Dynamic Programming and Optimal Growth

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Outline

Dynamic Programming under Certainty

Most of the problems in dynamics economics require us to find optimal paths...but how?

- If problem is finite in discrete time: Convex optimization (what you learned in undergrad calc)

$$\begin{array}{c} \max U(c_0, \ldots, c_T) \\ \text{st } G(c_0, \ldots, c_T) \epsilon B \end{array} \implies \begin{array}{c} \text{FOC: } \nabla U = \lambda \nabla G \\ \text{SOC: } U - \lambda G \text{ quasi-concave} \end{array}$$

- If problem is infinite:
  - Dynamic Programming
  - Optimal Control
    - Discrete time: Bellman's equation
    - Continuous time: Hamiltonian

Section 1

Discrete-Time Infinite-Horizon Optimization

Subsection 1

Problem

Dynamic Programming I

- Canonical dynamic optimization program in discrete time:

$$
\begin{aligned}
\sup_{\{x_t, y_t\}_{t=\mathbf{0}}^{\infty}} \sum_{t=0}^{\infty} \beta^t \tilde{U}(t, x_t, y_t)& \\
\text{subject to}& \\
y_t \in \tilde{G}(t, x_t) & \qquad \text{for all } t \geq 0 \\
x_{t+1} = \tilde{f}(t, x_t, y_t) & \qquad \text{for all } t \geq 0 \\
x_0 \text{ given,}&
\end{aligned}
$$

Dynamic Programming II

- (cont...) where
  - $\beta \in [0, 1]$ is the discount factor
  - $x_t \in X \subset \mathbb{R}^{K_x}$ and $y_t \in Y \subset \mathbb{R}^{K_y}$, for some $K_x, K_y \geq 1$.
  - x_t denotes the *state variables* and y_t denotes the *control variables*.
  - The real-valued function

    $$\tilde{U} : \mathbb{Z}_+ \times X \times Y \to \mathbb{R}$$

    is the instantaneous *payoff function* of this problem and $\sum_{t=0}^{\infty} \beta^t \tilde{U}(t, x_t, y_t)$ is the overall *objective function*.
  - Let $\tilde{G}(t, x)$ be a set-valued mapping or a correspondence, that is

    $$\tilde{G} : \mathbb{Z}_+ \times X \rightrightarrows Y.$$

Dynamic Programming III

- The previous problem, can be rewritten as follows:

$$
\begin{aligned}
\textbf{Problem 6.1} \quad & : \\
V^*(0, x_0) \quad &= \quad \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(t, x_t, x_{t+1}) \\
& \quad\ \text{subject to} \\
x_{t+1} \quad &\in \quad G(t, x_t), \qquad \text{for all } t \geq 0. \\
& \quad\ x_0 \text{ given.}
\end{aligned}
$$

- Remarks:
  - Constraint $x_{t+1} \in G(t, x_t)$: which x_{t+1} can be chosen given x_t.
  - Notice that x_{t+1} becomes the control variable, x_t is till our state variable.
  - sup rather than max: no guarantee that maximal value is attained by any feasible plan.

Dynamic Programming IV

- Remarks (cont...)
  - *Optimal plan:* when maximal value is attained by $\{x^*_{t+1}\}_{t=0}^{\infty} \in X^{\infty}$.
  - Problem is *non-stationary:* $U(x_t, x_{t+1}, t)$.
  - $V^* : \mathbb{Z}_+ \times X \to \mathbb{R}$ or *value function*: value of pursuing the optimal strategy starting with initial state x_0. It specifies the supremum (highest possible value) that the objective function can reach or approach (starting with some x_t at time t).

Subsection 2

Example

Dynamic Programming V

Example

Optimal Growth Problem
Consider the problem

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_{t+1} \leq f(k_t) - c_t + (1-\delta) k_t,$$

$k_t \geq 0$ and given k_0.

Dynamic Programming VI

Example

(cont...) Maps into the general formulation:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1} + (1-\delta)\, k_t\right)$$

subject to $k_t \geq 0$. Here we have

- $x_t = k_t$, $x_{t+1} = k_{t+1}$,
- $U\left(k_t, k_{t+1}\right) = u\left(f(k_t) - k_{t+1} + (1-\delta)\, k_t\right)$ and
- $G\left(k_t\right)$ given by $k_{t+1} \in [0, f\left(k_t\right) + (1-\delta)\, k_t]$.

Section 2

Stationary Dynamic Programming

Subsection 1

Problem

Stationary Dynamic Programming I

- The stationary form of Problem 6.1 is

$$\begin{aligned}
\textbf{Problem 6.2} &: \\
V^*(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \\
& \quad \text{subject to} \\
x_{t+1} &\in G(x_t), \qquad \text{for all } t \geq 0. \\
& \quad x_0 \text{ given.}
\end{aligned}$$

Stationary Dynamic Programming I

- Assumed discounted objective function, not $\sup_{\{x_{t+1}\}_{t=0}^{\infty}} U(x_0, x_1, ...)$.
- Discounted objective function ensures *time-consistency*.
- Problem 6.2 or *sequence problem*:
  - choosing an infinite sequence $\{x_t\}_{t=0}^{\infty}$ from some (vector) space of infinite sequences.
  - E.g. $\{x_t\}_{t=0}^{\infty} \in X^{\infty} \subset \mathcal{L}^{\infty}$, where $\mathcal{L}^{\infty}$: vector space of infinite sequences bounded with the $\|\cdot\|_{\infty}$ norm, which we will denote throughout by $\|\cdot\|$).
- Sequence problems solutions often difficult to characterize both analytically and numerically.
- Idea of dynamic programming: transform the problem into one of finding a function rather than a sequence

Stationary Dynamic Programming II

- The basic idea of dynamic programming is to turn the sequence problem into a functional equation; that is, to transform the problem into one of finding a function rather than a sequence. The relevant functional equation can be written as follows.

Problem 6.3 :

$$V(x) = \sup_{y \in G(x)} \{U(x,y) + \beta V(y)\}, \text{ for all } x \in X, \quad (2.1)$$

where $V : X \to \mathbb{R}$

- Remarks:
  - Instead of $\{x_t\}_{t=0}^{\infty}$, in (2.1) choose a *policy*: what x_{t+1} should be for a given x_t.
  - Since $U(\cdot,\cdot)$ does not depend on time, no reason for policy to be time-dependent either.
  - Denote control vector by y and state vector by x: problem is choosing right y for any x.
  - Mathematically, corresponds to maximizing $V(x)$ for any $x \in X$.

Stationary Dynamic Programming III

- Remarks (cont...)
  - Only subtlety in (2.1) is *recursive formulation*: $V(\cdot)$ on the right-hand side.
  - Functional equation in Problem 6.3 also called the *Bellman equation*.
  - Functional equation easy to work with in many instances.
  - In applied mathematics and engineering: computationally convenient.
  - In economics: gives better economic insights, similar to the logic of comparing today to tomorrow.
  - In some special but important cases: solution to Problem 3 simpler to characterize analytically than solution of 2.

Stationary Dynamic Programming IV

- Form of Problem 3 suggests itself naturally from Problem 2.
- Suppose Problem 2 has a maximum starting at x_0 attained by $\{x_t^*\}_{t=0}^{\infty}$ with $x_0^* = x_0$.
- Then under some relatively weak technical conditions:

$$\begin{aligned} V^*(x_0) &= \sum_{t=0}^{\infty} \beta^t U(x_t^*, x_{t+1}^*) \\ &= U(x_0, x_1^*) + \beta \sum_{s=0}^{\infty} \beta^s U(x_{s+1}^*, x_{s+2}^*) \\ &= U(x_0, x_1^*) + \beta V^*(x_1^*). \end{aligned}$$

- Encapsulates basic idea of dynamic programming: *Principle of Optimality*.
- Break optimal plan into two parts: what is optimal to do today, and the optimal continuation path.

Stationary Dynamic Programming V

- Solution can be represented by time invariant *policy function* determining x_{t+1} for a given x_t
$$\pi : X \to X.$$
- Two complications in general:
  1. a control reaching the optimal value may not exist
  2. there may be more than one maximizer: not a policy function but a correspondence $\Pi : X \rightrightarrows X$.
- Ignoring complications, once value function V is determined, if optimal policy is given by a policy function $\pi(x)$, then
$$V(x) = U(x, \pi(x)) + \beta V(\pi(x)), \text{ for all } x \in X,$$
- Provides one way of determining the policy function.

Section 3

Stationary Dynamic Programming Theorems

Stationary Dynamic Programming

- Consider a sequence $\{x_t^*\}_{t=0}^{\infty}$ which attains the supremum in Problem 2.
- Main purpose is to ensure this sequence satisfies recursive equation:

$$\begin{aligned} V(x_t^*) &= U(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*), \\ \text{for all } t &= 0, 1, 2, ..., \end{aligned} \tag{3.1}$$

  and that any solution to (3.1) will also be a solution to Problem 2.
- Define the set of feasible sequences or *plans* starting with an initial value x_t as:

$$\mathbf{\Phi}(x_t) = \{\{x_s\}_{s=t}^{\infty} : x_{s+1} \in G(x_s), \text{ for } s = t, t+1, ...\}.$$

- Denote a typical element of the set $\Phi(x_0)$ by $\mathbf{x} = (x_0, x_1, ...) \in \Phi(x_0)$.

Subsection 1

Assumptions

Assumptions I

Assumption 6.1

$G(x)$ is nonempty for all $x \in X$; and for all $x_0 \in X$ and $\mathbf{x} \in \Phi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x_t, x_{t+1})$ exists and is finite.

- Stronger than necessary: sufficient that the limit exists.
- But if households or firms achieve infinite value, mathematically typically not well defined and essence of economics, tradeoffs in the face of scarcity, would be absent.
- Could use "overtaking criteria:" compare sequences by looking at whether one of them gives higher utility than the other one at each date after some finite threshold.

Some Definitions I

Definition (Upper hemicontinuity)

A correspondence $G : X \rightrightarrows Y$ is said to be *upper hemicontinuous* at the point $x \in X$, if for any open neighborhood A of $G(x)$, $A \subset Y$, there exists a neighborhood $B(x) \subset X$ of x such that for all $\tilde{x}$ in $B(x)$, $G(\tilde{x})$ is a subset of A.

Equivalently, a correspondence $G : X \rightrightarrows Y$ is said to be *upper hemicontinuous* at the point $x \in X$, if for any sequence $\{x_n, y_n\}$ such that $y_n \in G(x_n)$, $x_n \to x$ and $y_n \to y \in Y$, it follows that $y \in G(x)$.

Some Definitions II

Definition (Lower hemicontinuity)

A correspondence $G : X \rightrightarrows Y$ is said to be *lower hemicontinuous* at the point $x \in X$, if for any open set $A \subset Y$ such that $A \cap G(x) \neq \emptyset$, there exists a neighborhood $B(x) \subset X$ of x such that for all $\tilde{x}$ in $B(x)$, $A \cap G(\tilde{x}) \neq \emptyset$.

Equivalently, a correspondence $G : X \rightrightarrows Y$ is said to be *lower hemicontinuous* at the point $x \in X$, if for any sequence $\{x_n\}$ such that $x_n \to x$, for any $y \in G(x)$, there exists a subsequence $\{x_{n_k}\}$ and a sequence $\{y_k\}$ such that $y_k \in G(x_{n_k})$ and $y_k \to y \in Y$.

Some Definitions III

Definition (Continuity)

A correspondence $G : X \rightrightarrows Y$ is said to be *continuous* at the point $x \in X$, if it is both upper and lower hemicontinuous at x.

Definition (Graph)

The *graph* of a correspondence $G : X \rightrightarrows Y$ is the set $X_G = \{(x, y) \in X \times Y \mid y \in G(x)\}$.

Assumptions II

Assumption 6.2

X is a compact subset of $\mathbb{R}^K$, G is nonempty, compact-valued and continuous. Moreover, $U : \mathbf{X}_G \to \mathbb{R}$ is continuous, where $\mathbf{X}_G = \{(x, y) \in X \times X : y \in G(x)\}$.

- Need $G(x)$ compact-valued: optimization problems with choices from non–compact sets are not well behaved
- U continuous leads to little loss of generality for most economic applications.
- Most restrictive assumption is X is compact.
- Most important results can be generalized to X not compact, but requires additional notation and more difficult analysis.
- Note since X is compact, $G(x)$ is continuous and compact-valued, $\mathbf{X}_G$ is also compact.
- Since a continuous function from a compact domain is also bounded, Assumption 6.2 also implies that U is bounded.
- Assumptions 6.1 and 6.2 together ensure that in both Problems 2 and 3, the supremum (the maximal value) is attained at a finite value for some feasible plan $\mathbf{x}^*$.

Assumptions III

Assumption 6.3

G is convex: for any $\alpha \in [0,1]$, and $x, x' \in X$, whenever $y \in G(x)$ and $y' \in G(x')$

$$\alpha y + (1-\alpha)y' \in G\left(\alpha x + (1-\alpha)x'\right).$$

Additionally, U is strictly concave: for any $\alpha \in (0,1)$ and any (x,y), $(x',y') \in \mathbf{X}_G$

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y'\right) \geq \alpha U(x,y) + (1-\alpha)U(x',y'),$$

and if $x \neq x'$,

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y'\right) > \alpha U(x,y) + (1-\alpha)U(x',y').$$

Assumptions IV

Assumption 6.4

For each $y \in X$, $U(\cdot, y)$ is strictly increasing in each of its first K arguments, and G is monotone in the sense that $x \leq x'$ implies $G(x) \subset G(x')$.

Assumption 6.5

U is continuously differentiable on the interior of its domain $\mathbf{X}_G$.

Subsection 2

Theorems

Dynamic Programming Theorems I

Theorem 6.1 (Equivalence of Values)

Suppose Assumptions 6.1 and 6.2 hold. Then for any $x \in X$, $V^*(x)$ defined in Problem 2 is also a solution to Problem 3. Moreover, any $V(x)$ defined in Problem 3 that satisfies $\lim_{t\to\infty} \beta^t V(x_t) = 0$ for all $(x, x_1, x_2, ...) \in \Phi(x)$ is also a solution to Problem 2, so that $V^*(x) = V(x)$ for all $x \in X$.

Theorem 6.2 (Principle of Optimality)

Suppose Assumption 6.1 holds. Let $\mathbf{x}^* \in \Phi(x_0)$ be a feasible plan that attains $V^*(x_0)$ in Problem 2. Then for $t = 0, 1, ...$ with $x_0^* = x_0$,

$$V^*(x_t^*) = U(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*) \tag{3.2}$$

Moreover, if any $\mathbf{x}^* \in \Phi(x_0)$ satisfies (3.2), then it attains the optimal value in Problem 2.

Dynamic Programming Theorems II

- Returns from an optimal plan (sequence) $\mathbf{x}^* \in \Phi(x_0)$ can be broken into the current return, $U(x_t^*, x_{t+1}^*)$, and the continuation return $\beta V^*(x_{t+1}^*)$, identically given by the discounted value of a problem starting from x_{t+1}^*.
- Since V^* in Problem 2 and V in Problem 3 are identical from the Equivalence of Values Theorem, (3.2) also implies

$$V(x_t^*) = U(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*).$$

- Second part equally important: if any feasible plan $\mathbf{x}^*$ starting with x_0, $\mathbf{x}^* \in \Phi(x_0)$, satisfies (3.2), then $\mathbf{x}^*$ attains $V^*(x_0)$.
- We can go from the solution of the recursive problem to the solution of the original problem and vice versa under Assumptions 6.1 and 6.2.

Dynamic Programming Theorems III

Theorem 6.3 (Existence of Solutions)

Suppose that Assumptions 6.1 and 6.2 hold. Then there exists a unique continuous and bounded function $V : X \to \mathbb{R}$ *that satisfies (2.1). Moreover, an optimal plan* $\mathbf{x}^* \in \Phi(x_0)$ *exists for any* $x_0 \in X$.

- Uniqueness of the value function combined with Equivalence of Values Theorem implies an optimal solution achieves supremum V^* in Problem 2 and also that like V, V^* is continuous and bounded.
- But optimal plan that solves Problem 2 or 3 may not be unique.

Theorem 6.4 (Concavity of the Value Function)

Suppose that Assumptions 6.1, 6.2 and 6.3 hold. Then the unique $V : X \to \mathbb{R}$ *that satisfies (2.1) is strictly concave.*

Dynamic Programming Theorems IV

Corollary 6.1

Suppose that Assumptions 6.1, 6.2 and 6.3 hold. Then there exists a unique optimal plan $\mathbf{x}^* \in \Phi(x_0)$ *for any* $x_0 \in X$*. Moreover, the optimal plan can be expressed as* $x^*_{t+1} = \pi(x^*_t)$*, where* $\pi : X \to X$ *is a continuous policy function.*

- I.e., policy function π is indeed a function, not a correspondence because x^* is uniquely determined.
- Also implies π is continuous in the state vector.
- Moreover, if a vector of parameters $\mathbf{z}$ continuously affects either Φ or U, same argument establishes π is also continuous in $\mathbf{z}$.

Dynamic Programming Theorems V

Theorem 6.5 (Monotonicity of the Value Function)
Suppose that Assumptions 6.1, 6.2 and 6.4 hold and let $V : X \to \mathbb{R}$ be the unique solution to (2.1). Then V is strictly increasing in all of its arguments.

- Difficulty to characterize solution using differential calculus with (2.1): right-hand side includes V.

Theorem 6.6 (Differentiability of the Value Function)

Suppose that Assumptions 6.1, 6.2, 6.3 and 6.5 hold. Let π be the policy function defined above and assume that $x' \in \text{Int}\, X$ and $\pi(x') \in \text{Int}\, G(x')$, then $V(x)$ is continuously differentiable at x', with derivative given by

$$DV(x') = D_x U(x', \pi(x')) . \tag{3.3}$$

Section 4

The Contraction Mapping Theorem and Applications*

Subsection 1

Contraction Mapping Theorem

Contraction Mapping Theorem and Applications* I

- Recall (S, d) is a metric space, if S is a non-empty set and d is a metric defined over this space with the usual properties.
- *Operators* or *mappings*: "functions" from the metric space into itself, denoted by T and writing Tz for the image of a point $z \in S$ under T, and $T(Z)$ when T is applied to a subset Z of S.

Definition Let (S, d) be a metric space and $T : S \to S$ be an operator mapping S into itself. T is a *contraction mapping* (with *modulus* β) if for some $\beta \in (0, 1)$,

$$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2), \text{ for all } z_1, z_2 \in S.$$

Contraction Mapping Theorem and Applications* II

- **Example:** Take a simple interval of the real line, $S = [a, b]$, with usual metric $d(z_1, z_2) = |z_1 - z_2|$. Then $T : S \to S$ is a contraction if for some $\beta \in (0, 1)$,

$$\frac{|Tz_1 - Tz_2|}{|z_1 - z_2|} \leq \beta < 1, \quad \text{all } z_1, z_2 \in S \text{ with } z_1 \neq z_2.$$

Definition A *fixed point* of T is any element of S satisfying $Tz = z$.

- Recall (S, d) is complete if every Cauchy sequence (whose elements are getting closer) in S converges to an element in S.

Theorem **(Contraction Mapping Theorem)** Let (S, d) be a complete metric space and suppose that $T : S \to S$ is a contraction. Then T has a unique fixed point, $\hat{z}$, i.e., there exists a unique $\hat{z} \in S$ such that

$$T\hat{z} = \hat{z}.$$

Subsection 2

Proof

Proof of Contraction Mapping Theorem I

- *(Existence)* Note $T^n z = T(T^{n-1}z)$ for any $n = 1, 2, ...$ Choose $z_0 \in S$, and construct a sequence $\{z_n\}_{n=0}^{\infty}$ with each element in S, such that $z_{n+1} = Tz_n$ so that
$$z_n = T^n z_0.$$
- Since T is a contraction:
$$d(z_2, z_1) = d(Tz_1, Tz_0) \leq \beta d(z_1, z_0).$$
- Repeating this argument
$$d(z_{n+1}, z_n) \leq \beta^n d(z_1, z_0), \quad n = 1, 2, ... \tag{4.1}$$
- Hence, for any $m > n$,
$$\begin{aligned} d(z_m, z_n) &\leq d(z_m, z_{m-1}) + ... + d(z_{n+2}, z_{n+1}) + d(z_{n+1}, z_n) \qquad (4.2) \\ &\leq \left(\beta^{m-1} + ... + \beta^{n+1} + \beta^n\right) d(z_1, z_0) \\ &= \beta^n \left(\beta^{m-n-1} + ... + \beta + 1\right) d(z_1, z_0) \leq \frac{\beta^n}{1-\beta} d(z_1, z_0), \end{aligned}$$

Proof of Contraction Mapping Theorem II

- Above: first inequality uses the triangle inequality, second uses (4.1), last uses $1/(1-\beta) = 1 + \beta + \beta^2 + ... > \beta^{m-n-1} + ... + \beta + 1$.
- Inequalities in (4.2) imply as $n \to \infty$, $m \to \infty$, z_m and z_n will be approaching each other, so that $\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence.
- Since S is complete, every Cauchy sequence in S has a limit point in S, therefore:

$$z_n \to \hat{z} \in S.$$

- Note that for any $z_0 \in S$ and any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(T\hat{z}, \hat{z}) &\leq d(T\hat{z}, T^n z_0) + d(T^n z_0, \hat{z}) \\ &\leq \beta d(\hat{z}, T^{n-1} z_0) + d(T^n z_0, \hat{z}), \end{aligned}$$

- First relationship uses the triangle inequality, and second that T is a contraction.
- Since $z_n \to \hat{z}$, both of the terms on the right tend to zero as $n \to \infty$, which implies that $d(T\hat{z}, \hat{z}) = 0$, and therefore $T\hat{z} = \hat{z}$, so $\hat{z}$ is a fixed point.

Proof of Contraction Mapping Theorem III

- *(Uniqueness)* Suppose, to obtain a contradiction, that there exist $\hat{z}, z \in S$, such that $Tz = z$ and $T\hat{z} = \hat{z}$ with $\hat{z} \neq z$.
- This implies

$$0 < d(\hat{z}, z) = d(T\hat{z}, Tz) \leq \beta d(\hat{z}, z),$$

which delivers a contradiction in view of the fact that $\beta < 1$.

Example: Difference Equation

- Consider the following difference equation:

$$x_{t+1} = ax_t + b$$

where $x_t \epsilon \mathbb{R}$ for all $t \geq 0$. Then

$$T(x) = ax + b$$

and

$$\|T(x) - T(x')\| = \|(ax + b) - (ax' + b)\| = \|a(x - x')\| \leq |a| \, |x - x'| \, .$$

So, $T(x)$ is a contraction if $|a| < 1$, in which case there exists a unique fixed point $x^* = T(x^*)$ and $x_t \to x^*$ as $t \to \infty$.

Example: Differential Equation I

- Consider the following one-dimensional differential equation

$$\dot{x}(t) = f(x(t)), \tag{4.3}$$

with a boundary condition $x(0) = c \in \mathbb{R}$.

- Suppose that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous: it is continuous and for some $M < \infty$, it satisfies $|f(x'') - f(x')| \leq M |x'' - x'|$ for all $x', x'' \in \mathbb{R}$.
- Contraction Mapping Theorem (CMT) can be used to prove the existence of a continuous function $x^*(t)$ that is the unique solution to this differential equation on any compact interval $[0, s]$ for some $s \in \mathbb{R}_+$.
- Consider the space of continuous functions on $[0, s]$, $\mathbf{C}[0, s]$, and define the operator T such that for any $g \in \mathbf{C}[0, s]$,

$$Tg(z) = c + \int_0^z f(g(x))\, dx.$$

Notice that a fixed point of T is the solution we need.

Example: Differential Equation II

- T is a mapping from the space of continuous functions on $[0, s]$ into itself, i.e., $T : \mathbf{C}[0, s] \to \mathbf{C}[0, s]$.
- Moreover, T is a contraction for some s because for any $z \in [0, s]$, by the Lipschitz continuity of $f(\cdot)$.

$$\left| \int_0^z f(g(x))\, dx - \int_0^z f(\tilde{g}(x))\, dx \right| \leq \int_0^z M \left| g(x) - \tilde{g}(x) \right| dx \tag{4.4}$$

- This implies that

$$\| Tg(z) - T\tilde{g}(z) \| \leq M \times s \times \| g - \tilde{g} \| ,$$

- Choosing $s < 1/M$, T is indeed a contraction.
- Applying the Contraction Mappting Theorem there exists a unique fixed point of T over $\mathbf{C}[0, s]$.
- This fixed point is the unique solution to the differential equation and it is also continuous.

Applications of Contraction Mapping Theorem I

- Main use of the CMT for us: it can be applied to space of functions, so applying it to equation (2.1) will establish the existence of a unique V in Problem 6.2.
- Thus must prove that the recursion in (2.1) defines a contraction mapping.
- Recall that if (S, d) is a complete metric space and S' is a closed subset of S, then (S', d) is also a complete metric space.

Theorem **(Applications of Contraction Mappings)** Let (S, d) be a complete metric space, $T : S \to S$ be a contraction mapping with $T\hat{z} = \hat{z}$.

1. If S' is a closed subset of S, and $T(S') \subset S'$, then $\hat{z} \in S'$.
2. Moreover, if $T(S') \subset S'' \subset S'$, then $\hat{z} \in S''$.

Applications of Contraction Mapping Theorem II

- **Proof:**
  - Take $z_0 \in S'$, and construct the sequence $\{T^n z_0\}_{n=0}^{\infty}$.
  - Each element of this sequence is in S' by the fact that $T(S') \subset S'$.
  - CMT implies that $T^n z_0 \to \hat{z}$.
  - Since S' is closed, $\hat{z} \in S'$, proving part 1.
  - We know that $\hat{z} \in S'$.
  - Then the fact that $T(S') \subset S'' \subset S'$ implies that $\hat{z} = T\hat{z} \in T(S') \subset S''$, establishing part 2.
- Second part very important to prove results such as strict concavity or that a function is strictly increasing
  - The set of strictly concave functions or the set of the strictly increasing functions are not closed (and complete).
  - Thus cannot apply the CMT to these spaces of functions.
- Second part enables us to circumvent this problem.

Blackwell's Sufficient Conditions

- Difficult to check whether an operator is indeed a contraction, especially with spaces whose elements correspond to functions.
- For a real valued function $f(\cdot)$ and some constant $c \in \mathbb{R}$ we define $(f+c)(x) \equiv f(x)+c$.

Theorem **(Blackwell's Sufficient Conditions For a Contraction)** Let $X \subseteq \mathbb{R}^K$, and $\mathbf{B}(X)$ be the space of bounded functions $f : X \to \mathbb{R}$ defined on X. Suppose that $T : \mathbf{B}(X) \to \mathbf{B}(X)$ is an operator satisfying the following two conditions:

1. **(monotonicity)** For any $f, g \in \mathbf{B}(X)$ and $f(x) \leq g(x)$ for all $x \in X$ implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.
2. **(discounting)** There exists $\beta \in (0,1)$ such that for all $f \in B(X)$, $c \geq 0$ and $x \in X$

$$[T(f+c)](x) \leq (Tf)(x) + \beta c.$$

Then, T is a contraction with modulus β.

Proof of Blackwell's Sufficient Conditions

- Let $\|\cdot\|$ denote the sup norm, so that $\|f - g\| = \sup_{x\in X} |f(x) - g(x)|$. Then, by definition for any $f, g \in \mathbf{B}(X)$,

$$
\begin{array}{rcll}
f(x) & \leq & g(x) + \|f - g\| & \text{for any } x \in X, \\
(Tf)(x) & \leq & T[g + \|f - g\|](x) & \text{for any } x \in X, \\
(Tf)(x) & \leq & (Tg)(x) + \beta \|f - g\| & \text{for any } x \in X,
\end{array}
$$

- the second line applies T on both sides and uses monotonicity, the third uses discounting ($\|f - g\|$ is simply a number).
- By the converse argument,

$$
\begin{array}{rcll}
g(x) & \leq & f(x) + \|g - f\| & \text{for any } x \in X, \\
(Tg)(x) & \leq & T[f + \|g - f\|](x) & \text{for any } x \in X, \\
(Tg)(x) & \leq & (Tf)(x) + \beta \|g - f\| & \text{for any } x \in X.
\end{array}
$$

- Combining the last two inequalities:

$$
\|Tf - Tg\| \leq \beta \|f - g\|.
$$

Section 5

Proofs of the Main Dynamic Programming Theorems*

Subsection 1

Proofs of Theorems

Proofs of the Main Dynamic Programming Theorems* I

- For a feasible infinite sequence $\mathbf{x} = (x_0, x_1, ...) \in \Phi(x_0)$ starting at x_0, let the value of choosing this potentially non-optimal infinite feasible sequence be

$$\mho(\mathbf{x}) \equiv \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1})$$

- Assumption 6.1 implies $\mho(\mathbf{x})$ exists and is finite.
- $\mho(\mathbf{x})$ can be separated into two parts: current return and the continuation return.

Lemma Suppose that Assumption 6.1 holds. Then for any $x_0 \in X$ and any $\mathbf{x} \in \Phi(x_0)$, we have that

$$\mho(\mathbf{x}) = U(x_0, x_1) + \beta \mho(\mathbf{x}')$$

where $\mathbf{x}' = (x_1, x_2, ...)$.

Proofs of the Main Dynamic Programming Theorems* II

- **Proof:** Since under Assumption 6.1 $\mho(\mathbf{x})$ exists and is finite, we have

$$
\begin{aligned}
\mho(\mathbf{x}) &= \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \\
&= U(x_0, x_1) + \beta \sum_{s=0}^{\infty} \beta^s U(x_{s+1}, x_{s+2}) \\
&= U(x_0, x_1) + \beta \mho(\mathbf{x}')
\end{aligned}
$$

- To prove the theorems, useful to be more explicit about what it means for V and V^* to be solutions to Problems 6.2 and 6.3.
- Problem 6.2: for any $x_0 \in X$,

$$
V^*(x_0) = \sup_{\mathbf{x} \in \mathbf{\Phi}(x_0)} \mho(\mathbf{x}).
$$

Proofs of the Main Dynamic Programming Theorems* III

- Assumption 6.1 ensures that all values are bounded, so

$$V^*(x_0) \geq \mathbf{U}(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{\Phi}(x_0), \tag{5.1}$$

- However, if some function $\tilde{V}$ satisfies condition (5.1), so will $\alpha \tilde{V}$ for $\alpha > 1$.
- Therefore, this condition is not sufficient; also require that

$$\text{for any } \varepsilon > 0, \text{ there exists } \mathbf{x}' \in \mathbf{\Phi}(x_0) \text{ s.t. } V^*(x_0) \leq \mathbf{U}(\mathbf{x}') + \varepsilon, \tag{5.2}$$

- Similarly: for $V(\cdot)$ to be a solution to Problem 6.2, for any $x_0 \in X$,

$$V(x_0) \geq U(x_0, y) + \beta V(y), \quad \text{all } y \in G(x_0), \tag{5.3}$$

$$\begin{aligned} \text{for any } \varepsilon &> 0, \text{ there exists } y' \in G(x_0) \\ \text{s.t. } V(x_0) &\leq U(x_0, y') + \beta V(y') + \varepsilon. \end{aligned} \tag{5.4}$$

Proof of Equivalence of Values Theorem I

- If $\beta = 0$, Problems 6.1 and 6.2 are identical, thus the result follows immediately.
- Suppose $\beta > 0$ and take an arbitrary $x_0 \in X$ and some $x_1 \in G(x_0)$.
- The objective function in Problem 6.2 is continuous in the product topology in view of Assumptions 6.1 and 6.2.
- Moreover, the constraint set $\Phi(x_0)$ is a closed subset of X^∞.
- From Assumption 6.2, X is compact. By Tychonoff's Theorem X^∞ is compact in the product topology.
- A closed subset of a compact set is compact, so $\Phi(x_0)$ is compact.
- Apply Weierstrass' Theorem to Problem 6.2: there exists $\mathbf{x} \in \Phi(x_0)$ attaining $V^*(x_0)$.
- Moreover, the constraint set is a continuous correspondence (again in the product topology).

Proof of Equivalence of Values Theorem II

- Apply Berge's Maximum Theorem: $V^*(x_0)$ is continuous.
- Since $x_0 \in X$ and X is compact, this implies $V^*(x_0)$ is bounded.
- A similar reasoning implies that there exists $\mathbf{x}' \in \Phi(x_1)$ attaining $V^*(x_1)$.
- Next, since $(x_0, \mathbf{x}') \in \Phi(x_0)$ and $V^*(x_0)$ is the supremum in Problem 6.2 starting with x_0, the Lemma above implies

$$\begin{aligned} V^*(x_0) &\geq U(x_0, x_1) + \beta \mho(\mathbf{x}'), \\ &= U(x_0, x_1) + \beta V^*(x_1), \end{aligned}$$

  thus verifying (5.3).
- Next, take an arbitrary $\varepsilon > 0$. By (5.2), there exists $\mathbf{x}'_\varepsilon = (x_0, x'_{\varepsilon 1}, x'_{\varepsilon 2}, ...) \in \Phi(x_0)$ such that

$$\mho(\mathbf{x}'_\varepsilon) \geq V^*(x_0) - \varepsilon.$$

Proof of Equivalence of Values Theorem III

- Now since $\mathbf{x}''_{\varepsilon} = (x'_{\varepsilon 1}, x'_{\varepsilon 2}, ...) \in \Phi(x'_{\varepsilon 1})$ and $V^*(x'_{\varepsilon 1})$ is the supremum in Problem 6.3 starting with $x'_{\varepsilon 1}$, the Lemma above implies

$$\begin{aligned} U(x_0, x'_{\varepsilon 1}) + \beta \bar{U}(\mathbf{x}''_{\varepsilon}) &\geq V^*(x_0) - \varepsilon \\ U(x_0, x'_{\varepsilon 1}) + \beta V^*(x'_{\varepsilon 1}) &\geq V^*(x_0) - \varepsilon, \end{aligned}$$

- The last inequality verifies (5.4) since $x'_{\varepsilon 1} \in G(x_0)$ for any $\varepsilon > 0$.
- *Thus, any solution to Problem 6.2 satisfies (5.3) and (5.4), and is thus a solution to Problem 6.3.*
- To establish the reverse, note (5.3) implies that for any $x_1 \in G(x_0)$,

$$V(x_0) \geq U(x_0, x_1) + \beta V(x_1).$$

Proof of Equivalence of Values Theorem IV

- Substituting recursively for $V(x_1)$, $V(x_2)$, etc., and defining $\mathbf{x} = (x_0, x_1, ...)$:

$$V(x_0) \geq \sum_{t=0}^{n} \beta^t U(x_t, x_{t+1}) + \beta^{n+1} V(x_{n+1}).$$

- Since $n \to \infty$, $\sum_{t=0}^{n} \beta^t U(x_t, x_{t+1}) \to \mathbf{U}(\mathbf{x})$ and $\beta^{n+1} V(x_{n+1}) \to 0$ (by hypothesis), we have that

$$V(x_0) \geq \mathbf{U}(\mathbf{x}),$$

  for any $\mathbf{x} \in \Phi(x_0)$, thus verifying (5.1).
- Next, let $\varepsilon > 0$ be a positive scalar. From (5.4), for any $\varepsilon' = \varepsilon(1-\beta) > 0$, there exists $x_{\varepsilon 1} \in G(x_0)$ such that

$$V(x_0) \leq U(x_0, x_{\varepsilon 1}) + \beta V(x_{\varepsilon 1}) + \varepsilon'.$$

Proof of Equivalence of Values Theorem V

- Let $x_{\varepsilon t} \in G\left(x_{\epsilon t-1}\right)$, with $x_{\varepsilon 0} = x_0$, and define $\mathbf{x}_\varepsilon \equiv \left(x_0, x_{\varepsilon 1}, x_{\varepsilon 2}, ...\right)$.
- Again substituting recursively for $V\left(x_{\varepsilon 1}\right)$, $V\left(x_{\varepsilon 2}\right)$,...,

$$
\begin{aligned}
V\left(x_0\right) &\leq \sum_{t=0}^{n} \beta^t U\left(x_{\varepsilon t}, x_{\varepsilon t+1}\right) + \beta^{n+1} V\left(x_{n+1}\right) \\
&\quad + \varepsilon' + \varepsilon' \beta + ... + \varepsilon' \beta^n \\
&\leq \mho\left(\mathbf{x}_\varepsilon\right) + \varepsilon,
\end{aligned}
$$

- Last line uses definition of ε ($\varepsilon = \varepsilon' \sum_{t=0}^{\infty} \beta^t$) and that as $n \to \infty$, $\sum_{t=0}^{n} \beta^t U\left(x_{\varepsilon t}, x_{\varepsilon t+1}\right) \to \mho\left(\mathbf{x}_\varepsilon\right)$.
- This establishes that $V(x_0)$ satisfies (5.2), and completes the proof. $\square$

Proof of the Principle of Optimality Theorem I

- By hypothesis $\mathbf{x}^* \equiv (x_0, x_1^*, x_2^*, ...)$ is a solution to Problem 6.2, i.e., it attains the supremum, $V^*(x_0)$ starting from x_0.
- Let $\mathbf{x}_t^* \equiv (x_t^*, x_{t+1}^*, ...)$.
- First show by induction that for any $t \geq 0$, $\mathbf{x}_t^*$ attains the supremum starting from x_t^*, so that
$$\mathbf{U}(\mathbf{x}_t^*) = V^*(x_t^*). \tag{5.5}$$
- Base step of induction for $t = 0$: by definition, $\mathbf{x}_0^* = \mathbf{x}^*$ attains $V^*(x_0)$.
- Suppose (5.5) is true for t, and we will establish it for $t + 1$.
- Equation (5.5) implies that
$$\begin{aligned} V^*(x_t^*) &= \mathbf{U}(\mathbf{x}_t^*) \\ &= U(x_t^*, x_{t+1}^*) + \beta\mathbf{U}(\mathbf{x}_{t+1}^*). \end{aligned} \tag{5.6}$$

Proof of the Principle of Optimality Theorem II

- Let $\mathbf{x}_{t+1} = \left(x_{t+1}^*, x_{t+2}, ...\right) \in \Phi\left(x_{t+1}^*\right)$ be any feasible plan starting with x_{t+1}^*.
- By definition, $\mathbf{x}_t = (x_t^*, \mathbf{x}_{t+1}) \in \Phi(x_t^*)$. Since $V^*\left(x_t^*\right)$ is the supremum starting with x_t^*:

$$
\begin{aligned}
V^*\left(x_t^*\right) &\geq \mathbf{U}(\mathbf{x}_t) \\
&= U(x_t^*, x_{t+1}^*) + \beta \mathbf{U}(\mathbf{x}_{t+1}).
\end{aligned}
$$

- Combining this inequality with (5.6), we obtain for all $\mathbf{x}_{t+1} \in \Phi\left(x_{t+1}^*\right)$

$$
V^*\left(x_{t+1}^*\right) = \mathbf{U}(\mathbf{x}_{t+1}^*) \geq \mathbf{U}(\mathbf{x}_{t+1})
$$

- This establishes that $\mathbf{x}_{t+1}^*$ attains the supremum starting from x_{t+1}^* and completes the induction step.
- Thus (5.5) holds for all $t \geq 0$.

Proof of the Principle of Optimality Theorem III

- Equation (5.5) then implies that

$$\begin{aligned} V^*\left(x_t^*\right) &= \mathbf{U}(\mathbf{x}_t^*) \\ &= U(x_t^*, x_{t+1}^*) + \beta \mathbf{U}(\mathbf{x}_{t+1}^*) \\ &= U(x_t^*, x_{t+1}^*) + \beta V^*\left(x_{t+1}^*\right), \end{aligned}$$

  establishing (3.2) and thus completing the proof of the first part of the theorem.
- Now suppose that (3.2) holds for $\mathbf{x}^* \in \Phi(x_0)$. Substituting repeatedly for $\mathbf{x}^*$:

$$V^*\left(x_0\right) = \sum_{t=0}^{n} \beta^t U\left(x_t^*, x_{t+1}^*\right) + \beta^{n+1} V^*\left(x_{n+1}\right).$$

Proof of the Principle of Optimality Theorem IV

- In view of the fact that $V^*(\cdot)$ is bounded:

$$\begin{aligned} \mathbf{U}(\mathbf{x}^*) &= \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U\left(x_t^*, x_{t+1}^*\right) \\ &= V^*(x_0), \end{aligned}$$

- Thus $\mathbf{x}^*$ attains the optimal value in Problem 6.2, completing the proof of the second part. □

Proof of Existence Theorem I

- Existence can be reached either by looking at Problem 6.2 or at Problem 6.3, and then exploiting their equivalence.

Version 1:

- Consider Problem 6.2:
  - The argument at the beginning of the proof of the Equivalence of Values Theorem again enables us to apply Weierstrass's Theorem, to conclude that an optimal path $\mathbf{x} \in \Phi_0$ exists. □

Version 2

- Let $\mathbf{C}(X)$ be the set of continuous functions defined on X, endowed with the sup norm, $\|f\| = \sup_{x \in X} |f(x)|$.
- In view of Assumption 6.2, X is compact and therefore all functions in $\mathbf{C}(X)$ are bounded since they are continuous and X is compact.

Berge's Maximum Theorem

Theorem

Let X and Y be metric spaces and $f : X \times Y \to \mathbb{R}$ be a function jointly continuous in its two arguments, and $G : X \rightrightarrows Y$ be a a compact-valued correspondence. Let

$$f^*(x) = \max_{y \epsilon G(x)} f(x,y) \quad \textit{and} \quad \Pi(x) = \arg \max_{y \epsilon G(x)} f(x,y)$$

If G is continuous at some $x \epsilon X$, then f^* is continous at x and Π is non-empty, compact-valued and continuous at x.

Proof of Existence Theorem II

- For $V \in \mathbf{C}(X)$, define the operator T as

$$TV(x) = \max_{y \in G(x)} \{U(x,y) + \beta V(y)\}. \tag{5.7}$$

- A fixed point of this operator, $V = TV$, will be a solution to Problem 6.3.
- First prove that such a fixed point (solution) exists:
  - T is well-defined: By Weierstrass's Theorem maximization on (5.7) has a solution– maximizing a continuous function over a compact set.
  - Recall $G(x)$ is a nonempty and continuous correspondence by Assumption 6.1 and $U(x,y)$ and $V(y)$ are continuous by hypothesis.
  - Thus Berge's Maximum Theorem implies $\max_{y \in G(x)} \{U(x,y) + \beta V(y)\}$ is continuous in x, thus $TV(x) \in \mathbf{C}(X)$ and T maps $\mathbf{C}(X)$ into itself.
  - T satisfies Blackwell's sufficient conditions for a contraction.
  - Thus a unique fixed point $V \in \mathbf{C}(X)$ to (5.7) exists and is also the unique solution to Problem 6.3.

Proof of Existence Theorem III

- Now consider the maximization in Problem 6.3.
- Weierstrass's Theorem once more: $y \in G(x)$ achieving the maximum exists since U and V are continuous and $G(x)$ is compact-valued.
- This defines the set of maximizers $\Pi(x)$ for Problem 6.3.
- Let $\mathbf{x}^* = (x_0, x_1^*, ...)$ with $x_{t+1}^* \in \Pi(x_t^*)$ for all $t \geq 0$.
- Then from the Equivalence of Values and Principle of Optimality Theorems, $\mathbf{x}^*$ is also an optimal plan for Problem 6.2. $\square$
- Additional result that follows from second version: Correspondence of maximizing values

$$\Pi : X \rightrightarrows X.$$

  is a upper hemi-continuous and compact-valued correspondence by Theorem of the Maximum.

Proof of Concavity Theorem I

- $\mathbf{C}(X)$: set of continuous (and bounded) functions over the compact set X.
- $\mathbf{C}'(X) \subset \mathbf{C}(X)$: set of bounded, continuous, (weakly) concave functions on X.
- $\mathbf{C}''(X) \subset \mathbf{C}'(X)$: set of strictly concave functions.
- $\mathbf{C}'(X)$ is a closed subset of the complete metric space $\mathbf{C}(X)$, but $\mathbf{C}''(X)$ is not a closed subset.
- Let T be as defined in (5.7).
- Since T is a contraction, it has a unique fixed point in $\mathbf{C}(X)$.
- By the Applications of Contraction Mappings Theorem, proving that $T[\mathbf{C}'(X)] \subset \mathbf{C}''(X) \subset \mathbf{C}'(X)$ would be sufficient to establish that this unique fixed point is in $\mathbf{C}''(X)$ and hence the value function is strictly concave.

Proof of Concavity Theorem II

- Let $V \in \mathbf{C}'(X)$ and for $x' \neq x''$ and $\alpha \in (0,1)$, let

$$x_\alpha \equiv \alpha x' + (1-\alpha)x''.$$

- Let $y' \in G(x')$ and $y'' \in G(x'')$ be solutions to Problem 6.2 with state vectors x' and x''. This implies:

$$\begin{aligned} TV(x') &= U\left(x', y'\right) + \beta V\left(y'\right) \text{ and} \\ TV(x'') &= U\left(x'', y''\right) + \beta V\left(y''\right). \end{aligned} \tag{5.8}$$

- In view of Assumption 6.3 (that G is convex valued) $y_\alpha \equiv \alpha y' + (1-\alpha)\, y'' \in G(x_\alpha)$, so that

$$\begin{aligned} TV(x_\alpha) &\geq U(x_\alpha, y_\alpha) + \beta V(y_\alpha), \\ &> \alpha\left[U\left(x', y'\right) + \beta V\left(y'\right)\right] \\ &\quad +(1-\alpha)\left[U\left(x'', y''\right) + \beta V\left(y''\right)\right] \\ &= \alpha\, TV(x') + (1-\alpha)\, TV(x''), \end{aligned}$$

Proof of Concavity Theorem III

- The first line follows by the fact that $y_\alpha \in G(x_\alpha)$ is not necessarily the maximizer, the second uses Assumption 6.3 (strict concavity of U), and the third the definition introduced in (5.8).
- Thus for any $V \in \mathbf{C}'(X)$, TV is strictly concave, thus $T[\mathbf{C}'(X)] \subset \mathbf{C}''(X)$.
- Then the Theorem Applications of Contraction Mappings implies that unique fixed point V^* is in $\mathbf{C}''(X)$, and hence it is strictly concave. $\square$

Proof of Corollary to the Existence Theorem

- Assumption 6.3 implies that $U(x, y)$ is concave in y: thus Concavity Theorem implies $V(y)$ is strictly concave in y.
- Sum of a concave function and a strictly concave function is strictly concave, thus the right-hand side of Problem 6.3 is strictly concave in y.
- Since $G(x)$ is convex for each $x \in X$ (again Assumption 6.3), there exists a unique maximizer $y \in G(x)$ for each $x \in X$.
- Thus the policy correspondence $\Pi(x)$ is single-valued, thus a function, and can thus be expressed as $\pi(x)$.
- Since $\Pi(x)$ is upper hemi-continuous as observed above, so is $\pi(x)$.
- An upper hemi-continuous function is continuous, thus the corollary follows.$\square$

Proof of Increasing Value Theorem

- $\mathbf{C}'(X) \subset \mathbf{C}(X)$: set of bounded, continuous, nondecreasing functions on X.
- $\mathbf{C}''(X) \subset \mathbf{C}'(X)$: set of strictly increasing functions.
- Since $\mathbf{C}'(X)$ is a closed subset of the complete metric space $\mathbf{C}(X)$ the Applications of Contraction Mappings Theorem implies:
  - if $T[\mathbf{C}'(X)] \subset \mathbf{C}''(X)$, then V, the fixed point to (5.7) is in $\mathbf{C}''(X)$, and therefore, it is a strictly increasing function.
- To see that this is the case, consider any $V \in \mathbf{C}'(X)$.
- Assumption 6.4 implies, $\max_{y \in G(x)} \{U(x, y) + \beta V(y)\}$ is strictly increasing.
- Thus $TV \in \mathbf{C}''(X)$. $\square$

Proof of Differentiability of Value Theorem I

- From the Corollary to the Existence Theorem, $\Pi(x)$ is single-valued, thus a function that can be represented by $\pi(x)$.
- By hypothesis, $\pi(x_0) \in \text{Int}\, G(x_0)$ and from Assumption 6.2 G is continuous.
- Therefore, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that $\pi(x_0) \in \text{Int}\, G(x)$, for all $x \in N(x_0)$.
- Define $W(\cdot)$ on $\mathcal{N}(x_0)$ by

$$W(x) = U(x, \pi(x_0)) + \beta V(\pi(x_0)).$$

- In view of Assumptions 6.3 and 6.5, the fact that $V[\pi(x_0)]$ is a number (independent of x), and the fact that U is concave and differentiable, $W(\cdot)$ is concave and differentiable.

Proof of Differentiability of Value Theorem II

- Moreover, since $\pi(x_0) \in G(x)$ for all $x \in \mathcal{N}(x_0)$:

$$W(x) \leq \max_{y \in G(x)} \{U(x,y) + \beta V(y)\} = V(x), \qquad \text{for all } x \in \mathcal{N}(x_0) \tag{5.9}$$

  with equality at x_0.
- Since $V(\cdot)$ is concave, $-V(\cdot)$ is convex, and by a standard result in convex analysis, it possesses subgradients.
- Moreover, any subgradient p of $-V$ at x_0 must satisfy for all $x \in \mathcal{N}(x_0)$,

$$p \cdot (x - x_0) \geq V(x) - V(x_0) \geq W(x) - W(x_0),$$

- The first inequality uses the definition of a subgradient and the second that $W(x) \leq V(x)$, with equality at x_0 as in (5.9).
- Thus every subgradient p of $-V$ is also a subgradient of $-W$.

Proof of Differentiability of Value Theorem III

- Since W is differentiable at x_0, its subgradient p must be unique, and another standard result in convex analysis implies that any convex function with a unique subgradient at an interior point x_0 is differentiable at x_0.
- This establishes that $-V(\cdot)$, thus $V(\cdot)$, is differentiable as desired.
- The expression for the gradient (3.3) is derived in detail below.

Section 6

Applications of Stationary Dynamic Programming

Subsection 1

Euler Equations

Basic Equations I

- Recall from Problem 6.3,

$$\textbf{Problem 6.3} \quad : \quad V(x) = \sup_{y \in G(x)} \{U(x,y) + \beta V(y)\}, \text{ for all } x \in X, \tag{6.1}$$

and assume Assumptions 6.1-6.5 hold (From Theorem 6.4, the maximization problem in (6.1) is strictly concave and from Theorem 6.6 the maximand is also differentiable).

Basic Equations II

- For any interior solution $y \in \text{Int}\, G(x)$, the first-order conditions are necessary and sufficient for an optimum (taking $V(\cdot)$ as given). In particular, (optimal) solutions can be characterized by the following convenient *Euler equations*:

$$D_y U(x, y^*) + \beta DV(y^*) = 0, \tag{6.2}$$

which are sufficient to solve for the optimal policy, y^*.

- The equivalent *Envelope Theorem* for dynamic programming: differentiate (6.1) with respect to x to obtain

$$DV(x) = D_x U(x, y^*). \tag{6.3}$$

Basic Equations III

- Using the fact that $y^* = \pi(x)$, and that $D_x V(y) = D_x U(\pi(x), \pi(\pi(x)))$, equation (6.2) can be expressed as follows

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0. \tag{6.4}$$

  - $D_x U$: gradient vector of U with respect to its first K arguments,
  - $D_y U$: gradient with respect to the second K arguments.

- Intuition: This equation is intuitive; it requires the sum of the marginal gain today from increasing y and the discounted marginal gain from increasing y on the value of all future returns to be equal to zero.
- Euler equation is not sufficient for optimality. It is necessary to have a transversality condition. It is important in infinite-dimensional problems, because it ensures that there are no beneficial simultaneous changes in an infinite number of choice variables. In the general case,

$$\lim_{t \to \infty} \beta^t D_x U(x_t^*, x_{t+1}^*) \cdot x_t^* = 0. \tag{6.5}$$

Basic Equations IV

- Simpler and more transparent when both x and y are scalars; (6.2) becomes

$$\frac{\partial U(x, y^*)}{\partial y} + \beta V'(y^*) = 0, \tag{6.6}$$

- Intuitive: sum of marginal gain today from increasing y and the discounted marginal gain from increasing y on the value of all future returns to be equal to zero.
  - Optimal Growth Example: U decreasing in y and increasing in x
  - (6.6) requires current cost of increasing y to be compensated by higher values tomorrow.
  - I.e. current cost of reducing consumption must be compensated by higher consumption tomorrow.
- As in (6.2), value of higher consumption in (6.6) is expressed in terms of unknown $V'(y^*)$.
- Use the one-dimensional version of (6.3) to find:

$$V'(x) = \frac{\partial U(x, y^*)}{\partial x}. \tag{6.7}$$

Basic Equations V

- Combining (6.7) with (6.6):

$$\frac{\partial U(x, \pi(x))}{\partial y} + \beta \frac{\partial U(\pi(x), \pi(\pi(x)))}{\partial x} = 0$$

- Alternatively:

$$\frac{\partial U(x_t, x_{t+1}^*)}{\partial x_{t+1}} + \beta \frac{\partial U(x_{t+1}^*, x_{t+2}^*)}{\partial x_{t+1}} = 0. \tag{6.8}$$

- But this Euler equation is not sufficient for optimality.
- Also need the *transversality condition*: essential in infinite-dimensional problems, makes sure there are no beneficial simultaneous changes in an infinite number of choice variables.

Basic Equations VI

- In general, transversality condition takes the form:

$$\lim_{t\to\infty} \beta^t D_{x_t} U(x_t^*, x_{t+1}^*) \cdot x_t^* = 0, \tag{6.9}$$

where "$\cdot$" denotes the inner product operator.

- One-dimensional case:

$$\lim_{t\to\infty} \beta^t \frac{\partial U(x_t^*, x_{t+1}^*)}{\partial x_t} \cdot x_t^* = 0. \tag{6.10}$$

- I.e., product of the marginal return from x times the value of this state variable does not increase asymptotically faster than $1/\beta$.

Necessity and Sufficiency of Euler Equations and Transversality Condition

Theorem 6.10 (Euler Equations and the Transversality Condition)
Let $X \subset \mathbb{R}_{+}^{K}$, and suppose that Assumptions 6.1-6.5 hold. Then a sequence $\{x_{t+1}^{*}\}_{t=0}^{\infty}$, with $x_{t+1}^{*} \in \operatorname{Int} G(x_{t}^{*})$, $t = 0, 1, \ldots$, is optimal for Problem 2 given x_0, if and only if it satisfies (6.4) and (6.5).

- Note: A stronger version applies even when the problem is nonstationary.

Proof of Theorem: Sufficiency of Euler Equations and Trasversality Condition II

- From Assumptions 6.2 and 6.5, U is continuous, concave, and differentiable. By concavity,

$$\mathbf{U}(\mathbf{x}^*) - \mathbf{U}(\mathbf{x}) \equiv \Delta_{\mathbf{x}} \geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [D_x U(x_t^*, x_{t+1}^*) \cdot (x_t^* - x_t) + D_y U(x_t^*, x_{t+1}^*) \cdot (x_{t+1}^* - x_{t+1})]$$

for any $\mathbf{x} \in \Phi(x_0)$.

- Using $x_0^* = x_0$ and rearranging terms

$$\begin{aligned} \Delta_{\mathbf{x}} \geq & \\ & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[\begin{array}{c} D_y U(x_t^*, x_{t+1}^*) \\ +\beta D_x U(x_{t+1}^*, x_{t+2}^*) \end{array} \right] \cdot \left(\begin{array}{c} x_{t+1}^* \\ -x_{t+1} \end{array} \right) \\ & - \lim_{T\to\infty} \beta^T D_x U(x_{T+1}^*, x_{T+2}^*) \cdot x_{T+1}^* \\ & + \lim_{T\to\infty} \beta^T D_x U(x_{T+1}^*, x_{T+2}^*) \cdot x_{T+1}). \end{aligned}$$

Proof of Theorem: Sufficiency of Euler Equations and Trasversality Condition III

- Since $\mathbf{x}^*$ satisfies *(6.4)*, the terms in first line are all equal to zero.
- Moreover, since it satisfies *(6.5)*, the second line is also equal to zero.
- From Assumption 6.4, U is increasing in x, i.e., $D_x U \geq 0$ and $x \geq 0$, so the last term is nonnegative, establishing that $\Delta_{\mathbf{x}} \geq 0$ for any $\mathbf{x} \in \Phi(x_0)$.
- Consequently, $\mathbf{x}^*$ yields higher value than any feasible $\mathbf{x} \in \Phi(x_0)$ and is therefore optimal.
- Proof of necessity is similar (see book).

Subsection 2

Optimal Growth

Problem of Optimal Growth I

- Let there be a normative representative agent who maximizes her utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad k_{t+1} \leq f(k_t) + (1-\delta)k_t - c_t \tag{6.11}$$

$$c_t \geq 0,\ \ k_t \geq 0,\ \ k_0 \text{ is given.}$$

Let us impose structure on this problem, so that we can apply our newly learned theorems.

Assumption 3'

$u : [\underline{c}, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and strictly concave for $\underline{c} \in [0, \infty)$.

Problem of Optimal Growth II

- Other assumptions:
  - $u(\cdot)$ is Neoclassical, i.e. continuous, strictly concave and strictly increasing. $u : \mathbb{R}_+ \to \mathbb{R}_+$.
  - $f(k_t)$ is also Neoclassical.
  - $\beta \in (0, 1)$.

Question: Are there capital and consumption paths, $\{k_t, c_t\}_{t=0}^{\infty}$, that maximize social welfare?

Problem of Optimal Growth II

Notice that since $u(\cdot)$ is strictly increasing, restriction holds under equality, that is $k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$.

Dynamic Programming Formulation: Let $k_t = k = x$, $k^{'} = k_{t+1} = y$ so that $k^{'}$ is the control variable and k is the state variable.

$$V(k) = \sup_{k^{'} \in G(k)} u(f(k) + (1-\delta)k - k^{'}) + \beta V(k^{'})$$

where $G(k) = \left\{ k^{'} \in \mathbb{R}_+ : 0 \leq k^{'} \leq f(k) + (1-\delta)k \right\}$.

We have the tools to show that the solution to this Dynamic Programming Problem is the solution to the central planner problem.

Problem of Optimal Growth III

- Assumption 6.1
  $G(k)$ is nonempty for all $k \geq 0$.
  Assumption holds since $G(k) = [0, f(k) + (1-\delta)k]$ and $\{0\} \subseteq G(k) \neq \varnothing$.
  Moreover, $\lim_{t\to\infty} \sum_{t=0}^{\infty} \beta^t u(c) < +\infty$. To see this, notice that $k_t \in [0, \max\{k_s^*, k_0\}]$, which is compact. Since u is continuous and strictly increasing,

$$u(c) = u(f(k) + (1-\delta)k - k^{'}) < u(f(k) + (1-\delta)k) \leq \bar{u},$$

then

$$\lim_{t\to\infty} \sum_{t=0}^{T} \beta^t u(c) \leq \sum_{t=0}^{\infty} \beta^t \bar{u} = \frac{\bar{u}}{1-\beta}.$$

Solution of the social planner is a solution of the Dynamic Programming Problem (Theorem 6.1 and 6.2). Then

$$V(k) = \sup_{k^{'} \in [0, f(k)+(1-\delta)k]} u(f(k) + (1-\delta)k^{'}) + \beta V(k^{'}).$$

Problem of Optimal Growth IV

- Assumption 6.2
  - $k_t \in [0, \max\{k_s^*, k_0\}]$, which is compact and convex.
  - $G(k) = [0, f(k) + (1-\delta)k]$ is nonempty for all $k \geq 0$. It is also bounded and closed (compact).
  - $G(k)$ is continuous.
    - $G(k)$ is upper-hemicontinuous: Any sequence $\{k_n, k_n'\}$ s.t. $k_n \to k$, $k_n^{'} \in [0, f(k_n) + (1-\delta)k_n]$, and $k_n^{'} \to k^{'}$, then $k^{'} \in [0, f(k) + (1-\delta)k]$.
    - $G(k)$ is lower-hemicontinuous: For any $(k, k^{'})$ and $\{k_n\}$ s.t. $k_n \to k$ there exists $\{k_n'\}$ s.t. $\{k_n' \in G(k_n)\}$ and $k_n^{'} \to k'$.
  - In this case, $\mathbf{X}_G = \left\{(k, k^{'}) \in \mathbb{R}_+^2 : k^{'} \in G(k)\right\}$. Since $u : X \to \mathbb{R}$ is continuous, and $c = f(k) + (1-\delta)k - k^{'}$, then $u : \mathbf{X}_G \to \mathbb{R}$ is continuous.

Problem of Optimal Growth V

- Assumption 6.3
  $G(k)$ is convex and we assumed $u(\cdot)$ is strictly concave.
- Assumption 6.4
  Since $f(k)$ is Neoclassical, $f^{'}(k) > 0$. If $k_1 \leq k_2$, then $f(k_1) + (1-\delta)k_1 \leq f(k_2) + (1-\delta)k_2$, then $G(k_1) \subseteq G(k_2)$. $u(f(k) + (1-\delta)k - k^{'})$ is clearly increasing in k, since $u(\cdot)$ is strictly increasing as well as $f(k) + (1-\delta)k$.
- Assumption 6.5
  Since $f(\cdot)$ and $u(\cdot)$ are twice differentiable, they are continuously differentiable.

Problem of Optimal Growth VI

- We can apply Theorems 6.1-6.6!

Proposition

There exists a unique value function such that

$$\begin{aligned} V(k) &= \sup_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t. \quad & c_t = f(k_t) + (1-\delta)k_t - k_{t+1} \\ & k_0 = k \end{aligned}$$

By strict concavity, there exists a unique policy function $\pi(k)$ such that $k_{t+1}^* = \pi(k_t^*)$, $k_0^* = k_0$, attains the maximum value $V(k_0)$. We also know that $V(k)$ is strictly increasing, strictly concave, and differentiable.

Problem of Optimal Growth VII

- One can show that $\pi(k) = s(k) = f(k) + (1-\delta)k - c(k)$ is non-decreasing in k.
- **Euler Equation**

$$\begin{aligned} D_y u(x, \pi(x)) + \beta D_x u(\pi(x), \pi(\pi(x))) &= 0 \\ u'(c)(-1) + \beta V'(k') &= 0 \\ u'(c) &= \beta V'(k'). \end{aligned}$$

- **Envelope Theorem**

$$\begin{aligned} D_x V &= D_x u(x, \pi(x)) \\ V'(k) &= u'(c)(f'(k) + (1-\delta)) \\ V'(k') &= u'(c')(f'(k') + (1-\delta)) \end{aligned}$$

Problem of Optimal Growth VIII

- Then,

$$u'(c) = \beta u'(c')(f'(k') + (1-\delta)).$$

- **Transversality Condition**

$$\begin{aligned}\lim_{t\to\infty} \beta^t D_x u(x_t^*, \pi(x_t^*))k_t &= 0\\ \lim_{t\to\infty} \beta^t \left[f'(k_t) + (1-\delta)\right] u'(c_t)k_t &= 0\end{aligned}$$

- In steady state, $c_t^* = c_{t+1}^*$, then

$$\begin{aligned}1 &= \beta[f^{'}(k^*) + (1-\delta)] \qquad (6.12)\\ f^{'}(k^*) &= \frac{1-\beta(1-\delta)}{\beta}.\end{aligned}$$

Then, there exists a unique $k^* > 0$. The form of the utility function does not affect k^*. Using the implicit function theorem, $k^* = k(\beta, \delta)$, and

$$k_\beta^* > 0 \qquad k_\delta^* < 0.$$

Problem of Optimal Growth IX

- $c^* = f(k^*) - \delta k^*$. We know that $\max c^*$is such that $f'(k_g^*) = \delta$. In this case,

$$\begin{aligned} \delta + \frac{1-\beta}{\beta} = f'(k^*) &> f'(k_g^*) = \delta \\ \implies k^* &< k_g^*, \end{aligned}$$

  which is called *modified golden rule.*

Proposition

In the neoclassical optimal growth model specified in (6.11) with standard assumptions on the production function and Assumption 3$'$, there exists a unique steady-state capital-labor ratio k^* given by (6.12), and starting from any initial $k_0 > 0$, the economy monotonically converges to this unique steady state, i.e., if $k_0 < k^*$, then the equilibrium capital stock sequence $k_t \uparrow k^*$ and if $k_0 > k^*$, then the equilibrium capital stock sequence $k_t \downarrow k^*$.

Problem of Optimal Growth X

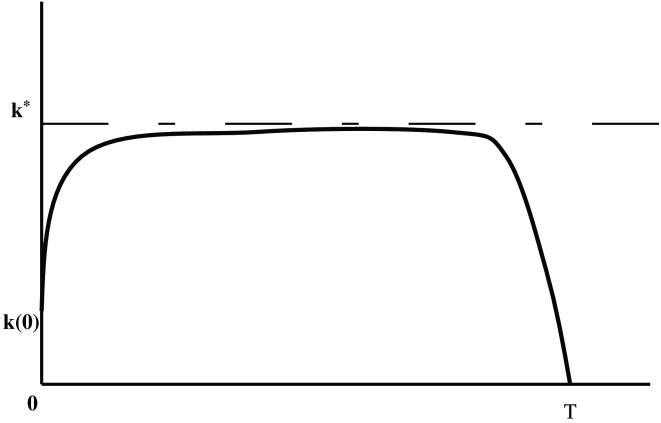
Proposition

$c(k)$ is nondecreasing. Moreover, if $k_0 < k^*$, then the equilibrium consumption sequence $c_t \uparrow c^*$ and if $k_0 > k^*$, then $c_t \downarrow c^*$, where c^* is given by

$$c^* = f(k^*) - \delta k^*.$$

- Optimal growth model very tractable: can incorporate population growth and technological change as in Solow model.
- No immediate counterpart of saving rate, depends on the utility function, and steady state capital-labor ratio and steady state income do not depend on saving rate anyway.
- Results concerning the convergence of optimal growth model are sometimes referred to as the "*Turnpike Theorem*".
- Suppose that the economy ends at some date $T > 0$.
- As $T \to \infty$, $\{k_t\}_{t=0}^{T}$ would become arbitrarily close to k^* as defined by (6.12), but in the last few periods would sharply decline to satisfy transversality condition.

Turnpike dynamics in a finite-horizon (T-periods) neoclassical growth model starting with initial capital-labor ratio k_0.



Example: Optimal Growth I

- Consider the following optimal growth, with log preferences, Cobb-Douglas technology and full depreciation of capital stock

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$
$$\text{subject to}$$
$$k_{t+1} = k_t^{\alpha} - c_t$$
$$k_0 = k_0 > 0.$$

- Canonical examples which admits an explicit-form characterization.
- Set up the maximization problem in its recursive form as

$$V(x) = \max_{y \geq 0} \{\ln(x^{\alpha} - y) + \beta V(y)\},$$

with x corresponding to today's capital stock and y to tomorrow's capital stock.

Example: Optimal Growth II

- Objective is to find the policy function $y = \pi(x)$.
- This problem satisfies Assumptions 6.1-6.5 (only non-obvious feature is whether x and y indeed belong to a compact set).
- Consequently, Theorems apply and in particular, since $V(\cdot)$ is differentiable, the Euler equation (6.4) implies

$$\frac{1}{x^{\alpha} - y} = \beta V'(y).$$

- Envelope condition, (6.3) gives:

$$V'(x) = \frac{\alpha x^{\alpha-1}}{x^{\alpha} - y}.$$

Example: Optimal Growth III

- Using the notation $y = \pi(x)$ and combining:

$$\frac{1}{x^{\alpha} - \pi(x)} = \beta \frac{\alpha \pi(x)^{\alpha-1}}{\pi(x)^{\alpha} - \pi(\pi(x))} \text{ for all } x,$$

- Functional equation in a single function, $\pi(x)$.
- No straightforward ways of solving functional equations; guess-and-verify type methods are most fruitful. Conjecture:

$$\pi(x) = ax^{\alpha}. \tag{6.13}$$

- Substituting for this in the previous expression:

$$\begin{aligned} \frac{1}{x^{\alpha} - ax^{\alpha}} &= \beta \frac{\alpha a^{\alpha-1} x^{\alpha(\alpha-1)}}{a^{\alpha} x^{\alpha^2} - a^{1+\alpha} x^{\alpha^2}}, \\ &= \frac{\beta}{a} \frac{\alpha}{x^{\alpha} - ax^{\alpha}}, \end{aligned}$$

Example: Optimal Growth IV

- Implies with the policy function (6.14), $a = \beta\alpha$ satisfies this equation.
- From the Corollary to the Existence Theorem there is a unique policy function. Since

$$\pi(x) = \beta\alpha x^{\alpha}$$

satisfies the necessary and sufficient conditions, it must be the unique policy function.
- Thus the law of motion of the capital stock is

$$k_{t+1} = \beta\alpha k_t^{\alpha} \tag{6.14}$$

- Optimal consumption level is

$$c_t = (1 - \beta\alpha) k_t^{\alpha}.$$

Example: Intertemporal Consumption Choice I

- Infinitely-lived consumer with instantaneous utility function over consumption $u(c)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, continuously differentiable and strictly concave.
- Discounts the future exponentially with the constant discount factor $\beta \in (0, 1)$.
- Faces a certain (nonnegative) labor income stream of $\{w_t\}_{t=0}^{\infty}$, and starts life with a given amount of assets a_0.
- Receives a constant net rate of interest $r > 0$ on his asset holdings (gross rate of return is $1 + r$).
- Suppose that wages are constant, that is, $w_t = w$.

Example: Intertemporal Consumption Choice II

- Utility maximization problem of the individual can be written as

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$a_{t+1} = (1+r)(a_t + w - c_t),$$

with $a_0 > 0$ given.

- In addition,impose assumption that $a_t \geq 0$ for all t.
- Common application of dynamic optimization, but notice feasible set for state variable a_t is not necessarily compact.
- Strengthen theorems, or make use of the economic structure of the model.

Example: Intertemporal Consumption Choice III

- In particular, choose some $\bar{a}$ and limit a_t to lie in the set $[0, \bar{a}]$, solve the problem and then verify that indeed a_t is in the interior of this set.
- In this example, choose $\bar{a} \equiv a_0 + w/r$ and assume it to be finite.
- Remarks:
  1. Budget constraint could have been written as $a_{t+1} = (1+r)\, a_t + w - c_t$.
     - Difference is timing of interest payments: a_t as asset holdings at the beginning of time t or at the end of time t.
  2. Flow budget constraint does not capture all the constraints
     - e.g. can satisfy flow budget constraint, but run assets position to $-\infty$.
- Focus on the case where $a_0 < \infty$ and $w/r < \infty$.
- Consumption can be expressed as

$$c_t = a_t + w - (1+r)^{-1}\, a_{t+1}.$$

Example: Intertemporal Consumption Choice IV

- Recursive formulation with state variable a_t: denoting current value of the state variable by a and its future value by a':

$$V(a) = \max_{a' \in [0,\bar{a}]} \left\{ u\left(a + w - (1+r)^{-1} a'\right) + \beta V(a') \right\}.$$

- Clearly $u(\cdot)$ is strictly increasing in a, continuously differentiable in a and a' and is strictly concave in a.
- Moreover, since $u(\cdot)$ is continuously differentiable in $a \in (0, \bar{a})$ and the individual's wealth is finite, $V(a_0)$ is also finite.
- Thus all Theorems apply and imply that $V(a)$ is differentiable and a continuous solution $a' = \pi(a)$ exists.
- Moreover, we can use the Euler equation (6.2) or (6.4):

$$\begin{aligned} u'\left(a + w - (1+r)^{-1} a'\right) &= \\ u'(c) &= \beta(1+r) V'(a'). \end{aligned} \tag{6.15}$$

Example: Intertemporal Consumption Choice V

- "Consumption Euler": captures economic intuition of dynamic programming, reduces complex infinite-dimensional optimization problem to one of comparing today to "tomorrow".
- Only difficulty here is tomorrow itself will involve a complicated maximization problem.
- But again envelope condition, (6.3):

$$V'(a') = u'(c'),$$

where c' refers to next period's consumption.

Example: Intertemporal Consumption Choice VI

- Consumption Euler equation becomes

$$u'(c) = \beta(1+r)\,u'(c'). \tag{6.16}$$

- I.e., marginal utility of consumption today must be equal to the marginal utility of consumption tomorrow multiplied by the product of the discount factor and the gross rate of return.
- Since we have assumed that β and $(1+r)$ are constant:

$$\begin{array}{lll} \text{if } r = \beta^{-1} - 1 & c = c' & \text{and consumption is constant over time} \\ \text{if } r > \beta^{-1} - 1 & c < c' & \text{and consumption increases over time} \\ \text{if } r < \beta^{-1} - 1 & c > c' & \text{and consumption decreases over time.} \end{array} \tag{6.17}$$

- Note no reference to the initial level of asset holdings a_0 and w.

Example: Intertemporal Consumption Choice VII

- "Slope" of the optimal consumption path is independent of the wealth of the individual.
- To determine the level of initial consumption use the transversality condition and the intertemporal budget constraint.
- May also verify that whenever $r \leq \beta^{-1} - 1$, $a_t \in (0, \bar{a})$ for all t (so artificial bounds on asset holdings have no bearing on the results).
- What if instead there is an arbitrary sequence of wages $\{w_t\}_{t=0}^{\infty}$?
- Assume no uncertainty: all of the results derived, in particular, the characterization in (6.17), still apply.
- But additional care is necessary since budget constraint, i.e. correspondence G, is no longer "autonomous" (independent of time).

Example: Intertemporal Consumption Choice VIII

- Two approaches are possible
  1. Introduce an additional state variable, e.g. $h_t = \sum_{s=0}^{\infty} (1+r)^{-s} w_{t+s}$
     - Budget constraint becomes:

$$a_{t+1} + h_{t+1} \leq (1+r)(a_t + h_t - c_t),$$

     - Similar analysis can be applied with the value function over two state variables, $V(a, h)$.
     - Economically meaningful, but does not always solve our problems: h_t is now a state variable that has its own non-autonomous evolution and in many problems it is difficult to find an economically meaningful additional state variable.
  2. One can directly apply the Theorem on the sufficiency of the Euler equations and Transversality condition, even when the Dynamic Programming Theorems do not hold.
- Result: exact shape of this labor income sequence has no effect on the slope or level of the consumption profile.

Subsection 3

Relating to the sequence problem

Dynamic Programming Versus the Sequence Problem I

- Return to the sequence problem.
- Suppose that x is one dimensional and that there is a finite horizon T:

$$\max_{\{x_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U(x_t, x_{t+1})$$

subject to $x_{t+1} \geq 0$ with x_0 as given.

- Moreover, let $U(x_T, x_{T+1})$ be the last period's utility, with x_{T+1} as the state variable left after the last period ("salvage value" for example).
- Finite-dimensional optimization problem: can simply look at first-order conditions.
- Moreover, assume optimal solution lies in the interior of the constraint set, i.e., $x_t^* > 0$.

Dynamic Programming Versus the Sequence Problem II

- First-order conditions are exactly as the above Euler equation: for any $0 \leq t \leq T-1$,

$$\frac{\partial U(x_t^*, x_{t+1}^*)}{\partial x_{t+1}} + \beta \frac{\partial U\left(x_{t+1}^*, x_{t+2}^*\right)}{\partial x_{t+1}} = 0,$$

- For x_{T+1}, we have the following boundary condition

$$x_{T+1}^* \geq 0, \text{ and } \beta^T \ \frac{\partial U(x_T^*, x_{T+1}^*)}{\partial x_{T+1}} x_{T+1}^* = 0. \tag{6.18}$$

- Intuitively, x_{T+1}^* should be positive only if an interior value of it maximizes the salvage value at the end.

Dynamic Programming Versus the Sequence Problem III

- **Example: Optimal growth problem,**

$$U\left(x_t, x_{t+1}\right) = u\left(f\left(x_t\right) + (1-\delta)\, x_t - x_{t+1}\right),$$

with $x_t = k_t$ and $x_{t+1} = k_{t+1}$.

  - Suppose world comes to an end at date T. Then at T,

$$\frac{\partial U(x_T^*, x_{T+1}^*)}{\partial x_{T+1}} = -u'\left(c_{T+1}^*\right) < 0.$$

  - From (6.18) and the fact that U is increasing in its first argument (Assumption 6.4), an optimal path must have $k_{T+1}^* = x_{T+1}^* = 0$.
  - Intuitively: no capital left at the end of the world, if it were left, utility could be increased by consuming them either at the last date or earlier.

Dynamic Programming Versus the Sequence Problem IV

- Heuristically, we can derive the transversality condition as an extension of condition (6.18) to $T \to \infty$:

$$\lim_{T \to \infty} \beta^T \frac{\partial U(x_T^*, x_{T+1}^*)}{\partial x_{T+1}} x_{T+1}^* = 0.$$

- Moreover, we have the Euler equation

$$\frac{\partial U(x_T^*, x_{T+1}^*)}{\partial x_{T+1}} + \beta \frac{\partial U(x_{T+1}^*, x_{T+2}^*)}{\partial x_{T+1}} = 0.$$

- Substituting this relationship into the previous equation:

$$-\lim_{T \to \infty} \beta^{T+1} \frac{\partial U(x_{T+1}^*, x_{T+2}^*)}{\partial x_{T+1}} x_{T+1}^* = 0.$$

Dynamic Programming Versus the Sequence Problem V

- Canceling the negative sign, and without loss of any generality, changing the timing:

$$\lim_{T\to\infty} \beta^T \frac{\partial U(x_T^*, x_{T+1}^*)}{\partial x_T} x_T^* = 0,$$

  which is exactly (6.5).

- This also highlights that alternatively we could have had the transversality condition as

$$\lim_{T\to\infty} \beta^T \frac{\partial U(x_T^*, x_{T+1}^*)}{\partial x_{T+1}} x_{T+1}^* = 0,$$

- Thus no unique transversality condition, but boundary condition at infinity to rule out variations that change an infinite number of control variables.
- Different boundary conditions at infinity can play this role.

Section 7

Nonstationary Infinite-Horizon Optimization

Nonstationary Problems

- Let us now return to Problem 6.1.
- Let us again define the set of feasible sequences or plans starting with an initial value x_t at time t as:

$$\Phi(t, x_t) = \{\{x_s\}_{s=t}^{\infty} : x_{s+1}) \in G(s, x_s), \text{ for } s = t, t+1, ...\}.$$

Subsection 1

Assumptions

Assumptions I

Assumption 6.1N

$G(t, x)$ is nonempty for all $x \in X$ and $t \in \mathbb{Z}_+$ and $U(t, x, y)$ is uniformly bounded (from above); that is, there exists $M < \infty$ such that $U(t, x, y) \leq M$ for all $t \in \mathbb{Z}_+$, $x \in X$, and $y \in G(t, x)$.

Assumption 6.2N

X is a compact subset of $\mathbb{R}^K$, G is nonempty-valued, compact-valued and continuous. Moreover, $U : \mathbf{X}_G \to \mathbb{R}$ is continuous in x and y, where $\mathbf{X}_G = \{(t, x, y) \in \mathbb{Z}_+ \times X \times X : y \in G(t, x)\}$.

Assumptions II

Assumption 6.3N

U is strictly concave: for any $\alpha \in (0,1)$ and any (t,x,y), $(t,x',y') \in \mathbf{X}_G$

$$U\left(t, \alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y'\right) \geq \alpha U(t,x,y) + (1-\alpha)U(t,x',y'),$$

and if $x \neq x'$,

$$U\left(t, \alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y'\right) > \alpha U(t,x,y) + (1-\alpha)U(t,x',y').$$

Moreover, G is convex: for any $\alpha \in [0,1]$, and $x, x' \in X$, whenever $y \in G(t,x)$ and $y' \in G(t,x')$

$$\alpha y + (1-\alpha)y' \in G\left(t, \alpha x + (1-\alpha)x'\right).$$

Assumptions III

Assumption 6.4N

For each $t \in \mathbb{Z}_+$ and $y \in X$, $U(t,x,y)$ is strictly increasing in each of x, and G is monotone in x in the sense that $x \leq x'$ implies $G(t,x) \subset G(t,x')$ for any $t \in \mathbb{Z}_+$.

Assumption 6.5N

U is continuously differentiable in x and y on the interior of its domain $\mathbf{X}_G$.

Main Results

Theorem 6.11 (Existence of Solutions)

Suppose Assumptions 6.1N and 6.2N hold. Then there exists a unique function $V^* : \mathbb{Z}_+ \times X \to \mathbb{R}$ *that is a solution to Problem 6.1.* V^* *is continuous in* x *and bounded. Moreover, for any* $x_0 \in X$*, an optimal plan* $x^*[x_0, 0] \in \Phi(0, x_0)$ *exists.*

Theorem 6.12 (Euler Equations and the Transversality Condition)

Let $X \subset \mathbb{R}^K_+$ *, and suppose that Assumptions 6.1N–6.5N hold. Then a sequence* $\{x^*_{t+1}\}_{t=0}^{\infty}$*, with* $x^*_{t+1} \in \operatorname{Int} G(t, x^*_t)$*,* $t = 0, 1, \ldots$ *, is optimal for Problem 6.1 given* x_0 *if and only if it satisfies the Euler equation*

$$D_y U(t, x^*_t, x^*_{t+1}) + \beta D_x U(t+1, x^*_{t+1}, x^*_{t+2}) = 0, \tag{7.1}$$

and the transversality condition

$$\lim_{t\to\infty} \beta^t D_x U(t, x^*_t, x^*_{t+1}) x^*_t = 0. \tag{7.2}$$

Subsection 2

Competitive Growth

Competitive Equilibrium Growth I

- Second Welfare Theorem: optimal growth path also corresponds to an equilibrium growth path (can be decentralized as a competitive equilibrium).
- Most straightforward competitive allocation: symmetric one where all households, each with $u(c)$, make the same decisions and receive the same allocations.
- Each household starts with an endowment of capital stock K_0.
- Mass 1 of households.
- Large number of competitive firms, which are modeled using the aggregate production function.

Competitive Equilibrium Growth II

Definition *A competitive equilibrium consists of paths of consumption, capital stock, wage rates and rental rates of capital, $\{C_t, K_t, w_t, R_t\}_{t=0}^{\infty}$, such that the representative household maximizes its utility given initial capital stock K_0 and the time path of prices $\{w_t, R_t\}_{t=0}^{\infty}$, and the time path of prices $\{w_t, R_t\}_{t=0}^{\infty}$ is such that given the time path of capital stock and labor $\{K_t, L_t\}_{t=0}^{\infty}$ all markets clear.*

- Households rent their capital to firms and receive the competitive rental price

$$R_t = f'(k_t),$$

- Thus face gross rate of return for renting one unit of capital at time t in terms of date $t+1$ goods:

$$1 + r_{t+1} = f'(k_{t+1}) + (1 - \delta) \tag{7.3}$$

Competitive Equilibrium Growth III

- In addition, to capital income, households receive wage income

$$w_t = f(k_t) - k_t f'(k_t).$$

- Maximization problem of the representative household:

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the flow budget constraint

$$a_{t+1} = (1 + r_t) a_t - c_t + w_t \quad (7.4)$$

$$a_0 > 0 \text{ given.} \quad (7.5)$$

Competitive Equilibrium Growth IV

- Set up of the problem in Dynamic Programming framework:

$$V(t, a_t) = \sup_{a_{t+\mathbf{1}} \in G(t, a_t)} u((1 + r_t)a_t + w_t - a_{t+1}) + \beta V(t + 1, a_{t+1}),$$

  where $G(t, a_t) = \{a_{t+1} \in \mathbb{R} : a_{t+1} \leq (1 + r_t)a_t + w_t\}$.

- From now on $a_t = x$ and $a_{t+1} = y$.

Competitive Equilibrium Growth V

Verifying Assumptions

- Assumption 6.1N
  $G(t,x) \neq \emptyset$, $G(t,x) = (-\infty, (1+r_t)x + w_t]$.
  From (7.4),

$$a_{t+k} = \prod_{s=0}^{k-1}(1+r_{t+s})a_t + \sum_{j=0}^{k-1}\prod_{s=0}^{j}(1+r_{t+s})(w_{t+j} - c_{t+j}).$$

  Since $u(c_t)$ is increasing in c_t, without any requirements, $a_{t+1} \to -\infty$, which is a contradiction because $V(0, a_0) \to +\infty$.
  Hence, it is necessary to impose conditions on the bounds of a_{t+1}.

Competitive Equilibrium Growth VI

Verifying Assumptions

- Assumption 6.1N (cont...)
  - Liquidity constraints: $a_t \geq 0$ for all t. The household cannot borrow.
  - Natural debt limit: level of a_t such that if household owes a_t and it never consumes again, then it will be able to repay the debt. We impose $a_t \geq -B$, with $0 < B < \infty$. Then, it is necessary that $\lim_{t\to\infty} a_t = -B$.
    Notice that given $\{r_t, w_t\}_{t=0}^{\infty}$, there is a maximum the household is able to repay in its lifetime (or from any period t onwards). If the household does not consume ($c_s = 0$ for all $s \geq t$) then

$$\frac{a_{t+1}}{1+r_t} - \frac{w_t}{1+r_t} = a_t$$

Competitive Equilibrium Growth VII

Verifying Assumptions

- Assumption 6.1N (cont...)
  - Natural debt limit (cont...):

$$
\begin{aligned}
\frac{a_{t+2}}{(1+r_{t+1})(1+r_t)} - \frac{w_{t+1}}{(1+r_{t+1})(1+r_t)} - \frac{w_t}{1+r_t} &= a_t \\
&\vdots \\
a_{t+T}\prod_{s=0}^{T-1}\frac{1}{1+r_{t+s}} - \sum_{s=0}^{T-1}\prod_{j=0}^{s}\frac{1}{1+r_{t+j}}w_{t+s} &= a_t.
\end{aligned}
$$

Since the household must be able to repay, $\lim_{T\to\infty} a_{t+T} \geq 0$, then

$$
\underline{a}_t \geq -\sum_{s=0}^{\infty}\prod_{j=0}^{s}\frac{1}{(1+r_{t+j})}w_{t+s} \equiv -\overline{W}.
$$

Assume that $\exists \overline{W} : \overline{W}_t \leq \overline{W} \leq \infty$ for all $t \geq 0$ (problem: if there is growth of wages, w_t is increasing and $\overline{W}$ may not be finite).
Assumption: $a_t \in [-\overline{W}, \overline{W} + a_0]$. In particular, if $r_t = r$ and $w_t = w$ for all t, $\overline{W} = \frac{w}{r}$.

Competitive Equilibrium Growth VII

Verifying Assumptions

- Assumption 6.1N (cont...)
  - No-Ponzi Condition (NPC): $\lim_{t\to\infty} a_t \prod_{s=0}^{t-1} \frac{1}{1+r_s} = 0$. Dying without debts or a way to ensure that same result as in A-D markets. The life time budget constraint is equal to that in the A-D economy,

$$
\begin{aligned}
a_t \prod_{s=0}^{t-1} \frac{1}{1+r_s} + \sum_{s=0}^{t-1} \prod_{j=0}^{t} \frac{1}{1+r_j} c_s &\leq a_0 + \sum_{s=0}^{t-1} \prod_{j=0}^{t} \frac{1}{1+r_j} w_s \\
\sum_{s=0}^{\infty} \prod_{j=0}^{t} \frac{1}{1+r_j} c_s &\leq a_0 + \sum_{s=0}^{\infty} \prod_{j=0}^{t} \frac{1}{1+r_j} w_s .
\end{aligned}
$$

Competitive Equilibrium Growth VIII

Verifying Assumptions

With any of those conditions, assumptions 6.1N-6.5N hold

- Solution under Natural Debt Limit:
  - $G(t,x) = [-\overline{W}, \overline{W} + a_0]$ is convex, non-empty, compact and continuous.
  - $u(\cdot)$ is continuous, differentiable, strictly increasing, strictly concave.
  - $u(\cdot)$ is uniformly bounded since

$$u((1+r_t)a_t + w_t - a_{t+1}) < u((1+r_t)a_t + w_t - (-\overline{W})) < u(\overline{W} + a_0 + \overline{W}) < +\infty$$

and $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(\overline{W} + a_0 + \overline{W}) = \frac{\bar{u}}{1-\beta} < +\infty$.

Competitive Equilibrium Growth IX

- Characterizing the solution:
  - the first order condition is

$$-u'((1+r_t)x + w_t - y) + \beta V'(t+1, y) = 0.$$

  - Envelope theorem

$$V'(t, x) = (1+r_t)u'((1+r_t)x + w_t - y)$$

  - Euler equation

$$u'(c_t^*) = \beta(1+r_{t+1})u'(c_{t+1}^*) \tag{7.6}$$

  - Transversality condition

$$\lim_{t\to\infty} \beta^t (1+r_t)u'(c_t^*)a_t = 0$$

Competitive Equilibrium Growth IX

- Notice that
  - $c_t = c_{t+1}$ iff $\beta(1 + r_{t+1}) = 1$
  - $c_t > c_{t+1}$ iff $\beta(1 + r_{t+1}) < 1$
  - $c_t < c_{t+1}$ iff $\beta(1 + r_{t+1}) > 1$

  where it does not depend on u, w, etc. Only on β and r_{t+1}.

Competitive Equilibrium Growth X

- Also, from the Euler equation, equation (7.1), $\beta(1+r_t)u'(c_t) = u'(c_{t-1})$ and $\beta(1+r_{t-1})u'(c_{t-1}) = u'(c_{t-2})$, then

$$\begin{aligned} u'(c_t) &= \frac{1}{\beta^2(1+r_t)(1+r_{t-1})} u' c_{t-2} \\ &= \cdots \\ u'(c_t) &= \beta^{-t} \prod_{s=0}^{t-1} \frac{1}{1+r_{t-s}} u'(c_0), \qquad (7.7) \end{aligned}$$

therefore

$$c_t = (u')^{-1} \left(\beta^{-t} \prod_{s=0}^{t-1} \frac{1}{1+r_{t-s}} u'(c_0) \right). \qquad (7.8)$$

In particular, if $r_t = r$ and $w_t = w$ for all t,

$$c_t = (u')^{-1} \left([\beta(1+r)]^{-t} u'(c_0) \right).$$

Competitive Equilibrium Growth XI

- Working with the budget constraint, we know that

$$a_t^* = \prod_{s=0}^{t-1}(1+r_s)a_0 + \sum_{s=0}^{t-1}\prod_{j=s}^{t-1}(1+r_j)(w_s - c_s^*),$$

and using (7.7),

$$\begin{aligned} \beta^t u'(c_t)(1+r_t) &= \prod_{s=1}^{t-1}\frac{1}{1+r_s}u'(c_0) \\ a_t^*\beta^t u'(c_t)(1+r_t) &= a_t^*\prod_{s=1}^{t-1}\frac{1}{1+r_s}u'(c_0). \end{aligned}$$

For transversality condition to hold, we need that as $t \to \infty$ LHS$\to 0$. Notice that this would be satisfied in No-Ponzi Condition since RHS is NPC. With NDL Transversality implies NPC.

Competitive Equilibrium Growth XII

- Using the budget constraint again, it is true that

$$u'(c_t)(1+r_t)\beta^t a_t^* = (1+r_0)u'(c_0)a_0 + u'(c_0)\prod_{s=1}^{t-1}\frac{1}{1+r_s}\sum_{s=0}^{t-1}\prod_{j=s}^{t-1}(1+r_j)(w_s - c_s^*),$$

and taking the limit as $t \to \infty$,

$$\sum_{s=0}^{\infty}\prod_{j=0}^{s}\frac{1}{1+r_j}c_s^* = a_0 + \sum_{s=0}^{\infty}\prod_{j=0}^{s}\frac{1}{1+r_j}w_s,$$

which implicitly determines c_0.

Competitive Equilibrium Growth XIII

- (cont...)
  If $r_t = r$ and $w_t = w$ for all t,

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^{t+1}} c_t^* = a_0 + \frac{w}{r}.$$

In particular, if $\beta(1+r) = 1$, then

$$c_0 = ra_0 + w.$$

If $\beta(1+r) \leq 1$, then $c_0 \geq c_1 \geq c_2 \geq \ldots$, so

$$\begin{aligned} a_0 + \frac{w}{r} = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^{t+1}} &\leq \frac{c_0}{r} \\ c_0 &\geq ra_0 + w, \end{aligned}$$

Competitive Equilibrium Growth XIII

- (cont...)
  and at any point of the time t, given a_t

$$a_t + \frac{w}{r} = \sum_{s=t}^{\infty} \frac{c_s}{(1+r)^{s+1}} \leq \frac{c_s}{r} \leq \frac{c_0}{r}$$

$$a_t \leq \frac{c_0 - w}{r}$$

additionally, the flow budget constraint implies that

$$\begin{aligned} a_t - a_{t-1} =& r\left(a_{t-1} + \frac{w - c_t}{r}\right) \leq 0 \\ \implies a_t \leq & a_{t-1} \leq \cdots \leq a_0 < a_0 + \bar{W} \\ \implies a_t <& a_0 \bar{W}. \end{aligned}$$

Competitive Equilibrium Growth XIV

- Profit Maximization: $R_t = f'(k_t) = r_t + \delta$ and $w_t = f(k_t) - k_t f'(k_t)$.
- Equilibrium: Using the fact that in a closed economy $a_t = k_t$, and replacing into the budget constraint, it is obtained

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t.$$

This implies that competitive equilibrium in this economy generates the same paths as the optimal growth model:

	Competitive Growth	**Optimal Growth**
Euler equation	$u'(c_t) = \beta(\mathbf{1} + r_{t+\mathbf{1}})u'(c_{t+\mathbf{1}})$	$u'(c_t) = \beta(\mathbf{1} + f'(k_{t+\mathbf{1}}) - \delta)u'(c_{t+\mathbf{1}})$
Transversality Condition	$\lim_{t\to\infty} \beta^t(\mathbf{1} + r_t)u'(c_t)a_t = \mathbf{0}$	$\lim_{t\to\infty} \beta^t(\mathbf{1} + f'(k_t) - \delta)u'(c_t)k_t = \mathbf{0}$

Competitive Equilibrium Growth XV

- First Welfare Theorem holds since competitive growth equilibrium is Pareto Optimal.
- Recall that the Golden Rule implies $f^{'}(k_g^*) = \delta$. In this case, in SS $f'(k^*) = \frac{1-\beta}{\beta} + \delta$. Since by assumption $\beta < 1$

$$\begin{aligned} f^{'}(k_g^*) &< f^{'}(k^*) \\ k_g^* &> k^*, \end{aligned}$$

so that the level of capital in steady state it is called *Modified Golden Rule* level of capital.

Conclusions

- Dynamic programming techniques are not only essential for the study of economic growth, but are widely used in many diverse areas of macroeconomics and economics.
- Number of applications of dynamic programming.
- Assumed away a number of difficult technical issues.
- Discounted problems, which are simpler than undiscounted problems.
- Payoffs are bounded and the state vector x belongs to a compact subset of the Euclidean space, X.
    - rules out many interesting problems, such as endogenous growth models, where the state vector grows over time.