4.18

Eks. Finne egenverdien til matrisen

$$A = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}$$

Viser at

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 4 & 1 \\ -5 & \lambda + 2 \end{vmatrix}$$

$$= (\lambda - 4)(\lambda + 2) - (-5) \cdot 1$$

$$= \lambda^2 - 2\lambda - 3$$

Egenverdien er altså løsningen til annengradslikningen

$$\lambda^2 - 2\lambda - 3 = 0$$

Løser:

$$\lambda_1, \lambda_2 = 3, -1$$

Setter inn egenvektoren i ligningen:

$$A v_1 = 3 v_1$$

Setter inn $v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ i tr. uttrykket for å finne v_1

$$4x - 1y = 3x$$
$$5x - 2y = 3y$$

$$\begin{vmatrix} 4 & -1 \\ 5 & -2 \end{vmatrix} = A$$

Omformet

$$\left.\begin{aligned} x - y &= 0 \\ 5x - 5y &= 0 \end{aligned}\right\}$$ forteller oss at $x = y$

Vi velger å sette y som en variabel. Hvis vi velger $y = 1$, gir det $x = 1$

dermed får vi $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Så må vi finne v_2 $(\lambda_2 = -1)$

$$A v_2 = (-1) v_2 = -v_2$$

$$\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} = A$$

Setter inn $v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

$$4x - 1y = -x$$
$$5x - 2y = -y$$

Skriver om

$$5x - y = 0$$
$$5x - y = 0$$

Vi ser at x og y ikke bestemt. Vi velger $y = 5$ som gir $x = 1$, $v_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

Dermed har vi to egenvektorer $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ og $v_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

1. Finne egenverdien og egenvektorene

a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\det(\lambda I_2 - A) = \begin{pmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{pmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3$$

finner "0"punktene

$$\frac{-(-4) \pm 2}{2} = \frac{4 \pm 2}{2} \Rightarrow \lambda = 3 \lor 1 = \lambda$$

Egenvektoren må tilfredstille $A v_1 = 3 v_1$

$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{aligned} 2x + y &= x \\ x + 2y &= y \end{aligned}\right\} \qquad \left.\begin{aligned} x + y &= 0 \\ x + y &= 0 \end{aligned}\right\}$$

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\left\{\begin{pmatrix} -x - y = 0 \\ -x - y = 0 \end{pmatrix}^{II - I} \sim \Rightarrow \left.\begin{aligned} -x - y &= 0 \\ 0 + 0 &= 0 \end{aligned}\right\} \quad x = -y\right.$$

Finne v_2, $Av_2 =$

$$\left.\begin{array}{l} -2x - 1y = \lambda x \\ -1x - 2y = \lambda y \end{array}\right\}$$

$1x - y = 0 \;/ + \lambda x$

$4x \cdot y = 3x$

$$\left.\begin{array}{l} 2x + y = \lambda x \\ x + 2y = \lambda y \end{array}\right\} \quad \begin{array}{l} -x + y = 0 \\ x - y = 0 \end{array}$$

c) $\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \Rightarrow \det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -4 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - (-1 \cdot -4)$

$= \lambda^2 - \lambda + 1 - 4$

$= \lambda^2 - 2\lambda - 3$

$= (x - 3)(x + 1)$

$\lambda_1, \lambda_2 = 3, -1$

Starter med λ_1, $Av_1 = 3v_1$

$$\left.\begin{array}{l} 1x + 4y = 3x \\ 1x + y = 3y \end{array}\right\} \quad \left.\begin{array}{l} -2x + 4y = 0 \\ x - 2y = 0 \end{array}\right\} \quad \begin{array}{l} 2x = 4y \\ x = 2y \end{array}$$

$\Rightarrow$ Velger vi $y = 1$ får vi $x = 2$ derved:

$\Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Så λ_2, $Av_2 = -v_1$

$$\left.\begin{array}{l} x + 4y = -x \\ x + y = -y \end{array}\right\} \quad \left.\begin{array}{l} 2x + 4y = 0 \\ x + 2y = 0 \end{array}\right\} \quad \begin{array}{l} 2x = -4y \\ x = -2y \end{array}$$

Velger vi $y = 1$ får vi $x = -2$

derved $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \Rightarrow \det(\lambda I_2 - A)$

$= (\lambda - 4) \cdot (\lambda - 2) - (3 \cdot 1)$

$\underline{\underline{\lambda_1, \lambda_2 = 1, 5}}$

$\lambda_1 \Rightarrow \left.\begin{matrix} 4x + 3y = 1x \\ x + 2y = 1y \end{matrix}\right\} \left.\begin{matrix} 3x + 3y = 0 \\ x + y = 0 \end{matrix}\right\} \begin{matrix} x = -y \\ x = -y \end{matrix}$

$\underline{\underline{v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}}$

$\lambda_2 \Rightarrow \left.\begin{matrix} 4x + 3y = 5x \\ x + 2y = 5y \end{matrix}\right\} \left.\begin{matrix} -x + 3y = 0 \\ x - 3y = 0 \end{matrix}\right\} \begin{matrix} 3y = x \\ 3y = x \end{matrix}$

velger $y = 1$ og det gir oss

$\underline{\underline{v_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}}}$

d) $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 3 & 1 \\ 1 & \lambda - 1 \end{vmatrix} = (\lambda - 3)(\lambda - 1) - (1 \cdot 1)$

$= \lambda^2 - \lambda - 3\lambda + 3$

$= \lambda^2 - 4\lambda + 3$

$\lambda_1, \lambda_2 = 3, 1$

$\lambda_1 \Rightarrow 3x$

4. a) $A=\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

$\det(\lambda I_2 - A)$

$\Rightarrow \ = (\lambda-2)(\lambda+1)-4$

$= \lambda^2 + \lambda - 2\lambda - 2 - 4$

$= \lambda^2 - \lambda - 6$

$\lambda_1, \lambda_2 = -2, 3$

$\lambda_1 \Rightarrow \quad \left.\begin{matrix} 2x+2y=-2x \\ 2x-y=-2y \end{matrix}\right\} \quad \left.\begin{matrix} 4x+2y=0 \\ 2x+y=0 \end{matrix}\right\} \quad \begin{matrix} 2x=-y \\ 2x=-y \end{matrix}$

Velger $x=1$

$\Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 \Rightarrow \quad \left.\begin{matrix} 2x+2y=3x \\ 2x-y=3y \end{matrix}\right\} \quad \left.\begin{matrix} -x+2y=0 \\ 2x-4y=0 \end{matrix}\right\} \quad \begin{matrix} 2y=x \\ 2y=x \end{matrix}$

velger $x=1$

$\Rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

skriver egenvektorene som en lineærkombinasjon

Radredusere

$$\begin{array}{ccc} v_1 & v_2 & x \end{array}$$

$\begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 5 \end{pmatrix} \overset{II+2I}{\sim} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{pmatrix} \overset{II:5}{\sim} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & \frac{3}{5} \end{pmatrix} \overset{I-II}{\sim}$

$\begin{pmatrix} 1 & 0 & -\frac{11}{5} \\ 0 & 1 & \frac{3}{5} \end{pmatrix}$

da kan vi skrive: $x = \underline{\underline{-\frac{11}{5} v_1 + \frac{3}{5} v_2}}$

7. Nyfødte x_n

voksne y_n

gamle z_n

t-regnes i uker

Nyfødte

$$\left.\begin{aligned} x_{n+1} &= 3y_n + 4z_n \\ y_{n+1} &= x_n \\ z_{n+1} &= \frac{1}{2} y_n \end{aligned}\right\} \quad \begin{array}{c c} \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} & \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \\ & \begin{matrix} 1 & \quad 2 & \quad 2 \end{matrix} \end{array}$$

$\begin{matrix} + & - \\ + & - \end{matrix}$

$$\begin{vmatrix} \lambda-0 & -3 & -4 \\ -1 & \lambda-0 & 0 \\ 0 & -\frac{1}{2} & \lambda-0 \end{vmatrix} \Rightarrow -\frac{1}{2}(-1)^{3+2}\begin{vmatrix} \lambda & -4 \\ -1 & 0 \end{vmatrix} + \lambda(-1)^{3+3}\begin{vmatrix} \lambda & -3 \\ -1 & \lambda \end{vmatrix}$$

$$= (\lambda+1)^2(\lambda-2)$$

$\lambda_1 = -1$