

# Overlapping Generation Models

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Macroeconomics II

## Section 1

# Growth with Overlapping Generations

## Subsection 1

# Growth with Overlapping Generations

## Growth with Overlapping Generations

- In many situations, the assumption of a *representative household* is not appropriate because
  - 1 households do not have an infinite planning horizon
  - 2 new households arrive (or are born) over time.
- New economic interactions: decisions made by older “generations” will affect the prices faced by younger “generations”.
- *Overlapping generations models*
  - 1 Capture potential interaction of different generations of individuals in the marketplace;
  - 2 Provide tractable alternative to infinite-horizon representative agent models;
  - 3 Some key implications different from neoclassical growth model;
  - 4 Dynamics in some special cases quite similar to Solow model rather than the neoclassical model;
  - 5 Generate new insights about the role of national debt and Social Security in the economy.

## Subsection 2

### **Problems of Infinity**

# Problems of Infinity I

- Static economy with countably infinite number of households,  $i \in \mathbb{N}$
- Countably infinite number of commodities,  $j \in \mathbb{N}$ .
- All households behave competitively (alternatively, there are  $M$  households of each type,  $M$  is a large number).
- Household  $i$  has preferences:

$$u_i = c_i^j + c_{i+1}^j,$$

- $c_i^j$  denotes the consumption of the  $j$ th type of commodity by household  $i$ .
- Endowment vector  $\omega$  of the economy: each household has one unit endowment of the commodity with the same index as its index.
- Choose the price of the first commodity as the numeraire, i.e.,  $p_0 = 1$ .

## Problems of Infinity II

**Proposition** In the above-described economy, the price vector  $\bar{p}$  such that  $\bar{p}_j = 1$  for all  $j \in \mathbb{N}$  is a competitive equilibrium price vector and induces an equilibrium with no trade, denoted by  $\bar{x}$ .

• **Proof:**

- At  $\bar{p}$ , each household has income equal to 1.
- Therefore, the budget constraint of household  $i$  can be written as

$$c_i^j + c_{i+1}^j \leq 1.$$

- This implies that consuming own endowment is optimal for each household,
- Thus  $\bar{p}$  and no trade,  $\bar{x}$ , constitute a competitive equilibrium.

## Problems of Infinity III

- However, this competitive equilibrium is not Pareto optimal. Consider alternative allocation,  $\tilde{x}$ :
  - Household  $i = 0$  consumes its own endowment and that of household 1.
  - All other households, indexed  $i > 0$ , consume the endowment of their neighboring household,  $i + 1$ .
  - All households with  $i > 0$  are as well off as in the competitive equilibrium  $(\bar{p}, \bar{x})$ .
  - Individual  $i = 0$  is strictly better-off.

**Proposition** In the above-described economy, the competitive equilibrium at  $(\bar{p}, \bar{x})$  is not Pareto optimal.



## Problems of Infinity IV

- Source of the problem must be related to the infinite number of commodities.
- Extended version of the First Welfare Theorem covers infinite number of commodities, but only assuming  $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$  (written with the aggregate endowment  $\omega_j$ ).
- Here the only endowment is the good, and thus  $p_j^* = 1$  for all  $j \in \mathbb{N}$ , so that  $\sum_{j=0}^{\infty} p_j^* \omega_j = \infty$  (why?).
- This abstract economy is “isomorphic” to the baseline overlapping generations model.
- The Pareto suboptimality in this economy will be the source of potential inefficiencies in overlapping generations model.

## Problems of Infinity V

- Second Welfare Theorem did not assume  $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$ .
- Instead, it used convexity of preferences, consumption sets and production possibilities sets.
- This exchange economy has convex preferences and convex consumption sets:
  - Pareto optima must be decentralizable by some redistribution of endowments.

**Proposition** In the above-described economy, there exists a reallocation of the endowment vector  $\omega$  to  $\tilde{\omega}$ , and an associated competitive equilibrium  $(\bar{p}, \tilde{x})$  that is Pareto optimal where  $\tilde{x}$  is as described above, and  $\bar{p}$  is such that  $\bar{p}_j = 1$  for all  $j \in \mathbb{N}$ .

## Proof of Proposition

- Consider the following reallocation of  $\omega$ : endowment of household  $i \geq 1$  is given to household  $i - 1$ .
  - At the new endowment vector  $\tilde{\omega}$ , household  $i = 0$  has one unit of good  $j = 0$  and one unit of good  $j = 1$ .
  - Other households  $i$  have one unit of good  $i + 1$ .
- At the price vector  $\bar{p}$ , household 0 has a budget set

$$c_0^0 + c_0^1 \leq 2,$$

thus chooses  $c_0^0 = c_0^1 = 1$ .

- All other households have budget sets given by

$$c_i^i + c_{i+1}^i \leq 1,$$

- Thus it is optimal for each household  $i > 0$  to consume one unit of the good  $c_{i+1}^i$
- Thus  $\tilde{x}$  is a competitive equilibrium.

## Section 2

# **The Baseline OLG Model**

## Subsection 1

### **Environment**

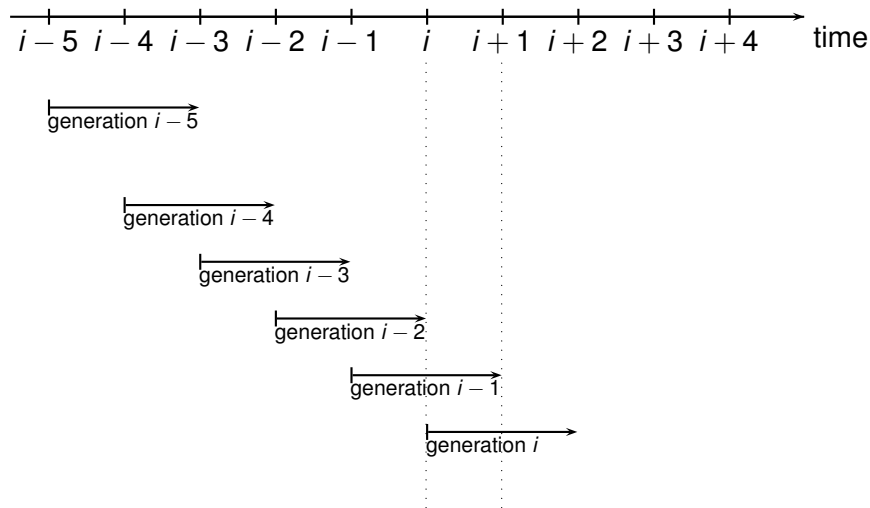
# The Baseline Overlapping Generations Model

- Time is discrete and runs to infinity.
- Each individual lives for two periods.
- Individuals born at time  $t$  live for dates  $t$  and  $t + 1$ .
- Assume a general (separable) utility function for individuals born at date  $t$ ,

$$U_t = u(c_{1t}) + \beta u(c_{2t+1}), \quad (1)$$

- $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfies the usual Assumptions on utility.
- $c_{1t}$ : consumption of the individual born at  $t$  when young (at date  $t$ ).
- $c_{2t+1}$ : consumption when old (at date  $t + 1$ ).
- $\beta \in (0, 1)$  is the discount factor.

## Structure of population across time



## Demographics, Preferences and Technology I

- Exponential population growth,

$$L_t = (1 + n)^t L(0) . \quad (2)$$

- Production side same as before: competitive firms, constant returns to scale aggregate production function, satisfying Assumptions 1 and 2:

$$Y_t = F(K_t, L_t) .$$

- Factor markets are competitive.
- Individuals can only work in the first period and supply one unit of labor inelastically, earning  $w_t$ .



## Demographics, Preferences and Technology II

- Assume that  $\delta = 1$ .
- $k \equiv K/L$ ,  $f(k) \equiv F(k, 1)$ , and the (gross) rate of return to saving, which equals the rental rate of capital, is

$$1 + r_t = R_t = f'(k_t), \quad (3)$$

- As usual, the wage rate is

$$w_t = f(k_t) - k_t f'(k_t). \quad (4)$$

## Subsection 2

# Consumption Decisions

# Consumption Decisions I

- Savings by an individual of generation  $t$ ,  $s_t$ , is determined as a solution to

$$\max_{c_{1t}, c_{2t+1}, s_t} u(c_{1t}) + \beta u(c_{2t+1})$$

subject to

$$c_{1t} + s_t \leq w_t$$

and

$$c_{2t+1} \leq R_{t+1} s_t,$$

- Old individuals rent their savings of time  $t$  as capital to firms at time  $t + 1$ , and receive gross rate of return  $R_{t+1} = 1 + r_{t+1}$ .
- Second constraint incorporates notion that individuals only spend money on their own end of life consumption (no altruism or bequest motive).

## Consumption Decisions II

- No need to introduce  $s_t \geq 0$ , since negative savings would violate second-period budget constraint (given  $c_{2t+1} \geq 0$ ).
- Since  $u(\cdot)$  is strictly increasing, both constraints will hold as equalities.
- Thus first-order condition for a maximum can be written in the familiar form of the consumption Euler equation,

$$u'(c_{1t}) = \beta R_{t+1} u'(c_{2t+1}). \quad (5)$$

- Problem of each individual is strictly concave, so this Euler equation is sufficient.
- Solving for consumption and thus for savings,

$$s_t = s(w_t, R_{t+1}), \quad (6)$$

# Consumption Decisions

- From the FOC and the BC

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t)$$

which implicitly defines

$$s_t = s(w_t, R_{t+1}).$$

- One can show that  $s : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  satisfies  $s_w > 0$ , but  $s_R \geq 0$ .

## Consumption Decisions III

- Total savings in the economy will be equal to

$$S_t = s_t L_t,$$

- $L_t$  denotes the size of generation  $t$ , who are saving for time  $t + 1$ .
- Since capital depreciates fully after use and all new savings are invested in capital,

$$K_{t+1} = L_t s(w_t, R_{t+1}). \quad (7)$$

## Subsection 3

### **Equilibrium**

# Equilibrium I

**Definition** A competitive equilibrium can be represented by a sequence of aggregate capital stocks, individual consumption and factor prices,  $\{K_t, c_{1t}, c_{2t}, R_t, w_t\}_{t=0}^{\infty}$ , such that the factor price sequence  $\{R_t, w_t\}_{t=0}^{\infty}$  is given by (3) and (4), individual consumption decisions  $\{c_{1t}, c_{2t}\}_{t=0}^{\infty}$  are given by (5) and (6), and the aggregate capital stock,  $\{K_t\}_{t=0}^{\infty}$ , evolves according to (7).

- Steady-state equilibrium defined as usual: an equilibrium in which  $k \equiv K/L$  is constant.
- To characterize the equilibrium, divide (7) by  $L_{t+1} = (1 + n) L_t$ ,

$$k_{t+1} = \frac{s(w_t, R_{t+1})}{1 + n}.$$



## Equilibrium II

- Now substituting for  $R_{t+1}$  and  $w_t$  from (3) and (4),

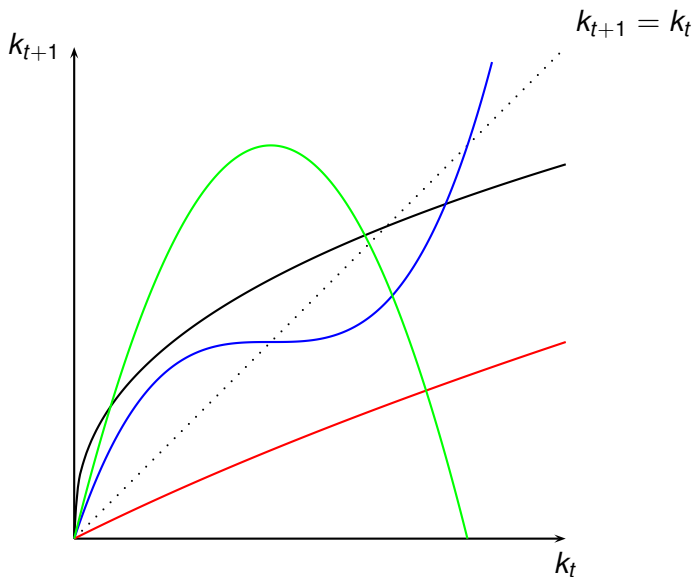
$$k_{t+1} = \frac{s\left(f(k_t) - k(t)f'(k_t), f'(k_{t+1})\right)}{1+n} \quad (8)$$

- This is the fundamental law of motion of the overlapping generations economy.
- A steady state is given by a solution to this equation such that  $k_{t+1} = k_t = k^*$ , i.e.,

$$k^* = \frac{s\left(f(k^*) - k^*f'(k^*), f'(k^*)\right)}{1+n} \quad (9)$$

- Since the savings function  $s(\cdot, \cdot)$  can take any form, the difference equation (8) can lead to quite complicated dynamics, and multiple steady states are possible.

# Possible Laws of Motion



## Subsection 4

### **Special Cases**

## Restrictions on Utility and Production Functions I

- Suppose that the utility functions take the familiar CRRA form:

$$U_t = \frac{c_{1t}^{1-\theta} - 1}{1-\theta} + \beta \left( \frac{c_{2t+1}^{1-\theta} - 1}{1-\theta} \right), \quad (10)$$

where  $\theta > 0$  and  $\beta \in (0, 1)$ .

- Technology is Cobb-Douglas,

$$f(k) = k^\alpha$$

- The rest of the environment is as described above.
- The CRRA utility simplifies the first-order condition for consumer optimization,

$$\frac{c_{2t+1}}{c_{1t}} = (\beta R_{t+1})^{1/\theta}.$$

## Restrictions on Utility and Production Functions II

- This Euler equation can be alternatively expressed in terms of savings as

$$s_t^{-\theta} \beta R_{t+1}^{1-\theta} = (w_t - s_t)^{-\theta}, \quad (11)$$

- Gives the following equation for the saving rate:

$$s_t = \frac{w_t}{\psi_{t+1}}, \quad (12)$$

where

$$\psi_{t+1} \equiv [1 + \beta^{-1/\theta} R_{t+1}^{-(1-\theta)/\theta}] > 1,$$

- Ensures that savings are always less than earnings.

## Restrictions on Utility and Production Functions III

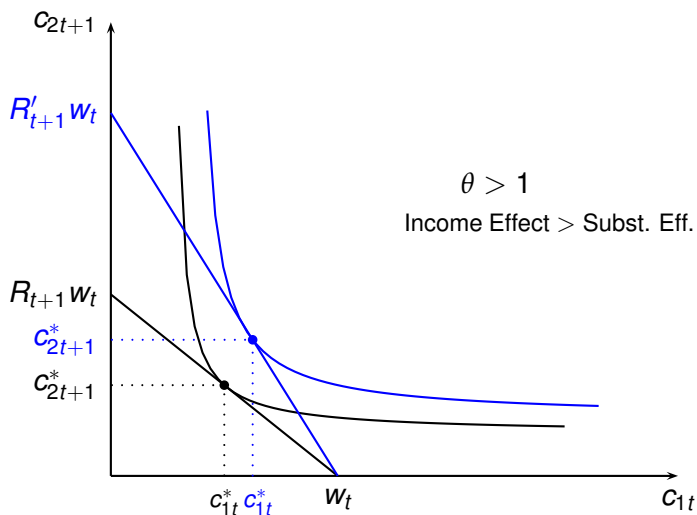
- The impact of factor prices on savings is summarized by the following and derivatives:

$$s_w \equiv \frac{\partial s_t}{\partial w_t} = \frac{1}{\psi_{t+1}} \in (0, 1),$$

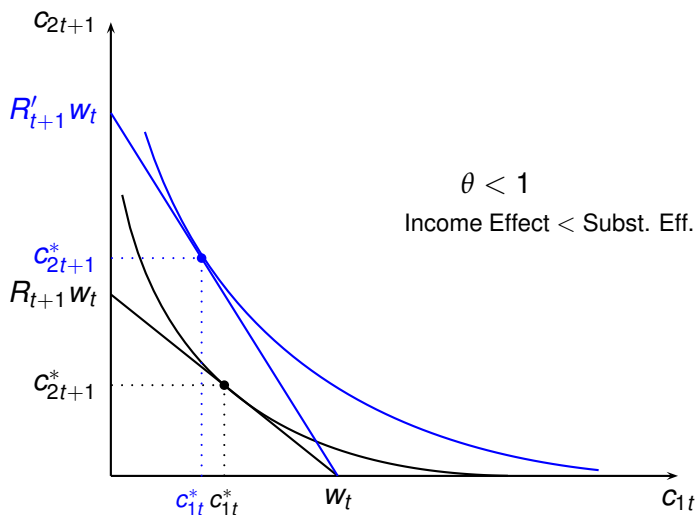
$$s_R \equiv \frac{\partial s_t}{\partial R_{t+1}} = \left( \frac{1 - \theta}{\theta} \right) (\beta R_{t+1})^{-1/\theta} \frac{s_t}{\psi_{t+1}}.$$

- Since  $\psi_{t+1} > 1$ , we also have that  $0 < s_w < 1$ .
- Moreover, in this case  $s_R < 0$  if  $\theta > 1$ ,  $s_R > 0$  if  $\theta < 1$ , and  $s_R = 0$  if  $\theta = 1$ .
- Reflects counteracting influences of income and substitution effects.
- Case of  $\theta = 1$  (log preferences) is of special importance, may deserve to be called the *canonical overlapping generations model*.

# RRA coefficient, income and substitution effects I

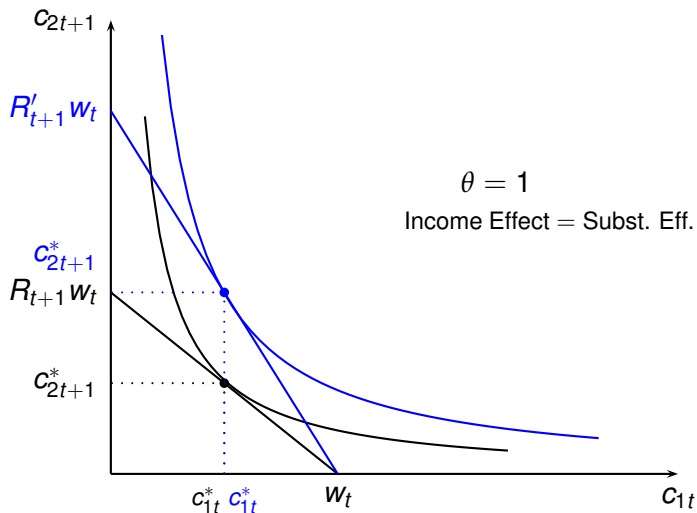


# RRA coefficient, income and substitution effects I





# RRA coefficient, income and substitution effects I



# Restrictions on Utility and Production Functions IV

- Equation (8) implies

$$\begin{aligned} k_{t+1} &= \frac{s_t}{(1+n)} \\ &= \frac{w_t}{(1+n)\psi_{t+1}}, \end{aligned} \quad (13)$$

- Or more explicitly,

$$k_{t+1} = \frac{f(k_t) - k_t f'(k_t)}{(1+n) [1 + \beta^{-1/\theta} f'(k_{t+1})^{-(1-\theta)/\theta}]} \quad (14)$$

- The steady state then involves a solution to the following implicit equation:

$$k^* = \frac{f(k^*) - k^* f'(k^*)}{(1+n) [1 + \beta^{-1/\theta} f'(k^*)^{-(1-\theta)/\theta}]}.$$

## Restrictions on Utility and Production Functions V

- Now using the Cobb-Douglas formula, steady state is the solution to the equation

$$(1+n) \left[ 1 + \beta^{-1/\theta} \left( \alpha (k^*)^{\alpha-1} \right)^{(\theta-1)/\theta} \right] = (1-\alpha)(k^*)^{\alpha-1}. \quad (15)$$

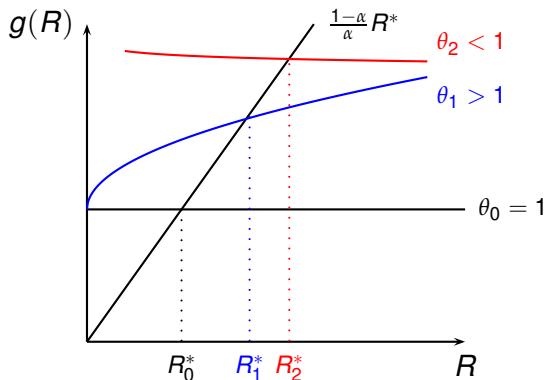
- For simplicity, define  $R^* \equiv \alpha (k^*)^{\alpha-1}$  as the marginal product of capital in steady-state, in which case, (15) can be rewritten as

$$(1+n) \left[ 1 + \beta^{-1/\theta} (R^*)^{(\theta-1)/\theta} \right] = \frac{1-\alpha}{\alpha} R^*. \quad (16)$$

- Steady-state value of  $R^*$ , and thus  $k^*$ , can now be determined from equation (16), which always has a unique solution.

# Steady State

- Notice that the steady state depends on  $\theta$  (compare with Ramsey!)
- Existence and uniqueness follow from figure

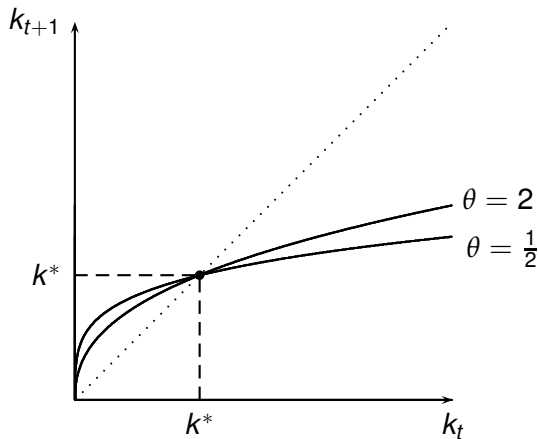


## RRA coefficient, steady state and dynamics

- To investigate the stability, substitute for the Cobb-Douglas production function in (14)

$$k_{t+1} = \frac{(1 - \alpha) k_t^\alpha}{(1 + n) [1 + \beta^{-1/\theta} (\alpha k_{t+1}^{\alpha-1})^{-(1-\theta)/\theta}]} \quad (17)$$

# RRA coefficient, steady state and dynamics



# Stability

Using (17) we can define

$$k_t = \left[ \frac{1+n}{1-\alpha} \left( k_{t+1} + \beta^{-1/\theta} \alpha^{(\theta-1)/\theta} k_{t+1}^{(1-\alpha)/\theta+\alpha} \right) \right]^{1/\alpha} \equiv \Gamma(k_{t+1}),$$

then

$$\frac{dk_t}{dk_{t+1}} = \frac{1}{\alpha} \Gamma(k_{t+1})^{1-\alpha} \left[ \frac{1+n}{1-\alpha} \left( 1 + \left( \frac{1-\alpha}{\theta} + \alpha \right) \beta^{-1/\theta} \alpha^{(\theta-1)/\theta} k_{t+1}^{(1-\alpha)/\theta+\alpha-1} \right) \right]$$

and at the steady state  $k^* = \Gamma(k^*)$ , so

$$\left. \frac{dk_t}{dk_{t+1}} \right|_{k^*} = \frac{1}{\alpha} k^{*-\alpha} \left[ \frac{1+n}{1-\alpha} \left( k^* + \left( \frac{1-\alpha}{\theta} + \alpha \right) \beta^{-1/\theta} \alpha^{(\theta-1)/\theta} k^{*(1-\alpha)/\theta+\alpha} \right) \right]$$

# Stability

- $\theta \leq 1 \implies \frac{1-\alpha}{\theta} + \alpha \geq 1$  so that

$$\left. \frac{dk_t}{dk_{t+1}} \right|_{k^*} \geq \frac{1}{\alpha} k^{*-\alpha} \Gamma(k^*) \alpha = \frac{1}{\alpha} \implies \left. \frac{dk_{t+1}}{dk_t} \right|_{k^*} \leq \alpha$$

- $\theta > 1 \implies \frac{1-\alpha}{\theta} + \alpha < 1$  so that

$$\begin{aligned} \left. \frac{dk_t}{dk_{t+1}} \right|_{k^*} &> \frac{1}{\alpha} k^{*-\alpha} \left( \frac{1-\alpha}{\theta} + \alpha \right) \Gamma(k^*) \alpha = \frac{1}{\alpha} \left( \frac{1-\alpha}{\theta} + \alpha \right) \implies \\ \left. \frac{dk_{t+1}}{dk_t} \right|_{k^*} &\leq \frac{1}{\frac{1-\alpha}{\alpha\theta} + 1} < 1 \end{aligned}$$



## Restrictions on Utility and Production Functions VI

**Proposition** In the overlapping-generations model with two-period lived households, Cobb-Douglas technology and CRRA preferences, there exists a unique steady-state equilibrium with the capital-labor ratio  $k^*$  given by (15), this steady-state equilibrium is globally stable for all  $k_0 > 0$ .

- In this particular (well-behaved) case, equilibrium dynamics are very similar to the basic Solow model
- Figure shows that convergence to the unique steady-state capital-labor ratio,  $k^*$ , is monotonic.

## Section 3

# Canonical OLG Model

## Subsection 1

# Canonical Model

## Canonical Model I

- Even the model with CRRA utility and Cobb-Douglas production function is relatively messy.
- Many of the applications use log preferences ( $\theta = 1$ ).
- Income and substitution effects exactly cancel each other: changes in the interest rate (and thus in the capital-labor ratio of the economy) have no effect on the saving rate.
- Structure of the equilibrium is essentially identical to the basic Solow model.
- Utility of the household and generation  $t$  is,

$$U_t = \log c_{1t} + \beta \log c_{2t+1}, \quad (18)$$

- $\beta \in (0, 1)$  (even though  $\beta \geq 1$  could be allowed).
- Again  $f(k) = k^\alpha$ .

## Canonical Model II

- Consumption Euler equation:

$$\frac{c_{2t+1}}{c_{1t}} = \beta R_{t+1} \implies c_{1t} = \frac{1}{1 + \beta} w_t$$

- Savings should satisfy the equation

$$s_t = \frac{\beta}{1 + \beta} w_t, \quad (19)$$

- Constant saving rate, equal to  $\beta / (1 + \beta)$ , out of labor income for each individual.

## Canonical Model III

- Combining this with the capital accumulation equation (8),

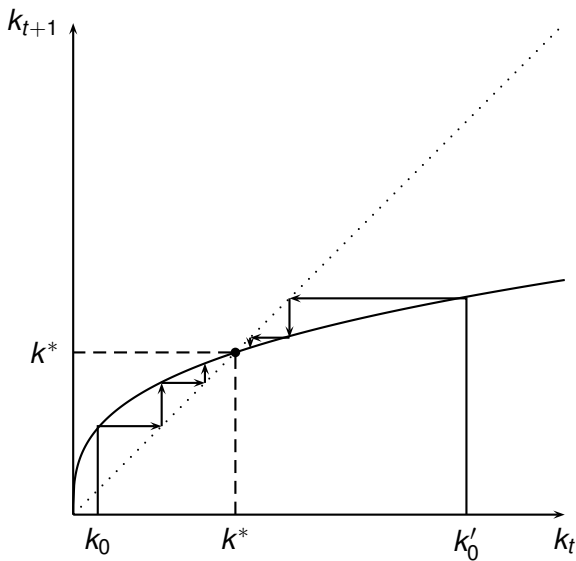
$$\begin{aligned}
 k_{t+1} &= \frac{s_t}{(1+n)} \\
 &= \frac{\beta w_t}{(1+n)(1+\beta)} \\
 &= \frac{\beta(1-\alpha)k_t^\alpha}{(1+n)(1+\beta)},
 \end{aligned}$$

- Second line uses (19) and last uses that, given competitive factor markets,  $w_t = (1-\alpha)[k_t]^\alpha$ .
- There exists a unique steady state with

$$k^* = \left[ \frac{\beta(1-\alpha)}{(1+n)(1+\beta)} \right]^{\frac{1}{1-\alpha}}. \quad (20)$$

- Equilibrium dynamics are identical to those of the basic Solow model and monotonically converge to  $k^*$ .

# Equilibrium dynamics in canonical OLG model



## Canonical Model IV

**Proposition** In the canonical overlapping generations model with log preferences and Cobb-Douglas technology, there exists a unique steady state, with capital-labor ratio  $k^*$  given by (20). Starting with any  $k_0 \in (0, k^*)$ , equilibrium dynamics are such that  $k_t \uparrow k^*$ , and starting with any  $k'_0 > k^*$ , equilibrium dynamics involve  $k_t \downarrow k^*$ .



## Section 4

# Overaccumulation and Policy

## Subsection 1

# Overaccumulation and Pareto Optimality

# Overaccumulation I

- Compare the overlapping-generations equilibrium to the choice of a social planner wishing to maximize a weighted average of all generations' utilities.
- Suppose that the social planner maximizes

$$\sum_{t=0}^{\infty} \zeta_t U_t$$

- $\zeta_t$  is the discount factor of the social planner, which reflects how she values the utilities of different generations.

# Overaccumulation II

- Substituting from (1), this implies:

$$\sum_{t=0}^{\infty} \xi_t (u(c_{1t}) + \beta u(c_{2t+1}))$$

subject to the resource constraint

$$F(K_t, L_t) = K_{t+1} + L_t c_{1t} + L_{t-1} c_{2t}.$$

- Dividing this by  $L_t$  and using (2),

$$f(k_t) = (1 + n) k_{t+1} + c_{1t} + \frac{c_{2t}}{1 + n}.$$

## Overaccumulation IIB

- Assume  $\sum \tilde{\xi}_t < \infty$  (Why?)
- Clearly, Assumption 6.1N-6.5N in the book hold and we can apply our dynamic programming theorems...

$$\begin{aligned} V(t, k_t) = & \max_{(c_{1t}, k_{t+1})} \{ \tilde{\xi}_t u(c_{1t}) \\ & + \tilde{\xi}_{t-1} \beta u((1+n)f(k_t) - (1+n)^2 k_{t+1} - (1+n)c_{1t}) \\ & + V(t+1, k_{t+1}) \} \end{aligned}$$

## Overaccumulation IIC

Euler equation implies

$$\begin{aligned}\tilde{\zeta}_t u'(c_{1t}) &= \beta \tilde{\zeta}_{t-1} (1+n) u'(c_{2t}) \\ (1+n)^2 \beta \tilde{\zeta}_{t-1} u'(c_{2t}) &= V_k(t+1, k_{t+1})\end{aligned}$$

and the envelope theorem

$$V_k(t+1, k_{t+1}) = \beta \tilde{\zeta}_t (1+n) f'(k_{t+1}) u'(c_{2t+1})$$

which together generate

$$\begin{aligned}\tilde{\zeta}_t u'(c_{1t}) &= \beta \tilde{\zeta}_{t-1} (1+n) u'(c_{2t}) = \frac{V_k(t+1, k_{t+1})}{1+n} \\ &= \beta \tilde{\zeta}_t f'(k_{t+1}) u'(c_{2t+1})\end{aligned}$$

- Transversality condition:  $\lim_{t \rightarrow \infty} k_t^* \beta^t \tilde{\zeta}_{t-1} f'(k_t) u'(c_{2t}) = 0$ .

## Overaccumulation III

- Social planner's maximization problem then implies the following first-order necessary condition:

$$u'(c_{1t}) = \beta f'(k_{t+1}) u'(c_{2t+1}).$$

- Since  $R_{t+1} = f'(k_{t+1})$ , this is identical to (5).
- Not surprising: allocate consumption of a given individual in exactly the same way as the individual himself would do.
- No “market failures” in the over-time allocation of consumption at given prices.
- However, the allocations across generations may differ from the competitive equilibrium: planner is giving different weights to different generations
- In particular, competitive equilibrium is **Pareto suboptimal** when  $k^* > k_{gold}$ ,

## Overaccumulation IV

- When  $k^* > k_{gold}$ , reducing savings can increase consumption for every generation.
- More specifically, note that in steady state

$$\begin{aligned} f(k^*) - (1+n)k^* &= c_1^* + (1+n)^{-1} c_2^* \\ &\equiv c^*, \end{aligned}$$

- First line follows by national income accounting, and second defines  $c^*$  (aggregate per capita consumption).
- Therefore

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n)$$

- $k_{gold}$  is defined as

$$f'(k_{gold}) = 1 + n \left( = (n + g + \delta) \right).$$



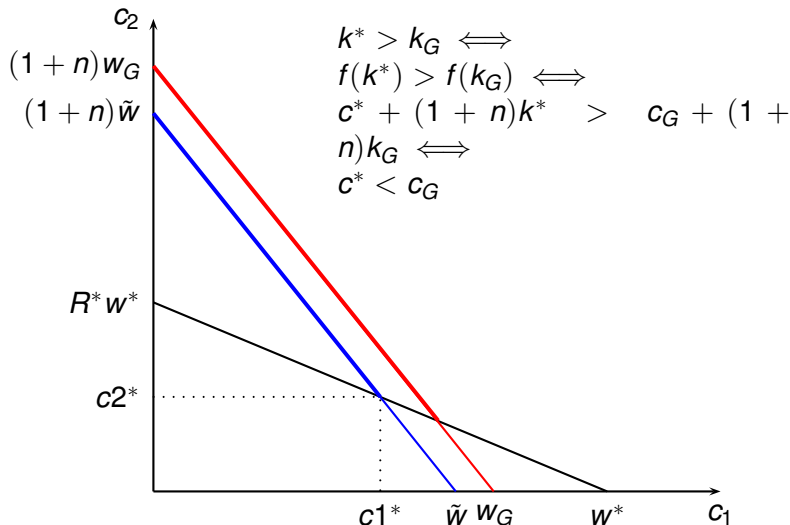
## Overaccumulation V

- Now if  $k^* > k_{gold}$ , then  $\partial c^* / \partial k^* < 0$ : reducing savings can increase (total) consumption for everybody.
- If this is the case, the economy is referred to as *dynamically inefficient*—it involves overaccumulation.
- Another way of expressing dynamic inefficiency is that

$$r^* < n,$$

- Recall in infinite-horizon Ramsey economy, transversality condition required that  $r > g + n$ .
- Dynamic inefficiency arises because of the heterogeneity inherent in the overlapping generations model, which removes the transversality condition.
- Suppose we start from steady state at time  $T$  with  $k^* > k_{gold}$ .

# Inefficiency: graphical analysis



## Overaccumulation VI

- Consider the following variation: change next period's capital stock by  $-\Delta k$ , where  $\Delta k > 0$ , and from then on, we immediately move to a new steady state (clearly feasible).
- This implies the following changes in consumption levels:

$$\Delta c_t = (1 + n) \Delta k > 0$$

$$\Delta c_t = - (f'(k^* - \Delta k) - (1 + n)) \Delta k \text{ for all } t > T$$

- The first expression reflects the direct increase in consumption due to the decrease in savings.
- In addition, since  $k^* > k_{gold}$ , for small enough  $\Delta k$ ,  $f'(k^* - \Delta k) - (1 + n) < 0$ , thus  $\Delta c(t) > 0$  for all  $t \geq T$ .
- The increase in consumption for each generation can be allocated equally during the two periods of their lives, thus necessarily increasing the utility of all generations.

# Pareto Optimality and Suboptimality in the OLG Model

**Proposition** In the baseline overlapping-generations economy, the competitive equilibrium is not necessarily Pareto optimal. More specifically, whenever  $r^* < n$  and the economy is dynamically inefficient, it is possible to reduce the capital stock starting from the competitive steady state and increase the consumption level of all generations.

- Pareto inefficiency of the competitive equilibrium is intimately linked with *dynamic inefficiency*.

# Interpretation

- Intuition for dynamic inefficiency:
  - Individuals who live at time  $t$  face prices determined by the capital stock with which they are working.
  - Capital stock is the outcome of actions taken by previous generations.
  - Pecuniary externality from the actions of previous generations affecting welfare of current generation.
  - Pecuniary externalities typically second-order and do not matter for welfare.
  - But not when an infinite stream of newborn agents joining the economy are affected.
  - It is possible to rearrange in a way that these pecuniary externalities can be exploited.

## Further Intuition

- Complementary intuition:
  - Dynamic inefficiency arises from overaccumulation.
  - Results from current young generation needs to save for old age.
  - However, the more they save, the lower is the rate of return and may encourage to save even more.
  - Effect on future rate of return to capital is a pecuniary externality on next generation
  - If alternative ways of providing consumption to individuals in old age were introduced, overaccumulation could be ameliorated.

## Subsection 2

### **Role of Social Security**

## Role of Social Security in Capital Accumulation

- Social Security as a way of dealing with overaccumulation
- Fully-funded system: young make contributions to the Social Security system and their contributions are paid back to them in their old age.
- Unfunded system or a *pay-as-you-go*: transfers from the young directly go to the current old.
- Pay-as-you-go (unfunded) Social Security discourages aggregate savings.
- With dynamic inefficiency, discouraging savings may lead to a Pareto improvement.



## Subsection 3

### **Fully Funded Social Security**

# Fully Funded Social Security I

- Government at date  $t$  raises some amount  $d_t$  from the young, funds are invested in capital stock, and pays workers when old  $R_{t+1} d_t$ .
- Thus individual maximization problem is,

$$\max_{c_{1t}, c_{2t+1}, s_t} u(c_{1t}) + \beta u(c_{2t+1})$$

subject to

$$c_{1t} + s_t + d_t \leq w_t$$

and

$$c_{2t+1} \leq R_{t+1} (s_t + d_t),$$

for a given choice of  $d_t$  by the government.

- Notice that now the total amount invested in capital accumulation is  $s_t + d_t = (1 + n) k_{t+1}$ .

## Fully Funded Social Security II

- Given the solution when  $d_t = 0$  for all  $t$ ,  $(\tilde{c}_{1t}, \tilde{c}_{2t+1})$ , (original problem), agents choose to save  $s_t = w_t - d_t - \tilde{c}_{1t}$
- No longer the case that individuals will always choose  $s_t > 0$ .
- As long as  $s_t$  is free, whatever  $\{d_t\}_{t=0}^{\infty}$ , the competitive equilibrium applies.
- When  $s_t \geq 0$  is imposed as a constraint, competitive equilibrium applies if given  $\{d_t\}_{t=0}^{\infty}$ , privately-optimal  $\{s_t\}_{t=0}^{\infty}$  is such that  $s_t > 0$  for all  $t$ .

## Fully Funded Social Security III

**Proposition** Consider a fully funded Social Security system in the above-described environment whereby the government collects  $d_t$  from young individuals at date  $t$ .

- 1 Suppose that  $s_t \geq 0$  for all  $t$ . If given the feasible sequence  $\{d_t\}_{t=0}^{\infty}$  of Social Security payments, the utility-maximizing sequence of savings  $\{s_t\}_{t=0}^{\infty}$  is such that  $s_t > 0$  for all  $t$ , then the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.
  - 2 Without the constraint  $s_t \geq 0$ , given any feasible sequence  $\{d_t\}_{t=0}^{\infty}$  of Social Security payments, the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.
- Moreover, even when there is the restriction that  $s_t \geq 0$ , a funded Social Security program cannot lead to the Pareto improvement.

## Subsection 4

# Unfunded Social Security

# Unfunded Social Security I

- Government collects  $d_t$  from the young at time  $t$  and distributes to the current old with per capita transfer  $b_t = (1 + n) d_t$
- Individual maximization problem becomes

$$\max_{c_{1t}, c_{2t+1}, s_t} u(c_{1t}) + \beta u(c_{2t+1})$$

subject to

$$c_{1t} + s_t + d_t \leq w_t$$

and

$$c_{2t+1} \leq R_{t+1} s_t + (1 + n) d_{t+1},$$

for a given feasible sequence of Social Security payment levels  $\{d_t\}_{t=0}^{\infty}$ .

- Rate of return on Social Security payments is  $n$  rather than  $r_{t+1} = R_{t+1} - 1$ , because unfunded Social Security is a pure transfer system.

# Unfunded Social Security

- Lifetime budget becomes

$$c_{1t} + \frac{c_{2t+1}}{R_{t+1}} + d_t - \frac{1+n}{R_{t+1}}d_{t+1} = w_t$$

so agent will be better off as long as

$$d_t - \frac{1+n}{R_{t+1}}d_{t+1} < 0 \quad (21)$$

- agent chooses  $s_t$  such that

$$u'(w_t - s_t - d_t) = \beta u'(R_{t+1}s_t + (1+n)d_{t+1})$$

- If  $d_t = d_{t+1} = d$ , and agent can choose  $d$ , then  $s_t = 0$  if  $R_{t+1} < (1+n)!$

## Unfunded Social Security II

- Only  $s_t$ —rather than  $s_t$  plus  $d_t$  as in the funded scheme—goes into capital accumulation.
- It is possible that  $s_t$  will change in order to compensate, but such an offsetting change does not typically take place.
- Thus unfunded Social Security reduces capital accumulation.
- Discouraging capital accumulation can have negative consequences for growth and welfare.
- In fact, empirical evidence suggests that there are many societies in which the level of capital accumulation is suboptimally low.
- But here reducing aggregate savings may be good when the economy exhibits dynamic inefficiency.



## Unfunded Social Security III

**Proposition** Consider the above-described overlapping generations economy and suppose that the decentralized competitive equilibrium is dynamically inefficient. Then there exists a feasible sequence of unfunded Social Security payments  $\{d_t\}_{t=0}^{\infty}$  which will lead to a competitive equilibrium starting from any date  $t$  that Pareto dominates the competitive equilibrium without Social Security.

- Similar to way in which the Pareto optimal allocation was decentralized in the example economy above.
- Social Security is transferring resources from future generations to initial old generation.
- But with *no* dynamic inefficiency, any transfer of resources (and any unfunded Social Security program) would make some future generation worse-off. (follows from (21))

## Section 5

# OLG with Impure Altruism

## Subsection 1

# **Impure Altruism**

# Overlapping Generations with Impure Altruism I

- Exact form of altruism within a family matters for whether the representative household would provide a good approximation.

$$U(c_t, b_t) = u(c_t) + U^b(b_t)$$

$$U(c_t, b_t) = u(c_t) + \beta V(b_t + w)$$

- Parents care about certain dimensions of the consumption vector of their offspring instead of their total utility or “impure altruism.”
- A particular type, “warm glow preferences”: parents derive utility from their bequest.

# Overlapping Generations with Impure Altruism I

- Production side given by the standard neoclassical production function, satisfying Assumptions 1 and 2,  $f(k)$ .
- Economy populated by a continuum of individuals of measure 1.
- Each individual lives for two periods, childhood and adulthood.
- In second period of her life, each individual begets an offspring, works and then her life comes to an end.
- No consumption in childhood (or incorporated in the parent's consumption).

## Overlapping Generations with Impure Altruism II

- No new households, so population is constant at 1.
- Each individual supplies 1 unit of labor inelastically during adulthood.
- Preferences of individual  $(i, t)$ , who reaches adulthood at time  $t$ , are

$$\log(c_{it}) + \beta \log(b_{it}), \quad (22)$$

where  $c_{it}$  denotes the consumption of this individual and  $b_{it}$  is bequest to her offspring.

- Offspring starts the following period with the bequest, rents this out as capital to firms, supplies labor, begets her own offspring, and makes consumption and bequests decisions.
- Capital fully depreciates after use.

## Overlapping Generations with Impure Altruism III

- Maximization problem of a typical individual can be written as

$$\max_{c_{it}, b_{it}} \log(c_{it}) + \beta \log(b_{it}), \quad (23)$$

subject to

$$c_{it} + b_{it} \leq y_{it} \equiv w_t + R_t b_{it-1}, \quad (24)$$

where  $y_{it}$  denotes the income of this individual.

- Equilibrium wage rate and rate of return on capital

$$w_t = f(k_t) - k_t f'(k_t) \quad (25)$$

$$R_t = f'(k_t) \quad (26)$$

- Capital-labor ratio at time  $t + 1$  is:

$$k_{t+1} = \int_0^1 b_{it} di, \quad (27)$$

## Overlapping Generations with Impure Altruism IV

- Measure of workers is 1, so that the capital stock and capital-labor ratio are identical.
- Denote the distribution of consumption and bequests across households at time  $t$  by  $[c_{it}]_{i \in [0,1]}$  and  $[b_{it}]_{i \in [0,1]}$ .
- Assume the economy starts with the distribution of wealth (bequests) at time 0 given by  $[b_{i0}]_{i \in [0,1]}$ , which satisfies  $\int_0^1 b_{i0} di > 0$ .

**Definition** An equilibrium in this overlapping generations economy with warm glow preferences is a sequence of consumption and bequest levels for each household,  $\left\{ [c_{it}]_{i \in [0,1]}, [b_{it}]_{i \in [0,1]} \right\}_{t=0}^{\infty}$ , that solve (23) subject to (24), a sequence of capital-labor ratios,  $\{k_t\}_{t=0}^{\infty}$ , given by (27) with some initial distribution of bequests  $[b_{i0}]_{i \in [0,1]}$ , and sequences of factor prices,  $\{w_t, R_t\}_{t=0}^{\infty}$ , that satisfy (25) and (26).



## Overlapping Generations with Impure Altruism V

- Solution of (23) subject to (24) is straightforward because of the log preferences,

$$\begin{aligned} b_{it} &= \frac{\beta}{1 + \beta} y_{it} \\ &= \frac{\beta}{1 + \beta} [w_t + R_t b_{it-1}] , \end{aligned} \tag{28}$$

for all  $i$  and  $t$ .

- Bequest levels will follow non-trivial dynamics.
- $b_{it}$  can alternatively be interpreted as “wealth” level: distribution of wealth that will evolve endogenously.
- This evolution will depend on factor prices.
- To obtain factor prices, aggregate bequests to obtain the capital-labor ratio of the economy via equation (27).

## Overlapping Generations with Impure Altruism VI

- Integrating (28) across all individuals,

$$\begin{aligned}k_{t+1} &= \int_0^1 b_{it} di \\&= \frac{\beta}{1+\beta} \int_0^1 [w_t + R_t b_{it-1}] di \\&= \frac{\beta}{1+\beta} f(k_t) .\end{aligned}\tag{29}$$

- The last equality follows from the fact that  $\int_0^1 b_{it-1} di = k_t$  and because by Euler's Theorem,  $w_t + R_t k_t = f(k_t)$ .
- Thus dynamics are straightforward and again closely resemble Solow growth model.
- Moreover dynamics do *not* depend on the distribution of bequests or income across households.

## Overlapping Generations with Impure Altruism VII

- Solving for the steady-state equilibrium capital-labor ratio from (29),

$$k^* = \frac{\beta}{1 + \beta} f(k^*), \quad (30)$$

- Uniquely defined and strictly positive in view of Assumptions 1 and 2.
- Moreover, equilibrium dynamics again involve monotonic convergence to this unique steady state.
- We know that  $k_t \rightarrow k^*$ , so the ultimate bequest dynamics are given by steady-state factor prices.
- Let these be denoted by  $w^* = f(k^*) - k^* f'(k^*)$  and  $R^* = f'(k^*)$ .
- Once the economy is in the neighborhood of the steady-state capital-labor ratio,  $k^*$ ,

$$b_{it} = \frac{\beta}{1 + \beta} [w^* + R^* b_{it-1}].$$

## Overlapping Generations with Impure Altruism VIII

- When  $R^* < (1 + \beta) / \beta$ , starting from any level  $b_{it}$  will converge to a unique bequest (wealth) level

$$b^* = \frac{\beta w^*}{1 + \beta(1 - R^*)}. \quad (31)$$

- Moreover, it can be verified that  $R^* < (1 + \beta) / \beta$ ,

$$\begin{aligned} R^* &= f'(k^*) \\ &< \frac{f(k^*)}{k^*} \\ &= \frac{1 + \beta}{\beta}, \end{aligned}$$

- Second line exploits the strict concavity of  $f(\cdot)$  and the last line uses the definition of  $k^*$  from (30).

## Overlapping Generations with Impure Altruism IX

**Proposition** Consider the overlapping generations economy with warm glow preferences described above. In this economy, there exists a unique competitive equilibrium. In this equilibrium the aggregate capital-labor ratio is given by (29) and monotonically converges to the unique steady-state capital-labor ratio  $k^*$  given by (30). The distribution of bequests and wealth ultimately converges towards full equality, with each individual having a bequest (wealth) level of  $b^*$  given by (31) with  $w^* = f(k^*) - k^* f'(k^*)$  and  $R^* = f'(k^*)$ .

## Section 6

# Overlapping Generations with Perpetual Youth

## Subsection 1

### **Perpetual Youth in Discrete time**

## Perpetual Youth in Discrete time I

- Production side given by the standard neoclassical production function, satisfying Assumptions 1 and 2,  $f(k)$ .
- Individuals are finitely lived and they are not aware of when they will die.
- Each individual faces a constant probability of death equal to  $\nu \in (0, 1)$ . This is a simplification, since likelihood of survival is not constant.
- This last assumption implies that individuals have an expected lifespan of  $\frac{1}{\nu} < \infty$  periods.
- Expected lifetime of an individual in this model is:

$$\text{Expected life} = \nu + 2(1 - \nu)\nu + 3(1 - \nu)^2\nu + \dots = \frac{1}{\nu} \quad (32)$$

This equation captures the fact that with probability  $\nu$  the individual will have a total life of length 1, with probability  $(1 - \nu)\nu$ , she will have a life of length 2, and so on.



## Perpetual Youth in Discrete time II

- Perpetual youth: even though each individual has a finite expected life, all individuals who have survived up to a certain date have exactly the same expectation of further life.
- Each individual supplies 1 unit of labor inelastically each period she is alive.
- Expected utility of an individual with a pure discount factor  $\beta$  is given by

$$\sum_{t=0}^{\infty} (\beta(1 - v))^t u(c_t),$$

where  $u(\cdot)$  is a standard instantaneous utility function satisfying Assumption 3, with the additional normalization that  $u(0) = 0$ .

## Perpetual Youth in Discrete time III

- Individual  $i$ 's flow budget constraint is

$$a_{it+1} = (1 + r_t)a_{it} - c_{it} + w_t + z_{it} \quad (33)$$

where  $z_{it}$  reflects transfers to the individual. Since individuals face an uncertain time of death, there may be accidental bequests. Government can collect and redistribute. However, this needs  $a_{it} \geq 0$  to avoid debts. An alternative is to introduce life insurance or annuity markets. Assuming competitive life insurance firms, their profits will be

$$\pi(a, t) = va - (1 - v)z(a).$$

With free entry,  $\pi(a, t) = 0$ , thus

$$z(a_t) = \frac{v}{1 - v} a_t. \quad (34)$$

## Perpetual Youth in Discrete time IV

- Demographics: There is an exogenous natural force toward decreasing population ( $\nu > 0$ ). But there also new agents who are born into a dynasty. The evolution of the total population is given by

$$L_{t+1} = (1 + n - \nu)L_t, \quad (35)$$

with the initial value  $L_0 = 1$ , and with  $n > \nu$ .

- Pattern of demographics in this economy: at  $t > 0$  there will be:
  - 1-year-olds =  $nL_{t-1} = n(1 + n - \nu)^{t-1}L_0$ .
  - 2-year-olds =  $nL_{t-2}(1 - \nu) = n(1 + n - \nu)^{t-2}(1 - \nu)L_0$ .
  - k-year-olds =  $nL_{t-k}(1 - \nu)^{k-1} = n(1 + n - \nu)^{t-k}(1 - \nu)^{k-1}L_0$ .

## Perpetual Youth in Discrete time V

- Maximization problem of a typical individual of generation  $\tau$  can be written as

$$\max_{\{c_{t|\tau}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\beta(1-\nu))^t u(c_{t|\tau}), \quad (36)$$

subject to

$$a_{t+1|\tau} = \left(1 + r_t + \frac{\nu}{1-\nu}\right) a_{t|\tau} - c_{t|\tau} + w_t \quad (37)$$

- Equilibrium wage rate and rate of return on capital

$$w_t = f(k_t) - k_t f'(k_t) \quad (38)$$

$$R_t = f'(k_t) \quad (39)$$

## Perpetual Youth in Discrete time VI

**Definition** An equilibrium in this overlapping generations economy with perpetual youth is a sequence of capital stocks, wage rates, and rental rates of capital,  $\{K_t, w_t, R_t\}_{t=0}^{\infty}$ , and paths of consumption for each generation,  $\{c_{t|\tau}\}_{t=0, \tau \leq t}^{\infty}$ , such that each individual maximizes utility, and the time path of factor prices,  $\{w_t, R_t\}_{t=0}^{\infty}$ , is such that given the time paths of the capital stock and labor,  $\{K_t, L_t\}_{t=0}^{\infty}$ , all markets clear.

## Perpetual Youth in Discrete time VII

- Solution of (36) subject to (37), using Bellman equation, and assuming logarithmic utility function:

$$V(a_{t|\tau}) = \max_{a_{t+1|\tau}} \log [c_{t|\tau}] + \beta(1 - \nu) V(a_{t+1|\tau})$$

The Euler equation is:

$$c_{t+k|\tau} = [\beta(1 - \nu)]^k \sigma_t^k c_{t|\tau}$$

where  $\sigma_t^k = \prod_{j=1}^k (1 + r_{t+j} + \frac{\nu}{1-\nu})$  and the transversality condition is

$$\lim_{t \rightarrow \infty} [\beta(1 - \nu)]^k \frac{a_{t|\tau}}{c_{t|\tau}} = 0$$

## Perpetual Youth in Discrete time VIII

- Consider the individual born in period  $\tau$  during period  $t$  has assets  $a_{t|\tau}$ , it is true that

$$\sum_{k=0}^{\infty} \frac{c_{t+k|\tau}}{\sigma_t^k} = \sum_{k=0}^{\infty} \frac{w_{t+k}}{\sigma_t^k} + a_{t|\tau}$$

where we use the No Ponzi condition given by  $\lim_{t \rightarrow \infty} \frac{a_{t|\tau}}{\sigma_0^t} = 0$ .  
Using the Euler equation, we obtain

$$\sum_{k=0}^{\infty} (\beta(1 - \nu))^k c_{t|\tau} = \frac{1}{\beta(1 - \nu)} c_{t|\tau} = (\omega_t + a_{t|\tau})$$

or

$$c_{t|\tau} = \beta(1 - \nu)(\omega_t + a_{t|\tau})$$

## Perpetual Youth in Discrete time IX

- Average consumption:

$$c_t = \sum_{\tau=0}^t \frac{L_{t|\tau}}{L_t} c_{t|\tau} = \beta(1 - \nu)(\omega_t + \bar{a}_t) \quad (40)$$

where  $\bar{a}_t = \sum_{\tau=0}^t \frac{L_{t|\tau}}{L_t} a_{t|\tau}$ .

- Additionally, the average stock of capital is

$$\frac{K_{t+1}}{L_{t+1}} \equiv k_{t+1} = \sum_{\tau=0}^t \frac{L_{t|\tau}}{L_{t+1}} a_{t|\tau} = \frac{\bar{a}_t}{1 + n - \nu}$$

or

$$k_{t+1} = \frac{f(k_t) - c_t + (1 - \delta)k_t}{1 + n - \nu}$$



## Perpetual Youth in Discrete time X

- In a steady state,  $c_{t|\tau} = c_{t+1|\tau}$  i.e.

$$\left[ \beta(1 - \nu) \left( 1 + r^* + \frac{\nu}{1 - \nu} \right) \right] = 1$$

and using  $r^* = f'(k^*) - \delta$ , we get

$$f'(k^*) = \frac{1 - \beta(1 - \nu)}{\beta(1 - \nu)} + \delta.$$

This equation implies that  $k^*$  is unique.

## Perpetual Youth in Discrete time XI

- Additionally,  $k_t = k_{t+1}$  i.e.

$$k^* = \frac{f(k^*) - c^* + (1 - \delta)k^*}{1 + n - v}$$

which implies

$$c^* = f(k^*) - (n + \delta - v)k^*$$

- Clearly Golden rule level in this setting requires

$$f'(k_G^*) = n + \delta - v.$$

So that the economy can be dynamically inefficient depending on

$$n - v \begin{matrix} \geq \\ < \end{matrix} \frac{1 - \beta(1 - v)}{\beta(1 - v)}.$$

Also, we may conclude that  $k^*$  is not necessarily equal to the modified golden rule stock of capital level.

## Subsection 2

### **Perpetual Youth in Continuous time**

# Perpetual Youth in Continuous time I

- The solution in this version is a closed-form solution for aggregate consumption and capital stock dynamics.
- Poisson rate of death,  $\nu \in (0, \infty)$ . Time of death follows an exponential distribution,  $g(t) = \nu e^{-\nu t}$ . So that the probability the agent dies before time  $t$  is  $\int_0^t \nu e^{-\nu s} ds = 1 - e^{-\nu t}$  and the probability she is alive is  $e^{-\nu t}$ .

## Perpetual Youth in Continuous time II

- Preferences:  $u(c(t | \tau)) = \log(c(t | \tau))$ , where  $c(t | \tau)$  is the consumption of generation  $\tau$  at moment  $t$ .
- Expected utility of an individual is given by  $\int_0^\infty e^{-(\rho+\nu)(t-\tau)} \log(c(t | \tau)) dt$  or

$$e^{(\rho+\nu)\tau} \int_0^\infty e^{-(\rho+\nu)t} \log(c(t | \tau)) dt, \quad (41)$$

where  $\rho$  is the discount rate.

- Demographic: As in the discrete time, assuming that  $n > \nu$ , the evolution of the total population is given by

$$\dot{L}(t) = (n - \nu)L(t). \quad (42)$$

It is also assumed that  $n - \nu < \rho$ .

- The number of individuals of the cohort born at time  $\tau < t$  is

$$L(t | \tau) = ne^{-\nu(t-\tau)+(n-\nu)\tau}, \quad (43)$$

where  $L(0) = 1$ .

## Perpetual Youth in Continuous time III

- As in the discrete case, individual  $i$ 's flow budget constraint is

$$\dot{a}(t | \tau) = r(t)a(t | \tau) - c(t | \tau) + w(t) + z(a(t | \tau)|t, \tau), \quad (44)$$

where again  $z(a(t | \tau)|t, \tau)$  reflects transfers to the individual. Since individuals face an uncertain time of death, there may be accidental bequests. Introduce a life insurance or annuity markets. Assuming competitive life insurance firms, their profits will be

$$\pi(a(t | \tau)|t, \tau) = \nu a(t | \tau) - z(a(t | \tau)|t, \tau),$$

since the individual will die and leave his assets to the life insurance company at the flow rate  $\nu$ . With free entry,  $\pi(a(t | \tau)|t, \tau) = 0$ , thus

$$z(a(t | \tau)|t, \tau) = \nu a(t | \tau).$$

## Perpetual Youth in Continuous time IV

- Maximization problem of a typical individual of generation  $\tau$  can be written as

$$\max_{c(t|\tau)} \int_0^{\infty} e^{-(\rho+\nu)t} \log(c(t|\tau)) dt, \quad (45)$$

subject to

$$\dot{a}(t|\tau) = (r(t) + \nu)a(t|\tau) - c(t|\tau) + w(t). \quad (46)$$

- Equilibrium wage rate and rate of return on capital

$$w(t) = f(k(t)) - k(t)f'(k(t)) \quad (47)$$

$$R(t) = f'(k(t)) \quad (48)$$

## Perpetual Youth in Continuous time V

- The law of motion of the capital-labor ratio is given by

$$\dot{k}(t) = f(k(t)) - (n - v + \delta)k(t) - c(t) \quad (49)$$

where

$$\begin{aligned} c(t) &= \frac{\int_{-\infty}^t c(t | \tau) L(t | \tau) d\tau}{\int_{-\infty}^t L(t | \tau) d\tau} \\ &= \frac{\int_{-\infty}^t c(t | \tau) L(t | \tau) d\tau}{L(t)} \end{aligned}$$

recalling that  $L(t|\tau)$  is the size of the cohort born at  $\tau$  at time  $t$ , and the lower limit of the integral is set to  $-\infty$  to include all cohorts, even those born in the distant past.



## Perpetual Youth in Continuous time VI

**Definition** An equilibrium in this overlapping generations economy with perpetual youth is a sequence of capital stock, wage rates, and rental rates of capital,  $\{K(t), w(t), R(t)\}_{t=0}^{\infty}$ , and paths of consumption for each generation,  $\{c(t|\tau)\}_{t=0, \tau \leq t}^{\infty}$ , such that each individual maximizes utility (45) subject to (46), and the time path of factor prices,  $\{w(t), R(t)\}_{t=0}^{\infty}$  given by (47) and (48), is such that given the time path of capital stock and labor,  $\{K(t), L(t)\}_{t=0}^{\infty}$ , all markets clear.

## Perpetual Youth in Continuous time VII

- Solution of (45) subject to (46), using Hamiltonian:

$$H(\cdot) = \max_{c(t|\tau)} \log [c(t|\tau)] + \mu(t|\tau) ((r(t) + v)a(t|\tau) + w(t) - c(t|\tau))$$

The first order conditions are

$$\frac{1}{c(t|\tau)} = \mu(t|\tau)$$

$$-\dot{\mu}(t|\tau) + (\rho + v)\mu(t|\tau) = (r(t) + v)\mu(t|\tau)$$

$$\dot{a}(t|\tau) = (r(t) + v)a(t|\tau) - c(t|\tau) + w(t).$$

From the first FOC,  $\frac{\dot{c}(t|\tau)}{c(t|\tau)} = -\frac{\dot{\mu}(t|\tau)}{\mu(t|\tau)}$ .

## Perpetual Youth in Continuous time VIII

- From the second FOC we have

$$-\frac{\dot{\mu}(t | \tau)}{\mu(t | \tau)} = r(t) - \rho$$

$$\mu(t | \tau) = \mu(\tau | \tau) e^{-(\bar{r}(t, \tau) - \rho)(t - \tau)}$$

where  $\bar{r}(t, \tau) \equiv \frac{1}{t - \tau} \int_{\tau}^t r(s) ds$ .

- The Euler equation is:

$$\frac{\dot{c}(t | \tau)}{c(t | \tau)} = r(t) - \rho$$

and the transversality condition is

$$\lim_{t \rightarrow \infty} e^{-(\rho + \nu)(t - \tau)} \mu(t | \tau) a(t | \tau) = 0.$$

# Perpetual Youth in Continuous time IX

- Using the transversality condition and combining with the solution of  $\mu(t | \tau)$  we have

$$\lim_{t \rightarrow \infty} e^{-(\rho+\nu)t} \mu(\tau|\tau) e^{-(\bar{r}(t,\tau)-\rho)(t-\tau)} a(t | \tau) = 0$$

$$\lim_{t \rightarrow \infty} e^{-(\bar{r}(t,\tau)+\nu)(t-\tau)} \mu(\tau|\tau) a(t | \tau) = 0$$

$$\lim_{t \rightarrow \infty} e^{-(\bar{r}(t,\tau)+\nu)(t-\tau)} a(t | \tau) = 0$$

which is the NPC in this case.

# Perpetual Youth in Continuous time X

- Integrating the FBC

$$\int_t^\infty [\dot{a}(s|\tau) - (r(s) + \nu)a(s|\tau)] e^{-(\bar{r}(s,t)+\nu)(s-t)} ds =$$

$$[a(s|\tau) e^{-(\bar{r}(s,t)+\nu)(s-t)}]_t^\infty =$$

$$-a(t | \tau) = \omega(t) - \int_t^\infty c(s|\tau) e^{-(\bar{r}(s,t)+\nu)(s-t)} ds$$

$$-a(t | \tau) = \omega(t) - \int_t^\infty c(t | \tau) e^{-(\rho+\nu)(s-t)} ds$$

$$-a(t | \tau) = \omega(t) - c(t | \tau) \left[ -\frac{e^{-(\rho+\nu)(s-t)}}{\rho + \nu} \right]_t^\infty$$

$$-a(t | \tau) = \omega(t) - c(t | \tau) \frac{1}{\rho + \nu}$$

# Perpetual Youth in Continuous time XI

- Thus,

$$c(t | \tau) = (\rho + \nu)(\omega(t) + a(t | \tau)) \quad (50)$$

where  $\omega(t) = \int_t^\infty e^{-(\bar{r}(s,t)+\nu)(s-t)} w(s) ds$

- In the aggregate,  $a(t) = k(t)$  and  $a(t) = \int_{-\infty}^t \frac{a(t|\tau)L(t|\tau)}{L(t)} d\tau$ , so

$$\begin{aligned} c(t) &= \frac{\int_{-\infty}^t c(t | \tau) L(t | \tau) d\tau}{L(t)} \\ &= (\rho + \nu) \frac{\int_{-\infty}^t (\omega(t) + a(t | \tau)) L(t | \tau) d\tau}{L(t)} \\ &= (\rho + \nu) \left[ \omega(t) + \frac{\int_{-\infty}^t a(t | \tau) L(t | \tau) d\tau}{L(t)} \right] \\ &= (\rho + \nu)(\omega(t) + a(t)) \end{aligned}$$

In the aggregate,  $\dot{c}(t) = (\rho + \nu)(\dot{\omega}(t) + \dot{a}(t))$

## Perpetual Youth in Continuous time XII

- From FBC,  $\dot{a}(t | \tau) = (r(t) + v)a(t | \tau) + w(t) - c(t | \tau)$ , thus  $\dot{a}(t) = (r(t) + v - n)a(t) + w(t) - c(t)$ . Moreover,

$$\dot{a}(t) = \frac{[\int_{-\infty}^t a(t|\tau) \dot{L}(t|\tau) d\tau]}{L(t)}, \text{ then}$$

$$\begin{aligned} \dot{a}(t) &= \frac{L(t|t)}{L(t)} a(t|t) + \int_{-\infty}^t \frac{\dot{L}(t|\tau)}{L(t|\tau)} \frac{L(t|\tau)}{L(t)} a(t|\tau) d\tau \\ &\quad - \int_{-\infty}^t \frac{L(t|\tau)}{L(t)} \frac{\dot{L}(t)}{L(t)} a(t|\tau) d\tau \\ &\quad + \int_{-\infty}^t \frac{L(t|\tau)}{L(t)} \dot{a}(t|\tau) d\tau \\ &= -va(t) - (n-v)a(t) + (r(t) + v)a(t) + w(t) - c(t) \end{aligned}$$

$$\text{and } \dot{\omega} + w(t) = (r(t) + v)\omega(t)$$

## Perpetual Youth in Continuous time XIII

• So

$$\begin{aligned}
 \dot{c}(t) &= (\rho + \nu) [(r(t) + \nu - n)a(t) + (r(t) + \nu)\omega(t) - c(t)] \\
 &= (\rho + \nu) [(r(t) + \nu)(a(t) + \omega(t)) - na(t) - c(t)] \\
 &= (\rho + \nu) \left[ \frac{(r(t) + \nu)}{\rho + \nu} c(t) - na(t) - c(t) \right] \\
 &= (r(t) - \rho)c(t) - (\rho + \nu)na(t) \\
 \frac{\dot{c}(t)}{c(t)} &= f'(k(t)) - \delta - \rho - (\rho + \nu)n \frac{k(t)}{c(t)}.
 \end{aligned}$$

where the last term reflects the addition of new agents who are less wealthy than the average agent.



## Perpetual Youth in Continuous time XIV

- The equilibrium dynamics are characterized by

$$\dot{k}(t) = f(k(t)) - (n + \delta - \nu)k(t) - c(t) \quad (51)$$

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu)n \frac{k(t)}{c(t)} \quad (52)$$

with initial condition  $k(0) > 0$  given and the transversality condition.

- In steady state  $\dot{k}(t) = 0$  and  $\dot{c}(t) = 0$ , so that

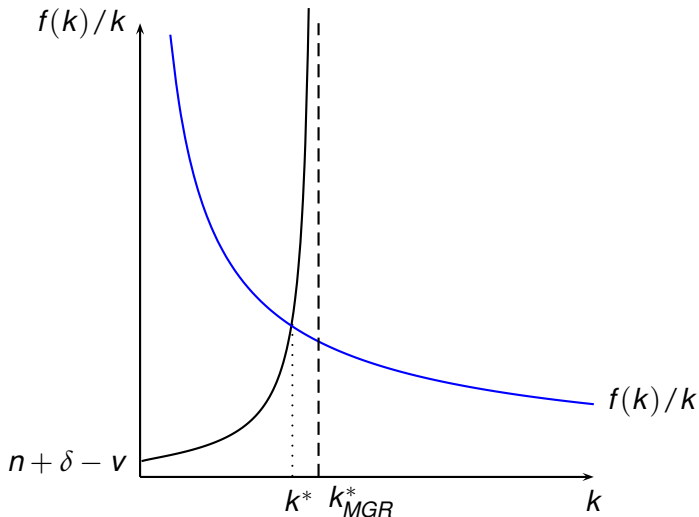
$$\frac{f'(k^*) - \delta - \rho}{(\rho + \nu)n} = \frac{k^*}{c^*} \quad (53)$$

$$\frac{f(k^*)}{k^*} - (n + \delta - \nu) = \frac{c^*}{k^*} \quad (54)$$

that is

$$\frac{f(k^*)}{k^*} = (n + \delta - \nu) + \frac{(\rho + \nu)n}{f'(k^*) - \delta - \rho} \quad (55)$$

# Perpetual Youth in Continuous time XV



## Perpetual Youth in Continuous time XVI

**Proposition** In the continuous-time perpetual youth model, there exists a unique steady state  $(k^*, c^*)$  given by (53) and (54). The steady-state capital-labor ratio  $k^*$  (equation 55) is less than the level of capital-labor ratio that satisfies the modified golden rule,  $k_{MGR}^*$ . Starting with any  $k(0) > 0$ , equilibrium dynamics monotonically converge to this unique steady state.

## Perpetual Youth in Continuous time XVII

- Unlike the Ramsey model, in this model it is possible to have over-accumulation of capital. Assume that every generation is born having 1 unit of labor, but this labor decreases at a rate  $\xi$ , i.e. labor income for generation  $\tau$  in period  $t$  becomes  $w(t)e^{-\xi(t-\tau)}$ . From above,  $c(t | \tau) = (\rho + \nu)(a(t | \tau) + \omega(t | \tau))$ , where  $\omega(t | \tau)$  is now  $\omega(t | \tau) = \int_t^\infty e^{-(\tilde{r}(s,t)+\nu)(s-t)} e^{-\xi(s-\tau)} w(s) ds$ . Then  $\dot{\omega}(t | \tau) = \omega(t | \tau)(r(t) + \nu - \xi) - w(t)e^{-\xi(t-\tau)}$ .
- The equation governing the dynamic of consumption is

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho + \xi - (\rho + \nu)(n - \xi) \frac{k(t)}{c(t)}$$

# Perpetual Youth in Continuous time XVIII

- The new dynamic system is

$$\begin{aligned}\dot{k}(t) &= f(k(t)) - (n + \delta - \nu)k(t) - c(t) \\ \frac{\dot{c}(t)}{c(t)} &= f'(k(t)) - \delta - \rho + \xi - (\rho + \nu)(n + \xi) \frac{k(t)}{c(t)}\end{aligned}$$

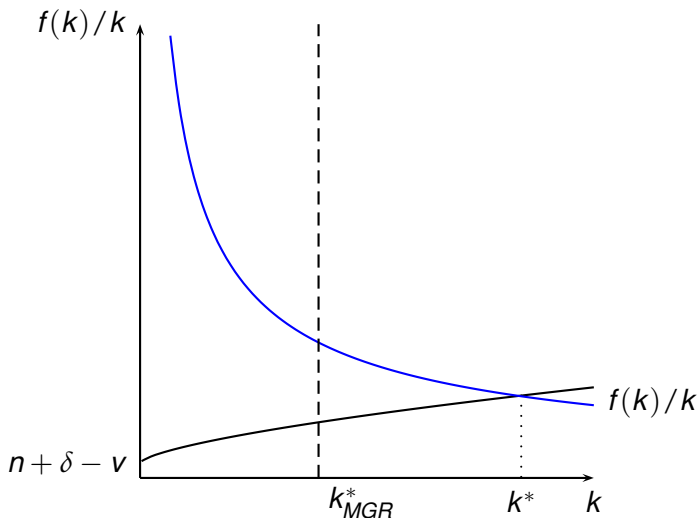
- In steady state  $\dot{k}(t) = 0$  and  $\dot{c}(t) = 0$ , so that

$$\frac{f'(k^{**}) - \delta - \rho + \xi}{(\rho + \nu)(n + \xi)} = \frac{k^{**}}{c^{**}} \quad (56)$$

that is

$$\frac{f(k^{**})}{k^{**}} = (n + \delta - \nu) + \frac{(\rho + \nu)(n + \xi)}{f'(k^{**}) - \delta - \rho + \xi} \quad (57)$$

# Perpetual Youth in Continuous time XIX



## Section 7

# Conclusions

## Subsection 1

### **Conclusions**



# Conclusions

- Overlapping generations often are more realistic than infinity-lived representative agents.
- Models with overlapping generations fall outside the scope of the First Welfare Theorem:
  - they were partly motivated by the possibility of Pareto suboptimal allocations.
- Equilibria may be “dynamically inefficient” and feature overaccumulation: unfunded Social Security (other assets/bubbles) can ameliorate the problem.

## Conclusions

- Declining path of labor income important for overaccumulation, and what matters is not finite horizons but arrival of new individuals.
- Overaccumulation and Pareto suboptimality: pecuniary externalities created on individuals that are not yet in the marketplace.
- Not overemphasize dynamic inefficiency: major question of economic growth is why so many countries have so little capital.