Overlapping Generation Models

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Macroeconomics II

Section 1

Growth with Overlapping Generations

Subsection 1

Growth with Overlapping Generations

Growth with Overlapping Generations

- In many situations, the assumption of a *representative household* is not appropriate because
  1. households do not have an infinite planning horizon
  2. new households arrive (or are born) over time.
- New economic interactions: decisions made by older "generations" will affect the prices faced by younger "generations".
- *Overlapping generations models*
  1. Capture potential interaction of different generations of individuals in the marketplace;
  2. Provide tractable alternative to infinite-horizon representative agent models;
  3. Some key implications different from neoclassical growth model;
  4. Dynamics in some special cases quite similar to Solow model rather than the neoclassical model;
  5. Generate new insights about the role of national debt and Social Security in the economy.

Subsection 2

Problems of Infinity

Problems of Infinity I

- Static economy with countably infinite number of households, $i \in \mathbb{N}$
- Countably infinite number of commodities, $j \in \mathbb{N}$.
- All households behave competitively (alternatively, there are M households of each type, M is a large number).
- Household i has preferences:

$$u_i = c_i^i + c_{i+1}^i,$$

- c_j^i denotes the consumption of the jth type of commodity by household i.
- Endowment vector ω of the economy: each household has one unit endowment of the commodity with the same index as its index.
- Choose the price of the first commodity as the numeraire, i.e., $p_0 = 1$.

Problems of Infinity II

Proposition In the above-described economy, the price vector $\bar{p}$ such that $\bar{p}_j = 1$ for all $j \in \mathbb{N}$ is a competitive equilibrium price vector and induces an equilibrium with no trade, denoted by $\bar{x}$.

- **Proof:**
  - At $\bar{p}$, each household has income equal to 1.
  - Therefore, the budget constraint of household i can be written as

$$c_i^i + c_{i+1}^i \leq 1.$$

  - This implies that consuming own endowment is optimal for each household,
  - Thus $\bar{p}$ and no trade, $\bar{x}$, constitute a competitive equilibrium.

Problems of Infinity III

- However, this competitive equilibrium is not Pareto optimal. Consider alternative allocation, $\tilde{x}$:
  - Household $i = 0$ consumes its own endowment and that of household 1.
  - All other households, indexed $i > 0$, consume the endowment of their neighboring household, $i + 1$.
  - All households with $i > 0$ are as well off as in the competitive equilibrium $(\bar{p}, \bar{x})$.
  - Individual $i = 0$ is strictly better-off.

Proposition In the above-described economy, the competitive equilibrium at $(\bar{p}, \bar{x})$ is not Pareto optimal.

Problems of Infinity IV

- Source of the problem must be related to the infinite number of commodities.
- Extended version of the First Welfare Theorem covers infinite number of commodities, but only assuming $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$ (written with the aggregate endowment ω_j).
- Here the only endowment is the good, and thus $p_j^* = 1$ for all $j \in \mathbb{N}$, so that $\sum_{j=0}^{\infty} p_j^* \omega_j = \infty$ (why?).
- This abstract economy is "isomorphic" to the baseline overlapping generations model.
- The Pareto suboptimality in this economy will be the source of potential inefficiencies in overlapping generations model.

Problems of Infinity V

- Second Welfare Theorem did not assume $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$.
- Instead, it used convexity of preferences, consumption sets and production possibilities sets.
- This exchange economy has convex preferences and convex consumption sets:
  - Pareto optima must be decentralizable by some redistribution of endowments.

Proposition In the above-described economy, there exists a reallocation of the endowment vector ω to $\tilde{\omega}$, and an associated competitive equilibrium $(\bar{p}, \tilde{x})$ that is Pareto optimal where $\tilde{x}$ is as described above, and $\bar{p}$ is such that $\bar{p}_j = 1$ for all $j \in \mathbb{N}$.

Proof of Proposition

- Consider the following reallocation of ω: endowment of household $i \geq 1$ is given to household $i - 1$.
  - At the new endowment vector $\tilde{\omega}$, household $i = 0$ has one unit of good $j = 0$ and one unit of good $j = 1$.
  - Other households i have one unit of good $i + 1$.
- At the price vector $\bar{p}$, household 0 has a budget set

$$c_0^0 + c_1^1 \leq 2,$$

  thus chooses $c_0^0 = c_1^0 = 1$.
- All other households have budget sets given by

$$c_i^i + c_{i+1}^i \leq 1,$$

- Thus it is optimal for each household $i > 0$ to consume one unit of the good c_{i+1}^i
- Thus $\tilde{x}$ is a competitive equilibrium.

Section 2

The Baseline OLG Model

Subsection 1

Environment

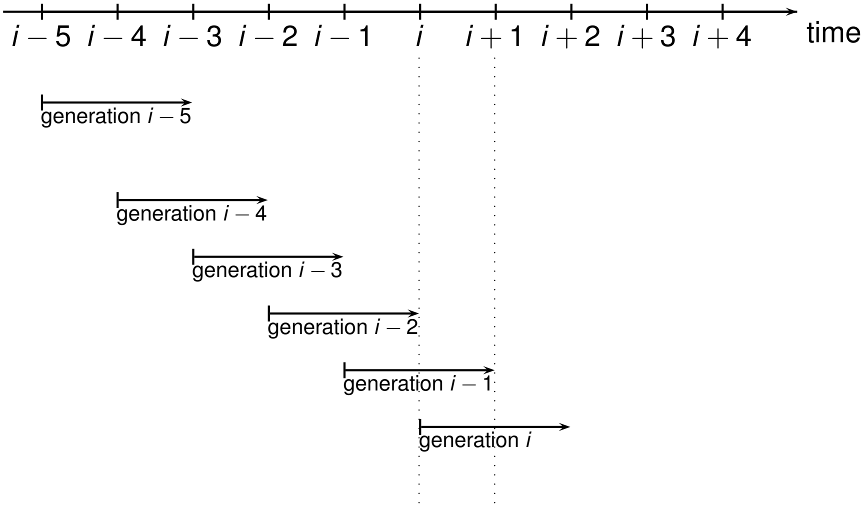
The Baseline Overlapping Generations Model

- Time is discrete and runs to infinity.
- Each individual lives for two periods.
- Individuals born at time t live for dates t and $t+1$.
- Assume a general (separable) utility function for individuals born at date t,

$$U_t = u\left(c_{1t}\right) + \beta u\left(c_{2t+1}\right), \tag{1}$$

- $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies the usual Assumptions on utility.
- c_{1t}: consumption of the individual born at t when young (at date t).
- c_{2t+1}: consumption when old (at date $t+1$).
- $\beta \in (0, 1)$ is the discount factor.

Structure of population across time



Demographics, Preferences and Technology I

- Exponential population growth,

$$L_t = (1+n)^t L(0). \tag{2}$$

- Production side same as before: competitive firms, constant returns to scale aggregate production function, satisfying Assumptions 1 and 2:

$$Y_t = F(K_t, L_t).$$

- Factor markets are competitive.
- Individuals can only work in the first period and supply one unit of labor inelastically, earning w_t.

Demographics, Preferences and Technology II

- Assume that $\delta = 1$.
- $k \equiv K/L$, $f(k) \equiv F(k, 1)$, and the (gross) rate of return to saving, which equals the rental rate of capital, is

$$1 + r_t = R_t = f'(k_t), \tag{3}$$

- As usual, the wage rate is

$$w_t = f(k_t) - k_t f'(k_t). \tag{4}$$

Subsection 2

Consumption Decisions

Consumption Decisions I

- Savings by an individual of generation t, s_t, is determined as a solution to

$$\max_{c_{1t}, c_{2t+1}, s_t} u(c_{1t}) + \beta u(c_{2t+1})$$

subject to

$$c_{1t} + s_t \leq w_t$$

and

$$c_{2t+1} \leq R_{t+1} s_t,$$

- Old individuals rent their savings of time t as capital to firms at time $t+1$, and receive gross rate of return $R_{t+1} = 1 + r_{t+1}$.
- Second constraint incorporates notion that individuals only spend money on their own end of life consumption (no altruism or bequest motive).

Consumption Decisions II

- No need to introduce $s_t \geq 0$, since negative savings would violate second-period budget constraint (given $c_{2t+1} \geq 0$).
- Since $u(\cdot)$ is strictly increasing, both constraints will hold as equalities.
- Thus first-order condition for a maximum can be written in the familiar form of the consumption Euler equation,

$$u'(c_{1t}) = \beta R_{t+1} u'(c_{2t+1}). \tag{5}$$

- Problem of each individual is strictly concave, so this Euler equation is sufficient.
- Solving for consumption and thus for savings,

$$s_t = s(w_t, R_{t+1}), \tag{6}$$

Consumption Decisions

- From the FOC and the BC

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t)$$

which implicitly defines

$$s_t = s(w_t, R_{t+1}).$$

- One can show that $s : \mathbb{R}^2_+ \to \mathbb{R}$ satisfies $s_w > 0$, but $s_R \gtrless 0$.

Consumption Decisions III

- Total savings in the economy will be equal to

$$S_t = s_t L_t,$$

- L_t denotes the size of generation t, who are saving for time $t+1$.
- Since capital depreciates fully after use and all new savings are invested in capital,

$$K_{t+1} = L_t s\left(w_t, R_{t+1}\right). \tag{7}$$

Subsection 3

Equilibrium

Equilibrium I

Definition A competitive equilibrium can be represented by a sequence of aggregate capital stocks, individual consumption and factor prices, $\{K_t, c_{1t}, c_{2t}, R_t, w_t\}_{t=0}^{\infty}$, such that the factor price sequence $\{R_t, w_t\}_{t=0}^{\infty}$ is given by (3) and (4), individual consumption decisions $\{c_{1t}, c_{2t}\}_{t=0}^{\infty}$ are given by (5) and (6), and the aggregate capital stock, $\{K_t\}_{t=0}^{\infty}$, evolves according to (7).

- Steady-state equilibrium defined as usual: an equilibrium in which $k \equiv K/L$ is constant.
- To characterize the equilibrium, divide (7) by $L_{t+1} = (1+n) L_t$,

$$k_{t+1} = \frac{s(w_t, R_{t+1})}{1+n}.$$

Equilibrium II

- Now substituting for R_{t+1} and w_t from (3) and (4),

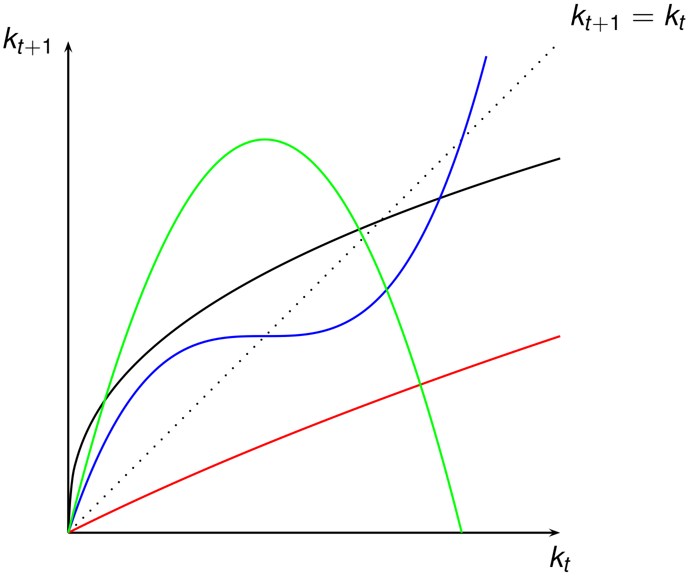
$$k_{t+1} = \frac{s\Big(f\left(k_t\right) - k\left(t\right) f'\left(k_t\right), f'\left(k_{t+1}\right)\Big)}{1+n} \tag{8}$$

- This is the fundamental law of motion of the overlapping generations economy.
- A steady state is given by a solution to this equation such that $k_{t+1} = k_t = k^*$, i.e.,

$$k^* = \frac{s\Big(f\left(k^*\right) - k^* f'\left(k^*\right), f'\left(k^*\right)\Big)}{1+n} \tag{9}$$

- Since the savings function $s\left(\cdot,\cdot\right)$ can take any form, the difference equation (8) can lead to quite complicated dynamics, and multiple steady states are possible.

Possible Laws of Motion



Subsection 4

Special Cases

Restrictions on Utility and Production Functions I

- Suppose that the utility functions take the familiar CRRA form:

$$U_t = \frac{c_{1t}^{1-\theta} - 1}{1-\theta} + \beta \left(\frac{c_{2t+1}^{1-\theta} - 1}{1-\theta} \right), \tag{10}$$

  where $\theta > 0$ and $\beta \in (0, 1)$.
- Technology is Cobb-Douglas,

$$f(k) = k^{\alpha}$$

- The rest of the environment is as described above.
- The CRRA utility simplifies the first-order condition for consumer optimization,

$$\frac{c_{2t+1}}{c_{1t}} = (\beta R_{t+1})^{1/\theta}.$$

Restrictions on Utility and Production Functions II

- This Euler equation can be alternatively expressed in terms of savings as
$$s_t^{-\theta} \beta R_{t+1}^{1-\theta} = \left(w\left(t\right) - s_t\right)^{-\theta}, \tag{11}$$
- Gives the following equation for the saving rate:
$$s_t = \frac{w_t}{\psi_{t+1}}, \tag{12}$$

  where
$$\psi_{t+1} \equiv [1 + \beta^{-1/\theta} R_{t+1}^{-(1-\theta)/\theta}] > 1,$$
- Ensures that savings are always less than earnings.

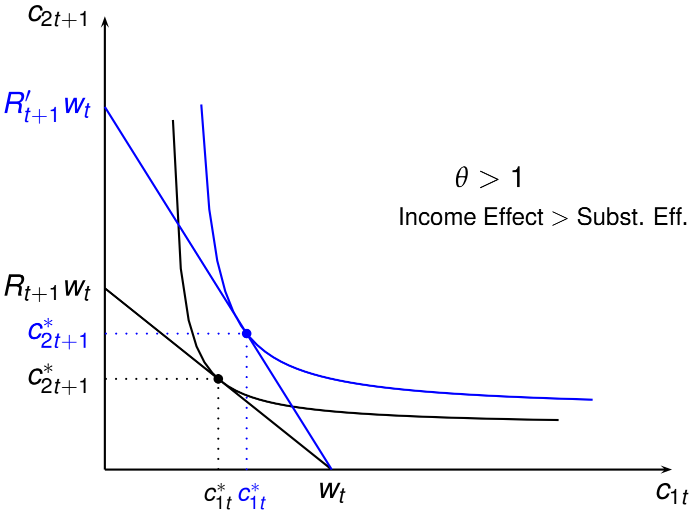
Restrictions on Utility and Production Functions III

- The impact of factor prices on savings is summarized by the following and derivatives:

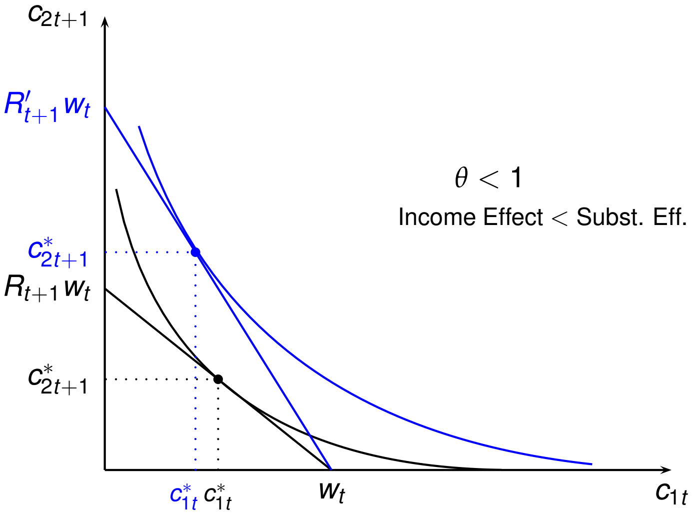
$$
\begin{aligned}
s_w &\equiv \frac{\partial s_t}{\partial w_t} = \frac{1}{\psi_{t+1}} \in (0,1)\,, \\
s_R &\equiv \frac{\partial s_t}{\partial R_{t+1}} = \left(\frac{1-\theta}{\theta}\right)\left(\beta R_{t+1}\right)^{-1/\theta}\frac{s_t}{\psi_{t+1}}.
\end{aligned}
$$

- Since $\psi_{t+1} > 1$, we also have that $0 < s_w < 1$.
- Moreover, in this case $s_R < 0$ if $\theta > 1$, $s_R > 0$ if $\theta < 1$, and $s_R = 0$ if $\theta = 1$.
- Reflects counteracting influences of income and substitution effects.
- Case of $\theta = 1$ (log preferences) is of special importance, may deserve to be called the *canonical overlapping generations model*.

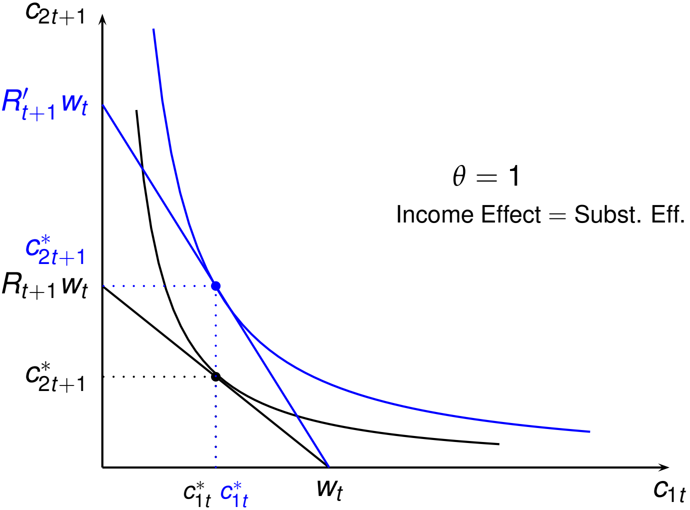
RRA coefficient, income and substitution effects I



RRA coefficient, income and substitution effects I



RRA coefficient, income and substitution effects I



Restrictions on Utility and Production Functions IV

- Equation (8) implies

$$\begin{aligned} k_{t+1} &= \frac{s_t}{(1+n)} \\ &= \frac{w_t}{(1+n)\psi_{t+1}}, \end{aligned} \tag{13}$$

- Or more explicitly,

$$k_{t+1} = \frac{f(k_t) - k_t f'(k_t)}{(1+n)\left[1+\beta^{-1/\theta} f'(k_{t+1})^{-(1-\theta)/\theta}\right]} \tag{14}$$

- The steady state then involves a solution to the following implicit equation:

$$k^* = \frac{f(k^*) - k^* f'(k^*)}{(1+n)\left[1+\beta^{-1/\theta} f'(k^*)^{-(1-\theta)/\theta}\right]}.$$

Restrictions on Utility and Production Functions V

- Now using the Cobb-Douglas formula, steady state is the solution to the equation

$$(1+n)\left[1+\beta^{-1/\theta}\left(\alpha(k^*)^{\alpha-1}\right)^{(\theta-1)/\theta}\right]=(1-\alpha)(k^*)^{\alpha-1}. \quad (15)$$

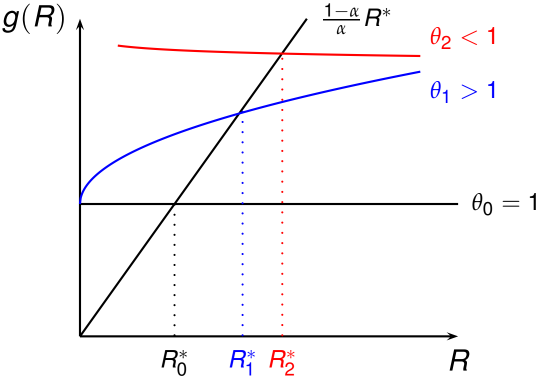
- For simplicity, define $R^* \equiv \alpha(k^*)^{\alpha-1}$ as the marginal product of capital in steady-state, in which case, (15) can be rewritten as

$$(1+n)\left[1+\beta^{-1/\theta}\left(R^*\right)^{(\theta-1)/\theta}\right]=\frac{1-\alpha}{\alpha}R^*. \quad (16)$$

- Steady-state value of R^*, and thus k^*, can now be determined from equation (16), which always has a unique solution.

Steady State

- Notice that the steady state depends on θ (compare with Ramsey!)
- Existence and uniqueness follow from figure

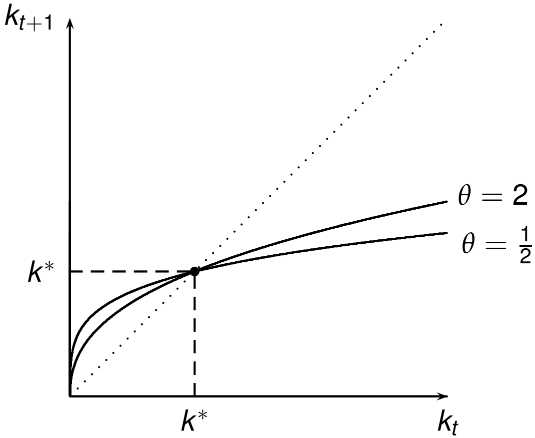


RRA coefficient, steady state and dynamics

- To investigate the stability, substitute for the Cobb-Douglas production function in (14)

$$k_{t+1} = \frac{(1-\alpha)\, k_t^{\alpha}}{(1+n)\left[1+\beta^{-1/\theta}\left(\alpha k_{t+1}^{\alpha-1}\right)^{-(1-\theta)/\theta}\right]}. \tag{17}$$

RRA coefficient, steady state and dynamics



Stability

Using (17) we can define

$$k_t = \left[\frac{1+n}{1-\alpha}\left(k_{t+1} + \beta^{-1/\theta}\alpha^{(\theta-1)/\theta}k_{t+1}^{(1-\alpha)/\theta+\alpha}\right)\right]^{1/\alpha} \equiv \Gamma(k_{t+1}),$$

then

$$\frac{dk_t}{dk_{t+1}} = \frac{1}{\alpha}\Gamma(k_{t+1})^{1-\alpha}\left[\frac{1+n}{1-\alpha}\left(1 + \left(\frac{1-\alpha}{\theta}+\alpha\right)\beta^{-1/\theta}\alpha^{(\theta-1)/\theta}k_{t+1}^{(1-\alpha)/\theta+\alpha-1}\right)\right]$$

and at the steady state $k^* = \Gamma(k^*)$, so

$$\left.\frac{dk_t}{dk_{t+1}}\right|_{k^*} = \frac{1}{\alpha}k^{*-\alpha}\left[\frac{1+n}{1-\alpha}\left(k^* + \left(\frac{1-\alpha}{\theta}+\alpha\right)\beta^{-1/\theta}\alpha^{(\theta-1)/\theta}k^{*(1-\alpha)/\theta+\alpha)}\right)\right]$$

Stability

- $\theta \leq 1 \Longrightarrow \frac{1-\alpha}{\theta} + \alpha \geq 1$ so that

$$\left.\frac{dk_t}{dk_{t+1}}\right|_{k^*} \geq \frac{1}{\alpha} k^{*-\alpha} \Gamma(k^*) \alpha = \frac{1}{\alpha} \Longrightarrow \left.\frac{dk_{t+1}}{dk_t}\right|_{k^*} \leq \alpha$$

- $\theta > 1 \Longrightarrow \frac{1-\alpha}{\theta} + \alpha < 1$ so that

$$\left.\frac{dk_t}{dk_{t+1}}\right|_{k^*} > \frac{1}{\alpha} k^{*-\alpha} \left(\frac{1-\alpha}{\theta} + \alpha\right) \Gamma(k^*) \alpha = \frac{1}{\alpha} \left(\frac{1-\alpha}{\theta} + \alpha\right) \Longrightarrow$$
$$\left.\frac{dk_{t+1}}{dk_t}\right|_{k^*} \leq \frac{1}{\frac{1-\alpha}{\alpha\theta} + 1} < 1$$

Restrictions on Utility and Production Functions VI

Proposition In the overlapping-generations model with two-period lived households, Cobb-Douglas technology and CRRA preferences, there exists a unique steady-state equilibrium with the capital-labor ratio k^* given by (15), this steady-state equilibrium is globally stable for all $k(0) > 0$.

- In this particular (well-behaved) case, equilibrium dynamics are very similar to the basic Solow model
- Figure shows that convergence to the unique steady-state capital-labor ratio, k^*, is monotonic.

Section 3

Canonical OLG Model

Subsection 1

Canonical Model

Canonical Model I

- Even the model with CRRA utility and Cobb-Douglas production function is relatively messy.
- Many of the applications use log preferences ($\theta = 1$).
- Income and substitution effects exactly cancel each othe: changes in the interest rate (and thus in the capital-labor ratio of the economy) have no effect on the saving rate.
- Structure of the equilibrium is essentially identical to the basic Solow model.
- Utility of the household and generation t is,

$$U_t = \log c_{1t} + \beta \log c_{2t+1}, \tag{18}$$

- $\beta \in (0, 1)$ (even though $\beta \geq 1$ could be allowed).
- Again $f(k) = k^{\alpha}$.

Canonical Model II

- Consumption Euler equation:

$$\frac{c_{2t+1}}{c_{1t}} = \beta R_{t+1} \Longrightarrow c_{1t} = \frac{1}{1+\beta} w_t$$

- Savings should satisfy the equation

$$s_t = \frac{\beta}{1+\beta} w_t, \tag{19}$$

- Constant saving rate, equal to $\beta / (1+\beta)$, out of labor income for each individual.

Canonical Model III

- Combining this with the capital accumulation equation (8),

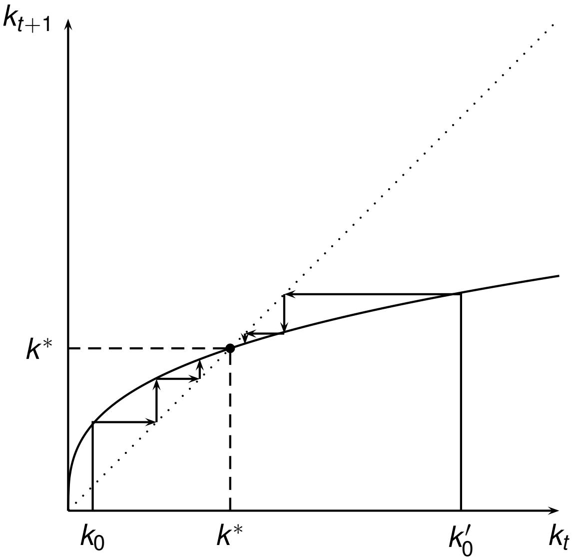
$$\begin{aligned} k_{t+1} &= \frac{s_t}{(1+n)} \\ &= \frac{\beta w_t}{(1+n)(1+\beta)} \\ &= \frac{\beta(1-\alpha)[k_t]^\alpha}{(1+n)(1+\beta)}, \end{aligned}$$

- Second line uses (19) and last uses that, given competitive factor markets, $w_t = (1-\alpha)[k_t]^\alpha$.
- There exists a unique steady state with

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}. \tag{20}$$

- Equilibrium dynamics are identical to those of the basic Solow model and monotonically converge to k^*.

Equilibrium dynamics in canonical OLG model



Canonical Model IV

Proposition In the canonical overlapping generations model with log preferences and Cobb-Douglas technology, there exists a unique steady state, with capital-labor ratio k^* given by (20). Starting with any $k(0) \in (0, k^*)$, equilibrium dynamics are such that $k_t \uparrow k^*$, and starting with any $k'(0) > k^*$, equilibrium dynamics involve $k_t \downarrow k^*$.

Section 4

Overaccumulation and Policy

Subsection 1

Overaccumulation and Pareto Optimality

Overaccumulation I

- Compare the overlapping-generations equilibrium to the choice of a social planner wishing to maximize a weighted average of all generations' utilities.
- Suppose that the social planner maximizes

$$\sum_{t=0}^{\infty} \xi_t U_t$$

- ξ_t is the discount factor of the social planner, which reflects how she values the utilities of different generations.

Overaccumulation II

- Substituting from (1), this implies:

$$\sum_{t=0}^{\infty} \xi_t \left(u\left(c_{1t}\right) + \beta u\left(c_{2t+1}\right)\right)$$

subject to the resource constraint

$$F\left(K_t, L_t\right) = K_{t+1} + L_t c_{1t} + L_{t-1} c_2\left(t\right).$$

- Dividing this by L_t and using (2),

$$f\left(k_t\right) = \left(1+n\right) k_{t+1} + c_{1t} + \frac{c_{2t}}{1+n}.$$

Overaccumulation IIB

- Assume $\sum \xi_t < \infty$ (Why?)
- Clearly, Assumption 6.1N-6.5N in the book hold and we can apply our dynamic programming theorems...

$$
\begin{aligned}
V(t, k_t) = \max_{(c_{1t}, k_{t+1})} \{ & \xi_t u(c_{1t}) \\
& + \xi_{t-1} \beta u\Big((1+n) f(k_t) - (1+n)^2 k_{t+1} - (1+n) c_{1t}\Big) \\
& + V(t+1, k_{t+1}) \}
\end{aligned}
$$

Overaccumulation IIC

Euler equation implies

$$\begin{aligned} \xi_t u'(c_{1t}) =& \beta \xi_{t-1} (1+n) u'(c_{2t}) \\ (1+n)^2 \beta \xi_{t-1} u'(c_{2t}) =& V_k(t+1, k_{t+1}) \end{aligned}$$

and the envelope theorem

$$V_k(t+1, k_{t+1}) = \beta \xi_t (1+n) f'(k_{t+1}) u'(c_{2t+1})$$

which together generate

$$\begin{aligned} \xi_t u'(c_{1t}) =& \beta \xi_{t-1} (1+n) u'(c_{2t}) = \frac{V_k(t+1, k_{t+1})}{1+n} \\ =& \beta \xi_t f'(k_{t+1}) u'(c_{2t+1}) \end{aligned}$$

- Transversality condition: $\lim_{t\to\infty} k_t^* \beta^t \xi_{t-1} f'(k_t) u'(c_{2t}) = 0$.

Overaccumulation III

- Social planner's maximization problem then implies the following first-order necessary condition:

$$u'\left(c_{1t}\right) = \beta f'\left(k_{t+1}\right) u'\left(c_{2t+1}\right).$$

- Since $R_{t+1} = f'\left(k_{t+1}\right)$, this is identical to (5).
- Not surprising: allocate consumption of a given individual in exactly the same way as the individual himself would do.
- No "market failures" in the over-time allocation of consumption at given prices.
- However, the allocations across generations may differ from the competitive equilibrium: planner is giving different weights to different generations
- In particular, competitive equilibrium is **Pareto suboptimal** when $k^* > k_{gold}$,

Overaccumulation IV

- When $k^* > k_{gold}$, reducing savings can increase consumption for every generation.
- More specifically, note that in steady state

$$\begin{aligned} f(k^*) - (1+n)k^* &= c_1^* + (1+n)^{-1} c_2^* \\ &\equiv c^*, \end{aligned}$$

- First line follows by national income accounting, and second defines c^* (aggregate per capita consumption).
- Therefore

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n)$$

- k_{gold} is defined as

$$f'(k_{gold}) = 1 + n \Big(= (n + g + \delta) \Big).$$

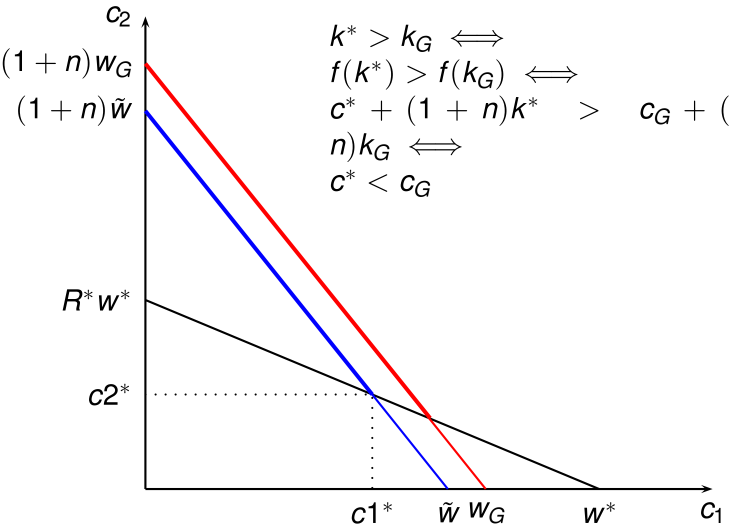
Overaccumulation V

- Now if $k^* > k_{gold}$, then $\partial c^* / \partial k^* < 0$: reducing savings can increase (total) consumption for everybody.
- If this is the case, the economy is referred to as *dynamically inefficient*—it involves overaccumulation.
- Another way of expressing dynamic inefficiency is that

$$r^* < n,$$

- Recall in infinite-horizon Ramsey economy, transversality condition required that $r > g + n$.
- Dynamic inefficiency arises because of the heterogeneity inherent in the overlapping generations model, which removes the transversality condition.
- Suppose we start from steady state at time T with $k^* > k_{gold}$.

Inefficiency: graphical analysis



$k^* > k_G \iff$
$f(k^*) > f(k_G) \iff$
$c^* + (1+n)k^* > c_G + (1+n)k_G \iff$
$c^* < c_G$

Overaccumulation VI

- Consider the following variation: change next period's capital stock by $-\Delta k$, where $\Delta k > 0$, and from then on, we immediately move to a new steady state (clearly feasible).
- This implies the following changes in consumption levels:

$$\begin{aligned}
\Delta c_t &= (1+n)\,\Delta k > 0 \\
\Delta c_t &= -\left(f'\left(k^* - \Delta k\right) - (1+n)\right)\Delta k \text{ for all } t > T
\end{aligned}$$

- The first expression reflects the direct increase in consumption due to the decrease in savings.
- In addition, since $k^* > k_{gold}$, for small enough Δk, $f'\left(k^* - \Delta k\right) - (1+n) < 0$, thus $\Delta c\left(t\right) > 0$ for all $t \geq T$.
- The increase in consumption for each generation can be allocated equally during the two periods of their lives, thus necessarily increasing the utility of all generations.

Pareto Optimality and Suboptimality in the OLG Model

Proposition In the baseline overlapping-generations economy, the competitive equilibrium is not necessarily Pareto optimal. More specifically, whenever $r^* < n$ and the economy is dynamically inefficient, it is possible to reduce the capital stock starting from the competitive steady state and increase the consumption level of all generations.

- Pareto inefficiency of the competitive equilibrium is intimately linked with *dynamic inefficiency.*

Interpretation

- Intuition for dynamic inefficiency:
  - Individuals who live at time t face prices determined by the capital stock with which they are working.
  - Capital stock is the outcome of actions taken by previous generations.
  - Pecuniary externality from the actions of previous generations affecting welfare of current generation.
  - Pecuniary externalities typically second-order and do not matter for welfare.
  - But not when an infinite stream of newborn agents joining the economy are affected.
  - It is possible to rearrange in a way that these pecuniary externalities can be exploited.

Further Intuition

- Complementary intuition:
  - Dynamic inefficiency arises from overaccumulation.
  - Results from current young generation needs to save for old age.
  - However, the more they save, the lower is the rate of return and may encourage to save even more.
  - Effect on future rate of return to capital is a pecuniary externality on next generation
  - If alternative ways of providing consumption to individuals in old age were introduced, overaccumulation could be ameliorated.

Subsection 2

Role of Social Security

Role of Social Security in Capital Accumulation

- Social Security as a way of dealing with overaccumulation
- Fully-funded system: young make contributions to the Social Security system and their contributions are paid back to them in their old age.
- Unfunded system or a *pay-as-you-go:* transfers from the young directly go to the current old.
- Pay-as-you-go (unfunded) Social Security discourages aggregate savings.
- With dynamic inefficiency, discouraging savings may lead to a Pareto improvement.

Subsection 3

Fully Funded Social Security

Fully Funded Social Security I

- Government at date t raises some amount d_t from the young, funds are invested in capital stock, and pays workers when old $R_{t+1}d_t$.
- Thus individual maximization problem is,

$$\max_{c_{1t},c_{2t+1},s_t} u\left(c_{1t}\right)+\beta u\left(c_{2t+1}\right)$$

  subject to

$$c_{1t}+s_t+d_t\leq w_t$$

  and

$$c_{2t+1}\leq R_{t+1}\left(s_t+d_t\right),$$

  for a given choice of d_t by the government.
- Notice that now the total amount invested in capital accumulation is $s_t+d_t=\left(1+n\right)k_{t+1}$.

Fully Funded Social Security II

- Given the solution when $d_t = 0$ for all t, $(\tilde{c}_{1t}, \tilde{c}_{2t+1})$, (original problem), agents choose to save $s_t = w_t - d_t - \tilde{c}_{1t}$
- No longer the case that individuals will always choose $s(t) > 0$.
- As long as s_t is free, whatever $\{d(t)\}_{t=0}^{\infty}$, the competitive equilibrium applies.
- When $s_t \geq 0$ is imposed as a constraint, competitive equilibrium applies if given $\{d_t\}_{t=0}^{\infty}$, privately-optimal $\{s_t\}_{t=0}^{\infty}$ is such that $s_t > 0$ for all t.

Fully Funded Social Security III

Proposition Consider a fully funded Social Security system in the above-described environment whereby the government collects d_t from young individuals at date t.

1. Suppose that $s_t \geq 0$ for all t. If given the feasible sequence $\{d_t\}_{t=0}^{\infty}$ of Social Security payments, the utility-maximizing sequence of savings $\{s_t\}_{t=0}^{\infty}$ is such that $s_t > 0$ for all t, then the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.
2. Without the constraint $s_t \geq 0$, given any feasible sequence $\{d_t\}_{t=0}^{\infty}$ of Social Security payments, the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.

- Moreover, even when there is the restriction that $s_t \geq 0$, a funded Social Security program cannot lead to the Pareto improvement.

Subsection 4

Unfunded Social Security

Unfunded Social Security I

- Government collects d_t from the young at time t and distributes to the current old with per capita transfer $b_t = (1+n)\, d_t$
- Individual maximization problem becomes

$$\max_{c_{1t}, c_{2t+1}, s_t} u\left(c_{1t}\right) + \beta u\left(c_{2t+1}\right)$$

subject to

$$c_{1t} + s_t + d_t \leq w_t$$

and

$$c_{2t+1} \leq R_{t+1} s_t + (1+n)\, d_{t+1},$$

for a given feasible sequence of Social Security payment levels $\{d_t\}_{t=0}^{\infty}$.

- Rate of return on Social Security payments is n rather than $r_{t+1} = R_{t+1} - 1$, because unfunded Social Security is a pure transfer system.

Unfunded Social Security

- Lifetime budget becomes

$$c_{1t} + \frac{c_{2t+1}}{R_{t+1}} + d_t - \frac{1+n}{R_{t+1}} d_{t+1} = w_t$$

so agent will be better off as long as

$$d_t - \frac{1+n}{R_{t+1}} d_{t+1} < 0 \tag{21}$$

- agent chooses s_t such that

$$u'(w_t - s_t - d_t) = \beta u'(R_{t+1} s_t + (1+n) d_{t+1})$$

- If $d_t = d_{t+1} = d$, and agent can choose d, then $s_t = 0$ if $R_{t+1} < (1+n)$!

Unfunded Social Security II

- Only s_t—rather than s_t plus d_t as in the funded scheme—goes into capital accumulation.
- It is possible that s_t will change in order to compensate, but such an offsetting change does not typically take place.
- Thus unfunded Social Security reduces capital accumulation.
- Discouraging capital accumulation can have negative consequences for growth and welfare.
- In fact, empirical evidence suggests that there are many societies in which the level of capital accumulation is suboptimally low.
- But here reducing aggregate savings may be good when the economy exhibits dynamic inefficiency.

Unfunded Social Security III

Proposition Consider the above-described overlapping generations economy and suppose that the decentralized competitive equilibrium is dynamically inefficient. Then there exists a feasible sequence of unfunded Social Security payments $\{d_t\}_{t=0}^{\infty}$ which will lead to a competitive equilibrium starting from any date t that Pareto dominates the competitive equilibrium without Social Security.

- Similar to way in which the Pareto optimal allocation was decentralized in the example economy above.
- Social Security is transferring resources from future generations to initial old generation.
- But with *no* dynamic inefficiency, any transfer of resources (and any unfunded Social Security program) would make some future generation worse-off. (follows from (21))

Section 5

OLG with Impure Altruism

Subsection 1

Impure Altruism

Overlapping Generations with Impure Altruism I

- Exact form of altruism within a family matters for whether the representative household would provide a good approximation.

$$
\begin{aligned}
U(c_t, b_t) =& u(c_t) + U^b(b_t) \\
U(c_t, b_t) =& u(c_t) + \beta V(b_t + w)
\end{aligned}
$$

- Parents care about certain dimensions of the consumption vector of their offspring instead of their total utility or "impure altruism."
- A particular type, "warm glow preferences": parents derive utility from their bequest.

Overlapping Generations with Impure Altruism I

- Production side given by the standard neoclassical production function, satisfying Assumptions 1 and 2, $f(k)$.
- Economy populated by a continuum of individuals of measure 1.
- Each individual lives for two periods, childhood and adulthood.
- In second period of his life, each individual begets an offspring, works and then his life comes to an end.
- No consumption in childhood (or incorporated in the parent's consumption).

Overlapping Generations with Impure Altruism II

- No new households, so population is constant at 1.
- Each individual supplies 1 unit of labor inelastically during is adulthood.
- Preferences of individual (i, t), who reaches adulthood at time t, are

$$\log(c_{it}) + \beta \log(b_{it}), \tag{22}$$

  where c_{it} denotes the consumption of this individual and b_{it} is bequest to his offspring.
- Offspring starts the following period with the bequest, rents this out as capital to firms, supplies labor, begets his own offspring, and makes consumption and bequests decisions.
- Capital fully depreciates after use.

Overlapping Generations with Impure Altruism III

- Maximization problem of a typical individual can be written as

$$\max_{c_{it}, b_{it}} \log(c_{it}) + \beta \log(b_{it}), \tag{23}$$

subject to

$$c_{it} + b_{it} \leq y_{it} \equiv w_t + R_t b_{it-1}, \tag{24}$$

where y_{it} denotes the income of this individual.

- Equilibrium wage rate and rate of return on capital

$$w_t = f(k_t) - k_t f'(k_t) \tag{25}$$

$$R_t = f'(k_t) \tag{26}$$

- Capital-labor ratio at time $t+1$ is:

$$k_{t+1} = \int_0^1 b_{it} di, \tag{27}$$

Overlapping Generations with Impure Altruism IV

- Measure of workers is 1, so that the capital stock and capital-labor ratio are identical.
- Denote the distribution of consumption and bequests across households at time t by $[c_{it}]_{i\in[0,1]}$ and $[b_{it}]_{i\in[0,1]}$.
- Assume the economy starts with the distribution of wealth (bequests) at time t given by $[b_{i0}]_{i\in[0,1]}$, which satisfies $\int_0^1 b_{i0} di > 0$.

Definition An equilibrium in this overlapping generations economy with warm glow preferences is a sequence of consumption and bequest levels for each household, $\left\{[c_{it}]_{i\in[0,1]}, [b_{it}]_{i\in[0,1]}\right\}_{t=0}^{\infty}$, that solve (23) subject to (24), a sequence of capital-labor ratios, $\{k_t\}_{t=0}^{\infty}$, given by (27) with some initial distribution of bequests $[b_{i0}]_{i\in[0,1]}$, and sequences of factor prices, $\{w_t, R_t\}_{t=0}^{\infty}$, that satisfy (25) and (26).

Overlapping Generations with Impure Altruism V

- Solution of (23) subject to (24) is straightforward because of the log preferences,

$$\begin{aligned} b_{it} &= \frac{\beta}{1+\beta} y_{it} \\ &= \frac{\beta}{1+\beta} \left[w_t + R_t b_{it-1} \right], \end{aligned} \tag{28}$$

  for all i and t.
- Bequest levels will follow non-trivial dynamics.
- b_{it} can alternatively be interpreted as “wealth” level: distribution of wealth that will evolve endogenously.
- This evolution will depend on factor prices.
- To obtain factor prices, aggregate bequests to obtain the capital-labor ratio of the economy via equation (27).

Overlapping Generations with Impure Altruism VI

- Integrating (28) across all individuals,

$$
\begin{aligned}
k_{t+1} &= \int_0^1 b_{it} di \\
&= \frac{\beta}{1+\beta} \int_0^1 \left[w_t + R_t b_{it-1} \right] di \\
&= \frac{\beta}{1+\beta} f\left(k_t\right). \qquad (29)
\end{aligned}
$$

- The last equality follows from the fact that $\int_0^1 b_i \left(t-1\right) di = k_t$ and because by Euler's Theorem, $w_t + R_t k_t = f\left(k_t\right)$.
- Thus dynamics are straightforward and again closely resemble Solow growth model.
- Moreover dynamics do *not* depend on the distribution of bequests or income across households.

Overlapping Generations with Impure Altruism VII

- Solving for the steady-state equilibrium capital-labor ratio from (29),

$$k^* = \frac{\beta}{1+\beta} f(k^*), \tag{30}$$

- Uniquely defined and strictly positive in view of Assumptions 1 and 2.
- Moreover, equilibrium dynamics again involve monotonic convergence to this unique steady state.
- We know that $k_t \to k^*$, so the ultimate bequest dynamics are given by steady-state factor prices.
- Let these be denoted by $w^* = f(k^*) - k^* f'(k^*)$ and $R^* = f'(k^*)$.
- Once the economy is in the neighborhood of the steady-state capital-labor ratio, k^*,

$$b_{it} = \frac{\beta}{1+\beta} \left[w^* + R^* b_{it-1} \right].$$

Overlapping Generations with Impure Altruism VIII

- When $R^* < (1+\beta)/\beta$, starting from any level b_{it} will converge to a unique bequest (wealth) level

$$b^* = \frac{\beta w^*}{1+\beta(1-R^*)}. \tag{31}$$

- Moreover, it can be verified that $R^* < (1+\beta)/\beta$,

$$\begin{aligned} R^* &= f'(k^*) \\ &< \frac{f(k^*)}{k^*} \\ &= \frac{1+\beta}{\beta}, \end{aligned}$$

- Second line exploits the strict concavity of $f(\cdot)$ and the last line uses the definition of k^* from (30).

Overlapping Generations with Impure Altruism IX

Proposition Consider the overlapping generations economy with warm glow preferences described above. In this economy, there exists a unique competitive equilibrium. In this equilibrium the aggregate capital-labor ratio is given by (29) and monotonically converges to the unique steady-state capital-labor ratio k^* given by (30). The distribution of bequests and wealth ultimately converges towards full equality, with each individual having a bequest (wealth) level of b^* given by (31) with $w^* = f(k^*) - k^* f'(k^*)$ and $R^* = f'(k^*)$.

Section 6

Overlapping Generations with Perpetual Youth

Subsection 1

Perpetual Youth in Discrete time

Perpetual Youth in Discrete time I

- Production side given by the standard neoclassical production function, satisfying Assumptions 1 and 2, $f(k)$.
- Individuals are finitely lived and they are not aware of when they will die.
- Each individual faces a constant probability of death equal to $v \in (0, 1)$. This is a simplification, since likelihood of survival is not constant.
- This last assumption implies that individuals have an expected lifespan of $\frac{1}{v} < \infty$ periods.
- Expected lifetime of an individual in this model is:

$$\text{Expected life} = v + 2(1-v)v + 3(1-v)^2 v + \cdots = \frac{1}{v} \qquad (32)$$

This equation captures the fact that with probability v the individual will have a total life of length 1, with probability $(1-v)v$, she will have a life of length 2, and so on.

Perpetual Youth in Discrete time II

- Perpetual youth: even though each individual has a finite expected life, all individuals who have survived up to a certain date have exactly the same expectation of further life.
- Each individual supplies 1 unit of labor inelastically each period she is alive.
- Expected utility of an individual with a pure discount factor β is given by

$$\sum_{t=0}^{\infty} (\beta(1-v))^t u(c_t),$$

where $u(\cdot)$ is a standard instantaneous utility function satisfying Assumption 3, with the additional normalization that $u(0) = 0$.

Perpetual Youth in Discrete time III

- Individual i's flow budget constraint is

$$a_{it+1} = (1 + r_t)a_{it} - c_{it} + w_t + z_{it} \tag{33}$$

where z_{it} reflects transfers to the individual. Since individuals face an uncertain time of death, there may be accidental bequests. Government can collect and redistribute. However, this needs $a_{it} \geq 0$ to avoid debts. An alternative is to introduce life insurance or annuity markets. Assuming competitive life insurance firms, their profits will be

$$\pi(a, t) = va - (1 - v)z(a).$$

With free entry, $\pi(a, t) = 0$, thus

$$z(a(t)) = \frac{v}{1 - v}a(t). \tag{34}$$

Perpetual Youth in Discrete time IV

- Demographics: There is an exogenous natural force toward decreasing population ($v > 0$). But there also new agents who are born into a dynasty. The evolution of the total population is given by

$$L_{t+1} = (1 + n - v)L_t, \tag{35}$$

  with the initial value $L_0 = 1$, and with $n > v$.
- Pattern of demographics in this economy: at $t > 0$ there will be:
  - 1-year-olds=$nL_{t-1} = n(1 + n - v)^{t-1}L_0$.
  - 2-year-olds=$nL_{t-2}(1 - v) = n(1 + n - v)^{t-2}(1 - v)L_0$.
  - k-year-olds=$nL_{t-k}(1 - v)^{k-1} = n(1 + n - v)^{t-k}(1 - v)^{k-1}L_0$.

Perpetual Youth in Discrete time V

- Maximization problem of a typical individual of generation τ can be written as

$$\max_{\{c_{t|\tau}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\beta(1-v))^t u(c_{t|\tau}), \tag{36}$$

subject to

$$a_{t+1|\tau} = \left(1 + r_t + \frac{v}{1-v}\right) a_{t|\tau} - c_{t|\tau} + w_t \tag{37}$$

- Equilibrium wage rate and rate of return on capital

$$w_t = f(k_t) - k_t f'(k_t) \tag{38}$$

$$R_t = f'(k_t) \tag{39}$$

Perpetual Youth in Discrete time VI

Definition An equilibrium in this overlapping generations economy with perpetual youth is a sequence of of capital stocks, wage rates, and rental rates of capital, $\{K_t, w_t, R_t\}_{t=0}^{\infty}$, and paths of consumption for each generation, $\{c_{t|\tau}\}_{t=0,\tau\leq t}^{\infty}$, such that each individual maximizes utility, and the time path of factor prices, $\{w_t, R_t\}_{t=0}^{\infty}$, is such that given the time paths of the capital stock and labor, $\{K_t, L_t\}_{t=0}^{\infty}$, all markets clear.

Perpetual Youth in Discrete time VII

- Solution of (36) subject to (37), using Bellman equation, and assuming logarithmic utility function:

$$V(a_{t|\tau}) = \max_{\left\{a_{t+1|\tau}\right\}_{t=0}^{\infty}} \log\left[c_{t|\tau}\right] + \beta(1-v)V(a_{t+1|\tau})$$

The Euler equation is:

$$c_{t+k|\tau} = \left[\beta(1-v)\right]^k \sigma_t^k c_{t|\tau}$$

where $\sigma_t^k = \prod_{j=1}^{k}\left(1 + r_{t+j} + \frac{v}{1-v}\right)$ and the transversality condition is

$$\lim_{t\to\infty}\left[\beta(1-v)\right]^k \frac{a_{t|\tau}}{c_{t|\tau}} = 0$$

Perpetual Youth in Discrete time VIII

- Consider the individual born in period τ during period t has assets $a_{t|\tau}$, it is true that

$$\sum_{k=0}^{\infty} \frac{c_{t+k|\tau}}{\sigma_t^k} = \sum_{k=0}^{\infty} \frac{w_{t+k}}{\sigma_t^k} + a_{t|\tau}$$

where we use the No Ponzi condition given by $\lim_{t\to\infty} \frac{a_{t|\tau}}{\sigma_0^t} = 0$. Using the Euler equation, we obtain

$$\sum_{k=0}^{\infty} (\beta(1-v))^k c_{t|\tau} = \frac{1}{\beta(1-v)} c_{t|\tau} = (\omega_t + a_{t|\tau})$$

or

$$c_{t|\tau} = \beta(1-v) c_{t|\tau} (\omega_t + a_{t|\tau})$$

Perpetual Youth in Discrete time IX

- Average consumption:

$$c(t) = \sum_{\tau=0}^{t} \frac{L_{t|\tau}}{L_t} c_{t|\tau} = \beta(1-v)(\omega_t + \bar{a}_t) \tag{40}$$

where $\bar{a}_t = \sum_{\tau=0}^{t} \frac{L_{t|\tau}}{L_t} a_{t|\tau}$.

- Additionally, the average stock of capital is

$$\frac{K_{t+1}}{L_{t+1}} \equiv k_{t+1} = \sum_{\tau=0}^{t} \frac{L_{t|\tau}}{L_{t+1}} a_{t|\tau} = \frac{\bar{a}_t}{1+n-v}$$

or

$$k_{t+1} = \frac{f(k_t) - c_t + (1-\delta)k_t}{1+n-v}$$

Perpetual Youth in Discrete time X

- In a steady state, $c_{t|\tau} = c_{t+1|\tau}$ i.e.

$$\left[\beta(1-v)\left(1+r^*+\frac{v}{1-v}\right)\right] = 1$$

and using $r^* = f'(k^*) - \delta$, we get

$$f'(k^*) = \frac{1-\beta(1-v)}{\beta(1-v)} + \delta.$$

This equation implies that k^* is unique.

Perpetual Youth in Discrete time XI

- Additionally, $k_t = k_{t+1}$ i.e.

  $$k^* = \frac{f(k^*) - c^* + (1-\delta)k^*}{1+n-v}$$

  which implies

  $$c^* = f(k^*) - (n + \delta - v)k^*$$

- Clearly Golden rule level in this setting requires

  $$f'(k_G^*) = n + \delta - v.$$

  So that the economy can be dynamically inefficient depending on

  $$n - v \gtreqless \frac{1-\beta(1-v)}{\beta(1-v)}.$$

  Also, we may conclude that k^* is not necessarily equal to the modified golden rule stock of capital level.

Subsection 2

Perpetual Youth in Continuous time

Perpetual Youth in Continuous time I

- The solution in this version is a closed-form solution for aggregate consumption and capital stock dynamics.
- Poisson rate of death, $v \in (0, \infty)$. Time of death follows an exponential distribution, $g(t) = ve^{-vt}$. So that the probability the agent dies before time t is $\int_0^t ve^{-vs}ds = 1 - e^{-vt}$ and the probability she is alive is e^{-vt}.

Perpetual Youth in Continuous time II

- Preferences: $u(c(t|\tau)) = \log(c(t|\tau))$, where $c(t|\tau)$ is the consumption pf generation τ at moment t.
- Expected utility of an individual is given by $\int_0^\infty e^{-(\rho+v)(t-\tau)} \log(c(t|\tau))$ or

$$e^{(\rho+v)\tau} \int_0^\infty e^{-(\rho+v)t} \log(c(t|\tau)), \tag{41}$$

  where ρ is the discount rate.
- Demographic: As in the discrete time, assuming that $n > v$, the evolution of the total population is given by

$$\dot{L}(t) = (n - v)L(t). \tag{42}$$

  It is also assumed that $n - v < \rho$.
- The number of individuals of the cohort born at time $\tau < t$ is

$$L(t|\tau) = ne^{-v(t-\tau)+(n-v)\tau}, \tag{43}$$

  where $L(0) = 1$.

Perpetual Youth in Continuous time III

- As in the discrete case, individual *i*'s flow budget constraint is

$$\dot{a}(t|\tau) = r(t)a(t|\tau) - c(t|\tau) + w(t) + z(a(t|\tau)|t,\tau), \tag{44}$$

where again $z(a(t|\tau)|t,\tau)$ reflects transfers to the individual. Since individuals face an uncertain time of death, there may be accidental bequests. Introduce a life insurance or annuity markets. Assuming competitive life insurance firms, their profits will be

$$\pi(a(t|\tau)|t,\tau) = va(t|\tau) - z(a(t|\tau)|t,\tau),$$

since the individual will die and leave his assets to the life insurance company at the flow rate v. With free entry, $\pi(a(t|\tau)|t,\tau) = 0$, thus

$$z(a(t|\tau)|t,\tau) = va(t|\tau).$$

Perpetual Youth in Continuous time IV

- Maximization problem of a typical individual of generation τ can be written as

$$\max_{c(t|\tau)} \int_0^\infty e^{-(\rho+v)t} \log(c(t|\tau)), \tag{45}$$

subject to

$$\dot{a}(t|\tau) = (r(t) + v)a(t|\tau) - c(t|\tau) + w(t). \tag{46}$$

- Equilibrium wage rate and rate of return on capital

$$w(t) = f(k(t)) - k(t) f'(k(t)) \tag{47}$$

$$R(t) = f'(k(t)) \tag{48}$$

Perpetual Youth in Continuous time V

- The law of motion of the capital-labor ratio is given by

$$\dot{k}(t) = f(k(t)) - (n - v + \delta)k(t) - c(t) \tag{49}$$

where

$$\begin{aligned} c(t) &= \frac{\int_{-\infty}^{t} c(t|\tau)L(t|\tau)d\tau}{\int_{-\infty}^{t} L(t|\tau)d\tau} \\ &= \frac{\int_{-\infty}^{t} c(t|\tau)L(t|\tau)d\tau}{L(t)} \end{aligned}$$

recalling that $L(t|\tau)$ is the size of the cohort born at τ at time t , and the lower limit of the integral is set to $-\infty$ to include all cohorts, even those born in the distant past.

Perpetual Youth in Continuous time VI

Definition An equilibrium in this overlapping generations economy with perpetual youth is a sequence of of capital stock, wage rates, and rental rates of capital, $\{K(t), w(t), R(t)\}_{t=0}^{\infty}$, and paths of consumption for each generation, $\{c(t|\tau)\}_{t=0,\tau\leq t}^{\infty}$, such that each individual maximizes utility (45) subject to (46), and the time path of factor prices, $\{w(t), R(t)\}_{t=0}^{\infty}$ given by (47) and (48), is such that given the time path of capital stock and labor, $\{K(t), L(t)\}_{t=0}^{\infty}$, all markets clear.

Perpetual Youth in Continuous time VII

- Solution of (45) subject to (46), using Hamiltonian:

$$H(\cdot) = \max_{c(t|\tau)} \log\left[c(t|\tau)\right] + \mu(t|\tau)\left((r(t)+v)a(t|\tau) + w(t) - c(t|\tau)\right)$$

The first order conditions are

$$\begin{aligned} \frac{1}{c(t|\tau)} &= \mu(t|\tau) \\ -\dot{\mu}(t|\tau) + (\rho+v)\mu(t|\tau) &= (r(t)+v)\mu(t|\tau) \\ \dot{a}(t|\tau) &= (r(t)+v)a(t|\tau) - c(t|\tau) + w(t). \end{aligned}$$

From the first FOC, $\frac{\dot{c}(t|\tau)}{c(t|\tau)} = -\frac{\dot{\mu}(t|\tau)}{\mu(t|\tau)}$.

Perpetual Youth in Continuous time VIII

- From the second FOC we have

$$-\frac{\dot{\mu}(t|\tau)}{\mu(t|\tau)} = r(t) - \rho$$
$$\mu(t|\tau) = \mu(\tau|\tau)e^{-(\bar{r}(t|\tau)-\rho)(t-\tau)}$$

  where $\bar{r}(t,\tau) \equiv \frac{1}{t-\tau}\int \tau^t r(s)d(s)$.
- The Euler equation is:

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = r(t) - \rho$$

  and the transversality condition is

$$\lim_{t\to\infty} e^{-(\rho+v)(t-\tau)}\mu(t|\tau)a(t|\tau) = 0.$$

Perpetual Youth in Continuous time IX

- Using the transversality condition and combining with the solution of $\mu(t|\tau)$ we have

$$\begin{aligned}
\lim_{t\to\infty} e^{-(\rho+v)t}\mu(\tau|\tau)e^{-(\bar{r}(t|\tau)-\rho)(t-\tau)}a(t|\tau) &= 0 \\
\lim_{t\to\infty} e^{-(\bar{r}(t|\tau)+v)(t-\tau)}\mu(\tau|\tau)a(t|\tau) &= 0 \\
\lim_{t\to\infty} e^{-(\bar{r}(t|\tau)+v)(t-\tau)}a(t|\tau) &= 0
\end{aligned}$$

which is the NPC in this case.

Perpetual Youth in Continuous time X

- Integrating the FBC

$$\int_t^\infty \left[\dot{a}(s|\tau) - (r(s) + v)a(s|\tau)\right] e^{-(\bar{r}(s,\tau)+v)(s-t)} ds =$$

$$\left[a(s|\tau)e^{-(\bar{r}(s,\tau)+v)(s-t)}\right]_t^\infty =$$

$$-a(t|\tau) = \omega(t) - \int_t^\infty c(s|\tau)e^{-(\bar{r}(s,\tau)+v)(s-t)} ds$$

$$-a(t|\tau) = \omega(t) - \int_t^\infty c(t|\tau)e^{-(\rho+v)(s-t)} ds$$

$$-a(t|\tau) = \omega(t) - c(t|\tau)\left[-\frac{e^{-(\rho+v)(s-t)}}{\rho + v}\right]_t^\infty$$

$$-a(t|\tau) = \omega(t) - c(t|\tau)\frac{1}{\rho + v}$$

Perpetual Youth in Continuous time XI

- Thus,

$$c(t|\tau) = (\rho + v)(\omega(t) + a(t|\tau)) \tag{50}$$

where $\omega(t) = \int_t^\infty e^{-(\bar{r}(s,\tau)+v)(s-t)} w(s) ds$

- In the aggregate, $a(t) = k(t)$ and $a(t) = \int_{-\infty}^t \frac{a(t|\tau)L(t|\tau)}{L(t)} d\tau$, so

$$\begin{aligned} c(t) &= \frac{\int_{-\infty}^t c(t|\tau)L(t|\tau)d\tau}{L(t)} \\ &= (\rho + v)\frac{\int_{-\infty}^t (\omega(t) + a(t|\tau))L(t|\tau)d\tau}{L(t)} \\ &= (\rho + v)\left[\omega(t) + \frac{\int_{-\infty}^t a(t|\tau)L(t|\tau)d\tau}{L(t)}\right] \\ &= (\rho + v)(\omega(t) + a(t)) \end{aligned}$$

In the aggregate, $\dot{c}(t) = (\rho + v)(\dot{\omega}(t) + \dot{a}(t))$

Perpetual Youth in Continuous time XII

- From FBC, $\dot{a}(t|\tau) = (r(t) + v)a(t|\tau) + w(t) - c(t|\tau)$, thus $\dot{a}(t) = (r(t) + v - n)a(t) + w(t) - c(t)$. Moreover, $\dot{a}(t) = \frac{\dot{\left[\int_{-\infty}^{t} a(t|\tau)L(t|\tau)d\tau\right]}}{L(t)}$, then

$$
\begin{aligned}
\dot{a}(t) =& \frac{L(t|t)}{L(t)}a(t|t) + \int_{-\infty}^{t} \frac{\dot{L}(t|\tau)}{L(t|\tau)} \frac{L(t|\tau)}{L(t)} a(t|\tau)d\tau \\
& - \int_{-\infty}^{t} \frac{L(t|\tau)}{L(t)} \frac{\dot{L}(t)}{L(t|\tau)} a(t|\tau)d\tau \\
& + \int_{-\infty}^{t} \frac{L(t|\tau)}{L(t)} \dot{a}(t|\tau)d\tau \\
=& - va(t) - (n - v)a(t) + (r(t) + v)a(t) + w(t) - c(t)
\end{aligned}
$$

and $\dot{\omega} + w(t) = (r(t) + v)\omega(t)$

Perpetual Youth in Continuous time XIII

- So

$$
\begin{aligned}
\dot{c}(t) =& (\rho+v)\left[(r(t)+v-n)a(t)+(r(t)+v)\omega(t)-c(t)\right] \\
=& (\rho+v)\left[(r(t)+v)(a(t)+\omega(t))-na(t)-c(t)\right] \\
=& (\rho+v)\left[\frac{(r(t)+v)}{\rho+v}(a(t)+\omega(t))-na(t)-c(t)\right] \\
=& (r(t)-\rho)c(t)-(\rho+v)na(t) \\
\frac{\dot{c}(t)}{c(t)} =& f'(k(t))-\delta-\rho-(\rho+v)n\frac{k(t)}{c(t)}.
\end{aligned}
$$

where the last term reflects the addition of new agents who are less wealthy than the average agent.

Perpetual Youth in Continuous time XIV

- The equilibrium dynamics are characterized by

$$\dot{k}(t) = f(k(t) - (n+\delta-v)k(t) - c(t) \tag{51}$$

$$\frac{\dot{c}(t)}{c(t)} = f^{'}(k(t)) - \delta - \rho - (\rho+v)n\frac{k(t)}{c(t)} \tag{52}$$

  with initial condition $k(0) > 0$ given and the transversality condition.
- In steady state $k(\dot{t}) = 0$ and $c(\dot{t}) = 0$, so that

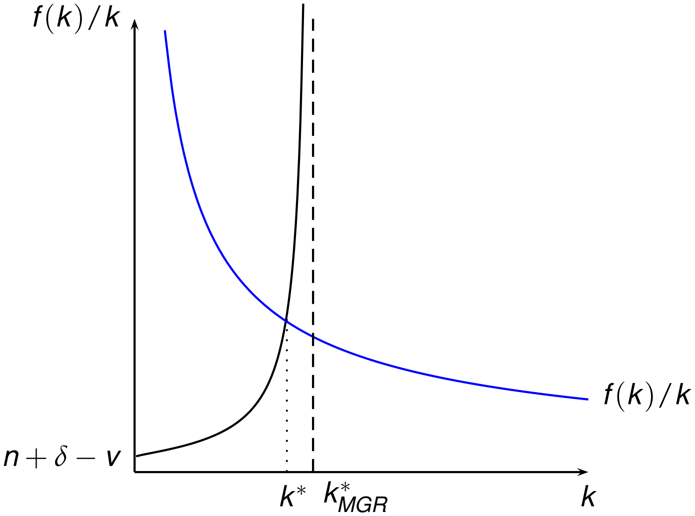
$$\frac{f^{'}(k^{*}) - \delta - \rho}{(\rho+v)n} = \frac{k^{*}}{c^{*}} \tag{53}$$

$$\frac{f(k^{*})}{k^{*}} - (n+\delta-v) = \frac{c^{*}}{k^{*}} \tag{54}$$

  that is

$$\frac{f(k^{*})}{k^{*}} = (n+\delta-v) + \frac{(\rho+v)n}{f^{'}(k^{*}) - \delta - \rho} \tag{55}$$

Perpetual Youth in Continuous time XV



Perpetual Youth in Continuous time XVI

Proposition In the continuous-time perpetual youth model, there exists a unique steady state (k^*, c^*) given by (53) and (54). The steady-state capital-labor ratio k^* (equation 55) is less than the level of capital-labor ratio that satisfies the modified golden rule, k^*_{MGR}. Starting with any $k(0) > 0$, equilibrium dynamics monotonically converge to this unique steady state.

Perpetual Youth in Continuous time XVII

- Unlike the Ramsey model, in this model it is possible to have over-accumulation of capital. Assume that every generation is born having 1 unit of labor, but this labor decreases at a rate ξ, i.e. labor income for generation τ in period t becomes $w(t)e^{-\xi(t-\tau)}$. From above, $c(t|\tau) = (\rho + v)(a(t|\tau) + \omega(t|\tau))$, where $\omega(t|\tau)$ is now $\omega(t|\tau) = \int_t^\infty e^{-(\bar{r}(s,\tau)+v)(s-t)}e^{-\xi(s-\tau)}w(s)ds$. Then $\dot{\omega}(t|\tau) = \omega(t|\tau)(r(t) + v - \xi) - w(t)e^{-\xi(t-\tau)}$.
- The equation governing the dynamic of consumption is

$$\frac{\dot{c}(t)}{c(t)} = f^{'}(k(t)) - \delta - \rho + \xi - (\rho + v)(n + \xi)\frac{k(t)}{c(t)}$$

Perpetual Youth in Continuous time XVIII

- The new dynamic system is

$$\begin{aligned}\dot{k}(t) =& f(k(t) - (n+\delta-v)k(t) - c(t)\\ \frac{\dot{c}(t)}{c(t)} =& f^{'}(k(t)) - \delta - \rho + \xi - (\rho+v)(n+\xi)\frac{k(t)}{c(t)}\end{aligned}$$

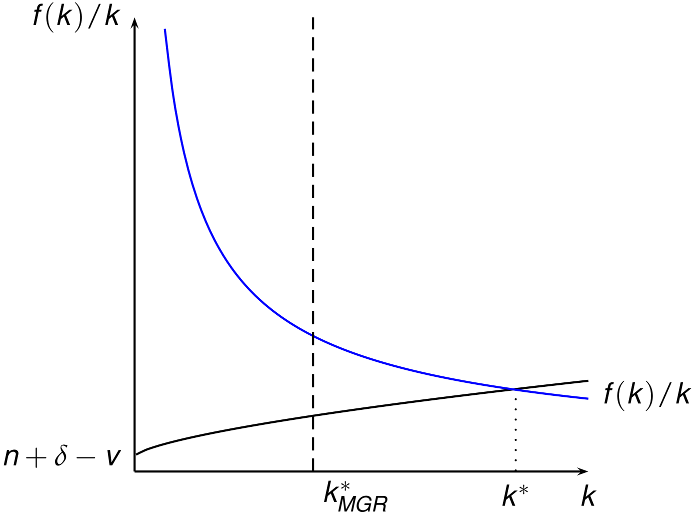
- In steady state $k(\dot{t}) = 0$ and $c(\dot{t}) = 0$, so that

$$\frac{f^{'}(k^{**}) - \delta - \rho + \xi}{(\rho+v)(n+\xi)} = \frac{k^{**}}{c^{**}} \tag{56}$$

that is

$$\frac{f(k^{**})}{k^{**}} = (n+\delta-v) + \frac{(\rho+v)(n+\xi)}{f'(k^{*}) - \delta - \rho + \xi} \tag{57}$$

Perpetual Youth in Continuous time XIX



Section 7

Conclusions

Subsection 1

Conclusions

Conclusions

- Overlapping generations often are more realistic than infinity-lived representative agents.
- Models with overlapping generations fall outside the scope of the First Welfare Theorem:
  - they were partly motivated by the possibility of Pareto suboptimal allocations.
- Equilibria may be “dynamically inefficient” and feature overaccumulation: unfunded Social Security (other assets/bubbles) can ameliorate the problem.

Conclusions

- Declining path of labor income important for overaccumulation, and what matters is not finite horizons but arrival of new individuals.
- Overaccumulation and Pareto suboptimality: pecuniary externalities created on individuals that are not yet in the marketplace.
- Not overemphasize dynamic inefficiency: major question of economic growth is why so many countries have so little capital.