

# Introduction to Modern Economic Growth: Chapter 3

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# Solow Growth Model and the Data

- Use Solow model or extensions to interpret both economic growth over time and cross-country output differences.
- Focus on *proximate causes* of economic growth.

# Growth Accounting I

- Aggregate production function in its general form:

$$Y(t) = F[K(t), L(t), A(t)].$$

- Combined with competitive factor markets, gives Solow (1957) *growth accounting framework*.
- Continuous-time economy and differentiate the aggregate production function with respect to time.
- Dropping time dependence,

$$\frac{\dot{Y}}{Y} = \frac{F_A A}{Y} \frac{\dot{A}}{A} + \frac{F_K K}{Y} \frac{\dot{K}}{K} + \frac{F_L L}{Y} \frac{\dot{L}}{L}. \quad (1)$$

# Growth Accounting II

- Denote growth rates of output, capital stock and labor by  $g \equiv \dot{Y}/Y$ ,  $g_K \equiv \dot{K}/K$  and  $g_L \equiv \dot{L}/L$ .
- Define the contribution of technology to growth as

$$x \equiv \frac{F_A A}{Y} \frac{\dot{A}}{A}$$

- Recall with competitive factor markets,  $w = F_L$  and  $R = F_K$ .
- Define factor shares as  $\alpha_K \equiv RK/Y$  and  $\alpha_L \equiv wL/Y$ .
- Putting all these together, (1) the *fundamental growth accounting* equation

$$x = g - \alpha_K g_K - \alpha_L g_L. \quad (2)$$

- Gives estimate of contribution of technological progress, *Total Factor Productivity* (TFP) or Multi Factor Productivity.

# Growth Accounting III

- Denoting an estimate by “ $\hat{\cdot}$ ”:

$$\hat{x}(t) = \hat{g}(t) - \hat{\alpha}_K(t) \hat{g}_K(t) - \hat{\alpha}_L(t) \hat{g}_L(t). \quad (3)$$

- All terms on right-hand side are “estimates” obtained with a range of assumptions from national accounts and other data sources.
- If interested in  $\dot{A}/A$  rather than  $\hat{x}$ , need further assumptions. For example, if we assume

$$Y(t) = \tilde{F}[K(t), A(t)L(t)],$$

then

$$\frac{\dot{A}}{A} = \frac{1}{\alpha_L} [g - \alpha_K g_K - \alpha_L g_L],$$

- But not particularly useful, the economically interesting object is  $\hat{x}$  in (3).

# Growth Accounting IV

- In continuous time, equation (3) is exact.
- With discrete time, potential problem in using (3): over the time horizon factor shares can change.
- Use beginning-of-period or end-of-period values of  $\alpha_K$  and  $\alpha_L$ ?
  - Either might lead to seriously biased estimates.
  - Best way of avoiding such biases is to use as high-frequency data as possible.
  - Typically use factor shares calculated as the average of the beginning and end of period values.
- In discrete time, the analog of equation (3) becomes

$$\hat{x}_{t,t+1} = g_{t,t+1} - \bar{\alpha}_{K,t,t+1}g_{K,t,t+1} - \bar{\alpha}_{L,t,t+1}g_{L,t,t+1}, \quad (4)$$

- $g_{t,t+1}$  is the growth rate of output between  $t$  and  $t + 1$ ; other growth rates defined analogously.

# Growth Accounting V

- Moreover,

$$\bar{\alpha}_{K,t,t+1} \equiv \frac{\alpha_{Kt} + \alpha_{Kt+1}}{2}$$
$$\text{and } \bar{\alpha}_{L,t,t+1} \equiv \frac{\alpha_{Lt} + \alpha_{Lt+1}}{2}$$

- Equation (4) would be a fairly good approximation to (3) when the difference between  $t$  and  $t + 1$  is small and the capital-labor ratio does not change much during this time interval.
- Solow (1957) applied this framework to US data: a large part of the growth was due to technological progress.
- From early days, however, a number of pitfalls were recognized.
  - Moses Abramovitz (1956): dubbed the  $\hat{x}$  term “the measure of our ignorance”.
  - If we mismeasure  $g_L$  and  $g_K$  we will arrive at inflated estimates of  $\hat{x}$ .

# Growth Accounting VI

- Reasons for mismeasurement:
  - what matters is not labor hours, but effective labor hours
    - important—though difficult—to make adjustments for changes in the *human capital* of workers.
  - measurement of capital inputs:
    - in the theoretical model, capital corresponds to the final good used as input to produce more goods.
    - in practice, capital is machinery, need assumptions about how relative prices of machinery change over time.
    - typical assumption was to use capital expenditures but if machines become cheaper would severely underestimate  $g_K$



# Solow Model and Regression Analyses I

- Another popular approach of taking the Solow model to data: *growth regressions*, following Barro (1991).
- Return to basic Solow model with constant population growth and labor-augmenting technological change in continuous time:

$$y(t) = A(t) f(k(t)), \quad (5)$$

and

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - \delta - g - n, \quad (6)$$

# Solow Model and Regression Analyses II

- Differentiating (5) with respect to time and dividing both sides by  $y(t)$ ,

$$\frac{\dot{y}(t)}{y(t)} = g + \varepsilon_f(k(t)) \frac{\dot{k}(t)}{k(t)}, \quad (7)$$

where

$$\varepsilon_f(k(t)) \equiv \frac{f'(k(t)) k(t)}{f(k(t))} \in (0, 1)$$

is the elasticity of the  $f(\cdot)$  function.

- $\varepsilon_f(k(t))$  is between 0 and 1 follows from Assumption 1. For example, with Cobb-Douglas  $\varepsilon_f(k(t)) = \alpha$ , but generally a function of  $k(t)$ .

# Solow Model and Regression Analyses III

- First-order Taylor expansion of (6) with respect to  $\log k(t)$  around  $k^*$  (and recall that  $\partial y / \partial \log x = (\partial y / \partial x) \cdot x$ ):

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &\simeq \left( \frac{sf(k^*)}{k^*} - \delta - g - n \right) \\ &\quad + \left( \frac{f'(k^*)k^*}{f(k^*)} - 1 \right) s \frac{f(k^*)}{k^*} (\log k(t) - \log k^*) . \\ &\simeq (\varepsilon_f(k^*) - 1) (\delta + g + n) (\log k(t) - \log k^*) . \end{aligned}$$

- First term in the first line is zero by definition of the steady-state value  $k^*$ .
- Also used definition of  $\varepsilon_f(k(t))$  and the fact that  $sf(k^*)/k^* = \delta + g + n$ .
- Substituting into (7),

$$\frac{\dot{y}(t)}{y(t)} \simeq g - \varepsilon_f(k^*) (1 - \varepsilon_f(k^*)) (\delta + g + n) (\log k(t) - \log k^*) .$$

# Solow Model and Regression Analyses IV

- Define  $y^*(t) \equiv A(t) f(k^*)$ ; refer to  $y^*(t)$  as the “steady-state level of output per capita” even though it is not constant.
- First-order Taylor expansions of  $\log y(t)$  with respect to  $\log k(t)$  around  $\log k^*(t)$ :

$$\log y(t) - \log y^*(t) \simeq \varepsilon_f(k^*) (\log k(t) - \log k^*).$$

- Combining this with the previous equation, “convergence equation”:

$$\frac{\dot{y}(t)}{y(t)} \simeq g - (1 - \varepsilon_f(k^*)) (\delta + g + n) (\log y(t) - \log y^*(t)). \quad (8)$$

- Two sources of growth in Solow model:  $g$ , the rate of technological progress, and “convergence”.

# Solow Model and Regression Analyses V

- Latter source, convergence:
  - Negative impact of the gap between current level and steady-state level of output per capita on rate of capital accumulation (recall  $0 < \varepsilon_f(k^*) < 1$ ).
  - The lower is  $y(t)$  relative to  $y^*(t)$ , hence the lower is  $k(t)$  relative to  $k^*$ , the greater is  $f(k^*)/k^*$ , and this leads to faster growth in the effective capital-labor ratio.
- Speed of convergence in (8), measured by the term  $(1 - \varepsilon_f(k^*))(\delta + g + n)$ , depends on:
  - $\delta + g + n$ : determines rate at which effective capital-labor ratio needs to be replenished.
  - $\varepsilon_f(k^*)$ : when  $\varepsilon_f(k^*)$  is high, we are close to a linear— $AK$ —production function, convergence should be slow.

## Example: Cobb-Douglas production function and convergence I

- Consider Cobb-Douglas production function
$$Y(t) = A(t) K(t)^\alpha L(t)^{1-\alpha}.$$
- Implies that  $y(t) = A(t) k(t)^\alpha$ ,  $\varepsilon_f(k(t)) = \alpha$ . Therefore, (8) becomes

$$\frac{\dot{y}(t)}{y(t)} \simeq g - (1 - \alpha) (\delta + g + n) (\log y(t) - \log y^*(t)).$$

- Enables us to “calibrate” the speed of convergence in practice
- Focus on advanced economies
  - $g \simeq 0.02$  for approximately 2% per year output per capita growth,
  - $n \simeq 0.01$  for approximately 1% population growth and
  - $\delta \simeq 0.05$  for about 5% per year depreciation.
  - Share of capital in national income is about 1/3, so  $\alpha \simeq 1/3$ .

## Example: Cobb-Douglas production function and convergence II

- Thus convergence coefficient would be around 0.054 ( $\simeq 0.67 \times 0.08$ ).
- Very rapid rate of convergence:
  - gap of income between two similar countries should be halved in little more than 10 years
- At odds with the patterns we saw before.

# Solow Model and Regression Analyses VI

- Using (8), we can obtain a growth regression similar to those estimated by Barro (1991).
- Using discrete time approximations, equation (8) yields:

$$g_{i,t,t-1} = b^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t}, \quad (9)$$

- $\varepsilon_{i,t}$  is a stochastic term capturing all omitted influences.
- If such an equation is estimated in the sample of core OECD countries,  $b^1$  is indeed estimated to be negative.
- But for the whole world, no evidence for a negative  $b^1$ . If anything,  $b^1$  would be positive.
- I.e., there is no evidence of world-wide convergence,
- Barro and Sala-i-Martin refer to this as “unconditional convergence.”



# Solow Model and Regression Analyses VII

- Unconditional convergence may be too demanding:
  - requires income gap between any two countries to decline, irrespective of what types of technological opportunities, investment behavior, policies and institutions these countries have.
  - If countries do differ, Solow model would *not* predict that they should converge in income level.
- If countries differ according to their characteristics, a more appropriate regression equation may be:

$$g_{i,t,t-1} = b_i^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t}, \quad (10)$$

- Now the constant term,  $b_i^0$ , is country specific.
- Slope term, measuring the speed of convergence,  $b^1$ , should also be country specific.
- May then model  $b_i^0$  as a function of certain country characteristics.

# Solow Model and Regression Analyses VII

- If the true equation is (10), (9) would not be a good fit to the data.
- I.e., there is no guarantee that the estimates of  $b^1$  resulting from this equation will be negative.
- In particular, it is natural to expect that  $\text{Cov}(b_i^0, \log y_{i,t-1}) < 0$ :
  - economies with certain growth-reducing characteristics will have low levels of output.
  - Implies a negative bias in the estimate of  $b^1$  in equation (9), when the more appropriate equation is (10).
- With this motivation, Barro (1991) and Barro and Sala-i-Martin (2004) favor the notion of “conditional convergence:”
  - convergence effects should lead to negative estimates of  $b^1$  once  $b_i^0$  is allowed to vary across countries.

# Solow Model and Regression Analyses VIII

- Barro (1991) and Barro and Sala-i-Martin (2004) estimate models where  $b_i^0$  is assumed to be a function of:
  - male schooling rate, female schooling rate, fertility rate, investment rate, government-consumption ratio, inflation rate, changes in terms of trades, openness and institutional variables such as rule of law and democracy.
- In regression form,

$$g_{i,t,t-1} = \mathbf{X}'_{i,t}\boldsymbol{\beta} + b^1 \log y_{i,t-1} + \varepsilon_{i,t}, \quad (11)$$

- $\mathbf{X}_{i,t}$  is a (column) vector including the variables mentioned above (and a constant).
- Imposes that  $b_i^0$  in equation (10) can be approximated by  $\mathbf{X}'_{i,t}\boldsymbol{\beta}$ .
- Conditional convergence: regressions of (11) tend to show a negative estimate of  $b^1$ .
- But the magnitude is much lower than that suggested by the computations in the Cobb-Douglas Example.

# Drawbacks of Growth Regressions I

- Regressions similar to (11) have not only been used to support “conditional convergence,” but also to estimate the “determinants of economic growth”.
- Coefficient vector  $\beta$ : information about *causal effects* of various variables on economic growth.
- Several problematic features with regressions of this form. These include:
- **Many variables in  $X_{i,t}$  and  $\log y_{i,t-1}$ , are econometrically endogenous: jointly determined  $g_{i,t,t-1}$ .**
  - May argue  $b^1$  is of interest even without “causal interpretation”.
  - But if  $\mathbf{X}_{i,t}$  is econometrically endogenous, estimate of  $b^1$  will also be inconsistent (unless  $\mathbf{X}_{i,t}$  is independent from  $\log y_{i,t-1}$ ).

# Drawbacks of Growth Regressions II

- Even if  $X_{i,t}$ 's were econometrically exogenous, a negative  $b^1$  could be by measurement error or other transitory shocks to  $y_{i,t}$ .
- For example, suppose we only observe  $\tilde{y}_{i,t} = y_{i,t} \exp(u_{i,t})$ .

- Note

$$\log \tilde{y}_{i,t} - \log \tilde{y}_{i,t-1} = \log y_{i,t} - \log y_{i,t-1} + u_{i,t} - u_{i,t-1}.$$

- Since measured growth is  
 $\tilde{g}_{i,t,t-1} \approx \log \tilde{y}_{i,t} - \log \tilde{y}_{i,t-1} = \log y_{i,t} - \log y_{i,t-1} + u_{i,t} - u_{i,t-1}$ ,  
 when we look at the growth regression

$$\tilde{g}_{i,t,t-1} = \mathbf{X}'_{i,t} \boldsymbol{\beta} + b^1 \log \tilde{y}_{i,t-1} + \varepsilon_{i,t},$$

- measurement error  $u_{i,t-1}$  will be part of both  $\varepsilon_{i,t}$  and  $\log \tilde{y}_{i,t-1} = \log y_{i,t-1} + u_{i,t-1}$ : negative bias in the estimation of  $b^1$ .
- Thus can end up negative estimate of  $b^1$ , *even when* there is no conditional convergence.

# Drawbacks of Growth Regressions III

- **Interpretation of regression equations like (11) is not always straightforward**
  - Investment rate in  $\mathbf{X}_{i,t}$ : in Solow model, differences in investment rates are *the* channel for convergence.
  - Thus conditional on investment rate, there should be no further effect of gap between current and steady-state level of output.
  - Same concern for variables in  $\mathbf{X}_{i,t}$  that would affect primarily by affecting investment or schooling rate.
- **Equation for (8) is derived for closed Solow economy.**

# The Solow Model with Human Capital I

- Labor hours supplied by different individuals do not contain the same efficiency units.
- Focus on the continuous time economy and suppose:

$$Y = F(K, H, AL), \quad (12)$$

where  $H$  denotes “human capital”.

- Assume throughout that  $A > 0$ .
- Assume  $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  in (12) is twice continuously differentiable in  $K$ ,  $H$  and  $L$ , and satisfies the equivalent of the neoclassical assumptions.
- Households save a fraction  $s_k$  of their income to invest in physical capital and a fraction  $s_h$  to invest in human capital.
- Human capital also depreciates in the same way as physical capital, denote depreciation rates by  $\delta_k$  and  $\delta_h$ .

# The Solow Model with Human Capital III

- Assume constant population growth and a constant rate of labor-augmenting technological progress, i.e.,

$$\frac{\dot{L}(t)}{L(t)} = n \text{ and } \frac{\dot{A}(t)}{A(t)} = g.$$

- Defining effective human and physical capital ratios as

$$k(t) \equiv \frac{K(t)}{A(t)L(t)} \text{ and } h(t) \equiv \frac{H(t)}{A(t)L(t)},$$

- Using the constant returns to scale, output per effective unit of labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left(\frac{K(t)}{A(t)L(t)}, \frac{H(t)}{A(t)L(t)}, 1\right) \\ &\equiv f(k(t), h(t)). \end{aligned}$$



# The Solow Model with Human Capital IV

- Law of motion of  $k(t)$  and  $h(t)$  can then be obtained as:

$$\begin{aligned}\dot{k}(t) &= s_k f(k(t), h(t)) - (\delta_k + g + n) k(t), \\ \dot{h}(t) &= s_h f(k(t), h(t)) - (\delta_h + g + n) h(t).\end{aligned}$$

- Steady-state equilibrium: effective human and physical capital ratios,  $(k^*, h^*)$ , which satisfy:

$$s_k f(k^*, h^*) - (\delta_k + g + n) k^* = 0, \quad (13)$$

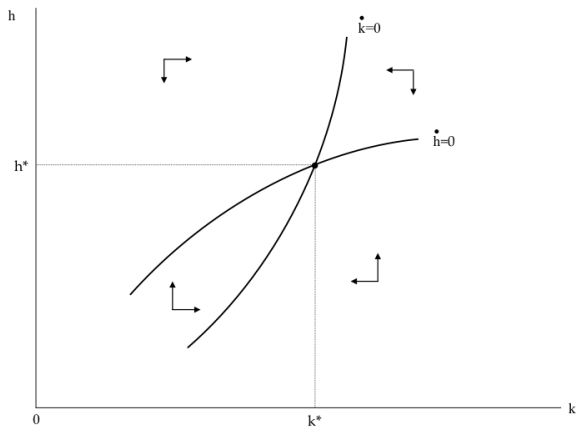
and

$$s_h f(k^*, h^*) - (\delta_h + g + n) h^* = 0. \quad (14)$$

# The Solow Model with Human Capital V

- Focus on steady-state equilibria with  $k^* > 0$  and  $h^* > 0$  (if  $f(0,0) = 0$ , then there exists a trivial steady state with  $k = h = 0$ , which we ignore it).
- Can first prove that steady-state equilibrium is unique. To see this heuristically, consider the Figure in the  $(k, h)$  space.
- Both lines are upward sloping, but proof of next proposition shows (14) is always shallower in the  $(k, h)$  space, so the two curves can only intersect once.

**Proposition** In the augmented Solow model with human capital, there exists a unique, globally stable steady-state equilibrium  $(k^*, h^*)$ .



**Figure:** Dynamics of physical capital-labor and human capital-labor ratios in the Solow model with human capital.

## Example: Augmented Solow model with Cobb-Douglas production function I

- Aggregate production function is

$$Y(t) = K(t)^\alpha H(t)^\beta (A(t)L(t))^{1-\alpha-\beta}, \quad (15)$$

where  $0 < \alpha < 1$ ,  $0 < \beta < 1$  and  $\alpha + \beta < 1$ .

- Output per effective unit of labor can then be written as

$$\hat{y}(t) = k^\alpha(t) h^\beta(t),$$

with the same definition of  $\hat{y}(t)$ ,  $k(t)$  and  $h(t)$  as above.

## Example: Augmented Solow model with Cobb-Douglas production function II

- Using this functional form, (13) and (14) give the unique steady-state equilibrium:

$$\begin{aligned} k^* &= \left( \left( \frac{s_k}{n+g+\delta_k} \right)^{1-\beta} \left( \frac{s_h}{n+g+\delta_h} \right)^\beta \right)^{\frac{1}{1-\alpha-\beta}} \\ h^* &= \left( \left( \frac{s_k}{n+g+\delta_k} \right)^\alpha \left( \frac{s_h}{n+g+\delta_h} \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha-\beta}}, \end{aligned} \quad (16)$$

- Higher saving rate in physical capital not only increases  $k^*$ , but also  $h^*$ .
- Same applies for a higher saving rate in human capital.
- Reflects that higher  $k^*$  raises overall output and thus the amount invested in schooling (since  $s_h$  is constant).

## Example: Augmented Solow model with Cobb-Douglas production function III

- Given (16), output per effective unit of labor in steady state is obtained as

$$\hat{y}^* = \left( \frac{s_k}{n + g + \delta_k} \right)^{\frac{\beta}{1-\alpha-\beta}} \left( \frac{s_h}{n + g + \delta_h} \right)^{\frac{\alpha}{1-\alpha-\beta}}. \quad (17)$$

- Relative contributions of the saving rates depends on the shares of physical and human capital:
  - the larger is  $\beta$ , the more important is  $s_k$  and the larger is  $\alpha$ , the more important is  $s_h$ .

# A World of Augmented Solow Economies I

- Mankiw, Romer and Weil (1992) used regression analysis to take the augmented Solow model, with human capital, to data.
- Use the Cobb-Douglas model and envisage a world consisting of  $j = 1, \dots, N$  countries.
- “Each country is an island”: countries do not interact (perhaps except for sharing some common technology growth).
- Country  $j = 1, \dots, N$  has the aggregate production function:

$$Y_j(t) = K_j(t)^\alpha H_j(t)^\beta (A_j(t) L_j(t))^{1-\alpha-\beta}.$$

- Nests the basic Solow model without human capital when  $\beta = 0$ .
- Countries differ in terms of their saving rates,  $s_{k,j}$  and  $s_{h,j}$ , population growth rates,  $n_j$ , and technology growth rates  $\dot{A}_j(t) / A_j(t) = g_j$ .
- Define  $k_j \equiv K_j / A_j L_j$  and  $h_j \equiv H_j / A_j L_j$ .

# A World of Augmented Solow Economies II

- Focus on a world in which each country is in their steady state
- Equivalents of equations (16) apply here and imply:

$$k_j^* = \left( \left( \frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^{1-\beta} \left( \frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^\beta \right)^{\frac{1}{1-\alpha-\beta}}$$

$$h_j^* = \left( \left( \frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^\alpha \left( \frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha-\beta}}.$$

- Consequently, using (17), the “steady-state” /balanced growth path income per capita of country  $j$  can be written as

$$y_j^*(t) \equiv \frac{Y(t)}{L(t)} \quad (18)$$

$$= A_j(t) \left( \frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left( \frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^{\frac{\beta}{1-\alpha-\beta}}.$$



# A World of Augmented Solow Economies II

- Here  $y_j^*(t)$  stands for output per capita of country  $j$  along the balanced growth path.
- Note if  $g_j$ 's are not equal across countries, income per capita will diverge.
- Mankiw, Romer and Weil (1992) make the following assumption:

$$A_j(t) = \bar{A}_j \exp(gt).$$

- Countries differ according to technology *level*, (initial level  $\bar{A}_j$ ) but they share the same common technology growth rate,  $g$ .

# A World of Augmented Solow Economies III

- Using this together with (18) and taking logs, equation for the balanced growth path of income for country  $j = 1, \dots, N$ :

$$\ln y_j^*(t) = \ln \bar{A}_j + gt + \frac{\beta}{1 - \alpha - \beta} \ln \left( \frac{s_{k,j}}{n_j + g + \delta_k} \right) + \frac{\alpha}{1 - \alpha - \beta} \ln \left( \frac{s_{h,j}}{n_j + g + \delta_h} \right). \quad (19)$$

- Mankiw, Romer and Weil (1992) take:
  - $\delta_k = \delta_h = \delta$  and  $\delta + g = 0.05$ .
  - $s_{k,j}$ =average investment rates (investments/GDP).
  - $s_{h,j}$ =fraction of the school-age population that is enrolled in secondary school.

# A World of Augmented Solow Economies IV

- Even with all of these assumptions, (19) can still not be estimated consistently.
- $\ln \bar{A}_j$  is unobserved (at least to the econometrician) and thus will be captured by the error term.
- Most reasonable models would suggest  $\ln \bar{A}_j$ 's should be correlated with investment rates.
- Thus an estimation of (19) would lead to omitted variable bias and inconsistent estimates.
- Implicitly, MRW make another *crucial* assumption, the **orthogonal technology assumption**:

$$\bar{A}_j = \varepsilon_j A, \text{ with } \varepsilon_j \text{ orthogonal to all other variables.}$$

# Cross-Country Income Differences: Regressions I

- MRW first estimate equation (19) without the human capital term for the cross-sectional sample of non-oil producing countries

$$\ln y_j^* = \text{constant} + \frac{\beta}{1-\beta} \ln(s_{kj}) - \frac{\beta}{1-\beta} \ln(n_j + g + \delta_k) + \varepsilon_j.$$

# Cross-Country Income Differences: Regressions II

## Estimates of the Basic Solow Model

|                       | MRW<br>1985    | Updated data<br>1985    2000 |                |
|-----------------------|----------------|------------------------------|----------------|
| $\ln(s_k)$            | 1.42<br>(.14)  | 1.01<br>(.11)                | 1.22<br>(.13)  |
| $\ln(n + g + \delta)$ | -1.97<br>(.56) | -1.12<br>(.55)               | -1.31<br>(.36) |
| Adj $R^2$             | .59            | .49                          | .49            |
| Implied $\beta$       | .59            | .50                          | .55            |
| No. of observations   | 98             | 98                           | 107            |

# Cross-Country Income Differences: Regressions III

- Their estimates for  $\beta / (1 - \beta)$ , implies that  $\beta$  must be around  $2/3$ , but should be around  $1/3$ .
- The most natural reason for the high implied values of  $\beta$  is that  $\varepsilon_j$  is correlated with  $\ln(s_{k,j})$ , either because:
  - ① the orthogonal technology assumption is not a good approximation to reality or
  - ② there are also human capital differences correlated with  $\ln(s_{k,j})$ .
- Mankiw, Romer and Weil favor the second interpretation and estimate the augmented model,

$$\ln y_j^* = \text{cst} + \frac{\beta}{1 - \alpha - \beta} \ln(s_{k,j}) - \frac{\beta}{1 - \alpha - \beta} \ln(n_j + g + \delta_k) + \frac{\alpha}{1 - \alpha - \beta} \ln(s_{h,j}) - \frac{\alpha}{1 - \alpha - \beta} \ln(n_j + g + \delta_h) + \varepsilon_j. \quad (20)$$

# Estimates of the Augmented Solow Model

|                       | MRW<br>1985    | Updated data<br>1985      2000 |                |
|-----------------------|----------------|--------------------------------|----------------|
| $\ln(s_k)$            | .69<br>(.13)   | .65<br>(.11)                   | .96<br>(.13)   |
| $\ln(n + g + \delta)$ | -1.73<br>(.41) | -1.02<br>(.45)                 | -1.06<br>(.33) |
| $\ln(s_h)$            | .66<br>(.07)   | .47<br>(.07)                   | .70<br>(.13)   |
| Adj $R^2$             | .78            | .65                            | .60            |
| Implied $\beta$       | .30            | .31                            | .36            |
| Implied $\alpha$      | .28            | .22                            | .26            |
| No. of observations   | 98             | 98                             | 107            |

# Cross-Country Income Differences: Regressions IV

- If these regression results are reliable, they give a big boost to the augmented Solow model.
  - Adjusted  $R^2$  suggests that three quarters of income per capita differences across countries can be explained by differences in their physical and human capital investment.
- Immediate implication is technology (TFP) differences have a somewhat limited role.
- But this conclusion should not be accepted without further investigation.



# Challenges to the Regression Analyses I

- **Technology differences across countries are not orthogonal to all other variables.**
- $\bar{A}_j$  is correlated with measures of  $s_j^h$  and  $s_j^k$  for two reasons.
  - ① *omitted variable bias*: societies with high  $\bar{A}_j$  will be those that have invested more in technology for various reasons; same reasons likely to induce greater investment in physical and human capital as well.
  - ② *reverse causality*: complementarity between technology and physical or human capital imply that countries with high  $\bar{A}_j$  will find it more beneficial to increase their stock of human and physical capital.
- In terms of (20), implies that key right-hand side variables are correlated with the error term,  $\varepsilon_j$ .
- OLS estimates of  $\alpha$  and  $\beta$  and  $R^2$  are biased upwards.

# Challenges to the Regression Analyses II

- $\alpha$  is too large relative to what we should expect on the basis of microeconomic evidence.
- The working age population enrolled in school ranges from 0.4% to over 12% in the sample of countries.
- Predicted log difference in incomes between these two countries is

$$\frac{\alpha}{1 - \alpha - \beta} (\ln 12 - \ln (0.4)) = 0.66 \times (\ln 12 - \ln (0.4)) \approx 2.24.$$

- Thus a country with schooling investment of over 12 should be about  $\exp(2.24) - 1 \approx 8.5$  times richer than one with investment of around 0.4.

# Challenges to the Regression Analyses III

- Take Mincer regressions of the form:

$$\ln w_i = \mathbf{X}_i' \gamma + \phi S_i, \quad (21)$$

- Microeconometrics literature suggests that  $\phi$  is between 0.06 and 0.10.
- Can deduce how much richer a country with 12 if we assume:
  - 1 That the micro-level relationship as captured by (21) applies identically to all countries.
  - 2 That there are no *human capital externalities*.

# Challenges to the Regression Analyses IV

- Suppose that each firm  $f$  in country  $j$  has access to the production function

$$y_{fj} = K_f^{1-\alpha} (A_j H_f)^\alpha,$$

- Suppose also that firms in this country face a cost of capital equal to  $R_j$ . With perfectly competitive factor markets,

$$R_j = (1 - \alpha) \left( \frac{K_f}{A_j H_f} \right)^{-\alpha}. \quad (22)$$

- Implies all firms ought to function at the same physical to human capital ratio.
- Thus all workers, irrespective of level of schooling, ought to work at the same physical to human capital ratio.

# Challenges to the Regression Analyses V

- Another direct implication of competitive labor markets is that in country  $j$ ,

$$w_j = \alpha (1 - \alpha)^{(1-\alpha)/\alpha} A_j R_j^{-(1-\alpha)/\alpha}.$$

- Consequently, a worker with human capital  $h_i$  will receive a wage income of  $w_j h_i$ .
- Next, substituting for capital from (22), we have total income in country  $j$  as

$$Y_j = (1 - \alpha)^{(1-\alpha)/\alpha} R_j^{-(1-\alpha)/\alpha} A_j H_j,$$

where  $H_j$  is the total efficiency units of labor in country  $j$ .

# Challenges to the Regression Analyses V

- Implies that *ceteris paribus* (in particular, holding constant capital intensity corresponding to  $R_j$  and technology,  $A_j$ ), a doubling of human capital will translate into a doubling of total income.
- It may be reasonable to keep technology,  $A_j$ , constant, but  $R_j$  may change in response to a change in  $H_j$ .
  - Maybe, but second-order:
    - 1 International capital flows may work towards equalizing the rates of returns across countries.
    - 2 When capital-output ratio is constant, which Uzawa Theorem established as a requirement for a balanced growth path, then  $R_j$  will indeed be constant
- So in the absence of *human capital externalities*: a country with 12 more years of average schooling should have between  $\exp(0.10 \times 12) \simeq 3.3$  and  $\exp(0.06 \times 12) \simeq 2.05$  times the stock of human capital of a county with fewer years of schooling.

# Challenges to the Regression Analyses VI

- Thus holding other factors constant, this country should be about 2-3 times as rich as the country with zero years of average schooling.
- Much less than the 8.5 fold difference implied by the Mankiw-Romer-Weil analysis.
- Thus  $\beta$  in MRW is too high relative to the estimates implied by the microeconomic evidence and thus likely upwardly biased.
- Overestimation of  $\alpha$  is, in turn, most likely related to correlation between the error term  $\varepsilon_j$  and the key right-hand side regressors in (20).

# Calibrating Productivity Differences I

- Suppose each country has access to the Cobb-Douglas aggregate production function:

$$Y_j = K_j^{1-\alpha} (A_j H_j)^\alpha, \quad (23)$$

- Each worker in country  $j$  has  $S_j$  years of schooling.
- Then using the Mincer equation (21) ignoring the other covariates and taking exponents,  $H_j$  can be estimated as

$$H_j = \exp(\phi S_j) L_j,$$

- Does not take into account differences in other “human capital” factors, such as experience.



# Calibrating Productivity Differences II

- Let the rate of return to acquiring the  $S$ th year of schooling be  $\phi(S)$ .
- A better estimate of the stock of human capital can be constructed as

$$H_j = \sum_S \exp \{ \phi(S) S \} L_j(S)$$

- $L_j(S)$  now refers to the total employment of workers with  $S$  years of schooling in country  $j$ .
- Series for  $K_j$  can be constructed from Summers-Heston dataset using investment data and the perpetual inventory method.

$$K_j(t+1) = (1 - \delta) K_j(t) + I_j(t),$$

- Assume, following Hall and Jones that  $\delta = 0.06$ .
- With same arguments as before, choose a value of  $2/3$  for  $\alpha$ .

# Calibrating Productivity Differences III

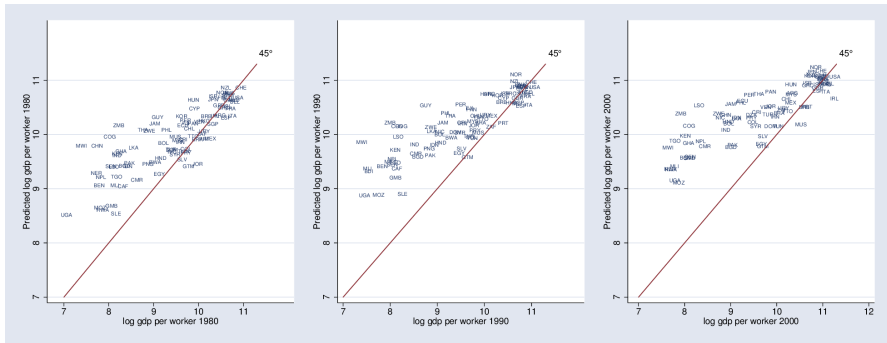
- Given series for  $H_j$  and  $K_j$  and a value for  $\alpha$ , construct “predicted” incomes at a point in time using

$$\hat{Y}_j = K_j^{1/3} (A_{US} H_j)^{2/3}$$

- $A_{US}$  is computed so that  $Y_{US} = K_{US}^{1/3} (A_{US} H_{US})^{2/3}$ .
- Once a series for  $\hat{Y}_j$  has been constructed, it can be compared to the actual output series.
- Gap between the two series represents the contribution of technology.
- Alternatively, could back out country-specific technology terms (relative to the United States) as

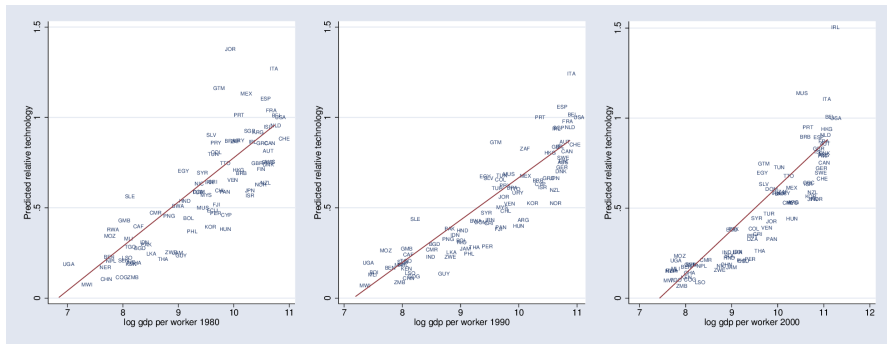
$$\frac{A_j}{A_{US}} = \left( \frac{Y_j}{Y_{US}} \right)^{3/2} \left( \frac{K_{US}}{K_j} \right)^{1/2} \left( \frac{H_{US}}{H_j} \right).$$

## Calibrating Productivity Differences IV



**Figure:** Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

# Calibrating Productivity Differences V



**Figure:** Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000

# Calibrating Productivity Differences VI

The following features are noteworthy:

- 1 Differences in physical and human capital still matter a lot.
- 2 However, differently from the regression analysis, this exercise also shows significant *technology (productivity) differences*.
- 3 Same pattern visible in the next three figures for the estimates of the technology differences,  $A_j/A_{US}$ , against log GDP per capita in the corresponding year.
- 4 Also interesting is the pattern that the empirical fit of the neoclassical growth model seems to deteriorate over time.

# Challenges to Calibration I

- In addition to the standard assumptions of competitive factor markets, we had to assume :
  - no human capital externalities, a Cobb-Douglas production function, and a range of approximations to measure cross-country differences in the stocks of physical and human capital.
- The calibration approach is in fact a close cousin of the growth-accounting exercise (sometimes referred to as “levels accounting”).
- Imagine that the production function that applies to all countries in the world is

$$F(K_j, H_j, A_j),$$

- Assume countries differ according to their physical and human capital as well as technology—but not according to  $F$ .

# Challenges to Calibration II

- Rank countries in descending order according to their physical capital to human capital ratios,  $K_j/H_j$  Then

$$\hat{x}_{j,j+1} = g_{j,j+1} - \bar{\alpha}_{K,j,j+1}g_{K,j,j+1} - \bar{\alpha}_{L,j,j+1}g_{H,j,j+1}, \quad (24)$$

- where:
  - $g_{j,j+1}$ : proportional difference in output between countries  $j$  and  $j+1$ ,
  - $g_{K,j,j+1}$ : proportional difference in capital stock between these countries and
  - $g_{H,j,j+1}$ : proportional difference in human capital stocks.
  - $\bar{\alpha}_{K,j,j+1}$  and  $\bar{\alpha}_{L,j,j+1}$ : average capital and labor shares between the two countries.
- The estimate  $\hat{x}_{j,j+1}$  is then the proportional TFP difference between the two countries.

# Challenges to Calibration III

- Levels-accounting faces two challenges.
  - ① Data on capital and labor shares across countries are not widely available. Almost all exercises use the Cobb-Douglas approach (i.e., a constant value of  $\alpha_K$  equal to  $1/3$ ).
  - ② The differences in factor proportions, e.g., differences in  $K_j/H_j$ , across countries are large. An equation like (24) is a good approximation when we consider small (infinitesimal) changes.



# Conclusions

- Message is somewhat mixed.
  - On the positive side, despite its simplicity, the Solow model has enough substance that we can take it to data in various different forms, including TFP accounting, regression analysis and calibration.
  - On the negative side, however, no single approach is entirely convincing.
- Complete agreement is not possible, but safe to say that consensus favors the interpretation that cross-country differences in income per capita cannot be understood solely on the basis of differences in physical and human capital
- Differences in TFP are not necessarily due to technology in the narrow sense.
- Have not examined *fundamental causes* of differences in prosperity: why some societies make choices that lead them to low physical capital, low human capital and inefficient technology and thus to relative poverty.