Problem Set 3
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Problem 1

1. The sequence problem is defined as,

$$\begin{aligned} V(k_0) &= \max_{\{C_t\in[0,F(k_t)]\}_{t=0}^{t=\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t. k_{t+1} - k_t &= F(k_t) - c_t, k_0 \text{ is given} \\ \text{where } F(k_t) &= ak_t - \frac{b}{2}k_t^2, u(c_t) = \ln(c_t), \beta = \frac{1}{1+\rho} \end{aligned}$$

2. The Bellman equation is

$$\begin{aligned} V(k_t) &= \max_{c_t\in[0,F(k_t)]} u(c_t) + \beta V(k_{t+1}) \\ s.t. k_{t+1} - k_t &= F(k_t) - c_t, k_0 \text{ is given} \end{aligned}$$

3. The interpretation of value function is a function of this period's state variable's maximum PDV that the agent can optimally obtained. It consists of two parts: (1) a current period reward function; (2) the discounted streams of future payoff (continuation value: the PDV of being in the next period's state).

4. Notice the relationship we have

$$\begin{aligned} c_t &= F(k_t) + k_t - k_{t+1} \\ 0 &\leq k_{t+1} = F(k_t) + k_t - c_t \leq F(k_t) + k_t \end{aligned}$$

   Hence the Bellman equation,

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}\in[0,F(k_t)+k_t]} u(F(k_t) + k_t - k_{t+1}) + \beta V(k_{t+1}) \\ &= \max_{k_{t+1}\in[0,F(k_t)+k_t]} \ln\left((a+1)k_t - \frac{b}{2}k_t^2 - k_{t+1}\right) + \beta V(k_{t+1}) \end{aligned}$$

5. Policy function:

6. Graph: See next page. Since we have the policy function, for each $k_t$ we can get $k_{t+1}$, then from budget constraint we have $c_{t+1}$. Repeat this process in a loop helps us to plot the trajectory.

7. Interpretation: following the policy function table, we can find the state state level of capital is $k^* = c^*$. Starting from $k_0 = 5$ it takes only 1 period to reach the next node $k_1 = k^*$. Also by definition we have the steady state level of consumption. We only discretize the space into 30 nodes, which make the numerical error quite large. If we furthere divide the nodes, it may take longer to get to steady state.

Table 1: Policy function for certainty cases: finite and infinite horizon

| | (1) k_t+1:infinite | (2) k_t+1: finite | | | | |
|---|---|---|---|---|---|---|
| k_t | | t=1 | t=2 | t=3 | t=4 | t=5 |
| 1 | 13 | 13 | 13 | 13 | 13 | 1 |
| 2 | 15 | 15 | 15 | 15 | 15 | 1 |
| 3 | 15 | 15 | 15 | 15 | 15 | 1 |
| 4 | 15 | 15 | 15 | 15 | 15 | 1 |
| 5 | 16 | 16 | 16 | 16 | 16 | 1 |
| 6 | 16 | 16 | 16 | 16 | 16 | 1 |
| 7 | 16 | 16 | 16 | 16 | 16 | 1 |
| 8 | 16 | 16 | 16 | 16 | 16 | 1 |
| 9 | 16 | 16 | 16 | 16 | 16 | 1 |
| 10 | 16 | 16 | 16 | 16 | 16 | 1 |
| 11 | 16 | 16 | 16 | 16 | 16 | 1 |
| 12 | 16 | 16 | 16 | 16 | 16 | 1 |
| 13 | 16 | 16 | 16 | 16 | 16 | 1 |
| 14 | 16 | 16 | 16 | 16 | 16 | 1 |
| 15 | 16 | 16 | 16 | 16 | 16 | 1 |
| 16 | 16 | 16 | 16 | 16 | 16 | 1 |
| 17 | 16 | 16 | 16 | 16 | 16 | 1 |
| 18 | 16 | 16 | 16 | 16 | 16 | 1 |
| 19 | 16 | 16 | 16 | 16 | 16 | 1 |
| 20 | 16 | 16 | 16 | 16 | 16 | 1 |
| 21 | 16 | 16 | 16 | 16 | 16 | 1 |
| 22 | 16 | 16 | 16 | 16 | 16 | 1 |
| 23 | 16 | 16 | 16 | 16 | 16 | 1 |
| 24 | 16 | 16 | 16 | 16 | 16 | 1 |
| 25 | 16 | 16 | 16 | 16 | 16 | 1 |
| 26 | 16 | 16 | 16 | 16 | 16 | 1 |
| 27 | 16 | 16 | 16 | 16 | 16 | 1 |
| 28 | 16 | 16 | 16 | 16 | 16 | 1 |
| 29 | 15 | 15 | 15 | 15 | 15 | 1 |
| 30 | 15 | 15 | 15 | 15 | 15 | 1 |

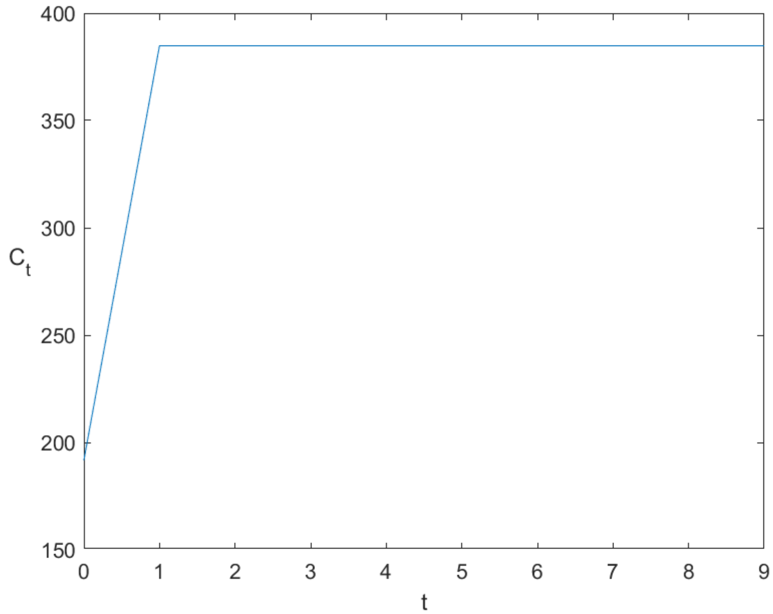


Figure 1: Consumption trajectory

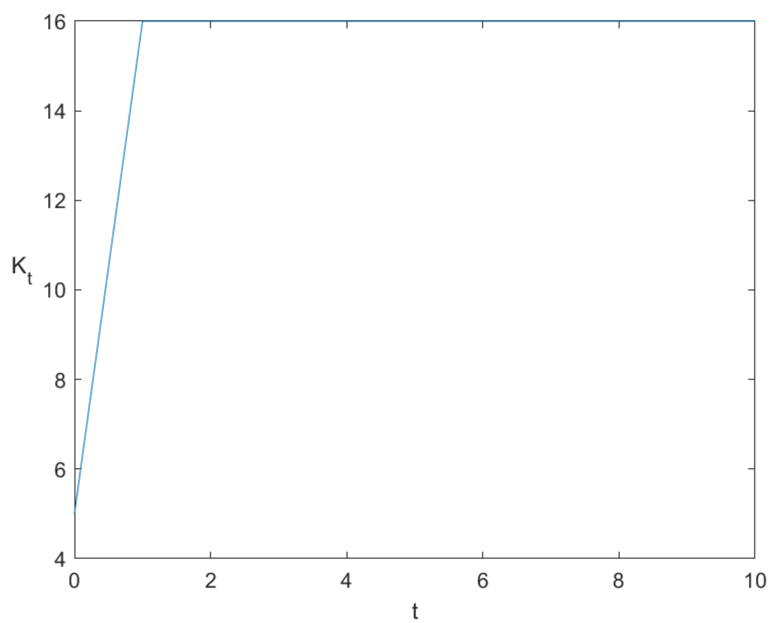


Figure 2: Capital trajectory

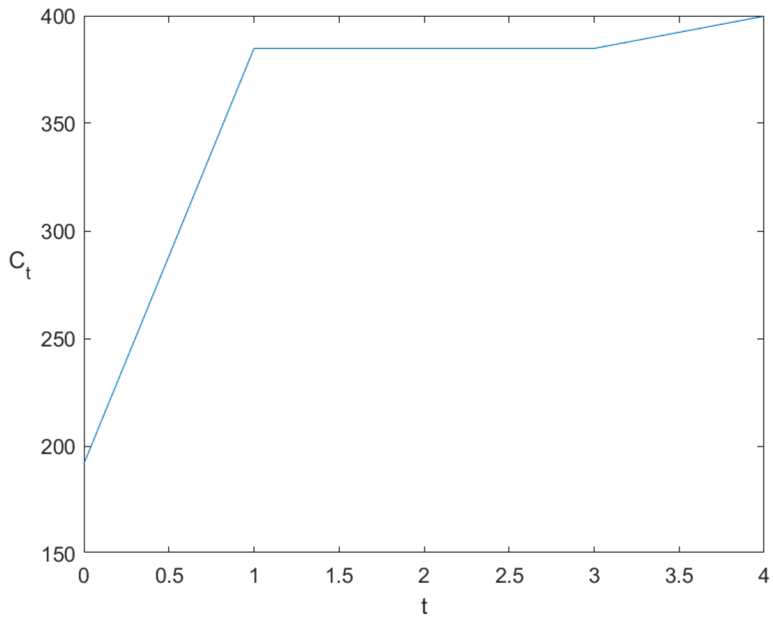


Figure 3: Consumption trajectory

Problem 2- Finite version with certainty

1. Bellman equation:

$$
\begin{aligned}
V_t\left(k_t\right) &= \begin{cases} \max_{c_t\in[0,F(k_t)+k_t]} u\left(F\left(k_t\right)+k_t-k_{t+1}\right)+\beta V_{t+1}\left(k_{t+1}\right), 0\leq t\leq T \\ \max_{c_t\in[0,F(k_t)+k_t]} u\left(c_t\right), t=T \end{cases} \\
&= \begin{cases} \max_{c_t\in[0,F(k_t)+k_t]} u\left(F\left(k_t\right)+k_t-k_{t+1}\right)+\beta V_{t+1}\left(k_{t+1}\right), 0\leq t\leq T \\ \max_{c_t\in[0,F(k_t)+k_t]} u\left(c_t\right), t=T \end{cases}
\end{aligned}
$$

2. The value function at period t is the maximum PDV of the entire stream of payoff from period t to T. The continuation value is the discounted expected value of the next period's value function. For the last period there is no continuation value.

3. Now we solve the problem using backward induction.

4. See Table 1.

5. Since we have the policy function $k_{t+1} = U_t(k_t)$, for each $k_t$ we can get $k_{t+1}$, then from budget constraint we have $c_{t+1}$. Repeat this process in a loop helps us to plot the trajectory.

6. The interpretation is similar to the finite case. First the capital reach the steady state, but at period t=4 we are close to the end of the game, hence we will eat all the capital and invest the minimal amount possible (k=1). The last period consumption is obtained accordingly.

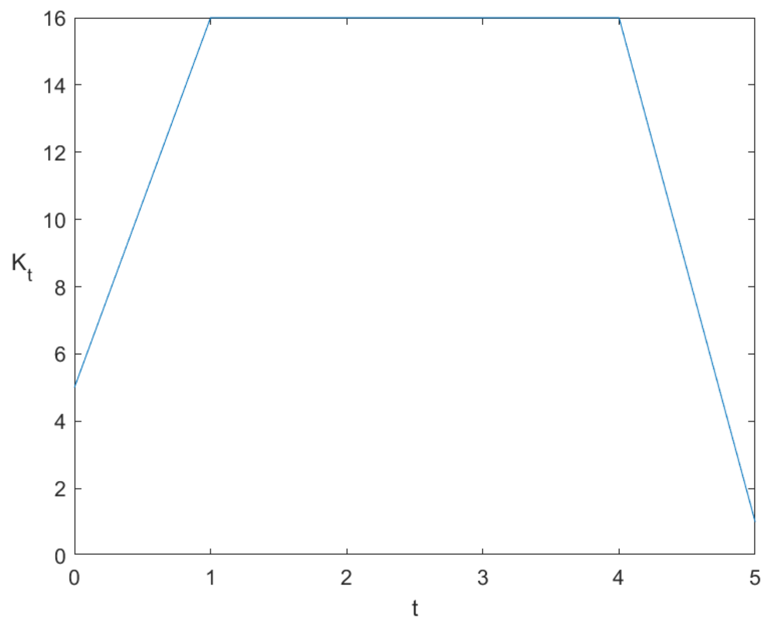


Figure 4: Capital trajectory

Problem 2- Stochastic version

1. The bellman equation,

$$
\begin{aligned}
V\left(k_t\right) &= \max_{c_t \in [0, F(k_t)+k_t]} u\left(c_t\right) + \beta E\left[V\left(k_{t+1}\right) | k_t, \varepsilon_t\right] \\
&= \max_{c_t \in [0, F(k_t)+k_t]} u\left(F\left(k_t\right) - k_{t+1} + k_t + \varepsilon_t\right) + \beta E\left[V\left(k_{t+1}\right) | k_t, \varepsilon_t\right] \\
s.t. k_{t+1} - k_t &= F\left(k_t\right) - c_t + \varepsilon_t, k_0 \text{ is given} \\
& \quad \text{error } \varepsilon \text{ follows a i.i.d distribution}
\end{aligned}
$$

2. The interpretation of value function is a function of this period's state variable's maximum PDV that the agent can optimally obtained. It consists of two parts: (1) a current period reward function; (2) the discounted streams of future payoff (continuation value: the PDV of being in the next period's state).

3. Redefine the control variable as $I = F(K) + K - C$. Then the next period state $K_{t+1} = I_t + \varepsilon_t$. in order for $K$ always on the grid, $I$ can take values:

$$I_t + \varepsilon_t \in [1, 30] \Rightarrow I_t \in [2, 29].$$

$$V\left(k_t\right) = \max_{I_t \in [2, \min\{F(k_t)+k_t, 29\}]} u\left(F\left(K_t\right) + K_t - I_t\right) + \beta E\left[V\left(I_t + \varepsilon_t\right) | I_t, \varepsilon_t\right]$$

4. See Table 2.

5. Since we have the policy function $k_{t+1} = U_t(k_t)$, for each $k_t$ we can get $I_{t+1}$, then adding the simulated random shock we have the realized $K_{t+1}$, then from budget constraint we have $c_{t+1}$. Repeat this process in a loop helps us to plot the trajectory.

6. Essentially, the bad/good draw of $\varepsilon$ could affect the next period capital shortly but would not affect the investment decision.

Table 2: Policy function: stochastic version finite and infinite

| | (3) k_t+1:infinite | (4) k_t+1: finite | | | | |
|---|---|---|---|---|---|---|
| k_t | | t=1 | t=2 | t=3 | t=4 | t=5 |
| 1 | 16 | 16 | 16 | 16 | 16 | 2 |
| 2 | 20 | 20 | 20 | 20 | 20 | 2 |
| 3 | 22 | 22 | 22 | 22 | 22 | 2 |
| 4 | 23 | 23 | 23 | 23 | 22 | 2 |
| 5 | 23 | 23 | 23 | 23 | 23 | 2 |
| 6 | 23 | 23 | 23 | 23 | 23 | 2 |
| 7 | 23 | 23 | 23 | 23 | 23 | 2 |
| 8 | 23 | 23 | 23 | 23 | 23 | 2 |
| 9 | 24 | 24 | 24 | 24 | 24 | 2 |
| 10 | 24 | 24 | 24 | 24 | 24 | 2 |
| 11 | 24 | 24 | 24 | 24 | 24 | 2 |
| 12 | 24 | 24 | 24 | 24 | 24 | 2 |
| 13 | 24 | 24 | 24 | 24 | 24 | 2 |
| 14 | 24 | 24 | 24 | 24 | 24 | 2 |
| 15 | 24 | 24 | 24 | 24 | 24 | 2 |
| 16 | 24 | 24 | 24 | 24 | 24 | 2 |
| 17 | 24 | 24 | 24 | 24 | 24 | 2 |
| 18 | 24 | 24 | 24 | 24 | 24 | 2 |
| 19 | 24 | 24 | 24 | 24 | 24 | 2 |
| 20 | 24 | 24 | 24 | 24 | 24 | 2 |
| 21 | 24 | 24 | 24 | 24 | 24 | 2 |
| 22 | 24 | 24 | 24 | 24 | 24 | 2 |
| 23 | 24 | 24 | 24 | 24 | 24 | 2 |
| 24 | 24 | 24 | 24 | 24 | 24 | 2 |
| 25 | 24 | 24 | 24 | 24 | 24 | 2 |
| 26 | 24 | 24 | 24 | 24 | 24 | 2 |
| 27 | 24 | 24 | 24 | 24 | 24 | 2 |
| 28 | 24 | 24 | 24 | 24 | 24 | 2 |
| 29 | 24 | 24 | 24 | 24 | 24 | 2 |
| 30 | 24 | 24 | 24 | 24 | 24 | 2 |

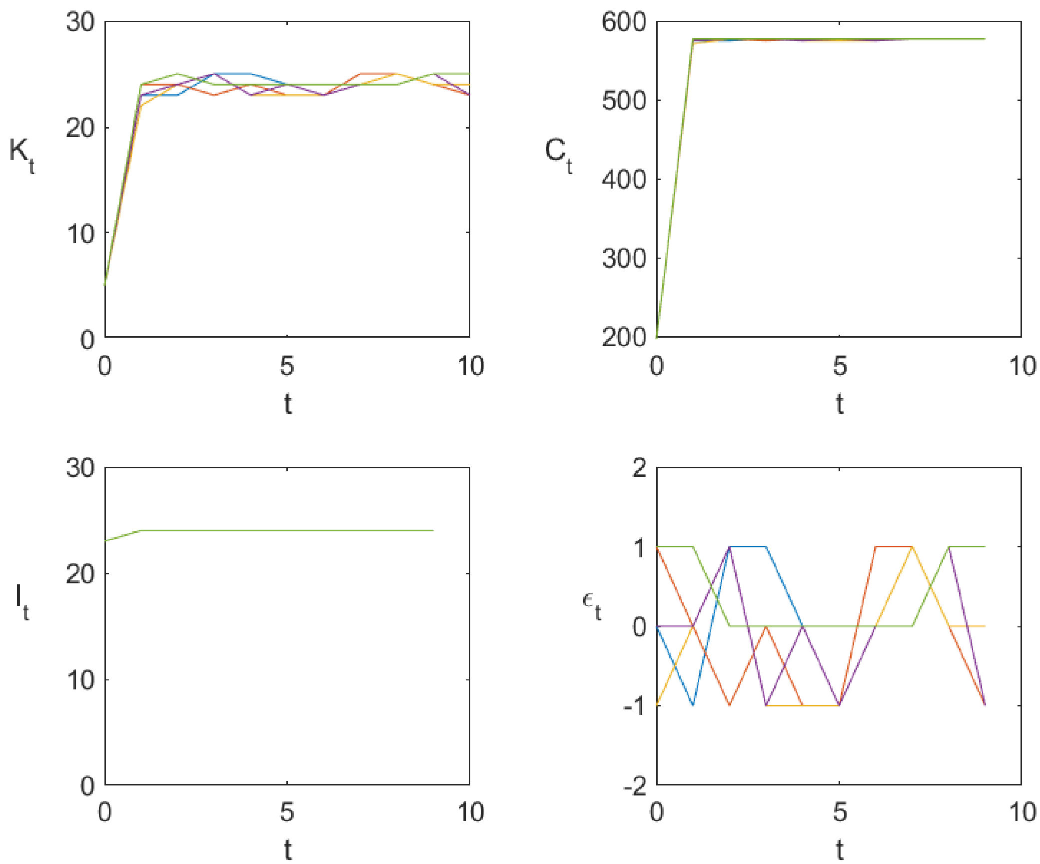


Figure 5: Five simulated trajectory upon different realizations of $\epsilon$

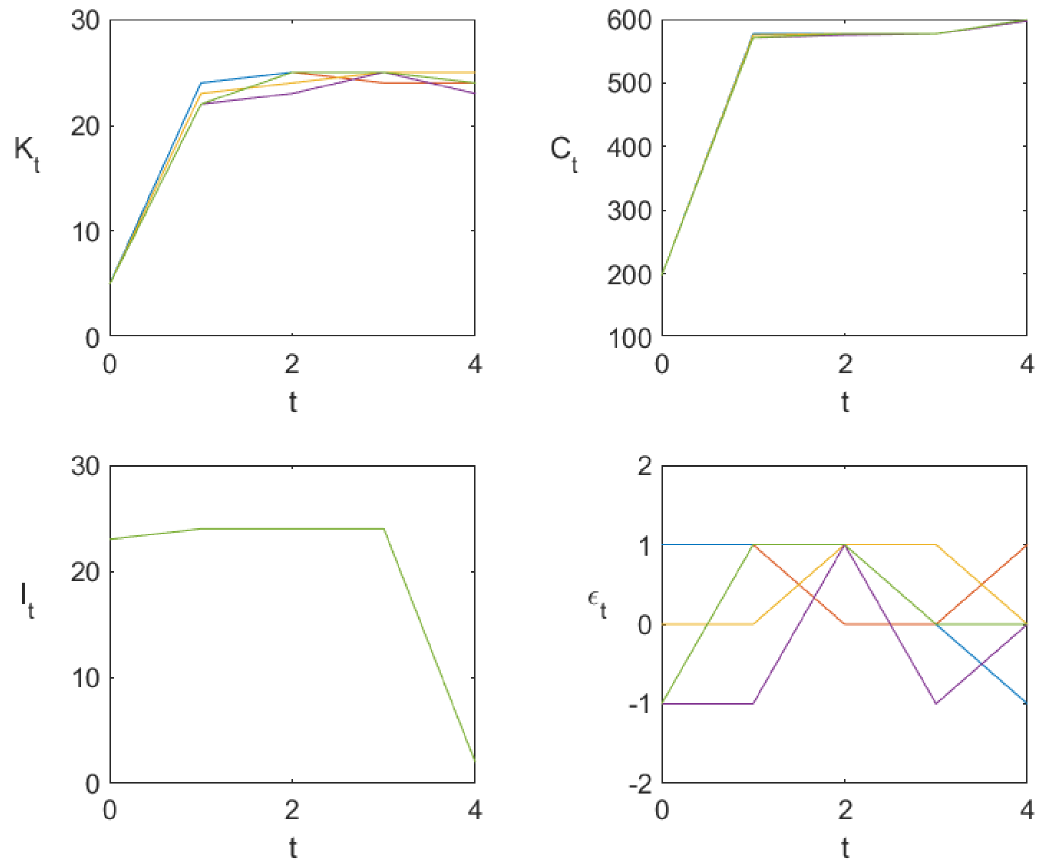


Figure 6: Five simulated trajectory upon different realizations of $\epsilon$

Problem 4- Stochastic finite version

1. The Bellman equation:

$$V_t(k_t) = \begin{cases} \max_{I_t \in [2, \min\{F(k_t)+k_t, 29\}]} u\left(F(k_t) + k_t - I_t\right) + \beta E V_{t+1}\left(I_t + \varepsilon_t\right), 0 \leq t \leq T \\ \max_{I_t \in [0, \min\{F(k_t)+k_t, 29\}]} u\left(F(k_t) + k_t - I_t\right), t = T \end{cases}$$

2. The value function at period t is the maximum PDV of the entire stream of payoff from period t to T. The continuation value is the discounted expected value of the next period's value function. For the last period there is no continuation value.

3. See Table 2.

4. Since we have the policy function $k_{t+1} = U_t(k_t)$, for each $k_t$ we can get $I_{t+1}$, then adding the simulated random shock we have the realized $K_{t+1}$, then from budget constraint we have $c_{t+1}$. Repeat this process in a loop helps us to plot the trajectory.

5. Similar to the last question, the optimal choice of $I_t$ is close to 24 and clearly in the last period the person will eat out all the capital stock.

Problem 5 - Investment theory

1. Orthodox theory of investment does not recognize three properties of investment and follows the positive NPV rule of investment.

2. The three rules are (1) irreversibility of investment; (2) uncertainty over future rewards from investment; (3) Leeway over timing of investment.

3. The real option approach recognizes the opportunity cost of the investment, and proposes the opninion approach rule, i.e. invest if the NPV is greater than the value of the opportunity cost of the investment.