The Monetary Policy Model

Prof. Raymond J. Hawkins, Econ 100B

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In this memo we develop the AD/AS model presented in Mishkin (2014) by considering the actions of a central bank subject to the constraint of the Philips curve. The derivation in this memo is based on the presentation in Carlin and Soskice (2014) of a closely related model of monetary policy.

The mandate of a central bank is to keep inflation π_t as close as possible to target inflation π^{T} and aggregate output Y_t as close as possible to potential output Y_t^{P}. These two goals can be expressed as a central bank loss function[1]

$$L = \left(Y_t - Y_t^{\mathsf{P}}\right)^2 + \beta\left(\pi_t - \pi^{\mathsf{T}}\right)^2 \tag{1}$$

[1] This loss function is mentioned in passing in the fourth footnote of chapter 13 in Mishkin. The inflation term in this loss function is an example of the symmetrical inflation objective seen in recent FOMC statements.

where β is a weight with which one can differentiate central bank mandates: $\beta > 1$ for mandates that preference inflation control, $0 \leq \beta < 1$ for those that preference the control of output.

Since a central bank can control Y_t through the policy rate via the IS curve

$$Y_t = \overline{Y}_t - \zeta_Y r_t \tag{2}$$

it is clear that the first term on the right-hand side of Eq. (1), known as the output gap, can be set to zero by setting the real interest rate to the *natural real rate, r^**, which is the rate that causes the IS curve to yield potential output

$$Y_t^{\mathsf{P}} = \overline{Y}_t - \zeta_Y r_t^* \,. \tag{3}$$

Inflation can be similarly controlled by the central bank via the Phillips curve[2]

$$\pi_t = \pi_{t-1} + \gamma\left(Y_t - Y_t^{\mathsf{P}}\right)\ , \tag{4}$$

[2] Mishkin refers to this as the short-run aggregate supply (SRAS) curve, but it is often referred to as the Phillips curve; this, despite the fact that Phillips established a relationship between unemployment and inflation, not output and inflation.

but setting the output gap to zero when $\pi_{t-1} \neq \pi^{\mathsf{T}}$ would shut off the inflation adjustment driver of the Phillips curve and thus fix inflation at a level different from target inflation π^{T}. Thus we see that in minimizing loss function the central bank faces a *constrained optimization problem:* the central bank would like to close the output gap to minimize that portion of the loss function but is constrained from doing so because it needs a finite output gap to close the inflation gap via the Phillips curve. Another way of expressing this is that a central bank seeks to minimize its loss function subject to the constraint of the Phillips curve.

One way to implement this constrained optimization is to use the Phillips curve to convert the loss function above into an equation that depends only upon current output[3]

[3] A more general approach to this constrained optimization problem is shown in the Appendix.

$$L = \left(Y_t - Y_t^{\mathsf{P}}\right)^2 + \beta \left(\underbrace{\pi_{t-1} + \gamma \left(Y_t - Y_t^{\mathsf{P}}\right)}_{\pi_t} - \pi^{\mathsf{T}} \right)^2 . \tag{5}$$

Having transformed the inflation gap into a function of the output gap, the value of Y_t that minimizes[4] the central bank loss function is the one associated with

[4] One can show that this is indeed a minimum by taking the second derivative of the loss function with respect to Y_t and observing that it is positive.

$$\frac{\partial L}{\partial Y_t} = 2\left(Y_t - Y_t^{\mathsf{P}}\right) + 2\beta\gamma\left(\pi_{t-1} + \gamma\left(Y_t - Y_t^{\mathsf{P}}\right) - \pi^{\mathsf{T}}\right) = 0 \tag{6}$$

or

$$\left(Y_t - Y_t^{\mathsf{P}}\right) = -\beta\gamma\left(\pi_t - \pi^{\mathsf{T}}\right) . \tag{7}$$

This equation specifies the optimal output gap that the central bank should set for a given inflation gap. It also shows through the country-specific terms of $\overline{Y}_t$ and ζ_Y which appear in Y_t, Y_t^{P}, β, and γ how the actual response of a central bank to an inflation gap will depend on the country in which the gap is experienced.

The optimal policy rate that a central bank should set in response to fluctuations of the loss function away from zero can be derived from the IS curve (Eq. (2)) by substituting it into Eq. (7):

$$\overline{Y}_t - \zeta_Y r_t - Y_t^{\mathsf{P}} = \beta\gamma\pi^{\mathsf{T}} - \beta\gamma\pi_t \tag{8}$$

from which

$$r_t = \frac{1}{\zeta_Y}\left(\overline{Y}_t - Y_t^{\mathsf{P}} - \beta\gamma\pi^{\mathsf{T}}\right) + \frac{\beta\gamma}{\zeta_Y}\pi_t . \tag{9}$$

This provides clear guidance to a central bank on how to respond to a variety of economic shocks. If inflation, π_t or the autonomous portion of the IS curve, $\overline{Y}_t$, increase the central bank should respond by increasing the real interest rate r_t. Similarly, if potential output, Y_t^{P} or target inflation, π^{T}, increase the central bank should respond by decreasing the real interest rate r_t.

The Taylor rule that is implicit in this policy model follows by writing Eq. (9) in terms of the inflation gap

$$r_t = r^* + \frac{\beta\gamma}{\zeta_Y}\left(\pi_t - \pi^{\mathsf{T}}\right) \tag{10}$$

We can show this to be the MP curve of Mishkin

$$r_t = \overline{r} + \lambda\pi_t \tag{11}$$

if we define $\overline{r}$ and λ as

$$\overline{r} = \frac{1}{\zeta_Y}\left(\overline{Y}_t - Y_t^{\mathsf{P}} - \beta\gamma\pi^{\mathsf{T}}\right) \tag{12}$$

or

$$\overline{r} = r^* - \lambda\pi^T \tag{13}$$

where r^* is the natural real interest rate and where

$$\lambda = \frac{\beta\gamma}{\zeta_Y} \,. \tag{14}$$

Substituting Eqs. (12) and (14) into (7) one can also show that (7) is the same as the aggregate demand curve presented in Mishkin (2014).

An advantage of this derivation is that the dependencies of $\overline{r}$ and λ are clear. The autonomous real rate, $\overline{r}$, is a function of autonomous output, potential output, and target inflation: all items that are known to shift the AD curve. The sensitivity of the real rate to inflation, λ, is a function of γ which relates the output gap to the change in inflation and ζ_Y which relates inflation to output in the AD curve. We also see that the mandate of the central bank features in the MP curve as β appears in both terms. The β of a central bank, however, is set by the mandate and is not something that the central bank can change with ease. This, together with the macroeconomic nature of the other terms in $\overline{r}$ and λ suggests that the MP curve is not something over which the central bank has control; rather, it is an economic relationship that the central bank can use as it seeks to minimize the inflation and output gaps.

In summary, with this approach the aggregate demand and MP curves of Mishkin (2014) appear as a natural consequence of a central bank trying to keep the output and inflation gaps as close to zero as possible. The only independent coefficients of the model are those that appear in the central bank loss function, the IS curve, and the (Okun's-law transformed) Phillips curve.

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References

Carlin, W. and Soskice, D. (2014). *Macroeconomics: Institutions, Instability, and the Financial System*. Oxford University Press, Oxford, UK.

Mishkin, F. S. (2014). *Macroeconomics: Policy and Practice*. Prentice Hall, New York, NY, second edition.

Appendix

The optimization employed above is a simplified version of a more general approach to constrained optimization using Lagrange multipliers.[5] With this approach the minimum of the central bank loss function subject to the constraint of the Phillips curve is obtained by forming the Lagrangian[6]

$$\mathcal{L} = L - \lambda \left[\pi_{t-1} + \gamma \left(Y_t - Y_t^{\mathsf{P}} \right) - \pi_t \right] \tag{15}$$

where the coefficient λ is the Lagrange multiplier. One then sets to zero the derivatives of $\mathcal{L}$ with respect to Y_t and to π_t :

$$\frac{\partial L}{\partial Y_t} = 2 \left(Y_t - Y_t^{\mathsf{P}} \right) - \lambda\gamma = 0 \tag{16}$$

and

$$\frac{\partial L}{\partial \pi_t} = 2\beta \left(\pi_t - \pi^{\mathsf{T}} \right) + \lambda = 0 \ . \tag{17}$$

Clearing λ we obtain

$$2 \left(Y_t - Y_t^{\mathsf{P}} \right) + 2\gamma\beta \left(\pi_t - \pi^{\mathsf{T}} \right) = 0 \tag{18}$$

or

$$\left(Y_t - Y_t^{\mathsf{P}} \right) = -\beta\gamma \left(\pi_t - \pi^{\mathsf{T}} \right) \tag{19}$$

which is our Eq. (7) above.

[5] Lagrange multipliers are not generally encountered in the prerequisites for Econ 100B. Some of you will have encountered them in the context of constrained optimization and the rest of you will encounter them in Econ 100A.

[6] For multiple constraints one introduces more Lagrange multipliers; one for each constraint. See, for example, the discussion of Lagrange multipliers at Wolfram MathWorld.