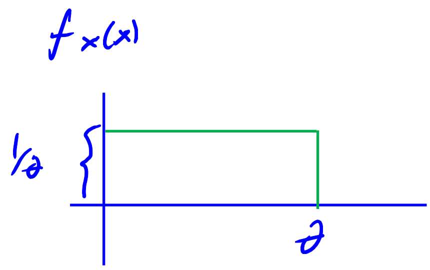
X antall minutter en kunde er i butikken ~ uniformt fordelt $[0, \theta]$

θ = ukjent parameter

Sannsynlighetstettheten er gitt ved

$$f_X(x) = \begin{cases} 1/\theta & 0 \leq x \leq \theta \\ 0 & \text{ellers} \end{cases}$$



a) Vis $E(X) = \frac{\theta}{2}$

$$E(X) = \int_0^\theta x \cdot 1/\theta \, dx = \frac{1}{\theta} \cdot \frac{\theta^2}{2}$$

$$= \underline{\underline{\frac{\theta}{2}}}$$

for å regne ut $V(X)$ trenger vi $E(X^2)$

$$E(X^2) = \int_0^\theta x^2 \cdot \frac{1}{\theta} \, dx = \frac{1}{\theta}\left[\frac{x^3}{3}\right]_0^\theta = \frac{\theta^2}{3}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{\theta^2}{3} - \left(\frac{\theta}{2}\right)^2$$

$$= \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{4\theta^2}{12} - \frac{3\theta^2}{12} = \frac{\theta^2}{12}$$

b) Anta at $X_1, \dots, X_n$ er n uavhengige tilfeldige variable

n kunder i butikken

Hver med same sannsynlighetstetthet som X

La $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\bar{x}$ er gjennomsnittlig tid i butikken

Beregne $E(\bar{X})$ og $V(\bar{X})$

$$E(\bar{x}) = E\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n}\underbrace{\sum_{i=1}^{n} E(x_i)}_{\theta/2 \cdot n} = \frac{1}{n}\cdot\frac{\theta \cdot n}{2} = \frac{\theta}{2}$$

$$V(\bar{x}) = V\left(\frac{1}{n}\sum x_i\right) = \left(\frac{1}{n}\right)^2 V\left(\sum x_i\right) = \frac{1}{n^2}\sum \underbrace{V(x_i)}_{\frac{\theta^2}{12}} = \frac{1}{n^2}\cdot n \cdot \frac{\theta^2}{12}$$

$$= \frac{\theta^2}{12n}$$

c) Foreslå en forventningsrett estimator for θ?

Ser her på

$$\hat{\theta} = 2\bar{x}$$

Har

$E(\hat{\theta}) = E(2\bar{x}) = 2\frac{\theta}{2} = \theta$ (dvs forventningsrett) Løse $E(\bar{x})$ for θ

$$V(\hat{\theta}) = V(2\bar{x}) = 2^2 V(\bar{x}) = 4\cdot\frac{\theta^2}{12n} = \frac{\theta}{3n}$$

d) antar $n = 192$, sentralgrenseteoremet, $P(\bar{x} \leq 20)$ ved θ

Kommenter $\theta = 40$

Har ved sentralgrensesetningen

$$\frac{\bar{x} - E(\bar{x})}{\sqrt{V(\bar{x})}} \Rightarrow \frac{\bar{x} - \theta/2}{\sqrt{\theta^2/12n}} \overset{\text{tilnærmet}}{\sim} N(0, 2)$$

Mer presist

$$P\left(\frac{\bar{x} - E(\bar{x})}{\sqrt{V(\bar{x})}} \leq z\right) \Rightarrow P\left(\frac{\bar{x} - \theta/2}{\sqrt{\theta^2/12n}} \leq z\right) \approx P(Z \leq z) \quad \text{der } Z \sim N(0,1)$$

Med $\Phi(z) = P(Z \leq z)$ Har vi da at $(n = 192)$

standardisert

$$P(\bar{x} \leq 20) = P\left(\frac{\bar{x} - \theta/2}{\sqrt{\theta^2/(12 \cdot 192)}} \leq \frac{20 - \theta/2}{\sqrt{\theta^2/12 \cdot 192}}\right)$$

$$\approx \Phi\left(\frac{20 - \theta/2}{\theta/48}\right) = \Phi\left(\frac{960}{\theta} - 24\right)$$

For $\theta = 40$

$$\Phi\left(\frac{960}{40} - 24\right) = \Phi(0) = 0.5$$

e) $n = 192$, $\bar{x} = 20$ 95 % konfidensintervall

La $x_1, \ldots, x_{192}$ uniform fordelt

$$P\left(-1.96 \leq \frac{\bar{x} - \theta/2}{\sqrt{\theta^2/(12 \cdot 192)}} \leq 1.96\right) \approx 0.95$$

dvs.

$$P\left(-1.96 \leq \frac{\bar{x} - \theta/2}{\theta/48} \leq 1.96\right) \approx 0.95$$

Løser på hensyn med θ

$$-1.96 \le \frac{\bar{X} - \theta/2}{\theta/48} \le 1.96$$ Løser θ

$\Leftrightarrow$

$$-1.96\,\frac{\theta}{48} \le \bar{x} - \theta/2 \le 1.96\,\frac{\theta}{48}$$

$$\Leftrightarrow \frac{\theta}{2} - 1.96 \le \bar{x} \le \frac{\theta}{2} + 1.96\,\frac{\theta}{48}$$

$$\frac{\bar{x}}{1/2 + 1.96/48} \le \theta \le \frac{\bar{x}}{1/2 - 1.96/48}$$

gir

$$P\left(\frac{\bar{x}}{1/2 + 1.96/48} \le \theta \le \frac{\bar{x}}{1/2 - 1.96/48}\right) \approx 0.95$$

Altså 95% konfedensintervall

$$\left[\frac{\bar{x}}{1/2 + 1.96/48}, \frac{\bar{x}}{1/2 - 1.96/48}\right]$$

med tall

$$\left[\frac{20}{1/2 + 1.96/48}, \frac{20}{1/2 - 1.96/48}\right] = \left[\frac{20}{0.5408}, \frac{20}{0.4592}\right]$$

dvs.

$$\underline{\underline{[36.98, 43{,}56]}}$$

f) Vis at den momentgenererende funksjonen til X er gitt ved

$$M_X(t) = E\, e^{tX} = \begin{cases} \frac{e^{\theta t}-1}{\theta t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

$X \sim$ uniform

Momentgenererende funksjon

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

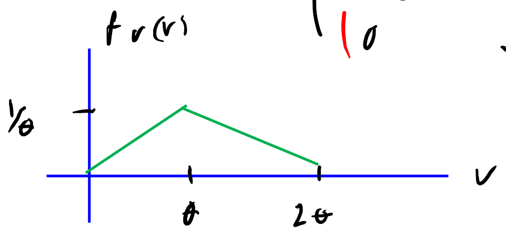
dvs.

$$\int_0^\theta e^{tx} \frac{1}{\theta}\, dx = \frac{1}{\theta} \int_0^\theta e^{tx}\, dx = \frac{1}{\theta} \left[e^{tx} \frac{1}{t} \right]_{x=0}^{x=\theta}$$

$$= \frac{e^{\theta t} - 1}{\theta t} \quad t \neq 0$$

g) $V = X_1 + X_2$, $X_1, X_2 \sim$ uavhengige uniform $[0, \theta]$

$$f_V(v) = \begin{cases} \frac{v}{\theta^2} & 0 \leq v \leq \theta \\ \frac{2\theta - v}{\theta^2} & \theta \leq v \leq 2\theta \\ 0 & \text{ellers} \end{cases}$$



Finne

$$E(V) = \int_{-\infty}^{\infty} v f_V(v)\,dv = \int_0^{\theta} v \frac{v}{\theta^2}\,dv + \int_{\theta}^{2\theta} v \frac{2\theta - v}{\theta^2}$$

$$= \frac{1}{\theta^2}\int_0^{\theta} v^2\,dv + \frac{1}{\theta^2}\int_{\theta}^{2\theta} (2\theta v - v^2)\,dv$$

$$= \theta$$

Dette stemmer med

$$E(V) = E(X_1 + X_2) = E(X_1) + E(X_2) = \frac{\theta}{2} + \frac{\theta}{2} = \theta$$

h) Finn $M_V(t)$ ved $f_V(v)$

$$M_V(t) = E(e^{tV}) = \int_0^{\theta} e^{tv} \frac{v}{\theta^2}\,dv + \int_{\theta}^{2\theta} e^{tv} \frac{2\theta v - v^2}{\theta^2}\,dv$$

$u = 2\theta - v \qquad v = \theta \Rightarrow u = \theta$

$du = -dv \qquad v = 2\theta \Rightarrow u = 0$

$$= \frac{1}{\theta^2}\int_0^{\theta} v e^{tv}\,dv + \int_{\theta}^{0} e^{(2t-u)} \frac{u}{\theta^2}(-du)$$

$$= \frac{1}{\theta}\int_0 v e^{tv}\,dv + \frac{1}{\theta^2} e^{2\theta t}\int u e^{u}\,du$$

$= (\text{delvis})$

$$= \frac{1}{\theta^2 t^2}\left(1 - 2e^{\theta t} + e^{2\theta t}\right)$$

$$= \frac{(e^{\theta t} - 1)^2}{(\theta t)^2} = [M_X(t)]^2$$

$$= M_X(t) = \frac{e^{\theta t} - 1}{\theta t} \text{ fra punkt f}$$

Dette stemmer med at

$$M_V(t) = E(e^{tV}) = E(e^{t(X_1 + X_2)}) = E(e^{tX_1} \cdot e^{tX_2})$$

$$= E(e^{tX_1})\,E(e^{tX_2}) = M_X(t) \cdot M_X(t) \text{ - Hvis avhengig}$$

$$= [M_X(t)]^2$$