Existence, uniqueness and computation of solutions to dynamic models with occasionally binding constraints.

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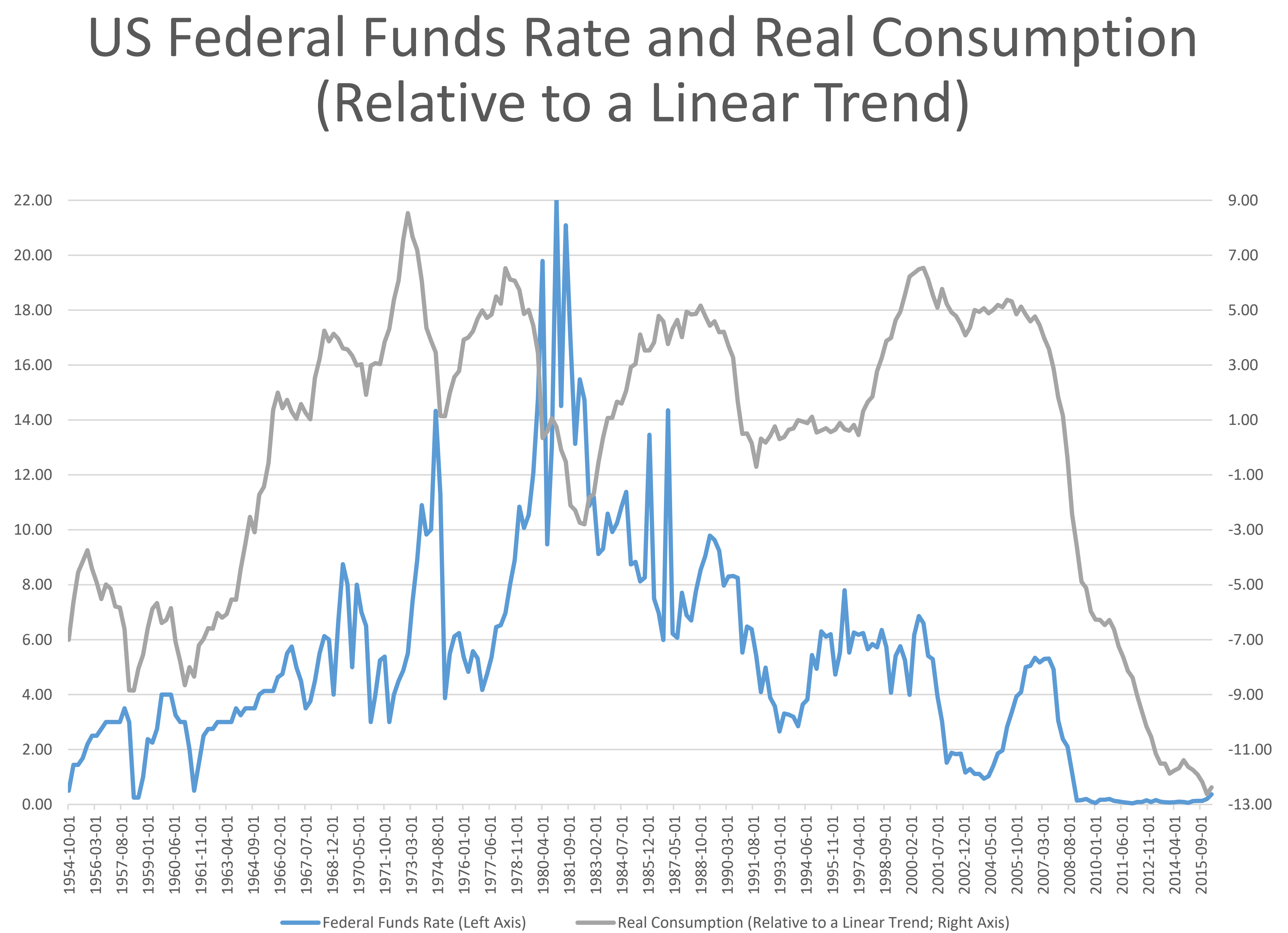

US Federal Funds Rate and Real Consumption
(Relative to a Linear Trend)
22.00
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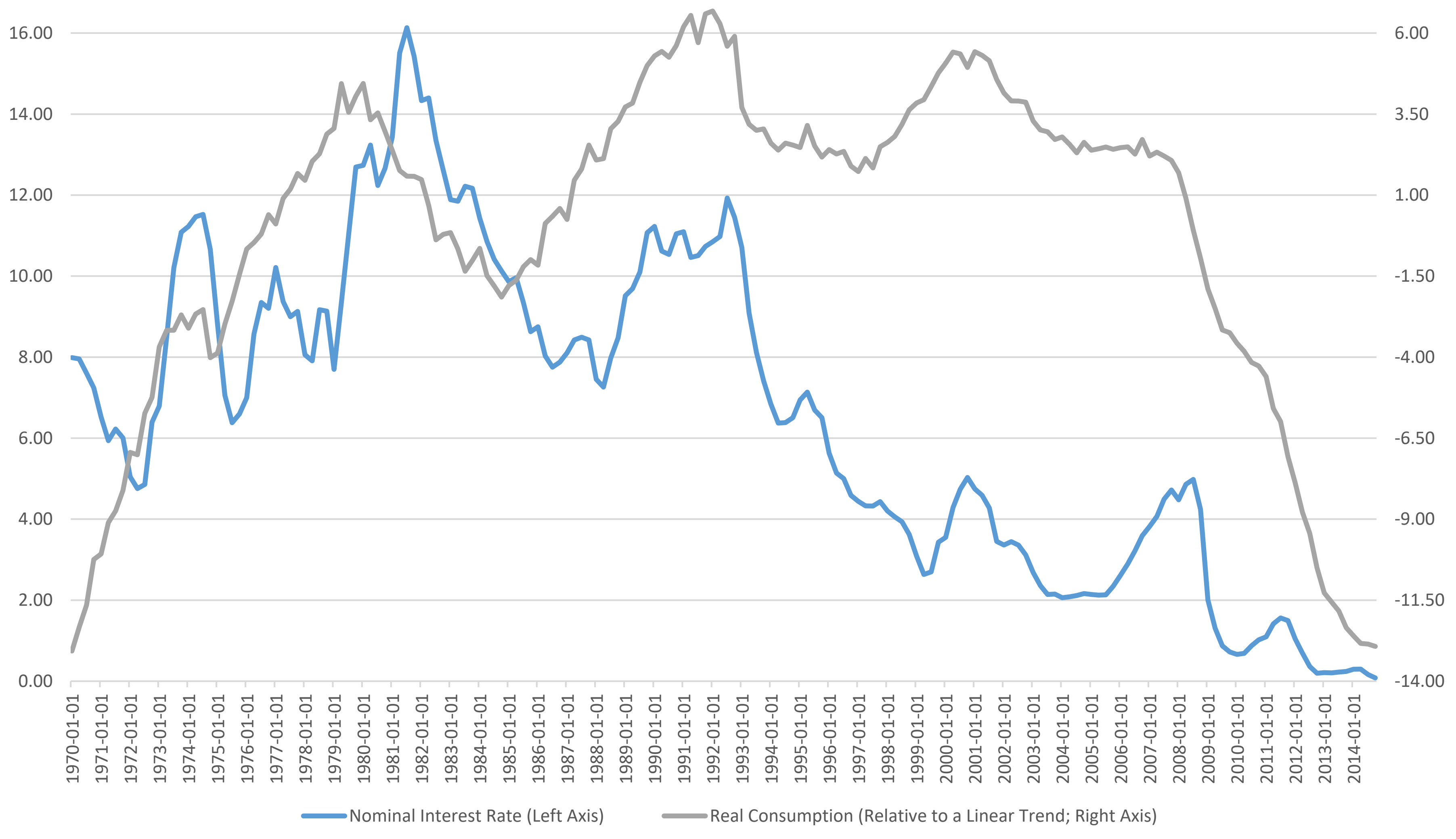
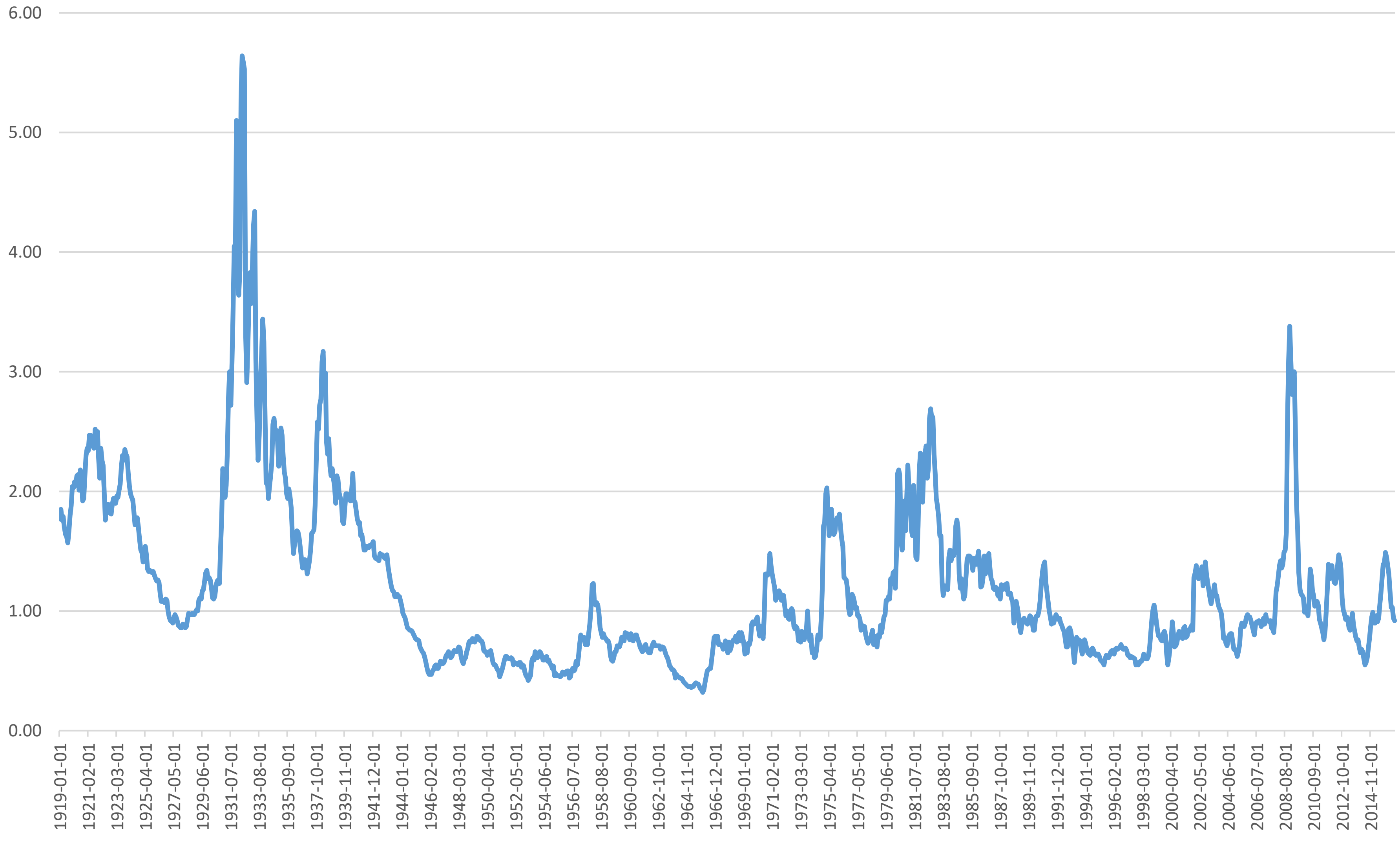
2012-11-01
2014-04-01
2015-09-01
Federal Funds Rate (Left Axis)
Real Consumption (Relative to a Linear Trend; Right Axis)

Euro Area Nominal Interest Rates and Real Consumption (Relative to a Linear Trend)



Moody's BAA-AAA Spread



Overview

This project (now two papers)

- "Blanchard and Kahn (1980) with occasionally binding constraints."

- Two elements to this:
  - Paper 1 (Theory): Existence and uniqueness results for otherwise linear models under perfect foresight.
  - Paper 2 (Computational): A new computational algorithm designed to be robust, accurate and scalable.

- All procedures implemented in my DynareOBC toolkit, which is one stop shop for all things OBC.
  - Designed to be super easy to use.
  - Also incorporates code supporting a third paper on estimating non-linear models, including those with OBCs.
  - Available from https://github.com/tholden/dynareOBC.

A note on terminal conditions

- All results are conditional on agents believing that the economy will eventually return to a given (locally-determinate) steady-state.
  - In line with the approach of Brendon, Paustian, and Yates (2013; 2016).
- Accords with a belief in the credibility of the long-run inflation target.
  - Christiano and Eichenbaum (2012) argue deflationary equilibria may be ruled out by switching to a money growth rule following severe deflation, along the lines of Christiano and Rostagno (2001).
  - Belief in credibility of the long-run target is in line with the evidence of Gürkaynak, Levin, and Swanson (2010).
- Contrary to the approach of, e.g. Benhabib, Schmitt-Grohe, and Uribe (2001a,b), Schmitt-Grohe and Uribe (2012), Mertens and Ravn (2014), Aruoba, Cuba-Borda and Schorfheide (2013).

Theory paper core contributions:

- Necessary and sufficient conditions for existence of a unique perfect-foresight solution to otherwise-linear models with OBCs (satisfying the terminal condition).
  - I will omit mentioning the terminal condition in the following.
- Necessary and sufficient conditions for existence of a unique solution when away from the bound.
  - E.g. suppose that the impulse response to some shock does not hit the bound. Must it be the unique solution?
- Some necessary conditions, and some sufficient conditions for existence of any solution.

A note on rational expectations solutions

- The theory paper focusses on results for perfect-foresight solutions, i.e. solutions to models without future uncertainty.

- Nonetheless, these results with have implications for solutions under rational expectations.
  - E.g. if a solution satisfying the terminal condition does not exist in all states of the world, then 100% belief in the terminal condition cannot be consistent with rationality.
  - Thus, the backwards induction arguments used by e.g. Mertens and Ravn (2014) imply global indeterminacy.

- Additionally, I show that:
  - Under mild assumptions, with sufficiently small shocks, there are at least as many solutions to the rational expectations model as there are to the model without future uncertainty.

A note on linearization

- Braun, Körber, and Waki (2012) (BKW) show multiplicity in a non-linear New Keynesian model.

- However, the linearized version of the model with the bound reintroduced posses a unique equilibrium.

- The multiplicity we find is strictly in addition to the BKW type multiplicity.

- Perhaps lucky given that:
  - Christiano and Eichenbaum (2012) brand the BKW multiple-equilibria as mere “mathematical curiosities” due to learnability considerations.
  - The BKW approach to finding multiple equilibria does not readily generalize to models with endogenous state variables.

Theory paper applications:

- Multiplicity is the rule in New Keynesian models, once they have at least one state variable, e.g. price dispersion.

- For example, augment the Smets and Wouters (2003) and (2007) models with a zero lower bound, at their estimated modes.
- For both, there are states and sequences of predicted future shocks for which:
  1. There are multiple solutions to the model, including combinations for which one solution features strictly positive nominal interest rates.
  2. There are zero solutions to the model.
- Both models are determinate under price level targeting.

- Across a variety of models, level targeting appears to be a magic bullet.

Computational paper core contributions:

- First perfect foresight solver for linear models with occasionally binding constraints that is guaranteed to give an answer in finite time.
  - Guaranteed to find a solution in finite time if one exists.
  - If no solution exists, will also report this in finite time too.
  - Allows easy selection of particular equilibria in the presence of multiplicity.
  - Generally very efficient.
  - Trick is representing the bound as providing endogenous news.

- Accurate solver for general non-linear models with OBCs based on leveraging the underlying perfect foresight solver.
  - Based upon a high order pruned perturbation solution to the underlying model.
  - Account for risk of hitting the bound by integrating over future uncertainty following the original Stochastic Extended Path of Adjemian and Juillard (2013).
    - Rather than the new 2016 version.

Outline for today

- Representation results behind both the perfect foresight solver and the theoretical results.

- Intuition for multiplicity.

- A little on the theory.

- Computational algorithm.

- Accuracy tests (time permitting).

Representation result

The set-up without bounds (1/3)

- Suppose for $t \in \mathbb{N}^+$:

$$\left(\hat{A} + \hat{B} + \hat{C}\right)\hat{\mu} = \hat{A}\hat{x}_{t-1} + B\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \widehat{D}\varepsilon_t,$$

  - where $\mathbb{E}_{t-1}\varepsilon_t = 0$ for all $t \in \mathbb{N}^+$,
  - $\varepsilon_t = 0$ for $t > 1$, (impulse response/perfect foresight simulation).

- $\hat{x}_0$ is given as an initial condition.
- Terminal condition: $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$.

The set-up without bounds (2/3)

- For $t \in \mathbb{N}^+$, define:

$$x_t := \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}, \qquad \mu := \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}, \qquad A := \begin{bmatrix} \hat{A} & \widehat{D} \\ 0 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}, \qquad C := \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$$

- then, for $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1},$$

- and $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, $x_t \to \mu$ as $t \to \infty$.

- Take this as the form of our problem without bounds in the following.

The set-up without bounds (3/3)

- **Problem 1**
- Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}.$$

- **Assumption:** For any given $x_0 \in \mathbb{R}^n$, Problem 1 has a unique solution, of the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in \mathbb{N}^+$, where $F = -(B + CF)^{-1}A$, and all of the eigenvalues of F are weakly inside the unit circle.

- **Assumption:** $\det(A + B + C) \neq 0$.

The set-up with bounds

- **Problem 2**
- Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\left\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + \left(B_{1,\cdot} + I_{1,\cdot}\right)(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\right\},$$

$$\left(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot}\right)\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

- and such that $x_{1,t} > 0$ for $t > T$. (Existence of such a T is WLOG.)
  - Ruling out solutions that get stuck at another steady-state by assumption.

- Note: KKT type conditions e.g. $y \geq 0$, $\lambda \geq 0$, $y\lambda = 0$ may be encoded as: $0 = \min\{y, \lambda\}$, so $y = \max\{0, y - \lambda\}$.

The news shock set-up

- **Problem 3**
- Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$, $y_t \to 0$, as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1}y_{1,t-1},$$
$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t} = y_{i+1,t-1},$$
$$y_{T,t} = 0.$$

- A version of Problem 1 with news shocks up to horizon T added to the first equation.
  - The value of $y_{t,0}$ gives the news shock that hits in period t.
  - I.e. $y_{1,t-1} = y_{t,0}$ for $t \le T$, and $y_{1,t-1} = 0$ for $t > T$.

A representation of solutions to Problem 3

- **Lemma**: There is a unique solution to Problem 3 that is linear in x_0 and y_0.

- Let $x_t^{(3,k)}$ be the solution to Problem 3 when $x_0 = \mu$, $y_0 = I_{\cdot,k}$.
- Let $M \in \mathbb{R}^{T\times T}$ satisfy:
$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t,k \in \{1,..,T\},$$
  - i.e. M horizontally stacks the (column-vector) relative impulse responses to the news shocks.
- Let $x_t^{(1)}$ be the solution to Problem 1 for some given x_0.
- Then the solution to Problem 3 for given x_0, y_0 satisfies:
$$\left(x_{1,1\dots T}\right)' = q + My_0,$$
  - where $q \coloneqq \left(x_{1,1\dots T}^{(1)}\right)'$, i.e. the path of the first variable in the absence of the bound.

The links between the solutions to Problem 2 and the solution to Problem 3 (1/2)

- Let $x_t^{(2)}$ be a solution to Problem 2 given an arbitrary x_0.
- Define:

$$e_t := \begin{cases} -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + \left(B_{1,\cdot} + I_{1,\cdot}\right)\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right] & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}$$

- **Lemma:** The following statements hold:
  - $e_{1\ldots T} \geq 0$, $x_{1,1\ldots T}^{(2)} \geq 0$ and $x_{1,1\ldots T}^{(2)} \circ e_{1\ldots T} = 0$,
  - $x_t^{(2)}$ is the unique solution to Problem 3 when started with $x_0 = x_0^{(2)}$ and with $y_0 = e_{1\ldots T}'$.
  - If $x_t^{(2)}$ solves Problem 3 when started with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e_{1\ldots T}'$.

The links between the solutions to Problem 2 and the solution to Problem 3 (2/2)

- **Proposition:** The following statements hold:

  - Let $x_t^{(3)}$ be the unique solution to Problem 3 when initialized with some x_0, y_0. Then $x_t^{(3)}$ is a solution to Problem 2 when initialized with x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$ with $t > T$.

  - Let $x_t^{(2)}$ be any solution to Problem 2 when initialized with x_0. Then there exists a $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, such that $x_t^{(2)}$ is the unique solution to Problem 3 when initialized with x_0, y_0.

Linear complementarity problems (LCPs)

- The previous proposition establishes that solving the model with occasionally binding constraints is equivalent to solving the following "linear complementarity problem".

- **Problem 4**
- Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. Find $y \in \mathbb{R}^T$ such that: $y \geq 0$, $y \circ (q + My) = 0$ and $q + My \geq 0$.

- We call this the linear complementarity problem (q, M).

Generalisations

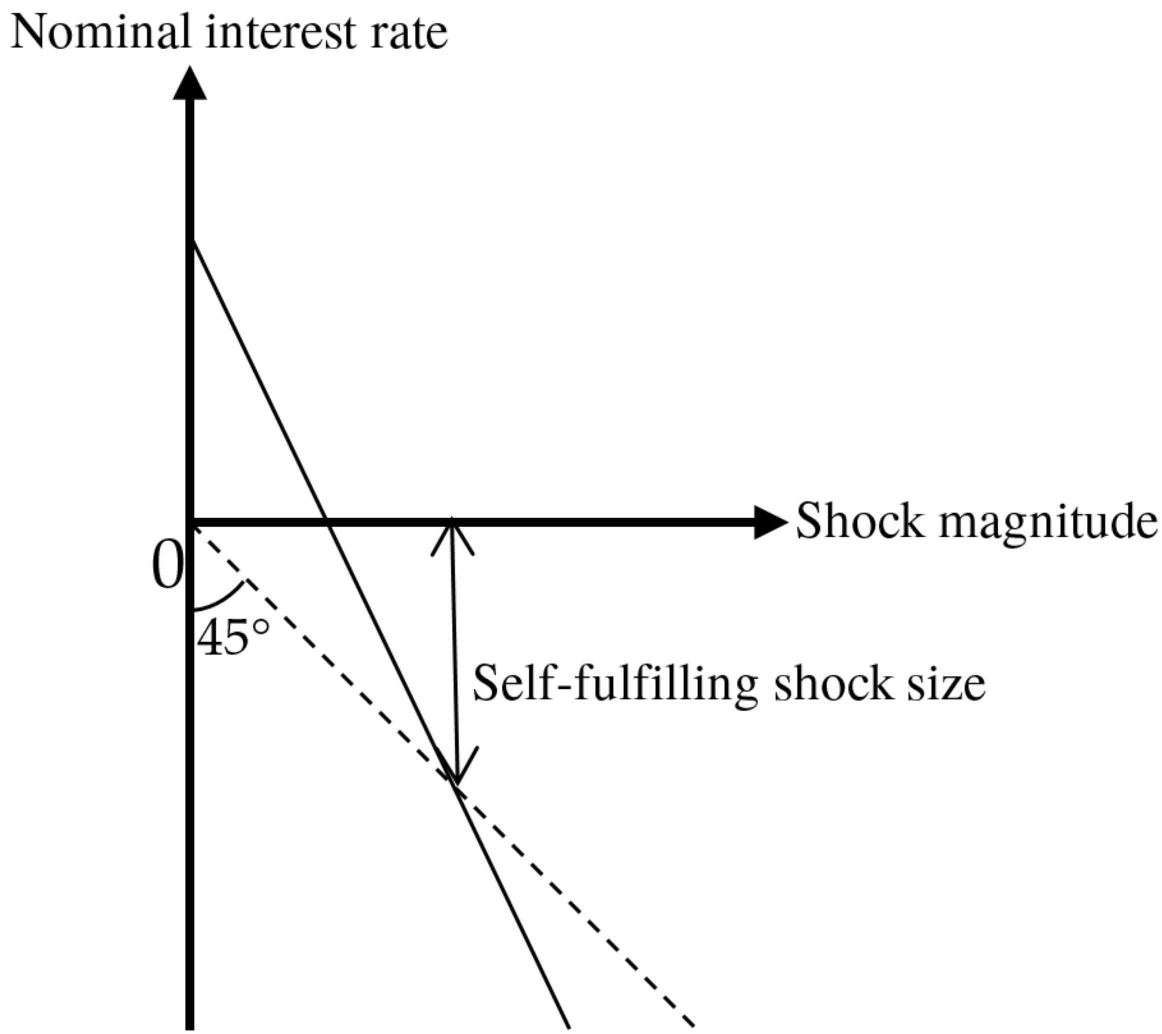
- For multiple bounds:
  - We stack the impulse responses of the bounded variables ignoring bounds into q.
  - We stack the vectors of news shocks to each variable into y.
  - M is a block matrix of each bounded variable's responses to each bounded variable's news shocks.
  - Then the stacked solution for the paths of the bounded variables is $q + My$, and we again have an LCP, so results go through as before.

- For bounds not at zero:
  - If $z_{1,t} = \max\{z_{2,t}, z_{3,t}\}$, then $z_{1,t} - z_{2,t} = \max\{0, z_{3,t} - z_{2,t}\}$.

- For minimums:
  - If $z_{1,t} = \min\{z_{2,t}, z_{3,t}\}$, then $-z_{1,t} = \max\{-z_{2,t}, -z_{3,t}\}$.

Intuition

Intuition for multiplicity: Self-fulfilling news shocks



Intuition for multiplicity: The negative response need not be on impact (1/2)

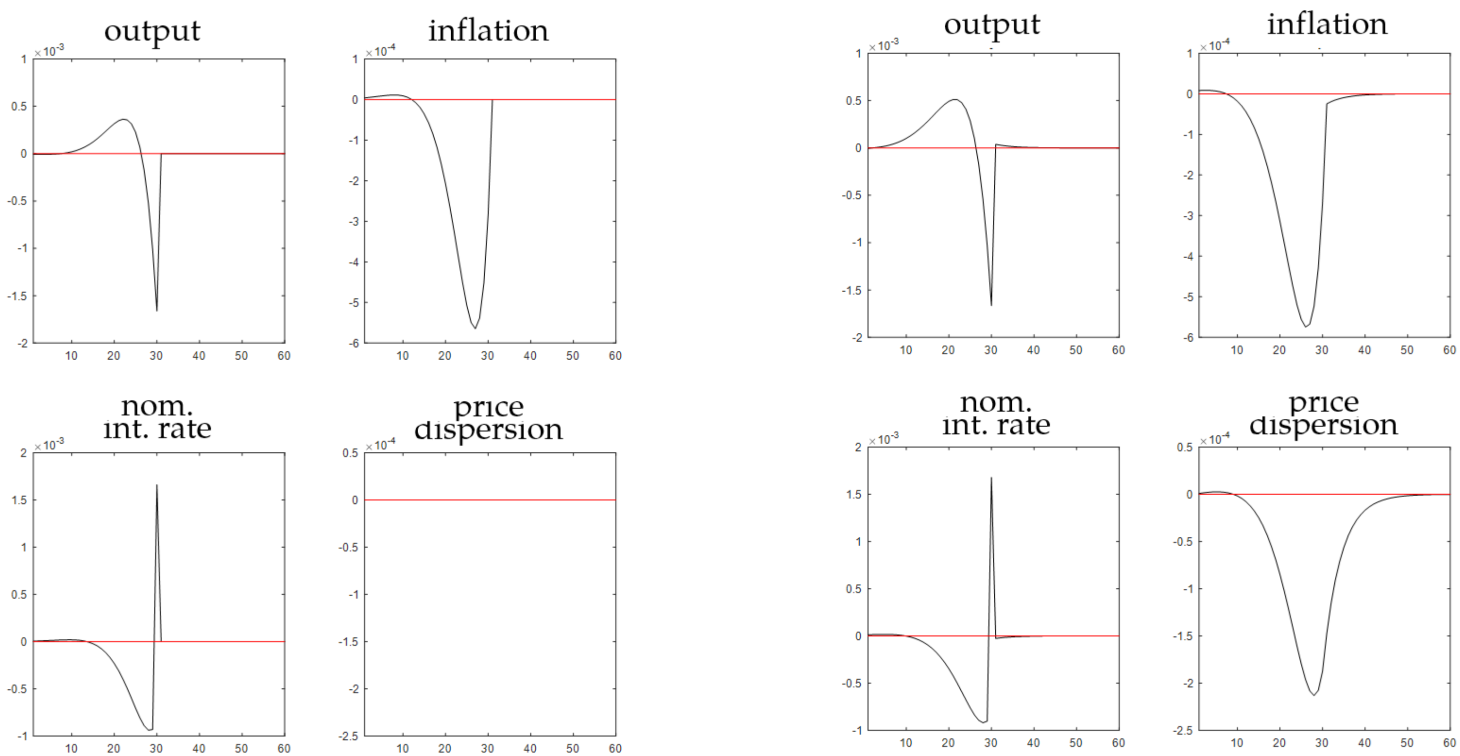


Figure: Impulse responses to a shock announced in period 1, but hitting in period 30, in basic New Keynesian models with (left) and without (right) indexation to steady-state inflation.
(Model of Fernández-Villaverde et al. 2015, modified to include indexation on left).
All variables are in logarithms.

Intuition for multiplicity: The negative response need not be on impact (2/2)

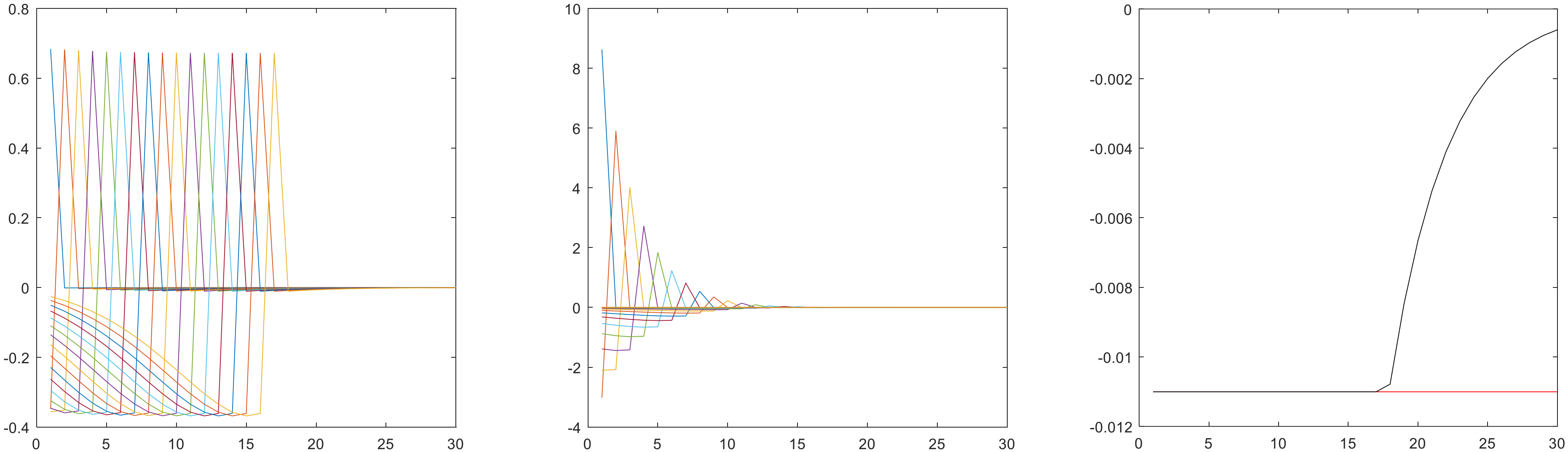


Figure: Construction of multiple equilibria in a basic NK model with price dispersion (model of Fernández-Villaverde et al. 2015).

The left-hand plot shows the impulse responses to news shocks arriving zero to sixteen quarters after becoming known. The middle plot shows the same impulse responses scaled appropriately. The right-hand plot shows the sum of the scaled impulse responses shown in the central figure, where the red line gives the ZLB's location, relative to steady-state.

Theoretical results

Is our M matrix special?

- The properties of solutions to LCPs (existence, uniqueness, computational difficulty) are determined by the properties of the M matrix.
  - One might think that ours would have “nice” properties because of where it came from.

- Unfortunately:

- **Proposition:** For any matrix $\mathcal{M} \in \mathbb{R}^{T\times T}$, there exists a model in the form of Problem 2 with a number of state variables given by a quadratic in T, such that $M = \mathcal{M}$ for that model.

Overview of the theoretical results

- Properties of solutions of LCPs have been extensively studied in the linear algebra and optimization literatures.
- As previously mentioned, all existence and uniqueness results are given in terms of the properties of the matrix M.
- Unfortunately, the required properties are rather harder to state (and check) than just looking at a few eigenvalues.
- In this presentation I will skip the definitions unless there is a desperate demand to see them.
    - All conditions can be checked numerically though.

Uniqueness results (1/3)

- We would ideally like a unique solution to exist for all possible q, since there are predicted shocks which can bring about any such q.

- **Proposition:** The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix.
- If M is not a P-matrix, then the LCP (q, M) has multiple solutions for some q.
- (Samelson, Thrall, and Wesler 1958; Cottle, Pang, and Stone 2009)

- This is the analogue of the Blanchard-Kahn conditions for models with occasionally binding constraints.
  - It tends to be satisfied in efficient models, but is rarely satisfied in New Keynesian models with zero lower bounds.

Uniqueness results (2/3)

- While for many models, the previous condition does not hold, we would hope that at least for $q \geq 0$ there ought to still be a unique solution.

- **Proposition:** The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \gg 0$ if and only if M is semi-monotone. (Cottle, Pang, and Stone 2009)

- **Proposition:** The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \geq 0$ if and only if M is strictly semi-monotone. (Cottle, Pang, and Stone 2009)

Uniqueness results (3/3)

- Thus, if M is not semi-monotone, there are some $q \gg 0$ such that the LCP (q, M) has multiple solutions.

- I.e., in certain states, if agents today got appropriate signals about future shocks, then the economy could jump to the bound, even though the bound would not have been violated had it not been there at all.

- This remains true even if shocks are arbitrarily small, and even if the steady-state is arbitrarily far away from the bound.

Finite T existence results (1/4)

- Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. The LCP corresponding to M and q is called **feasible** if there exists $y \in \mathbb{R}^T$ such that $y \geq 0$ and $q + My \geq 0$.

- **Proposition:** The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. (Cottle, Pang, and Stone 2009)

- If M is not an S-matrix, there are positive measure of q for which no solution exists.

Finite T existence results (2/4)

- **Proposition:** The LCP (q, M) is solvable if it is feasible and, either:
  - M is row-sufficient, or,
  - M is copositive and all the non-singular principal submatrices of M satisfy a further technical condition.
- (Cottle, Pang, and Stone 2009; Väliaho 1986)

- This gives sufficient conditions for existence for feasible q.
  - Checking feasibility just requires solving a linear programming problem, which is possible in time polynomial in T.

Finite T existence results (3/4)

- **Proposition**: The LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if at least one of the following conditions holds:
  - M is an S-matrix, and either of the conditions of the previous proposition are satisfied.
  - M is copositive with no zero principal minors.
  - M is a P-matrix, a strictly copositive matrix or a strictly semi-monotone matrix.
- (Cottle, Pang, and Stone 2009)

- This gives sufficient conditions for existence for all q.

Finite T existence results (4/4)

- In the special case in which M has nonnegative entries, we have both necessary and sufficient conditions:

- **Proposition:** If M is a matrix with nonnegative entries, then the LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if and only if M has a strictly positive diagonal. (Cottle, Pang, and Stone 2009)

Large T existence results

- Define:

$$\varsigma \coloneqq \sup_{\substack{y \in [0,1]^{\mathbb{N}^+} \\ \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y \,,$$

- Then M is an S-matrix for sufficiently large T if and only if $\varsigma > 0$.

- We show that there exists $\underline{\varsigma}_T, \overline{\varsigma}_T \geq 0$, both computable in time polynomial in T, such that $\underline{\varsigma}_T \leq \varsigma \leq \overline{\varsigma}_T$ and $\left|\underline{\varsigma}_T - \overline{\varsigma}_T\right| \to 0$ as $T \to \infty$.

- The proof relies on deriving constructive bounds on M.
  - Of independent interest to the news shock literature.

Applications to NK models

Examples from New Keynesian models

- First look at a three equation NK model with a response to output growth, following Brendon, Paustian, and Yates (BPY) (2012) .

- Then the Fernandez-Villaverde et al. (2012) model with price dispersion.

- Then Smets Wouters (2003) and (2007).

Simple Brendon, Paustian, and Yates (BPY) (2012) model (1/3)

- Equations:

$$x_{i,t} = \max\{0, 1 - \beta + \alpha_{\Delta y}(x_{y,t} - x_{y,t-1}) + \alpha_\pi x_{\pi,t}\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1}),$$
$$x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t},$$

- $\beta \in (0,1)$, $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$, $\alpha_\pi \in (1, \infty)$.
- Unique stationary solution in the absence of bounds.

- If $T = 1$, then:

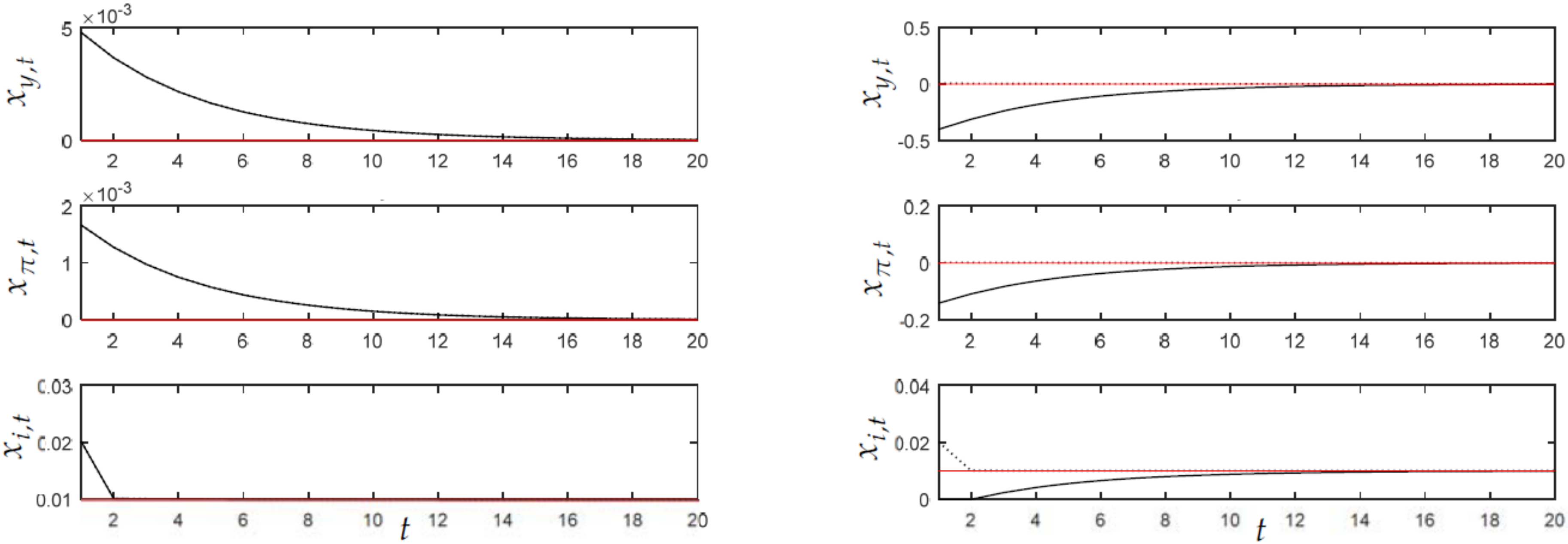
$$M = \frac{\beta\sigma f^2 - ((1+\beta)\sigma + \gamma) f + \sigma}{\beta\sigma f^2 - \left((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\right) f + \sigma + \alpha_{\Delta y} + \gamma\alpha_\pi},$$

- M is negative if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$. M is zero if and only if $\alpha_{\Delta y} = \sigma\alpha_\pi$.

Simple BPY (2012) model (2/3)

- When $T = 1$:
  - If $\alpha_{\Delta y} < \sigma\alpha_{\pi}$ then the model has a unique solution for all q.
  - When $\alpha_{\Delta y} > \sigma\alpha_{\pi}$, for any positive q, there exists $y > 0$ such that $q + My = 0$, so the model has multiple solutions.
  - When $\alpha_{\Delta y} > \sigma\alpha_{\pi}$, for any negative q, there is no $y \geq 0$ such that $q + My \geq 0$, so the model has no solutions.
- When $T > 1$:
  - If $\alpha_{\Delta y} > \sigma\alpha_{\pi}$ then at least for some $q \gg 0$, the model has multiple solutions.
- Adding shadow interest rate persistence doesn't change things much as long as T is large.

Simple BPY (2012) model (3/3)



Minimum $\|y\|_\infty$ solution[12] **Minimum $\|q + My\|_\infty$ solution[13]**

Figure 1: Alternative solutions following a magnitude 1 impulse to ε_t

Linearized Fernandez-Villaverde et al. (2012) model

- A basic NK model without investment, but with positive steady-state inflation, and hence price dispersion.

- With $T \leq 14$, M is a P-matrix, but with $T \geq 15$, M is not a P-matrix.
  - Thus this model always has multiple solutions (as we saw earlier).

- With $T = 1000$, from our upper bound on ς, we have that: $\varsigma \leq 0 +$ numerical error.
  - Provides strong evidence that M is not an S-matrix for large T either.

Linearized Fernandez-Villaverde et al. (2012) model with price level targetting

- With nominal GDP targeting (unit coefficients), with $T = 200$, our lower bound gives $\varsigma > 0.0048175$.

- Hence, for all sufficiently large T, M is an S-matrix, so there is always a feasible solution.

- For all T tested, M is a P-matrix.

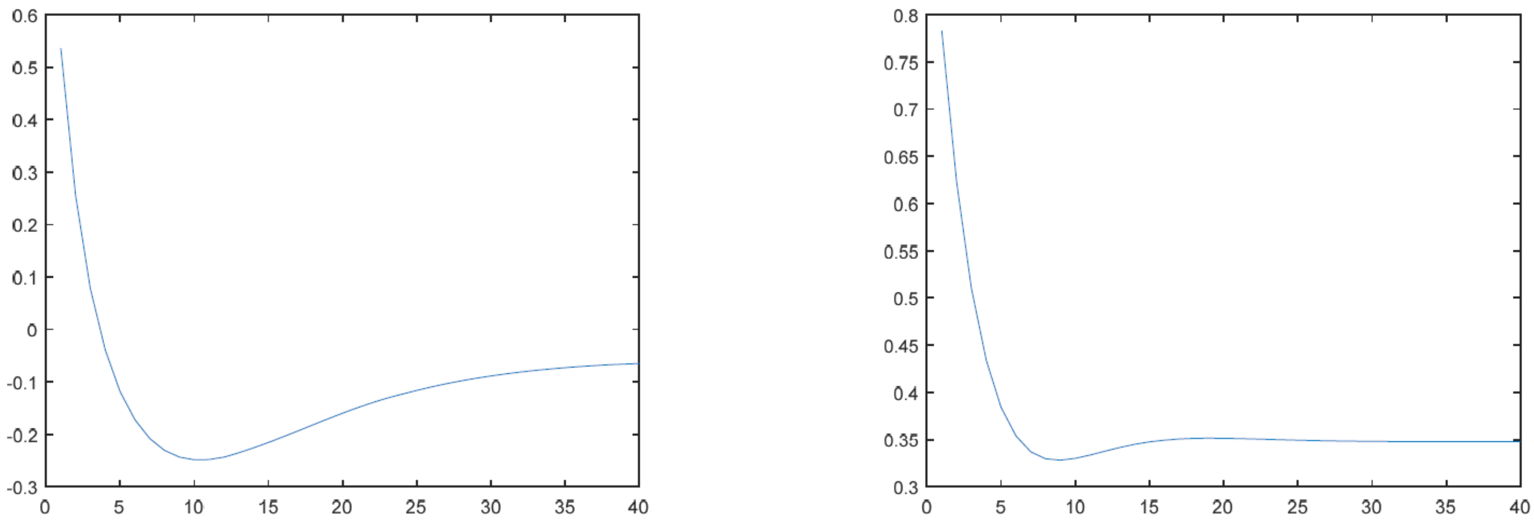
Smets & Wouters (2003; 2007) (1/3)

- Both models have:
  - assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation, general monetary policy reaction functions
- We augment both models with nominal interest rate rules of the form:

$$r_t = \max\{0, (1-\rho)(\cdots) + \rho r_{t-1} + \cdots\}$$

- Recall that the 2003 model is estimated on Euro area data, and the 2007 one is estimated on US data.
  - We use the posterior modes.
- Fairly similar models, except that the 2007 one:
  - Contains trend growth (permitting its estimation on non-detrended data),
  - Has a slightly more general aggregator across industries.

Smets & Wouters (2003; 2007) (2/3)



The Smets and Wouters (2003) model

The Smets and Wouters (2007) model

Figure 2: The diagonals of the M matrices for the Smets and Wouters (2003) and Smets and Wouters (2007) models

Smets & Wouters (2003; 2007) (3/3)

- Perhaps surprising that these graphs are so different.
- Negative diagonal for the Euro area model implies that the model does not always have a unique solution, even when $q \gg 0$.
- In fact, providing $T \geq 9$, the US model also does not always have a unique solution, even when $q \gg 0$.
- Furthermore, for both countries, there are some q for which the model has no solution.
  - Suggests that only solutions converging to the "bad" steady-state exist in those cases.

- Increasing the coefficient on inflation results in a positive diagonal for M even for the Euro area model, but does not result in a P-matrix.
- However, replacing inflation by the price-level minus a linear trend in the Taylor rule produces a P-matrix.

An example of multiplicity in Smets-Wouters (2007)

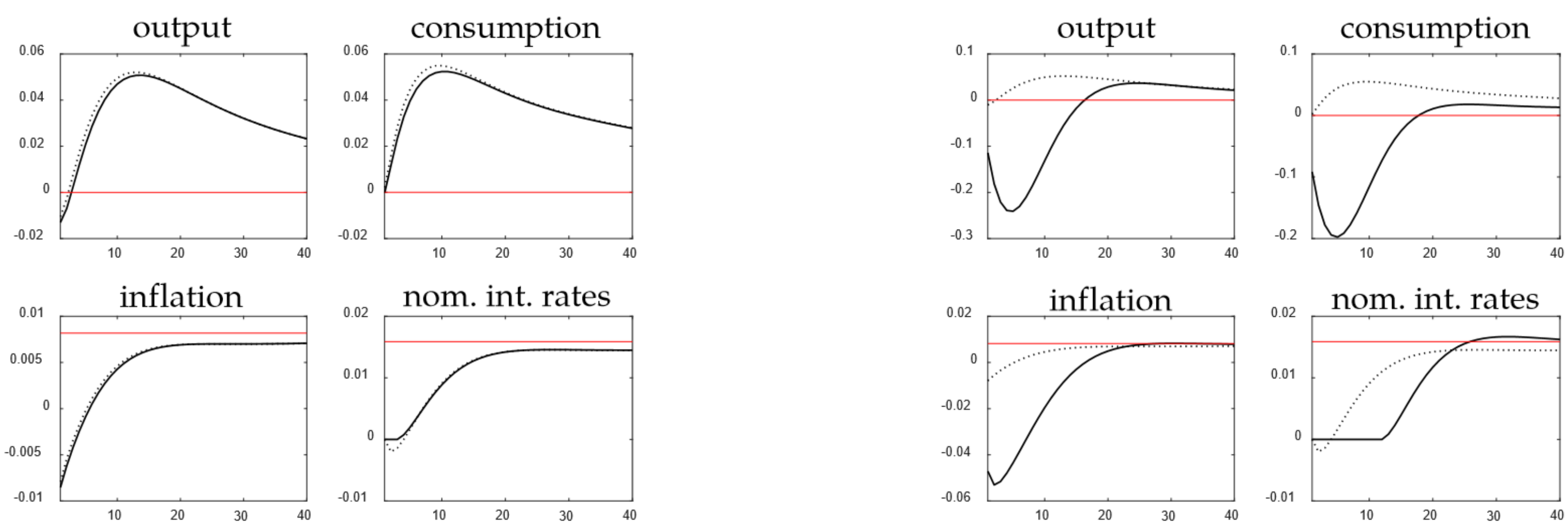


Figure 7: Two alternative solutions following a combination of shocks to the Smets and Wouters (2007) model
All variables are in logarithms. The precise combination of shocks is detailed in footnote 23.

Empirically plausible multiplicity in Smets-Wouters (2003)

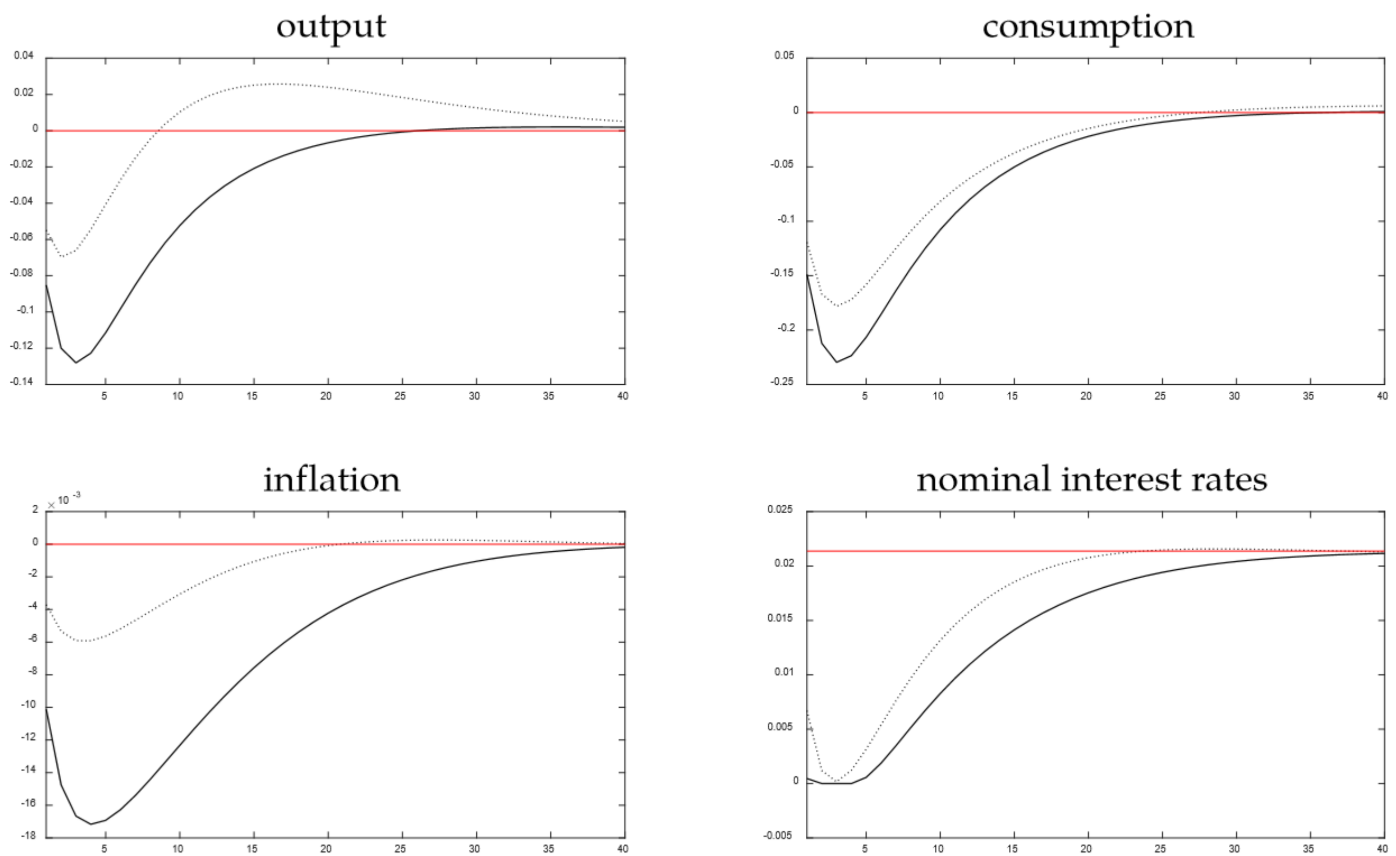


Figure 8: Two solutions following a preference shock in the Smets and Wouters (2003) model.
All variables are in logarithms.

Computational algorithm

Efficient computation of solutions (1/2)

- If M is unrestricted, or M is a P_0-matrix, then finding a single solution to the LCP (q, M) is "strongly NP complete".

- If we could do this efficiently (i.e. in polynomial time), we could also solve in polynomial time any problem whose solution could be efficiently verified.
  - This includes, for example, breaking all standard forms of cryptography.

- Since there is a model corresponding to any M matrix, with quadratic in T states, if there were a solution algorithm for DSGE models with OBCs that worked in time polynomial in the number of states, then it could also be used to defeat all known forms of cryptography.
  - So there almost certainly can't be such an algorithm!

Efficient computation of solutions (2/2)

- Polynomial time algorithms exist for special cases, but checking whether the relevant ones apply is not possible in polynomial time.

- This means that there cannot be an algorithm for checking if a model e.g. has a unique solution, that runs in time polynomial in the number of states.

Our computational approach to the perfect foresight problem (1/2)

- There is no way of escaping solving an NP-complete problem if we wish to simulate DSGE models with OBCs.
- Any algorithm we invented for the problem is likely to be inefficient, and possibly even non-finite.
- A better approach is to map our problem into another to which smart computer scientists have devoted a lot of time.

- It turns out that the solution to an LCP can be represented as a mixed integer linear programming problem.
  - One of the best studied problems in computer science.
  - Extremely well optimised, fully global, solvers exist.

Our computational approach to the perfect foresight problem (2/2)

- **Problem 7**
- Suppose $\widetilde{\omega} > 0$, $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T\times T}$ are given.
- Find $\alpha \in \mathbb{R}$, $\hat{y} \in \mathbb{R}^T$, $z \in \{0,1\}^T$ to maximise α subject to the following constraints: $\alpha \geq 0$, $0 \leq \hat{y} \leq z$, $0 \leq \alpha q + M\hat{y} \leq \widetilde{\omega}(1_{T\times 1} - z)$.

- **Proposition**: If α, $\hat{y}$, z solve Problem 7, then if $\alpha = 0$, the LCP (q, M) has no solution, and if $\alpha > 0$, then $y \coloneqq \frac{\hat{y}}{\alpha}$ solves it. (Partial converse in paper.)

- As $\widetilde{\omega} \to 0$, the solution to Problem 7 is the solution to the LCP which minimises $\|q + My\|_\infty$.
- As $\widetilde{\omega} \to \infty$, the solution to Problem 7 is the solution to the LCP which minimises $\|y\|_\infty$.

Application to models with uncertainty

- To convert the perfect foresight solver into a solver for stochastic models, we start by using a variant of the extended path algorithm of Fair and Taylor (1983).
  - Each period we draw a shock, and then solve for the expected future path of the model, ignoring the impact of the OBC on expectations (for now).
  - From this expected path, we can solve for the news shocks necessary to impose the bound.
  - We then add those news shocks to today's variables, and step the model forward using the model's transition matrix.
  - Not consistent with rational expectations but this will be (partially) rectified.

The non-linear problem (1/2)

- **Problem 6**
- Suppose that $x_0 \in \mathbb{R}^n$ is given and that $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^n$, $g, h: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^c$ are given continuously $d \in \mathbb{N}^+$ times differentiable functions.

- Find $x_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^c$ for $t \in \mathbb{N}^+$ such that for all $t \in \mathbb{N}^+$:

$$\begin{aligned}0 &= \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t),\\ r_t &= \mathbb{E}_t \max\{h(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t), g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\}\end{aligned}$$

- where $\varepsilon_t \sim \text{NIID}(0, \Sigma)$, where the $\max$ operator acts elementwise on vectors, and where the information set is such that for all $t \in \mathbb{N}^+$, $\mathbb{E}_{t-1}\varepsilon_t = 0$ and $\mathbb{E}_t \varepsilon_t = \varepsilon_t$.

The non-linear problem (2/2)

- **Assumption:** There exists $\mu_x \in \mathbb{R}^n$ and $\mu_r \in \mathbb{R}^c$ such that:

$$0 = f(\mu_x, \mu_x, \mu_x, \mu_r, 0),$$

$$\mu_r = \max\{h(\mu_x, \mu_x, \mu_x, \mu_r, 0), g(\mu_x, \mu_x, \mu_x, \mu_r, 0)\},$$

- and such that for all $a \in \{1, \dots, c\}$:

$$\big(h(\mu_x, \mu_x, \mu_x, \mu_r, 0)\big)_a \neq \big(g(\mu_x, \mu_x, \mu_x, \mu_r, 0)\big)_a.$$

Application via linearization (1/2)

- Without loss of generality, suppose our model is:

$$0 = \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t),$$
$$r_t = \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\},$$

  - where $g(\mu_x, \mu_x, \mu_x, \mu_r, 0) \gg 0$.

- Linearizing around the steady-state gives:

$$r_t = \mu_r + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(r_t - \mu_r) + g_5 \varepsilon_t.$$

- We replace this with the more accurate:

$$r_t = \max\{0, \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(r_t - \mu_v) + g_5 \varepsilon_t\}.$$

Application via linearization (2/2)

- For our algorithm, we replace this in turn with:

$$r_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)},$$

- for all $a \in \{1, \dots, c\}$, where, for all $a \in \{1, \dots, c\}$:

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \eta_{i,t}^{(a)}$$
$$y_{T,t}^{(a)} = \eta_{T,t}^{(a)}.$$

Application via higher order pruned perturbation

- We first take a pruned perturbation approximation to the source non-linear model.

- A convenient property of pruned perturbation solutions of order d is that they are linear in additive shocks of the form η_t^d.
  - So using shocks of this form preserves the tractable linearity.
  - In fact the M matrix we get at second or higher order is equal to the M matrix at first order, (at least in the limit as the variance of the news shocks goes to zero).

- While this is still treating the bound in a perfect-foresight manner (for now), by taking a higher order approximation we at least capture other risk channels.

Accounting for the risk of hitting the bound

- In period t, our approach approximates the value of x_t in the model of Problem 6 by the solution to the system:

$$
\begin{aligned}
0 &= \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t), \\
r_t &= \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, r_t, \varepsilon_t)\}, \\
\forall s \in \mathbb{N}^+, \qquad 0 &= f(x_{t+s-1}, x_{t+s}, x_{t+s+1}, r_{t+s}, \kappa_s \varepsilon_{t+s}), \\
\forall s \in \mathbb{N}^+, \qquad r_t &= \max\{0, g(x_{t+s-1}, x_{t+s}, x_{t+s+1}, r_{t+s}, \kappa_s \varepsilon_{t+s})\},
\end{aligned}
$$

  - where $\kappa_1, \kappa_2, \ldots \in [0,1]$ control the degree of future uncertainty considered.

- Equivalent to supposing that in period t agents believe that in period $t+1$ they will be told the value of all future shocks (i.e. $\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots$).
  - From the perspective of period t, all future shocks are uncertain, meaning that this should capture well the effect of risk.
  - If the model is linear then by the law of iterated expectations, there is no approximation at all.

Integrating over future uncertainty (1/3)

- Following Adjemian and Juillard (2013) the procedure is as follows:
  - Draw shocks for a certain number of future periods, $t+1, \dots, t+S$.
  - Solve for the perfect foresight path assuming they were known at t.
  - Repeat many times to get expectations.

- In their very general non-linear set-up, doing this integration requires p^{mS} solutions of the perfect foresight problem,
  - for some $p > 1$, m is the number of shocks, S is the integration horizon.

- Solving their general perfect foresight problem is also orders of magnitude slower than solving our LCP.

Integrating over future uncertainty (2/3)

- Let $w_{t,s}$ be the value the bounded variables would take at s if the constraints did not apply from period t onwards.

- By the properties of pruned perturbation solutions, we can evaluate $\operatorname{cov}_t\left(w_{t,t+i}, w_{t,t+j}\right)$, for $t, i, j \in \mathbb{N}$ in closed form.
  - So we can take a Gaussian approximation to the joint distribution of $w_{t,t}, w_{t,t+1}, \dots$, and efficiently integrate over these variables via Gaussian cubature techniques.
  - Rather than exponential in both m and S evaluations, we just need polynomial in S evaluations.

- For each draw of $w_{t,t}, w_{t,t+1}, \dots$, we solve the bounds problem to get the cumulated news shocks (i.e. y).

Integrating over future uncertainty (3/3)

- Unlike Adjemian and Juillard (2013) we do not just consider full variance shocks up to some horizon, and then nothing beyond.

- Instead, we apply a windowing function to the shock variances, to ensure that the covariance is a smooth function of time.
  - This reduces artefacts caused by the sudden change at horizon S.

- In particular, we scale the shock variance at horizon k by:

$$\kappa_k^2 = \frac{1}{2}\left(1 + \cos\left(\pi \frac{k-1}{S}\right)\right).$$

- The cosine form has some desirable frequency domain properties.

Three alternative Gaussian cubature methods

- With $\hat{S} \leq S$ the integration dimension, these are:
  - A degree 3 monomial rule with $2\hat{S}+1$ nodes and positive weights.
    - Positive weights give robustness. Evaluates far from steady-state though.
  - The Genz and Keister (1996) Gaussian cubature rules with $\mathrm{O}\left(\hat{S}^K\right)$ nodes.
    - $2K+1$ is the degree of monomial integrated exactly.
    - Since the rules are nested, adaptive degree is possible.
  - Quasi-Monte Carlo.
    - Much less efficient than the others on well behaved functions, but is much better behaved on non-differentiable ones.

The DynareOBC toolkit (1/2)

- Complete code to test your own models is available under an open source license from:

  https://github.com/tholden/dynareOBC

- To use DynareOBC, just include a max, min or abs in your mod file, then type "dynareOBC modfilename.mod".

- Assorted command line options are documented on the home page and in the ReadMe.pdf.

- Provides accurate simulation under rational expectations even for large models, as documented in the computational companion paper.

- Early support for estimation.

The DynareOBC toolkit (2/2)

- Even if you do not have OBCs in your model, DynareOBC may be useful since it can:
  - Simulate MLVs, including integrating over ones with +1 terms, which makes checking accuracy very easy.
  - Perform exact, faster, average IRF calculation without Monte Carlo.
  - Estimate non-linear models at 3^{rd} order using the cubature Kalman filter.

- DynareOBC is covered in the advanced DSGE modelling summer course run at the University of Surrey.
  - Please tell your PhD students to sign up!

Accuracy

Accuracy results

- By way of conclusion, we give some accuracy results.

- We examine three simple models, to ensure accuracy tests are reliable.

- These are:
  - A very simple model with an analytic solution.
  - A model for which log-linearization gives the exact answer in the absence of bounds.
  - An otherwise linear open-economy model.

A simple model with an analytic solution

- Closed economy, no capital, inelastic unit labour supply.
- Households maximise:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

- Subject to the budget constraint:

$$A_t + R_{t-1} B_{t-1} = C_t + B_t$$

  - A_t is productivity. R_t is the real interest rate.
  - B_t is the household's holdings of zero net supply bonds.

- Define $g_t \coloneqq \log A_t - \log A_{t-1}$. Evolves according to:

$$g_t = \max\{0, (1-\rho)\bar{g} + \rho g_{t-1} + \sigma \varepsilon_t\},$$

- $\varepsilon_t \sim \mathrm{NIID}(0,1)$, $\beta \coloneqq 0.99$, $\gamma \coloneqq 5$, $\bar{g} \coloneqq 0.05$, $\rho \coloneqq 0.95$, $\sigma \coloneqq 0.07$.

Accuracy in the simple model, along simulated paths of length 1000 (after 100 periods dropped)

Bound in Model	*Order*	*Cubature*	*Seconds*	*Log_{10} Mean Abs Error*	*Log_{10} Root M.S.E.*	*Log_{10} Max Abs Error*	*Log_{10} Mean Abs Error at Bound*
No	*1*	*N/A*	66	-3.213	-3.213	-3.213	
No	*2*	*N/A*	62	-16.82	-16.63	-15.78	
No	*3*	*N/A*	53	-16.70	-16.57	-15.95	
Yes	1	No	141	-2.435	-2.218	-1.882	-1.882
Yes	2	No	139	-2.425	-2.194	-1.862	-1.862
Yes	3	No	140	-2.425	-2.194	-1.862	-1.862
Yes	1	Monomial, Degree 3	274	-3.136	-3.073	-2.725	-3.131
Yes	2	Monomial, Degree 3	1537	-3.378	-3.172	-2.706	-3.893
Yes	3	Monomial, Degree 3	1397	-3.378	-3.172	-2.706	-3.893
Yes	2	Sparse, Degree 3	1794	-3.016	-2.777	-2.415	-2.415
Yes	2	Sparse, Degree 5	1840	-3.016	-2.777	-2.415	-2.415
Yes	2	Sparse, Degree 7	2009	-3.280	-3.032	-2.663	-2.663
Yes	2	QMC, 15 Points	1965	-3.040	-2.895	-2.664	-2.664
Yes	2	QMC, 63 Points	3184	-3.394	-3.260	-3.020	-3.020
Yes	2	QMC, 1023 Points	5197	-3.804	-3.638	-3.351	-3.351

[1] All timings are "wall" time, and include time spent starting the parallel pool, time spent compiling code (although written in MATLAB, DynareOBC generates and compiles C code for key routines), and time spent calculating accuracy. Code was run on one of the following (very similar) twenty core machines: 2x E5-2670 v2 2.5GHz, 64GB RAM; 2x E5-2660 v3 2.6GHz, 128GB RAM. Use of machines with network attached storage means that there may be some additional variance in these timings.

[2] Errors conditional on the bounded variable being less than *0.0001.* The numbers for this column would be identical had we used root mean squared errors or maximum absolute errors, conditional on being at the bound.

A model for which log-linearization is exact without bounds

- The social planner chooses consumption, C_t, L_t, and K_t, to maximise:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left[\log C_{t+k} - \frac{L_{t+k}^{1+\nu}}{1+\nu}\right],$$

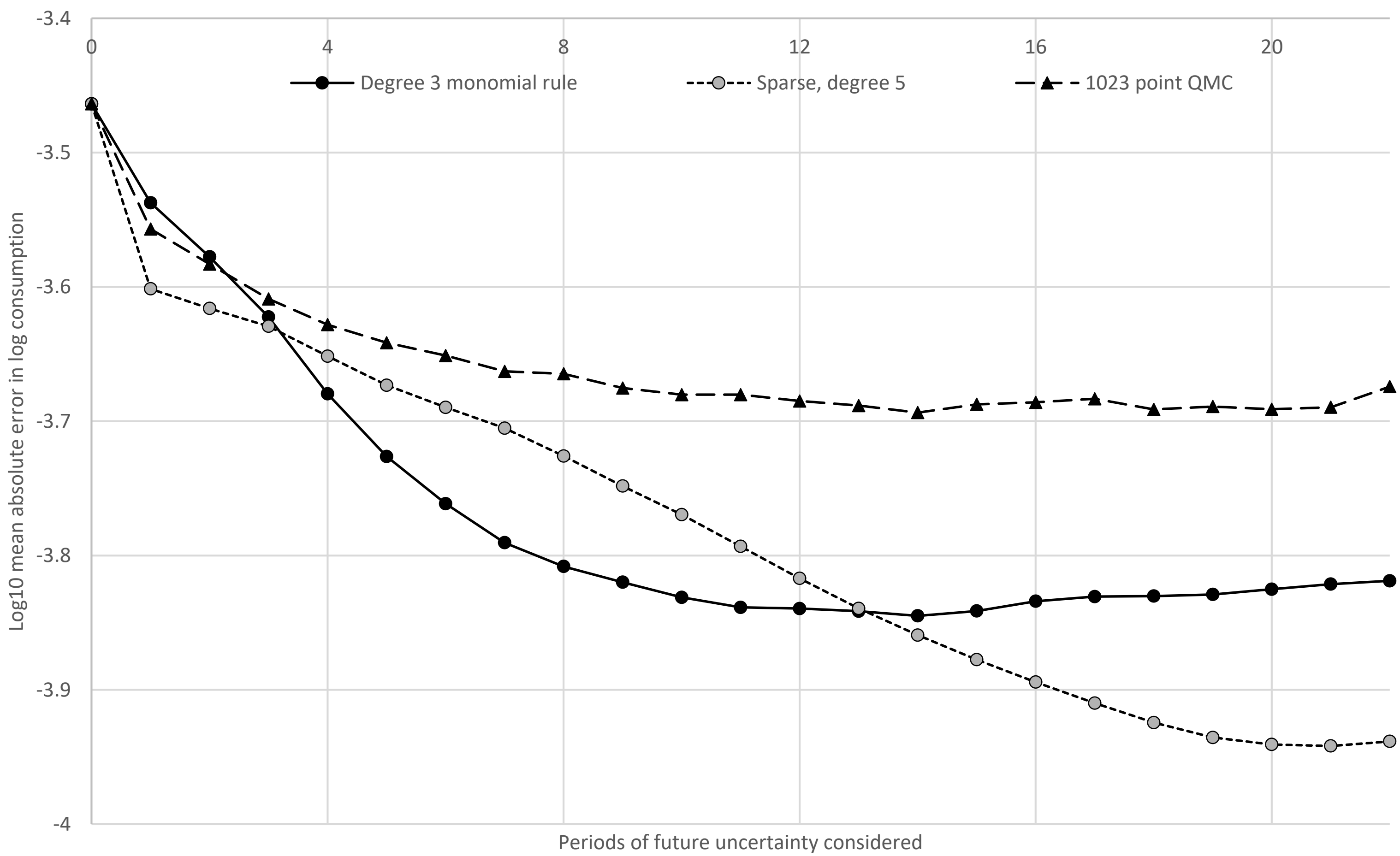
- subject to the capital constraint: $K_t \geq \theta K_{t-1}$,
- and the budget constraint $C_t + K_t = Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$.

- Productivity, A_t, evolves according to $A_t = A_{t-1}^{\rho} \exp \varepsilon_t$, where $\varepsilon_t \sim \mathrm{N}(0, \sigma^2)$.

- Set $\alpha = 0.3$, $\beta = 0.99$, $\nu = 2$, $\theta = 0.99$, $\rho = 0.95$ and $\sigma = 0.01$.

- Compare to a full global solution.

Effect of increasing periods of uncertainty on accuracy, along simulated paths



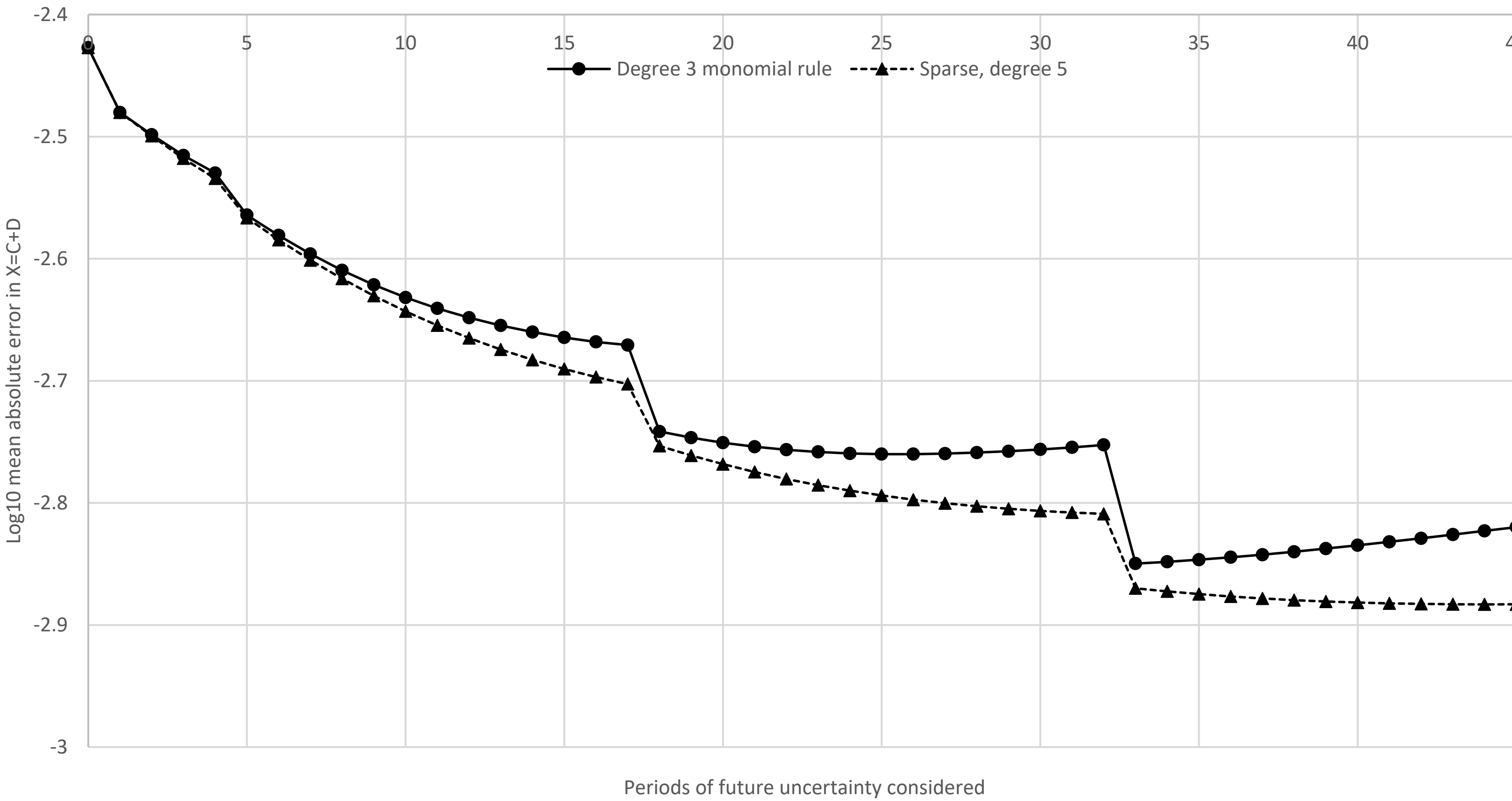
An otherwise linear open-economy model

- The social planner chooses C_t, D_t and B_t to maximise:
$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left[-\frac{1}{2}(1 - C_t)^2 - \frac{\phi}{2} B_t^2 \right],$$
- subject to the budget constraint: $C_t + D_t + B_t - RB_{t-1} = Y_t = \max\{\underline{Y}, A_t\}$,
- the positivity constraints: $0 \le C_t,\ 0 \le D_t$,
- and the certain repayment of interest constraint:
$$\forall k \in \mathbb{N}^+, \qquad \Pr_t\big((R-1)B_t \le Y_{t+k}\big) = 1.$$

- Productivity evolves according to $A_t = (1-\rho)\mu + \rho A_{t-1} + \sigma\varepsilon_t$, where $\varepsilon_t \sim \mathrm{NIID}(0,1)$.

- Set $\beta = 0.99$, $\mu = 0.5$, $\rho = 0.95$, $\sigma = 0.05$, $\underline{Y} = 0.25$, $R = \beta^{-1}$ and $\phi = R - 1$.

Effect of increasing periods of uncertainty on accuracy, along simulated paths (as before)



Conclusion

- Theory paper (Holden 2016a) proves completely general "Blanchard Kahn conditions" for models with occasionally binding constraints, and shows that multiplicity is to be expected in models with a ZLB.
- The theory paper presents a powerful argument for level targeting rules. If you think the Taylor principle should be followed to rule out indeterminacy, then you should support a level target in the presence of the ZLB.
- Computational paper (Holden 2016b) provides an efficient perfect foresight solver that is guaranteed to finish in finite time, and which can detect solution non-existence.
- Computational paper also gives an accurate solver for even large models with occasionally binding constraints.

Appendices

Principal sub matrices and principal minors

- For a matrix $M \in \mathbb{R}^{T \times T}$, the principal sub-matrices of M are the matrices:
$$\left\{ \left[M_{i,j}\right]_{i,j=k_1,\ldots,k_S} \middle| S, k_1, \ldots, k_S \in \{1, \ldots, T\}, k_1 < k_2 < \cdots < k_S \right\},$$
  - i.e. the principal sub-matrices of M are formed by deleting the same rows and columns.

- The principal minors of M are the collection of values:
$$\left\{ \det\left(\left[M_{i,j}\right]_{i,j=k_1,\ldots,k_S} \right) \middle| S, k_1, \ldots, k_S \in \{1, \ldots, T\}, k_1 < k_2 < \cdots < k_S \right\},$$
  - i.e. the principal minors of M are the determinants of the principal sub-matrices of M.

P($_0$)-matrices and General positive (semi-)definite matrices

- A matrix $M \in \mathbb{R}^{T\times T}$ is called a **P-matrix** if the principal minors of M are all strictly positive. M is called a **P$_0$-matrix** if the principal minors of M are all non-negative.
  - *Note: for symmetric M, M is a P($_0$)-matrix if and only if all of its eigenvalues are strictly (weakly) positive.*

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **general positive definite** if $M + M'$ is a P-matrix. If $M + M'$is a P$_0$-matrix, then M is called **general positive semi-definite**.
  - *Note: that we do not require that M is symmetric in either case, but the definitions coincide with the standard ones for symmetric M.*

Intuition for the relevance of principal sub-matrices

- Recall that the LCP complementary-slackness-type condition states that $y \circ (q + My) = 0$.

- Equivalently:

$$0 = y'(q + My) = \frac{1}{2}[y'(q + My) + (q' + y'M')y] = y'q + \frac{1}{2}y'(M + M')y$$

- The non-zero elements of y thus select a principal sub-matrix of $M + M'$.

- If this sub-matrix is positive-definite, then $y'(M + M')y$ will be positive, which is desirable as $y'q$ is likely to be negative (as the bound usually binds when q is negative).

S($_0$)-matrices, (Strictly) Semi-monotone matrices and (Strictly) Copositive matrices

- A matrix $M \in \mathbb{R}^{T\times T}$ is called an **S-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$. M is called an **S$_0$-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \geq 0$.

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly semi-monotone** if each of its principal sub-matrices is an **S-matrix**. M is called **semi-monotone** if each of its principal sub-matrices is an **S$_0$-matrix**.

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly copositive** if $M + M'$ is strictly semi-monotone. If $M + M'$ is semi-monotone then M is called **copositive**.

Sufficient matrices

- Let $M \in \mathbb{R}^{T\times T}$. M is called **column sufficient** if M is a P_0-matrix, and the principal sub-matrices of M with zero determinant satisfy a further technical condition.

- M is called **row sufficient** if M' is column sufficient.

- M is called **sufficient** if it is column sufficient and row sufficient.

Relationships between the matrix classes (Cottle, Pang and Stone 2009)

- All general p.s.d. matrices are copositive and sufficient.
- P_0 includes skew-symmetric matrices, general p.s.d. matrices, sufficient matrices and P-matrices.
- All P_0-matrices, and all copositive matrices are semi-monotone.
- All P-matrices, and all strictly copositive matrices are strictly semi-monotone (and hence S-matrices).

- All general p.s.d., semi-monotone, sufficient, P_0 and copositive matrices have non-negative diagonals.
- All general p.d., strictly semi-monotone, P and strictly copositive matrices have strictly positive diagonals.

Bounds on M

- **Lemma**
- The difference equation $A\hat{d}_{k+1} + B\hat{d}_k + C\hat{d}_{k-1} = 0$ for all $k \in \mathbb{N}^+$ has a unique solution satisfying the terminal condition $\hat{d}_k \to 0$ as $k \to \infty$, given by $\hat{d}_k = H\hat{d}_{k-1}$, for all $k \in \mathbb{N}^+$, for some H with eigenvalues in the unit circle.

- Define $d_0 \coloneqq -(AH + B + CF)^{-1} I_{\cdot,1}$, $d_k = Hd_{k-1}$, for all $k \in \mathbb{N}^+$, and $d_{-t} = Fd_{-(t-1)}$, for all $t \in \mathbb{N}^+$.
- The rows and columns of M are converging to 0 (with constructive bounds).
- The k^{th} diagonal of the M matrix is converging to the value $d_{1,k}$.
  - Diagonals are indexed such that the principal diagonal is index 0, and indices increase as one moves up and to the right in the M matrix.

Results from dynamic programming (1/2)

- Here we introduce a class of problems that our algorithm will solve arbitrarily accurately, and give alternative uniqueness results.

- **Problem 5**
- Solve a concave quadratic dynamic programming problem subject to linear inequality constraints. (Full definition in the paper.)

- **Proposition:** If either:
  - $\tilde{\Gamma}(x)$ (the choice set) is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \tilde{X}$ (the state space),
  - or, $\tilde{X}$ (the state space) is compact,
- then for all $x_0 \in \tilde{X}$, there is a unique path $(x_t)_{t=0}^{\infty}$ which solves Problem 5.

Results from dynamic programming (2/2)

- **Proposition:**
- The KKT conditions of Problem 5 may be placed into the form of the multiple-bound generalisation of Problem 2.
- Let (q_{x_0}, M) be the infinite LCP corresponding to this representation, given initial state $x_0 \in \tilde{X}$.
- If y is a solution to the LCP, $q_{x_0} + My$ gives the stacked paths of the bounded variables in a solution to Problem 5.
- If, further, either condition of the previous proposition holds, then:
  - Both Problem 5 and this LCP have a unique solution for all $x_0 \in \tilde{X}$.
  - For sufficiently large T, the finite LCP also has a unique solution.