , such that  $M = \mathcal{M}$  for that model, where  $M$  is defined as in equation (2), and such that for all  $q \in \mathbb{R}^T$ , there exists an initial state for which  $q = q$ , where  $q$  is the path of the bounded variable when constraints are ignored. (Holden 2016)

---

### 2.4. A simple special case and motivating example

When  $T = 1$ , it is particularly easy to characterise the properties of LCPs. This amounts to considering the behaviour of an economy in which everyone believes there will be at most one period at the bound. In this case,  $y$  gives the "shock" to the bounded equation necessary to impose the bound, and  $M$  gives the contemporaneous response of the bounded variable to an unanticipated shock: i.e. in a ZLB context,  $M$

gives the initial movement of nominal interest rates from steady-state following a standard monetary policy shock.

First, suppose that  $M$  (a scalar as  $T = 1$  for now) is positive. Then, if  $q > 0$ , for any  $y \geq 0$ ,  $q + My > 0$ , so by the complementary slackness condition, in fact  $y = 0$ . Conversely, if  $q \leq 0$ , then there is a unique  $y$  satisfying the complementary slackness condition given by  $y = -\frac{q}{M} \geq 0$ . Thus, with  $M > 0$ , there is always a unique solution to the  $T = 1$  LCP. With  $M = 0$ ,  $q + My = q$ , so a solution to the LCP exists if and only if  $q \geq 0$ . It will be unique providing  $q > 0$  (by the complementary slackness condition), but when  $q = 0$ , any  $y \geq 0$  gives a solution. Finally, suppose that  $M < 0$ . Then, if  $q > 0$ , there are precisely two solutions. The “standard” solution has  $y = 0$ , but there is an additional solution featuring a jump to the bound in which  $y = -\frac{q}{M} > 0$ . If  $q = 0$ , then there is a unique solution ( $y = 0$ ) and if  $q < 0$ , then with  $y \geq 0$ ,  $q + My < 0$ , so there is no solution at all. Hence, the  $T = 1$  LCP already provides examples of cases of uniqueness, non-existence and multiplicity.

Brendon, Paustian, and Yates (2013), henceforth BPY, provide a simple New Keynesian model that we can use to illustrate these cases. Its equations are as follows:

$$\begin{aligned} x_{i,t} &= \max\{0, 1 - \beta + \alpha_{\Delta y}(x_{y,t} - x_{y,t-1}) + \alpha_{\pi}x_{\pi,t}\}, \\ x_{y,t} &= \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1}), \quad x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t}, \end{aligned}$$

where  $x_{i,t}$  is the nominal interest rate,  $x_{y,t}$  is the deviation of output from steady-state,  $x_{\pi,t}$  is the deviation of inflation from steady-state, and  $\beta \in (0,1)$ ,  $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$ ,  $\alpha_{\pi} \in (1, \infty)$  are parameters. The model’s only departure from the textbook three equation NK model is the presence of an output growth rate term in the Taylor rule. This merely serves to introduce an endogenous state variable in a tractable manner.

In online appendix D, we prove the following:

---

**Proposition 3** The BPY model is in the form of Problem 2, and satisfies Assumptions 1 and 2. With  $T = 1$ ,  $M < 0$  ( $M = 0$ ) if and only if  $\alpha_{\Delta y} > \sigma\alpha_{\pi}$  ( $\alpha_{\Delta y} = \sigma\alpha_{\pi}$ ).

---

Hence, when all agents believe the bound will be escaped after at most one period, if  $\alpha_{\Delta y} < \sigma\alpha_{\pi}$ , the model has a unique solution for all  $q$ , i.e. no matter what value the nominal interest rate would take that period if there were no ZLB. If  $\alpha_{\Delta y} = \sigma\alpha_{\pi}$ , then, again by the above, the model has a unique solution whenever  $q > 0$ , infinitely many solutions when  $q = 0$ , and no solutions leaving the ZLB after one period when  $q < 0$ . Finally, if  $\alpha_{\Delta y} > \sigma\alpha_{\pi}$  then the above implies that the model has two solutions when  $q > 0$ , one solution when  $q = 0$  and no solution escaping the ZLB next period when  $q < 0$ . Here, multiplicity is due to the mechanism discussed in section 1.3: a jump to the ZLB may be sustained given a sufficiently strong response to growth rates since this implies low expected inflation, and thus higher real interest rates, consistent with low consumption today.

## 2.5. Relevant matrix properties

We now give definitions of the matrix properties that are required for the statement of our key existence and uniqueness results for  $T > 1$ . The properties of the solutions to the OBC model are determined by which of these matrix properties  $M$  possesses. In each case, we give the definitions in a constructive form which makes clear both how the property might be verified computationally, and the links between definitions. These are not necessarily in the form which is standard in the original literature, however. For the original definitions, and the proofs of equivalence between the ones below and the originals, see Cottle, Pang, and Stone (2009a) and Xu (1993).

---

**Definition 1 (Principal sub-matrix, Principal minor)** For a matrix  $M \in \mathbb{R}^{T \times T}$ , the **principal sub-matrices** of  $M$  are the matrices:

$$\left\{ [M_{i,j}]_{i,j=k_1, \dots, k_S} \mid S, k_1, \dots, k_S \in \{1, \dots, T\}, k_1 < k_2 < \dots < k_S \right\},$$

i.e. the **principal sub-matrices** of  $M$  are formed by deleting the same rows and columns. The **principal minors** of  $M$  are the collection of values:

$$\left\{ \det \left( [M_{i,j}]_{i,j=k_1, \dots, k_S} \right) \mid S, k_1, \dots, k_S \in \{1, \dots, T\}, k_1 < k_2 < \dots < k_S \right\},$$

i.e. the **principal minors** of  $M$  are the determinants of  $M$ 's principal sub-matrices.

**Definition 2 ( $P_{(0)}$ -matrix)** A matrix  $M \in \mathbb{R}^{T \times T}$  is called a **P-matrix** ( **$P_0$ -matrix**) if the principal minors of  $M$  are all strictly (weakly) positive. *Note: for symmetric  $M$ ,  $M$  is a  $P_{(0)}$ -matrix if and only if it is positive (semi-)definite.*

**Definition 3 (General positive (semi-)definite)** A matrix  $M \in \mathbb{R}^{T \times T}$  is called **general positive (semi-)definite** if  $M + M'$  is a P-matrix ( $P_0$ -matrix). *Note: if  $M$  is symmetric, then,  $M$  is general positive (semi-)definite if and only if it is positive (semi-)definite.*

---

For intuition on the relevance of these properties, recall that the definition of a linear complementarity problem (Problem 4) contained the complementary slackness type condition,  $y \circ (q + My) = 0$ . Equivalently then,  $0 = y'(q + My) = y'q + y'My$ . Now, if there is no multiplicity,  $y'q$  is likely to be negative as the bound usually binds when  $q$  (the path in the absence of the bound) is negative. Thus, for the equation to be satisfied,  $y'My = \frac{1}{2}y'(M + M')y$  should be positive, which certainly holds when  $M$  is general positive definite (so  $M + M'$  is positive definite). This is in line with the positivity condition for existence of a unique solution we found with  $T = 1$  in the previous section. More generally,  $y$  will usually have many zeros, since  $y$  is zero whenever the model is away from the bound. The remaining non-zero elements of  $y$  select a principal sub-matrix of  $M$ , which will be a P-matrix if  $M$  is a P-matrix. Since being a P-matrix is an alternative generalisation of positive-definiteness to non-symmetric matrices, this turns out to be sufficient for there to be a unique solution to the original equation. In the  $T = 1$  special case, being a P-matrix coincides with being positive.

We now present two further definitions, both of which are relevant as they again help ensure that  $My$  can be made positive:

---

**Definition 4 ( $S_0$ -matrix)** A matrix  $M \in \mathbb{R}^{T \times T}$  is called an **S-matrix** ( **$S_0$ -matrix**) if there exists  $y \in \mathbb{R}^T$  such that  $y > 0$  and  $My \gg 0$  ( $My \geq 0$ ).<sup>12</sup>

---

**Definition 5 ((Strictly) Semi-monotone)** A matrix  $M \in \mathbb{R}^{T \times T}$  is called **(strictly) semi-monotone** if each of its principal sub-matrices is an  **$S_0$ -matrix** (**S-matrix**).

---

In the  $T = 1$  case, being an S-matrix and being strictly semi-monotone coincide with positivity of  $M$ , so these conditions may once more be interpreted as generalisations of the positivity we required for existence with  $T = 1$ . Additionally, in appendix 6.4 we go on to definite (non-)degenerate, (row/column) sufficient matrices, and (strictly) copositive matrices, which are of secondary importance. Cottle, Pang, and Stone (2009a) note the following relationships between these classes (amongst others):

---

**Lemma 3** The following hold:

- 1) All general positive semi-definite matrices are copositive and sufficient.
  - 2)  $P_0$  includes skew-symmetric, general positive semi-definite, sufficient and P matrices.
  - 3) All  $P_0$ -matrices, and all copositive matrices are semi-monotone, and all P-matrices, and all strictly copositive matrices are strictly semi-monotone (so also S-matrices).
- 

Furthermore, from considering the  $1 \times 1$  principal sub-matrices of  $M$ , we have the following restrictions on the diagonal of  $M$ :

---

**Lemma 4** All general positive semi-definite, semi-monotone, sufficient,  $P_0$  and copositive matrices have non-negative diagonals, and all general positive definite, strictly semi-monotone, P and strictly copositive matrices have positive diagonals.

---

For many macroeconomic models, this simple condition is sufficient to rule out membership of these matrix classes, as medium-scale DSGE models<sup>13</sup> with a ZLB frequently have negative elements on the diagonal of their  $M$  matrix, when  $T$  is large enough. Thus, following the intuition of Figure 1, such models will satisfy the conditions to have multiple equilibria, though they will not be the only such models.

A common “intuition” is that in models without state variables,  $M$  must be both a P matrix, and an S matrix. In fact, this is not true. Indeed, there are even purely static models for which  $M$  is not in either of these classes. For example, in online appendix

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<sup>12</sup> These condition may be rewritten as  $\sup\{\zeta \in \mathbb{R} | \exists y \geq 0 \text{ s.t. } \forall t \in \{1, \dots, T\}, (My)_t \geq \zeta \wedge y_t \leq 1\} > 0$  , and  $\sup\{\sum_{t=1}^T y_t | y \geq 0, My \geq 0 \wedge \forall t \in \{1, \dots, T\}, y_t \leq 1\} > 0$ , respectively. As linear programming problems, these may be solved in time polynomial in  $T$  using the methods described in e.g. Roos, Terlaky, and Vial (2006). Alternatively, by Ville’s theorem of the alternative (Cottle, Pang, and Stone 2009b),  $M$  is not an  $S_0$ -matrix if and only if  $-M'$  is an S-matrix.

<sup>13</sup> This applies, for example, to the Smets and Wouters (2003) model, as we will show in section 3.4.

B, we construct a purely static model for which  $M_{1:\infty,1:\infty} = -I_{\infty \times \infty}$ , which is neither a P-matrix, nor an S-matrix, for any  $T$ .

## 2.6. Existence results

We start by considering necessary or sufficient conditions for the existence of a solution to a model with occasionally binding constraints. Ideally, we would like the solution to exist for any possible path the bounded variable might have taken in the future were there no OBC, i.e. for any possible  $q$ . To see this, note that under a perfect foresight exercise we are ignoring the fact that shocks might hit the economy in future. More properly, we ought to take future uncertainty into account. One way to do this would be to follow the original stochastic extended path approach of Adjemian and Juillard (2013) by drawing lots of samples of future shocks for periods  $1, \dots, S$ , and averaging over these draws.<sup>14</sup> However, in a linear model with shocks with unbounded support, providing at least one shock has an impact on a given variable, the distribution of future paths of that variable has positive support over the entirety of  $\mathbb{R}^S$ . Thus, ideally, we would like  $M$  to be such that for any  $q$ , the linear complementarity problem  $(q, M)$  has a solution. For clarity, we remind the reader that the matrix  $M$  is independent of the initial state, so existence for all possible initial states does not require considering more than one  $M$  matrix.

---

**Definition 6 (Feasible LCP)** Suppose  $q \in \mathbb{R}^T$  and  $M \in \mathbb{R}^{T \times T}$  are given. The LCP  $(q, M)$  is called **feasible** if there exists  $y \in \mathbb{R}^T$  such that  $y \geq 0$  and  $q + My \geq 0$ .

---

By construction, if an LCP  $(q, M)$  has a solution, then it is feasible, so being feasible is necessary for existence. Checking feasibility is straightforward for any particular  $(q, M)$ , as to find a feasible solution we just need to solve a linear programming problem, which can be completed in an amount of time that is polynomial in  $T$ .

Note that if the LCP  $(q, M)$  is not feasible, then for any  $\hat{q} \leq q$ , if  $y \geq 0$ , then  $\hat{q} + My \leq q + My < 0$  since  $(q, M)$  is not feasible, so the LCP  $(\hat{q}, M)$  is also not feasible. Consequently, if there are any  $q$  for which the LCP is non-feasible, then there is a positive measure of such  $q$ . Thus, in a model in which  $q$  is uncertain, if there are some  $q$  for which the model has no solution satisfying the terminal condition, even with arbitrarily large  $T$ , then the model will have no solution satisfying the terminal condition with positive probability. Hence it is not consistent with rationality for agents to believe that our terminal condition is satisfied with certainty, so they would have to place some positive probability on getting stuck in an alternative steady-state.

The following proposition gives an easily verified necessary condition for the global existence of a solution to the model with occasionally binding constraints, given some fixed horizon  $T$ :

---

<sup>14</sup> See footnote 4 for caveats to this procedure.

---

**Proposition 4** The LCP  $(q, M)$  is feasible for all  $q \in \mathbb{R}^T$  if and only if  $M$  is an S-matrix. (Cottle, Pang, and Stone 2009a)<sup>15</sup>

---

Of course, it may be the case that the  $M$  matrix is only an S-matrix when  $T$  is very large, so we must be careful in using this condition to imply non-existence of a solution. Furthermore, it may be the case that although there exists some  $y \in \mathbb{R}^T$  with  $y \geq 0$  such that  $M_{1:T,1:T}y \gg 0$  (indexing the  $M$  matrix by its size for clarity), for any such  $y$ ,  $\inf_{t \in \mathbb{N}^+} M_{t,1:T}y < 0$ , so for some  $q \in \mathbb{R}^{\mathbb{N}^+}$ , the infinite LCP  $(q, M_{1:\infty,1:\infty})$  is not feasible under the additional restriction that  $y_t = 0$  for  $t > T$ . It is this infinite LCP which we ought to be solving, subject to the additional constraint that  $y$  has only finitely many non-zero elements, as implied by our terminal condition.

From Proposition 3, we immediately have the following result on feasibility of the infinite problem:

---

**Corollary 1** The infinite LCP  $(q, M_{1:\infty,1:\infty})$  is feasible if and only if:

$$\varsigma := \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty}y > 0.$$


---

$\exists T \in \mathbb{N}$  s.t.  $\forall t > T, y_t = 0$

Consequently, if  $\varsigma > 0$  then for every  $q \in \mathbb{R}^{\mathbb{N}^+}$ , for sufficiently large  $T$ , the finite problem  $(q_{1:T}, M_{1:T,1:T})$  will be feasible, which is a sufficient condition for solvability. To evaluate this limit, we first need to derive constructive bounds on the  $M$  matrix for large  $T$ . We do this in the online appendix C, where we prove that the rows and columns of  $M$  are converging to zero (with constructive bounds), and that the  $k^{\text{th}}$  diagonal of the  $M$  matrix is converging to the value  $d_{1,k}$ , to be defined (again with constructive bounds), where diagonals are indexed such that the principal diagonal is index zero, and indices increase as one moves up and to the right in the  $M$  matrix.

To explain the origins of  $d_{1,k}$  we note the following lemma proved in online appendix C:

---

**Lemma 5** The (time-reversed) difference equation  $A\hat{d}_{k+1} + B\hat{d}_k + C\hat{d}_{k-1} = 0$  for all  $k \in \mathbb{N}^+$  has a unique solution satisfying the terminal condition  $\hat{d}_k \rightarrow 0$  as  $k \rightarrow \infty$ , given by  $\hat{d}_k = H\hat{d}_{k-1}$ , for all  $k \in \mathbb{N}^+$ , for some  $H$  with eigenvalues in the unit circle.

---

Then, we define  $d_0 := -(AH + B + CF)^{-1}I_{\cdot,1}$ ,  $d_k = Hd_{k-1}$ , for all  $k \in \mathbb{N}^+$ , and  $d_{-t} = Fd_{-(t-1)}$ , for all  $t \in \mathbb{N}^+$ , so  $d_k$  follows the time reversed difference equation for positive indices, and the original difference equation for negative indices. This is opposite to what one might expect since time is increasing as one descends the rows of  $M$ , but diagonal indices are decreasing as one descends in  $M$ .

---

<sup>15</sup> Most of the results on LCPs in both this and the following section are restatements of (assorted) results contained in Cottle, Pang, and Stone (2009a) and Väliaho (1986) (for the characterisation of “copositive-plus” matrices), and the reader is referred to those works for proofs and further references.

Using the resulting bounds on  $M$ , we can construct upper and lower bounds on  $\zeta$ , which are described in the following proposition, also proven in online appendix C:

---

**Proposition 5** There exists  $\underline{\zeta}_T, \bar{\zeta}_T \geq 0$ , defined in the online appendix C, computable in time polynomial in  $T$ , such that  $\underline{\zeta}_T \leq \zeta \leq \bar{\zeta}_T$ , and  $|\underline{\zeta}_T - \bar{\zeta}_T| \rightarrow 0$  as  $T \rightarrow \infty$ .

---

This condition gives a simple test for feasibility with any sufficiently large  $T$ . It also provides a test giving strong numerical evidence of non-feasibility, since if  $\bar{\zeta}_T = 0$  + numerical error, then  $\zeta = 0$  is likely.

We now turn to sufficient conditions for existence of a solution for some finite  $T$ .

---

**Proposition 6** The LCP  $(q, M)$  is solvable if it is feasible and, either:

1.  $M$  is row-sufficient, or,
2.  $M$  is copositive and for all non-singular principal sub-matrices  $W$  of  $M$ , all non-negative columns of  $W^{-1}$  possess a non-zero diagonal element.

(Cottle, Pang, and Stone 2009a; Väliäho 1986)

---

If either condition 1 or condition 2 of Proposition 5 is satisfied, then to check existence for any particular  $q$ , we only need to solve a linear programming problem to see if a solution exists for a particular  $q$ . As this will be faster than solving the particular LCP, this may be helpful in practice.

---

**Proposition 7** The LCP  $(q, M)$  is solvable for all  $q \in \mathbb{R}^T$ , if at least one of the following conditions holds:

1.  $M$  is an S-matrix, and either condition 1 or condition 2 of Proposition 5 is satisfied.
2.  $M$  is copositive and non-degenerate.
3.  $M$  is a P-matrix, a strictly copositive matrix or a strictly semi-monotone matrix.

(Cottle, Pang, and Stone 2009a)

---

If condition 1, 2 or 3 of Proposition 6 is satisfied, then we know that the LCP will always have a solution. Therefore, for any path of the bounded variable in the absence of the bound, we will also be able to solve the model when the bound is imposed. Monetary policy makers should always choose a policy rule that produces a model that satisfies one of these three conditions, if they can, since otherwise there is a positive probability that only solutions converging to the “bad” steady-state will exist for some values of state variables and shock realisations.

Ideally, we might have liked conditions for the existence of a solution that are both necessary and sufficient, but unfortunately at present no such conditions exist in full generality. However, in the special case of nonnegative  $M$  matrices, we obtain:

---

**Proposition 8** If  $M$  is a nonnegative matrix, then the LCP  $(q, M)$  is solvable for all  $q \in \mathbb{R}^T$  if and only if  $M$  has a strictly positive diagonal. (Cottle, Pang, and Stone 2009a)

---

## 2.7. Uniqueness results

While no fully general necessary and sufficient conditions have been derived for existence, such conditions are available for uniqueness, in particular:

---

**Proposition 9** The LCP  $(q, M)$  has a unique solution for all  $q \in \mathbb{R}^T$ , if and only if  $M$  is a P-matrix. If  $M$  is not a P-matrix, then for some  $q$  the LCP  $(q, M)$  has multiple solutions. (Samelson, Thrall, and Wesler 1958; Cottle, Pang, and Stone 2009a)

---

This proposition is the equivalent for models with OBCs of the key proposition of Blanchard and Kahn (1980). By testing whether our matrix  $M$  is a P-matrix we can immediately determine if the model possesses a unique solution no matter what the initial state is, and no matter what shocks (if any) are predicted to hit the model in future, for a fixed  $T$ . In our experience, this condition is satisfied in efficient models, such as models of irreversible investment, as one would expect, but is not usually satisfied in medium-scale NK models with a ZLB on nominal interest rates. Given that if  $M$  is a P-matrix, so too are all its principal sub-matrices, if we see that  $M$  is not a P-matrix for some  $T$ , then we know that with larger  $T$  it would also not be a P-matrix. Thus, if for some  $T$ ,  $M$  is not a P-matrix, then we know that the model does not have a unique solution, even for arbitrarily large  $T$ . Alternatively, we can prove that with large  $T$  some  $M$  is not a P-matrix by using the analytic formula for the limit of its diagonal given in the previous section, i.e.  $d_{0,1} = -I_{1,\cdot}(AH + B + CF)^{-1}I_{\cdot,1}$ . If this value is negative, then we know that with sufficiently large  $T$ ,  $M$  will not be a P-matrix.

Since some classes of models almost never possess a unique solution when at the zero lower bound, we might reasonably require a lesser condition, namely that at least when the solution to the model without a bound is a solution to the model with the bound, then it ought to be the unique solution. This is equivalent to requiring that when  $q$  is non-negative, the LCP  $(q, M)$  has a unique solution. Conditions for this are given in the following proposition:

---

**Proposition 10** The LCP  $(q, M)$  has a unique solution for all  $q \in \mathbb{R}^T$  with  $q \gg 0$  ( $q \geq 0$ ) if and only if  $M$  is (strictly) semi-monotone. (Cottle, Pang, and Stone 2009a)

---

Hence, by verifying that  $M$  is (strictly) semi-monotone, we can reassure ourselves that merely introducing the bound will not change the solution away from the bound. When this condition is violated, even when the economy is a long way from the bound, there may be solutions which jump to the bound. Again, since principal sub-matrices of (strictly) semi-monotone are (strictly) semi-monotone, a failure of (strict) semi-monotonicity for some  $T$  implies a failure for all larger  $T$ . Furthermore, if  $d_{0,1} < 0$  then again for sufficiently large  $T$ ,  $M$  cannot be semi-monotone.



Where there are multiple solutions, we might like to be able to select one via some objective function. This is tractable when either the number of solutions is finite, or the solution set is convex. Conditions for this are given in the appendix, section 6.5.

## 2.8. Checking the existence and uniqueness conditions in practice

This section has presented many results on existence and uniqueness, but the practical details of what one should test and in what order may still be unclear. Luckily, a lot of the decisions are automated by the author's DynareOBC toolkit, but we present a suggested testing procedure here in any case. This also serves to give an overview of our results and their limitations.

For checking feasibility and existence, the most powerful result is the combination of Corollary 1 and Proposition 5. If the lower bound from Proposition 5 is positive, then Corollary 1 guarantees that for all sufficiently high  $T$ , the LCP is always feasible. If further conditions are satisfied for a given  $T$ , (see Proposition 6 and Proposition 7) then this guarantees existence for that particular  $T$ . However, since the additional conditions are sufficient and not necessary, in practice it may not be worth checking them, since we have never encountered a problem without a solution that was nonetheless feasible. Finding a  $T$  for which Proposition 5 produces a positive lower bound on  $\underline{\zeta}$  requires a bit of trial and error.  $T$  will need to be big enough that the asymptotic approximation is accurate, which usually requires  $T$  to be bigger than the time it takes for the model's dynamics to die out. However, if  $T$  is too large, then DynareOBC's conservative approach to handling numerical error means that it can be difficult to reject  $\underline{\zeta} = 0$ . Usually though, an intermediary value for  $T$  can be found at which we can establish  $\underline{\zeta} > 0$ , even with this conservative approach to numerical error.

For checking non-existence, Corollary 1 and Proposition 5 can still be useful, though in this case, they do not provide definitive proof of non-feasibility, due to inescapable numerical inaccuracies. For a particular  $T$ , we may test if  $M$  is not an S-matrix in time polynomial in  $T$  by solving a simple linear programming problem. If  $M$  is not an S-matrix, then by Proposition 4, there are some  $q$  for which there is no solution which finally escapes the bound after at most  $T$  periods. With  $T$  larger than the time it takes for the model's dynamics to die out, this provides further evidence of non-existence for arbitrarily large  $T$ . In any case, given that only having a solution that stays at the bound for 250 years is arguably as bad as having no solution at all, for medium scale models, we suggest to just check if  $M$  is an S-matrix with  $T = 1000$ .

For checking uniqueness vs multiplicity, it is important to remember that while we can prove uniqueness for a given finite  $T$  by proving that the  $M$  matrix is a P-matrix, once we have found one  $T$  for which  $M$  is not a P-matrix (so there are multiple solutions, by Proposition 9), we know the same is true for all higher  $T$ . Additionally, it is much easier to prove a matrix is not a P-matrix than to prove that it is, as the former

just requires us to find one principal sub-matrix with negative determinant, while the latter requires us to check all such sub-matrices. Thus, in practice, checking that  $M$  is a P-matrix for very large  $T$  may not be computationally feasible, though finding a counter-example usually is.

If we wish to prove multiplicity by finding a principal sub-matrix with negative determinant, it is often sensible to begin by checking the contiguous principal sub-matrices.<sup>16</sup> These correspond to a single spell at the ZLB which is natural given that impulse responses in DSGE models tend to be single peaked. This is so reliable a diagnostic (and so fast) that DynareOBC reports it automatically for all models. Additionally, DynareOBC always checks the sign of  $d_{0,1}$ , and a condition on the arguments of the  $M$  matrix's eigenvalues,<sup>17</sup> which can also quickly help rule out being a P-matrix. If none of these conditions are informative, then it is sensible to start by checking all the  $2 \times 2$  principal sub-matrices, then the  $3 \times 3$  ones, and so on. With  $T$  around the half-life of the model's dynamics, usually one of these tests will quickly produce a counter-example. A similar search strategy can be used to rule out semi-monotonicity, implying multiplicity when away from the bound, by Proposition 10.

Given the computational challenge of verifying whether  $M$  is a P-matrix, it may be tempting to wonder if our results really enable one to accomplish anything that could not have been accomplished by a naïve brute force approach. For example, given  $T$  and an initial state, one could check for multiple equilibria by considering all of the  $2^T$  possible combinations of periods at which the model could be at the bound and testing if each guess is consistent with the model, following, for example, the solution algorithms of Fair and Taylor (1983) or Guerrieri and Iacoviello (2012). Since there are  $2^T$  principal sub-matrices of  $M$ , it might seem likely that this will be computationally very similar to checking if  $M$  is a P-matrix. However, our uniqueness results are not conditional on  $q$  or the initial state. Thus, while the brute force approach may tell you about uniqueness given an initial state in a reasonable amount of time, using our results, in a comparable amount of time you will learn whether there are multiple solutions for any possible  $q$ . A brute force approach to checking for all possible initial conditions would require one to consider each pair of possible sets of periods at the bound, of which there are  $2^{2T-1} - 2^{T-1}$ .<sup>18</sup> This is far more computationally demanding than our approach, and becomes intractable for even very small  $T$ . Additionally, our approach is numerically more robust, allows the easy management of the effects of numerical error in order to avoid false positives and false negatives, and requires less

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<sup>16</sup> Some care must be taken though as checking the signs of determinants of large matrices is numerically unreliable.

<sup>17</sup> All of the eigenvalues of a  $T \times T$  P-matrix have complex arguments in the interval  $(-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T})$ , and all of the eigenvalues of a  $T \times T$  P<sub>0</sub>-matrix have complex arguments in the interval  $[-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T}]$  (Fang 1989).

<sup>18</sup> For each set of periods at the bound, one has a linear restriction that must be satisfied by the path. Imposing that  $x_0$  is common to both paths then gives a linear system where any solution implies multiplicity, if the regimes agree with the guesses along both paths.

work in each step. Finally, we stress that several of the properties used in our existence results may be checked in polynomial time in  $T$ , so for these properties, our advantage over a brute force approach is even clearer.

### 3. Applications to New Keynesian models

Since one of the great successes of the original Blanchard and Kahn (1980) result has been the development of the Taylor principle, and since the zero lower bound remains of great interest to policy makers, we now seek to apply our theoretical results to New Keynesian models with a ZLB. We stress though that our results are likely to have application to many other classes of model, including, for example, models with financial frictions, or models with overlapping generations and borrowing constraints.

In the first subsection, here, we examine the simple Brendon, Paustian, and Yates (2013) (henceforth: BPY) model in further detail, looking at its response to shocks, and at its behaviour when we do not constrain agents to believe that they will escape the bound in one period. We go on to consider a variant of the BPY model with price targeting, and show that it produces determinacy. In the BPY model, as previously noted, multiplicity and non-existence stemmed from a response to growth rates in the Taylor rule. However, we do not want to give the impression that multiplicity and non-existence are only caused by such a response, or that they are only a problem in carefully constructed theoretical examples. In subsection 3.3, we show that a standard NK model with positive steady-state inflation and a ZLB possesses multiple equilibria in some states, and no solutions in others, even with an entirely standard Taylor rule. We also show that here too price level targeting is sufficient to restore determinacy. Next, we show that these conclusions also carry through to the posterior-modes of the Smets and Wouters (2003; 2007) models. We conclude this section by discussing the economic significance of multiplicity, arguing that self-fulfilling recessions could well explain some aspects of recent economic outcomes in the US, Europe, and Japan.

#### 3.1. The simple Brendon, Paustian, and Yates (2013) (BPY) model

To obtain a better understanding of the potential economic consequences of the multiplicity we found in the BPY (2013) model, here we present impulse responses to a shock to the Euler equation under two different solutions. With the shock added to the Euler equation, it now takes the form:

$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma} (x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1} - (0.01)\varepsilon_t).$$

All other equations remain as given in section 2.4. We take the parameterisation  $\sigma = 1$ ,  $\beta = 0.99$ ,  $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85} (2 + \sigma)$ , following BPY, and we additionally set  $\alpha_\pi = 1.5$  and  $\alpha_{\Delta y} = 1.6$ , to ensure we are in the region with multiple solutions. In Figure 5, we show two alternative solutions to the impulse response to a magnitude 1 shock to  $\varepsilon_t$ . The solid line in the left plot gives the solution which minimises  $\|y\|_\infty$ . This solution

never hits the bound, and is moderately expansionary. The solid line in the right plot gives the solution which minimises  $\|q + My\|_\infty$ . (The dotted line in the right plot repeats the left plot, for comparison.) This solution stays at the bound for two periods, and is strongly contractionary, with a magnitude around 100 times larger than the other solution.

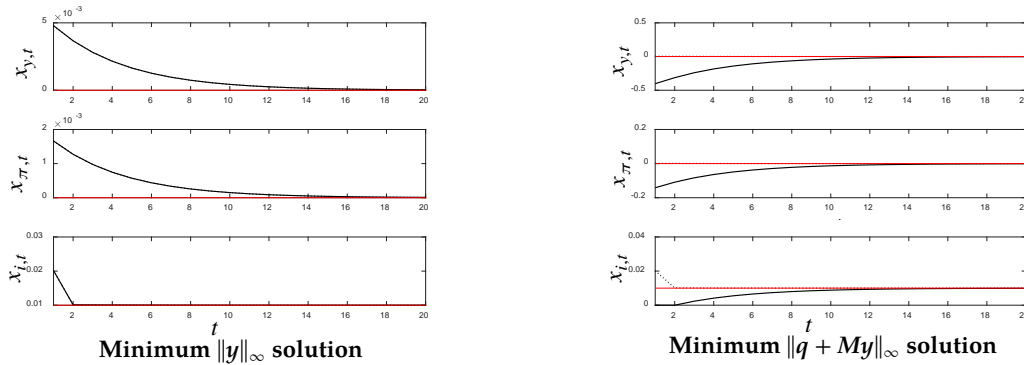


Figure 5: Alternative solutions following a magnitude 1 impulse to  $\varepsilon_t$

When  $T > 1$ , the previous results imply that if  $\alpha_{\Delta y} > \sigma\alpha_\pi$ , then  $M$  is neither  $P_0$ , general positive semi-definite, semi-monotone, co-positive, nor sufficient, by Lemma 4, since the top-left  $1 \times 1$  principal sub-matrix of  $M$  is the same as when  $T = 1$ . Thus, if anything, when  $T > 1$ , the parameter region in which there are multiple solutions (when away from the bound or at it) is larger. However, numerical experiments suggest that this parameter region in fact remains the same as  $T$  increases, which is unsurprising given the weak persistence of this model. Thus, if we want more interesting results with higher  $T$ , we need to consider a model with a stronger persistence mechanism. One obvious possibility is to consider models with either persistence in the interest rate, or persistence in the “shadow” rate that would hold were it not for the ZLB. However, unsurprisingly, in online appendix E we find that persistence in the shadow interest rate does not change the determinacy region providing  $T$  is large enough. Identical results can be shown for persistence in the actual interest rate.

### 3.2. The BPY model with price targeting

One way to introduce persistence to shadow interest rates is to set:

$$\begin{aligned} x_{d,t} &= (1 - \rho) \left( 1 - \beta + \frac{\alpha_{\Delta y}}{1 - \rho} (x_{y,t} - x_{y,t-1}) + \frac{\alpha_\pi}{1 - \rho} x_{\pi,t} \right) + \rho x_{d,t-1} \\ &= (1 - \rho)(1 - \beta) + (\alpha_{\Delta y}(x_{y,t} - x_{y,t-1}) + \alpha_\pi x_{\pi,t}) + \rho x_{d,t-1}, \end{aligned}$$

where  $x_{i,t} = \max\{0, x_{d,t}\}$ . If the second bracketed term was multiplied by  $(1 - \rho)$ , then this would be entirely standard, however as written here, we have that in the limit as  $\rho \rightarrow 1$ , this tends to:

$$x_{d,t} = 1 - \beta + \alpha_{\Delta y} x_{y,t} + \alpha_\pi x_{p,t}$$

where  $x_{p,t}$  is the price level, so  $x_{\pi,t} = x_{p,t} - x_{p,t-1}$ . This is a level targeting rule, with nominal GDP targeting as a special case with  $\alpha_{\Delta y} = \alpha_{\pi}$ . Note that the omission of the  $(1 - \rho)$  coefficient on  $\alpha_{\Delta y}$  and  $\alpha_{\pi}$  is akin to having a “true” response to output growth of  $\frac{\alpha_{\Delta y}}{1-\rho}$  and a “true” response to inflation of  $\frac{\alpha_{\pi}}{1-\rho}$ , so in the limit as  $\rho \rightarrow 1$ , we effectively have an infinitely strong response to these quantities. It turns out that this is sufficient to produce determinacy for all  $\alpha_{\Delta y}, \alpha_{\pi} \in (0, \infty)$ . In particular, given the model:

$$\begin{aligned} x_{i,t} &= \max\{0, 1 - \beta + \alpha_{\Delta y} x_{y,t} + \alpha_{\pi} x_{p,t}\}, \\ x_{y,t} &= \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma} (x_{i,t} + \beta - 1 - \mathbb{E}_t x_{p,t+1} + x_{p,t}), \\ x_{p,t} - x_{p,t-1} &= \beta \mathbb{E}_t x_{p,t+1} - \beta x_{p,t} + \gamma x_{y,t}, \end{aligned}$$

we prove in online appendix F that the following proposition holds:

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**Proposition 11** The BPY model with price targeting is in the form of Problem 2, and satisfies Assumptions 1 and 2. With  $T = 1, M > 0$  for all  $\alpha_{\pi} \in (0, \infty), \alpha_{\Delta y} \in [0, \infty)$ .

---

Furthermore, with  $\sigma = 1, \beta = 0.99, \gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85} (2 + \sigma)$ , as before, and  $\alpha_{\Delta y} = 1, \alpha_{\pi} = 1$ , if we check our lower bound on  $\varsigma$  with  $T = 20$ , we find that  $\varsigma > 0.042$ . Hence, this model is always feasible for any sufficiently large  $T$ . Given that  $d_0 > 0$  for this model, and that for  $T = 20, M$  is a P-matrix, this is strongly suggestive of the existence of a unique solution for any  $q$  and for arbitrarily large  $T$ .

### 3.3. The linearized Fernández-Villaverde et al. (2015) model

The discussion of the BPY (2013) model might lead one to believe that multiplicity and non-existence is solely a consequence of overly aggressive monetary responses to output growth, and overly weak monetary responses to inflation. However, it turns out that in basic New Keynesian models with positive inflation in steady-state, and hence price dispersion, even without any monetary response to output growth, and even with extremely aggressive monetary responses to inflation, there are still multiple equilibria in some states of the world (i.e. for some  $q$ ), and no solutions in others. Price level targeting is again sufficient to fix these problems though.

We show these results in the Fernández-Villaverde et al. (2015) model, which is a basic non-linear New Keynesian model without capital or price indexation of non-resetting firms, but featuring (non-valued) government spending and steady-state inflation (and hence price-dispersion). We refer the reader to the original paper for the model’s equations. After substitutions, the model has four non-linear equations which are functions of gross inflation, labour supply, price dispersion and an auxiliary variable introduced from the firms’ price-setting first order condition. Of these variables, only price dispersion enters with a lag. We linearize the model around its steady-state, and then reintroduce the “max” operator which linearization removed

from the Taylor rule.<sup>19</sup> All parameters are set to the values given in Fernández-Villaverde et al. (2015). There is no response to output growth in the Taylor rule, so any multiplicity cannot be a consequence of the mechanism highlighted by BPY (2013).

For this model, numerical calculations reveal that with  $T \leq 14$ ,  $M$  is a P-matrix. However, with  $T \geq 15$ ,  $M$  is not a P matrix, and thus there are certainly some states of the world (some  $q$ ) in which the model has multiple solutions. One example was given in Figure 4. Furthermore, with  $T = 1000$ , our upper bound on  $\varsigma$  from Proposition 4 implies that  $\varsigma \leq 0 + \text{numerical error}$ , suggesting that  $M$  is not an S-matrix for arbitrarily large  $T$ , by Corollary 1. If this is correct, then even for arbitrarily large  $T$ , there are some  $q$  for which no solution exists.

However, if we replace inflation in the monetary rule with the price level relative to its linear trend, which evolves according to:

$$x_{p,t} = x_{p,t-1} + x_{\pi,t} - x_{\pi}, \quad (5)$$

then with  $T = 200$ , we have that  $M$  is an S-matrix, and the lower bound from Proposition 4 implies that  $\varsigma > 0.003$ , and hence that for all sufficiently large  $T$ ,  $M$  is an S-matrix (by Corollary 1), so there is always a feasible solution. Furthermore, with  $T = 20$ ,  $M$  is a P-matrix, and even with  $T = 200$ ,  $M$  has no contiguous sub-matrices with negative determinant. This is strongly suggestive of uniqueness even for arbitrarily large  $T$ , given the reasonably short lived dynamics of the model.

### 3.4. The Smets and Wouters (2003) and Smets and Wouters (2007) models

Smets and Wouters (2003) and Smets and Wouters (2007) are the canonical medium-scale linear DSGE models, featuring assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation and quite general monetary policy reaction functions. The former model is estimated on Euro area data, while the latter is estimated on US data. The latter model also contains trend growth (permitting its estimation on non-detrended data), and a slightly more general aggregator across industries. However, overall, they are quite similar models, and any differences in their behaviour chiefly stems from differences in the estimated parameters. Since both models are incredibly well known in the literature, we omit their equations here, referring the reader to the original papers for further details.

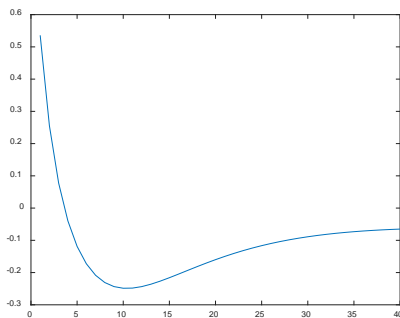
To assess the likelihood of multiple equilibria at or away from the zero lower bound, we augment each model with a ZLB on nominal interest rates, and evaluate the properties of each model's  $M$  matrix at the estimated posterior-modes from the original papers. To minimise the deviation from the original papers, we do not

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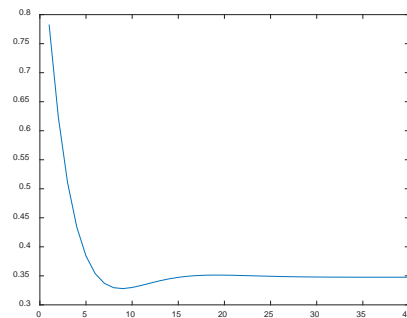
<sup>19</sup> Prior to linearization, we first transform the model's variables so that the transformed variables may take values on the entire real line. I.e. we work with the logarithms of labour supply, price dispersion and the auxiliary variable. For inflation, we note that inflation is always less than  $\theta^{\frac{1}{1-\varepsilon}}$  (in the notation of Fernández-Villaverde et al. (2015)). Thus, we work with a logit transformation of inflation over  $\theta^{\frac{1}{1-\varepsilon}}$ . This is generally more accurate than working with the logarithm of inflation.

introduce an auxiliary variable for shadow nominal interest rates, so the monetary rules take the form of  $x_{r,t} = \max\{0, (1 - \rho_r)(\dots) + \rho_r x_{r,t-1} + \dots\}$ , in both cases. Our results would be essentially identical with a shadow nominal interest rate though.

As shown in Lemma 4, if the diagonal of the  $M$  matrix ever goes negative, then the  $M$  matrix cannot be general positive semi-definite, semi-monotone, sufficient,  $P_0$  or copositive, and hence the model will sometimes have multiple solutions even when away from the zero lower bound (i.e. for some strictly positive  $q$ ), by Proposition 10. In Figure 6, we plot the diagonal of the  $M$  matrix for each model in turn,<sup>20</sup> i.e. the impact on nominal interest rates in period  $t$  of news in period 1 that a positive, magnitude one shock will hit nominal interest rates in period  $t$ . Immediately, we see that while in the US model, these impacts remain positive at all horizons, in the Euro area model, these impacts turn negative after just a few periods, and remain so at least up to period 40. Therefore, in the ZLB augmented Smets and Wouters (2003) model, there is not always a unique equilibrium. Furthermore, if a run of future shocks was drawn from a distribution with unbounded support, then the value of these shocks was revealed to the model's agents (as in the stochastic extended path), then there would be a positive probability that the model without the ZLB would always feature positive interest rates, but that the model with the ZLB could hit zero.



The Smets and Wouters (2003) model



The Smets and Wouters (2007) model

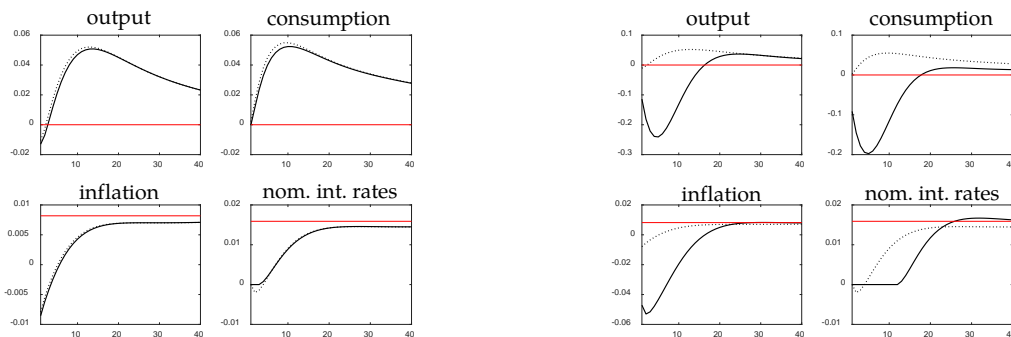
**Figure 6: The diagonals of the  $M$  matrices for the Smets and Wouters (2003; 2007) models**

It remains for us to assess whether  $M$  is a  $P_0$ -matrix or (strictly) semi-monotone for the Smets and Wouters (2007) model. Numerical calculations reveal that for  $T < 9$ ,  $M$  is a  $P$ -matrix, and hence is strictly semi-monotone. However, with  $T \geq 9$ , the top-left  $9 \times 9$  sub-matrix of  $M$  has negative determinant and is not an  $S$  or  $S_0$  matrix. Thus, for  $T \geq 9$ ,  $M$  is not a  $P_0$ -matrix or (strictly) semi-monotone, and hence this model also has multiple equilibria, even when away from the bound. Given that the US has been at the ZLB for over eight years, that  $T$  ought to be greater than eight quarters seems uncontroversial. While placing a larger coefficient on inflation in the Taylor rule can make the Euro area picture more like the US one, with a strictly positive diagonal to

<sup>20</sup> The MOD files for the Smets and Wouters (2003) and (2007) models were derived, respectively, from the Macro Model Database (Wieland et al. 2012) and files provided by Johannes Pfeifer here: <http://goo.gl/CP53x5>.

the  $M$  matrix, even with incredibly large coefficients,  $M$  remains a non-P-matrix for both models. Hence, in both the Euro area and the US, we ought to take seriously the possibility that the existence of the ZLB produces non-uniqueness.

As an example of such non-uniqueness, in Figure 7 we plot two different solutions following the most likely combination of shocks to the Smets and Wouters (2007) model that would produce negative interest rates for a year in the absence of a ZLB.<sup>21</sup> In both cases, the dotted line shows the response in the absence of the ZLB. Particularly notable is the flip in sign, since the shocks most likely to take the model to the ZLB for a year are expansionary ones reducing prices (i.e. positive productivity and negative mark-up shocks). The following section shows an example of multiplicity in the Smets and Wouters (2003) model, and discusses the economic relevance of such multiplicity.



**Figure 7: Two alternative solutions following a combination of shocks to the Smets and Wouters (2007) model**  
All variables are in logarithms. The precise combination of shocks is detailed in footnote 21.

In addition, it turns out that for neither model is  $M$  an S-matrix even with  $T = 1000$ , and thus for both models there are some  $q \in \mathbb{R}^{1000}$  for which no solution exists. This is strongly suggestive of non-existence for some  $q$  even for arbitrarily large  $T$ , something that is reinforced by the fact that for the Smets and Wouters (2007) model, with  $T = 1000$ , Proposition 4 gives that  $\zeta \leq 0 + \text{numerical error}$ .

Alternatively, suppose we replace the monetary rule in both models by:

$$x_{r,t} = \max\{0, (1 - \rho_r)(x_{y,t} + x_{p,t}) + \rho_r x_{r,t-1}\}$$

where  $\rho_r$  is as in the respective original model, where the price level  $x_{p,t}$  again evolves per equation (16), and where  $x_{y,t}$  is output relative to its linear trend. Then, for both models, with  $T = 20$ ,  $M$  was a P-matrix, and there are no contiguous principal submatrices with negative determinant even for  $T = 200$ . Furthermore, from Proposition 4, with  $T = 1000$ , for the Euro area model we have that  $\zeta > 3 \times 10^{-7}$  and for the US model we have that  $\zeta > 0.002$ , so Corollary 1 implies that a solution always exists to both models for sufficiently large  $T$ . As one would expect, this result is also robust to

<sup>21</sup> We find the vector  $w$  that minimises  $w'w$  subject to  $\bar{r} + Zw \leq 0$ , where  $\bar{r}$  is the steady-state interest rate, and columns of  $Z$  give four periods of the impulse response of interest rates to a given shocks. This gives the following shock magnitudes: productivity, 3.56; risk premium, -2.70; government, -1.63; investment, -4.43; monetary, -2.81; price mark-up, -3.19; wage mark-up, -4.14.



departures from equal, unit, coefficients. Thus, price level targeting again appears to be sufficient for determinacy in the presence of the ZLB.

### 3.5. Economic significance of multiplicity

There are two reasons why one might be sceptical about the economic significance of the multiple equilibria caused by the presence of the ZLB. Firstly, as with any non-fundamental equilibrium, the coordination of beliefs needed to sustain the equilibrium may be difficult. Secondly, as we have seen, self-fulfilling jumps to the zero lower bound may feature implausibly large falls in output and inflation. This reflects the implausibly large response to news about future policy innovations, a problem that has been termed the “forward guidance puzzle” in the literature (Carlstrom, Fuerst, and Paustian 2015; Del Negro, Giannoni, and Patterson 2015).<sup>22</sup>

However, if the economy is already in a recession, then both problems are substantially ameliorated. If interest rates are already low, then it does not seem too great a stretch to suggest that a drop in confidence may lead people to expect to hit the ZLB. Even more plausibly, if the economy is already at the ZLB, then small changes in confidence could easily select an equilibrium featuring a longer spell at the ZLB than in the equilibrium with the shortest time there. Indeed, there is no good reason people should coordinate on the equilibrium with the shortest time at the ZLB. Furthermore, with interest rates already low, the size of the required self-fulfilling news shock is much smaller, meaning that the additional drop in output and inflation caused by a jump to the ZLB will be much more moderate.

As an example, in Figure 8 we plot the impulse response to a large magnitude preference shock (scaling utility), in the Smets and Wouters (2003) model.<sup>23</sup> The shock is not quite large enough to send the economy to the zero lower bound<sup>24</sup> in the fundamental solution, shown with a dotted line. However, there is an alternative solution in which the economy jumps to the bound one period after the initial shock, remaining there for three periods. While the alternative solution features larger drops in output and inflation, they are less than twice the magnitude. Indeed, the falls are

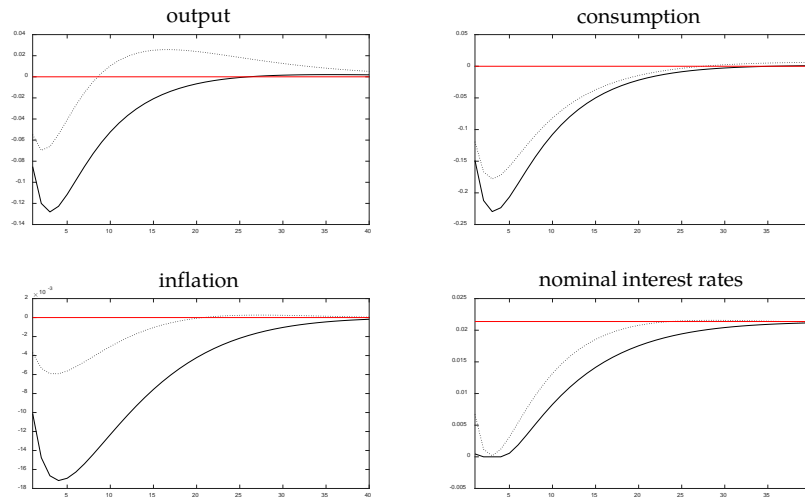
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<sup>22</sup> McKay, Nakamura, and Steinsson (2016) point out that these implausibly large responses to news are muted in models with heterogeneous agents, and give a simple “discounted Euler” approximation that produces similar results to a full heterogeneous agent model. While including a discounted Euler equation makes it harder to generate multiplicity (e.g. reducing the parameter space with multiplicity in the BPY (2013) model), when there is multiplicity, the resulting responses are much larger, as the weaker response to news means the required endogenous news shocks need to be much greater in order to drive the model to the bound.

<sup>23</sup> The shock is 22.5 standard deviations. While this is implausibly large, the economy could be driven to the bound with a run of much smaller shocks. It is also worth recalling that the model was estimated on the great moderation period, and so the estimated standard deviations may be too low. Finally, recent evidence (Cúrdia, del Negro, and Greenwald 2014) suggests that the shocks in DSGE models are better modelled as being fat tailed, making large shocks more likely.

<sup>24</sup> Since the Smets and Wouters (2003) model does not include trend growth, it is impossible to produce a steady-state value for nominal interest rates that is consistent with both the model and the data. We choose to follow the data, setting the steady-state of nominal interest rates to its mean level over the same sample period used by Smets and Wouters (2003), using data from the same source (Fagan, Henry, and Mestre 2005).

broadly in line with the magnitude of the crisis, with Eurozone GDP and consumption now being about 20% below a pre-crisis log-linear trend, and the largest drop in Eurozone consumption inflation from 2008q3 to 2008q4 being around 1%.<sup>25</sup> Considering this, we view it as plausible that multiplicity of equilibria could be a significant component of the explanation for the great recession.



**Figure 8: Two solutions following a preference shock in the Smets and Wouters (2003) model.**

All variables are in logarithms. The dotted line is a solution which does not hit the bound. The solid line is an alternative solution which does hit the bound.

## 4. Conclusion

This paper provides the first theoretical results on existence and uniqueness for otherwise linear models with occasionally binding constraints, given terminal conditions. As such, it may be thought of as doing for models with occasionally binding constraints what Blanchard and Kahn (1980) did for linear models.

We provided necessary and sufficient conditions for the existence of a unique equilibrium, as well as such conditions for uniqueness when away from the bound, all conditional on the economy returning to a specific steady-state. We also provide a toolkit (“DynareOBC”) which verifies these conditions. In our application to New Keynesian models, we showed that these conditions were violated in entirely standard models, rather than just being a consequence of policy rules responding to growth rates. In the presence of multiplicity, there is the potential for additional endogenous volatility from sunspots, so the welfare benefits of avoiding multiplicity may be substantial. Additionally, we saw that the additional equilibria may feature huge drops in output, providing further reasons for their avoidance. The possibility of self-fulfilling jumps and returns from the ZLB also gives a new rationale for the neo-Fisherian view that argues that raising interest rates may raise inflation at the ZLB.<sup>26</sup>

<sup>25</sup> Data was again from the area-wide model database (Fagan, Henry, and Mestre 2005).

<sup>26</sup> Theoretical and empirical evidence for this view is presented in Cochrane (2015).

Luckily, our results suggest that given belief in an eventual return to inflation, a determinate equilibrium may be produced in standard New Keynesian models if the central bank switches to targeting the price level, rather than the inflation rate. There is of course a large literature advocating price level targeting already. Vestin (2006) made an important early contribution by showing that its history dependence mimics the optimal rule, a conclusion reinforced by Giannoni (2010). Eggertsson and Woodford (2003) showed the particular desirability of price level targeting in the presence of the ZLB, since it produces inflation after the bound is escaped. A later contribution by Nakov (2008) showed that this result survived taking a fully global solution, and Coibion, Gorodnichenko, and Wieland (2012) showed that it still holds in a richer model. More recently, Basu and Bundick (2015) have argued that a response to the price level avoids the kinds of equilibrium non-existence problems stressed by Mendes (2011), while also solving the contractionary bias caused by the ZLB, drastically improving welfare. Our argument is distinct from all of these; we showed that in the presence of the ZLB, inflation targeting rules are indeterminate, even conditional on an eventual return to inflation, whereas price level targeting rules produce determinacy, in the sense of the existence of a unique path returning to the inflationary steady state.<sup>27</sup> Consequently, if one believes the arguments for the Taylor principle in the absence of the ZLB, then one should advocate price level targeting if the ZLB constraint is inescapable.

In addition, we provided conditions for existence of any solution that converges to the “good” steady-state, conditions which again may be verified using our DynareOBC toolkit. We showed that under inflation targeting, standard New Keynesian models again failed to satisfy these conditions over all the space of state variables and shocks. Whereas the literature started by Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b) showed that the existence of a “bad” steady-state may imply additional volatility if agents long-run beliefs are not pinned down by the inflation target, here we showed that under inflation targeting, there was positive probability of arriving in a state from which there was no way for the economy to converge to the “good” steady-state. This in turn implies that agents should not place prior certainty on converging to the “good” steady-state, thus rationalising the beliefs required to get

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<sup>27</sup> Our results are also distinct from those of Adão, Correia, and Teles (2011) who showed that if the central bank is not constrained to respect the ZLB out of equilibrium, and if the central bank uses a rule that responds to the right hand side of the Euler equation, then a globally unique equilibrium may be produced, even without ruling out explosive beliefs about prices. Their rule has the flavour of a (future) price-targeting rule, due to the presence of future prices in the right-hand side of the Euler equation. Here though, we are assuming that the central bank must satisfy the ZLB even out of equilibrium, which makes it harder to produce uniqueness. Additionally, we do not require that the central bank can choose a knife-edge value for its response to the (future) price-level, or that it knows the precise form of agents’ utility function, both of which are apparently required by the rule of Adão, Correia, and Teles and which may be difficult in practice. However, in line with the New Keynesian literature, we maintain the standard assumption that explosive paths for inflation are ruled out, an assumption which the knife-edge rules of Adão, Correia, and Teles do not require. (Note that the unstable solutions under price level targeting feature exponential growth in the logarithm of the price level, which also implies explosions in inflation rates.)

the kind of global multiplicity at the zero lower bound that these and other authors have focussed on. Once again though, we showed that price level targeting is sufficient to restore existence and determinacy.

## 5. References

- Adão, Bernardino, Isabel Correia, and Pedro Teles. 2011. "Unique Monetary Equilibria with Interest Rate Rules." *Review of Economic Dynamics* 14 (3) (July): 432–442.
- Adjemian, Stéphane, and Michel Juillard. 2013. "Stochastic Extended Path Approach." CEIS Tor Vergata.
- . 2016. "The Stochastic Extended Path Approach." National Bank of Belgium.
- Albanesi, Stefania, V. V. Chari, and Lawrence J. Christiano. 2003. "Expectation Traps and Monetary Policy." *Review of Economic Studies* 70 (4): 715–741.
- Aliprantis, C.D., and K. Border. 2013. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Studies in Economic Theory. Springer Berlin Heidelberg.
- Anderson, Gary. 2015. "A Solution Strategy for Occasionally Binding Constraints." Unpublished.
- Armenter, Roc. 2016. *The Perils of Nominal Targets*. Federal Reserve Bank of Philadelphia.
- Aruoba, S. Borağan, Pablo Cuba-Borda, and Frank Schorfheide. 2014. *Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries*. Penn Institute for Economic Research, Department of Economics, University of Pennsylvania.
- Ascari, Guido, and Tiziano Ropele. 2009. "Trend Inflation, Taylor Principle, and Indeterminacy." *Journal of Money, Credit and Banking* 41 (8). *Journal of Money, Credit and Banking*: 1557–1584.
- Basu, Susanto, and Brent Bundick. 2015. "Endogenous Volatility at the Zero Lower Bound: Implications for Stabilization Policy." *National Bureau of Economic Research Working Paper Series* No. 21838.
- Beaudry, Paul, and Franck Portier. 2006. "Stock Prices, News, and Economic Fluctuations." *American Economic Review* 96 (4): 1293–1307.
- Benhabib, Jess, Stephanie Schmitt-Grohé, and Martín Uribe. 2001a. "Monetary Policy and Multiple Equilibria." *American Economic Review* 91 (1). *American Economic Review*: 167–186.
- . 2001b. "The Perils of Taylor Rules." *Journal of Economic Theory* 96 (1–2). *Journal of Economic Theory*: 40–69.
- Benigno, Pierpaolo, and Michael Woodford. 2012. "Linear-Quadratic Approximation of Optimal Policy Problems." *Journal of Economic Theory* 147 (1). *Journal of Economic Theory*: 1–42.
- Blanchard, Olivier Jean, and Charles M. Kahn. 1980. "The Solution of Linear Difference Models under Rational Expectations." *Econometrica* 48 (5): 1305–1311.
- Bodenstein, Martin. 2010. "Trade Elasticity of Substitution and Equilibrium Dynamics." *Journal of Economic Theory* 145 (3) (May): 1033–1059.
- Bodenstein, Martin, Luca Guerrieri, and Christopher J. Gust. 2013. "Oil Shocks and the Zero Bound on Nominal Interest Rates." *Journal of International Money and Finance* 32 (February): 941–967.
- Braun, R. Anton, Lena Mareen Körber, and Yuichiro Waki. 2012. *Some Unpleasant Properties of Loglinearized Solutions When the Nominal Rate Is Zero*. FRB Atlanta Working Paper. Federal Reserve Bank of Atlanta.
- Brendon, Charles, Matthias Paustian, and Tony Yates. 2013. "The Pitfalls of Speed-Limit Interest Rate Rules at the Zero Lower Bound." Bank of England.
- . 2016. "Self-Fulfilling Recessions at the Zero Lower Bound."
- Calvo, Guillermo A. 1983. "Staggered Prices in a Utility-Maximizing Framework." *Journal of Monetary Economics* 12 (3): 383–398.
- Carlstrom, Charles T., Timothy S. Fuerst, and Matthias Paustian. 2015. "Inflation and Output in New Keynesian Models with a Transient Interest Rate Peg." *Journal of Monetary Economics* 76 (November): 230–243.
- Christiano, Lawrence J., and Martin Eichenbaum. 2012. "Notes on Linear Approximations, Equilibrium Multiplicity and E-Learnability in the Analysis of the Zero Lower Bound." *Manuscript, Northwestern University*.
- Christiano, Lawrence J., and Massimo Rostagno. 2001. *Money Growth Monitoring and the Taylor Rule*. Working Paper. National Bureau of Economic Research.
- Cochrane, John H. 2015. "Do Higher Interest Rates Raise or Lower Inflation?" Unpublished.
- Coibion, Olivier, Yuriy Gorodnichenko, and Johannes Wieland. 2012. "The Optimal Inflation Rate in New Keynesian Models: Should Central Banks Raise Their Inflation Targets in Light of the Zero Lower Bound?" *The Review of Economic Studies* 79 (4) (October 1): 1371–1406.
- Cottle, Richard W. 2009. "Linear Complementarity Problem." In *Encyclopedia of Optimization*, edited by Christodoulos A. Floudas and Panos M. Pardalos, 1873–1878. Springer US.

- Cottle, Richard W., Jong-Shi Pang, and Richard E. Stone. 2009a. "3. Existence and Multiplicity." In *The Linear Complementarity Problem*, 137–223. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.
- . 2009b. "2. Background." In *The Linear Complementarity Problem*, 43–136. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.
- Cúrdia, Vasco, Marco del Negro, and Daniel L. Greenwald. 2014. "Rare Shocks, Great Recessions." *Journal of Applied Econometrics* 29 (7): 1031–1052.
- Davig, Troy, and Eric M. Leeper. 2007. "Generalizing the Taylor Principle." *American Economic Review* 97 (3): 607–635.
- Del Negro, Marco, Marc Giannoni, and Christina Patterson. 2015. *The Forward Guidance Puzzle*. Staff Reports. Federal Reserve Bank of New York.
- Eggertsson, Gauti B., and Michael Woodford. 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy." *Brookings Papers on Economic Activity* 34 (1): 139–235.
- Fagan, Gabriel, Jerome Henry, and Ricardo Mestre. 2005. "An Area-Wide Model for the Euro Area." *Economic Modelling* 22 (1): 39–59.
- Fair, Ray C, and John B Taylor. 1983. "Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models." *Econometrica* 51 (4) (July): 1169–85.
- Fang, Li. 1989. "On the Spectra of P- and P0-Matrices." *Linear Algebra and Its Applications* 119 (July): 1–25.
- Farmer, Roger E. A., Daniel F. Waggoner, and Tao Zha. 2010. "Generalizing the Taylor Principle: Comment." *American Economic Review* 100 (1): 608–17.
- Farmer, Roger E.A., Daniel F. Waggoner, and Tao Zha. 2011. "Minimal State Variable Solutions to Markov-Switching Rational Expectations Models." *Frontiers in Structural Macroeconomic Modeling* 35 (12) (December): 2150–2166.
- Fernández-Villaverde, Jesús, Grey Gordon, Pablo Guerrón-Quintana, and Juan F. Rubio-Ramírez. 2015. "Nonlinear Adventures at the Zero Lower Bound." *Journal of Economic Dynamics and Control* 57 (C). *Journal of Economic Dynamics and Control*: 182–204.
- Gavin, William T., Benjamin D. Keen, Alexander W. Richter, and Nathaniel A. Throckmorton. 2015. "The Zero Lower Bound, the Dual Mandate, and Unconventional Dynamics." *Journal of Economic Dynamics and Control* 55 (June): 14–38.
- Giannoni, Marc P. 2010. *Optimal Interest-Rate Rules in a Forward-Looking Model, and Inflation Stabilization versus Price-Level Stabilization*. NBER Working Papers. National Bureau of Economic Research, Inc.
- Guerrieri, Luca, and Matteo Iacoviello. 2012. "A Toolkit to Solve Models with Occasionally Binding Constraints Easily."
- Gürkaynak, Refet S., Andrew Levin, and Eric Swanson. 2010. "Does Inflation Targeting Anchor Long-Run Inflation Expectations? Evidence from the U.S., UK, and Sweden." *Journal of the European Economic Association* 8 (6) (December): 1208–1242.
- Hebden, James, Jesper Lindé, and Lars Svensson. 2011. "Optimal Monetary Policy in the Hybrid New Keynesian Model under the Zero Lower Bound." Unpublished.
- Holden, Tom D. 2010. *Products, Patents and Productivity Persistence: A DSGE Model of Endogenous Growth*. Economics Series Working Paper. University of Oxford, Department of Economics.
- . 2016. "Computation of Solutions to Dynamic Models with Occasionally Binding Constraints." Unpublished.
- Holden, Tom D., and Michael Paetz. 2012. "Efficient Simulation of DSGE Models with Inequality Constraints." School of Economics Discussion Papers.
- Kamihigashi, Takashi. 2014. "Elementary Results on Solutions to the Bellman Equation of Dynamic Programming: Existence, Uniqueness, and Convergence." *Economic Theory* 56 (2) (June 1): 251–273.
- Kamihigashi, Takashi, and Santanu Roy. 2003. *A Nonsmooth, Nonconvex Model of Optimal Growth*. Discussion Paper Series. Research Institute for Economics & Business Administration, Kobe University.
- Levine, Paul, Joseph Pearlman, and Richard Pierse. 2008. "Linear-Quadratic Approximation, External Habit and Targeting Rules." *Journal of Economic Dynamics and Control* 32 (10). *Journal of Economic Dynamics and Control*: 3315–3349.
- Marx, Magali, and Jean Barthelemy. 2013. *State-Dependent Probability Distributions in Non Linear Rational Expectations Models*. Society for Economic Dynamics.
- McKay, Alisdair, Emi Nakamura, and Jón Steinsson. 2016. *The Discounted Euler Equation: A Note*. National Bureau of Economic Research, Inc.
- Mendes, Rhys R. 2011. *Uncertainty and the Zero Lower Bound: A Theoretical Analysis*. University Library of Munich, Germany.

- Mertens, Karel RSM, and Morten O. Ravn. 2014. "Fiscal Policy in an Expectations-Driven Liquidity Trap." *The Review of Economic Studies*: rdu016.
- Nakata, Taisuke, and Sebastian Schmidt. 2014. *Conservatism and Liquidity Traps*. Board of Governors of the Federal Reserve System (U.S.).
- Nakov, Anton. 2008. "Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate." *International Journal of Central Banking* 4 (2). *International Journal of Central Banking*: 73–127.
- Richter, Alexander W., and Nathaniel A. Throckmorton. 2015. "The Zero Lower Bound: Frequency, Duration, and Numerical Convergence." *The B.E. Journal of Macroeconomics* 15 (1): 157–182.
- Roos, C., T. Terlaky, and J. P. Vial. 2006. *Interior Point Methods for Linear Optimization*. Springer.
- Samelson, Hans, Robert M. Thrall, and Oscar Wesler. 1958. "A Partition Theorem for Euclidean N-Space." *Proceedings of the American Mathematical Society* 9 (5): 805–807.
- Schmitt-Grohé, Stephanie, and Martín Uribe. 2012. *The Making Of A Great Contraction With A Liquidity Trap and A Jobless Recovery*. National Bureau of Economic Research, Inc.
- Sims, Christopher A. 2002. "Solving Linear Rational Expectations Models." *Computational Economics* 20 (1–2) (October 1): 1–20.
- Smets, Frank, and Rafael Wouters. 2003. "An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area." *Journal of the European Economic Association* 1 (5): 1123–1175.
- . 2007. "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach." *American Economic Review* 97 (3) (June): 586–606.
- Stokey, N., R. J. Lucas, and E. Prescott. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press.
- Trefethen, L.N., and M. Embree. 2005. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*. Princeton University Press.
- Väliäho, Hannu. 1986. "Criteria for Copositive Matrices." *Linear Algebra and Its Applications* 81 (September 1): 19–34.
- Vestin, David. 2006. "Price-Level versus Inflation Targeting." *Journal of Monetary Economics* 53 (7) (October): 1361–1376.
- Wieland, Volker, Tobias Cwik, Gernot J. Müller, Sebastian Schmidt, and Maik Wolters. 2012. "A New Comparative Approach to Macroeconomic Modeling and Policy Analysis." *The Great Recession: Motivation for Re-Thinking Paradigms in Macroeconomic Modeling* 83 (3) (August): 523–541.
- Wright, Thomas G, and Lloyd N Trefethen. 2001. "Large-Scale Computation of Pseudospectra Using ARPACK and Eigs." *SIAM Journal on Scientific Computing* 23 (2): 591–605.
- Xu, Song. 1993. "Notes on Sufficient Matrices." *Linear Algebra and Its Applications* 191 (September 15): 1–13.

## 6. Appendices

### 6.1. Proof of Lemma 1

Since  $y_{1,t-1} = 0$  for  $t > T$ , and using Assumption 1,  $(x_{T+1} - \mu) = F(x_T - \mu)$ , so with  $t = T$ , defining  $s_{T+1} := 0$ ,  $(x_{t+1} - \mu) = s_{t+1} + F(x_t - \mu)$ . Proceeding now by backwards induction on  $t$ , note that  $0 = A(x_{t-1} - \mu) + B(x_t - \mu) + CF(x_t - \mu) + Cs_{t+1} + I_{.,1}y_{t,0}$ , so:

$$\begin{aligned}(x_t - \mu) &= -(B + CF)^{-1}[A(x_{t-1} - \mu) + Cs_{t+1} + I_{.,1}y_{t,0}] \\ &= F(x_{t-1} - \mu) - (B + CF)^{-1}(Cs_{t+1} + I_{.,1}y_{t,0}),\end{aligned}$$

i.e., if we define:  $s_t := -(B + CF)^{-1}(Cs_{t+1} + I_{.,1}y_{t,0})$ , then  $(x_t - \mu) = s_t + F(x_{t-1} - \mu)$ . By induction then, this holds for all  $t \in \{1, \dots, T\}$ , as required.<sup>28</sup>

### 6.2. Proof of Lemma 2

From the definition of Problem 2, we also have that for all  $t \in \mathbb{N}^+$ ,  $0 = A(x_{t-1}^{(2)} - \mu) + B(x_t^{(2)} - \mu) + C(x_{t+1}^{(2)} - \mu) + I_{.,1}e_t$ . Furthermore, if  $t > T$ , then  $t > T'$ , and hence  $e_t = 0$ . Hence, by Assumption 1,  $(x_{T+1}^{(2)} - \mu) = F(x_T^{(2)} - \mu)$ . Thus, much as before, with  $t = T$ , defining  $\tilde{s}_{T+1} := 0$ ,  $(x_{t+1}^{(2)} - \mu) = \tilde{s}_{t+1} + F(x_t^{(2)} - \mu)$ . Consequently,  $0 = A(x_{t-1}^{(2)} - \mu) + B(x_t^{(2)} - \mu) + CF(x_t^{(2)} - \mu) + C\tilde{s}_{t+1} + I_{.,1}e_t$ , so  $(x_t^{(2)} - \mu) = F(x_{t-1}^{(2)} - \mu) - (B + CF)^{-1}(C\tilde{s}_{t+1} + I_{.,1}e_t)$ , i.e., if we define:  $\tilde{s}_t := -(B + CF)^{-1}(C\tilde{s}_{t+1} + I_{.,1}e_t)$ , then  $(x_t^{(2)} - \mu) = \tilde{s}_t + F(x_{t-1}^{(2)} - \mu)$ . As before, by induction this must hold for all  $t \in \{1, \dots, T\}$ . By comparing the definitions of  $s_t$  and  $\tilde{s}_t$ , and the laws of motion of  $x_t$  under both problems, we then immediately have that if Problem 3 is started with  $x_0 = x_0^{(2)}$  and  $y_0 = e'_{1:T}$ , then  $x_t^{(2)}$  solves Problem 3. Conversely, if  $x_t^{(2)}$  solves Problem 3 for some  $y_0$ , then from the laws of motion of  $x_t$  under both problems it must be the case that  $\tilde{s}_t = s_t$  for all  $t \in \mathbb{N}$ , and hence from the definitions of  $s_t$  and  $\tilde{s}_t$ , we have that  $y_0 = e'_{1:T}$ . This establishes the result.

### 6.3. Proof of additional result for Proposition 1

Suppose that  $y_0 \in \mathbb{R}^T$  is such that  $y_0 \geq 0$ ,  $x_{1,1:T}^{(3)} \circ y'_0 = 0$  and  $x_{1,t}^{(3)} \geq 0$  for all  $t \in \mathbb{N}$ , where  $x_t^{(3)}$  is the unique solution to Problem 3 when started at  $x_0, y_0$ . We would like to prove that in this case  $x_t^{(3)}$  must also be a solution to Problem 2. I.e., we must prove that for all  $t \in \mathbb{N}^+$ :

$$\begin{aligned}x_{1,t}^{(3)} &= \max\{0, I_{1,.}\mu + A_{1,.}(x_{t-1}^{(3)} - \mu) + (B_{1,.} + I_{1,.})(x_t^{(3)} - \mu) + C_{1,.}(x_{t+1}^{(3)} - \mu)\}, \\ (A_{-1,.} + B_{-1,.} + C_{-1,.})\mu &= A_{-1,.}x_{t-1}^{(3)} + B_{-1,.}x_t^{(3)} + C_{-1,.}x_{t+1}^{(3)}.\end{aligned}\quad (6)$$

By the definition of Problem 3, the latter equation must hold with equality. Hence, we just need to prove that equation (6) holds for all  $t \in \mathbb{N}^+$ . So, let  $t \in \mathbb{N}^+$ . Now, if  $x_{1,t}^{(3)} > 0$ , then  $y_{t,0} = 0$ , by the complementary slackness type condition ( $x_{1,1:T}^{(3)} \circ y'_0 = 0$ ). Thus, from the definition of Problem 3:

$$x_{1,t}^{(3)} = I_{1,.}\mu + A_{1,.}(x_{t-1}^{(3)} - \mu) + (B_{1,.} + I_{1,.})(x_t^{(3)} - \mu) + C_{1,.}(x_{t+1}^{(3)} - \mu)$$

<sup>28</sup> This representation of the solution to Problem 3 was inspired by that of Anderson (2015).

$$= \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu)\},$$
 as required. The only remaining case is that  $x_{1,t}^{(3)} = 0$  (since  $x_{1,t}^{(3)} \geq 0$  for all  $t \in \mathbb{N}$ , by assumption), which implies that:

$$\begin{aligned} x_{1,t}^{(3)} = 0 &= A_{1,\cdot}(x_{t-1} - \mu) + B_{1,\cdot}(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0} \\ &= I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0}, \end{aligned}$$

by the definition of Problem 3. Thus:

$$I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) = -y_{t,0} \leq 0.$$

Consequently, equation (6) holds in this case too, completing the proof.

#### 6.4. Additional matrix property definitions

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**Definition 7 ((Non-)Degenerate matrix)** A matrix  $M \in \mathbb{R}^{T \times T}$  is called a **non-degenerate matrix** if the principal minors of  $M$  are all non-zero.  $M$  is called a **degenerate matrix** if it is not a non-degenerate matrix.

**Definition 8 (Sufficient matrices)** Let  $M \in \mathbb{R}^{T \times T}$ .  $M$  is called **column sufficient** if  $M$  is a  $P_0$ -matrix, and for each principal sub-matrix  $W := [M_{i,j}]_{i,j=k_1, \dots, k_S}$  of  $M$ , with zero determinant, and for each proper principal sub-matrix  $[W_{i,j}]_{i,j=l_1, \dots, l_R}$  of  $W$  ( $R < S$ ), with zero determinant, the columns of  $[W_{i,j}]_{\substack{i=1, \dots, S \\ j=l_1, \dots, l_R}}$  do not form a basis for the column space of  $W$ .<sup>29</sup>  $M$  is called **row sufficient** if  $M'$  is column sufficient.  $M$  is called **sufficient** if it is column sufficient and row sufficient.

**Definition 9 ((Strictly) Copositive)** A matrix  $M \in \mathbb{R}^{T \times T}$  is called **(strictly) copositive** if  $M + M'$  is (strictly) semi-monotone.<sup>30</sup>

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#### 6.5. Other properties of the solution set

Conditions for having a finite or convex set of solutions are given in the following propositions:

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**Proposition 12** The LCP  $(q, M)$  has a finite (possibly zero) number of solutions for all  $q \in \mathbb{R}^T$  if and only if  $M$  is non-degenerate. (Cottle, Pang, and Stone 2009a)

**Proposition 13** The LCP  $(q, M)$  has a convex (possibly empty) set of solutions for all  $q \in \mathbb{R}^T$  if and only if  $M$  is column sufficient. (Cottle, Pang, and Stone 2009a)

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#### 6.6. Results from dynamic programming

Alternative existence and uniqueness results for the infinite  $T$  problem can be established via dynamic programming methods, under the assumption that Problem 2 comes from the first order conditions solution of a social planner problem. These have the advantage that their conditions are potentially much easier to evaluate,

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<sup>29</sup> This may be checked via the singular value decomposition.

<sup>30</sup> Väliäho (1986) contains an alternative characterisation which avoids solving any linear programming problems.



though they also have somewhat limited applicability. We focus here on uniqueness results, since these are of greater interest.

Suppose that the social planner in some economy solves the following problem:

---

**Problem 5** Suppose  $\mu \in \mathbb{R}^n$ ,  $\Psi^{(0)} \in \mathbb{R}^{c \times 1}$  and  $\Psi^{(1)} \in \mathbb{R}^{c \times 2n}$  are given, where  $c \in \mathbb{N}$ . Define  $\tilde{\Gamma}: \mathbb{R}^n \rightarrow \mathbb{P}(\mathbb{R}^n)$  (where  $\mathbb{P}$  denotes the power-set operator) by:

$$\tilde{\Gamma}(x) = \left\{ z \in \mathbb{R}^n \mid 0 \leq \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix} \right\}, \quad (7)$$

for all  $x \in \mathbb{R}^n$ . (Note:  $\tilde{\Gamma}(x)$  will give the set of feasible values for next period's state if the current state is  $x$ . Equality constraints may be included by including an identical lower bound and upper bound.) Define:

$$\tilde{X} := \{x \in \mathbb{R}^n \mid \tilde{\Gamma}(x) \neq \emptyset\}, \quad (8)$$

and suppose without loss of generality that for all  $x \in \mathbb{R}^n$ ,  $\tilde{\Gamma}(x) \cap \tilde{X} = \tilde{\Gamma}(x)$ . (Note: this means that the linear inequalities bounding  $\tilde{X}$  are already included in those in the definition of  $\tilde{\Gamma}(x)$ . It is without loss of generality as the planner will never choose an  $\tilde{x} \in \tilde{\Gamma}(x)$  such that  $\tilde{\Gamma}(\tilde{x}) = \emptyset$ .) Further define  $\tilde{\mathcal{F}}: \tilde{X} \times \tilde{X} \rightarrow \mathbb{R}$  by:

$$\tilde{\mathcal{F}}(x, z) = u^{(0)} + u^{(1)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}' \tilde{u}^{(2)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}, \quad (9)$$

for all  $x, z \in \tilde{X}$ , where  $u^{(0)} \in \mathbb{R}$ ,  $u^{(1)} \in \mathbb{R}^{1 \times 2n}$  and  $\tilde{u}^{(2)} = \tilde{u}^{(2)'} \in \mathbb{R}^{2n \times 2n}$  are given. Finally, suppose  $x_0 \in \tilde{X}$  is given and  $\beta \in (0, 1)$ , and choose  $x_1, x_2, \dots$  to maximise:

$$\liminf_{T \rightarrow \infty} \sum_{t=1}^T \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \quad (10)$$

subject to the constraints that for all  $t \in \mathbb{N}^+$ ,  $x_t \in \tilde{\Gamma}(x_{t-1})$ .

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To ensure the problem is well behaved, we make the following assumption:

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**Assumption 3**  $\tilde{u}^{(2)}$  is negative-definite.

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In online appendix G, we establish the following (unsurprising) result:

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**Proposition 14** If either  $\tilde{X}$  is compact, or,  $\tilde{\Gamma}(x)$  is compact valued and  $x \in \tilde{\Gamma}(x)$  for all  $x \in \tilde{X}$ , then for all  $x_0 \in \tilde{X}$ , there is a unique path  $(x_t)_{t=0}^\infty$  which solves Problem 5.

---

We wish to use this result to establish the uniqueness of the solution to the first order conditions. The Lagrangian for our problem is given by:

$$\sum_{t=1}^{\infty} \beta^{t-1} \left[ \tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda'_{\Psi, t} \left[ \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] \right], \quad (11)$$

for some KKT-multipliers  $\lambda_t \in \mathbb{R}^c$  for all  $t \in \mathbb{N}^+$ . Taking the first order conditions leads to the following necessary KKT conditions, for all  $t \in \mathbb{N}^+$ :

$$0 = u^{(1)}_{\cdot, 2} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot, 2} + \lambda'_t \Psi^{(1)}_{\cdot, 2} + \beta \left[ u^{(1)}_{\cdot, 1} + \begin{bmatrix} x_t - \mu \\ x_{t+1} - \mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot, 1} + \lambda'_{t+1} \Psi^{(1)}_{\cdot, 1} \right], \quad (12)$$

$$0 \leq \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}, \quad 0 \leq \lambda_t, \quad 0 = \lambda_t \circ \left[ \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right], \quad (13)$$

where subscripts 1 and 2 refer to blocks of rows or columns of length  $n$ . Additionally, for  $\mu$  to be the steady-state of  $x_t$  and  $\bar{\lambda}$  to be the steady-state of  $\lambda_t$ , we require:

$$0 = u_{:,2}^{(1)} + \bar{\lambda}' \Psi_{:,2}^{(1)} + \beta[u_{:,1}^{(1)} + \bar{\lambda}' \Psi_{:,1}^{(1)}], \quad (14)$$

$$0 \leq \Psi^{(0)}, \quad 0 \leq \bar{\lambda}, \quad 0 = \bar{\lambda} \circ \Psi^{(0)}. \quad (15)$$

In online appendix H we prove the following result:

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**Proposition 15** Suppose that for all  $t \in \mathbb{N}$ ,  $(x_t)_{t=1}^\infty$  and  $(\lambda_t)_{t=1}^\infty$  satisfy the KKT conditions given in equations (10) and (11), and that as  $t \rightarrow \infty$ ,  $x_t \rightarrow \mu$  and  $\lambda_t \rightarrow \bar{\lambda}$ , where  $\mu$  and  $\lambda$  satisfy the steady-state KKT conditions given in equations (12) and (13). Then  $(x_t)_{t=1}^\infty$  solves Problem 5. If, further, either condition of Proposition 10 is satisfied, then  $(x_t)_{t=1}^\infty$  is the unique solution to Problem 5, and there can be no other solutions to the KKT conditions given in equations (10) and (11) satisfying  $x_t \rightarrow \mu$  and  $\lambda_t \rightarrow \bar{\lambda}$  as  $t \rightarrow \infty$ .

---

Now, it is possible to convert the KKT conditions given in equations (10) and (11) into a problem in the form of the multiple-bound generalisation of Problem 2 quite generally. To see this, first note that we may rewrite equation (10) as:

$$0 = u_{:,2}^{(1)'} + \tilde{u}_{2,1}^{(2)}(x_{t-1} - \mu) + \tilde{u}_{2,2}^{(2)}(x_t - \mu) + \Psi_{:,2}^{(1)'} \lambda_t \\ + \beta[u_{:,1}^{(1)'} + \tilde{u}_{1,1}^{(2)}(x_t - \mu) + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{:,1}^{(1)'} \lambda_{t+1}].$$

Now,  $\tilde{u}_{2,2}^{(2)} + \beta \tilde{u}_{1,1}^{(2)}$  is negative definite, hence it is valid to define:

$$\mathcal{U} := \Psi_{:,2}^{(1)} [\tilde{u}_{2,2}^{(2)} + \beta \tilde{u}_{1,1}^{(2)}]^{-1}.$$

Then, equation (9) implies that:

$$\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \\ = \Psi^{(0)} + (\Psi_{:,1}^{(1)} - \mathcal{U} \tilde{u}_{2,1}^{(2)})(x_{t-1} - \mu) - \mathcal{U} \left[ u_{:,2}^{(1)'} + \beta[u_{:,1}^{(1)'} + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{:,1}^{(1)'} \lambda_{t+1}] \right] \\ - \Psi_{:,2}^{(1)} [\tilde{u}_{2,2}^{(2)} + \beta \tilde{u}_{1,1}^{(2)}]^{-1} \Psi_{:,2}^{(1)'} \lambda_t. \quad (16)$$

Moreover, equation (11) implies that if the  $k^{\text{th}}$  element of  $\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$  is strictly positive, then the  $k^{\text{th}}$  element of  $\lambda_t$  is zero, so:

$$\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} = \max\{0, z_t\}, \quad (17)$$

where:

$$z_t := \Psi^{(0)} + (\Psi_{:,1}^{(1)} - \mathcal{U} \tilde{u}_{2,1}^{(2)})(x_{t-1} - \mu) \\ - \mathcal{U} \left[ u_{:,2}^{(1)'} + \beta[u_{:,1}^{(1)'} + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{:,1}^{(1)'} \lambda_{t+1}] \right] \\ - \left[ \Psi_{:,2}^{(1)} [\tilde{u}_{2,2}^{(2)} + \beta \tilde{u}_{1,1}^{(2)}]^{-1} \Psi_{:,2}^{(1)'} + \mathcal{U} \right] \lambda_t,$$

and  $\mathcal{U} \in \mathbb{R}^{c \times c}$  is an arbitrary, strictly positive diagonal matrix. A natural choice is:

$$\mathcal{U} := -\text{diag} \text{diag} \left[ \Psi_{:,2}^{(1)} [\tilde{u}_{2,2}^{(2)} + \beta \tilde{u}_{1,1}^{(2)}]^{-1} \Psi_{:,2}^{(1)'} \right],$$

providing this is strictly positive (it is weakly positive at least as  $\tilde{u}_{2,2}^{(2)} + \beta \tilde{u}_{1,1}^{(2)}$  is negative definite), where the diag operator maps matrices to a vector containing their diagonal,

and maps vectors to a matrix with the given vector on the diagonal, and zeros elsewhere.

We claim that we may replace equation (11) with equation (15) without changing the model. We have already shown that equation (11) implies equation (15), so we just have to prove the converse. We continue to suppose equation (9) holds, and thus, so too does equation (14). Then, from subtracting equation (14) from equation (15), we have that:

$$\mathcal{W}\lambda_t = \max\{-z_t, 0\}.$$

Hence, as  $\mathcal{W}$  is a strictly positive diagonal matrix, and the right-hand side is weakly positive,  $\lambda_t \geq 0$ . Furthermore, the  $k$ th element of  $\lambda_t$  is non-negative if and only if the  $k$ th element of  $z_t$  is non-positive (as  $\mathcal{W}$  is a strictly positive diagonal matrix), which in turn holds if and only if the  $k$ th element of  $\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$  is equal to zero, by equation (15). Thus equation (11) is satisfied.

Combined with our previous results, this gives the following proposition:

---

**Proposition 16** Suppose we are given a problem in the form of Problem 5. Then, the KKT conditions of that problem may be placed into the form of the multiple-bound generalisation of Problem 2. Let  $(q_{x_0}, M)$  be the infinite LCP corresponding to this representation, given initial state  $x_0 \in \tilde{X}$ . Then, if  $y$  is a solution to the LCP,  $q_{x_0} + My$  gives the stacked paths of the bounded variables in a solution to Problem 5. If, further, either condition of Proposition 10 is satisfied, then this LCP has a unique solution for all  $x_0 \in \tilde{X}$ , which gives the unique solution to Problem 5, and, for sufficiently large  $T^*$ , the finite LCP  $(q_{x_0}^{(T^*)}, M^{(T^*)})$  has a unique solution  $y^{(T^*)}$  for all  $x_0 \in \tilde{X}$ , where  $q_{x_0}^{(T^*)} + M^{(T^*)}y^{(T^*)}$  gives the first  $T^*$  periods of the stacked paths of the bounded variables in a solution to Problem 5.

---

This proposition provides some evidence that the LCP will have a unique solution when it is generated from a dynamic programming problem with a unique solution. In online appendix I, we derive similar results for models with more general constraints and objective functions. The proof of this proposition also showed how one can convert KKT conditions into equations of the form handled by our methods.

# Online appendices to: “Existence and uniqueness of solutions to dynamic models with occasionally binding constraints.”

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## A. Relationship between multiplicity under perfect-foresight, and multiplicity under rational expectations

By augmenting the state-space appropriately, the first order conditions of a general, non-linear, rational expectations, DSGE model may always be placed in the form:

$$0 = \mathbb{E}_t \hat{f}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \sigma \varepsilon_t),$$

for all  $t \in \mathbb{Z}$ , where  $\sigma \in [0,1]$ ,  $\hat{f}: (\mathbb{R}^{\hat{n}})^3 \times \mathbb{R}^m \rightarrow \mathbb{R}^{\hat{n}}$ , and where for all  $t \in \mathbb{Z}$ ,  $\hat{x}_t \in \mathbb{R}^{\hat{n}}$ ,  $\varepsilon_t \in \mathbb{R}^m$ ,  $\mathbb{E}_{t-1} \varepsilon_t = 0$ , and  $\mathbb{E}_t \hat{x}_t = \hat{x}_t$ . Since  $f$  is arbitrary, without loss of generality we may further assume that  $\varepsilon_t \sim \text{NIID}(0, I)$ . We further assume:

---

**Assumption 4**  $\hat{f}$  is everywhere continuous.

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The continuity of  $\hat{f}$  does rule out some models, but all models in which the only source of non-differentiability is a max or min operator (like those studied in this paper and its computational companion (Holden 2016)) will have a continuous  $\hat{f}$ .

Now, by further augmenting the state space, we can then find a continuous function  $f: (\mathbb{R}^n)^3 \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that for all  $t \in \mathbb{Z}$ :

$$0 = f(x_{t-1}, x_t, \mathbb{E}_t x_{t+1}, \sigma \varepsilon_t),$$

where for all  $t \in \mathbb{Z}$ ,  $x_t \in \mathbb{R}^n$  and  $\mathbb{E}_t x_t = x_t$ .<sup>31</sup> A solution to this model is given by a policy function. Given  $f$  is continuous, it is natural to restrict attention to continuous policy functions.<sup>32</sup> Furthermore, given the model's transversality conditions, we are usually only interested in stationary, Markov solutions, so the policy function will not be a function of  $t$  or of lags of the state. Additionally, in this paper we are only interested in solutions in which the deterministic model converges to some particular steady-state  $\mu$ . Thus, we make the following assumption:

---

**Assumption 5** The policy function is given by a continuous function:  $g: [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ , such that for all  $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$ :

$$0 = f(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g(\sigma, g(\sigma, x, e), \sigma \varepsilon), e),$$

where  $\varepsilon \sim N(0, I)$  and  $\mathbb{E}_\varepsilon$  denotes an expectation with respect to  $\varepsilon$ . Furthermore, for all  $x_0 \in \mathbb{R}^n$ , the recurrence  $x_t = g(0, x_{t-1}, 0)$  satisfies  $x_t \rightarrow \mu$  as  $t \rightarrow \infty$ .

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<sup>31</sup> For example, we may use the equations:  $\hat{x}_t^o = \hat{x}_{t-1}$ ,  $\hat{\varepsilon}_t = \varepsilon_t$ ,  $z_t = \hat{f}(\hat{x}_{t-1}^o, \hat{x}_{t-1}, \hat{x}_t, \sigma \hat{\varepsilon}_{t-1})$ ,  $0 = \mathbb{E}_t z_{t+1}$ , with  $x_t := [\hat{x}_t' \ \hat{x}_t^{o'} \ \hat{\varepsilon}_t' \ z_t']'$ .

<sup>32</sup> Note also that in standard dynamic programming applications, the policy function will be continuous. See e.g. Theorem 9.8 of Stokey, Lucas, and Prescott (1989).

To produce a lower bound on the number of policy functions satisfying Assumption 5, we need two further assumptions. The first assumption just gives the existence of the “time iteration” (a.k.a. “policy function iteration”) operator  $\mathcal{T}$ , and ensures that it maps fixed points to fixed points.

---

**Assumption 6** Let  $\mathcal{G}$  denote the space of all continuous functions  $[0,1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ . We assume there exists a function  $\mathcal{T}: \mathcal{G} \rightarrow \mathcal{G}$  such that for all  $(g, \sigma, x, e) \in \mathcal{G} \times [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$ :

$$0 = f(x, \mathcal{T}(g)(\sigma, x, e), \mathbb{E}_\varepsilon g(\sigma, \mathcal{T}(g)(\sigma, x, e), \sigma\varepsilon), e).$$

We further assume that if there exists some  $(g, \sigma) \in \mathcal{G} \times [0,1]$  such that for all  $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$ :

$$0 = f(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g(\sigma, g(\sigma, x, e), \sigma\varepsilon), e),$$

then for all  $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$ ,  $\mathcal{T}(g)(\sigma, x, e) = g(\sigma, x, e)$ .

---

The second assumption ensures that time iteration always converges when started from a solution to the model with no uncertainty after the current period. This is a weak assumption since the policy functions under uncertainty are invariably close to the policy function in the absence of uncertainty.

---

**Assumption 7** Let  $h: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a continuous function giving a solution to the model in which there is no future uncertainty, i.e. for all  $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$ :

$$0 = f(x, h(x, e), h(h(x, e), 0), e).$$

Further, define  $g_{h,0} \in \mathcal{G}$  by  $g_{h,0}(\sigma, x, e) = h(x, e)$  for all  $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$ , and define  $g_{h,k} \in \mathcal{G}$  inductively by  $g_{h,k+1} = \mathcal{T}(g_{h,k})$  for all  $k \in \mathbb{N}$ . Then there exists some  $g_{h,\infty} \in \mathcal{G}$  such that  $g_{h,\infty} = \mathcal{T}(g_{h,\infty})$  and for all  $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$ ,  $g_{h,k}(\sigma, x, e) \rightarrow g_{h,\infty}(\sigma, x, e)$  as  $k \rightarrow \infty$ .

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Note, by construction, if  $h$  is as in Assumption 7, then for all  $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$ :

$$0 = f(x, g_{h,0}(0, x, e), \mathbb{E}_\varepsilon g_{h,0}(0, g_{h,0}(0, x, e), 0\varepsilon), e).$$

Hence, by Assumption 6, for all  $k \in \mathbb{N}$ , all  $x(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$ ,  $g_{h,k}(0, x, e) = g_{h,0}(0, x, e)$ . Consequently, for all  $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$ ,  $g_{h,\infty}(0, x, e) = g_{h,0}(0, x, e) = h(x, e)$ .

Now suppose that  $h_1$  and  $h_2$  were as in Assumption 7, and that there exists  $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$ , such that  $h_1(x, e) \neq h_2(x, e)$ . Then, by the continuity of  $g_{h_1,\infty}$  and  $g_{h_2,\infty}$ , there is some  $\mathcal{J} \subseteq [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$  of positive measure, with  $(0, x, e) \in \mathcal{J}$ , such that for all  $(\sigma, x, e) \in \mathcal{J}$ ,  $g_{h_1,\infty}(\sigma, x, e) \neq g_{h_2,\infty}(\sigma, x, e)$ . Hence, the rational expectations policy functions differ, at least for small  $\sigma$ . Thus, if Assumption 6 and Assumption 7 are satisfied, there are at least as many policy functions satisfying Assumption 5 as there are solutions to the model in which there is no future uncertainty.

## B. Construction of a static model with no dynamic solution in some states

Consider the model:

$$a_t = \max\{0, b_t\}, \quad a_t = 1 - c_t, \quad c_t = a_t - b_t.$$

The model has steady-state  $a = b = 1, c = 0$ . Furthermore, in the model's Problem 3 type equivalent, in which for  $t \in \mathbb{N}^+$ :

$$a_t = \begin{cases} b_t + y_{t,0} & \text{if } t \leq T \\ b_t & \text{if } t > T' \end{cases}$$

where  $y_{\cdot,0}$  is defined as in Problem 3, we have that:

$$c_t = \begin{cases} y_{t,0} & \text{if } t \leq T \\ 0 & \text{if } t > T' \end{cases}$$

so:

$$b_t = \begin{cases} 1 - 2y_{t,0} & \text{if } t \leq T \\ 1 & \text{if } t > T' \end{cases}$$

implying:

$$a_t = \begin{cases} 1 - y_{t,0} & \text{if } t \leq T \\ 1 & \text{if } t > T' \end{cases}$$

thus,  $M = -I$  for this model.

## C. Proof of sufficient conditions for feasibility with $T = \infty$

First, define  $G := -C(B + CF)^{-1}$ , and note that if  $L$  is the lag (right-shift) operator, the model from Problem 1 can be written as:

$$L^{-1}(ALL + BL + C)(x - \mu) = 0.$$

Furthermore, by the definitions of  $F$  and  $G$ :

$$(L - G)(B + CF)(I - FL) = ALL + BL + C,$$

so, the stability of the model from Problem 1 is determined by the solutions for  $z \in \mathbb{C}$  of the polynomial:

$$0 = \det(Az^2 + Bz + C) = \det(Iz - G) \det(B + CF) \det(I - Fz).$$

Now by Assumption 1, all of the roots of  $\det(I - Fz)$  are strictly outside of the unit circle, and all of the other roots of  $\det(Az^2 + Bz + C)$  are weakly inside the unit circle (else there would be indeterminacy), thus, all of the roots of  $\det(Iz - G)$  are weakly inside the unit circle. Therefore, if we write  $\rho_M$  for the spectral radius of some matrix  $M$ , then, by this discussion and Assumption 2,  $\rho_G < 1$ .

Next, let  $s_t^*, x_t^* \in \mathbb{R}^{n \times \mathbb{N}^+}$  be such that for any  $y \in \mathbb{R}^{\mathbb{N}^+}$ , the  $k^{\text{th}}$  columns of  $s_t^* y$  and  $x_t^* y$  give the value of  $s_t$  and  $x_t$  following a magnitude 1 news shock at horizon  $k$ , i.e. when  $x_0 = \mu$  and  $y_0$  is the  $k^{\text{th}}$  row of  $I_{\mathbb{N}^+ \times \mathbb{N}^+}$ . Then:

$$\begin{aligned} s_t^* &= -(B + CF)^{-1} [I_{\cdot,1} I_{t,1:\infty} + G I_{\cdot,1} I_{t+1,1:\infty} + G^2 I_{\cdot,1} I_{t+2,1:\infty} + \dots] \\ &= -(B + CF)^{-1} \sum_{k=0}^{\infty} (GL)^k I_{\cdot,1} I_{t,1:\infty} \\ &= -(B + CF)^{-1} (I - GL)^{-1} I_{\cdot,1} I_{t,1:\infty}, \end{aligned}$$

where the infinite sums are well defined as  $\rho_G < 1$ , and where  $I_{t,1:\infty} \in \mathbb{R}^{1 \times \mathbb{N}^+}$  is a row vector with zeros everywhere except position  $t$  where there is a 1. Thus:

$$s_t^* = [0_{n \times (t-1)} \quad s_1^*] = L^{t-1} s_1^*.$$

Furthermore,

$$(x_t^* - \mu^*) = \sum_{j=1}^t F^{t-j} s_k^* = \sum_{j=1}^t F^{t-j} L^{j-1} s_1^*,$$

i.e.:

$$(x_t^* - \mu^*)_{\cdot,k} = \sum_{j=1}^t F^{t-j} s_{1,\cdot,k+1-j}^* = - \sum_{j=1}^{\min\{t,k\}} F^{t-j} (B + CF)^{-1} G^{k-j} I_{\cdot,1},$$

and so, the  $k^{\text{th}}$  offset diagonal of  $M$  ( $k \in \mathbb{Z}$ ) is given by the first row of the  $k^{\text{th}}$  column of:

$$L^{-t}(x_t^* - \mu^*) = L^{-1} \sum_{j=1}^t (FL^{-1})^{t-j} s_1^* = L^{-1} \sum_{j=0}^{t-1} (FL^{-1})^j s_1^*,$$

where we abuse notation slightly by allowing  $L^{-1}$  to give a result with indices in  $\mathbb{Z}$  rather than  $\mathbb{N}^+$ , with padding by zeros. Consequently, for all  $k \in \mathbb{N}^+$ ,  $M_{t,k} = O(t^n \rho_F^t)$ , as  $t \rightarrow \infty$ , for all  $t \in \mathbb{N}^+$ ,  $M_{t,k} = O(t^n \rho_G^k)$ , as  $k \rightarrow \infty$ , and for all  $k \in \mathbb{Z}$ ,  $M_{t,t+k} - \lim_{\tau \rightarrow \infty} M_{\tau,\tau+k} = O(t^{n-1} (\rho_F \rho_G)^t)$ , as  $t \rightarrow \infty$ . Hence,

$$\sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y$$

exists and is well defined, and so:

$$\zeta = \sup_{\substack{y \in [0,1]^{\mathbb{N}^+} \\ \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y = \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y,$$

since every point in  $[0,1]^{\mathbb{N}^+}$  is a limit (under the supremum norm) of a sequence of points in the set:

$$\{y \in [0,1]^{\mathbb{N}^+} \mid \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0\}.$$

Thus, we just need to provide conditions under which  $\sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y > 0$ .

To produce such conditions, we need constructive bounds on  $M$ , even if they have slightly worse convergence rates. For any matrix,  $M \in \mathbb{R}^{n \times n}$  with  $\rho_M < 1$ , and any  $\phi \in (\rho_M, 1)$ , let:

$$C_{M,\phi} := \sup_{k \in \mathbb{N}} \|(M\phi^{-1})^k\|_2.$$

Furthermore, for any matrix,  $M \in \mathbb{R}^{n \times n}$  with  $\rho_M < 1$ , and any  $\epsilon > 0$ , let:

$$\rho_{M,\epsilon} := \max\{|z| \mid z \in \mathbb{C}, \sigma_{\min}(M - zI) = \epsilon\},$$

where  $\sigma_{\min}(M - zI)$  is the minimum singular value of  $M - zI$ , and let  $\epsilon^*(M) \in (0, \infty]$  solve:

$$\rho_{M,\epsilon} = 1.$$

(This has a solution in  $(0, \infty]$  by continuity as  $\rho_M < 1$ .)

Then, by Theorem 16.2 of Trefethen and Embree (2005), for any  $K \in \mathbb{N}$  and  $k > K$ :

$$\|(M\phi^{-1})^k\|_2 \leq \|(M\phi^{-1})^K\|_2 \|(M\phi^{-1})^{k-K}\|_2 \leq \frac{\|(M\phi^{-1})^K\|_2}{\epsilon^*(M\phi^{-1})}.$$

Now,  $\|(M\phi^{-1})^K\|_2 \rightarrow 0$  as  $K \rightarrow \infty$ , hence, there exists some  $K \in \mathbb{N}$  such that:

$$\sup_{k=0,\dots,K} \|(M\phi^{-1})^k\|_2 \geq \frac{\|(M\phi^{-1})^K\|_2}{\epsilon^*(M\phi^{-1})} \geq \sup_{k>K} \|(M\phi^{-1})^k\|_2,$$

meaning  $C_{M,\phi} = \sup_{k=0,\dots,K} \|(M\phi^{-1})^k\|_2$ . The quantity  $\rho_{M,\epsilon}$  (and hence  $\epsilon^*(M)$ ) may be

efficiently computed using the methods described by Wright and Trefethen (2001), and implemented in their EigTool toolkit<sup>33</sup>. Thus,  $C_{M,\phi}$  may be calculated in finitely many operations by iterating over  $K \in \mathbb{N}$  until a  $K$  is found which satisfies:

$$\sup_{k=0,\dots,K} \|(M\phi^{-1})^k\|_2 \geq \frac{\|(M\phi^{-1})^K\|_2}{\epsilon^*(M\phi^{-1})}.$$

From the definition of  $C_{M,\phi}$ , we have that for any  $k \in \mathbb{N}$  and any  $\phi \in (\rho_M, 1)$ :

$$\|M^k\|_2 \leq C_{M,\phi} \phi^k.$$

Now, fix  $\phi_F \in (\rho_F, 1)$  and  $\phi_G \in (\rho_G, 1)$ ,<sup>34</sup> and define:

$$\mathcal{D}_{\phi_F, \phi_G} := C_{F, \phi_F} C_{G, \phi_G} \|(B + CF)^{-1}\|_2,$$

then, for all  $t, k \in \mathbb{N}^+$ :

$$\begin{aligned} |M_{t,k}| &= |(x_t^* - \mu^*)_{1,k}| \leq \|(x_t^* - \mu^*)_{\cdot,k}\|_2 \leq \sum_{j=1}^{\min\{t,k\}} \|F^{t-j}\|_2 \|(B + CF)^{-1}\|_2 \|G^{k-j}\|_2 \\ &\leq \mathcal{D}_{\phi_F, \phi_G} \sum_{j=1}^{\min\{t,k\}} \phi_F^{t-j} \phi_G^{k-j} = \mathcal{D}_{\phi_F, \phi_G} \phi_F^t \phi_G^k \frac{(\phi_F \phi_G)^{-\min\{t,k\}} - 1}{1 - \phi_F \phi_G}. \end{aligned}$$

Additionally, for all  $t \in \mathbb{N}^+$ ,  $k \in \mathbb{Z}$ :

$$\begin{aligned} |M_{t,t+k} - \lim_{\tau \rightarrow \infty} M_{\tau, \tau+k}| &= \left| (L^{-t}(x_t^* - \mu^*))_{1,k} - \left( \lim_{\tau \rightarrow \infty} L^{-t}(x_t^* - \mu^*) \right)_{1,k} \right| \\ &\leq \left\| \left( L^{-1} \sum_{j=0}^{t-1} (FL^{-1})^j s_1^* - L^{-1} \sum_{j=0}^{\infty} (FL^{-1})^j s_1^* \right)_{\cdot,k} \right\|_2 \\ &= \left\| \left( \sum_{j=\max\{t,-k\}}^{\infty} F^j s_{1,\cdot, j+k+1}^* \right)_{\cdot,0} \right\|_2 \\ &= \left\| \sum_{j=\max\{t,-k\}}^{\infty} F^j (B + CF)^{-1} G^{j+k} I_{\cdot,1} \right\|_2 \\ &\leq \sum_{j=\max\{t,-k\}}^{\infty} \|F^j\|_2 \|(B + CF)^{-1}\|_2 \|G^{j+k}\|_2 \\ &\leq \mathcal{D}_{\phi_F, \phi_G} \sum_{j=\max\{t,-k\}}^{\infty} \phi_F^j \phi_G^{j+k} = \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^{\max\{t,-k\}} \phi_G^{\max\{0,t+k\}}}{1 - \phi_F \phi_G}, \end{aligned}$$

<sup>33</sup> This toolkit is available from <https://github.com/eigtool/eigtool>, and is included in DynareOBC.

<sup>34</sup> In practice, we try a grid of values, as it is problem dependent whether high  $\phi_F$  and low  $\mathcal{K}(M\phi^{-1})$  is preferable to low  $\phi_F$  and high  $\mathcal{K}(M\phi^{-1})$ .



so, for all  $t, k \in \mathbb{N}^+$ :

$$|M_{t,k} - \lim_{\tau \rightarrow \infty} M_{\tau, \tau+k-t}| \leq \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G}.$$

To evaluate  $\lim_{\tau \rightarrow \infty} M_{\tau, \tau+k-t}$ , note that this limit is the top element from the  $(k-t)^{\text{th}}$  column of:

$$\begin{aligned} d &:= \lim_{\tau \rightarrow \infty} L^{-\tau} (x_\tau^* - \mu^*) = L^{-1} (I - FL^{-1})^{-1} s_1^* \\ &= -(I - FL^{-1})^{-1} (B + CF)^{-1} (I - GL)^{-1} I_{\cdot,1} I_{0,-\infty:\infty}, \end{aligned}$$

where  $I_{0,-\infty:\infty} \in \mathbb{R}^{1 \times \mathbb{Z}}$  is zero everywhere apart from index 0 where it equals 1. Hence, by the definitions of  $F$  and  $G$ :

$$AL^{-1}d + Bd + CLd = -I_{\cdot,1} I_{0,-\infty:\infty}.$$

In other words, if we write  $d_k$  in place of  $d_{\cdot,k}$  for convenience, then, for all  $k \in \mathbb{Z}$ :

$$Ad_{k+1} + Bd_k + Cd_{k-1} = - \begin{cases} I_{\cdot,1} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

I.e. the homogeneous part of the difference equation for  $d_{-t}$  is identical to that of  $x_t - \mu$ . The time reversal here is intuitive since we are indexing diagonals such that indices increase as we move up and to the right in  $M$ , but time is increasing as we move down in  $M$ .

It turns out that exploiting the possibility of reversing time is the key to easy evaluating  $d_k$ . First, note that for  $k < 0$ , it must be the case that  $d_k = Fd_{k+1}$ , since the shock has already “occurred” (remember, that we are going forwards in “time” when we reduce  $k$ ). Now consider the model in which we are going forwards time when we increase  $k$ , i.e. the model with:

$$L(AL^{-1}L^{-1} + BL^{-1} + C)d = 0,$$

subject to the terminal condition that  $d_k \rightarrow 0$  as  $k \rightarrow \infty$ , which must hold as we have already proved that the first row of  $M$  converges to zero. Now, let  $z \in \mathbb{C}, z \neq 0$  be a solution to:

$$0 = \det(Az^2 + Bz + C),$$

and define  $\tilde{z} = z^{-1}$ , so:

$$\begin{aligned} 0 &= \det(A + B\tilde{z} + C\tilde{z}^2) = z^{-2} \det(Az^2 + Bz + C) \\ &= \det(I - G\tilde{z}) \det(B + CF) \det(I\tilde{z} - F). \end{aligned}$$

By Assumption 1, all the roots of  $\det(I\tilde{z} - F)$  are inside the unit circle, thus they cannot contribute to the dynamics of the time reversed process, else the terminal condition would be violated. Thus, the time reversed model has a unique solution satisfying the terminal condition with a transition matrix with the same eigenvalues as  $G$ . Consequently, this solution can be calculated via standard methods for solving linear DSGE models, and it will be given by  $d_k = Hd_{k-1}$ , for all  $k > 0$ , where  $H = -(B + AH)^{-1}C$ , and  $\phi_H = \phi_G < 1$ , by Assumption 2.

It just remains to determine the value of  $d_0$ . By the previous results, this must satisfy:

$$-I_{\cdot,1} = Ad_1 + Bd_0 + Cd_{-1} = (AH + B + CF)d_0.$$

Hence:

$$d_0 = -(AH + B + CF)^{-1}I_{\cdot,1}.$$

This gives a readily computed solution for the limits of the diagonals of  $M$ . Lastly, note that:

$$|d_{-t,1}| \leq \|d_{-t}\|_2 = \|F^t d_0\|_2 \leq \|F^t\|_2 \|d_0\|_2 \leq C_{F,\phi_F} \phi_F^t \|d_0\|_2,$$

and:

$$|d_{t,1}| \leq \|d_t\|_2 = \|H^t d_0\|_2 \leq \|H^t\|_2 \|d_0\|_2 \leq C_{H,\phi_H} \phi_H^t \|d_0\|_2.$$

We will use these results in producing our bounds on  $\varsigma$ .

First, fix  $T \in \mathbb{N}^+$ , and define a new matrix  $\underline{M}^{(T)} \in \mathbb{R}^{\mathbb{N}^+ \times \mathbb{N}^+}$  by  $\underline{M}_{1:T,1:T}^{(T)} = M_{1:T,1:T}$ , and for all  $t, k \in \mathbb{N}^+$ , with  $\min\{t, k\} > T$ ,  $\underline{M}_{t,k}^{(T)} = d_{k-t,1} - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G}$ , then:

$$\begin{aligned} \varsigma &\geq \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} \left[ y_\infty 1_{\infty \times 1} \right] \geq \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \inf_{t \in \mathbb{N}^+} \underline{M}_{t,1:\infty}^{(T)} \left[ y_\infty 1_{\infty \times 1} \right] \\ &= \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{aligned} &\min_{t=1,\dots,T} \left[ M_{t,1:T} y + \sum_{k=T+1}^{\infty} \left( d_{k-t,1} - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_\infty \right], \\ &\inf_{t \in \mathbb{N}^+, t > T} \left[ \sum_{k=1}^T \left( d_{k-t,1} - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_k + \sum_{k=T+1}^{\infty} \left( d_{k-t,1} - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_\infty \right] \end{aligned} \right\} \\ &\geq \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{aligned} &\min_{t=1,\dots,T} \left[ M_{t,1:T} y + ((I-H)^{-1} d_{T+1-t})_1 y_\infty - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1 - \phi_F \phi_G)(1 - \phi_G)} y_\infty \right], \\ &\min_{t=T+1,\dots,2T} \left[ \sum_{k=1}^T \left( d_{-(t-k),1} - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1 - \phi_F \phi_G} \right) y_k + ((I-F)^{-1} (d_{-1} - d_{-(t-T)}))_1 y_\infty \right. \\ &\quad \left. + ((I-H)^{-1} d_0)_1 y_\infty - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1 - \phi_F \phi_G)(1 - \phi_G)} y_\infty \right], \\ &\inf_{t \in \mathbb{N}^+, t > 2T} \left[ \sum_{k=1}^T d_{-(t-k),1} y_k + ((I-F)^{-1} (d_{-1} - d_{-(t-T)}))_1 y_\infty \right. \\ &\quad \left. + ((I-H)^{-1} d_0)_1 y_\infty - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^{2T+1} \phi_G}{(1 - \phi_F \phi_G)(1 - \phi_G)} \right] \end{aligned} \right\}. \end{aligned}$$

Now, for  $t \geq T$ :

$$\begin{aligned} |((I-F)^{-1} d_{-(t-T)})_1| &\leq \|(I-F)^{-1} d_{-(t-T)}\|_2 \leq \|(I-F)^{-1}\|_2 \|d_{-(t-T)}\|_2 \\ &\leq C_{F,\phi_F} \phi_F^{t-T} \|(I-F)^{-1}\|_2 \|d_0\|_2, \end{aligned}$$

so:

$$\begin{aligned} &\sum_{k=1}^T d_{-(t-k),1} y_k - ((I-F)^{-1} d_{-(t-T)})_1 y_\infty \\ &\geq - \sum_{k=1}^T C_{F,\phi_F} \phi_F^{t-k} \|d_0\|_2 - C_{F,\phi_F} \phi_F^{t-T} \|(I-F)^{-1}\|_2 \|d_0\|_2 y_\infty \\ &= -C_{F,\phi_F} \frac{\phi_F^t (\phi_F^{-T} - 1)}{1 - \phi_F} \|d_0\|_2 - C_{F,\phi_F} \phi_F^{t-T} \|(I-F)^{-1}\|_2 \|d_0\|_2 y_\infty, \end{aligned}$$

thus  $\varsigma \geq \underline{\varsigma}$ , where:

$$\underline{\varsigma}_T := \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{aligned} & \min_{t=1,\dots,T} \left[ M_{t,1:T}y + ((I-H)^{-1}d_{T+1-t})_1 y_\infty - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F \phi_G)(1-\phi_G)} y_\infty \right], \\ & \min_{t=T+1,\dots,2T} \left[ \sum_{k=1}^T \left( d_{-(t-k),1} - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F \phi_G} \right) y_k + ((I-F)^{-1}(d_{-1} - d_{-(t-T)}))_1 y_\infty \right. \\ & \quad \left. + ((I-H)^{-1}d_0)_1 y_\infty - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F \phi_G)(1-\phi_G)} y_\infty \right], \\ & \left[ -C_{F, \phi_F} \frac{\phi_F^{2T+1}(\phi_F^{-T} - 1)}{1-\phi_F} \|d_0\|_2 - C_{F, \phi_F} \phi_F^{T+1} \|(I-F)^{-1}\|_2 \|d_0\|_2 y_\infty + ((I-F)^{-1}d_{-1})_1 y_\infty \right. \\ & \quad \left. + ((I-H)^{-1}d_0)_1 y_\infty - \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^{2T+1} \phi_G}{(1-\phi_F \phi_G)(1-\phi_G)} \right] \end{aligned} \right\}.$$

It is worth noting that as  $T \rightarrow \infty$ , the final minimand in this expression tends to:

$$((I-F)^{-1}d_{-1})_1 y_\infty + ((I-H)^{-1}d_0)_1 y_\infty,$$

i.e. a positive multiple of the doubly infinite sum of  $d_{1,k}$  over all  $k \in \mathbb{Z}$ . If this expression is negative, then our lower bound on  $\varsigma$  will be negative as well, and hence uninformative.

To construct an upper bound on  $\varsigma$ , fix  $T \in \mathbb{N}^+$ , and define a new matrix  $\bar{M}^{(T)} \in \mathbb{R}^{\mathbb{N}^+ \times \mathbb{N}^+}$  by  $\bar{M}_{1:T,1:T}^{(T)} = M_{1:T,1:T}$ , and for all  $t, k \in \mathbb{N}^+$ , with  $\min\{t, k\} > T$ ,  $\bar{M}_{t,k}^{(T)} = |d_{k-t,1}| + \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F \phi_G}$ . Then:

$$\begin{aligned} \varsigma &= \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y \leq \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} \bar{M}_{t,1:\infty} y \leq \sup_{y \in [0,1]^{\mathbb{N}^+}} \min_{t=1,\dots,T} \bar{M}_{t,1:\infty} y \\ &\leq \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \bar{M}_{t,1:\infty} \begin{bmatrix} y \\ 1_{\infty \times 1} \end{bmatrix} \\ &\leq \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \left[ M_{t,1:T}y + \sum_{k=T+1}^{\infty} |d_{k-t,1}| + \sum_{k=T+1}^{\infty} \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F \phi_G} \right] \\ &\leq \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \left[ M_{t,1:T}y + \sum_{k=T+1-t}^{\infty} |d_{k,1}| + \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^{T+1}}{1-\phi_F \phi_G} \sum_{k=0}^{\infty} \phi_G^k \right] \\ &\leq \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \left[ M_{t,1:T}y + C_{H, \phi_H} \|d_0\|_2 \phi_H^{T+1-t} \sum_{k=0}^{\infty} \phi_H^k + \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F \phi_G)(1-\phi_G)} \right] \\ &= \bar{\varsigma}_T := \max_{y \in [0,1]^T} \min_{t=1,\dots,T} \left[ M_{t,1:T}y + \frac{C_{H, \phi_H} \|d_0\|_2 \phi_H^{T+1-t}}{1-\phi_H} + \mathcal{D}_{\phi_F, \phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F \phi_G)(1-\phi_G)} \right]. \end{aligned}$$

## D. Proof of the properties of the BPY model

Defining  $x_t = [x_{i,t} \ x_{y,t} \ x_{\pi,t}]'$ , the BPY model is in the form of Problem 2, with:

$$A := \begin{bmatrix} 0 & -\alpha_{\Delta y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_{\pi} \\ -\frac{1}{\sigma} & -1 & 0 \\ 0 & \gamma & -1 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A + B + C) = \det \begin{bmatrix} -1 & 0 & \alpha_{\pi} \\ -\frac{1}{\sigma} & 0 & \frac{1}{\sigma} \\ 0 & \gamma & -1 \end{bmatrix} \neq 0$$

as  $\alpha_\pi \neq 1$  and  $\gamma \neq 0$ . Let  $f := F_{2,2}$ , where  $F$  is as in Assumption 1. Then:

$$F = \begin{bmatrix} 0 & \alpha_{\Delta y}(f-1) + \alpha_\pi \frac{\gamma f}{1-\beta f} & 0 \\ 0 & f & 0 \\ 0 & \frac{\gamma f}{1-\beta f} & 0 \end{bmatrix}.$$

Hence:

$$f = f^2 - \frac{1}{\sigma} \left( \alpha_{\Delta y}(f-1) + \alpha_\pi \frac{\gamma f}{1-\beta f} - \frac{\gamma f^2}{1-\beta f} \right),$$

i.e.:

$$\beta \sigma f^3 - \left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f^2 + \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma \right) f - \alpha_{\Delta y} = 0. \quad (18)$$

When  $f \leq 0$ , the left-hand side is negative, and when  $f = 1$ , the left-hand side equals  $(\alpha_\pi - 1)\gamma > 0$  (by assumption on  $\alpha_\pi$ ), hence equation (3) has either one or three solutions in  $(0,1)$ , and no solutions in  $(-\infty, 0]$ . We wish to prove there is a unique solution in  $(-1,1)$ . First note that when  $\alpha_\pi = 1$ , the discriminant of the polynomial is:

$$\left( (1 - \beta)(\alpha_{\Delta y} - \sigma) - \gamma \right)^2 \left( (\beta\alpha_{\Delta y})^2 + 2\beta(\gamma - \sigma)\alpha_{\Delta y} + (\gamma + \sigma)^2 \right).$$

The first multiplicand is positive. The second is minimised when  $\sigma = \beta\alpha_{\Delta y} - \gamma$ , at the value  $4\beta\gamma\alpha_{\Delta y} > 0$ , hence this multiplicand is positive too. Consequently, at least for small  $\alpha_\pi$ , there are three real solutions for  $f$ , so there may be multiple solutions in  $(0,1)$ .

Suppose for a contradiction that there were at least three solutions to equation (3) in  $(0,1)$  (double counting repeated roots), even for arbitrary large  $\beta \in (0,1)$ . Let  $f_1, f_2, f_3 \in (0,1)$  be the three roots. Then, by Vieta's formulas:

$$\begin{aligned} 3 &> f_1 + f_2 + f_3 = \frac{(\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma}{\beta\sigma}, \\ 3 &> f_1f_2 + f_1f_3 + f_2f_3 = \frac{(1 + \beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma}{\beta\sigma}, \\ 1 &> f_1f_2f_3 = \frac{\alpha_{\Delta y}}{\beta\sigma}, \end{aligned}$$

so:

$$\begin{aligned} (2\beta - 1)\sigma &> \beta\alpha_{\Delta y} + \gamma > \gamma > 0 \\ \beta &> \frac{1}{2}, \quad (2\beta - 1)\sigma &> \gamma, \\ \beta\sigma &> \beta\alpha_{\Delta y} + \gamma + \sigma(1 - \beta), \\ 2\beta\sigma &> (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma(1 - \beta), \\ \beta\sigma &> \alpha_{\Delta y}. \end{aligned}$$

Also, the first derivative of equation (3) must be strictly positive at  $f = 1$ , so:

$$(1 - \beta)(\alpha_{\Delta y} - \sigma) + (\alpha_\pi - 2)\gamma > 0.$$

Combining these inequalities gives the bounds:

$$0 < \alpha_{\Delta y} < 2\sigma - \frac{\gamma + \sigma}{\beta},$$

$$2 + \frac{(1 - \beta)(\sigma - \alpha_{\Delta y})}{\gamma} < \alpha_{\pi} < \frac{(3\beta - 1)\sigma - (1 + \beta)\alpha_{\Delta y}}{\gamma}.$$

Furthermore, if there are multiple solutions to equation (3), then the discriminant of its first derivative must be weakly positive, i.e.:

$$\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right)^2 - 3\beta\sigma \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma \right) \geq 0.$$

Therefore, we have the following bounds on  $\alpha_{\pi}$ :

$$2 + \frac{(1 - \beta)(\sigma - \alpha_{\Delta y})}{\gamma} < \alpha_{\pi} \leq \frac{\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right)^2 - 3\beta\sigma \left( (1 + \beta)\alpha_{\Delta y} + \sigma \right)}{3\beta\sigma\gamma}$$

since,

$$\begin{aligned} & \frac{(3\beta - 1)\sigma - (1 + \beta)\alpha_{\Delta y}}{\gamma} - \frac{\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right)^2 - 3\beta\sigma \left( (1 + \beta)\alpha_{\Delta y} + \sigma \right)}{3\beta\sigma\gamma} \\ &= \frac{\left( (2\sigma - \alpha_{\Delta y})\beta - \gamma - \sigma \right) \left( (4\sigma + \alpha_{\Delta y})\beta + \gamma + \sigma \right)}{3\beta\gamma\sigma} > 0 \end{aligned}$$

as  $\alpha_{\Delta y} < 2\sigma - \frac{\gamma + \sigma}{\beta}$ . Consequently, there exists  $\lambda, \mu, \kappa \in [0, 1]$  such that:

$$\begin{aligned} \alpha_{\pi} &= (1 - \lambda) \left[ 2 + \frac{(1 - \beta)(\sigma - \alpha_{\Delta y})}{\gamma} \right] \\ &+ \lambda \left[ \frac{\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right)^2 - 3\beta\sigma \left( (1 + \beta)\alpha_{\Delta y} + \sigma \right)}{3\beta\sigma\gamma} \right], \\ \alpha_{\Delta y} &= (1 - \mu)[0] + \mu \left[ 2\sigma - \frac{\gamma + \sigma}{\beta} \right], \\ \gamma &= (1 - \kappa)[0] + \kappa[(2\beta - 1)\sigma] \end{aligned}$$

These simultaneous equations have unique solutions for  $\alpha_{\pi}$ ,  $\alpha_{\Delta y}$  and  $\gamma$  in terms of  $\lambda$ ,  $\mu$  and  $\kappa$ . Substituting these solutions into the discriminant of equation (3) gives a polynomial in  $\lambda, \mu, \kappa, \beta, \sigma$ . As such, an exact global maximum of the discriminant may be found subject to the constraints  $\lambda, \mu, \kappa \in [0, 1]$ ,  $\beta \in [\frac{1}{2}, 1]$ ,  $\sigma \in [0, \infty)$ , by using an exact compact polynomial optimisation solver, such as that in the Maple computer algebra package. Doing this gives a maximum of 0 when  $\beta \in \{\frac{1}{2}, 1\}$ ,  $\kappa = 1$  and  $\sigma = 0$ . But of course, we actually require that  $\beta \in (\frac{1}{2}, 1)$ ,  $\kappa < 1$ ,  $\sigma > 0$ . Thus, by continuity, the discriminant is strictly negative over the entire domain. This gives the required contradiction to our assumption of three roots to the polynomial, establishing that Assumption 1 holds for this model.

Now, when  $T = 1$ ,  $M$  is equal to the top left element of the matrix  $-(B + CF)^{-1}$ , i.e.:

$$M = \frac{\beta\sigma f^2 - ((1 + \beta)\sigma + \gamma)f + \sigma}{\beta\sigma f^2 - ((1 + \beta)\sigma + \gamma + \beta\alpha_{\Delta y})f + \sigma + \alpha_{\Delta y} + \gamma\alpha_{\pi}}$$

Now, multiplying the denominator by  $f$  gives:

$$\begin{aligned} & \beta\sigma f^3 - \left( (1 + \beta)\sigma + \gamma + \beta\alpha_{\Delta y} \right) f^2 + (\sigma + \alpha_{\Delta y} + \gamma\alpha_{\pi})f \\ &= \left[ \beta\sigma f^3 - \left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f^2 + \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma \right) f - \alpha_{\Delta y} \right] \\ & - \left[ \beta\alpha_{\Delta y} f - \alpha_{\Delta y} \right] = (1 - \beta f)\alpha_{\Delta y} > 0, \end{aligned}$$

by equation (19). Hence, the sign of  $M$  is that of  $\beta\sigma f^2 - ((1 + \beta)\sigma + \gamma)f + \sigma$ . I.e.,  $M$  is negative if and only if:

$$\begin{aligned} & \frac{((1 + \beta)\sigma + \gamma) - \sqrt{((1 + \beta)\sigma + \gamma)^2 - 4\beta\sigma^2}}{2\beta\sigma} < f \\ & < \frac{((1 + \beta)\sigma + \gamma) + \sqrt{((1 + \beta)\sigma + \gamma)^2 - 4\beta\sigma^2}}{2\beta\sigma}. \end{aligned}$$

The upper limit is greater than 1, so only the lower is relevant. To translate this bound on  $f$  into a bound on  $\alpha_{\Delta y}$ , we first need to establish that  $f$  is monotonic in  $\alpha_{\Delta y}$ .

Totally differentiating equation (19) gives:

$$\begin{aligned} & \left[ 3\beta\sigma f^2 - 2\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f + \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma \right) \right] \frac{df}{d\alpha_{\Delta y}} = (1 - \beta f)(1 - f) \\ & > 0. \end{aligned}$$

Thus, the sign of  $\frac{df}{d\alpha_{\Delta y}}$  is equal to that of:

$$3\beta\sigma f^2 - 2\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f + \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma \right).$$

Note, however, that this expression is just the derivative of the left-hand side of equation (19) with respect to  $f$ .

To establish the sign of  $\frac{df}{d\alpha_{\Delta y}}$ , we consider two cases. First, suppose that equation (19) has three real solutions. Then, the unique solution to equation (19) in  $(0,1)$  is its lowest solution. Hence, this solution must be below the first local maximum of the left-hand side of equation (19). Consequently, at the  $f \in (0,1)$ , which solves equation (19),  $3\beta\sigma f^2 - 2\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f + \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma \right) > 0$ . Alternatively, suppose that equation (19) has a unique real solution. Then the left-hand side of this equation cannot change sign in between its local maximum and its local minimum (if it has any). Thus, at the  $f \in (0,1)$  at which it changes sign, we must have that  $3\beta\sigma f^2 - 2\left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f + \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma \right) > 0$ . Therefore, in either case  $\frac{df}{d\alpha_{\Delta y}} > 0$ , meaning that  $f$  is monotonic increasing in  $\alpha_{\Delta y}$ .

Consequently, to find the critical  $(f, \alpha_{\Delta y})$  at which  $M$  changes sign, it is sufficient to find the lowest solution with respect to both  $f$  and  $\alpha_{\Delta y}$  of the pair of equations:

$$\begin{aligned} & \beta\sigma f^2 - ((1 + \beta)\sigma + \gamma)f + \sigma = 0, \\ & \beta\sigma f^3 - \left( (\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f^2 + \left( (1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma \right) f - \alpha_{\Delta y} = 0. \end{aligned}$$

The former implies that:

$$\beta\sigma f^3 - ((1 + \beta)\sigma + \gamma)f^2 + \sigma f = 0,$$

so, by the latter:

$$\alpha_{\Delta y}\beta f^2 - ((1 + \beta)\alpha_{\Delta y} + \gamma\alpha_{\pi})f + \alpha_{\Delta y} = 0.$$

If  $\alpha_{\Delta y} = \sigma\alpha_{\pi}$ , then this equation holds if and only if:

$$\sigma\beta f^2 - ((1 + \beta)\sigma + \gamma)f + \sigma = 0.$$

Therefore, the critical  $(f, \alpha_{\Delta y})$  at which  $M$  changes sign are given by:

$$\alpha_{\Delta y} = \sigma\alpha_{\pi},$$

$$f = \frac{((1 + \beta)\sigma + \gamma) - \sqrt{((1 + \beta)\sigma + \gamma)^2 - 4\beta\sigma^2}}{2\beta\sigma}.$$

Thus,  $M$  is negative if and only if  $\alpha_{\Delta y} > \sigma\alpha_{\pi}$ , and  $M$  is zero if and only if  $\alpha_{\Delta y} = \sigma\alpha_{\pi}$ .

## E. The BPY model with shadow interest rate persistence

Following BPY (2013), we introduce persistence in the shadow interest rate by replacing the previous Taylor rule with  $x_{i,t} = \max\{0, x_{d,t}\}$ , where  $x_{d,t}$ , the shadow nominal interest rate is given by:

$$x_{d,t} = (1 - \rho)(1 - \beta + \alpha_{\Delta y}(x_{y,t} - x_{y,t-1}) + \alpha_{\pi}x_{\pi,t}) + \rho x_{d,t-1}.$$

It is easy to verify that this may be put in the form of Problem 2, and that with  $\beta \in (0,1)$ ,  $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$ ,  $\alpha_{\pi} \in (1, \infty)$ ,  $\rho \in (-1,1)$ , Assumption 2 is satisfied. For our numerical exercise, we again set  $\sigma = 1$ ,  $\beta = 0.99$ ,  $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2 + \sigma)$ ,  $\rho = 0.5$ , following BPY.

In Figure 9, we plot the regions in  $(\alpha_{\Delta y}, \alpha_{\pi})$  space in which  $M$  is a P-matrix (P<sub>0</sub>-matrix) when  $T = 2$  or  $T = 4$ . For this model, these correspond to the regions in which  $M$  is strictly semi-monotone (semi-monotone). As may be seen, in the smaller  $T$  case, the P-matrix region is much larger. This relationship appears to continue to hold for both larger and smaller  $T$ , with the equivalent  $T = 1$  plot being almost entirely shaded, and the large  $T$  plot tending to the equivalent plot from the model without monetary policy persistence. Intuitively, the persistence in the shadow nominal interest rate dampens the immediate response of nominal interest rates to inflation and output growth, making it harder to induce a ZLB episode over short-horizons.

Further evidence that the long-horizon behaviour is the same as in the model without persistence is provided by the fact that with  $\alpha_{\pi} = 1.5$  and  $\alpha_{\Delta y} = 1.05$ ,<sup>35</sup> then  $M$  is a P-matrix with  $T = 20$ . Moreover, from Proposition 4 with  $T = 50$ , we have that  $\varsigma > 6.385 \times 10^{-8}$ , so  $M$  is an S-matrix for all sufficiently large  $T$ , by Corollary 1.

<sup>35</sup> Results for larger  $\alpha_{\Delta y}$  were impossible due to numerical errors.

On the other hand, with  $\alpha_\pi = 1.5$  and  $\alpha_{\Delta y} = 1.51$ , then with  $T = 200$ ,  $M$  is not an S-matrix,<sup>36</sup> meaning that for all sufficiently large  $T$ ,  $M$  is not a P-matrix, so there are sometimes multiple solutions. Additionally, from Proposition 4 with  $T = 200$ ,  $\zeta \leq 0 +$  numerical error, meaning that it is likely that the model does not always possess a solution, no matter how high is  $T$ .

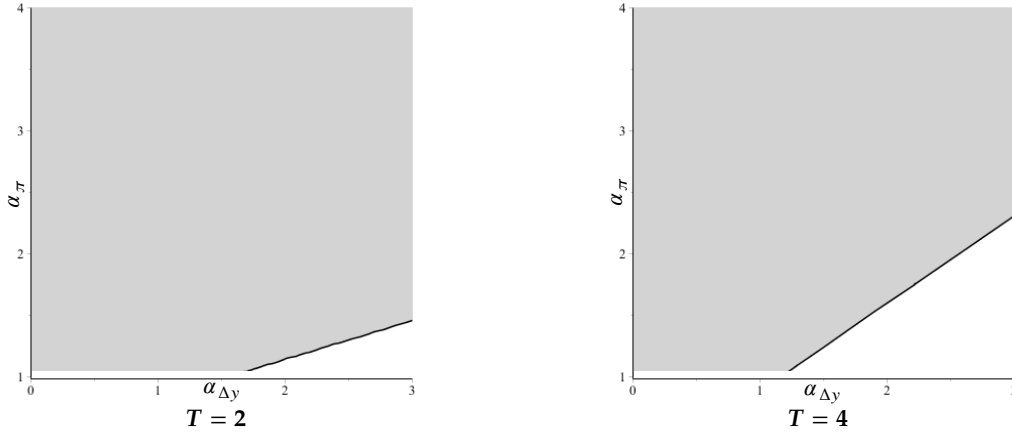


Figure 9: Regions in which  $M$  is a P-matrix (shaded grey) or a P<sub>0</sub>-matrix (shaded grey, plus the black line), when  $T = 2$  (left) or  $T = 4$  (right).

## F. Proof of the properties of the BPY model with level targeting

Defining  $x_t = [x_{i,t} \ x_{y,t} \ x_{p,t}]'$ , the model of section 3.3 is in the form of Problem 2, with:

$$A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & -1 & -\frac{1}{\sigma} \\ 0 & \gamma & -1 - \beta \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A + B + C) = \det \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & 0 & 0 \\ 0 & \gamma & -1 \end{bmatrix} \neq 0$$

as  $\alpha_{\Delta y} \neq 0$  and  $\alpha_\pi \neq 0$ . Let  $f := F_{3,3}$ , where  $F$  is as in Assumption 1. Then:

$$F = \begin{bmatrix} 0 & 0 & \frac{f(1-f)(\sigma\alpha_\pi - \alpha_{\Delta y})}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & \frac{f(1-f - \alpha_\pi)}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & f \end{bmatrix},$$

<sup>36</sup> This was verified a second way by checking that  $-M'$  was an S<sub>0</sub>-matrix, as discussed in footnote 12.



and so:

$$\beta\sigma f^3 - \left( (1+2\beta)\sigma + \beta\alpha_{\Delta y} + \gamma \right) f^2 + \left( (2+\beta)\sigma + (1+\beta)\alpha_{\Delta y} + (1+\alpha_\pi)\gamma \right) f - (\sigma + \alpha_{\Delta y}) = 0.$$

Now define:

$$\hat{\alpha}_{\Delta y} := \sigma + \alpha_{\Delta y}, \quad \hat{\alpha}_\pi := 1 + \alpha_\pi$$

so:

$$\beta\sigma f^3 - \left( (\hat{\alpha}_{\Delta y} + \sigma)\beta + \gamma + \sigma \right) f^2 + \left( (1+\beta)\hat{\alpha}_{\Delta y} + \gamma\hat{\alpha}_\pi + \sigma \right) f - \hat{\alpha}_{\Delta y} = 0.$$

This is identical to the equation for  $f$  in the previous section, apart from the fact that  $\hat{\alpha}_{\Delta y}$  has replaced  $\alpha_{\Delta y}$  and  $\hat{\alpha}_\pi$  has replaced  $\alpha_\pi$ . Hence, by the results of the previous section, Assumption 1 holds for this model as well.

Finally, for this model, with  $T = 1$ , we have that:

$$M = \frac{(1-f)(1+(1-f)\beta)\sigma^2 + \left( (1+(1-f)\beta)\alpha_{\Delta y} + ((1-f) + \alpha_\pi f)\gamma \right)\sigma + (1-f)\gamma\alpha_{\Delta y}}{\left( (1-f)(1+(1-f)\beta)\sigma + (1+(1-f)\beta)\alpha_{\Delta y} + ((1-f) + \alpha_\pi)\gamma \right)(\sigma + \alpha_{\Delta y})} > 0.$$

## G. Proof of the sufficient conditions for the existence of a unique solution to the dynamic programming problem

**Results when  $\tilde{X}$  is possibly non-compact, but  $\tilde{\Gamma}(x)$  is compact valued and  $x \in \tilde{\Gamma}(x)$  for all  $x \in \tilde{X}$**  We first note that for all  $x, z \in \tilde{X}$ :

$$\tilde{\mathcal{F}}(x, z) \leq u^{(0)} - \frac{1}{2}u^{(1)}\tilde{u}^{(2)-1}u^{(1)'},$$

thus, our objective function is bounded above without additional assumptions. For a lower bound, we assume that for all  $x \in \tilde{X}$ ,  $x \in \tilde{\Gamma}(x)$ , so holding the state fixed is always feasible. This is true in very many standard applications. Then, the value of setting  $x_t = x_0$  for all  $t \in \mathbb{N}^+$  provides a lower bound for our objective function.

More precisely, we define  $\mathbb{V} := \{v | v: \tilde{X} \rightarrow [-\infty, \infty)\}$  and  $\underline{v}, \bar{v} \in \mathbb{V}$  by:

$$\underline{v}(x) = \frac{1}{1-\beta} \tilde{\mathcal{F}}(x_0, x_0),$$

$$\bar{v}(x) = \frac{1}{1-\beta} \left[ u^{(0)} - \frac{1}{2}u^{(1)}\tilde{u}^{(2)-1}u^{(1)'} \right],$$

for all  $x \in \tilde{X}$ .

Finally, define  $\mathcal{B}: \mathbb{V} \rightarrow \mathbb{V}$  by:

$$\mathcal{B}(v)(x) = \sup_{z \in \tilde{\Gamma}(x)} \left[ \tilde{\mathcal{F}}(x, z) + \beta v(z) \right] \quad (19)$$

for all  $v \in \mathbb{V}$  and for all  $x \in \tilde{X}$ . Then  $\mathcal{B}(\underline{v}) \geq \underline{v}$  and  $\mathcal{B}(\bar{v}) \leq \bar{v}$ . Furthermore, if some sequence  $(x_t)_{t=1}^\infty$  satisfies the constraint that for all  $t \in \mathbb{N}^+$ ,  $x_t \in \tilde{\Gamma}(x_{t-1})$ , and the objective in (8) is finite for that sequence, then it must be the case that  $\|x_t\|_\infty t^{\frac{t}{2}} \rightarrow 0$  as  $t \rightarrow \infty$  (by the comparison test), so:

$$\liminf_{t \rightarrow \infty} \beta^t \underline{v}(x_t) = 0.$$

Additionally, for any sequence  $(x_t)_{t=1}^\infty$ :

$$\limsup_{t \rightarrow \infty} \beta^t \bar{v}(x_t) = 0.$$

Thus, our dynamic programming problem satisfies the assumptions of Theorem 2.1 of Kamihigashi (2014), and so  $\mathcal{B}$  has a unique fixed point in  $[\underline{v}, \bar{v}]$  to which  $\mathcal{B}^k(\underline{v})$  converges pointwise, monotonically, as  $k \rightarrow \infty$ , and which is equal to the function  $v^*: \tilde{X} \rightarrow \mathbb{R}$  defined by:

$$v^*(x_0) = \sup \left\{ \sum_{t=1}^\infty \beta^{t-1} \tilde{\mathcal{F}}_t(x_{t-1}, x_t) \mid \forall t \in \mathbb{N}^+, x_t \in \Gamma(x_{t-1}) \right\}, \quad (20)$$

for all  $x_0 \in \tilde{X}$ .

Furthermore, if we define:

$$\mathbb{W} := \{v \in V \mid v \text{ is continuous on } \tilde{X}, v \text{ is concave on } \tilde{X}\},$$

then as  $\tilde{u}^{(2)}$  is negative-definite,  $\underline{v} \in \mathbb{W}$ . Additionally, under the assumption that  $\tilde{\Gamma}(x)$  is compact valued, if  $v \in \mathbb{W}$ , then  $\mathcal{B}(v) \in \mathbb{W}$ , by the theorem of the maximum,<sup>37</sup> and, furthermore, there is a unique policy function which attains the supremum in the definition of  $\mathcal{B}(v)$ . Moreover,  $v^* = \lim_{k \rightarrow \infty} \mathcal{B}^k(\underline{v})$  is concave and lower semi-continuous on  $\tilde{X}$ .<sup>38</sup> We just need to prove that  $v^*$  is upper semi-continuous.<sup>39</sup> Suppose for a contradiction that it is not, so there exists  $x^* \in \tilde{X}$  such that:

$$\limsup_{x \rightarrow x^*} v^*(x) > \lim_{k \rightarrow \infty} v^*(x^k).$$

Then, there exists  $\delta > 0$  such that for all  $\epsilon > 0$ , there exists  $x_0^{(\epsilon)} \in \tilde{X}$  with  $\|x^* - x_0^{(\epsilon)}\|_\infty < \epsilon$  such that:

$$v^*(x_0^{(\epsilon)}) > \delta + v^*(x^*).$$

Now, by the definition of a supremum, for all  $\epsilon > 0$ , there exists  $(x_t^{(\epsilon)})_{t=1}^\infty$  such that for all  $t \in \mathbb{N}^+$ ,  $x_t^{(\epsilon)} \in \Gamma(x_{t-1}^{(\epsilon)})$  and:

$$v^*(x_0^{(\epsilon)}) < \delta + \sum_{t=1}^\infty \beta^{t-1} \tilde{\mathcal{F}}_t(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}).$$

Hence:

$$\sum_{t=1}^\infty \beta^{t-1} \tilde{\mathcal{F}}_t(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}) > v^*(x_0^{(\epsilon)}) - \delta > v^*(x^*).$$

Now, let  $\mathcal{S}_0 := \{x \in \tilde{X} \mid \|x^* - x\|_\infty \leq 1\}$ , and for  $t \in \mathbb{N}^+$ , let  $\mathcal{S}_t := \Gamma(\mathcal{S}_{t-1})$ . Then, since we are assuming  $\Gamma$  is compact valued, for all  $t \in \mathbb{N}$ ,  $\mathcal{S}_t$  is compact by the continuity of  $\Gamma$ . Furthermore, for all  $t \in \mathbb{N}$  and  $\epsilon \in (0, 1)$ ,  $x_t^{(\epsilon)} \in \mathcal{S}_t$ . Hence,  $\prod_{t=0}^\infty \mathcal{S}_t$  is sequentially compact in the product topology. Thus, there exists a sequence  $(\epsilon_k)_{k=1}^\infty$  with  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$  and such that  $x_t^{(\epsilon_k)}$  converges for all  $t \in \mathbb{N}$ . Let  $x_t := \lim_{k \rightarrow \infty} x_t^{(\epsilon_k)}$ , and note that  $x^* = x_0 \in \mathcal{S}_0 \subseteq \tilde{X}$ , and that for all  $t, k \in \mathbb{N}^+$ ,  $x_t^{(\epsilon_k)} \in \Gamma(x_{t-1}^{(\epsilon_k)})$ , so by the continuity of  $\Gamma$ ,  $x_t \in \Gamma(x_{t-1})$  for all  $t \in \mathbb{N}^+$ . Thus, by Fatou's Lemma:

<sup>37</sup> See e.g. Theorem 3.6 and following of Stokey, Lucas, and Prescott (1989).

<sup>38</sup> See e.g. Lemma 2.41 of Aliprantis and Border (2013).

<sup>39</sup> In the following, we broadly follow the proof of Lemma 3.3 of Kamihigashi and Roy (2003).

$$v^*(x^*) \geq \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \geq \limsup_{k \rightarrow \infty} \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}^{(\epsilon, k)}, x_t^{(\epsilon, k)}) > v^*(x^*),$$

which gives the required contradiction. Thus,  $v^*$  is continuous and concave, and there is a unique policy function attaining the supremum in the definition of  $\mathcal{B}(v^*) = v^*$ .

**Results when  $\tilde{X}$  is compact** If  $\tilde{X}$  is compact, then  $\Gamma$  is compact valued. Furthermore,  $\tilde{X}$  is clearly convex, and  $\Gamma$  is continuous. Thus assumption 4.3 of Stokey, Lucas, and Prescott (1989) (henceforth: SLP) is satisfied. Since the continuous image of a compact set is compact,  $\tilde{\mathcal{F}}$  is bounded above and below, so assumption 4.4 of SLP is satisfied as well. Furthermore,  $\tilde{\mathcal{F}}$  is concave and  $\Gamma$  is convex, so assumptions 4.7 and 4.8 of SLP are satisfied too. Thus, by theorem 4.6 of SLP, with  $\mathcal{B}$  defined as in equation (17) and  $v^*$  defined as in equation (18),  $\mathcal{B}$  has a unique fixed point which is continuous and equal to  $v^*$ . Moreover, by theorem 4.8 of SLP, there is a unique policy function which attains the supremum in the definition of  $\mathcal{B}(v^*) = v^*$ .

## H. Proof of the sufficiency of the KKT and limit conditions

Suppose that  $(x_t)_{t=1}^{\infty}, (\lambda_t)_{t=1}^{\infty}$  satisfy the KKT conditions given in equations (10) and (11), and that  $x_t \rightarrow \mu$  and  $\lambda_t \rightarrow \bar{\lambda}$  as  $t \rightarrow \infty$ . Let  $(z_t)_{t=0}^{\infty}$  satisfy  $z_0 = x_0$  and  $z_t \in \tilde{\Gamma}(z_{t-1})$  for all  $t \in \mathbb{N}^+$ . Then, by the KKT conditions and the concavity of:

$$(x_{t-1}, x_t) \mapsto \tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda'_t \left[ \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right],$$

we have that for all  $T \in \mathbb{N}^+$ :<sup>40</sup>

$$\begin{aligned} & \sum_{t=1}^T \beta^{t-1} [\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t)] \\ &= \sum_{t=1}^T \beta^{t-1} \left[ \tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda'_t \left[ \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \\ &\geq \sum_{t=1}^T \beta^{t-1} \left[ \tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda'_t \left[ \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right. \\ &\quad \left. - \lambda'_t \left[ \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} z_{t-1} - \mu \\ z_t - \mu \end{bmatrix} \right] \right] \\ &\geq \sum_{t=1}^T \beta^{t-1} \left[ \left[ u_{\cdot, 2}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot, 2}^{(2)} + \lambda'_t \Psi_{\cdot, 2}^{(1)} \right] (x_t - z_t) \right. \\ &\quad \left. + \left[ u_{\cdot, 1}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot, 1}^{(2)} + \lambda'_t \Psi_{\cdot, 1}^{(1)} \right] (x_{t-1} - z_{t-1}) \right] \\ &= \sum_{t=1}^T \beta^{t-1} \left[ \left[ u_{\cdot, 2}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot, 2}^{(2)} + \lambda'_t \Psi_{\cdot, 2}^{(1)} \right. \right. \\ &\quad \left. \left. + \beta \left[ u_{\cdot, 1}^{(1)} + \begin{bmatrix} x_t - \mu \\ x_{t+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot, 1}^{(2)} + \lambda'_{t+1} \Psi_{\cdot, 1}^{(1)} \right] \right] (x_t - z_t) \right] \\ &\quad + \beta^T \left[ u_{\cdot, 1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot, 1}^{(2)} + \lambda'_{T+1} \Psi_{\cdot, 1}^{(1)} \right] (z_T - x_T) \end{aligned}$$

<sup>40</sup> Here, we broadly follow the proof of Theorem 4.15 of Stokey, Lucas, and Prescott (1989).

$$= \beta^T \left[ u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda'_{T+1} \Psi_{\cdot,1}^{(1)} \right] (z_T - x_T).$$

Thus:

$$\begin{aligned} \sum_{t=1}^{\infty} \beta^{t-1} [\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t)] \\ \geq \lim_{T \rightarrow \infty} \beta^T \left[ u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda'_{T+1} \Psi_{\cdot,1}^{(1)} \right] (z_T - x_T) \\ = \lim_{T \rightarrow \infty} \beta^T [u_{\cdot,1}^{(1)} + \bar{\lambda}' \Psi_{\cdot,1}^{(1)}] (z_T - \mu) = \lim_{T \rightarrow \infty} \beta^T [u_{\cdot,1}^{(1)} + \bar{\lambda}' \Psi_{\cdot,1}^{(1)}] z_T. \end{aligned}$$

Now, suppose  $\lim_{T \rightarrow \infty} \beta^T z_T \neq 0$ , then since  $\tilde{u}^{(2)}$  is negative definite:

$$\sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(z_{t-1}, z_t) = -\infty,$$

so  $(z_t)_{t=0}^{\infty}$  cannot be optimal. Hence, regardless of the value of  $\lim_{T \rightarrow \infty} \beta^T z_T$ :

$$\sum_{t=1}^{\infty} \beta^{t-1} [\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t)] \geq 0,$$

which implies that  $(x_t)_{t=1}^{\infty}$  solves Problem 5.

## I. Results from and for general dynamic programming problems

Here we consider non-linear dynamic programming problems with general objective functions. Consider then the following generalisation of Problem 5:

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**Problem 6** Suppose  $\Gamma: \mathbb{R}^n \rightarrow \mathbb{P}(\mathbb{R}^n)$  is a given compact, convex valued continuous function. Define  $X := \{x \in \mathbb{R}^n | \Gamma(x) \neq \emptyset\}$ , and suppose without loss of generality that for all  $x \in \mathbb{R}^n$ ,  $\Gamma(x) \cap X = \Gamma(x)$ . Further suppose that  $\mathcal{F}: X \times X \rightarrow \mathbb{R}$  is a given twice continuously differentiable, concave function, and that  $x_0 \in X$  and  $\beta \in (0,1)$  are given.

Choose  $x_1, x_2, \dots$  to maximise:

$$\liminf_{T \rightarrow \infty} \sum_{t=1}^T \beta^{t-1} \mathcal{F}(x_{t-1}, x_t),$$

subject to the constraints that for all  $t \in \mathbb{N}^+$ ,  $x_t \in \Gamma(x_{t-1})$ .

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For tractability, we make the following additional assumption, which enables us to uniformly approximate  $\Gamma$  by a finite number of inequalities:

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**Assumption 8**  $X$  is compact.

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Then, by theorem 4.8 of Stokey, Lucas, and Prescott (1989), there is a unique solution to Problem 6 for any  $x_0$ . We further assume the following to ensure that there is a natural point to approximate around:<sup>41</sup>

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<sup>41</sup> If  $X$  is convex, then the existence of a fixed point of the policy function is a consequence of the Brouwer fixed point theorem, but there is no reason the fixed point guaranteed by Brouwer's theorem should be even locally attractive.

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**Assumption 9** There exists  $\mu \in X$  such that for any given  $x_0 \in X$ , in the solution to Problem 6 with that  $x_0$ , as  $t \rightarrow \infty$ ,  $x_t \rightarrow \mu$ .

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Having defined  $\mu$ , we can let  $\tilde{\mathcal{F}}$  be a second order Taylor approximation to  $\mathcal{F}$  around  $\mu$ , which will take the form of equation (7). Assumption 3 will be satisfied for this approximation thanks to the concavity of  $\mathcal{F}$ . To apply the previous results, we also then need to approximate the constraints.

Suppose first that the graph of  $\Gamma$  is convex, i.e. the set  $\{(x, z) | x \in X, z \in \Gamma(x)\}$  is convex. Since it is also compact, by Assumption 4, for any  $\epsilon > 0$ , there exists  $c \in \mathbb{N}$ ,  $\Psi^{(0)} \in \mathbb{R}^{c \times 1}$  and  $\Psi^{(1)} \in \mathbb{R}^{c \times 2n}$  such that with  $\tilde{\Gamma}$  defined as in equation (5) and  $\tilde{X}$  defined as in equation (6):

- 1)  $\mu \in \tilde{X} \subseteq X$ ,
- 2) for all  $x \in X$ , there exists  $\tilde{x} \in \tilde{X}$  such that  $\|x - \tilde{x}\|_2 < \epsilon$ ,
- 3) for all  $x \in \tilde{X}$ ,  $\tilde{\Gamma}(x) \subseteq \Gamma(x)$ ,
- 4) for all  $x \in \tilde{X}$ , and for all  $z \in \Gamma(x)$ , there exists  $\tilde{z} \in \tilde{\Gamma}(x)$  such that  $\|z - \tilde{z}\|_2 < \epsilon$ .

(This follows from standard properties of convex sets.) Then, by our previous results, the following proposition is immediate:

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**Proposition 17** Suppose we are given a problem in the form of Problem 6 (and which satisfies Assumption 4 and Assumption 5). If the graph of  $\Gamma$  is convex, then we can construct a problem in the form of the multiple-bound generalisation of Problem 2 which encodes a local approximation to the original dynamic programming problem around  $x_t = \mu$ . Furthermore, the LCP corresponding to this approximation will have a unique solution for all  $x_0 \in \tilde{X}$ . Moreover, the approximation is consistent for quadratic objectives in the sense that as the number of inequalities used to approximate  $\Gamma$  goes to infinity, the approximate value function converges uniformly to the true value function.

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Unfortunately, if the graph of  $\Gamma$  is non-convex, then we will not be able to derive similar results. To see the best we could do along similar proof lines, here we merely sketch the construction of an approximation to the graph of  $\Gamma$  in this case. We will need to assume that there exists  $z \in \text{int } \Gamma(x)$  for all  $x \in X$ , which precludes the existence of equality constraints.<sup>42</sup> We first approximate the graph of  $\Gamma$  by a polytope (i.e.  $n$  dimensional polygon) contained in the graph of  $\Gamma$  such that all points in the graph of  $\Gamma$  are within  $\frac{\epsilon}{2}$  of a point in the polytope. Then, providing  $\epsilon$  is sufficiently small, for each simplicial surface element of the polytope, indexed by  $k \in \{1, \dots, c\}$ , we can find a quadratic function  $q_k: X \times X \rightarrow \mathbb{R}$  with:

$$q_k = \Psi_k^{(0)} + \Psi_k^{(1)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix} + \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}' \Psi_k^{(2)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}$$

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<sup>42</sup> This is often not too much of a restriction, since equality constraints may be substituted out.

for all  $x, z \in X$  and such that  $q_k$  is zero at the corners of the simplicial surface element, such that  $q_k$  is weakly negative on its surface, such that  $\Psi_k^{(2)}$  is symmetric positive definite, and such that all points in the polytope are within  $\frac{\epsilon}{2}$  of a point in the set:

$$\{(x, z) \in X \times X | \forall k \in \{1, \dots, S\}, 0 \leq q_k(x, z)\}.$$

This gives a set of quadratic constraints that approximate  $\Gamma$ . If we then define:

$$\tilde{u}^{(2)} := u^{(2)} + \sum_{k=1}^c \bar{\lambda}'_{\Psi, k} \Psi_k^{(2)},$$

where  $u^{(2)}$  is the Hessian of  $\mathcal{F}$ , then the Lagrangian in equation (9) is the same as what would be obtained from taking a second order Taylor approximation to the Lagrangian of the problem of maximising our non-linear objective subject to the approximate quadratic constraints, suggesting it may perform acceptably well for  $x$  near  $\mu$ , along similar lines to the results of Levine, Pearlman, and Pierse (2008) and Benigno and Woodford (2012). However, existence of a unique solution to the original problem cannot be used to establish even the existence of a solution of the approximated problem, since only linear approximations to the quadratic constraints would be imposed by our algorithm, giving a reduced choice set (as the quadratic terms are positive definite).