1) Stokastisk variabel X har sannsynlighetstetthet 2015

$$f_X(x) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x>0 \\ 0 & \text{ellers} \end{cases}$$

anta at $X \sim \text{Gama}(\alpha, \beta)$

Gammafunksjonen

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} e^{-u}\, du$$

a) Vis $M_X(t)$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx = \int_0^{\infty} e^{tx} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dt$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(1-\beta t)x/\beta}\, dt$$

substitusjon $u = (1-\beta t)x/\beta$

$x = \left(\frac{\beta}{1-\beta t} u\right)$ For at dette skal gå opp

$1-\beta t > 0$

$t < \frac{1}{\beta}$

Dermed

$$M_X(t) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} \left(\frac{\beta}{1-\beta t}\right)^{\alpha-1} e^{u} \left(\frac{\beta}{1-\beta t} du\right)$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^{\alpha} \underbrace{\int_0^{\infty} u^{\alpha-1} e^{-u}\, du}_{\Gamma(\alpha)}$$

$$= \frac{1^{\alpha}}{(1-\beta^{\alpha})} = \frac{1}{1-\beta t^{\alpha}}$$

Gjelder når $t < 1/\beta$. Så er $M_X(t)$ er definert i et åpent intervall som inneholder 0

b) Finn $E(x)$ og $V(x)$

Har generelt

$$M'_X(t) = \frac{d}{dt} E(e^{tx}) = E(\frac{d}{dt} e^{tx}) = E(x e^{tx})$$

Tilsvarende

$$M''_X(t) = \frac{d}{dt} M_X'(t) = E(\frac{d}{dt} x e^{tx}) = E(x^2 e^{tx})$$

Herav

$$M'_X(0) = E(x) \qquad M''_X(0) = E(x^2)$$

Har for gamma fordelingen

$$M_X(t) = \frac{1}{(1-\beta t)^\alpha} = (1-\beta t)^{-\alpha}$$

Finner

$$M'_\alpha(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta) = \alpha\beta(1-\beta t)^{-\alpha-1}$$

$$M''_\alpha(t) = \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha-2}$$

$$E(x) = M'_\alpha(0) = \alpha\beta$$

$$E(x^2) = M''_X(0) = \alpha\beta^2(\alpha+1)$$

$$V(x) = E(x^2) - [E\,x]^2 = \alpha\beta^2$$

c) X_1, X_2 uavhengige

$$X_i \sim \text{Gamma}(\alpha_i, \beta) \qquad i = 1,2$$

Vise at $X_1 + X_2 \sim \text{gamma}(\alpha_1 + \alpha_2, \beta)$

Bruker at momentgenererende funksjon bestemer entydig fordelingen

Så holder å vise at

$$M_{X_1+X_2}(t) = \frac{1}{(1-\beta t)^{\alpha_1+\alpha_2}}$$

Finner

$$M_{x_1+x_2}(t) = E\left(e^{t(x_1+x_2)}\right) = E\left(\underbrace{e^{tx_1}}_{h_1(x_1)} \underbrace{e^{tx_2}}_{h_2(x_2)}\right)$$

$$= E\left(e^{tx_1}\right) E\left(x^{e\alpha_2}\right)$$

$$= M_{x_1}(t)\, M_{x_2}(t)$$

$$= \frac{1}{(1-\beta t)^{\alpha_1}} \cdot \frac{1}{(1-\beta t)^{\alpha_2}} = \frac{1}{(1-\beta t)^{\alpha_1+\alpha_2}}$$

Som vi ville vise

Hvis X_1, X_2, X_3 uavh

$x_i \sim \text{gamma}(\alpha_i, \beta) \quad i=1,2,3$

$Y = X_1 + X_2 \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

Ved det vi har vist

$X_1 + X_2 + X_3 = Y + X_3 \sim \text{Gamma}(\alpha_1+\alpha_2+\alpha_3, \beta)$

Ved resultatet vi har vist

2

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{ellers} \end{cases}$$

a) finn kumulativ fordelingen

$$F_{T}(t) \int_0^t \lambda e^{-\lambda t} dt = -\lambda \left[e^{-\lambda t} \right]_0^t = 1 - e^{-\lambda t}$$

b) median

For å finne median

$$F_T(m) = 0.5$$

$$1 - e^{-\lambda m} = 0.5$$

$$e^{-\lambda m} = 0.5$$

$$-\lambda m = \ln(1/2) = -\ln(2)$$

$$m = \frac{\ln(2)}{\lambda}$$

Har at $P(T < m) = 1/2 = P(T > m)$, dvs. like stor sjanse for tilbakefall før og etter m

c) $T_1, \dots, T_n$ tid til tilbakefall

Vi vil estimere m = medianen $= \frac{\ln(2)}{\lambda}$

Estimator

$$\hat{m} = \frac{\ln(2)}{n} \sum_{i=1}^{n} T_i = \ln(2) \cdot \bar{T} \qquad \bar{T} = \frac{1}{n} \sum T_i$$

Har at tettheten til T_i-ene er gitt ved

$$f_T(t) = \lambda e^{-\lambda t} = \frac{1}{(1/\lambda)^{\alpha} \Gamma(1)} t^{1-1} e^{-t/(1/\lambda)}$$

Ser at T_i-ene er gamma lik $(\underset{\alpha}{1}, \underset{\beta}{1/\lambda})$

Har fra oppgave 1b)

$$E(T_i) = \alpha\beta = 1 \cdot 1/\lambda = 1/\lambda$$

$$V(T_i) = \alpha\beta^2 = 1 \cdot (1/\lambda)^2 = 1/\lambda^2$$

Finner dermed at

$$E(\hat{m}) = E\left(\frac{\ln(2)}{n}\sum T_i\right) = \frac{\ln(2)}{n}\sum_{i=1}^{n} E(T_i)$$

$$= \frac{\ln(2)}{n} \cdot \frac{n}{\lambda} = \frac{\ln(2)}{\lambda}$$

fra oppg (2 a)

↓

dermed forventningsrett

Finner standardfeil

$$V(\hat{m}) = V\left(\frac{\ln 2}{n}\sum T_i\right) = \left(\frac{\ln(2)}{n}\right)^2 \sum_{i=1}^{n} \underbrace{V(T_i)}_{1/\lambda^2}$$

$$= \left(\frac{\ln(2)}{n}\right)^2 \frac{n}{\lambda^2} = \frac{\ln(2)^2}{n\lambda^2}$$

$$= \frac{m^2}{\lambda}$$ - fordi $\frac{\ln(2)}{\lambda} = m$