substitusjon / skifte av variabel

9.2 $$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

F er en antiderivert til f

Praksis:

$$\int f(g(x))\, g'(x)\, dx$$

$$\int f(u)\,du = F(u) + C = F(g(x)) + C$$

$\frac{du}{dx} = u' = g'(x)$

$\Rightarrow du = g'(x)\,dx$

Tøffere versjon: $\int f(g(x))\,dx$, uten $g'(x)$. Hva gjør vi nå? Anta at "h" er den omvendte funksjonen til "g". Da er

$$h'(g(x)) = h'(y) = \frac{1}{g'(x)} \quad \text{der } y = g'(x)$$

$$h'(g(x))\, g'(x) = 1$$

Altså $\int f(g(x))\,dx = \int f(g(x)) \cdot 1\,dx = \int f(g(x))\, \underbrace{h'(g(x))\, g'(x)}_{=1}\,dx$

$$= \int f(u)\, h'(u)\, du$$

$u = g(x)$

$du = g'(x)\,dx$

Vi har dermed vist at:

$$\int f(g(x))\,dx = \int f(u)\, h'(u)\,du \Big|_{u = g(x)}$$

der h er den omvendte funksjonen til " "

I praksis: $\int f(g(x))\,dx$

$\int f(u)\,h'(u)\,du$

$u = g(x)$

Løser for x

$x = h(u)$

$\frac{dx}{du} = x''(u) = h'(u)$

$dx = h'(u)\,du$

Eksempel: $\int \frac{dx}{1+\sqrt{x}}$, setter u = det lille dere $u = \sqrt{x}$

$= \int \frac{2u}{1+u}\,du$

$u = \sqrt{x}$

$x = u^2$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$= \int \frac{2u + 2 - 2}{1+u}\,du$

$= \int \frac{2(u+1) - 2}{u+1}\,du$

$= \int 2 - \frac{2}{u+1}\,du = 2u - 2\ln|u+1| + C$

$= \underline{\underline{2\sqrt{x} - 2\ln|\sqrt{x}+1| + C}}$

setter u = problemet

Alternativt:

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2(z-1)}{z}\, dz$$

$z = 1 + \sqrt{x}$

$\sqrt{x} = z - 1$

$x = (z-1)^2$

$dx = 2(z-1)\, dz$

$$= \int \left(2 - \frac{2}{z}\right) dz$$

$$= 2z - 2\ln|z| + C = 2(1+\sqrt{x}) - 2\ln 1 + \sqrt{x} + C$$

$$= 2 + 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$$

, kan slå 2 sammen med C

1a)b)c)g), 3a)b), 11, 23, 28

9,2,1 a) $\int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx$

$u = \sqrt{x}$

$x = u^2$

$dx = 2u\, du$

$\int \frac{\sin u}{u}\, 2u \qquad = \int \sin(u) \cdot u^{-1}$

$\int 2 \sin u$

$2 \cdot -\cos u + c$

$\underline{\underline{-2\cos\sqrt{x} + c}}$

b) $\int \frac{\sqrt{x}}{1+x}\, dx$

$u = \sqrt{x}$

$du = \frac{dx}{2\sqrt{x}}$

$dx = 2\sqrt{x}\, du$

$\int \frac{\sqrt{x}}{1+(\sqrt{x})^2}\, dx = \int \frac{u}{1+u^2}\, dx$

$\int \frac{u}{1+u^2}\, 2 \cdot u \cdot du = \int \frac{2u^2}{1+u^2}\, du$

$= 2\int \frac{u^2}{1+u^2}\, du = 2\int \frac{1+u^2-1}{1+u^2}\, du$

$= 2\int \frac{\cancel{1+u^2}}{\cancel{1+u^2}} - \frac{1}{1+u^2}$

$= 2\int 1 - \frac{1}{1+u^2} = 2 \cdot u - 2\arctan u$

$\underline{\underline{2\sqrt{x} - 2\arctan\sqrt{x}}}$

c) $\int \frac{x}{\sqrt{x}+1}\,dx = \int \frac{u^2}{u+1}\,2u$ $\qquad u=\sqrt{x}$

$x = u^2$

$= 3\int \frac{1+u^3-1}{u+1} = 3\int \frac{1+u^3}{1+u} - \frac{1}{u+1}$ $\qquad dx = 2u$

$= 3\int u^2 - \frac{1}{u+1} = 3\left(\frac{u^3}{3} - \arctan u\right)$

$= \sqrt{x}^3 - \arctan\sqrt{x} = x^{\frac{3}{2}} - \arctan x$

g) $\int \cos(\ln x)\,dx$ $\qquad u = \ln x$

$= \int \cos(u)\,e^u$ $\qquad e^u = e^{\ln x}$

Prøver delvis integrasjon $\qquad e^u = x$

$z = \cos(u)$, $y = e^u$ $\qquad dx = e^u$

$z' = -\sin(u)$ $\quad y' = e^u$

Delvis integrasjon

$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$

$\cos(u)\,e^u - \int -\sin(u)\,e^u$

$\cos(u)\,e^u - \sin u\,e^u - \int \cos(u)\,e^u = \int \cos(u)\,e^u \;/ + \int \cos(u)\,e^u$

$\cos(u)e^u - \sin(u)\,e^u = 2\int \cos u\,e^u \;/ :2$

$e^u(\cos(u) - \sin(u)) = \int \cos(u)\,e^u$

$e^{\ln x}(\cos(\ln x) - \sin(\ln x)) = \int \frac{\cos(\ln x)}{x}\cdot x$

$x(\cos(\ln x) - \sin(\ln x)) = \int \cos(\ln x)$

$x(\cos(\ln x) - \sin(\ln x)) + \int \cos(\ln x) = \int \cos(\ln x)$

$\frac{x(\cos(\ln x) - \sin(\ln x))}{2} = \frac{2\int \cos(\ln x)}{2}$

3) $\int_0^{\sqrt{2}} x e^{x^2} dx$ $\quad u$

2. run

1 a) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}$

$u = \sqrt{x}$

$x = u^2$

$dx = 2u$

$\int \frac{\sin u}{\cancel{x}} \cdot 2\cancel{x}$

$\int 2 \sin u$

$2 \int \sin u$

$-2 \cos u$

1 d) $\int \frac{e^x}{\sqrt{1 - e^{2x}}}$

$u = e^x$

$du = e^x$

$dx = \frac{1}{e^x}$

$= \int \frac{\cancel{u}}{\sqrt{1 - u^2}\,\cancel{x}}$

$= \int \frac{1}{\sqrt{1 - u^2}}$

$= \arcsin(u) + C$

$= \arcsin(e^x) + C$

e) $\int e^{\sqrt{x}}$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}$

$= \int e^u \, 2u$

$= 2\int u e^u$

$w = u \qquad v = e^u$

$w' = u' \qquad v' = e^u$

$= 2\left(u e^u - \int 1 \cdot e^u\right)$

$2(u e^u - e^u) + C$

$= 2e^u(u - 1) + C$

Setter inn $\sqrt{x}$ for u

$\frac{1}{3} \quad -\frac{2}{3}$

$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$

f) $\int \sin(\sqrt[3]{x})\, dx$

$u = \sqrt[3]{x} \qquad u' = \frac{1}{3x^{2/3}}$

$u = x^{\frac{1}{3}} \qquad dx = 3x^{2/3}$

$\int \sin(u) \cdot u^2$

$w = u^2 \qquad v = -\cos(u)$

$w' = 2u \qquad v' = \sin(u)$

$-3u^2 \cos u + 6\int u \cos(u)$

$F = u \qquad G = \sin(u)$

$F' = 1 \qquad G' = \cos(u)$

$-3u^2 \cos(u) + 6\left(u \sin(u) - \int 1 \sin(u)\right)$

$-3u^2 \cos(u) + 6(u \sin(u) + \cos u)$

$-3u^2 \cos u + 6u \sin u + 6\cos u$

Setter inn $\sqrt[3]{x}$

$-3x^{2/3} \cos \sqrt[3]{x} + 6\sqrt[3]{x} \sin \sqrt[3]{x} + 6 \cos \sqrt[3]{x}$

g) $\int \cos(\ln(x))\,dx = \int 1 \cdot \cos(\ln x)$

$e^{\ln x} = e^{x}$

$\int x \cos u\, du$

$u = \ln(x)$

$\boxed{u' = \frac{du}{dx}}$

$x \sin u - \int \cos u$

$e^{\ln x} = x$

$\frac{1}{x} = \frac{du}{dx} \Rightarrow du = \frac{1}{x} dx$

$x \sin u + \sin u$

$\frac{dx}{x}$

$du = \frac{1}{x} dx$

$x\,du = dx$

g) $\int \cos(\ln(x))$

$u = \ln(x)$

$x = e^{u}$

$dx = e^{u}$

$\boxed{\int e^{u} \cos(u)\, du}$

$w = \sin u \qquad v' = e^{u}$

$w' = \cos u \qquad v = e^{u}$

$e^{u} \cos u + \int e^{u} \sin u\, du$

$\left[e^{u} \cos u \right] + e^{u} \sin u - \int e^{u} \cos u\, du$

$2I = e^{u} \cos u + \sin u$

$I = \frac{e^{u} \cos u + \sin u}{2}$

$= \frac{x \cos(\ln x) + \sin x}{2}$