Existence and uniqueness of solutions to dynamic models with occasionally binding constraints.

Tom D. Holden, *Deutsche Bundesbank and the University of Surrey*[1]

Abstract: We present necessary and sufficient conditions for the global existence of a unique perfect-foresight solution to an otherwise linear model with occasionally binding constraints, given fixed terminal conditions. This gives determinacy conditions for models with occasionally binding constraints much as Blanchard & Kahn (1980) did for the linear case. We derive further conditions on multiplicity and (non-)existence for such models. We show standard New Keynesian models possess multiple perfect-foresight paths when there is a zero lower bound on nominal interest rates, even conditional on inflation eventually becoming positive. However, we demonstrate that price level targeting restores determinacy in these settings.

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The latest release of the toolkit accompanying this paper is available from:
https://github.com/tholden/dynareOBC/releases

[1] Contact address: Deutsche Bundesbank, Wilhelm-Epstein-Straße 14, 60431 Frankfurt am Main, Germany.
The views expressed in this paper are those of the author and do not represent the views of the Deutsche Bundesbank, the Eurosystem or its staff.
Contact email: thomas.holden@gmail.com Web-site: http://www.tholden.org/
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1. Introduction

This paper provides the first necessary and sufficient conditions for an otherwise linear model with occasionally binding constraints (OBCs) to have a unique perfect foresight solution, returning to a given steady state, for any shock sequence and any value of the initial state. Our results generalise those of Blanchard & Kahn (1980) for the linear case.

It is now well known that the zero lower bound (ZLB) can lead to multiplicity of steady states (see e.g. Benhabib, Schmitt-Grohé & Uribe 2001a; 2001b). However, multiple steady states do not automatically imply multiple dynamic solutions, as agents' beliefs may rule out paths converging to deflation.[2] Indeed, it is an open question under which circumstances New Keynesian (NK) models admit multiple solutions when agents believe the economy will eventually escape the ZLB.[3] We find that multiplicity is the rule even in a transient ZLB episode and with a monetary rule that satisfies the Taylor principle. However, we show that a weak response to the price level in the monetary rule is sufficient to restore determinacy.

To see why multiplicity is still possible even with the terminal condition fixed, suppose the model's agents knew that from next period onwards, the economy would be away from the bound. Then, in an otherwise linear model, expectations of next period's outcomes would be linear in today's variables. However, substituting out these expectations does not leave a linear system in today's variables, due to the OBC. For some models, this non-linear system will

[2] This is in line with the evidence of Gürkaynak, Levin & Swanson (2010). Christiano and Eichenbaum (2012) argue that deflation can be escaped by switching to a money growth rule, along the lines of Christiano & Rostagno (2001).

[3] Hebden, Lindé & Svensson (2011) and Brendon, Paustian & Yates (2013; 2016) provide some specific examples of NK models with multiple transition paths to the inflationary steady state in certain states.

have two solutions, with one featuring a slack constraint, and the other having a binding constraint. Thus, even though the rule for forming expectations is pinned down, multiple outcomes may be possible. Without the assumption that next period the economy is away from the bound, the scope for multiplicity is even richer, and there may be infinitely many solutions.

We prove that under mild assumptions there are at least as many solutions under rational expectations as under perfect foresight.[4] Thus, our results imply lower bounds on the number of solutions under rational expectations, even though the two solution concepts will not agree here given the OBC's non-linearity.

We further provide necessary conditions and sufficient conditions for the global[5] existence of perfect foresight solutions returning to a given steady state for otherwise linear models with OBCs. We also give existence conditions that are conditional on the economy's initial state. Non-existence of solutions returning to the "standard" steady state may rationalise the beliefs needed to sustain indeterminacy driven by multiple steady states, e.g. a belief in the possibility of converging to deflation. [6]

The next section presents simple examples of multiplicity and non-existence. In Section 3, we provide the key equivalence result enabling us to examine existence and uniqueness in models with OBCs via examining the properties of linear complementarity problems. Section 4 provides our main results on existence and uniqueness, which we use to revisit our first example in Section 5.

[4] Proven in Appendix I.
[5] I.e. independent of the value of state variables.
[6] The consequences of indeterminacy of this kind has been explored by Schmitt-Grohé & Uribe (2012), Mertens & Ravn (2014) and Aruoba, Cuba-Borda & Schorfheide (2018), amongst others.

2. Multiplicity in simple models

We begin by supplying examples of the type of multiplicity and solution non-existence on which we focus in this paper. This will make clear why these problems are so common in models with OBCs and will illustrate the idea behind our results.

2.1. A simple first example

Consider the simplest possible "NK" model: the flexible price limit. The model consists of the Fisher equation[7] and the Taylor rule:

$$i_t = r_t + \pi_{t+1},$$
$$i_t = \max\{0, r_t + \phi\pi_t\},$$

where the real interest rate $r_t = r + \varepsilon_t$, where $\varepsilon_t = 0$ for all $t > 1$, and where i_t is the nominal rate, π_t is inflation and $\phi > 1$. Although the model has two steady states (the "standard" one with $i = r$ and $\pi = 0$, plus the deflationary one with $i = 0$ and $\pi = -r$), we assume that the economy returns to the standard steady state away from the ZLB. This implies that $\pi_t = 0$ for $t > 1$.[8] Hence, in period 1:

$$r + \varepsilon_1 = i_1 = \max\{0, r + \varepsilon_1 + \phi\pi_1\}.$$

If $\varepsilon_1 > -r$, then $i_1 = r + \varepsilon_1 > 0$, so $\pi_1 = 0$. However, if $\varepsilon_1 < -r$, then $i_1 < 0$ according to the Fisher equation, which is never consistent with the Taylor rule. Thus, the model has no solution returning to the standard steady state in this case.[9] Finally, if $\varepsilon_1 = -r$, then $i_1 = 0$ and any $\pi_1 \leq 0$ is consistent with the model. To summarise: with $\varepsilon_1 > -r$, the model has a unique solution returning

[7] There is no expectation operator in the Fisher equation as there is no uncertainty beyond the initial period.

[8] Suppose $i_t = 0$ for some $t > 1$, so by the Fisher equation $\pi_{t+1} = -r$, meaning $i_{t+1} = 0$ by the Taylor rule. By induction, $i_s = 0$ for all $s \geq t$, contradicting our assumption of a return to the standard steady state. Thus, $i_t > 0$ for all $t > 1$, so $\pi_t = 0$ for $t > 1$.

[9] Indeed, the model has no bounded solution in this case. $r + \varepsilon_1 < 0$ means we must have $\pi_2 > 0$ to ensure $i_1 \geq 0$. Thus, since $\phi > 1$, $\pi_t \to \infty$ and $i_t \to \infty$ as $t \to \infty$.

to the standard steady state; with $\varepsilon_1 = -r$, the model has multiple such solutions; and with $\varepsilon_1 < -r$, the model has no such solutions.

2.2. An example with robust multiplicity

The previous example only has multiplicity in a knife-edge case, but in richer models, multiplicity is more widespread. For example, suppose the central bank responds to lagged as well as current inflation.[10, 11] This is the easiest way of generating some endogenous persistence, but almost any state variable would have a similar effect. Assuming r_t is now constant, the model is then:

$$r + \pi_{t+1} = i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1}\},$$

where $\phi - \psi > 1$ and $\psi > 0$.[12] The initial state, π_0, is given. To further simplify presentation, we set $\phi := 2$, so $\psi < 1$. Our results are not specific to this special case, however.

Away from the ZLB, the model's solution must take the form $\pi_t = A\pi_{t-1}$. Substituting this back into the model's equations gives that $A = 1 - \sqrt{1 - \psi}$, so the persistence is increasing in ψ. Note that this "fundamental" solution is away from the ZLB at t when $0 < r + \pi_{t+1} = r + A^2\pi_{t-1}$, i.e. if and only if $\pi_{t-1} > -\frac{r}{A^2}$.

Now, suppose that in period 1 the economy was at the ZLB, but that it was expected to escape next period, meaning that $\pi_2 = A\pi_1$. The Fisher equation then implies that $0 = i_1 = r + A\pi_1$, so $\pi_1 = -\frac{r}{A}$. This outcome is an equilibrium

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[10] This may be justified by noting that responding negatively to lagged inflation is optimal in the presence of inflation inertia coming from e.g. indexation to past inflation. See e.g. Giannoni & Woodford (2003).

[11] Brendon, Paustian & Yates (2013; 2016) consider an Euler + Phillips curve set-up in which the monetary policy maker responds to output growth as an alternative way of introducing an endogenous state.

[12] These assumptions are sufficient for there to be a real determinate solution in the absence of the ZLB.

only if it is consistent with the monetary rule in period 1 and 2, which is true if and only if $\pi_0 \geq -\frac{r}{A^2}$.[13]

To recap, the fundamental solution is always away from the ZLB when $\pi_0 > -\frac{r}{A^2}$ and being at the ZLB today but escaping next period is an equilibrium when $\pi_0 \geq -\frac{r}{A^2}$. So, if $\pi_0 > -\frac{r}{A^2}$, then there are two solutions: the usual fundamental one with $\pi_t = A\pi_{t-1}$ and $i_t > 0$ for all $t > 0$, plus an additional solution in which $\pi_1 = -\frac{r}{A}$ (so $\pi_1 < A\pi_0$) and $i_1 = 0$. This additional solution jumps to the bound in period 1 but escapes it next period, before gradually returning to the standard steady state. Crucially, the additional solution does not require any change in beliefs about the steady state to which the economy will converge.

Conversely, if $\pi_0 < -\frac{r}{A^2}$, the only remaining possibility is that the model is at the ZLB for more than one period. But if $i_{t+1} = 0$ with $i_{t+2} > 0$ for some $t > 0$, then by the Fisher equation, $\pi_{t+1} = -\frac{r}{A}$ and $i_t = r - \frac{r}{A} < 0$ which is inconsistent with the monetary rule. So, there cannot be a solution path returning to the standard steady state when $\pi_0 < -\frac{r}{A^2}$.

As we approach the canonical model with $\psi \to 0$ (but $\psi \neq 0$), the region of non-existence shrinks but the multiplicity region grows until it encompasses the entire state space.[14] Given that the Fisher equation and Taylor rule are the core of all NK models, it should then be unsurprising that there is non-knife-edge multiplicity in all NK models with endogenous state variables that we

[13] I.e. only if $r + \phi\pi_1 - \psi\pi_0 \leq 0$ and $r + \phi\pi_2 - \psi\pi_1 \geq 0$ with $\pi_1 = -\frac{r}{A}$ and $\pi_2 = -r$. The former holds if and only if $\pi_0 \geq \frac{r}{\psi}\left(1 - \frac{2}{A}\right) = -\frac{r}{A^2}$. The latter is equivalent to $0 \leq \left(\frac{\psi}{A} - 1\right)r = (1 - A)r$, which always holds.

[14] With $\psi = 0$ and constant r, there is a unique solution returning to the standard steady state (as with $\psi = 0$, if $i_t = 0$ for some $t > 0$, then $\pi_{t+1} = -r$, so $i_{t+1} = 0$ as well). This no longer holds once a shock is introduced, as seen above.

have analysed. Even price dispersion suffices as a state. Appendix F contains a compendium of examples.

2.3. The mechanics of our main results

Even in such a simple model, deriving these pen and paper results on multiplicity and non-existence is cumbersome. Our general theoretical results provide a convenient alternative. To understand how they work, it is helpful to begin by looking at the impact of a monetary policy shock in this simple model. I.e. consider the model:

$$r + \pi_{t+1} = i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + \nu_t\},$$

where $\nu_t = 0$ for $t > 1$ and π_0 is again given. The solution away from the ZLB must take the form $\pi_t = A\pi_{t-1} + F\nu_t$, with A as before and $F = -\frac{1}{\phi - A} < 0$. Thus, away from the ZLB, $i_1 = r + A^2\pi_0 + AF\nu_1$. With $\psi > 0$, $AF < 0$, so in the fundamental solution to this model, a positive monetary policy shock actually lowers nominal interest rates.

Now suppose that we choose $\nu_1 = -\frac{r + A^2\pi_0}{AF}$. Since $F < 0$, this is a positive shock if and only if $\pi_0 > -\frac{r}{A^2}$. With this value of ν_1, in the fundamental solution, $\pi_1 = A\pi_0 + F\nu_1 = -\frac{r}{A}$ and $i_1 = r + A^2\pi_0 + AF\nu_1 = 0$, so this shock is just the right magnitude to drive the economy to touch the ZLB. Observe too that the outcome for inflation is identical to that in the non-fundamental solution to the model without a shock considered previously. This coincidence is explained by the fact that if $\pi_0 > -\frac{r}{A^2}$, then:

$$0 = i_1 = r + \phi\pi_1 - \psi\pi_0 + \nu_1 = \max\{0, r + \phi\pi_1 - \psi\pi_0\}.$$

Given the ZLB and the positivity of ν_1, there is no observable evidence that a shock has arrived at all, since the ZLB implies that with these values of output and inflation, nominal interest rates should be zero even without a shock. Such a jump to the ZLB must then be a self-fulfilling prophecy: agents' beliefs and equilibrium outcomes are as if such a monetary policy shock had hit, whether or not it did in reality. Given $\psi > 0$, the condition for multiplicity ($\pi_0 > -\frac{r}{A^2}$) here is then precisely the same as the condition for there to be a positive shock that drives interest rates to zero in the absence of the ZLB ($\pi_0 > -\frac{r}{A^2}$). Likewise, the condition for there to be multiplicity for some π_0 ($\psi > 0$) is precisely the condition for a positive shock to have a negative effect ($\psi > 0$), which is what permits this censoring away of positive shocks.

This reveals a tight connection between multiplicity and positive shocks having negative effects. Indeed, our key condition for uniqueness will require that positive shocks to the bounded variable have positive effects.[15] It will also require that news today about a future positive shock to the bounded variable will result in the bounded variable being higher in the period the shock arrives. This is the natural generalisation for models in which the bound may be hit in future periods. More than this, it requires that the impact of news shocks to the bounded variable at different horizons be "jointly" positive, in a sense to be made clear.

3. Equivalence result

We now present the result that establishes an equivalence between solutions of a DSGE model with OBCs, and solutions of a linear complementarity problem (LCP). For now, we assume that there is a single OBC of the form $i_t = \max\{0, \dots\}$, where i_t is the constrained variable (not necessarily interest rates). This covers all OBCs one encounters in practice, possibly via a transformation.[16] For example, the Karush-Kuhn-Tucker type constraints $i_t \geq 0$,

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[15] The condition requires strict positivity precisely so cases like $\psi = 0$ are treated correctly as cases with multiple solutions. We will always assume that the shock and/or state space is sufficiently rich that the path in the absence of the bound is arbitrary. See Appendix D.2 for discussion of this assumption.
[16] See Appendix H.

$\lambda_t \geq 0$, $i_t \lambda_t = 0$ hold if and only if $0 = \min\{i_t, \lambda_t\}$ which in turn holds if and only if $i_t = \max\{0, i_t - \lambda_t\}$. Generalizations to multiple constraints are also straightforward.[16] We continue to look for perfect foresight solutions converging to a steady state at which $i_t > 0$,[17] taking as given the value of the initial state of the model's endogenous variables. We assume throughout that without the bound, the model would be determinate around a unique steady state.

Without loss of generality then, the equation containing the bound is of the form:

$$i_t = \max\{0, f(x_{t-1}, x_t, x_{t+1})\}, \tag{1}$$

where x_t contains the model's period t endogenous variables, including i_t, and f is some differentiable function (later restricted to be linear). The model's other equations are of the form:

$$0 = g(x_{t-1}, i_{t-1}, x_t, i_t, x_{t+1}, i_{t+1}),$$

for some differentiable function g (also later restricted to be linear). Now define:

$$y_t := \max\{0, f(x_{t-1}, x_t, x_{t+1})\} - f(x_{t-1}, x_t, x_{t+1}).$$

By construction, $y_t \geq 0$. Also:

$$i_t = f(x_{t-1}, x_t, x_{t+1}) + y_t. \tag{2}$$

Despite its simplicity (we have just added and subtracted a term), this result turns out to be crucial. It states that the value of the bounded variable is given by its value in the absence of the constraint (but given other endogenous variables), plus an additional positive "forcing" term capturing the effect of the constraint. Furthermore, by construction, if $i_t > 0$, then $y_t = 0$ and if $y_t > 0$, then $i_t = 0$. Thus, for all t, the bounded variable i_t and the forcing term y_t satisfy the complementary slackness condition, $i_t y_t = 0$. For further intuition,

[17] A constraint that binds in steady state can be transformed into one that does not. See Appendix H.

note that when the constraint originally came from the Karush-Kuhn-Tucker (KKT) conditions $i_t \geq 0$, $\lambda_t \geq 0$, $i_t \lambda_t = 0$ (so $i_t = \max\{0, i_t - \lambda_t\}$), then $y_t = \max\{0, i_t - \lambda_t\} - i_t + \lambda_t = \lambda_t$, meaning y_t recovers the original KKT multiplier. Finally, note that since we are assuming the model returns to a steady state where $i_t > 0$, there must be some period T such that for all $t > T$, $y_t = 0$.

In order to understand the behaviour of the model with OBCs, it is helpful to first consider the behaviour of a model without OBCs but with an exogenous forcing process in one equation. In particular, we consider replacing equation (1) with equation (2), where for now we treat y_t as an exogenous forcing process. Since we are working under perfect-foresight, we are assuming that the entire path of y_t is known in period 1. We also assume that there exists some period T such that for $t > T$, $y_t = 0$, as this always holds when y_t arises endogenously from an OBC. We now make the following key definitions:

Definition 1 Under the setup of the preceding text:

- $y := [y_1, \dots, y_T]'$ is a vector giving the path of the forcing variable.
- $i \colon \mathbb{R}^T \to \mathbb{R}^T$ is a function, where for all y, $i(y)$ is a vector containing the first T elements of the path of i_t for the given path of the forcing variable y.
- $q := i(0)$ is a vector giving the first T elements of the path of i_t when $y_t = 0$ for all t, i.e. q gives the path i_t would follow were there no bound in the model.
- M is a $T \times T$ matrix where the 1st column equals $\left.\frac{\partial i(y)}{\partial y_1}\right|_{y=0}$, the 2nd equals $\left.\frac{\partial i(y)}{\partial y_2}\right|_{y=0}$, and so on.

Then, by Taylor's theorem $i(y) = q + My + \mathrm{O}(y'y)$ for small y. Henceforth, we restrict f and g to be linear, in which case this approximation is exact and $i(y) = q + My$, with only q, not M, depending on the initial state. We prove this and establish expressions for the elements of M in Appendix E. The proof proceeds by backwards induction, starting from the known transition matrix in period $T+1$ from which point on the economy is away from the bound. Note that with f and g linear, the first column of M gives the impulse response to a contemporaneous shock to the bounded variable, the second column of M gives the impulse response to a one period ahead news shock to the bounded variable, and so on.[18]

Given the complementary slackness conditions for y_t already established, and the positivity of the path of the bounded variable, we then have that $y \geq 0$, $q + My \geq 0$ and $y'(q + My) = 0$. These conditions completely characterise the solution in the presence of OBCs:

Theorem 1

1) Suppose x_t is a solution to the model without an OBC in which equation (1) is replaced with equation (2), with y_t as an exogenous driving process. Suppose that there is some $T \geq 0$ such that $y_t = 0$ for $t > T$. Then x_t is also a solution to the original model with an OBC, permanently escaping the bound after at most T periods, if and only if $y \geq 0$, $q + My \geq 0$, $y'(q + My) = 0, f(x_{t-1}, x_t, x_{t+1}) \geq 0$ for $t > T$.
2) Suppose x_t is a solution to the model with an OBC which eventually escapes the bound. Then there exists $T \geq 0$ and a unique $T \times 1$ vector y such that: $y \geq 0$, $q + My \geq 0$, $y'(q + My) = 0, f(x_{t-1}, x_t, x_{t+1}) \geq 0$ for $t > T$ and such that x_t is the unique solution to the model without an OBC in which equation (1) is replaced with equation (2), with y_t exogenous.

[18] The idea of imposing an OBC by adding news shocks is also present in Holden (2010), Hebden et al. (2011), Holden & Paetz (2012) and Bodenstein et al. (2013). Laséen & Svensson (2011) use a similar technique to impose a path of nominal interest rates, in a non-ZLB context. None of these papers formally establish our equivalence result. News shocks were introduced by Beaudry & Portier (2006).

The proof (in Appendix E) again relies on backward induction arguments. This theorem establishes that in order to solve for the perfect-foresight solution of the model with OBCs, we just need to guess a sufficiently high T, then find a forcing process y which solves the following "linear complementarity problem" (LCP):

Definition 2 (LCP) We say $y \in \mathbb{R}^T$ solves the **LCP** (q, M) if and only if $y \geq 0$, $q + My \geq 0$ and $y'(q + My) = 0$.

LCPs have been extensively studied in mathematics. See Cottle (2009) for a brief introduction, and Cottle, Pang & Stone (2009a) for a definitive survey. Also see Appendix G, for direct results on the properties of small LCPs.

General LCPs can be solved via mixed-integer linear programming, for which highly optimised solvers exist. This approach is developed into a solution algorithm for models with OBCs in Holden (2016).

4. Existence and uniqueness results

We now turn to our main theoretical results on the existence and uniqueness of perfect foresight solutions to models that are linear apart from an OBC. Supplemental results are contained in Appendices C and J, with the latter relating our findings to models solvable via dynamic programming. Our results exploit the bijection between solutions of the model with an OBC and solutions to the LCP, which permits us to import the conclusions of the LCP literature. The LCP results all rest on the properties of the M matrix. Here we will focus on just two: that of being a P-matrix and that of being an S-matrix. The former will be key for uniqueness, and the latter for existence.

4.1. Uniqueness results

We start by looking at uniqueness. The main definition follows:

Definition 3 (P-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is a **P-matrix** if and only if for all $z \in \mathbb{R}^{T\times 1}$ with $z \neq 0$, there exists $t \in \{1, \dots, T\}$, such that $z_t (Mz)_t > 0$. (Cottle, Pang & Stone 2009b)

Clearly, all positive definite matrices are P-matrices, so this definition captures a broader notion of positivity for an arbitrary matrix. Recall that in Section 2.3 we found that multiplicity was driven by positive monetary policy shocks having negative effects. Thus, it is unsurprising that some type of positivity is key for uniqueness. In fact:

Theorem 2 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix. If M is not a P-matrix, then for some q the LCP (q, M) has multiple solutions.

(Samelson, Thrall & Wesler 1958; Cottle, Pang & Stone 2009b)

Applied to models with an OBC, this becomes:

Corollary 1 Consider an otherwise linear model with an OBC. Let $T > 0$. Then:

1) If M is a P-matrix, and $(x_t)_{t=1}^{\infty}$ is a solution to the model with an OBC that is away from the bound from period $T + 1$ onwards, then $(x_t)_{t=1}^{\infty}$ is the unique such solution.
2) If M is a P-matrix, then for any x_0 there exists a unique path $(x_t)_{t=1}^{\infty}$ with x_t satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T + 1$ onwards.

Furthermore, suppose that the model's state space is rich enough such that for any path $\tilde{q} \in \mathbb{R}^T$, there exists x_0 such that $q(x_0) = \tilde{q}$ (making explicit the dependency of q on x_0),[19] then:

3) If M is not a P-matrix then there exists x_0 such that there are multiple paths $(x_t)_{t=1}^{\infty}$ with x_t satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T+1$ onwards.

This result is the equivalent for models with OBCs of the key theorem of Blanchard & Kahn (1980). Its proof is immediate from Theorem 1. Note that if M is not a P-matrix for some T, then M will also not be a P-matrix for any larger T,[20] so to show general multiplicity it suffices to show that M is not a P-matrix for a small T. Parts 2) and 3) are of practical relevance despite the non-imposition of the bound from period $T+1$ onwards for two reasons. Firstly, with large T, any path is likely to be away from the bound by period $T+1$. Secondly, many real world OBCs are likely to be eventually made obsolete by technological developments.[21]

To see why being a P-matrix is the correct notion of positivity, suppose that y and $\tilde{y}$ both solved the LCP (q, M). Thus, for all $t \in \{1, \dots, T\}$, $0 = y_t(q + My)_t = \tilde{y}_t(q + M\tilde{y})_t$, so:

$$\begin{aligned}(y-\tilde{y})_t\big(M(y-\tilde{y})\big)_t &= (y-\tilde{y})_t\big((q+My)-(q+M\tilde{y})\big)_t \\ &= y_t(q+My)_t + \tilde{y}_t(q+M\tilde{y})_t - y_t(q+M\tilde{y})_t - \tilde{y}_t(q+My)_t \\ &\le 0\end{aligned}$$

[19] This can usually be achieved by adding "news shocks" to the bounded variable. See Appendix D.2.
[20] Immediate from the alternative definition in Appendix B. See also Cottle, Pang & Stone (2009b).
[21] E.g. a move to electronic cash will mean the ZLB is no longer a constraint.

as y_t, $\tilde{y}_t$, $q + My$ and $q + M\tilde{y}$ must all be weakly positive. Hence, if we define $z = y - \tilde{y}$, then we have that for all $t \in \{1, \dots, T\}$, $z_t(Mz)_t \le 0$. If M is a P-matrix, this implies that $z = 0$ so $y = \tilde{y}$, meaning the solution is unique.[22] Informally, M being a P-matrix guarantees positive shocks to i_t increase i_t enough on average that one cannot have the kinds of self-fulfilling jumps to the bound we saw in Section 2.

Checking whether M is a P-matrix requires checking the positivity of the determinants of all M's 2^T principal sub-matrices (see Appendix B). Since this is rather onerous, in Appendix C.1 we also present both easier to verify necessary conditions, and easier to verify sufficient conditions, which give a fast answer one way or the other in most cases.

4.2. Existence results

We now turn to existence conditions. In this case, the key property is being an S-matrix:

Definition 4 (S-matrix) A matrix $M \in \mathbb{R}^{T \times T}$ is called an **S-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$.[23] *Note: all P-matrices are S-matrices.*

Again, this captures a type of positivity of M. It is closely related to the feasibility of an LCP:

Definition 5 (Feasibility) We say $y \in \mathbb{R}^T$ is **feasible** for the LCP (q, M) if and only if $y \ge 0$ and $q + My \ge 0$. We say a path $(x_t)_{t=1}^{\infty}$ is **feasible** for a model with an OBC given initial state x_0, if when equation (1) is replaced by equation (2), with y_t exogenous, there is some $(y_t)_{t=1}^{\infty}$ with $y_t \ge 0$ for all t, such that $(x_t)_{t=1}^{\infty}$ solves the model with equation (2), and $i_t \ge 0$ for all t.

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[22] This argument just follows that of Cottle, Pang & Stone (2009b).

[23] This may be tested by solving a linear programming problem. See Appendix B.

By definition, if an LCP has a solution, then it is feasible. Likewise, if a model with an OBC has a solution, then it is feasible. If a monetary policy maker could make credible promises about (positive) future monetary policy shocks, then feasibility would be sufficient to allow the policy maker to ensure a solution. If M is an S-matrix then feasibility is guaranteed:

Proposition 1 The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. If the LCP (q, M) has a solution for all $q \in \mathbb{R}^T$, then M is an S-matrix. (Cottle, Pang & Stone 2009b)

Moreover, in most cases one encounters in practice, an LCP is solvable whenever it is feasible, i.e. whenever M is an S-matrix.[24] This has immediate practical consequences: if M is an S-matrix for some T then we are likely to be able to solve the size T LCPs we encounter in simulating the model, whatever the model's path without the bound, q. Additionally, we have:

Corollary 2 Let $T > 0$. Consider an otherwise linear model with an OBC where the model's state space is rich enough such that for any path $\tilde{q} \in \mathbb{R}^T$, there exists x_0 such that $q(x_0) = \tilde{q}$. Then if M is not an S-matrix, there exists x_0 such that there is no path $(x_t)_{t=1}^{\infty}$ with x_t satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T + 1$ onwards.

To obtain results on the feasibility of model paths, we need results on the $T = \infty$ case. Proposition 1 implies that the infinite LCP (q, M) is feasible for all $q \in \mathbb{R}^{\mathbb{N}^+}$ if and only if $\varsigma := \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} (My)_t > 0$. Furthermore, in Appendix L.1 we prove:

[24] Formal sufficient conditions for existence are provided in Appendix C.2.

Proposition 2 Given an otherwise linear model with an OBC, there exist potentially informative bounds $\underline{\varsigma}_S$, $\overline{\varsigma}_S$, computable in time polynomial in S, such that $\underline{\varsigma}_S \leq \varsigma \leq \overline{\varsigma}_S$.[25]

This enables us to derive results despite the infeasible infinite dimensional problem that defines ς. Relating this to our situation gives:

Corollary 3 Suppose that for some S, $\underline{\varsigma}_S > 0$. Then for any x_0 the model with an OBC has a feasible path (a necessary condition for existence of a solution). Conversely, suppose $\overline{\varsigma}_S = 0$. Then there is some path $\left(\tilde{q}_t\right)_{t=1}^{\infty}$ such that if $q_t = \tilde{q}_t$ for all t (i.e. in a version of the model without a bound, $i_t = \tilde{q}_t$ for all t),[26] then the model with the bound has no solution.

This result is important as it gives existence conditions without any dependence on T.

We note that the proof of Proposition 2 may be of independent interest for two reasons. Firstly, as it derives closed form expressions for the limits of the diagonals of M, via novel expressions for the impulse response to a news shock as the horizon goes to infinity. Secondly, because it derives constructive bounds on the elements of M using results on pseudospectra from Trefethen and Embree (2005), which are not well known in the economics profession.

[25] The practical informativeness of these bounds is made clear by the results for NK models in Appendix F.

[26] E.g. because x_0 was chosen appropriately, or the model was augmented with an exogenous forcing process.

5. Revisiting our first example

Section 2.1 showed that a model with flexible prices and a standard Taylor rule had multiple equilibria. To put this result into the context of our general theory, we derive the M matrix for this model in the $\psi = 0$ case. Since the M matrix stacks the impulse responses to news shocks at different horizons (ignoring the bound), we start by augmenting the model without bound by an exogenous forcing process, ν_t, giving:

$$r + \pi_{t+1} = i_t = r + \phi\pi_t + \nu_t.$$

Given the entire path of ν_t is known in period 1, the solution must take the infinite moving-average form $\pi_t = \sum_{j=0}^{\infty} F_j \nu_{t+j}$. Matching coefficients implies that $F_j = -\phi^{-(j+1)}$ for all $j \in \mathbb{N}$, so $i_t = r - \sum_{j=1}^{\infty} \phi^{-j} \nu_{t+j}$. From this, we can read off the columns of the M matrix. The first column is the path of $i_t - r$ when $\nu_1 = 1$ and $\nu_t = 0$ for $t \neq 1$, which is 0,0,.... The second column is the path of $i_t - r$ when $\nu_2 = 1$ and $\nu_t = 0$ for $t \neq 2$, which is ϕ^{-1},0,0,.... The third is ϕ^{-2}, ϕ^{-1}, 0,0,..., and so on. Thus, for any T, the M matrix has a zero diagonal, a strictly negative upper triangle, and a zero lower triangle. Consequently, all M's principal sub-matrices have zero determinant, so M cannot be a P-matrix (see Appendix B). Thus, as we already saw, this model does not always have a unique solution when augmented with appropriate shocks (in this case, a shock to the real interest rate).

Now suppose we augment the Taylor rule with a response to the price level, p_t, so:

$$r + p_{t+1} - p_t = i_t = \max\{0, r + \phi(p_t - p_{t-1}) + \chi p_t\},$$

where $p_0 = 0$. In this case, to find M we need to solve the model:

$$r + p_{t+1} - p_t = i_t = r + \phi(p_t - p_{t-1}) + \chi p_t + \nu_t,$$

which must have a solution in the form $p_t = \sum_{j=-\infty}^{\infty} G_j \nu_{t+j}$, where $\nu_t = 0$ for all $t < 0$. Again, by matching coefficients, we can derive closed form expressions for G_j, given in Appendix K. Furthermore, we show there that for any T, all of the elements of M are strictly increasing in χ for small χ. Thus, by Jacobi's formula, for any principal sub-matrix W of M with $W \in \mathbb{R}^{S \times S}$ ($S \le T$), if $\chi = 0$:

$$\frac{d \det W}{d\chi} = \frac{dW_{S,1}}{d\chi} (-1)^{S-1} \det W_{1:(S-1),2:S} = \frac{dW_{S,1}}{d\chi} \prod_{s=1}^{S-1} (-W_{s,s+1}) > 0,$$

as with $\chi = 0$, W must be strictly upper triangular with negative elements in the upper triangle. Thus, for any T, there exists $\overline{\chi}_T \in (0, \infty]$ such that for all $\chi \in (0, \overline{\chi}_T)$, M is a P-matrix. Consequently, a weak but positive response to the price level restores determinacy.

In Appendix F, we show numerically that this result generalises from this simple case. A response to the price level ensures determinacy in the presence of the ZLB across a wide range of NK models. The intuition again comes down to the sign of the response to monetary policy (news) shocks. With the price level in the Taylor rule, the reduction in prices brought about by a positive monetary policy (news) shock must be followed eventually by a counter-balancing increase. But if inflation is higher in future, then real rates are lower today, meaning that consumption, output, inflation and nominal rates will all be relatively higher today. This ensures that positive monetary policy (news) shocks have sufficiently positive effects on nominal rates to prevent self-fulfilling jumps to the bound. Thus, in the presence of the ZLB, a positive response to the price level is the equivalent of the Taylor principle. We discuss this and other results in the context of the broader literature in Appendix D.

6. Conclusion

This paper provides the first general theoretical results on existence and uniqueness for otherwise linear models with occasionally binding constraints, given terminal conditions. As such, it may be viewed as doing for models with OBCs what Blanchard & Kahn (1980) did for linear models. Applying our results, we showed that multiplicity is the norm in New Keynesian models, but that a response to the price level can restore determinacy. Our conditions may be easily checked numerically using the "DynareOBC" toolkit we provide.

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